The algebraic geometry of representation spaces associated to Seifert fibered homology 3-spheres

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THE ALGEBRAIC GEOMETRY OF REPRESENTATION SPACES ASSOCIATED TO SEIFERT FIBERED HOMOLOGY 3-SPHERES

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Introduction

In this paper we will study geometric properties of representation spaces associated to certain homology 3–spheres. Recall that a smooth 3–dimensional manifold M is a homology 3–sphere, if it has the same integral homology groups as S^3. Examples of homology spheres exist in abundance, e.g. the well known Poincaré sphere $SU(2)/I$, where $I \subset SU(2)$ is the binary icosahedral group. A rather large but still accessible class of homology 3–spheres are the Seifert fibered 3–spheres. These can be described as follows: Let $a_1, \ldots, a_n$ be integers greater that 1. For a sufficiently general $(n-2) \times n$–matrix $A = (\alpha_{ij})$ of complex numbers the complete intersection

$$V_A(a_1, \ldots, a_n) = \{z \in \mathbf{C}^n \mid \alpha_{i1} z_1^{a_1} + \ldots + \alpha_{in} z_n^{a_n} = 0; \quad i = 1, \ldots, n-2\}$$

of Brieskorn varieties is a complex surface which is non–singular except at the origin. The link

$$\Sigma_A(a_1, \ldots, a_n) = V_A(a_1, \ldots, a_n) \cap S^{2n-1}$$

of the singularity is a smooth 3–manifold whose diffeomorphism type is independent of A. $\Sigma_A(a_1, \ldots, a_n)$ is a homology 3–sphere if and only if the a_i are pairwise relatively prime. The natural $\mathbf{C}^*$–action on $V_A(a_1, \ldots, a_n)$ induces a fixed point free S^1–action on $\Sigma_A(a_1, \ldots, a_n)$ with orbit space S^2. The orbit map represents $\Sigma_A(a_1, \ldots, a_n)$ as a Seifert fibered space over S^2 with n exceptional (multiple) fibres of order $a_1, \ldots, a_n$. The set of diffeomorphism types $\Sigma(a_1, \ldots, a_n)$ of these neighborhood boundaries is exactly the set of all diffeomorphism types of Seifert fibered homology 3–spheres, as has been shown by Neumann and Raymond, [78]. This includes for $n = 3$ the well known Brieskorn Spheres $\Sigma(p, q, r)$ with $\Sigma(2, 3, 5)$ being the Poincaré sphere $SU(2)/I$. There exist several alternative ways to construct these Seifert fibered 3–spheres, e.g. using Dehn twists or as homogeneous spaces of either $SU(2)$ or $\widetilde{\mathbf{P}Sl}(2, \mathbf{R})$ as shown by Milnor [75], Neumann [77] and Neumann–Raymond [78]. The representation spaces which we will investigate are the spaces

$$\mathcal{R}(\Sigma) = Hom^*(\pi_1(\Sigma), SU(2))/ad\, SU(2)$$

of nontrivial $SU(2)$–representations of the fundamental group $\pi_1(\Sigma)$ of such a Seifert fibered homology 3–sphere $\Sigma = \Sigma(a_1, \ldots, a_n)$.

The motivation for analyzing these spaces arises from recent developments in 3- and 4–dimensional manifold theory, which we briefly indicate. For any (not necessarily Seifert fibered) oriented homology 3–sphere M, Casson has defined a $\mathbf{Z}$–valued invariant $\lambda(M)$

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which is computed using the space $\mathcal{R}(M)$ of irreducible $SU(2)$–representations of $\pi_1(M)$ (cf. Akbulut–McCarthy [87]). This invariant has been refined by Floer [88], who introduced the instanton homology of such a sphere M. His invariant is a $\mathbf{Z}/8$–graded abelian group $I_*(M)$, which is the homology group of a certain graded complex whose modules —in the simplest cases— are generated by the elements of $\mathcal{R}(M)$. These instanton homology groups have very interesting relations to 4–manifold theory: Donaldson has defined a series of $\mathcal{C}^\infty$–invariants for certain smooth closed 4–manifolds X which have been applied to detecting different $\mathcal{C}^\infty$–structures (cf. Donaldson [87] and [87]). His invariants are a sequence of $\mathbf{Z}$–valued polynomials Φ_k on $H_2(X;\mathbf{Z})$ defined for sufficiently large integers k. If now X is decomposed into two pieces $X = X^+ \cup X^-$ with intersection $X^+ \cap X^- = M$ a homology 3–sphere M, then the Donaldson polynomials factor in some sense through the instanton homology groups of M. This is nicely explained by Atiyah [88]. Both the definition of the Donaldson polynomials as well as of the Floer homology makes heavy use of gauge theory on 3- and 4–dimensional manifolds. These invariants are therefore in general difficult to compute explicitely.

In order to determine the Floer homology for Seifert fibered homology spheres, Fintushel and Stern [88] showed that any connected component of the compact space $\mathcal{R}(\Sigma)$ is a differentiable manifold of even dimension. The critical points of a Morse function $\mathcal{R}(\Sigma)\longrightarrow\mathbf{R}$ give a base of the instanton chain complex. The grading in the chain complex of such a critical point b is $-(R(b) - \mu(b) + 3)$. Here $\mu(b)$ denotes the Morse index of b and $R(b)$ is an invariant of the connected component of $b \in \mathcal{R}(\Sigma)$. The invariants $R(b)$ take only odd values. By describing $\mathcal{R}(\Sigma)$ as a space of "linkages", Fintushel and Stern determined the 2–dimensional components of $\mathcal{R}(\Sigma)$ to be spheres. So in the case of Seifert fibered spheres with no more than four exceptional fibers (otherwise there are components of dimension greater than 2!) the differentials in the Floer chain complex have to vanish and $I_*(\Sigma)$ can be computed from the numerical invariants $R(b)$ and the Betti numbers of the components of $\mathcal{R}(\Sigma)$. Fintushel and Stern finally conjectured that this nice description should hold for all Seifert fibered spheres. More precisely:

Conjecture(Fintushel–Stern): For a Seifert fibered homology sphere Σ the representation space $\mathcal{R}(\Sigma)$ admits a Morse function with only even indices.

The description of $\mathcal{R}(\Sigma)$ in terms of linkages in principle works in all dimensions, but gets quite involved even in low dimensions. In this paper we introduce a different approach, showing that the representation spaces $\mathcal{R}(\Sigma)$ of Seifert fibered homology spheres Σ admit a description in terms of algebraic geometry. We show that the $\mathcal{R}(\Sigma)$'s are complex algebraic varieties, which exhibit a surprisingly rich geometry. Making use of their geometric properties we prove the conjecture "up to torsion".

Let us explain this in more detail: The quotient group $\pi_1(\Sigma)/center$ is a Fuchsian group of genus zero. These groups arise as fundamental groups of certain algebraic surfaces, known as generalized Dolgachev surfaces. In particular there is for any Seifert fibered homology sphere Σ a Dolgachev surface X such that the space $\mathcal{R}(\Sigma)$ coincides with the representation space

$$\mathcal{R}(X) := Hom^*(\pi_1(X), SU(2))/ad\, SU(2).$$

The latter space on the other hand can be interpreted as a certain moduli space of algebraic vector bundles on X, using Donaldson's solution [85] to the Kobayashi–Hitchin conjecture.

This identification provides $\mathcal{R}(\Sigma)$ with the structure of a complex projective variety. We work this out in the first chapter.

In the second chapter we show that every connected component of this moduli space is a smooth rational variety. These varieties come with a natural stratification, the strata being locally closed smooth subvarieties isomorphic to Zariski open subsets of projective spaces. As a byproduct it turns out that representations with the same rotation numbers (cf. Fintushel–Stern [88]) form a connected component. A similar result is true for $\mathbf{P}Sl(2,\mathbf{R})$–representations, according to Jankins–Neumann [85]. The main result of the second chapter yields the conjecture of Fintushel–Stern for all components of $\mathcal{R}(\Sigma)$ of dimension ≤ 4. Furthermore —using a result of Smale [62]— it shows that the conjecture is equivalent to the following purely homological conjecture:

Equivalent conjecture: The cohomology groups $H^i(\mathcal{R}(\Sigma);\mathbf{Z})$ vanish in odd dimensions.

