Topology of cyclic configuration spaces and periodic trajectories of multi-dimensional billiards

Michael Farber * and Serge Tabachnikov**
School of Mathematical Sciences, Tel Aviv University
Ramat Aviv, Tel Aviv 69978, Israel
and
Department of Mathematics, Penn State University
University Park, PA 16802, USA
e-mail: *farber@math.tau.ac.il, tabachni@math.psu.edu*

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Abstract

We give lower bounds on the number of periodic trajectories in strictly convex smooth billiards in $\mathbf{R}^{m+1}$ for $m \geq 3$. For plane billiards (when $m = 1$) such bounds were obtained by G. Birkhoff in the 1920's. Our proof is based on topological methods of calculus of variations – equivariant Morse and Lusternik-Schnirelman theories. We compute the equivariant cohomology ring of the cyclic configuration space of the sphere S^m, i.e., the space of n-tuples of points $(x_1, \dots, x_n)$, where $x_i \in S^m$ and $x_i \neq x_{i+1}$ for $i = 1, \dots, n$.

Keywords: mathematical billiards, Morse and Lusternik-Schnirelman theories, cyclic configuration space, equivariant cohomology

1 Introduction

The billiard dynamical system describes the free motion of a mass-point in a domain in Euclidean space with a reflecting boundary: a point moves along a straight line with unit speed until it hits the boundary, at the impact point the normal component of the velocity instantaneously changes sign while the tangential component remains the same, and the rectilinear motion continues with a new unit velocity. We refer to [16], [19] for surveys of mathematical billiards.

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We address the following problem: how many periodic billiard trajectories are there in a smooth strictly convex domain in $\mathbf{R}^{m+1}$?

For plane billiards ($m = 1$) this problem was studied by G. Birkhoff in [4]. A closed billiard orbit of a plane billiard is an inscribed plane n-gon whose consecutive sides make equal angles with a closed convex plane curve X. Such a periodic orbit has a rotation number $0 < r < n/2$. Birkhoff proved that *for every $n \geq 2$ and $r \leq n/2$, coprime with n, there exist at least two distinct n-periodic billiard trajectories with the rotation number r* – see [4]. From contemporary view point, this result follows from the theory of area-preserving twist maps – see, e.g., [3] for a survey.

In the present paper we use a well-known variational reduction of the periodic billiard trajectories problem which is based on the observation that these trajectories are critical points of the perimeter length functional on the variety of inscribed polygons. This allows to apply topological methods (Morse and Lusternik-Schnirelman theories) and reduces the problem to studying the topology of *the cyclic configuration space.* Inscribed n-gons are in one-to-one correspondence with sequences

$$(x_1, x_2, \ldots, x_n) \in X^{\times n} = X \times \ldots \times X,$$

such that

$$x_1 \neq x_2,\ x_2 \neq x_3,\ \ldots, x_n \neq x_1;$$

here $X \subset \mathbf{R}^{m+1}$ denotes the boundary of the billiard domain, topologically, the sphere. The variety of all such configurations is the *cyclic configuration space.* Our main effort in this paper is in studying the cohomology ring of this configuration space. Note that the length functional, considered as a map $X^{\times n} \to \mathbf{R}$ on the total Cartesian power of X, fails to be smooth at the points with $x_i = x_{i+1}$ for some i. This explains why the cyclic configuration space is the natural topological object related to the billiard problem.

The dihedral group D_n acts on the cyclic configuration space; its action is generated by the cyclic permutation and the reflection

$$(x_1, x_2, \ldots, x_n) \mapsto (x_2, x_3, \ldots, x_n, x_1), \quad (x_1, x_2, \ldots, x_n) \mapsto (x_n, x_{n-1}, \ldots, x_1).$$

Two n-periodic billiard trajectories will be considered the same if they belong to the same orbit of D_n. In particular, the estimates in Theorem 1 below concern the number of distinct D_n-orbits in the cyclic configuration space.

Our main result is as follows.

Theorem 1 *Let $X \subset \mathbf{R}^{m+1}$ be a smooth strictly convex hypersurface, where $m \geq 3$. Fix an odd number $n \geq 3$. Then*

(A) The number of distinct D_n-orbits of n-periodic billiard trajectories inside X is not less than

$$[\log_2(n-1)] + m. \tag{1}$$

(B) For a generic X, the number of distinct D_n-orbits of n-periodic billiard trajectories inside X is not less than

$$(n-1)m. \tag{2}$$

Square brackets in (1) denote the floor function, i,e., the largest integer not exceeding $\log_2(n-1)$.

We deduce statement (A) of Theorem 1 from Lusternik-Schnirelman theory and statement (B) from Morse theory; this explains the genericity assumption in (B) (precise definition of the needed genericity is given in Section 4).

We conjecture that the estimate in case (A) can be improved to be $n+m-2$.

It is important to note that if the period n in Theorem 1 is not prime then n-periodic billiard trajectories, whose existence is asserted by this theorem, may be multiple ones; for example, instead of a genuine 9-periodic orbit one might have a 3-periodic orbit, traversed three times.

To the best of our knowledge, the only previous attempt to extend Birkhoff's results to multi-dimensional setting was made by I. Babenko in [1]. He applied the variational approach to periodic trajectories of convex billiards in 3-dimensional space. Unfortunately, the paper [1] contains an error (cf. Remark 3.4), and the situation with $m=2$ remains unclear. We would like to emphasize that ideas and results from [1] are substantially used in the present paper.

We would like to mention, in passing, the case of 2-periodic billiard trajectories. These are the diameters of the billiard hypersurface $X \subset \mathbf{R}^{m+1}$, that is, the chords, perpendicular to the hypersurface at both ends. It is well known that the least number of diameters is $m+1$, and this estimate is sharp as the example of a generic ellipsoid shows.

The arguments that prove Theorem 1 apply to a much wider class of billiards. Namely, let X be a smooth strictly geodesically convex billiard hypersurface in a Riemannian manifold such that there exists a unique geodesic line through every two points of X. For example, this condition holds when the ambient manifold is a spherical or hyperbolic space or, more generally, a Hadamard space.

Theorem 2 *For X as in the preceding paragraph, both statements of Theorem 1 hold true, provided* $\dim X \geq 3$.

The content of the paper is, briefly, as follows. In Section 2 we construct a general spectral sequence that computes the cohomology ring of the cyclic configuration space of a smooth manifold. In Section 3 we use it to compute the cohomology ring of the cyclic configuration space of the sphere. Section 4 is Morse theoretical: we describe the relations (mostly known) between the information about the periodic billiard trajectories and the topology of the cyclic configuration space of the billiard hypersurface. In the last Section 5 we compute the D_n-equivariant cohomology of the cyclic configuration space of the sphere; the computation makes use of the topological results from

Section 3 and the Morse-theoretical ones from Section 4. We finish the paper with a proof of our main result, Theorems 1 and 2.

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2 Cyclic configuration space of a smooth manifold

Let X be a smooth manifold. Consider the configurations of ordered n-tuples $(x_1, x_2, \ldots, x_n)$ of points of X satisfying:

$$x_i \neq x_{i+1} \quad \text{for} \quad i = 1, 2, \ldots, n. \tag{3}$$

Here and elsewhere below we understand the indices cyclically, that is $n + i = i$ (in particular, (3) contains the requirement $x_n \neq x_1$). The space of all such configurations will be called the *cyclic configuration space* and denoted by $G(X, n)$. The dihedral group D_n acts naturally on $G(X, n)$ (cf. above); this action is free if n is an odd prime.

Example 2.1 The simplest example of a cyclic configuration space is provided by $G(S^1, n)$, the cyclic configuration space of a circle, which plays an important role in Birkhoff's theory of convex plane billiards [4]. A configuration $(x_1, x_2, \ldots, x_n) \in G(S^1, n)$ can be uniquely described by the initial point $x_1 \in S^1$ and by the angles $0 < \phi_i < 1$, where $i = 1, 2, \ldots, n-1$, assuming that the sum $\phi_1 + \phi_2 + \ldots + \phi_{n-1}$ is not an integer. Namely, given x_1 and $\phi_1, \ldots, \phi_{n-1}$ we set

$$x_2 = x_1 \exp(2\pi i \phi_1),\ x_3 = x_2 \exp(2\pi i \phi_2),\ \ldots, x_n = x_{n-1} \exp(2\pi i \phi_{n-1}).$$

Here we identify S^1 with the unit circle in $\mathbf{C}$. The hyperplanes

$$\phi_1 + \phi_2 + \ldots + \phi_{n-1} = r, \quad \text{where} \quad r = 1, 2, \ldots, n-2,$$

divide the cube $(0, 1)^{n-1}$ into domains $N_1, N_2, \ldots, N_{n-1}$, where

$$N_r = \{(\phi_1, \phi_2, \ldots, \phi_{n-1}) : r - 1 < \sum_{i=1}^{n-1} \phi_i < r\}.$$

Each N_r is homeomorphic to $\mathbf{R}^{n-1}$. It follows that the cyclic configuration space $G(S^1, n)$ is homeomorphic to a disjoint union of $S^1 \times N_r$, where $r = 1, 2, \ldots, n-1$. Note that the number r, determining the connected component of a given configuration, is precisely the rotation number. Let us also indicate how the dihedral group D_n acts on $G(S^1, n)$. Given a point $(x_1, \phi_1, \ldots, \phi_{n-1}) \in S^1 \times N_r$, the cyclic permutation takes it to

$$(x_1 \exp(2\pi i \phi_1), \phi_2, \ldots, \phi_{n-1}, r - \sum_{i=1}^{n-1} \phi_i).$$

The reflection maps it to

$$(x_1, 1 - r + \sum_{i=1}^{n-1} \phi_i, 1 - \phi_{n-1}, 1 - \phi_{n-2}, \dots, 1 - \phi_2).$$

Hence the cyclic permutation preserves the rotation number, and the reflection maps configurations with rotation number r to configurations with rotation number $n - r$.

The cyclic configuration space $G(X, n)$ is not to be confused with the usual configuration space

$$F(X, n) = \{(x_1, \dots, x_n) \in X^{\times n} : x_i \neq x_j \quad \text{for} \quad i \neq j\}.$$

Recall a description of the cohomology ring $H^*(F(\mathbf{R}^m, n); \mathbf{Z})$, obtained by F. Cohen – see [10], [11]. The space $F(\mathbf{R}^m, 2)$ is homotopy equivalent to $m-1$-dimensional sphere; denote by ω its top-dimensional cohomology class. Let p_{ij} be the projection $F(\mathbf{R}^m, n) \to F(\mathbf{R}^m, 2)$ on the i-th and j-th components, and let $G_{i,j} = p_{ij}^*(\omega)$. Then the ring $H^*(F(\mathbf{R}^m, n); \mathbf{Z})$, where $m > 1$, is the graded-commutative algebra over $\mathbf{Z}$ with generators

$$G_{i,j}, \ 1 \leq i, j \leq n, \ i \neq j, \quad \deg G_{i,j} = m - 1$$

and relations

(a) $G_{i,j} = (-1)^m G_{j,i}$,

(b) $G_{i,j}^2 = 0$,

(c) $G_{i,j} G_{i,k} + G_{j,k} G_{j,i} + G_{k,i} G_{k,j} = 0$ for i, j, k distinct.

An additive basis for $H^{r(m-1)}(F(\mathbf{R}^m, n); \mathbf{Z})$ is given by monomials $G_{i_1,j_1} G_{i_2,j_2} \dots G_{i_r,j_r}$ with $i_1 < \dots < i_r$ and $i_k < j_k$ for all k.

We start the study of topology of cyclic configuration spaces with the space $G(\mathbf{R}^m, n)$. The following proposition is an analog of Cohen's result.

Proposition 2.2 *For $m > 1$ the ring $H^*(G(\mathbf{R}^m, n); \mathbf{Z})$ is a graded commutative algebra over $\mathbf{Z}$ with generators*

$$s_1, s_2, \dots, s_n, \quad \textit{with} \quad \deg s_i = m - 1$$

and the relations

$$s_1^2 = s_2^2 = \dots = s_n^2 = 0,$$

$$s_1 s_2 \dots s_{n-1} + \epsilon s_2 s_3 \dots s_n + \epsilon^2 s_3 s_4 \dots s_n s_1 + \dots + \epsilon^{n-1} s_n s_1 \dots s_{n-2} = 0, \tag{4}$$

where $\epsilon = (-1)^{(m-1)(n-1)}$. If ϕ denotes the inclusion of $F(\mathbf{R}^m, n)$ into $G(\mathbf{R}^m, n)$, then $\phi^*(s_i) = G_{i,i+1}$.

Proof. In general, the cohomology groups of the complement $\mathbf{R}^N - \cup V_i$ of a collection of affine subspaces $V_i \subset \mathbf{R}^N$, where $i = 1, \ldots, k$, is described in [15]. Recall this description, following [21], chapter 3, §6. For $I \subset \{1, \ldots, k\}$ let

$$V_I = \cap_{i \in I} V_i.$$

Two subsets $I, J \subset \{1, \ldots, k\}$ are equivalent, $I \sim J$, if $V_I = V_J$. Each equivalence class has a unique maximal element. Given such a maximal element I, consider the standard chain complex of the simplex Δ_I spanned by I. Factorize this complex by the subcomplex generated by the faces Δ_J with $J \subset I, J \nsim I$, and denote the resulting quotient complex by $C(I)$. Then the cohomology of the complement $\mathbf{R}^N - \cup V_i$ is given by

$$\tilde{H}^q(\mathbf{R}^N - \cup V_i; \mathbf{Z}) = \oplus \tilde{H}_{N - \dim\ V_I - q - 1}(C(I); \mathbf{Z}),$$

the sum taken over all the equivalence classes with maximal elements I, such that V_I are nonempty.

In our situation, $\mathbf{R}^N = (\mathbf{R}^m)^{\times n}$, and $V_i = \{x_i = x_{i+1}\}$ for $i = 1, 2, \ldots, n$. If $|I| \leq n-2$ then the equivalence class of I consists of one element only; on the other hand, all subsets of cardinality $n-1$ are equivalent to each other and to $\{1, \ldots, n\}$. Therefore, if $|I| \leq n-2$ then

$$\tilde{H}_*(C(I); \mathbf{Z}) = H_*(\Delta_I, \partial\Delta_I; \mathbf{Z}) = \tilde{H}_*(S^{|I|-1}; \mathbf{Z}),$$

and each subset I with $s = |I| \leq n-2$ makes a contribution of a copy of $\mathbf{Z}$ to $H^{s(m-1)}(G(\mathbf{R}^m, n); \mathbf{Z})$. Similarly, if $I = \{1, \ldots, n\}$ then

$$\tilde{H}_*(C(I); \mathbf{Z}) = H_*(\Delta_I, sk_{n-2}(\Delta_I); \mathbf{Z}) = \tilde{H}_*(\vee_{i=1}^{n-1} S_i^{n-1}; \mathbf{Z}),$$

(where $sk_{n-2}(\Delta_I)$ denotes the $(n-2)$-dimensional skeleton of the simplex Δ_I) and I makes a contribution of $\mathbf{Z}^{n-1}$ to $H^{(n-1)(m-1)}(G(\mathbf{R}^m, n); \mathbf{Z})$. This implies that $H^*(G(\mathbf{R}^m, n); \mathbf{Z})$ is free abelian of rank

$$\text{rk } H^{s(m-1)}(G(\mathbf{R}^m, n); \mathbf{Z}) = \begin{cases} \binom{n}{s}, & \text{for} \quad 0 \leq s \leq n-2, \\ \\ n-1, & \text{for} \quad s = n-1 \end{cases}$$

Hence, the Poincaré polynomial of the cyclic configuration space $G(\mathbf{R}^m, n)$ equals

$$(t^{m-1} + 1)^n - t^{(n-1)(m-1)} - t^{n(m-1)}.$$

For $i = 1, \ldots, n$ denote by s_i the generators of $H^{m-1}(G(\mathbf{R}^m, n); \mathbf{Z})$ coming from 1-element sets $\{i\}$. It is clear that s_i is the pull-back of the top-dimensional class of $G(\mathbf{R}^m, 2) = F(\mathbf{R}^m, 2) \approx S^{m-1}$ under the projection $p_{i,i+1}$. Therefore $s_i^2 = 0$ and $\phi^*(s_i) = G_{i,i+1}$.