In the third chapter we prove this modified conjecture up to torsion, i.e. we show that the odd Betti numbers of $\mathcal{R}(\Sigma)$ vanish. In order to achieve this we investigate the geometry of the stratification of the moduli spaces. The individual strata are complements of certain subvarieties in projective spaces. These subvarieties, though being singular, have a nice description as cones over secant varieties. Applying Deligne's solution to the Weil conjectures to these cones we obtain the vanishing of the odd Betti numbers. The birational invariance of the topological Brauer group finally proves the conjecture of Fintushel and Stern for the components of $\mathcal{R}(\Sigma)$ of dimension not greater than six.

In chapter four we illustrate how the results can be used for explicit computations. In particular we give formulae for the computation of the Betti numbers of the representation spaces $\mathcal{R}(\Sigma(a_1,\ldots,a_n))$ in terms of the initial data $a_1,\ldots,a_n$. This would give the Floer homology groups, if the conjecture of Fintushel and Stern turned out to be true in general. We illustrate the general situation with some complex 2–dimensional examples.

At the end of this introduction we comment upon the rôle of the Dolgachev surfaces. One should think both the Dogachev surfaces and the Seifert fibered spheres as being different geometric realizations of a 2–dimensional orbifold: Moding out the S^1–action on a Seifert sphere gives the same orbifold as does the elliptic fibration of a Dolgachev surface. The spaces $\mathcal{R}(\Sigma)$ and $\mathcal{R}(X)$ thus are nothing else than representation spaces of orbifold fundamental groups.

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Addendum: After finishing the paper, we learnt that P. Kirk and E. Klassen proved the conjecture of Fintushel and Stern directly. Using linkages they showed that there always is a Morse function with only even indices. Combining this result with the computation of the Betti numbers of the representation space in chapter 4 and the work of Fintushel and Stern, one gets the Floer homology for all Seifert fibered homology spheres. Finally Furuta and Steer have announced similar results: Using differential geometry they show that the components of the representation spaces are simply connected Kähler manifolds and compute the integral homology groups.

1. Representation spaces and moduli of stable bundles

In this chapter we will interpret the representation space $\mathcal{R}(\Sigma(a_1,\ldots,a_n))$ as a moduli space of stable algebraic bundles over a generalized Dolgachev surface.

A set $\{a_1,\ldots,a_n\}$ of pairwise coprime integers greater than 1 constitute the initial data. We will always assume these integers to be indexed in such a way that at most a_1 is even. A Dolgachev surface X is constructed as follows: Blowing up the projective plane $\mathbf{P}^2$ in the nine base points of a generic cubic pencil results in an elliptic fibration $\hat{\mathbf{P}}^2(x_0,\ldots,x_8)\longrightarrow\mathbf{P}^1$. Applying logarithmic transformations along n disjoint smooth fibres with multiplicities $a_1,\ldots,a_n$ gives a complex surface X together with an elliptic fibration $\pi: X\longrightarrow\mathbf{P}^1$. In order to emphasize the a_i, such a Dolgachev surface X sometimes will be denoted by $X(a_1,\ldots,a_n)$.

Let F denote a generic fiber of π and F_i the reduction of a multiple fiber with multiplicity a_i. The divisors F and a_iF_i are linearly equivalent. The canonical bundle formula (cf.Barth, Peters, Van de Ven [84], p.161) shows that a canonical divisor K_X is linearly equivalent to $-F+\sum_{i=1}^n(a_i-1)F_i$. In particular the geometric genus $p_g=h^0(K_X)=0$. Logarithmic transformations preserve the topological Euler characteristic, hence $e(X)=12$ and Noether's formula implies $\chi(\mathcal{O}_X)=1$. Thus $h^1(\mathcal{O}_X)=h^{0,1}(X)$ vanishes and the Picard group $Pic(X)$ is isomorphic to the second cohomology of X with integer coefficients. There is an ample line bundle $\mathcal{L}$ on X with $c_1(\mathcal{L})^2>0$. By Kodaira's ampleness criterion X is algebraic (cf. Barth et al., [84], p.116 and 126).

The following formulae for certain cohomology groups will be useful in the sequel. Note that a vertical divisor always is linearly equivalent to a divisor of the form $lF+\sum_i l_iF_i$ with $0\le l_i<a_i$.

1.1. Lemma: Let $\mathcal{L}$ denote the bundle $\mathcal{O}_X(lF+\sum_i l_iF_i)$ with $0\le l_i<a_i$.

i) $h^0(\mathcal{L})=max(l+1,0)$; $\quad h^1(\mathcal{L})=max(l,-l-1)$; $\quad h^2(\mathcal{L})=max(0,-l)$.

ii) The bundles $\mathcal{L}\otimes_{\mathcal{O}_X}\mathcal{O}_{nF}$ and $\mathcal{L}\otimes_{\mathcal{O}_X}\mathcal{O}_{n_iF_i}$ are isomorphic to $\mathcal{O}_{nF}$ and $\mathcal{O}_{n_iF_i}(l_iF_i)$, respectively. One has $h^0(\mathcal{O}_{nF})=n$ and $h^0(\mathcal{O}_{n_iF_i}(l_iF_i))=1+[a_i^{-1}(n_i-l_i-1)]$, where $[x]$ is the greatest integer satisfying $[x]\le x$.

iii) $h^0(\mathcal{O}_{D_1}\otimes\mathcal{L})=h^0(\mathcal{O}_{D_2}\otimes\mathcal{L})$, if D_1 and D_2 are linearly equivalent vertical divisors.

Proof: Suppose $l\ge 0$. Let $D_1,\ldots D_l$ denote pairwise disjoint generic fibers. Then $\mathcal{O}_{D_i}\otimes\mathcal{L}\cong\mathcal{O}_{D_i}$, since $\mathcal{L}$ admits a section which is nonvanishing on D_i. The sheaf $\mathcal{O}_X(lF)$ admits a section vanishing exactly on $D_1,\ldots,D_l$. The corresponding short exact sequence of sheaves

$$0\longrightarrow\mathcal{O}_X\longrightarrow\mathcal{O}_X(lF)\longrightarrow\oplus_{i=1}^l\mathcal{O}_{D_i}\longrightarrow 0$$

gives $h^0(\mathcal{O}_X(lF))=l+1$, since X has vanishing irregularity and the D_i are connected.

The map $0\longrightarrow\mathcal{O}_X(lF+l_iF_i)\longrightarrow\mathcal{O}_X((l+1)F)$, induced by multiplication with the divisor $(a_i-l_i)F_i$, yields an estimate $l+1\le h^0(lF+l_iF_i)\le l+2$. There are sections in $\mathcal{O}_X((l+1)F)$ not containing a_iF_i in its zero locus. Hence the second inequality has to be a strict one. Induction on the number of nonvanishing l_i implies $h^0(\mathcal{L}) = l + 1$. Serre duality shows

$$h^2(\mathcal{L}) = h^0(\mathcal{K}_X \otimes \mathcal{L}^*)$$
$$= h^0(\mathcal{O}_X((-l-1)F + \Sigma_i(a_i - l_i - 1)F_i) = 0.$$

Applying the Riemann Roch formula finally gives $\chi(\mathcal{L}) = 1$, which implies $h^1(\mathcal{L}) = l$. This shows i) in case $l \geq 0$.
The case $l < 0$ follows by Serre duality.
The Riemann–Roch formula gives

$$h^1(\mathcal{O}_X(l_iF_i)) = h^0(\mathcal{O}_X(l_iF_i)) + h^0(\mathcal{O}_X(K_X - l_iF_i)) - 1 = 0.$$

Applied to the cohomology sequence associated to the short exact sequence

$$0 \longrightarrow \mathcal{O}_X((l_i - n_i)F_i) \longrightarrow \mathcal{O}_X(l_iF_i) \longrightarrow \mathcal{O}_{n_iF_i}(l_iF_i) \longrightarrow 0$$

one gets:

$$\begin{aligned} h^0(\mathcal{O}_{n_iF_i}(l_iF_i)) &= h^1(\mathcal{O}_X((l_i - n_i)F_i)) + 1 - h^0(\mathcal{O}_X(l_i - n_i)F_i) \\ &= h^0(\mathcal{O}_X(K_X + (n_i - l_i)F_i) \\ &= h^0(\mathcal{O}_X(-F + (a_i - l_i + n_i - 1)F_i)) \\ &= \left[\frac{a_i + n_i - l_i - 1}{a_i}\right]. \end{aligned}$$

The third statement is a direct consequence of the first two using the linear equivalence $a_iF_i \sim F$. ♣

The choices involved in the construction of a Dolgachev surface give rise to a moduli space of algebraic surfaces. Nevertheless the diffeomorphism type of the underlying four dimensional manifolds is completely determined by the set $\{a_i\}$. In case $n \leq 1$ the surface is rational and diffeomorphic to $\hat{\mathbf{P}}^2(x_0, \ldots, x_8)$. The moduli space is irreducible for $n = 2$ according to Friedman and Morgan, [88], p. 318. For $n \geq 3$ we do not know, whether the moduli space is irreducible. However, by a result of Ue [86], p. 634, the diffeomorphism type is determined by the fundamental group, which in turn is determined by the $\{a_i\}$.

1.2. Proposition:

The fundamental group of a Dolgachev surface $X(a_1, \ldots, a_n)$ for a given base point has the presentation

$$\pi_1(X(a_1, \ldots, a_n)) = < t_1, \ldots, t_n \mid t_i^{a_i} = 1,\ t_1t_2 \cdot \ldots \cdot t_n = 1 > .$$

The closed paths t_i are freely homotopic, i.e. the base point is not fixed by the homotopy, to closed paths τ_i, contained in the exceptional fibers F_i. These paths satisfy the following property: Parallel transport along τ_i in the holomorphic flat normal bundle $N_{F_i/X}$ of F_i results in multiplication with $exp(\frac{-2\pi\sqrt{-1}}{a_i})$.

Proof: The computation of the fundamental group of X was done in principle by Dolgachev [81] II.3, more explicitely by Ue, [86] p 634 and p 639. Let U_i denote a small disk neighborhood of $\pi(F_i) \in \mathbf{P}^1$. Dolgachev shows that, restricted to $\mathbf{P}^1 \setminus \cup_i U_i$, there is a section of π containing the base point in its image. We will identify $\mathbf{P}^1 \setminus \cup_i U_i$ with its image. Choose paths σ_i in $\mathbf{P}^1 \setminus \cup_i U_i$from the base point to ∂U_i without self intersections and without crossings. Moving counterclockwise along the boundary of U_i defines a path w_i. By the computation of Dolgachev the fundamental group of X has the presentation of the proposition, where the t_i are given by $\sigma_i^{-1} w_i \sigma_i$. It remains to be shown that w_i is homotopic to a closed path as in the statement.
The normal bundle $N_{F_i/X}$ is isomorphic to $\mathcal{O}_{F_i}(F_i)$. The linear equivalence $a_i F_i \sim F$ and (1.1.ii) show that $\mathcal{O}_{F_i}(F_i)$ is a torsion bundle. In particular $N_{F_i/X}$ is flat.
We now use the explicit description of logarithmic transformation in Barth et al., [84], p. 164f. Set $m = a_i$ and $U_i = \Delta_t$ and denote $\pi^{-1}(\Delta_t)$ by Y'. The space Y' is the orbit space of the $\mathbf{Z}/m$-action on $\Delta_s \times \mathbf{C}/\mathbf{Z} + \mathbf{Z}\omega(s^m)$, given by

$$(s, c) \longmapsto (e(1/m) \cdot s, c + k/m).$$

Here $t = s^m$ and $e(x)$ denotes $exp(2\pi\sqrt{-1}x)$ and the integers k and m are coprime. For an element $s \in \Delta_s$ define τ_s to be the closed path $r \mapsto (e(r/m) \cdot s, kr/m)$. The space $Y' \setminus Y_0$, where Y_0 is the fiber over $0 \in \Delta_t$, is identified with $(\Delta_t \setminus 0) \times \mathbf{C}/\mathbf{Z} + \mathbf{Z}\omega(t)$ by the map

$$f : (s, c) \longmapsto (s^m, c - (k/(2\pi\sqrt{-1}))log(s)).$$

The image $f(\tau_s)$ is given by $r \longmapsto (e(r) \cdot s, log(s))$. In particular $f(\tau_s)$ is homotopic both to w_i and $\tau_{s=0} = \tau_i$.
By construction the normal bundle of F_i is the orbit space of $\mathbf{C} \times \mathbf{C}/\mathbf{Z} + \mathbf{Z}\omega(0)$ by the action $(z, c) \longmapsto (e(1/m) \cdot z, c + k/m)$. The claim on parallel transport is immediate. ♣

The groups described in the proposition above are well known: In general these groups are Fuchsian groups of genus zero. The only exceptions are: For $n \leq 2$ the fundamental group is trivial and $\pi_1(X(2,3,5)) \cong \mathcal{A}_5$ is an alternating group. We will exclude these exceptional cases from our discussion in the sequel.The results nevertheless extend with only minor modification of the arguments. These Fuchsian groups are related to Seifert fibered homology 3-spheres $\Sigma(a_1, \ldots, a_n)$ in the following way: The fundamental group of $\Sigma(a_1, \ldots, a_n)$ has the presentation

$$\pi_1(\Sigma(a_1, \ldots, a_n)) = < t_1, ..., t_n, h \mid t_i^{a_i} \cdot h^{b_i} = t_1 t_2 ... t_n h^{b_0} = [h, t_i] = 1 > .$$

Here $b_0, \ldots, b_n$ are arbitrary numbers subject to the condition $a(b_0 + \sum_{i=1}^n (b_i/a_i)) = 1$ where a is the product over all a_i (cf.Neumann–Raymond [78]); the space Σ is said to have Seifert invariants $(b_0; (a_1, b_1), (a_2, b_2), ..., (a_n, b_n))$. These fundamental groups are cocompact subgroups of $\widetilde{PSL}(2, \mathbf{R})$, as shown by Milnor [75] and Neumann [77]. The center of $\pi_1(\Sigma(a_1, \ldots, a_n))$ is generated by h and is infinite cyclic (cf.Burde–Zieschang [85] p. 199). In particular

$$\pi_1(X(a_1, \ldots, a_n)) \cong \pi_1(\Sigma(a_1, \ldots, a_n))/center.$$

Applying the Serre spectral sequence to the fiber sequence of Eilenberg MacLane spaces

$$S^1 \longrightarrow \Sigma(a_1, \ldots, a_n) \longrightarrow B\pi_1(X(a_1, \ldots, a_n))$$

one easily deduces $H^2(\pi_1(X); \mathbf{Z}) \cong \mathbf{Z}$. Let $\phi : \pi_1(\Sigma(a_1, \ldots, a_n)) \longrightarrow SU(2)$ be a nontrivial homomorphism. The image of ϕ is contained in the centralizer of $\phi(h)$, which is a torus, if $\phi(h) \neq \pm 1$. The fundamental group of a homology sphere being perfect thus forces $\phi(h) = \pm 1$. Factoring out centers gives a homomorphism $\overline{\phi} : \pi_1(X(a_1, \ldots, a_n)) \longrightarrow SO(3)$. To such a homomorphism we can associate a second Stiefel–Whitney class $w_2 \in H^2(\pi_1(X); \mathbf{Z}/2) \cong \mathbf{Z}/2$. A similar trick as used in the paper of Fintushel and Stern [88] p. 12, will help us to avoid separate discussions for the different Stiefel–Whitney classes. We use the homomorphism $\chi : \pi_1(X(2a_1, a_2, ..., a_n)) \longrightarrow \pi_1(X(a_1, \ldots, a_n))$ of fundamental groups defined by $\chi(s_i) = t_i$, where s_i and t_i are generators as in proposition (1.2).