It remains to show that an additive basis for $H^{r(m-1)}(G(\mathbf{R}^m, n); \mathbf{Z})$ with $r \leq n-1$, is given by the monomials $s_{i_1} \ldots s_{i_r}$, where $i_1 < \ldots < i_r$, with one additional relation (4).

Denote by $S = S^{m-1} \subset \mathbf{R}^m$ the unit sphere and let $S^{\times(n-1)}$ be the Cartesian power of S. For any $p = 1, 2, \dots, n$ we construct an embedding

$$g_p : S^{\times(n-1)} \to G(\mathbf{R}^m, n), \quad (y_1, y_2, \dots, y_{n-1}) \mapsto (x_1, x_2, \dots, x_n),$$

where

$$\begin{aligned}
&x_p = 0,\\
&x_{p+1} = y_1,\\
&x_{p+2} = y_1 + y_2,\\
&\dots\\
&x_n = y_1 + y_2 + \ldots + y_{n-p}\\
&x_1 = y_1 + y_2 + \ldots + y_{n-p} + y_{n-p+1}\\
&\dots\\
&x_{p-2} = y_1 + y_2 + \ldots + y_{n-2}\\
&x_{p-1} = y_1 + y_2 + \ldots + y_{n-2} + ny_{n-1}.
\end{aligned}$$

If $\overline{s}_i \in H^{m-1}(S^{\times(n-1)}; \mathbf{Z})$ denotes the obvious generator corresponding to the i-th factor (where $i = 1, 2, \dots, n-1$), we have

$$g_p^*(s_i) = \begin{cases} \overline{s}_{n-p+1+i}, & \text{for} \quad 1 \le i \le p-2, \\ \overline{s}_{i-p+1}, & \text{for} \quad p \le i \le n, \\ (-1)^m \overline{s}_{n-1}, & \text{for} \quad i = p-1. \end{cases}$$

All but the last relation being obvious, let us explain the relation $g_p^*(s_{p-1}) = (-1)^m \overline{s}_{n-1}$. Consider the homotopy $F_t : S^{\times(n-1)} \to S$, where $t \in [0,1]$, given by

$$F_t(y_1, \dots, y_{n-1}) = \frac{ty_1 + \ldots + ty_{n-2} + ny_{n-1}}{||ty_1 + \ldots + ty_{n-2} + ny_{n-1}||}, \quad y_i \in S.$$

It is clear from the definitions that $F_1^*(v) = (-1)^m g_p^*(s_{p-1})$, where $v \in H^{m-1}(S; \mathbf{Z})$ is the generator. On the other hand, the map F_0 is just the projection $(y_1, \dots, y_{n-1}) \mapsto y_{n-1}$ and hence $F_0^*(v) = \overline{s}_{n-1}$.

Let $a_i \in H^{(n-1)(m-1)}(G(\mathbf{R}^m, n); \mathbf{Z})$ denote the product $s_{i+1}s_{i+2} \dots s_n s_1 \dots s_{i-1}$. Then we obtain

$$g_p^*(a_i) = \begin{cases} \overline{s}_1 \overline{s}_2 \dots \overline{s}_{n-1}, & \text{for} \quad i = p-1, \\ -\epsilon \overline{s}_1 \overline{s}_2 \dots \overline{s}_{n-1}, & \text{for} \quad i = p-2, \\ 0, & \text{for} \quad i \ne p-2,\, p-1, \end{cases}$$

where $\epsilon = (-1)^{(m-1)(n-1)}$. Varying p, this shows that all a_i are nonzero and also that for any nontrivial relation $\sum \beta_i a_i$ in $H^{(n-1)(m-1)}(G(\mathbf{R}^m, n); \mathbf{Z})$ one must have $\beta_{i+1} = \epsilon \beta_i$. In other words, any relation of degree $n-1$ between the classes s_i must be a consequence of (4). The rank calculation above shows that at least one nontrivial relation between the classes $a_1, \dots, a_n$ exists, and so (4) holds.

Let us finally show that there exist no nontrivial relations between $s_1, \dots, s_n$ of degree $r < n-1$. Assume that there exists such a relation x containing a monomial $s_{i_1} s_{i_2} \dots s_{i_r}$. Choose a set of indices $\{j_1, \dots, j_{n-r-1}\}$, disjoint from the set $\{i_1, \dots, i_r\}$, and let $y = s_{j_1} \dots s_{j_{n-r-1}}$. Then xy is a relation of degree $n-1$, containing neither of the terms $a_{j_1}, \dots, a_{j_{n-r-1}}$; we have shown above that it is impossible. This completes the proof. □

Remark 2.3 The cyclic configuration space $G(\mathbf{R}^m, 3)$ coincides with $F(\mathbf{R}^m, 3)$. In this case relation (4) turns into F. Cohen's relation (c) in the cohomology $H^*(F(\mathbf{R}^m, 3); \mathbf{Z})$ after substituting $s_i = G_{i,i+1}$.

Next, we construct a spectral sequence computing the cohomology of the cyclic configuration space $G(X, n)$, where X is an arbitrary smooth orientable manifold. This spectral sequence is an analog of the one constructed by Totaro in [20] for the configuration space $F(X, n)$.

Denote by $p_j : X^{\times n} \to X$ the projection on the j-th component and by $q_j = p_{j,j+1}$ the projection $X^{\times n} \to X^{\times 2}$ on the j-th and $j+1$-th components. Also, let $\Delta \in H^m(X^{\times 2}; \mathbf{k})$ denote the cohomology class of the diagonal; here $m = \dim X$ and $\mathbf{k}$ is a field.

Theorem 3 *Let X be a smooth orientable manifold of dimension $m > 1$; let $\mathbf{k}$ be a field.*

(A) There exists a spectral sequence of bigraded differential algebras over $\mathbf{k}$ which converges to $H^*(G(X, n); \mathbf{k})$ whose E_2-term is the quotient of the bigraded commutative algebra

$$H^*(X^{\times n}; \mathbf{k}) \otimes H^*(G(\mathbf{R}^m, n); \mathbf{k}),$$

where $H^p(X^{\times n}; \mathbf{k})$ has bidegree $(p, 0)$ and $H^q(G(\mathbf{R}^m, n); \mathbf{k})$ has bidegree $(0, q)$, by the relations

$$p_i^*(v)s_i = p_{i+1}^*(v)s_i, \;\; i = 1, \ldots, n,$$

where $v \in H^*(X; \mathbf{k})$ is an arbitrary class.

(B) The dihedral group D_n acts on all terms of the spectral sequence E_r, $r \geq 2$. The group D_n commutes with the differentials and its action on E_{r+1} is determined by its action on E_r; the action of D_n on $H^*(G(X, n); \mathbf{k})$ is compatible with the action on E_∞. The action of the dihedral group D_n on the initial term E_2 is given by the action of D_n on $H^*(X^{\times n}; \mathbf{k})$, induced by its obvious action on $X^{\times n}$, and by

$$\tau^*(s_{i+1}) = (-1)^m s_{\tau(i)}, \qquad \tau^*(s_1) = (-1)^m s_{\tau(n)},$$

where $\tau \in D_n$ is a reflection and $i = 1, \ldots, n-1$.

(C) The first non-trivial differential is d_m, where $m = \dim X$. It is defined by the formulas

$$d_m s_i = q_i^*(\Delta) \quad \text{and} \quad d_m(H^*(X^{\times n}; \mathbf{k})) = 0.$$

Here is a comment concerning statement (B). We identify the dihedral group D_n with a subgroup of permutations of the set $\{1, 2, \ldots .n\}$. An element $\tau \in D_n$ is a *reflection* if it reverses the cyclic order of the set $\{1, 2, \ldots, n\}$. In other words, $\tau \in D_n$ is a reflection if for any $i = 1, 2, \ldots, n-1$ holds

$$\tau(i+1) = \tau(i) - 1 \bmod n \quad \text{and} \quad \tau(1) = \tau(n) - 1 \bmod n.$$

D_n contains n reflections, which generate D_n. Hence, knowing the action of reflections completely describes the action of D_n on E_2.

Proof. The proof is a modification of the arguments by B. Totaro in [20].

Consider the inclusion $\psi : G(X,n) \to X^{\times n}$ and the Leray spectral sequence of the continuous map ψ:

$$E_2^{p,q} = H^p(X^{\times n}; R^q\psi_*\mathbf{k}) \Rightarrow H^{p+q}(G(X,n);\mathbf{k})$$

where $R^q\psi_*\mathbf{k}$ is the sheaf on $X^{\times n}$ associated with the presheaf

$$U \mapsto H^q(U \cap G(X,n); \mathbf{k}).$$

We will consider partitions of the set of indices $\{1,\dots,n\}$ into intervals, that is, subsets of the form $\{i, i+1, i+2, \dots, i+j\}$; as usual, the indices are understood cyclically. For example, the following are interval partitions of the set $\{1,2,3,4,5\}$:

$$\{1,2\} \cup \{3,4,5\}, \quad \{2\} \cup \{3\} \cup \{4,5,1\}.$$

Let J be such a partition; denote by X_J the subset of $X^{\times n}$ consisting of configurations $c = (x_1, x_2, \dots, x_n) \in X^{\times n}$ satisfying the conditions $x_i = x_j$ if i and j lie in the same interval of the partition J. Given two interval partitions I and J, we say that J *refines* I and write $I \prec J$ if the intervals of I are unions of the intervals of J. Denote by $|J|$ the number of intervals in the partition J. X_J is naturally identified with the Cartesian power $X^{\times|J|}$. The case $|J| = 1$ corresponds to the deepest diagonal $\{(x,x,\dots,x)\} \subset X^{\times n}$. Note that $I \prec J$ implies $X_I \subset X_J$ and $|I| \le |J|$.

Denote by $D(X,n)$ the subset of $X^{\times n}$ satisfying the conditions $x_i \ne x_{i+1}$ for $i = 1,\dots,n-1$, but not necessarily for $i = n$. It is clear that $D(\mathbf{R}^m, n)$ is homotopy equivalent to the product $(S^{m-1})^{\times(n-1)}$ and so its cohomology admits a description similar to Proposition 2.2, but without relation (4).

Let J be a partition of $\{1,2,\dots,n\}$ on intervals of lengths $j_1, \dots, j_r$ as above, and let c be a configuration with $c \in X_J$ such that c does not belong to any X_I if J refines I. We claim that the stalk of the sheaf $R^q\psi_*\mathbf{k}$ at c equals

$$(R^q\psi_*\mathbf{k})_c = \begin{cases} H^q(D(\mathbf{R}^m, j_1) \times \ldots \times D(\mathbf{R}^m, j_r); \mathbf{k}) & \text{if} \quad r \ge 2 \\ \\ H^q(G(\mathbf{R}^m, n); \mathbf{k}) \quad \text{if} \quad r = 1. \end{cases}$$

Indeed, by definition, this stalk is $H^q(U \cap G(X,n); \mathbf{k})$, where U is a small open ball around c. If $c = (x_1, x_2, \dots x_n)$ then we may choose points $y_1, y_2, \dots, y_r \in X$, one for each interval of J, so that $x_i = y_{j_s}$ if i belongs to the j_s-th interval. Let $U_j \subset X$ be a small open neighborhood of y_j, so that each U_j is diffeomorphic to $\mathbf{R}^m$ and the sets U_j and $U_{j'}$ are disjoint when the points y_j and $y_{j'}$ are distinct. Then we may take $U = U_1^{\times j_1} \times U_2^{\times j_2} \times \ldots \times U_r^{\times j_r}$, and our claim follows.

In particular, we see that $R^q\psi_*\mathbf{k}$ vanishes unless q is a multiple of $m-1$.

If $|J| \ge 2$ then

$$(R^*\psi_*\mathbf{k})_c = H^*((S^{m-1})^{\times(n-|J|)}; \mathbf{k}),$$

and hence

$$\dim\,(R^{s(m-1)}\psi_*\mathbf{k})_c = \binom{n-|J|}{s}.$$

Note that there is a canonical top-dimensional class of dimension $(m-1)(n-|J|)$, which is uniquely determined by a choice of an orientation of the sphere S^{m-1}.

If $|J| = 1$ (which means that X_J is the deepest diagonal) then, by Proposition 2.2,

$$\dim\,(R^{r(m-1)}\psi_*\mathbf{k})_c = \begin{cases} \binom{n}{r} & \text{for} \quad r \leq n-2, \\ \\ n-1 & \text{for} \quad r = n-1, \end{cases}$$

and hence there exists $(n-1)$-dimensional space of top-dimensional cohomology classes; they have dimension $(n-1)(m-1)$. Note that the sheaf $R^{(n-1)(m-1)}\psi_*\mathbf{k}$ is a constant sheaf $\mathbf{k}^{n-1}$ with support on the deepest diagonal X_J. This follows repeating the arguments of Totaro [20], p. 1062. The fact that this sheaf is locally constant is easy since this sheaf can be represented as sheaf of cohomology of fibers of a fibration over X_J. A further statement that this sheaf is constant (i.e., has a trivial monodromy) follows from explicit calculation of the cohomology $H^*(G(\mathbf{R}^m, n); \mathbf{k})$ (which serves as the cohomology of the fiber), since we may explicitly label the cohomology classes by polynomials in $s_1, \dots, s_n$.

We wish to have a similar description for the sheaves $R^{r(m-1)}\psi_*\mathbf{k}$ with $r < n-1$. We will show that they can be represented as direct sums of constant sheaves supported on different diagonals X_J. For an interval partition J of $\{1, 2, \dots, n\}$ with $|J| > 1$ denote by ε_J the constant sheaf with stalk $\mathbf{k}$ and support X_J. We claim that *for $r < n-1$ the sheaf $R^{r(m-1)}\psi_*\mathbf{k}$ is isomorphic to the direct sum of sheaves*

$$R^{r(m-1)}\psi_*\mathbf{k} \simeq \bigoplus_{|J|=n-r} \varepsilon_J, \tag{5}$$

the sum taken over all interval partitions J with $|J| = n-r$.

To prove the above claim we first make the following remark. Let I be an interval partition of $\{1, 2, \dots, n\}$ into intervals of length $i_1, i_2, \dots, i_s$, where $s = |I| > 1$. Then for any interval partition J into intervals of length $j_1, j_2, \dots, j_{n-r}$, such that $I \prec J$, we have the canonical inclusion

$$\nu_{JI} : D(\mathbf{R}^m, i_1) \times \ldots \times D(\mathbf{R}^m, i_s) \to D(\mathbf{R}^m, j_1) \times \ldots \times D(\mathbf{R}^m, j_{n-r}).$$

Note that the target of ν_{JI} has a canonical $r(m-1)$-dimensional cohomology class (cf. above). The induced map on $r(m-1)$-dimensional cohomology with $\mathbf{k}$ coefficients

$$\nu_{JI}^* : H^{r(m-1)}(D(\mathbf{R}^m, j_1) \times \ldots \times D(\mathbf{R}^m, j_{n-r})) \to H^{r(m-1)}(D(\mathbf{R}^m, i_1) \times \ldots \times D(\mathbf{R}^m, i_s))$$

is a monomorphism. Let z_{JI} denote the image of the top-dimensional generator. Then we observe (similarly to Lemma 3 in [20]) that for a fixed I the classes $\{z_{JI}\}$ form a linear basis of the cohomology $H^{r(m-1)}(D(\mathbf{R}^m, i_1)\times \ldots \times D(\mathbf{R}^m, i_s);\mathbf{k})$, where J runs over all partitions with $I \prec J$ and $|J| = n - r$.