1.3. Lemma: Reduction modulo centers gives a bijective map

$$Hom(\pi_1(\Sigma(a_1, \ldots, a_n)), SU(2)) \longrightarrow Hom(\pi_1(X(a_1, \ldots, a_n)), SO(3)).$$

Moreover χ induces a bijection

$$Hom(\pi_1(X(a_1, \ldots, a_n)), SO(3)) \longrightarrow Hom(\pi_1(X(2a_1, a_2, ..., a_n)), SU(2)).$$

Proof: Abbreviate $\pi_1(\Sigma(a_1, \ldots, a_n))$ by π and $\pi_1(\Sigma(2a_1, a_2, \ldots, a_n))$ by π'. Since the cohomology group $H^2(\pi; \mathbf{Z}/2)$ vanishes, every homomorphism to $SO(3)$ can be lifted to $SU(2)$ and since $H^1(\pi; \mathbf{Z}/2)$ vanishes, this can be done uniquely. That proves the first claim.

If the tuple $(b_0; (2a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n))$ denotes the Seifert invariants of π', then π has $(2b_0; (a_1, b_1), (a_2, 2b_2), \ldots, (a_n, 2b_n))$ as Seifert invariants. Using the presentations of the groups, one can see that the homomorphism χ can be lifted to a homomorphism $\chi : \pi' \longrightarrow \pi$, which, restricted to the centers, has cokernel of order 2. Naturality of exact cohomology sequences associated to the fiber sequence

$$S^1 \longrightarrow \Sigma(a_1, \ldots, a_n) \longrightarrow B\pi_1(X(a_1, \ldots, a_n)),$$

(cf. Whitehead [78], p. 651), gives a commuting diagram

$$\begin{array}{ccc} H^1(S^1; \mathbf{Z}) & \xrightarrow{\cong} & H^2(\pi/center); \mathbf{Z}) \\ \cdot 2 \downarrow & & \downarrow \chi^* \\ H^1(S^1; \mathbf{Z}) & \xrightarrow{\cong} & H^2(\pi'/center; \mathbf{Z}). \end{array}$$

The group $H^2(\pi_1(X(a_1, \ldots, a_n)); \mathbf{Z}/2)$ thus is mapped to zero by χ^*. As a consequence there is for any homomorphism $\phi : \pi \longrightarrow SO(3)$ a unique lift of $\phi \circ \chi$ to $SU(2)$. On the other hand a homomorphism $\psi : \pi'/center \longrightarrow SO(3)$, which admits a lift $\hat{\psi}$ to $SU(2)$, factors through $\pi/center$: The square of $\hat{\psi}(s_1^{a_1})$ is trivial and hence $\hat{\psi}(s_1^{a_1}) = \pm 1$. ♣

Let $\mathcal{R}(Y) = Hom^*(\pi_1(Y), SU(2))/ad\,SU(2)$ denote the space of nontrivial $SU(2)$-representations of the fundamental group of a space Y. As usual we assume the tupel $(a_1, \ldots, a_n)$ to be ordered in such a way that $a_2, \ldots, a_n$ are odd.

1.4. Theorem:

The representation space $\mathcal{R}(\Sigma(a_1, \ldots, a_n))$ for a Seifert fibered homology 3-sphere $\Sigma(a_1, \ldots, a_n)$ is isomorphic to the representation space $\mathcal{R}(X(2a_1, a_2, ..., a_n))$ for a Dolgachev surface $X(2a_1, a_2, ..., a_n)$.
For an ample divisor H on a Dolgachev surface $X = X(a_1, \ldots, a_n)$ there is a diffeomorphism $\mathcal{R}(X) \cong \mathcal{M}_X^H(0,0)$ of the representation space with the moduli space of H-stable algebraic bundles on X of rank two with vanishing Chern classes.

Proof: The first statement in the theorem is just a reformulation of previous results. The second statement is a consequence of the solution of the Kobayashi–Hitchin conjecture by Donaldson, [85]. The divisor H induces a projective imbedding of X. Let g denote a Kähler metric on X with Kähler form ω_g, whose associated cohomology class is dual to a hyperplane section. The moduli space $\mathcal{M}_{HE}^{\omega_g}(0,0)$ of irreducible Hermite–Einstein bundles with vanishing Chern classes on X is by the result of Donaldson isomorphic to $\mathcal{M}_X^H(0,0)$. These bundles are flat according to Lübke [82] or Kobayashi [87], p. 115. In particular $\mathcal{M}_X^H(0,0)$ is isomorphic to the $U(2)$-representation space of the fundamental group of X. Since $\pi_1(X)$ is a perfect group, all $U(2)$-representations actually are $SU(2)$-representations. ♣

The trick of using the homomorphism χ of course is not essential for the argument. As an application of the work of Okonek–Van de Ven [88], one can show that $\mathcal{R}(\Sigma(a_1, \ldots, a_n))$ is diffeomorphic to a disjoint union of the moduli spaces

$$\mathcal{M}_X^H(K_X, 0) \amalg \mathcal{M}_X^H(0,0)$$

of algebraic bundles over $X = X(a_1, \ldots, a_n)$. The moduli space $\mathcal{M}_X^H(K_X, 0)$ is easier to compute than $\mathcal{M}_X^H(0,0)$, but one cannot avoid computing the latter.

2. Algebraic Description of the Moduli Spaces

We will show that the moduli space $\mathcal{M}_X^H(0,0)$ of H-stable bundles over a Dolgachev surface $X = X(a_1, \ldots, a_n)$ as defined in the preceding chapter admits the structure of a smooth projective algebraic variety over the complex numbers, which hasa natural stratification by Zariski open subsets of projective spaces.

As in the last chapter we will assume the integers a_i to be odd for $i \geq 2$. It will be important lateron to have some information on the differentiable structure of the representation spaces $\mathcal{R}(X(a_1, \ldots, a_n))$. A representation $\alpha : \pi_1(X) \longrightarrow SU(2)$ has associated to it some invariants: Let t_i denote the generators of $\pi_1(X)$ as in proposition (1.2). Up to conjugation the image $\alpha(t_i)$ is given by a matrix

$$\alpha(t_i) = \begin{pmatrix} \omega & 0 \\ 0 & \bar{\omega} \end{pmatrix}, \quad \omega = exp\left(\frac{2\pi\sqrt{-1}}{a_i}\right)^{l_i}.$$

The tuple $(\pm l_1, \dots, \pm l_n)$ with $l_i \in \mathbf{Z}/a_i$ actually is an invariant of the connected component of $\mathcal{R}(X(a_1, \dots, a_n))$ containing α. The following proposition was proved in a different way by Fintushel and Stern [88], prop. 2.7.

2.1. Proposition: The representation space $\mathcal{R}(X)$ for a Dolgachev surface X is a smooth compact manifold. A connected component with invariant $(\pm l_1, \dots, \pm l_n)$ has dimension $2(M-3)$, where M is the number of l_i satisfying $2l_i \neq 0$.

Proof: Let S_{l_i} denote the conjugacy class of $\alpha(t_i)$, which is a 2-sphere, if $2l_i \neq 0$ and a single point else. The homomorphisms with invariants $(\pm l_1, \dots, \pm l_n)$ are given as the preimage $\phi^{-1}(1)$ of the map

$$\begin{array}{rcl} \phi : S_{l_1} \times \dots \times S_{l_n} & \longrightarrow & SU(2) \\ (x_1, \dots, x_n) & \longmapsto & x_1 \cdot \dots \cdot x_n. \end{array}$$

The proposition is an immediate consequence of the following two claims:

1. The unit element $1 \in SU(2)$ is a regular value of ϕ.
2. $SU(2)/\pm 1$ acts freely on $\phi^{-1}(1)$.

ad 1: Let $x = (x_1, \dots, x_n)$ be an element in $\phi^{-1}(1)$. To show surjectivity of the differential $(D_\phi)_x$ at the point x, we consider the maps $\rho_i : SU(2) \longrightarrow SU(2)$, given by

$$y \longmapsto x_1 \cdot \dots \cdot x_{i-1} y x_{i+1} \cdot \dots \cdot x_n.$$

The differential

$$(*) \qquad (D\phi)_x = \prod_{i=1}^{n} (D\rho_i|_{S_{l_i}})_{x_i}$$

is not surjective if and only if the images of all the factors coincide. Suppose $x_i \neq \pm 1$. Then the unique maximal torus T_i containing x_i intersects S_{l_i} transversely. The map ρ_i as a composition of multiplications is an isometry. The differential $(*)$ thus is not surjective if and only if

$$\rho_i(T_i) = exp(Im(D_{\rho_i}|_{S_{l_i}})_{x_i}^{\perp})$$

coincides for all i. In this case one has

$$T_1 = x_1 T_2 x_1^{-1} = \dots = x_1 x_2 \dots x_{n-1} T_n x_{n-1}^{-1} \dots x_1^{-1}$$

and because of $x_i \in T_i$ this is equivalent to

$$T_1 = T_2 = \dots = T_n.$$

In particular α has abelian image and thus has to be trivial, since $\pi_1(X)$ is perfect. This contradicts our assumption $x_i \neq \pm 1$.

ad 2: Suppose there is an element $x \in SU(2)/\{\pm 1\}$ and a representation α with $x\alpha(t_i)x^{-1} = \alpha(t_i)$ for all i. Then $\alpha(t_i)$ has to be an element of the unique maximal torus containing x. Thus $\alpha(\pi_1(X))$ is contained in this torus and consequently trivial. ♣

In order to describe the stable bundles arising, we need a technical definition:

2.2. Definition: An algebraic bundle E of rank 2 over a Dolgachev surface X will be called *admissible*, if there is a nonsplitting short exact sequence

$$E: \qquad 0 \longrightarrow \mathcal{O}_X(-D) \xrightarrow{\alpha} \mathcal{E} \xrightarrow{\beta} \mathcal{O}_X(D) \longrightarrow 0,$$

satisfying the following conditions:

i) $D \sim dF + \sum d_i F_i$, with $0 \leq d_i < a_i$, is a vertical divisor such that for some $a >> 0$ the divisor aD is linearly equivalent to an effective divisor.

ii) $2d \leq -m$, where m denotes the number of nonvanishing d_i.

iii) In case $2d = -m$ the inequality
$$d_i \leq \frac{a_i - 1}{2}$$
holds for the maximal i with $d_i \neq 0$.

Let Z denote the finite subset $\mathbf{Z}$ of the Picard group $Pic(X)$ formed by the line bundles $\mathcal{O}_X(D)$ satisfying the three conditionsabove.

As an immediate consequence of conditions ii) and iii) the divisor $2D$ is not linearly equivalent to an effective divisor. Condition iii) of course depends on the chosen ordering of the set $\{a_i\}$. This condition is included high–handedly to achieve the uniqueness statement in the following characterization of stable bundles:
Let H^0 denote an ample divisor on X and $H = (H^0 \cdot K_X + 1)K_X + H^0$. Then the divisor H is ample.

2.3. Proposition: An admissible bundle $\mathcal{E}$ is simple and the presentation E as an extension is unique up to isomorphism. Furthermore any H–stable bundle of rank two with vanishing Chern classes over a Dolgachev surface is admissible.

Proof: Applying condition ii) to $\mathcal{E}(D)$ gives $h^0(\mathcal{E}(D)) = h^0(\mathcal{O}_X) = 1$. The nonsplitting of the short exact sequence

$$E \otimes \mathcal{O}_X(-D): \qquad 0 \longrightarrow \mathcal{O}_X(-2D) \longrightarrow \mathcal{E}(-D) \longrightarrow \mathcal{O}_X \longrightarrow 0$$

implies that the boundary map $H^0(\mathcal{O}_X) \longrightarrow H^1(\mathcal{O}_X(-2D))$ is nontrivial. In particular $h^0(\mathcal{E}(-D)) = 0$. The cohomology sequence associated to the short exact sequence

$$E \otimes \mathcal{E}: \qquad 0 \longrightarrow \mathcal{E}(-D) \longrightarrow \mathcal{E}nd(\mathcal{E}) \longrightarrow \mathcal{E}(D) \longrightarrow 0,$$

gives $h^0(\mathcal{E}nd(\mathcal{E})) \leq 1$, i. e. $\mathcal{E}$ is simple.
Let

$$E': \qquad 0 \longrightarrow \mathcal{O}_X(-C) \xrightarrow{\gamma} \mathcal{E} \xrightarrow{\delta} \mathcal{O}_X(C) \longrightarrow 0,$$

be another presentation of $\mathcal{E}$ as an admissible bundle with $C \sim cF + \sum_i c_i F_i$. The map $\delta\alpha$ corresponds to a section of $\mathcal{O}_X(C+D)$. Let m_d and m_c denote the numbers of nonvanishing

d_i and c_i, respectively. Furthermore let Δ be the number of indices i, for which the inequality $c_i + d_i \geq a_i$ holds. Condition ii) implies $2d \leq -m_d \leq -\Delta$ and $2c \leq -m_c \leq -\Delta$. Note that one of these inequalities has to be a strict one, since equality $m_d = m_c = \Delta$ contradicts condition iii). Summation leads to the strict inequality

$$d + c + \Delta < 0.$$

Invoking (1.1) gives $h^0(\mathcal{O}_X(C + D)) = 0$, i. e. α factors through γ. This implies up to linear equivalence the inequality $C \leq D$ of divisors. The argument is symmetric in C and D. Thus C and D are linear equivalent. In particular there is an isomorphism of the short exact sequences E and E'.

Now let $\mathcal{E}$ denote an H–stable bundle with vanishing Chern classes. The Riemann–Roch theorem gives

$$h^0(\mathcal{E}) + h^0(\mathcal{E}(K_X)) \geq 2.$$

In case $\mathcal{E}$ admits a nontrivial section, it cannot be H–stable. Hence there is a divisor $C' \geq 0$ and a short exact sequence

$$0 \longrightarrow \mathcal{O}_X(C' - K_X) \longrightarrow \mathcal{E} \longrightarrow \mathcal{J} \otimes \mathcal{O}_X(K_X - C') \longrightarrow 0,$$

where $\mathcal{J}$ is the ideal sheaf of a 0–dimensional subscheme of X. If C' were not vertical, then $C' \cdot K_X > 0$ and the estimate

$$H \cdot C' = (H^0 \cdot K_X + 1)C' \cdot K_X + H^0 \cdot C' > H^0 \cdot K_X + 1 = H \cdot K_X + 1$$

would contradict the stability condition $H \cdot C' < H \cdot K_X$. Set $C = K_X - C'$. The sheaf $\mathcal{J}$ is the structure sheaf $\mathcal{O}_X$, since

$$length(\mathcal{O}_X/\mathcal{J}) = c_2(\mathcal{E} \otimes \mathcal{O}_X(C)) = 0.$$

It remains to check our three conditions. The stability condition $H \cdot C$ for the vertical divisor C is equivalent with the first condition: just multiply with the product $a_1 \cdot \ldots \cdot a_n$.
(**) In particular there exists a presentation of the bundle $\mathcal{E}$

$$0 \longrightarrow \mathcal{O}_X(-C) \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_X(C) \longrightarrow 0$$

as an extension of line bundles with $h^0(\mathcal{O}_X(C)) = 0$ and $h^0(\mathcal{O}_X(nC)) \neq 0$ for some $n >> 0$.