A similar statement holds in the case $|I| = 1$. Namely, for any interval partition J into intervals of length $j_1, j_2, \ldots, j_{n-r}$, where $n-r > 1$, we have the canonical inclusion

$$\nu_{JI} : G(\mathbf{R}^m, n) \to D(\mathbf{R}^m, j_1)\times \ldots \times D(\mathbf{R}^m, j_{n-r});$$

denote by z_{JI} the image of the top-dimensional generator under the induced map on the cohomology, $z_{JI} \in H^{r(m-1)}(G(\mathbf{R}^m, n);\mathbf{k})$. Then the set $\{z_{JI}\}$ (where J runs over all partitions with $|J| = n - r$) forms a linear basis of $H^{r(m-1)}(G(\mathbf{R}^m, n);\mathbf{k})$. This follows from the explicit calculation of the cohomology of $G(\mathbf{R}^m, n)$, given in Proposition 2.2.

Given an interval partition J on intervals of length $j_1, \ldots, j_{n-r}$, where $n - r > 1$, consider the following commutative diagram

$$\begin{array}{ccc} G(X, n) & \to & X^{\times n} \\ \downarrow & & \downarrow \mathrm{id} \\ D(X, j_1)\times \ldots \times D(X, j_{n-r}) & \stackrel{g_J}{\to} & X^{\times n} \end{array}$$

formed by the natural inclusions. Define a sheaf ε'_J over $X^{\times n}$ by

$$\varepsilon'_J = R^{r(m-1)} g_{J*}(\mathbf{k}).$$

We want to show that ε'_J is isomorphic to ε_J, defined above. First, ε'_J vanishes outside X_J (since we are considering the cohomology of the top dimension). Let U be a small open neighborhood of a point $c \in X_J \subset X^{\times n}$, such that $U = \prod U_i$, where all U_i are small open disks and $U_i = U_j$ if i and j lie in the same interval of J. Then

$$\varepsilon'_J(U) = H^{r(m-1)}(D(U_{i_1}, j_1)\times \ldots \times D(U_{i_{n-r}}, j_{n-r});\mathbf{k}) = \mathbf{k}.$$

Hence ε'_J is a constant sheaf with stalk $\mathbf{k}$ supported on X_J, i.e., $\varepsilon'_J \simeq \varepsilon_J$.

The commutative diagram above gives a map of sheaves $\varepsilon_J \to R^{r(m-1)}\psi_*(\mathbf{k})$ and we obtain a map of sheaves

$$\bigoplus_{|J|=n-r} \varepsilon_J \to R^{r(m-1)}\psi_*(\mathbf{k}),$$

which, as we have seen above, is an isomorphism on stalks; hence it is an isomorphism, and the claim (5) follows.

We arrive at the following description of the term E_2 of the spectral sequence:

$$E_2^{p,r(m-1)} = \begin{cases} H^p(X_{J_0};\mathbf{k})\otimes \mathbf{k}^{n-1}, & \text{for} \quad r = n-1, \quad \text{where} \quad |J_0| = 1, \\ \bigoplus_{|I|=n-r} H^p(X_I;\mathbf{k}), & \text{for} \quad r < n-1 \end{cases}$$

We will now identify this description with the one given in the statement of the theorem. Consider first the case $r < n-1$. Assign to a monomial $s_{i_1}\ldots s_{i_r}$ with $i_1 < i_2 < \ldots < i_r$ the equivalence relation on the set of indices $\{1, \ldots, n\}$ generated by the relations:

$$i_1 \sim i_1 + 1, \ i_2 \sim i_2 + 1, \ldots, i_r \sim i_r + 1.$$

This equivalence relation defines a partition I of the set $\{1, \ldots, n\}$ on $n - r$ intervals. In view of the relations $p_i^*(v)s_i = p_{i+1}^*(v)s_i$, the term $H^p(X^{\times n}; \mathbf{k})s_{i_1} \ldots s_{i_r}$ is isomorphic to $H^p(X_I; \mathbf{k})$, and we are done in this case. Consider now the case $r = n-1$. There are $n-1$ linearly independent monomials of degree $n-1$ in classes $s_1, \ldots, s_n$, and for every such monomial the corresponding interval partition equals $J_0 = \{1, \ldots, n\}$. Hence the image of $H^p(X^{\times n}; \mathbf{k}) \otimes H^{(n-1)(m-1)}(G(\mathbf{R}^m, n); \mathbf{k})$ is isomorphic to $H^p(X_{J_0}; \mathbf{k}) \otimes \mathbf{k}^{n-1}$ after imposing the relations $p_i^*(v)s_i = p_{i+1}^*(v)s_i$, as stated.

Statement (B) follows since the dihedral group D_n acts naturally on $G(X, n)$ and on $X^{\times n}$ and the inclusion $\psi : G(X, n) \to X^{\times n}$ is D_n-equivariant. Hence, D_n acts on the Leray spectral sequence of map ψ.

To find the action of D_n on the generators $s_1, \ldots, s_n$, consider an embedding $\mathbf{R}^m \to X$ (for example, the interior of a small disk) and the induced embedding of the cyclic configuration spaces $\phi : G(\mathbf{R}^m, n) \to G(X, n)$; clearly, ϕ is D_n-equivariant. ϕ determines a D_n-equivariant homomorphism of the spectral sequence of Theorem 3 for X into a similar spectral sequence for $\mathbf{R}^m$. This homomorphism is an isomorphism on $E_2^{0,*}$. Therefore, it is enough to find the action of D_n on the cohomology classes $s_i \in H^{m-1}(G(\mathbf{R}^m, n); \mathbf{k})$. Each class s_i is the pullback of the fundamental class of S^{m-1} under the map

$$G(\mathbf{R}^m, n) \to S^{m-1}, \quad (x_1, \ldots, x_n) \mapsto \frac{x_{i+1} - x_i}{|x_{i+1} - x_i|}.$$

If $\tau \in D_n$ is a reflection then the class $\tau^* s_i$ is the pullback of the fundamental class of S^{m-1} under the map

$$(x_1, \ldots, x_n) \mapsto \frac{x_{j-1} - x_j}{|x_{j-1} - x_j|}, \quad \text{where} \quad \tau(j) = i.$$

Hence it follows that

$$\tau^* s_i = (-1)^m s_{j-1} = (-1)^m s_{\tau^{-1}(i)-1} = (-1)^m s_{\tau(i)-1},$$

as stated.

Now we consider the differentials of the spectral sequence. The first $m - 2$ of them, $d_2, \ldots, d_{m-1}$, vanish by the dimension considerations. To find d_m consider the inclusion $\phi : F(X, n) \hookrightarrow G(X, n)$. This inclusion induces a homomorphism of the spectral sequence for $G(X, n)$ to that for $F(X, n)$, constructed in [20]. In the latter spectral sequence the E_2-term is the quotient of the graded commutative algebra

$$H^*(X^{\times n}; \mathbf{k}) \otimes H^*(F(\mathbf{R}^m, n); \mathbf{k})$$

modulo the relations

$$p_i^*(v)G_{ij} = p_j^*(v)G_{ij} \quad \text{for} \quad i \neq j \quad \text{and} \quad v \in H^*(X; \mathbf{k}),$$

and the first non-trivial differential acts as follows: $d_m G_{ij} = p_{ij}^* \Delta$. According to Proposition 2.2, $\phi^*(s_i) = G_{i,i+1}$, and, obviously, ϕ^* is identical on $H^*(X^{\times n}; \mathbf{k})$. This implies that the differential d_m of our spectral sequence acts by $d_m(s_i) = q_i^*(\Delta)$, as stated in the theorem. $\square$

Remark 2.4 Without assumption $m > 1$ the arguments of the proof involving the top-dimensional cohomology classes do not work. In fact, Theorem 3 fails for $X = S^1$ (see Example 2.1 above and Remark 3.2 below). The assumption $m > 1$ should be also added to Theorem 1 of Totaro [20].

3 Cyclic configuration space of the sphere

In this section we will use Theorem 3 to calculate the cohomology ring of $G(S^m, n)$ with coefficients in $\mathbf{Z}_2$.

Theorem 4 *Let $m > 2$. The cohomology ring $H^*(G(S^m, n); \mathbf{Z}_2)$ is multiplicatively generated by cohomology classes*

$$\sigma_i \in H^{i(m-1)}(G(S^m, n); \mathbf{Z}_2) \quad \text{where} \quad i = 1, 2, \dots \quad \text{and} \quad u \in H^m(G(S^m, n); \mathbf{Z}_2).$$

These classes satisfy the following relations:

(i) $\sigma_i = 0$ *for* $i \geq n - 1$;

(ii) $\sigma_i \sigma_j = \binom{i+j}{i} \sigma_{i+j}$;

(iii) $u^2 = 0$.

The Poincaré polynomial of the space $G(S^m, n)$ with coefficients in $\mathbf{Z}_2$ equals

$$\frac{(t^m + 1)(t^{(n-1)(m-1)} - 1)}{(t^{m-1} - 1)},$$

and the sum of Betti numbers is $2(n - 1)$. The dihedral group D_n acts trivially on $H^*(G(S^m, n); \mathbf{Z}_2)$.

Remark 3.1 Relation (ii) implies $\sigma_i^2 = 0$, since the binomial coefficient $\binom{2i}{i}$ is always even.

Remark 3.2 The space $G(S^1, n)$ consists of $n - 1$ connected components and each is homotopy equivalent to S^1. Comparing with Theorem 4, we see that this theorem gives a correct additive structure of $H^*(G(S^1, n); \mathbf{Z}_2)$, although the multiplicative structure for $m = 1$ is different. In fact, for any zero-dimensional cohomology class x with $\mathbf{Z}_2$ coefficients one has $x^2 = x$, and so for $m = 1$ the classes σ_i with $\sigma_i^2 = 0$ do not exist. Nor is the action of the dihedral group D_n on $H^*(G(S^1, n); \mathbf{Z}_2)$ trivial since the reflection changes the rotation number (as we mentioned above) and so it acts nontrivially on $H^0(G(S^1); \mathbf{Z}_2)$.

Remark 3.3 Most of the statements of Theorem 4 hold true for $m = 2$. In particular, for $m = 2$, the Poincaré polynomial remains the same as stated in Theorem 4. There are only two distinctions. First, for $m = 2$, relation (ii) should be replaced by a more general one

(ii') $\sigma_i\sigma_j = \binom{i+j}{i}\sigma_{i+j} + \varepsilon_{i,j}\sigma_{i+j-2}u$, where $\varepsilon_{i,j} \in \mathbf{Z}_2$;

Secondly, our methods do not prove that the dihedral group D_n acts trivially on $H^*(G(S^2, n); \mathbf{Z}_2)$; instead we have a weaker statement that the cyclic group $\mathbf{Z}_n \subset D_n$ acts trivially on $H^*(G(S^2, n); \mathbf{Z}_2)$, if n is odd.

Remark 3.4 Paper [1] contains an error. Proposition 3.3 of [1] claims that $G(S^2, n)$ is simply connected and all further computations in [1] depend on this claim. In fact, the space $G(S^2, n)$ is simply connected only for $n = 1$ and $n = 2$, as follows from our previous remark 3.3.

Proof. In what follows we will omit the coefficients $\mathbf{Z}_2$ from the notation. We will apply the spectral sequence of Theorem 3 in the case when $X = S^m$. Let us describe its E_2-term.

Let $v \in H^m(S^m)$ be the generator. For $i = 1, 2, \ldots, n$ let $u_i \in H^m((S^m)^{\times n})$ denote the class $u_i = 1 \times \ldots \times v \times 1 \times \ldots \times 1$ (with v at the i-th position). The classes $u_1, \ldots, u_n$ generate the cohomology ring $H^*((S^m)^{\times n})$; they commute and satisfy the relations $u_i^2 = 0$.

The $E_2 = E_m$-term of the spectral sequence of Theorem 3 is the quotient of the bigraded commutative differential algebra $\mathbf{Z}_2[s_1, \ldots, s_n, u_1, \ldots, u_n]$, where each s_i has bidegree $(0, m-1)$ and each u_i has bidegree $(m, 0)$, by the ideal generated by the relations:

(a) $s_i^2 = 0$ for $i = 1, 2, \ldots, n$;

(b) $\sigma_{n-1}(s) = s_1s_2\ldots s_{n-1} + s_2s_3\ldots s_n + s_3s_4\ldots s_ns_1 + \ldots + s_ns_1\ldots s_{n-2} = 0$;

(c) $u_i^2 = 0$ for $i = 1, 2, \ldots, n$;

(d) $(u_i + u_{i+1})s_i = 0$ for $i = 1, 2, \ldots, n$.

Recall that we understand indices cyclically; for example, (d) contains the relation $(u_1 + u_n)s_n = 0$.

The first nontrivial differential is d_m and it acts as follows

$$d_ms_i = u_i + u_{i+1},\ d_mu_i = 0, \quad i = 1, ..., n.$$

Introduce new variables: $v_i = u_i + u_{i+1}$, where $i = 1, 2, \ldots, n$ and $u = u_1$. In the new variables the differential algebra (E_m, d_m) can be understood as the quotient of the polynomial ring $\mathbf{Z}_2[s_1, \ldots, s_n, v_1, \ldots, v_n, u]$ modulo the ideal generated by the relations (a), (b) and the following relations (c'), (d') and (e):

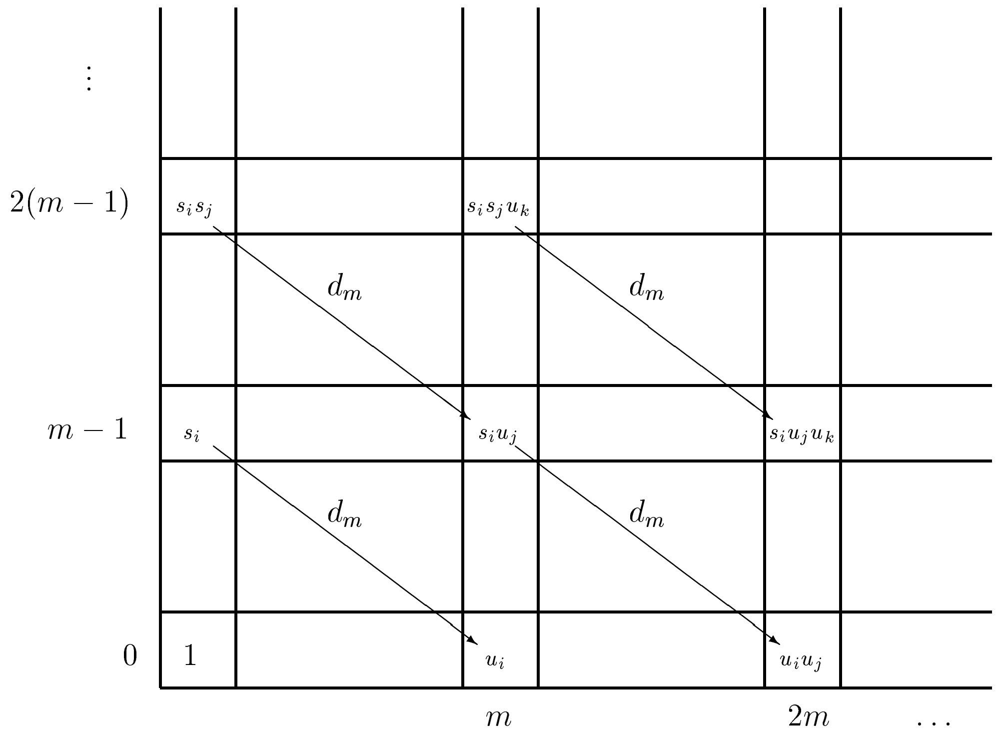


Figure 1: Term E_m of the spectral sequence

(c') $v_i^2 = 0$ for $i = 1, 2, \dots, n$ and $u^2 = 0$;

(d') $v_i s_i = 0$ for $i = 1, 2, \dots, n$;

(e) $\sum_{i=1}^n v_i = 0$.