However, the divisor $C \sim cF + \sum_i c_i F_i$ not necessarily satisfies the last two conditions. It has to be replaced by an admissible divisor.
Let $B \sim bF + \sum_i b_i F_i$ be a vertical divisor satisfying $B \cdot H > 0$ and $h^0(\mathcal{O}_X(B)) = 0$. Then the exact sequence

$$0 \longrightarrow \mathcal{O}_X(B - C) \longrightarrow \mathcal{E}(B) \longrightarrow \mathcal{O}_X(B + C) \longrightarrow 0$$

and lemma (1.1.) lead to an estimate

$$\begin{aligned} h^0(\mathcal{E}(B)) &\geq h^0(C+B) + h^0(B-C) - h^1(B-C) \\ &= h^0(B+C) + 1 - h^0(K_X - B + C) \\ &= (b + c + \#\{i \mid b_i + c_i \geq a_i\} + 1) + 1 - (-1 + c - b + \#\{i \mid b_i < c_i\} + 1) \\ &= 2b + 2 + \#\{i \mid b_i + c_i \geq a_i\} - \#\{i \mid b_i < c_i\} \end{aligned}$$

For the replacing procedure to work we need a divisor B satisfying the conditions i)–iii) such that $h^0(\mathcal{E}(B)) \geq 1$. Given such a divisor, there exists an effective divisor D' such that by the same argument as above a short exact sequence

$$0 \longrightarrow \mathcal{O}_X(D') \longrightarrow \mathcal{E}(B) \longrightarrow \mathcal{O}_X(2B - D') \longrightarrow 0$$

presents $\mathcal{E}$ as an admissible bundle with $\mathcal{O}_X(B - D') \in Z$. Such a divisor B can be specified the following way: Let m denote the number $\#\{i \mid c_i \neq 0\}$ of nonvanishing c_i. In case m is odd, set

$$b = \frac{-m-1}{2} \ ; \quad b_i = \begin{cases} 0 & \text{if } c_i = 0 \\ a_i - 1 & \text{else} \end{cases}$$

In case m is even, set

$$b = \frac{-m}{2} \ ; \quad b_i = \begin{cases} 0 & \text{if } c_i = 0 \\ a_i - c_i & \text{if } i \text{ is the maximal index with } c_i \neq 0 \\ a_i - 1 & \text{else} \end{cases}$$

The inequality $min(c_i, a_i - c_i) \leq \frac{a_i - 1}{2}$ holds, because by (**) the maximal index i with $c_i \neq 0$ is greater than 1. ♣

The isomorphism class of the extension E is determined by the image of the boundary homomorphism

$$d : H^0(\mathcal{O}_X) \longrightarrow H^1(\mathcal{O}_X(-2D)) = Ext^1(\mathcal{O}_X, \mathcal{O}_X(-2D))$$

in the exact cohomology sequence associated to the short exact sequence $E \otimes \mathcal{O}_X(-C)$. The proposition above shows that the isomorphism classes of admissible bundles correspond bijectively to elements of

$$\coprod_{\mathcal{O}_X(D) \in Z} \mathbf{P}(D) \overset{def}{=} \coprod_{\mathcal{O}_X(D) \in Z} \mathbf{P}(H^1(\mathcal{O}_X(-2D))).$$

Henceforth an admissible bundle will be identified with the corresponding element in $\coprod_Z \mathbf{P}(D)$.

So far we have found two invariants of representations: First the tuple $(\pm l_1, \ldots, \pm l_n)$, then the admissible line bundles in Z. As one might expect, these invariants are closely related. Before we can show this we need a lemma:

2.4. Lemma: Let $\mathcal{E} \in \mathbf{P}(D)$ be an admissible bundle. Then the restriction to F_i splits:

$$\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_{F_i} \cong \mathcal{O}_{F_i}(D) \oplus \mathcal{O}_{F_i}(-D).$$

In particular the holomorphic structure of $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_{F_i}$ is flat.

Proof: We consider the following diagram with exact columns and rows:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_X(-2D-F_i) & \longrightarrow & \mathcal{E}(-D-F_i) & \longrightarrow & \mathcal{O}_X(-F_i) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_X(-2D) & \longrightarrow & \mathcal{E}(-D) & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_{F_i}(-2D) & \longrightarrow & \mathcal{E} \otimes \mathcal{O}_{F_i}(-D) & \longrightarrow & \mathcal{O}_{F_i} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

The bundles $\mathcal{E}(-D)$ and $\mathcal{E} \otimes \mathcal{O}_{F_i}(-D)$ are classified by the subspaces

$$d(H^0(\mathcal{O}_X)) \subset H^1(\mathcal{O}_X(-2D)) \quad \text{and} \quad d(H^0(\mathcal{O}_{F_i})) \subset H^1(\mathcal{O}_{F_i}(-2D)).$$

Of course $d(H^0(\mathcal{O}_X))$ is mapped onto $d(H^0(\mathcal{O}_{F_i}))$ in the cohomology sequence associated to the first column of the diagram. In order to show that $d(H^0(\mathcal{O}_{F_i}))$ is trivial, we have to show that the map α in the exact sequence

$$\begin{array}{l}
0 \longrightarrow H^0(\mathcal{O}_X(-2D-F_i)) \longrightarrow H^0(\mathcal{O}_X(-2D)) \longrightarrow H^0(\mathcal{O}_{F_i}(-2D)) \longrightarrow \\
\longrightarrow H^1(\mathcal{O}_X(-2D-F_i)) \stackrel{\alpha}{\longrightarrow} H^1(\mathcal{O}_X(-2D)) \longrightarrow \quad \ldots
\end{array}$$

is surjective. We will do that by computing the ranks of the cohomology groups in the exact sequence above. Using (1.1.) and the conditions of definition (2.2.), we get:

$$h^0(\mathcal{O}_X(-2D-F_i)) = h^0(\mathcal{O}_X(-2D)) = 0,$$

$$h^0(\mathcal{O}_{F_i}(-2D)) = \begin{cases} 1 & \text{if } 2d_i = 0 \\ 0 & \text{else} \end{cases}$$

and

$$h^1(\mathcal{O}_X(-2D-F_i)) = \begin{cases} h^1(\mathcal{O}_X(-2D)) + 1 & \text{if } 2d_i = 0 \\ h^1(\mathcal{O}_X(-2D)) & \text{else.} \end{cases}$$

Combining these formulas gives the desired surjectivity of α. $\clubsuit$

In the first chapter we showed that an H–stable bundle $\mathcal{E} \in \mathbf{P}(D)$ is flat and is defined by a representation $\alpha(\mathcal{E}) \in \mathcal{R}(X)$.

2.5. Proposition: The representation $\alpha(\mathcal{E})$ for an H–stable bundle $\mathcal{E} \in \mathbf{P}(D)$ has invariants $(\pm l_1, \ldots, \pm l_n) = (\pm d_1, \ldots, \pm d_n)$, where $D \sim dF + \sum_i d_i F_i$.

Proof: We only have to put together (2.4.) and proposition (1.2.). To compute the invariants we have to know the effect of parallel transport in $\mathcal{E}$ along the path $\tau_i \subset F_i$ defined in (1.2.). Parallel transport in the normal bundle $N_{F_i/X} = \mathcal{O}_{F_i}(F_i)$ results in multiplication by $exp\left(\frac{-2\pi\sqrt{-1}}{a_i}\right)$. Applying this to $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_{F_i} \cong N_{F_i/X}^{\otimes d_i} \oplus N_{F_i/X}^{\otimes(a_i-d_i)}$ now proves the claim. ♣

2.6. Theorem: The moduli space $\mathcal{M}_X^H(0,0)$ of H–stable bundles of rank 2 with vanishing Chern classes on a Dolgachev surface $X = X(a_1, \ldots, a_n)$ is a smooth, projective, complex algebraic manifold. It admits a stratification by locally closed smooth subvarieties $U(D)$, each of which is isomorphic to a Zariski open subset of a projective space.
For an invariant $(\pm d_1, \ldots, \pm d_n)$ there exists at most one connected component of the moduli space $\mathcal{M}_X^H(0,0)$ with $d_i \in \mathbf{Z}/a_i$. The dimension of such a component is $M - 3$, where M is the number of indices i, $1 \le i \le n$, with $2d_i \ne 0$.