The differential d_m acts by

$$d_m(s_i) = v_i, \quad d_m(v_i) = d_m(u) = 0. \tag{6}$$

Our goal is to compute the cohomology of (E_m, d_m).

Let A denote the quotient of $\mathbf{Z}_2[s_1, \dots, s_n, v_1, \dots, v_n, u]$ by the ideal generated by the relations (a), (c'), (d'), i.e., we simply ignore relations (b) and (e). We consider A and E_m with the total grading, where each s_i has degree $m-1$ and the elements v_i and u have degree m. Let the differential $d_A : A \to A$ act by $d_A(s_i) = v_i$ and $d_A(v_i) = 0 = d_A(u)$. There is a canonical epimorphism of graded differential algebras $f : A \to E_m$, so that $f d_A = d_m f$. The kernel of f is the ideal of A generated by $\sum_{i=1}^n v_i$ and $\sigma_{n-1}(s)$.

We claim that

$$H^i(A, d_A) = \begin{cases} 0, & \text{if} \quad i \neq m, \\ \mathbf{Z}_2, & \text{if} \quad i = m, \end{cases} \tag{7}$$

and the only nontrivial cohomology class is represented by u. In order to prove this, consider the filtration $A_0 \subset A_1 \subset \ldots \subset A_n = A$, where A_0 is generated by u, and each A_k is generated by u, s_i, v_i for $i = 1, \ldots, k$. The differential d_A preserves this filtration. d_A, restricted to A_0, vanishes, and so (A_0, d_A) has a one-dimensional cohomology generated by the class of u. We will now show that all factors A_k/A_{k-1} are acyclic, where $k = 1, \ldots, n$. Indeed, any element $a \in A_k/A_{k-1}$ can be uniquely represented in the form $a = s_k x + v_k y$, where $x, y \in A_{k-1}$. If a is a cocycle then we have

$$d_A(a) = v_k x + s_k d_A(x) + v_k d_A(y) = s_k d_A(x) + v_k[x + d_A(y)] = 0$$

and hence $x = d_A(y)$. Therefore $a = d_A(s_k y)$. This proves that $H^*(A, d_A) = H^*(A_0, d_A)$ and our claim (7) follows.

Introduce a new differential $\delta_A : A \to A$ of degree m:

$$\delta_A(x) = (\sum_{i=1}^{n} v_i)x.$$

Clearly, $\delta_A^2 = 0$ and $\delta_A d_A = d_A \delta_A$; however δ_A does not obey the Leibnitz rule. We claim that

$$H^i(A, \delta_A) = \begin{cases} 0, & \text{if} \quad i \neq n(m-1), \quad i \neq n(m-1) + m, \\ \\ \mathbf{Z}_2, & \text{if} \quad i = n(m-1) \quad \text{or} \quad i = n(m-1) + m, \end{cases} \tag{8}$$

and the only nontrivial cohomology classes are represented by $s_1 s_2 \ldots s_n$ and by $s_1 s_2 \ldots s_n u$. Indeed, each element of A can be written as a sum of monomials in s_i, v_i and u. For $I \subset \{1, 2, \ldots n\}$, denote by s_I the product of all s_i for $i \in I$. Similarly, label monomials in variables v_i as v_J for $J \subset \{1, 2, \ldots n\}$. Note that the product $s_I v_J \in A$ is nontrivial if and only if I and J are disjoint subsets of $\{1, \ldots, n\}$. Note also that

$$\delta_A(s_I v_J) = \sum_{i \notin I \cup J} s_I v_{J \cup \{i\}}, \quad \delta_A(s_I v_J u) = \sum_{i \notin I \cup J} s_I v_{J \cup \{i\}} u.$$

We see that application of δ_A does not change the multi-index I or the factor u. Hence, the complex (A, δ_A) splits into a direct sum over different multi-indices I. Fix a set I and denote by k the cardinality of the set $\{1, \ldots, n\} - I$. Then the respective part of the complex (A, δ_A) is isomorphic to two copies (one with u and one without) of the standard cochain complex of the simplex with k vertices: the differential of an r-dimensional face (i.e., set J) is the sum of $r + 1$-dimensional faces that contain the given one (sets $J \cup \{i\}$). Note that empty set J is also allowed. This complex has zero cohomology, unless $k = 0$ (empty simplex), in which case the cohomology is $\mathbf{Z}_2$. This exceptional case corresponds to $I = \{1, \ldots, n\}$, and the claim follows.

Now we are ready to compute the cohomology of (E_m, d_m), i.e., the term E_{m+1} of the spectral sequence. Denote by $\sigma_k \in A$ the k-th elementary symmetric function in variables $s_1 \ldots, s_n$, i.e.,

$$\sigma_0 = 1, \quad \text{and} \quad \sigma_k = \sum_{1 \leq i_1 < \ldots < i_k \leq n} s_{i_1} s_{i_2} \ldots s_{i_k} \quad \text{for} \quad k = 1, 2, \ldots, n.$$

It is clear that

$$\sigma_i\sigma_j = \begin{cases} \binom{i+j}{i}\sigma_{i+1}, & \text{for } \ i+j \le n, \\ \\ 0, \quad \text{for } \ i+j > n. \end{cases}$$

Also note that

$$d_A(\sigma_{k+1}) = \delta_A(\sigma_k), \quad d_A(\sigma_{k+1}u) = \delta_A(\sigma_k u). \tag{9}$$

Therefore the images of the classes σ_i and $\sigma_i u$ under the projection $f : A \to E_m$ are cocycles. Note also that the classes $f(\sigma_n)$, $f(\sigma_{n-1})$ vanish due to the relation (b). All other classes $f(\sigma_0) = 1, f(\sigma_1), \dots, f(\sigma_{n-2})$ are nonzero elements of E_m; this follows since the kernel of f is generated by the image of δ_A and by the element σ_{n-1}, cf. above. It is clear from dimension considerations that σ_i is not a coboundary; likewise, $\sigma_i u$ is not a coboundary either. Hence we have nontrivial cohomology classes

$$f(\sigma_i) \in H^{i(m-1)}(E_m, d_m) \quad \text{and} \quad f(\sigma_i u) \in H^{i(m-1)+m}(E_m, d_m), \quad i = 0, 1, \dots, n-2.$$

Our aim now is to show that these classes constitute all the cohomology.

Let $K \subset A$ denote the kernel of f. Using long exact sequences, we deduce from (7) that

$$H^i(E_m, d_m) \simeq H^{i+1}(K, d_A) \quad \text{for} \quad i \ne m. \tag{10}$$

Note that K contains σ_n, and denote by $\overline{K}$ the factor of K by the ideal, generated by σ_n and $d_A(\sigma_n)$. This ideal is four-dimensional and is generated, as a vector space, by the classes σ_n, $d_A(\sigma_n)$, $\sigma_n u$ and $d_A(\sigma_n)u$; it is closed under d_A and is acyclic. Hence we have

$$H^*(K, d_A) \simeq H^*(\overline{K}, d_A).$$

Define a chain map $\phi : (E_m, d_m) \to (\overline{K}, d_A)$ increasing the degree by m. Given an element $x \in E_m$, consider a lift $\overline{x} \in A$ and set: $\phi(x) = \delta_A(\overline{x}) \in \overline{K}$. To see that ϕ is well-defined, note that $\overline{x}$ is determined up to summation with elements of the form $\delta_A(a) + \sigma_{n-1}b$, where $a, b \in A$. Since $\delta_A(\delta_A(a) + \sigma_{n-1}b) = d_A(\sigma_n)b = 0 \in \overline{K}$, we see that $\phi(x)$ does not depend on the choice of $\overline{x}$. It is clear that ϕ is a chain map, i.e., $\phi d_m = d_A\phi$. Due to (8), ϕ is a monomorphism. The cokernel of ϕ is linearly generated by the classes σ_{n-1} and $\sigma_{n-1}u$, and we find from the short exact sequence $0 \to E_m \xrightarrow{\phi} \overline{K} \to \mathrm{Coker}(\phi) \to 0$ that

$$\phi : H^i(E_m, d_m) \to H^{i+m}(\overline{K}, d_A) \tag{11}$$

is an isomorphism for all i except $i = (n-2)(m-1)$ and $i = (n-1)(m-1)+1$; for these two exceptional values of i the homomorphism (11) is an epimorphism with one dimensional kernel. To prove this claim about (11) notice that the induced differential on $\mathrm{Coker}(\phi)$ is trivial, as well as the induced map $H^*(\overline{K}) \to H^*(\mathrm{Coker}(\phi))$.

Combining isomorphisms (10) and (11), one finds that

$$H^i(E_m, d_m) \xrightarrow{\simeq} H^{i+m-1}(E_m, d_m), \tag{12}$$

where $i \neq 1$, $i \neq (n-2)(m-1)$ and $i \neq (n-1)(m-1)+1$. Note also that for $i=(n-2)(m-1)$ and $i=(n-1)(m-1)+1$ instead of isomorphism (12) we have an epimorphism with one dimensional kernel.

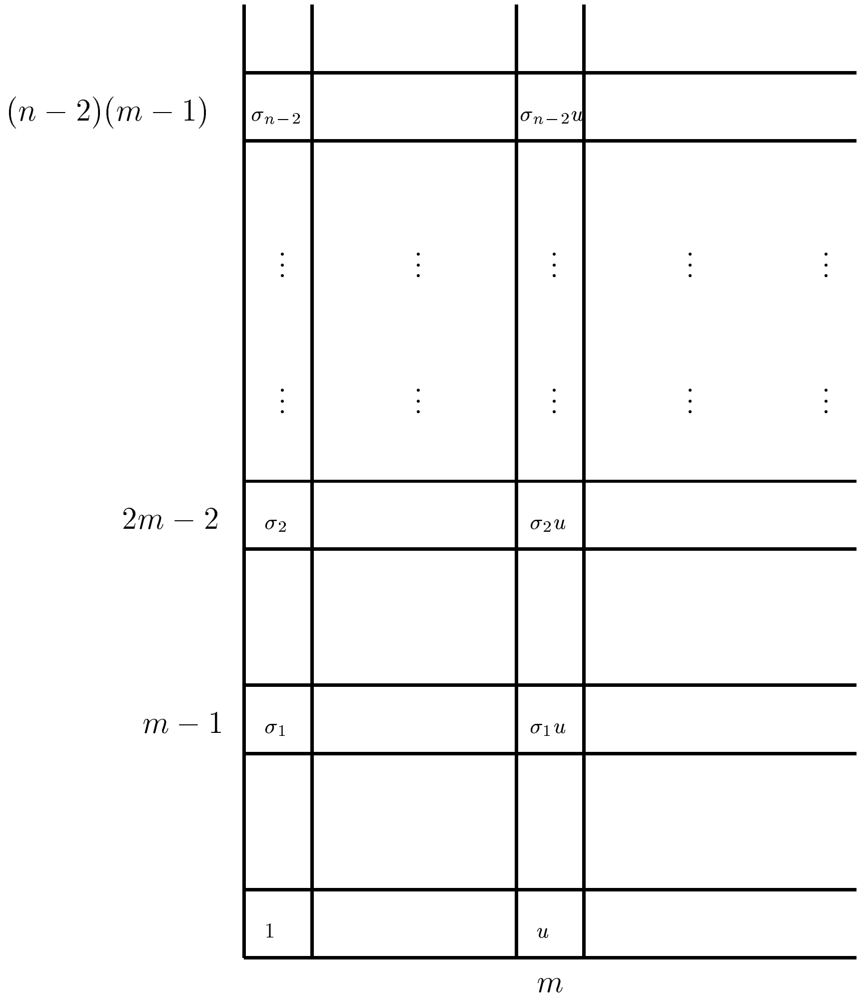


Figure 2: Term E_{m+1} of the spectral sequence.

We can finally show that the term E_{m+1} is linearly generated by the classes $\sigma_1, \ldots, \sigma_{n-2}$ and $u, \sigma_1 u, \ldots, \sigma_{n-2}u$, where, abusing notation, we identify $f(\sigma_i)$ and $f(u)$ with σ_i and u, respectively. Obviously, σ_1 and u are the only nontrivial classes of E_{m+1} of total degree $< 2(m-1)$. The periodicity isomorphism (12) implies that in each degree $i(m-1)$ (where $1 \leq i < n-1$) we have only one nontrivial class, which therefore must coincide with σ_i. Also, again from (12), one concludes that in each degree $i(m-1)+m$, where $1 \leq i < n-1$, there is a single nontrivial class, and so it must coincide with $\sigma_i u$. Hence the term E_{m+1} has the structure shown in Figure 2.

By dimension considerations, all further differentials d_r with $r > m$ vanish and thus $E_{m+1} = E_\infty$.

Now we want to reconstruct the cohomology algebra $H^*(G(S^m,n))$ from E_∞. The m-th column is an ideal in $H^*(G(S^m,n))$, and the factor of $H^*(G(S^m,n))$ by this ideal is the 0-th column. Since $m>2$, each diagonal $x+y=c$ contains at most one nonzero term of E_∞. Hence the generators σ_i and u admit unique lifts to $H^*(G(S^m,n))$, and we label the cohomology classes of $H^*(G(S^m,n))$ as σ_i and $\sigma_i u$. Since the multiplication in E_∞ is induced from the multiplication in $H^*(G(S^m,n))$, the product of $\sigma_i u$ and $\sigma_j u$ is trivial, and the product of σ_i and $\sigma_j u$ equals $\binom{i+j}{i}\sigma_{i+j}u$. The product of σ_i and σ_j may be equal to $\binom{i+j}{i}\sigma_{i+j}$ plus a term from the m-th column, but, in fact, this additional term must vanish since the m-th column has no nonzero terms in dimensions, divisible by $m-1$. We conclude that the cohomology algebra $H^*(G(S^m,n))$ has the structure described in Theorem 4.

The dihedral group D_n acts trivially on $H^*(G(S^m,n))$ since in each dimension we have at most one nonzero element.

The arguments in the two last paragraphs do not apply if $m=2$. This explains the modifications to the statement of Theorem 4 in case $m=2$, described in Remark 3.3 above. Let us show that $Z_n\subset D_n$ acts trivially on $H^*(G(S^2,n);\mathbf{Z}_2)$, assuming that n is odd. Indeed, it follows from the above calculation of the E_∞-term that $H^i(G(S^2,n);\mathbf{Z}_2)$ is at most two-dimensional for any i, and hence it has either 1 or 3 nonzero elements. $\mathbf{Z}_n$ acts by permutations of nonzero classes, and at least one of these classes is fixed, since D_n acts trivially on the E_∞-term. Therefore the action of $\mathbf{Z}_n$ is trivial provided n is odd. This completes the proof. □

Corollary 5 *Consider the cohomology classes*

$$\sigma_1,\,\sigma_2,\,\sigma_4,\dots,\sigma_{2^k}\;\in\;H^*(G(S^m,n);\mathbf{Z}_2),\qquad m>2,$$

where k is the largest integer, not exceeding $\log_2(n-2)$. *Then, for any $i\le n-2$, the class σ_i can be expressed as a product*

$$\sigma_i=\sigma_{2^{k_1}}\sigma_{2^{k_2}}\dots\sigma_{2^{k_r}}, \tag{13}$$

where $i=2^{k_1}+2^{k_2}+\dots+2^{k_r}$ is the binary expansion of i; here $k_1<k_2<\dots<k_r$.