Proof: Maruyama [77] has shown that for fixed Chern classes and fixed ample divisor H a (coarse) moduli space of H–stable bundles exists and is quasiprojective. The moduli space $\mathcal{M}_X^H(0,0)$ moreover is projective, since the underlying topological space $\mathcal{R}(X)$ is compact. Let $\mathcal{S}_X(0,0)$ denote the moduli space of simple bundles with vanishing Chern classes, as constructed by Altman–Kleiman [80]. The underlying analytic space of $\mathcal{S}_X(0,0)$, for which we will use the same notation, coincides with the moduli space of analytic simple bundles (cf. Norton [79], Kosarew–Okonek [87], Fujiki–Schumacher [87], Kobayashi [87], Lübke–Okonek [87] and Miyajima [88]). The underlying analytic space of $\mathcal{M}_X^H(0,0)$ is Hausdorff and is contained in the locally Hausdorff complex analytic space $\mathcal{S}_X(0,0)$ as an open subset (cf. Kobayashi [87], p. 266, Lübke–Okonek [87] or Kosarew–Okonek [87]). Let $\mathcal{E} \in \mathbf{P}(D)$ be a stable bundle. There is an estimate for the dimension of the Zariski tangent space $H^1(\mathcal{E}nd(\mathcal{E}))$: From the cohomology sequence associated to the exact sequence

$$0 \longrightarrow \mathcal{E}(-D) \longrightarrow \mathcal{E}nd(\mathcal{E}) \longrightarrow \mathcal{E}(D) \longrightarrow 0$$

of bundles we get

$$h^1(\mathcal{E}nd(\mathcal{E})) \le h^1(\mathcal{E}(-D)) + h^1(\mathcal{E}(D)).$$

The exact sequences

$$0 \longrightarrow \mathcal{O}_X(-2D) \longrightarrow \mathcal{E}(-D) \longrightarrow \mathcal{O}_X \longrightarrow 0$$

and

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{E}(-D) \longrightarrow \mathcal{O}_X(2D) \longrightarrow 0,$$

together with the properties of the admissible bundle $\mathcal{O}_X(2D)$, give:

$$h^1(\mathcal{E}(-D)) = h^1(\mathcal{O}_X(-2D)) - 1 \quad \text{and} \quad h^1(\mathcal{E}(D)) = h^1(\mathcal{O}_X(2D)).$$

A multiple of $2D \sim \delta F + \sum_i \delta_i F_i$ is linear equivalent to an effective divisor. This forces $-\delta < M$. The divisor $-2D$ is linear equivalent to $(-\delta - M)F + \sum_i \delta_i' F_i$ for $\delta_i' = -l_i$ mod a_i and $0 \le l_i' < a_i$. Hence $h^1(\mathcal{O}_X(-2D)) = M + \delta - 1$. Combined with the equality

$$(***) \qquad h^1(\mathcal{O}_X(2D)) = -\delta - 1$$

we obtain the estimate

$$h^1(\mathcal{E}nd(\mathcal{E})) \le M - 3.$$

The moduli space locally can be embedded in the Zariski tangent space, cf. Kobayashi [87], p. 261. Since the topological dimension of the moduli space is $2(M-3)$ by proposition (2.2), it has to be smooth.

The inclusion $\mathbf{P}(D) \longrightarrow \mathcal{S}_X(0,0)$ is a smooth closed embedding. To prove this claim one has to check that the inclusion is holomorphic and that the differential is injective. The former follows by the universal propery of $\mathcal{S}_X(0,0)$. For an affine neighborhood V of $\mathcal{E} \in \mathbf{P}(D)$ one can construct a bundle $\tilde{\mathcal{E}}$ over $V \times X$, such that the restricted bundles $\tilde{\mathcal{E}}_v$ over $v \times X$ are classified by v. Choose an open Stein covering $\mathcal{U}$ of X and represent $\mathcal{E}$ by a cocycle $(g_{ij}) \in Z(\mathcal{U}, Gl(2, \mathcal{O}_X))$. Because $\mathcal{E}$ is an extension of two line bundles, the g_{ij} can be chosen to be upper triangular matrices $\binom{1\ \ x}{0\ \ 1} \in T(2, \mathcal{O}_X)$. Let $(h_{ij})^\mu \in Z(\mathcal{U}, T(2, \mathcal{O}_X))$ represent a base of a hyperplane V_0 of $Ext^1(\mathcal{O}_X(D), \mathcal{O}_X(-D))$ not containing $\mathcal{E}$. The bundle $\tilde{\mathcal{E}}$ over $V \times X = (V_0 + \mathcal{E}) \times X$ then is represented by the cocycle $(g_{ij} \cdot \prod_\mu \alpha^\mu h_{ij}^\mu)$ for $\alpha^\mu \in \mathbf{C}$,

The differential, the Kodaira–Spencer map, can be computed as deformations over the double point and is given by the map

$$V_0 \hookrightarrow Ext^1(\mathcal{O}_X(D), \mathcal{O}_X(-D)) \xrightarrow{\alpha} Ext^1(\mathcal{E}, \mathcal{O}_X(-D)) \xrightarrow{\beta} Ext^1(\mathcal{E}, \mathcal{E})$$

induced by the projection $\mathcal{E} \longrightarrow \mathcal{O}_X(D)$ and the inclusion $\mathcal{O}_X(-D) \longrightarrow \mathcal{E}$. The map

$$\alpha : H^1(\mathcal{O}_X(-2D)) \longrightarrow H^1(\mathcal{E}(-D))$$

is contained in the long exact cohomology sequence associated to the extension

$$0 \longrightarrow \mathcal{O}_X(-2D) \longrightarrow \mathcal{E}(-D) \longrightarrow \mathcal{O}_X \longrightarrow 0$$

and thus is injective on V_0. Similarly the sequence

$$0 \longrightarrow \mathcal{E}(-D) \longrightarrow \mathcal{E} \otimes \mathcal{E} \longrightarrow \mathcal{E}(D) \longrightarrow 0$$

gives β in cohomology. In the proof of (2.3) it was shown that $H^0(\mathcal{E}nd(\mathcal{E})) \longrightarrow H^0(\mathcal{E}(D))$ is surjective. Hence the Kodaira–Spencer map $V_0 \longrightarrow Ext^1(\mathcal{E}, \mathcal{E})$ is injective.

Consider the pullback diagram

$$\begin{array}{ccc} \coprod_Z U(D) & \longrightarrow & \mathcal{M}_X^H(0,0) \\ \downarrow & & \downarrow j \\ \coprod_Z \mathbf{P}(D) & \overset{i}{\hookrightarrow} & \mathcal{S}_X(0,0). \end{array}$$

The image of j is contained in the image of i. Since j is an open embedding, this gives the claimed stratification.

Let $(\pm d_1, \ldots, \pm d_n)$ be a tuple with $d_i \in \mathbf{Z}/a_i$ and suppose $\mathcal{O}_X(C) \in Z$ is an admissible bundle satisfying the equality $h^1(\mathcal{O}_X(-2D)) = M - 2$ and $(\pm c_1, \ldots, \pm c_n) = (\pm d_1, \ldots, \pm d_n)$. Here we use the linear equivalence $C \sim cF + \sum_i c_i F_i$ with $0 \leq c_i < a_i$. Using $(***)$ we get $2C \sim -F + \sum_i l_i F_i$ with $0 \leq l_i < a_i$. Hence

$$(****) \qquad 2c + \#\{i \mid c_i \geq a_i/2\} = -1$$

. The inequality $2c \leq -m$ of definition (2.2.ii) forces

$$\{i \mid c_i \geq a_i/2\} \geq m - 1.$$

Condition (2.2.iii) finally determines uniquely whether c_i is in the interval $[0, a_i/2[$ or in $[a_i/2, a_i[$. The invariants c_i therefore are determined without any ambiguity by the congruences $\pm c_i = \pm d_i \bmod a_i$. Thus there is at most one component of maximal dimension contained in $\coprod_Z U(D)$ for any given invariant $(\pm d_1, \ldots, \pm d_n)$, i.e. representations with the same invariants do belong to the same connected component of $\mathcal{R}(X)$. ♣

2.7. Corollary: i) The moduli space $\mathcal{M}_X^H(0,0)$ is rational.

ii) The representation spaces $\mathcal{R}(\Sigma)$ of Seifert fibered homology 3–spheres are simply connected. ♣

Let $\mathcal{C}$ be a connected component of $\mathcal{R}(\Sigma)$ of (real) dimension ≤ 4. Then $\mathcal{C}$ is either diffeomorphic to $\mathbf{P}^1$ or to a rational surface. The rational surfaces are well known; they are blow–ups of $\mathbf{P}^2$ or of Hirzebruch surfaces and all admit Morse functions with only even indices. This proves the conjecture of Fintushel–Stern for all components of dimension ≤ 4, in particular for Seifert spheres $\Sigma(a_1, \ldots, a_n)$ with $n \leq 5$.