Proof. It follows from relation (ii) of Theorem 4 that

$$\sigma_{l_1}\dots\sigma_{l_r}=\binom{l_1+\dots+l_r}{l_1,\dots,l_r}\sigma_{l_1+\dots+l_r},\qquad l_1+\dots+l_r\le n-2.$$

Hence the statement of the Corollary will follow once we show that the multinomial coefficient

$$\binom{2^{k_1}+\dots+2^{k_r}}{2^{k_1},\dots,2^{k_r}}=\frac{(2^{k_1}+\dots+2^{k_r})!}{(2^{k_1})!\dots(2^{k_r})!}$$

is odd for $k_1<\dots<k_r$. This is a consequence of the well known fact that the maximal power of 2 that divides $l!$ equals $l-r$, where r is the number of 1's in the binary expansion of l. □

Corollary 6 *For $m > 1$ the cup-length of $H^*(G(S^m, n); \mathbf{Z}_2)$ equals*

$$[\log_2(n-1)] + 1.$$

Proof. Assume first that $m > 2$. Then, by Corollary 5, the length of the longest non-trivial product of classes σ_i equals the maximal number of 1's in the binary expansion, that integers, not exceeding $n-2$, may have. It is easy to see that the latter number equals $[\log_2(n-1)]$. This implies Corollary 6, since we also have the class u at our disposal. For $m = 2$ the arguments are similar using relation (ii') in Remark 3.3. □

4 Morse theory of closed billiard trajectories

In this section we describe Morse theory of closed billiard trajectories which is similar to the classical Morse theory of closed geodesics in Riemannian manifolds. Some of the results of this section are known to experts, but do not exist in the literature in an accessible form.

The proof of Theorem 1 will depend on the results of this section in two ways. First, Morse theory of closed billiard trajectories, in the simplest case when the billiard table is the round sphere (like in [1]), will provide a tool for computation of the D_n-equivariant cohomology ring of the cyclic configuration space $G(S^m, n)$. Second, we will invoke equivariant Morse theory [7] and a version of equivariant Lusternik-Schnirelman theory developed in [17] to deduce Theorem 1 concerning arbitrary smooth strictly convex billiards.

Let $X \subset \mathbf{R}^{m+1}$ be a smooth closed strictly convex hypersurface, topologically the sphere, which is the boundary of the billiard table. Denote by

$$L_X : G(X, n) \to \mathbf{R} \tag{14}$$

the perimeter length function, taken with the minus sign,

$$L_X(x_1, \dots, x_n) = -|x_1 - x_2| - |x_2 - x_3| - \dots - |x_n - x_1|,$$

where $(x_1, x_2, \dots, x_n) \in G(X, n)$ and the distance $|x_i - x_{i+1}|$ is measured in the ambient Euclidean space $\mathbf{R}^{m+1}$. The reason for the minus sign will become clear shortly.

It is well-known that n-periodic billiard orbits in X are precisely the critical points of the function L_X; this is the Maupertuis' principle of the classical mechanics as applied to billiards (see, e.g., [16, 19]). Clearly, the function L_X is smooth and D_n-equivariant. Identifying $G(X, n)$ with $G(S^m, n)$, we see that the shape of the billiard domain X becomes encoded in the function $L_X : G(S^m, n) \to \mathbf{R}$, and the problem of finding the closed billiard trajectories inside X turns into a problem of Morse-Lusternik-Schnirelman theory.

We encounter the following difficulty: one cannot apply Morse-Lusternik-Schnirelman theory directly to $G(X, n)$ since this manifold is not compact. The function L_X extends to a continuous function on the space of all n-tuples $X^{\times n}$ but this extension fails to be differentiable on the singular set Σ consisting of the points with $x_i = x_{i+1}$ for some i. A way around this difficulty is in replacing $G(X, n)$ by a compact manifold with boundary $G_\varepsilon(X, n) \subset G(X, n)$, where $\varepsilon > 0$ is small enough and

$$G_\varepsilon(X,n) = \{(x_1, \dots, x_n) \in X^{\times n} : \prod_{i=1}^{n} |x_i - x_{i+1}| \geq \varepsilon\}; \tag{15}$$

similar approach can be found in [1] and in [12], [16] for the two-dimensional case.

Proposition 4.1 *If $\varepsilon > 0$ is sufficiently small then:*

(a) $G_\varepsilon(X, n)$ is a smooth manifold with boundary;

(b) the inclusion $G_\varepsilon(X, n) \subset G(X, n)$ is a D_n-equivariant homotopy equivalence;

(c) all critical points of $L_X : G(X, n) \to \mathbf{R}$ are contained in $G_\varepsilon(X, n)$;

(d) at every point of $\partial G_\varepsilon(X, n)$, the gradient of L_X has the outward direction.

Proof. The function $\phi(x_1, \dots, x_n) = \prod_1^n |x_i - x_{i+1}|^2$ is smooth on $X^{\times n}$. Its zero level set $\phi^{-1}(0)$ is $\Sigma = X^{\times n} - G(X, n)$, which is a critical level. Statements (a) and (b) will follow once we show that *there exists a constant $\delta > 0$ such that the interval $(0, \delta)$ consists of regular values of ϕ.*

To prove the claim in italic we need to understand geometry of n-tuples $\bar{x} = (x_1, \dots, x_n) \in G(X, n)$ which are critical points of ϕ. An easy computation with Lagrange multipliers shows that $\bar{x}$ is critical if and only if

$$\frac{x_{i-1} - x_i}{|x_{i-1} - x_i|^2} + \frac{x_{i+1} - x_i}{|x_{i+1} - x_i|^2} = t_i \nu_i, \quad i = 1, \dots, n \tag{16}$$

where $\nu_i = \nu(x_i)$ is the unit normal vector to X at point x_i and $t_i \in \mathbf{R}$ is some constant. It will be convenient for us to assume that ν_i has the inward direction.

Since X is smooth and strictly convex one can find two positive numbers $r < R$ such that for every $x \in X$

(1) there is a sphere $s(x)$ of radius r, tangent to X at x and contained inside X; and

(2) there is a sphere $S(x)$ of radius R, tangent to X at x and containing X.

Let x_{i-1}, x_i, x_{i+1} be three consecutive points on X from a critical n-tuple $\bar{x}$ of the function ϕ. Note that the 2-plane through the points x_{i-1}, x_i, x_{i+1} contains the centers of both spheres $s(x_i)$ and $S(x_i)$. Let α_i be the angle between ν_i and $x_{i-1} - x_i$, and β_i the angle between ν_i and $x_{i+1} - x_i$. Set: $l_i = |x_i - x_{i-1}|$. Then (16) gives

$$\sin \alpha_i / l_i = \sin \beta_i / l_{i+1}. \tag{17}$$

Since x_{i-1} and x_{i+1} lie outside the sphere $s(x_i)$ and inside the sphere $S(x_i)$, we have

$$2r \cos \alpha_i < l_i < 2R \cos \alpha_i, \quad 2r \cos \beta_i < l_{i+1} < 2R \cos \beta_i, \tag{18}$$

and hence

$$\frac{r\cos\alpha_i}{R\cos\beta_i} \le \frac{l_i}{l_{i+1}} \le \frac{R\cos\alpha_i}{r\cos\beta_i}. \tag{19}$$

We claim that

$$\frac{r}{\sqrt{r^2+R^2}} \le \frac{l_i}{l_{i+1}} \le \frac{\sqrt{r^2+R^2}}{r} \tag{20}$$

for any $i = 1, 2, \ldots, n$. Indeed, if (20) fails and

$$\frac{l_i}{l_{i+1}} < \frac{r}{\sqrt{r^2+R^2}} \tag{21}$$

then, combining (21) with the left inequality in (19), we obtain

$$\cos\alpha_i < \frac{R\cos\beta_i}{\sqrt{r^2+R^2}} \le \frac{R}{\sqrt{r^2+R^2}}. \tag{22}$$

On the other hand, combining (21) with (17), we obtain

$$\sin\alpha_i < \frac{r\sin\beta_i}{\sqrt{r^2+R^2}} \le \frac{r}{\sqrt{r^2+R^2}}, \tag{23}$$

which leads to a contradiction since (22) and (23) are incompatible. This argument shows that the left inequality in (20) holds. The right inequality in (20) follows similarly.

It follows from (20) that for any two edges l_i and l_j of a critical n-gon one has:

$$\frac{l_i}{l_j} < \left(1 + \frac{R^2}{r^2}\right)^{n/2}.$$

In other words, if one of the edges of a critical n-gon is "short" then all others are also "short".

It remains to show that a critical n-gon cannot have all edges arbitrarily short. Indeed, assume that $\bar{x} = (x_1, \ldots, x_n)$ is a critical polygon such that all points x_i lie in a small neighborhood $U \subset X$ of a point $y \in U$. We will assume that U is so small that for any $x \in U$ the scalar product $\langle \nu(x), \nu(y)\rangle$ is positive (recall that $\nu(x)$ denotes the inward unit normal to X at point x). Taking scalar product of equation (16) with ν_i gives

$$\frac{\cos\alpha_i}{l_i} + \frac{\cos\beta_i}{l_{i+1}} = t_i,$$

and hence all numbers t_i, which appear in (16), are positive. Now, the scalar product of (16) with vector $\nu(y)$ gives

$$\frac{\langle (x_{i-1} - x_i), \nu(y)\rangle}{l_i^2} > \frac{\langle (x_i - x_{i+1}), \nu(y)\rangle}{l_{i+1}^2}, \quad i = 1, 2, \ldots, n. \tag{24}$$

Recall that in the last inequality the indices are understood cyclically, i.e. $x_{n+j} = x_j$. Hence (24) leads to a contradiction.

The above arguments prove statements (a) and (b).

Next we prove statement (c). The argument is similar. If $\bar{x} = (x_1, \ldots, x_n) \in G(X, n)$ is a billiard trajectory in X, then (instead of (16)) we have

$$\frac{x_{i-1} - x_i}{l_i} + \frac{x_{i+1} - x_i}{l_{i+1}} = t_i \nu_i, \quad i = 1, \ldots, n,$$

and (17) becomes the usual reflection law

$$\alpha_i = \beta_i.$$

We have the inequality

$$\frac{r}{R} \leq \frac{l_i}{l_{i+1}} \leq \frac{R}{r}$$

which is an analog of (20) in the present case. As above, this inequality implies that if one edge of an n-periodic billiard trajectory is "short" then so are all its edges. The preceding argument shows that billiard n-gons cannot lie entirely in a small neighborhood of a point of the hypersurface X.

Finally we prove claim (d). Since the gradient $\nabla \phi$ is orthogonal to the boundary $\partial G_\varepsilon(X, n)$ and points inside $G_\varepsilon(X, n)$, it suffices to show that the scalar product $\langle \nabla(\ln \phi), \nabla L_X \rangle$ is negative along $\partial G_\varepsilon(X, n)$ for every sufficiently small $\varepsilon > 0$.

Let us compute the gradients involved. Taking into account the decomposition $T_{\bar{x}} G(X, n) = T_{x_1} X \times \ldots \times T_{x_n} X$, where $\bar{x} \in G(X, n)$, one finds that the i-th components of the gradients ∇L_X and $\nabla(\ln \phi)$ are given by

$$(\nabla L_X)_i = -\frac{x_{i-1} - x_i}{l_i} - \frac{x_{i+1} - x_i}{l_{i+1}} + (\cos \alpha_i + \cos \beta_i) \nu_i, \tag{25}$$

$$\frac{1}{2} \, (\nabla \ln \phi)_i = \frac{x_{i-1} - x_i}{l_i^2} + \frac{x_{i+1} - x_i}{l_{i+1}^2} - \Big(\frac{\cos \alpha_i}{l_i} + \frac{\cos \beta_i}{l_{i+1}} \Big) \nu_i. \tag{26}$$

Denote by θ_i the angle between the vectors $x_{i-1} - x_i$ and $x_{i+1} - x_i$. Due to strict convexity of X, we have $\theta_i \in [0, \pi)$. A direct computation using (25) and (26) shows that

$$\langle \nabla L, \nabla \ln \phi \rangle = 2(S_1 + S_2), \tag{27}$$

where

$$S_1 = -\sum_{i=1}^{n} \Big(\frac{1}{l_i} + \frac{1}{l_{i+1}} \Big)(1 + \cos \theta_i), \quad S_2 = \sum_{i=1}^{n} (\cos \alpha_i + \cos \beta_i) \Big(\frac{\cos \alpha_i}{l_i} + \frac{\cos \beta_i}{l_{i+1}} \Big). \tag{28}$$

We want to show that the right-hand side of (27) is negative for all $\bar{x} \in \partial G_\varepsilon(X, n)$ with $\varepsilon > 0$ small enough. It follows from inequalities (18) that

$$\frac{\cos \alpha_i}{l_i} + \frac{\cos \beta_i}{l_{i+1}} < \frac{1}{r},$$

therefore $S_2 < 2n/r$. We will be done once we show that S_1 tends to $-\infty$ as ε goes to zero.

Assume, to the contrary, that there exists a constant C and an infinite sequence $\bar{x}_k \in G(X,n)$, where $k = 1, 2, \ldots$, such that $\phi(\bar{x}_k) = \varepsilon_k$ tends to 0 and $S_1(\bar{x}_k) > -C$. One has

$$\Big(\frac{1}{l_i} + \frac{1}{l_{i+1}}\Big)(1+\cos\theta_i) < C, \quad i = 1, \ldots, n, \tag{29}$$

and $l_1 \ldots l_n = \varepsilon_k^{1/2}$.

Suppose that for an n-gon $\bar{x}_k \in \partial G_{\varepsilon_k}(X,n)$ and some index j one has an inequality $l_i \leq b\epsilon_k^a$, where k is large enough, $a > 0$ and $b > 0$ (for example, for the smallest link l_i, one has: $l_i \leq (\varepsilon_k)^{1/2n}$. Then it follows from (29) that

$$1+\cos\theta_j < Cb\varepsilon_k^a, \quad \text{and hence} \quad \pi - \theta_j < \frac{2}{\sqrt{3}}\sqrt{Cb}\varepsilon_k^{a/2}.$$

Since $\theta_j \leq \alpha_j + \beta_j$, one concludes that

$$\pi/2 - \beta_j < \frac{2}{\sqrt{3}}\sqrt{Cb}\varepsilon_k^{a/2}, \quad \text{and} \quad \cos\beta_j = \sin(\frac{\pi}{2} - \beta_i) < \frac{2}{\sqrt{3}}\sqrt{Cb}\varepsilon_k^{a/2}.$$

It follows then from inequalities (18) that

$$l_{j+1} < \frac{4R}{\sqrt{3}}\sqrt{Cb}\varepsilon_k^{a/2}. \tag{30}$$

The argument that derives (30) from the initial assumption $l_i \leq b\epsilon_k^a$ can be repeated n times to conclude that there exist positive constants b, b' and a_0, so that for any $j = 1, 2, \ldots, n$ and for any sufficiently large k one has:

$$l_j < b\varepsilon_k^{a_0} \quad \text{and} \quad \pi - \theta_j < b'\varepsilon_k^{a_0/2}$$

(we may take $a_0 = 2^{-n}n^{-1}$).

Thus, for k large enough, $\sum_{i=1}^n(\pi - \theta_i)$ is close to zero. On the other hand, given a closed polygonal line in Euclidean space, the sum of its exterior angles, that is, the angles $\pi - \theta_i$, is at least 2π (a smooth version of this statement holds too: the total curvature of a closed curve is at least 2π). This is a contradiction. □

Remark 4.2 L_X and ϕ are particular cases of the following more general function on inscribed polygons:

$$F(x_1, \ldots, x_n) = \sum_1^n f(|x_{i+1} - x_i|)$$

where $f(t)$ is a function of one variable (we obtain L_X when $f(t) = t$ and $\ln\phi$ when $f(t) = 2\ln t$). For some of these functions it is not true that all the critical n-tuples lie off a neighborhood of the singular set Σ. The simplest example is provided by $f(t) = t^2$. An analog of the condition (16) reads

$$\overrightarrow{x_i x_{i-1}} + \overrightarrow{x_i x_{i+1}} = t_i \nu(x_i).$$

If X is a circle then this equation holds whenever either $x_{i-1}x_ix_{i+1}$ is a right angle or $|x_{i-1} - x_i| = |x_i - x_{i+1}|$. In this "billiard" we have closed trajectories in the form of arbitrary rectangles inscribed into the circle. Thus, we may have one pair of sides arbitrarily small.

Definition 4.3 *Let $X \subset \mathbf{R}^{m+1}$ be a smooth hypersurface. Then X is n-generic if $L_X : G(X, n) \to \mathbf{R}$ is a Morse function.*

A justification for this definition is provided by the next lemma.

Lemma 4.4 *There is a massive subset E in the space of embeddings $S^m \to \mathbf{R}^{m+1}$ such that for every $f \in E$ the hypersurface Im f is n-generic for all n.*

Recall that the space of smooth maps from one manifold to another is considered in the Whitney C^∞ topology; a massive set is a countable intersection of open dense sets. Due to the Baire property, a massive set is dense – see, [14].

Proof. Consider n germs of immersions

$$\phi_i : (S^m, s_i) \to (\mathbf{R}^{m+1}, x_i), \quad i = 1, \dots, n,$$

and assume that the targets $x_1, \dots, x_n$ satisfy $x_i \neq x_{i+1}$ for $i = 1, \dots, n$. Then a germ of the respective perimeter length function $(S^m, s_1) \times \dots \times (S^m, s_n) \to \mathbf{R}$ is defined:

$$L(t_1, \dots, t_n) = \sum_1^n |\phi_{i+1}(t_{i+1}) - \phi_i(t_i)|.$$

Clearly, the first partial derivatives of L depend on the first derivatives of ϕ_i. Therefore we may consider the situation on the level of 1-jets. Namely, let

$$\mathcal{U} \subset J^1(S^m, \mathbf{R}^{m+1}) \times \dots \times J^1(S^m, \mathbf{R}^{m+1})$$

consist of multi-jets of immersions satisfying the following three conditions:

(i) the targets $x_1, \dots, x_n$ satisfy $x_i \neq x_{i+1}$ for $i = 1, \dots, n$;

(ii) the vector

$$\nu_i = \frac{x_i - x_{i-1}}{|x_i - x_{i-1}|} + \frac{x_i - x_{i+1}}{|x_i - x_{i+1}|} \tag{31}$$

does not vanish for $i = 1, \dots, n$;

(iii) the sources $t_1, \dots, t_n$ satisfy $t_i \neq t_j$ for all $i \neq j$.

The first requirement is needed for L to be smooth and the third for the multi-jet transversality theorem to be applicable; the role of the second one will become clear shortly. Note that $\mathcal{U}$ is an open subset of the multi-jet space. Also consider the space of 1-jets of functions $J^1(S^m \times \ldots \times S^m)$, and let D be its submanifold of codimension mn consisting of the 1-jets with trivial differential. Assigning the 1-jet of the perimeter length function L to a multi-jet in $\mathcal{U}$ provides a map

$$\pi : \mathcal{U} \to J^1((S^m)^{\times n}).$$

Claim: *the map π is transversal to the submanifold D.*

Assuming the claim, the proof proceeds as follows. The 1-jet extension of the perimeter length function L determines the section $j^1(L)$ of the 1-jet bundle

$$J^1((S^m)^{\times n}) \to (S^m)^{\times n},$$

and the critical points of L are non-degenerate if and only if this section is transversal to D. The claim implies that $\Delta = \pi^{-1}(D)$ is a submanifold in $\mathcal{U}$. According to the multi-jet transversality theorem (see [14]), there exists a massive set E_n of embeddings $S^m \to \mathbf{R}^{m+1}$ such that for $f \in E_n$ the multi-jet extension $j^1(f) : G(S^m, n) \to \mathcal{U}$ is transversal to Δ. Since $j^1(L) = \pi \circ j^1(f)$, it follows from the claim that $j^1(L)$ is transversal to D for $f \in E_n$, that is, the critical points of the perimeter length function are non-degenerate. Setting $E = \cap_{n=2}^{\infty} E_n$ completes the proof.

It remains to prove the italized claim above. It is convenient to choose local coordinates in the 1-jet spaces involved. Consider a multi-jet $\bar{\phi} = (\phi_1, \ldots, \phi_n) \in \Delta$, and let $s_i \in S^m$ be the source of ϕ_i. For each i identify a neighborhood of s_i with an open disk $U \subset \mathbf{R}^m$. Then a neighborhood of $\bar{\phi}$ in $J^1(S^m, \mathbf{R}^{m+1})^{\times n}$ is identified with $J^1(U, \mathbf{R}^{m+1})^{\times n}$ and π becomes a map

$$\pi : J^1(U, \mathbf{R}^{m+1})^{\times n} \to J^1(U^{\times n}).$$

The 1-jet space $J^1(U, \mathbf{R}^{m+1})$ consists of triples (u, x, A) where $u \in U$ is the source, $x \in \mathbf{R}^{m+1}$ is the target and $A : \mathbf{R}^m \to \mathbf{R}^{m+1}$ is a linear map (derivative of a map $U \to \mathbf{R}^{m+1}$); the multi-jet space $J^1(U, \mathbf{R}^{m+1})^{\times n}$ consists of n-tuples $(u_i, x_i, A_i),\ i = 1, \ldots, n$, of such triples. The space $J^1(U^{\times n})$ consists of 1-jets of functions $\psi : U^{\times n} \to \mathbf{R}$, that is, of $2n + 1$-tuples $(u_i, p_i, z),\ i = 1, \ldots, n$, where $u_i \in U$, $p_i = \partial\psi/\partial u_i \in (\mathbf{R}^{m+1})^*$ is a covector and $z = \psi(u_1, \ldots, u_n) \in \mathbf{R}$. In these coordinates we explicitly describe the map $\pi : (u_i, x_i, A_i) \to (u_i, p_i, z)$:

$$z = \sum_{i=1}^{n} |x_{i+1} - x_i|, \quad p_i(v_i) = \langle \nu_i, A_i(v_i) \rangle$$

where $v_i \in \mathbf{R}^m$ is a test vector and the vector ν_i is as in (31). The first formula is obvious and the second was established in the proof of Proposition 4.1. Identifying vectors and covectors by the Euclidean structure, one has:

$$p_i = A_i^*(\nu_i). \tag{32}$$

Consider a multi-jet $\bar{\phi} \in \Delta \subset J^1(U, \mathbf{R}^{m+1})^{\times n}$; as before, $\phi_i = (u_i, x_i, A_i)$. We want to show that

$$d\pi(T_{\bar{\phi}} J^1(U, \mathbf{R}^{m+1})^{\times n}) + T_{\pi(\bar{\phi})} D = T_{\pi(\bar{\phi})} J^1(U^{\times n}). \tag{33}$$

The space $T_{\pi(\bar{\phi})} D$ consists of vectors whose p_i components vanish, while the u_i and z components are arbitrary. Thus the equality (33) will follow once we show that every vector in $T_{\pi(\bar{\phi})} J^1(U^{\times n})$ with trivial u_i and z components is in the image of $d\pi$.

Consider an infinitesimal deformation $(u_i, x_i, A_i + \varepsilon B_i)$ of $\bar{\phi} \in J^1(U, \mathbf{R}^{m+1})^{\times n}$ where $B_i : \mathbf{R}^m \to \mathbf{R}^{m+1}$ is a linear map; this deformation determines a tangent vector $\xi \in T_{\bar{\phi}} J^1(U, \mathbf{R}^{m+1})^{\times n}$. Since $\pi(\bar{\phi}) \in D$, it follows from (32) that $A_i^*(\nu_i) = 0$. Therefore $d\pi(\xi)$ is a vector in $T_{\pi(\bar{\phi})} J^1(U^{\times n})$ whose p_i component is $B_i^*(\nu_i)$, while the u_i and z components vanish. Since $\nu_i \neq 0$, the vector $B_i^*(\nu_i) \in \mathbf{R}^m$ can be made arbitrary by varying B_i, and the result follows. $\Box$

We will use Proposition 4.1 to tackle the problem of finding topological lower bounds on the number of closed billiard trajectories by applying the methods of Morse theory. The function L_X being D_n-invariant, we will use equivariant Morse and Lusternik-Schnirelman theories. Namely, one has the next result.

Proposition 4.5 *Let $X \subset \mathbf{R}^{m+1}$ be a smooth strictly convex hypersurface. Then for any odd $n \geq 3$ the number of D_n-orbits of n-periodic billiard trajectories in X is greater than the cup-length of*

$$H^*(G(S^m, n)/D_n; \mathbf{Z}_2). \tag{34}$$

If X is n-generic then the number of D_n-orbits of n-periodic billiard trajectories in X is not less than the sum of Betti numbers

$$\sum_i \dim_{\mathbf{Z}_2} H^i(G(S^m, n)/D_n; \mathbf{Z}_2). \tag{35}$$

Proof. Start with the following claim: for odd n the cohomology group of the quotient $G(S^m, n)/D_n$ coincides with the equivariant cohomology of $G(S^m, n)$:

$$H^j(G(S^m, n)/D_n; \mathbf{Z}_2) \simeq H^j(ED_n \times_{D_n} G(S^m, n), \mathbf{Z}_2) \tag{36}$$

where ED_n is a contractible space with a free D_n-action.

Indeed, consider the Leray spectral sequence of the projection $ED_n \times_{D_n} G(S^m, n) \to G(S^m, n)/D_n$, see [6]. The E_2-term has the form $E_2^{p,q} = H^p(G(S^m, n)/D_n; \mathcal{L}^q)$, where $\mathcal{L}^q$ is the Leray sheaf. Let $\bar{x} \in G(S^m, n)$ be an orbit. Since n is odd, no reflection in D_n belongs to the stabilizer $H_{\bar{x}}$. Thus the stabilizer of $\bar{x}$ is a cyclic subgroup $H_{\bar{x}} \subset D_n$ of odd order. The stalk of $\mathcal{L}^q$ over the orbit of $\bar{x}$ is

$$H^q(ED_n \times_{D_n} D_n/H_{\bar{x}}; \mathbf{Z}_2) = H^q(ED_n/H_{\bar{x}}; \mathbf{Z}_2).$$

It follows that $\mathcal{L}^q = 0$ for $q > 0$. Therefore the Leray spectral sequence is nonzero only along the p-axis. This implies (36).

Now we use Proposition 4.1. Fix a sufficiently small $\varepsilon > 0$ and consider the function $L_X : G_\varepsilon(X, n) \to \mathbf{R}$, determined by the billiard hypersurface $X \subset \mathbf{R}^{m+1}$. Since L_X is D_n-invariant, the set of critical points of L_X is also D_n-invariant. Our task is to estimate the number of critical D_n-orbits. We have to take into account the presence of the boundary. However, due to statement (d) of Proposition 4.1, the boundary points make no contribution to the topology of $G_\varepsilon(X, n)$ – cf. [2] and also [13], chapter 1. In other words, the principles of the critical point theory apply to $L_X : G_\varepsilon(X, n) \to \mathbf{R}$ the same way as if $G_\varepsilon(X, n)$ were a manifold without boundary.

Assume that X is n-generic. Then, using the negative gradient flow of L_X, we obtain a D_n-equivariant cell decomposition of $G_\varepsilon(X, n)$. The number of cells in the resulting cell decomposition of the quotient space $G_\varepsilon(X, n)/D_n$ equals the number of critical D_n-orbits of L_X, i.e., the number of D_n-orbits of n-periodic billiard trajectories in X. This proves the second statement of Proposition 4.5.

To prove the first statement, apply equivariant Lusternik - Schnirelman theory developed in [17], see also [9]. We use Theorems 3.2 and 1.13 from [17]. The first example in section 1.14 of [17] of singular multiplicative D_n-cohomology theory is given by $Y \mapsto H^*(Y/D_n; \mathbf{Z}_2)$; in our case $H^*(G(X, n)/D_n; \mathbf{Z}_2)$ coincides with the equivariant cohomology $H^*_{D_n}(G(X, n); \mathbf{Z}_2)$ due to (36) and statement (b) of Proposition 4.1. $\square$

5 Equivariant cohomology of the cyclic configuration space of the sphere

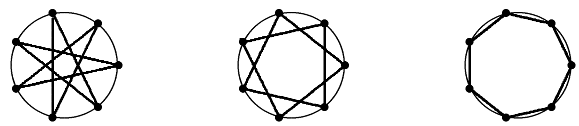
In view of Proposition 4.5, in order to find lower bounds on the number of periodic trajectories of billiards we need to compute the equivariant cohomology ring of the cyclic configuration space $G(S^m, n)$. This is the main goal of this section.

Our computation of the equivariant cohomology will be based on Theorem 4, giving the structure of the usual cohomology ring. We will also use the method which was first suggested by I. Babenko [1]. It consists in applying Morse theory in the opposite direction, that is, studying the topology of the cyclic configuration space $G(S^m, n)$ by examining the billiard inside the round ball in $\mathbf{R}^{m+1}$; in the latter case the closed trajectories are readily described. Note that a similar idea of studying the closed geodesics on the round sphere leads to a computation of the homology of the loop space of the sphere – see [7] and the references therein.

From now on we will assume that the number n is odd and that $m > 1$.

Let $X = S^m \subset \mathbf{R}^{m+1}$ be the unit sphere. Consider the corresponding length function $L_X : G(X, n) \to \mathbf{R}$. The critical points of L_X are precisely the closed billiard trajectories inside X having n reflections. Each such n-periodic trajectory lies in a two-dimensional plane P through the center of the sphere X. The intersection $P \cap X$

is a unit circle and the reflections go the same way as in the plane circular billiard $P \cap X$. Hence any billiard trajectory $(x_1, x_2, \ldots, x_n)$ is a plane regular n-gon, possibly star-shaped, inscribed into the circle $P \cap X$. If n is not prime then such a polygon may be multiple, i.e., it may traverse itself several times. The angle α between x_1 and x_2 is of the form $\alpha = 2\pi r/n$, where $1 \leq r \leq (n-1)/2$. This number is clearly related to the rotation number (cf. Example 2.1). The following picture shows the 7-periodic trajectories:



$r = 3$ $\qquad$ $r = 2$ $\qquad$ $r = 1$

Figure 3: Critical submanifolds for $G(S^n, 7)$

Note that any two n-periodic trajectories in X with the same rotation number r can be continuously deformed one to another. We conclude that the critical points of the function $L_X : G(X, n) \to \mathbf{R}$ form a disjoint union of connected submanifolds

$$V_0, V_1, \ldots, V_{(n-3)/2}, \tag{37}$$

where V_p denotes the set of all closed trajectories with the rotation number $(n-1-2p)/2$. Each V_p is diffeomorphic to the Stiefel manifold $V_{2,m+1}$ and hence is a closed manifold of dimension $2m-1$. The next result is due to Babenko [1].

Proposition 5.1 *If $X = S^m \subset \mathbf{R}^{m+1}$ is a round sphere then*

(a) The function $L_X : G(X, n) \to \mathbf{R}$ is nondegenerate in the sense of Bott.

(b) The index of the critical manifold V_p equals $2p(m-1)$.

(c) The critical values of the function L_X on the critical manifolds V_p increase: $L_X(V_p) < L_X(V_{p'})$ for $p < p'$.

As an important addition to Proposition 5.1 we make the following observation.

Proposition 5.2 *If X is a round sphere then $L_X : G(X, n) \to \mathbf{R}$ is a perfect Bott function with respect to the field $\mathbf{Z}_2$.*

Proof. Let us first explain the meaning of our statement. Choose $\varepsilon > 0$ as in Proposition 4.1 holds. We may find constants $c_0, \ldots, c_{(n-3)/2}$ with

$$L_X(V_p) < c_p < L_X(V_{p+1}).$$

Set: $F_p = L_X^{-1}((-\infty, c_p])$. We obtain a filtration

$$F_0 \subset F_1 \subset \ldots \subset F_{(n-3)/2} = G_\varepsilon(X, n), \tag{38}$$

and our statement means that the sum of the Poincaré polynomials of the pairs (F_p, F_{p-1}) with $\mathbf{Z}_2$ coefficients equals the Poincaré polynomial of $G_\varepsilon(X, n)$.

Indeed, the mod 2 Poincaré polynomial of (F_p, F_{p-1}) equals

$$\sum_j t^j \dim_{\mathbf{Z}_2} H^j(F_p, F_{p-1}; \mathbf{Z}_2) \;=\; t^{2p(m-1)}[t^{2m-1} + t^m + t^{m-1} + 1]. \tag{39}$$

Here we used the fact that (F_p, F_{p-1}) is homotopy equivalent to the Thom space of the negative normal bundle of V_p, the Thom isomorphism, the index computation, given by Proposition 5.1, and the fact that the Poincaré polynomial with coefficients in $\mathbf{Z}_2$ of the Stiefel manifold $V_{2,m+1}$ is $t^{2m-1} + t^m + t^{m-1} + 1$, cf. [5]. Summing formulae (39) for all $p = 0, \ldots, (n-3)/2$ we obtain

$$\sum_{p=0}^{(n-3)/2} \sum_j t^j \dim_{\mathbf{Z}_2} H^j(F_p, F_{p-1}; \mathbf{Z}_2) \;=$$

$$= \;(t^{2m-1} + t^m + t^{m-1} + 1) \cdot \frac{t^{(n-1)(m-1)} - 1}{t^{2(m-1)} - 1} \;=$$

$$= \;\frac{(t^m + 1)(t^{(n-1)(m-1)} - 1)}{t^{(m-1)} - 1},$$

which, according to Theorem 4, coincides with the Poincaré polynomial of $G(X, n)$. $\square$

Propositions 5.1 and 5.2 hold for even n as well, but then their statements are slightly different. We will not need these results in this paper.

Remark 5.3 For $m > 3$ there exists a different proof of Proposition 5.2, which does not use Theorem 4 and provides an independent computation of the Poincaré polynomial of the cyclic configuration space of the sphere. It is quite straightforward. One considers the spectral sequence of the filtration (38), where

$$E_1^{p,q} = H^{p+q}(F_p, F_{p-1}; \mathbf{Z}_2).$$

The calculation of the relative homology $H^{p+q}(F_p, F_{p-1}; \mathbf{Z}_2)$ as in the above proof and elementary geometric considerations show that all differentials must be zero provided $m > 3$.

This argument fails for $m = 2$ and for $m = 3$. The nonzero terms and the differential d_1 of the spectral sequence for $m = 2$ and $m = 3$ are shown in Figure 4. Unlike the case $m > 3$, this picture does not imply that the differentials vanish.

Note also that for $n = 3$ the proof of Proposition 5.2 trivializes: the filtration (38) consists of a single term only.

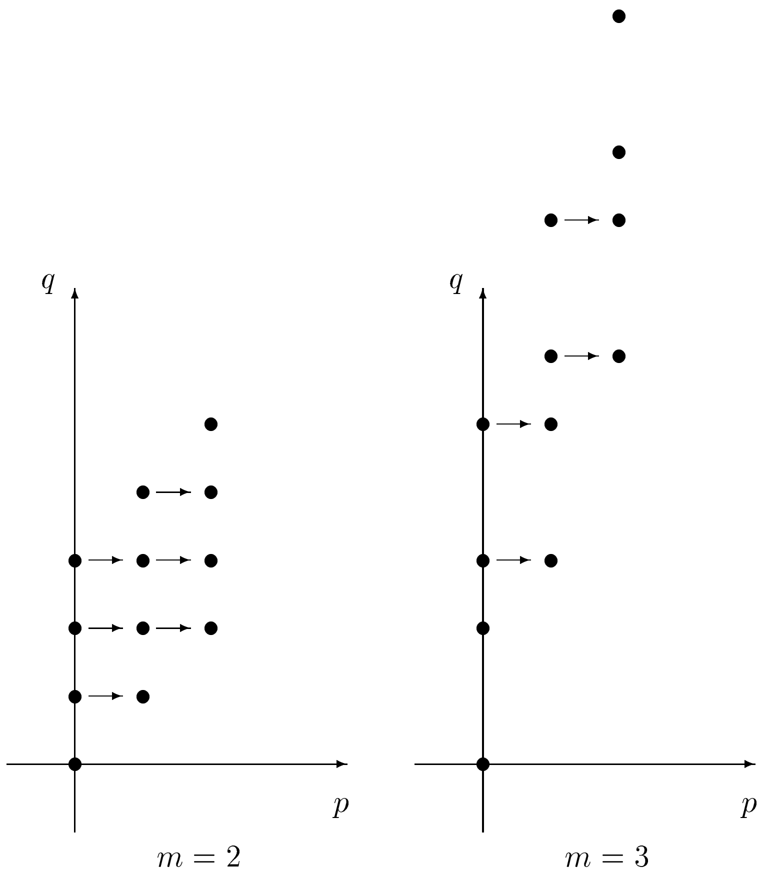


Figure 4: Nonzero terms of the spectral sequence for $m = 2$ and $m = 3$.

The main result of this section is the next theorem.

Theorem 7 *Let $m \geq 3$ and let n be an odd integer. Then the cohomology ring $H^*(G(S^m;n)/D_n;\mathbf{Z}_2)$ is multiplicatively generated by cohomology classes*

$$\sigma_{2i} \in H^{2i(m-1)}(G(S^m;n)/D_n;\mathbf{Z}_2), \quad \textit{where} \quad i = 1, 2, \ldots$$

and by classes

$$e \in H^1(G(S^m;n)/D_n;\mathbf{Z}_2) \quad \textit{and} \quad u \in H^m(G(S^m;n)/D_n;\mathbf{Z}_2).$$

These classes satisfy the following relations

(i) $\sigma_i = 0$ *for* $i \geq n-1$*;*

(ii) $\sigma_i\sigma_j = \binom{i+j}{i}\sigma_{i+j} + \epsilon_{ij}\sigma_{i+j-2}ue^{m-2}$*, where i and j are even and $\epsilon_{ij} \in \mathbf{Z}_2$;*

(iii) $e^m = 0$*;*

(iv) $u^2 = 0$.

The Poincaré polynomial of the quotient space $G(S^m, n)/D_n$ with coefficients in $\mathbf{Z}_2$ equals

$$\frac{(t^{(n-1)(m-1)}-1)}{(t^{2(m-1)}-1)}\cdot\frac{t^m-1}{t-1}\cdot(t^m+1),$$

and the sum of Betti numbers is $m(n-1)$.

The rest of this section consists of the proof of Theorem 7.

Start with the following lemma.

Lemma 5.4 *Let G be a finite group acting simplicially on a finite polyhedron Y such that the action of G on the cohomology $H^*(Y;\mathbf{Z}_2)$ is trivial. Suppose that $G' \subset G$ is a subgroup of odd index such that G acts trivially on $H^*(G';\mathbf{Z}_2)$. Then the induced homomorphism of the equivariant cohomology*

$$H^*_G(Y;\mathbf{Z}_2) \to H^*_{G'}(Y;\mathbf{Z}_2) \tag{40}$$

is an isomorphism.

The notation $H^*_G(Y;\mathbf{Z}_2)$ stands for the equivariant cohomology $H^*(EG\times_G Y;\mathbf{Z}_2)$.

Proof. We will use the comparison theorem for spectral sequences. Since G acts trivially on $H^*(Y;\mathbf{Z}_2)$, the Serre spectral sequence of the fibration $EG\times_G Y \to BG$ with fiber Y, converging to $H^*_G(Y;\mathbf{Z}_2)$, has the initial term

$$E_2^{p,q} \;=\; H^p(G;\mathbf{Z}_2)\otimes H^q(Y;\mathbf{Z}_2).$$

Similarly, we have a spectral sequence with the initial term

$$E'^{p,q}_2 \;=\; H^p(G';\mathbf{Z}_2)\otimes H^q(Y;\mathbf{Z}_2)$$

converging to $H^*_{G'}(Y;\mathbf{Z}_2)$. The inclusion $G' \to G$ induces a homomorphism of the spectral sequences $E \to E'$ which is an isomorphism of the E_2-terms (cf. [8], Proposition 10.4, chapter 3). Hence, by the comparison theorem for spectral sequences, (40) is an isomorphism. $\square$

Next we compute the D_n-equivariant cohomology of the critical manifolds of the function L_X.

Proposition 5.5 *Suppose that $m \geq 3$ and n is odd. Let V_p be any of the critical submanifolds (37) with the induced action of the dihedral group D_n. Then*

(a) the equivariant cohomology ring $H^*_{D_n}(V_p;\mathbf{Z}_2)$ has two multiplicative generators

$$e \in H^1_{D_n}(V_p;\mathbf{Z}_2) \quad \textit{and} \quad u \in H^m_{D_n}(V_p;\mathbf{Z}_2);$$

(b) they satisfy the relations $e^m = 0$ and $u^2 = 0$;

(c) *the classes $e^i u^j$, with $i = 0, 1, \ldots, m-1$ and $j = 0, 1$ form an additive basis of $H^*_{D_n}(V_p; \mathbf{Z}_2)$;*

(d) *the canonical homomorphism $H^*(V_p/D_n; \mathbf{Z}_2) \to H^*_{D_n}(V_p; \mathbf{Z}_2)$ is an isomorphism;*

(e) *the kernel of the canonical homomorphism $\Phi : H^*(V_p/D_n; \mathbf{Z}_2) \to H^*(V_p; \mathbf{Z}_2)$ coincides with the ideal generated by e, i.e., is the linear span of the classes $e^i u^j$ where $i \geq 1$.*

Proof. First note that (d) follows from the isomorphism (36) in the proof of Proposition 4.5. Indeed, the stabilizer of any orbit in V_p is a cyclic subgroup of odd order.

To compute the equivariant cohomology ring $H^*_{D_n}(V_p; \mathbf{Z}_2)$ we will apply Lemma 5.4 with $G = D_n$, $Y = V_p$ and $G' \simeq \mathbf{Z}_2$ being a subgroup of the dihedral group D_n generated by a reflection. Since V_p is homeomorphic to the Stiefel manifold $V_{2,m+1}$, the cohomology of V_p with $\mathbf{Z}_2$ coefficients is isomorphic to $\mathbf{Z}_2$ in dimensions $0, m-1, m, 2m-1$ and is trivial in all other dimensions. It follows that D_n acts trivially on $H^*(V_p; \mathbf{Z}_2)$; likewise, D_n acts trivially on $H^*(G'; \mathbf{Z}_2)$. Applying Lemma 5.4, we conclude that there exists a ring isomorphism

$$H^*_{D_n}(V_p; \mathbf{Z}_2) \simeq H^*_{G'}(V_p; \mathbf{Z}_2) \simeq H^*(V_p/G'; \mathbf{Z}_2). \tag{41}$$

The second isomorphism follows since the reflection acts freely on V_p for odd n. Formulae (41) show that, computing the equivariant cohomology $H^*_{D_n}(V_p; \mathbf{Z}_2)$, we may ignore a large part of the D_n-action and keep track only of the G'-action.

Note that V_p can be identified with the variety of ordered pairs (v_1, v_2) of unit vectors in $\mathbf{R}^{m+1}$ making the angle of $\alpha_p = \pi(n-1-2p)/n$. The reflection (i.e. the generator of G') acts on such pairs by sending (v_1, v_2) to (v_1, v_2'), where v_2' is the reflection of v_2 in the line spanned by v_1.

Applying the Gram - Schmidt orthogonalization, construct a diffeomorphism between the quotient space V_p/G' and the set of pairs (v_1, v_2) of mutually orthogonal unit vectors in $\mathbf{R}^{m+1}$ with identification $(v_1, v_2) \simeq (v_1, -v_2)$. In other words, V_p/G' is diffeomorphic to space of pairs (v, ℓ), where $e \in \mathbf{R}^{m+1}$ is a unit vector, and $\ell \subset \mathbf{R}^{m+1}$ is a one-dimensional linear subspace orthogonal to v. The projection $(v, \ell) \mapsto v$ identifies V_p/G' with the projective tangent bundle of S^m.

Alternatively, projecting $(v, \ell) \mapsto \ell$, we view V_p/G' as the space of a unit sphere bundle of a rank m vector bundle ξ over the projective space $\mathbf{RP}^m$. The fiber of ξ over a line $\ell \in \mathbf{RP}^m$ is the orthogonal complement $\ell^\perp$ of ℓ. The spectral sequence of this unit sphere bundle

$$E_2^{p\,q} \;=\; H^p(\mathbf{RP}^m; \mathbf{Z}_2) \otimes H^q(S^{m-1}; \mathbf{Z}_2) \Rightarrow H^{p+q}(V_p/G'; \mathbf{Z}_2)$$

has only two rows and the only possibly nontrivial differential is the transgression

$$d_m : E_m^{0,m-1} = H^{m-1}(S^{m-1}; \mathbf{Z}_2) \to E_m^{m,0} = H^m(\mathbf{RP}^m; \mathbf{Z}_2).$$

We claim that the differential $d_m : E_m^{0,m-1} \to E_m^{m,0}$ is an isomorphism. The image of the generator of $H^{m-1}(S^{m-1}; \mathbf{Z}_2)$ under d_m is the top Stiefel-Whitney class $w_m(\xi) \in$

$H^m(\mathbf{RP}^m;\mathbf{Z}_2)$. Let η be the tautological line bundle over $\mathbf{RP}^m$ whose fiber over a line ℓ is ℓ itself. Then the Whitney sum $\xi\oplus\eta$ is the trivial bundle of rank $m+1$. Since the total Stiefel-Whitney class of η is $1+e$, where $e\in H^1(\mathbf{RP}^m;\mathbf{Z}_2)$ is the generator, the Cartan's formula

$$(1+e)(1+w_1(\xi)+\ldots+w_m(\xi))=1 \tag{42}$$

gives $w_j(\xi)=e^j$, for all $j=1,\ldots,m$. In particular, $w_m(\xi)=e^m$.

These arguments completely describe the ring structure of $H^*(V_p/G';\mathbf{Z}_2)\simeq H^*_{D_n}(V_p;\mathbf{Z}_2)$. Namely, in the above spectral sequence, the classes $1,e,\ldots,e^{m-1}$ survive in the bottom row; also there is a class $u\in E_\infty^{1,m-1}$ such that the nonzero classes in row $q=m-1$, surviving in E_∞, are $u,ue,\ldots,ue^{m-1}$. The cohomology classes e and u lift uniquely from E_∞ to $H^*(V_p/G';\mathbf{Z}_2)$ and satisfy the same relations therein. This proves statements (a), (b), (c).

It remains to prove statement (e). Clearly, Φ is a ring homomorphism and $\Phi(e)=0$. Therefore (e) will follow once we show that $\Phi: H^m(V_p/G';\mathbf{Z}_2)\to H^m(V_p;\mathbf{Z}_2)$ is an isomorphism in degree m.

Consider the product $V_p\times[0,1]$ and identify the points $(x,1)$ and $(Tx,1)$ for all $x\in V_p$, where $T\in G'$ denotes the reflection. After the identification, we obtain a compact $2m$-dimensional manifold with boundary Y, so that Y is homotopy equivalent to V_p/G' and ∂Y is diffeomorphic to V_p. The restriction homomorphism $H^j(Y;\mathbf{Z}_2)\to H^j(\partial Y;\mathbf{Z}_2)$ coincides with Φ. Using the Poincaré duality we obtain

$$H^j(Y,\partial Y;\mathbf{Z}_2)\simeq H^{2m-j}(Y;\mathbf{Z}_2)\simeq\mathbf{Z}_2 \quad \text{for all } j. \tag{43}$$

Consider the exact cohomological sequence of $(Y,\partial Y)$. The homomorphisms $H^j(Y;\mathbf{Z}_2)\to H^j(\partial Y;\mathbf{Z}_2)$ are zero for $j=m-1$ and $j=m+1$, therefore the homomorphisms $\mathbf{Z}_2=H^{m-1}(\partial Y;\mathbf{Z}_2)\to H^m(Y,\partial Y;\mathbf{Z}_2)=\mathbf{Z}_2$ and $\mathbf{Z}_2=H^{m+1}(Y,\partial Y;\mathbf{Z}_2)\to H^{m+1}(Y;\mathbf{Z}_2)=\mathbf{Z}_2$ are isomorphisms. It follows that Φ is an isomorphism in the following exact sequence (we suppress the coefficients $\mathbf{Z}_2$ from notation):

$$H^{m-1}(\partial Y)\xrightarrow{\simeq}H^m(Y,\partial Y)\to H^m(Y)\xrightarrow{\Phi}H^m(\partial Y)\to H^{m+1}(Y,\partial Y)\xrightarrow{\simeq}H^{m+1}(Y).$$

The proposition is proved. $\square$

Proof of Theorem 7. Let $X\subset\mathbf{R}^{m+1}$ be a round sphere. Consider filtration (38) of the space $G_\varepsilon(X,n)$, where $\varepsilon>0$ is small enough, so that the claims of Proposition 4.1 hold. Denote the space F_i/D_n by F'_i. Hence we obtain a filtration

$$F'_0\subset F'_1\subset\ldots\subset F'_{(n-3)/2}=G_\varepsilon(X,n)/D_n \tag{44}$$

of the space of D_n-orbits.

Formulate the following *inductive hypothesis* $\mathcal{F}_p$, depending on a number $p=0,1,\ldots,(n-3)/2$:

The cohomology ring $H^*(F'_p;\mathbf{Z}_2)$ is multiplicatively generated by cohomology classes

$$\sigma_{2i}\in H^{2i(m-1)}(F'_p;\mathbf{Z}_2),\quad \textit{where}\quad i=1,2,\dots$$

and by classes

$$e\in H^1(F'_p;\mathbf{Z}_2)\quad \textit{and}\quad u\in H^m(F'_p;\mathbf{Z}_2).$$

These classes satisfy the following relations

(i) $\sigma_i=0$ *for* $i>2p$;

(ii) $\sigma_i\sigma_j=\binom{i+j}{i}\sigma_{i+j}+\epsilon_{ij}\sigma_{i+j-2}ue^{m-2}$, *where i and j are even and* $\epsilon_{ij}\in\mathbf{Z}_2$;

(iii) $e^m=0$;

(iv) $u^2=0$.

The Poincaré polynomial of the quotient space F'_p with coefficients in $\mathbf{Z}_2$ equals

$$\frac{(t^{2(p+1)(m-1)}-1)}{(t^{2(m-1)}-1)}\cdot\frac{t^m-1}{t-1}\cdot(t^m+1),$$

and the sum of Betti numbers is $2m(p+1)$.

The kernel of the canonical homomorphism $\Phi: H^*(F'_p;\mathbf{Z}_2)\to H^*(F_p;\mathbf{Z}_2)$ coincides with the ideal generated by e.

Our aim is to show that statement $\mathcal{F}_p$ holds for $p=(n-3)/2$. This would imply Theorem 7. Argue by induction. Proposition 5.5 implies that $\mathcal{F}_0$ holds. Hence we need to show that $\mathcal{F}_p$ implies $\mathcal{F}_{p+1}$.

Assuming that $\mathcal{F}_p$ is satisfied, consider the boundary homomorphism

$$\delta: H^i(F'_p;\mathbf{Z}_2)\to H^{i+1}(F'_{p+1},F'_p;\mathbf{Z}_2). \tag{45}$$

We claim that this homomorphism is trivial for all i. Indeed, the group $H^i(F'_p;\mathbf{Z}_2)$ is nonzero only for $i\le 2(p+1)(m-1)+1$ (by the assumption $\mathcal{F}_p$) and the group $H^{i+1}(F'_{p+1},F'_p;\mathbf{Z}_2)$ is nonzero only for $i+1\ge 2(p+1)(m-1)$. The latter follows since F'_{p+1}/F'_p is homotopy equivalent to the Thom space of a vector bundle of rank $2(p+1)(m-1)$ over V_{p+1}/D_n, the space of D_n-orbits of the critical manifold V_{p+1}. Therefore homomorphism (45) can be nonzero only for three values of i, namely for

$$i=2(p+1)(m-1)-1,\ i=2(p+1)(m-1)\quad\text{and}\quad i=2(p+1)(m-1)+1.$$

Let us first show that δ vanishes for $i=2(p+1)(m-1)-1$, i.e., in the lowest possible dimension. Consider the critical manifold $V_{p+1}\subset F_{p+1}$ and the decomposition of its normal bundle into the negative and positive parts with respect to the Hessian of function L_X. This decomposition is D_n-equivariant, and after factorization by the action of the dihedral group D_n it produces two bundles (the negative and the positive) over V_{p+1}/D_n. The space F'_{p+1}/F'_p is homotopy equivalent to the Thom space of the negative bundle. Consider the commutative diagram

$$\begin{array}{ccc} H^i(F'_p;\mathbf{Z}_2) & \xrightarrow{\delta} & H^{i+1}(F'_{p+1},F'_p;\mathbf{Z}_2) \\ \Phi\downarrow & & \downarrow\Phi_1 \\ H^i(F_p;\mathbf{Z}_2) & \xrightarrow{\delta_1} & H^{i+1}(F_{p+1},F_p;\mathbf{Z}_2). \end{array}$$

Here Φ_1 is defined similarly to Φ, i.e. it is induced by the canonical projection $(F_{p+1},F_p) \to (F'_{p+1},F'_p)$. We know that δ_1 vanishes (by Proposition 5.2). Hence, to show that δ vanishes for $i = 2(p+1)(m-1)-1$, it suffices to show that Φ_1 is a monomorphism for this value of i. We have the following commutative diagram with the vertical maps being the Thom isomomorphisms

$$\begin{array}{ccc} H^{i+1}(F'_{p+1},F'_p;\mathbf{Z}_2) & \xrightarrow{\Phi_1} & H^{i+1}(F_{p+1},F_p;\mathbf{Z}_2) \\ \simeq\ \downarrow & & \downarrow\ \simeq \\ H^0(V'_{p+1};\mathbf{Z}_2) & \xrightarrow{\Phi_2} & H^0(V_{p+1};\mathbf{Z}_2). \end{array}$$

It is clear that the homomorphism Φ_2 is an isomorphism (compare statement (e) of Proposition 5.5), and therefore Φ_1 is also an isomorphism for $i = 2(p+1)(m-1)-1$.

Hence δ vanishes for $i = 2(p+1)(m-1)-1$, in other words, $\delta(\sigma_{2p}ue^{m-3}) = 0$. Recall that we assume that $m \geq 3$. Now we will show that δ vanishes in the two other dimensions as well:

$$\delta(\sigma_{2p}ue^{m-2}) = 0, \quad \delta(\sigma_{2p}ue^{m-1}) = 0.$$

It is clear that there exists a class $\tilde{e} \in H^1(F'_{p+1};\mathbf{Z}_2)$ such that $\tilde{e}|_{F'_p} = e$. Using statement 12 in §6, chapter 5 of [18], we obtain

$$\delta(\sigma_{2p}ue^{m-2}) = \delta(\sigma_{2p}ue^{m-3}) \cdot \tilde{e} = 0.$$

Similarly, $\delta(\sigma_{2p}ue^{m-1}) = 0$.

The vanishing of the boundary homomorphism (45) means that filtration (44) is also perfect, that is, we have an isomorphism

$$H^*(F'_{p+1};\mathbf{Z}_2) \simeq H^*(F'_p;\mathbf{Z}_2) \oplus H^*(F'_{p+1}/F'_p;\mathbf{Z}_2) \tag{46}$$

The additive structure of the relative homology $H^*(F'_{p+1}/F'_p;\mathbf{Z}_2)$ is given by Proposition 5.5 with a shift of all degrees by $2(p+1)(m-1)$. Here we use the Thom isomorphism and the equality between the equivariant cohomology and the cohomology of the factor space V_{p+1}/D_n, which holds by statement (d) of Proposition 5.5. Hence (46) fully describes the additive structure of the cohomology $H^*(F'_{p+1};\mathbf{Z}_2)$, which coincides with the statement of the hypothesis $\mathcal{F}_{p+1}$.

Now we want to show that the multiplicative structure of $H^*(F'_{p+1};\mathbf{Z}_2)$ is as stated in the hypothesis $\mathcal{F}_{p+1}$. Notice first that the cohomology classes e, u, $\sigma_2,\dots,\sigma_{2p}$ in $H^*(F'_p;\mathbf{Z}_2)$ extend uniquely to cohomology classes in $H^*(F'_{p+1};\mathbf{Z}_2)$ and denote the extensions by the same symbols. Our next problem is to identify the class $\sigma_{2p+2}\in H^{2(p+1)(m-1)}(F'_{p+1};\mathbf{Z}_2)$; caution must be exercised since the cohomology of F'_{p+1} in this degree is two-dimensional.

Consider again the critical manifold $V_{p+1}\subset F_{p+1}$ and the D_n-equivariant decomposition of its normal bundle into the negative and positive parts with respect to the Hessian of the function L_X. After factorization by the action of the dihedral group, these bundles give two bundles (the negative and the positive) over V_{p+1}/D_n; the space F'_{p+1}/F'_p is homotopy equivalent to the Thom space of the negative bundle. The Thom class of the negative bundle lies in $H^{2(p+1)(m-1)}(F'_{p+1}/F'_p;\mathbf{Z}_2)$, but the last group is canonically embedded into $H^{2(p+1)(m-1)}(F'_{p+1};\mathbf{Z}_2)$ via (46). Hence we define the class σ_{2p+2} as representing this Thom class.

All the generators having been defined, we want to check that the hypothesis $\mathcal{F}_{p+1}$ is satisfied. First we note that $\sigma_{2p+2}u$ and $\sigma_{2p+2}e^{m-1}$ are nonzero cohomology classes in $H^*(F'_{p+1};\mathbf{Z}_2)$. This would follow from the Thom Isomorphism Theorem once we show that e^{m-1} and u restrict to nontrivial cohomology classes on V_{p+1}/D_n. Nontriviality of the restriction of e^{m-1} is almost obvious; indeed, for any $(m-2)$-connected D_n-invariant subset $A\subset F_{p+1}$ we have $e^{m-1}|_{A/D_n}\neq 0$, as follows by considering the Serre spectral sequence. In order to show that $u|_{V_{p+1}/D_n}\neq 0$, it is enough to show that $u|_{V_{p+1}}\neq 0$ (by statement (e) of Proposition 5.5). Let $W=V_{3,m+1}$ be the Stiefel manifold of triples (e_1,e_2,e_3) of mutually orthogonal unit vectors in $\mathbf{R}^{m+1}$. For any $p=0,1,\dots,(n-3)/2$ denote by $I_p: W\to G(S^m,n)$ the following map

$$I_p(e_1,e_2,e_3)=(x_1,x_2,\dots,x_n),\quad x_{j+1}=\cos(j\alpha_p)e_1+\sin(j\alpha_p)e_2,$$

where $j=0,\dots,n-1$ and $\alpha_p=\pi(n-1-2p)/n$. The image of I_p is the critical submanifold V_p. Hence, it is enough to show that $I_p^*(u)\neq 0$ for any p. Construct a homotopy between I_p and I_0. Let

$$H_\tau: W\to G(S^m,n),\quad \tau\in[0,1],$$

be defined by $H_\tau(e_1,e_2,e_3)=(x_1,x_2,\dots,x_n)$, where

$$x_{j+1}=\cos(j\alpha_\tau)e_1+\sin(j\alpha_\tau)e_2,\quad \text{and}\quad \alpha_\tau=\pi(n-1-2p(1-\tau))/n,$$

for $j=0,1,2,\dots,n-2$, while

$$x_n=\cos((n-1)\alpha_\tau)e_1+\sin((n-1)\alpha_\tau)e_2+\sin(\pi\tau)e_3.$$

It is clear that $H_0=I_p$ and $H_1=I_0$. Since $u|_{V_0}\neq 0$ we obtain $I_0^*(u)\neq 0$ as a consequence of Proposition 10.3 in [5]. This Proposition describes the cohomology of Stiefel manifolds with $\mathbf{Z}_2$ coefficients; it implies that $I_0^*: H^*(V_p;\mathbf{Z}_2)\to H^*(W;\mathbf{Z}_2)$ is a monomorphism. Hence, it follows $I_p^*(u)=I_0^*(u)\neq 0$ and thus $u|_{V_p}\neq 0$.

We want to show that for even i and j with $i + j = 2p + 2$ the following relation holds in the ring $H^*(F'_{p+1}; \mathbf{Z}_2)$:

$$\sigma_i \sigma_j = \binom{2p+2}{i} \sigma_{2p+2} + \epsilon_{ij} \sigma_{2p} u e^{m-2}, \tag{47}$$

where $\epsilon_{ij} \in \mathbf{Z}_2$. Note that for $r = 2(p+1)(m-1)$ the homomorphism

$$\Phi : H^r(F'_{p+1}; \mathbf{Z}_2) \to H^r(F_{p+1}; \mathbf{Z}_2)$$

has one-dimensional image and one-dimensional kernel. For any even $r = 2, 4, \ldots, 2p+2$ the image of the Thom class σ_r under Φ is nonzero and so it equals the class $\Phi(\sigma_r) \in H^*(F_{p+1}; \mathbf{Z}_2)$, which was denoted in Theorem 4 by σ_r. From Theorem 4 we know that

$$\Phi(\sigma_i)\Phi(\sigma_j) = \binom{2p+2}{i} \Phi(\sigma_{2p+2}).$$

This proves (47) since $\sigma_{2p} u e^{m-2}$ belongs to the kernel of Φ.

The rest of properties in hypothesis $\mathcal{F}_{p+1}$ are now obvious. Thus $\mathcal{F}_p$ implies $\mathcal{F}_{p+1}$, and the proof of Theorem 7 is complete. $\Box$

Remark 5.6 The above proof shows that the function $L_X : G(X, n) \to \mathbf{R}$ is perfect with respect to the field $\mathbf{Z}_2$ in the D_n-equivariant sense as well, compare Proposition 5.2.

We conclude the paper with a proof of Theorem 1 formulated in Introduction.

Proof of Theorem 1. Statement (B) follows from Proposition 4.5 and Theorem 7 that states that the sum of Betti numbers of the space $G(S^m; n)/D_n$ is $m(n-1)$.

To prove statement (A) of Theorem 1 one needs to find the cup-length of $H^*(G(S^m, n)/D_n; \mathbf{Z}_2)$. The argument is similar to the one in Corollaries 5 and 6. Namely, the length of the longest nontrivial product of classes σ_{2i} from Theorem 7 equals the maximal number of 1's in the binary expansion, that *even* integers, not exceeding $n-3$, may have. This number is equal to the maximal number of 1's in the binary expansion, that integers, not exceeding $(n-3)/2$, may have, that is, to

$$[\log_2((n-3)/2 + 1)] = [\log_2(n-1)] - 1.$$

We also have the classes u and e^{m-1} at our disposal; therefore the cup-length equals $[\log_2(n-1)] + m - 1$. Statement (A) follows now from Proposition 4.5. $\Box$

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