In order to treat the higher dimensional components we invoke a result of Smale [62], theorem 6.1, to the effect that 1–connected differentiable manifolds of dimension ≥ 6 always admit Morse functions with the minimum possible number of critical points. Combined with our description of $\mathcal{R}(\Sigma)$ this yields

2.8. Corollary: The conjecture of Fintushel–Stern is equivalent to the vanishing of $H^i(\mathcal{R}(\Sigma); \mathbf{Z})$ in odd dimensions. ♣

3. Cohomological Properties of the Moduli Spaces

In this chapter we analyze the stratification of the moduli spaces. Using Deligne's solution to the Weil conjecture we prove the vanishing of the odd Betti numbers of $\mathcal{R}(\Sigma)$, which then leads to a solution of the conjecture of Fintushel and Stern modulo torsion.

As we have seen in the proof of theorem (2.6), an admissible bundle may be simple, but unstable. This phenomenon is connected with the existence of certain "destabilizing" vertical curves in X. Let $\mathcal{E} \in \mathbf{P}(D)$ be an admissible bundle, classified by the image of the differential

$$d : H^0(\mathcal{O}_X) \longrightarrow H^1(\mathcal{O}_X(-2D))$$

in the cohomology sequence associated to the short exact sequence

$$0 \longrightarrow \mathcal{O}_X(-2D) \longrightarrow \mathcal{E}(-D) \xrightarrow{\beta} \mathcal{O}_X \longrightarrow 0.$$

The divisor $2D$ is linear equivalent to $\delta F + \sum_i \delta_i F_i$ with $0 \le \delta_i < a_i$.

3.1. Proposition: An admissible bundle $\mathcal{E} \in \mathbf{P}(D)$ is unstable if and only if there exists an effective vertical divisor $L = D - C$ in X such that

i) $\mathcal{O}_X(C) \in Z$ is admissible

ii) the image $d(H^0(\mathcal{O}_X))$ is contained in the kernel of the map

$$H^1(\mathcal{O}_X(-2D)) \longrightarrow H^1(\mathcal{O}_X(-D-C))$$

induced by multiplication with the section of $\mathcal{O}_X(L)$ with zero locus L.

Furthermore $L \sim lF + \sum_i l_i F_i$ can be chosen with $l_i \in \{0, \delta_i\}$.

Proof: Let $\rho : \mathcal{O}_X(C) \longrightarrow \mathcal{E}$ be a nontrivial map such that $C \cdot H \ge 0$. Then the map $\sigma = (\beta \otimes \mathcal{O}_X(D)) \circ \rho$ is nonvanishing, since otherwise $C \cdot H \le (-D) \cdot H < 0$. Hence $\sigma \otimes \mathcal{O}_X(-C)$ gives a section of $\mathcal{O}_X(L)$ with zero locus L, which is an effective divisor, since $\mathcal{E}$ doesn't split. In the proof of proposition (2.3) it was shown that the inequality $C \cdot H \ge 0$ forces the effective divisors $K - C$ and L to be vertical. In particular $C \cdot H \ne 0$, since K_X is not linearly equivalent to an effective divisor and thus $\mathcal{O}_X(C) \in Z$.

Suppose L is an effective divisor with $\mathcal{O}_X(C) \in Z$. Multiplication with the associated section of $\mathcal{O}_X(L)$ leads to a morphism of short exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_X(-2D) & \longrightarrow & \mathcal{E}(-D) & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & \sigma \swarrow & \downarrow & & \\ 0 & \longrightarrow & \mathcal{O}_X(-D-C) & \longrightarrow & \mathcal{E}(L-D) & \longrightarrow & \mathcal{O}_X(L) & \longrightarrow & 0 \end{array}$$

The existence of a map σ in the diagram is equivalent to the existence of a destabilizing map ρ and also is equivalent to the vanishing of the composed map in the commuting square

$$\begin{array}{ccc} H^0(\mathcal{O}_X) & \xrightarrow{d} & H^1(\mathcal{O}_X(-2D)) \\ \downarrow & & \downarrow \\ H^0(\mathcal{O}_X(L)) & \xrightarrow{d} & H^1(\mathcal{O}_X(-D-C)) \end{array}.$$

Suppose $L \sim lF + \sum_i l_i F_i$ with $0 \le l_i < a_i$ does not satisfy $l_i \in \{0, (a_i - \delta_i)\}$. Consider the effective divisor

$$L' = L - \sum_i \lambda_i F_i, \quad \text{where} \quad \lambda_i = \begin{cases} l_i & \text{if } \delta_i > l_i \\ l_i - \delta_i & \text{if } \delta_i \le l_i. \end{cases}$$

The line bundle $\mathcal{O}_X(D - L')$ is an element of Z, since both $\mathcal{O}_X(D - L)$ and $\mathcal{O}_X(D)$ are elements of Z. The commuting diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_X(-2D) & \xrightarrow{\cdot L'} & \mathcal{O}_X(L'-2D) & \longrightarrow & \mathcal{O}_{L'}(L'-2D) & \longrightarrow & 0 \\ & & \downarrow = & & \downarrow \sum_i \lambda_i F_i & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{O}_X(-2D) & \xrightarrow{\cdot L} & \mathcal{O}_X(L-2D) & \longrightarrow & \mathcal{O}_L(L-2D) & \longrightarrow & 0 \end{array}$$

induces on the cohomology level a diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^0(\mathcal{O}_X L'(L'-2D)) & \xrightarrow{d_{L'}} & H^1(\mathcal{O}_X(-2D)) & \xrightarrow{\cdot L'} & H^1(\mathcal{O}_X(L'-2D)) \\ & & \downarrow & & \downarrow = & & \downarrow \\ 0 & \longrightarrow & H^0(\mathcal{O}_X L(L-2D)) & \xrightarrow{d_L} & H^1(\mathcal{O}_X(-2D)) & \xrightarrow{\cdot L} & H^1(\mathcal{O}_X(L-2D)). \end{array}$$

The lines are exact, since $\mathcal{O}_X(L-2D)$ admits only the trivial section. Computation of dimensions shows

$$h^0(\mathcal{O}_{L'}(L'-2D)) = h^0(\mathcal{O}_L(L-2D)) = l + \#\{i \mid \delta_i \leq l_i\}$$

and hence multiplication by L and multiplication by L' give the same kernel. An identical argument as for L gives a nontrivial section of $\mathcal{E}(L'-D)$. ♣

This criterion for detecting unstable points in the moduli space of simple bundles can be used to describe the subvariety $\mathbf{P}(D) \setminus U(D)$ of unstable bundles globally. For our analysis it will be necessary to consider a more general class of curves than the destabilizing ones. So we fix an element $\mathcal{O}_X(D) \in Z$ with $D \sim dF + \sum_i d_i F_i$ and $2D \sim \delta F + \sum_i \delta_i F_i$ with $0 \leq \delta_i, d_i < a_i$.

3.2. Definition: An effective vertical divisor $L \sim lF + \sum_i l_i F_i$ with $0 \leq l_i < a_i$ is *associated to* $\mathbf{P}(D)$, if

i) $h^0(\mathcal{O}_L(L-2D)) \leq h^1(\mathcal{O}_X(-2D))$

ii) $l_i \in \{0, \delta_i\}$.

If furthermore the bundle $\mathcal{O}_X(D-L) \in Z$ is admissible, L will be called *destabilizing*.

3.3. Lemma: Let $\mathcal{E} \in \mathbf{P}(D)$ be an admissible bundle and L a vertical effective divisor associated to $\mathbf{P}(D)$. The short exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}_X(-2D) \longrightarrow \mathcal{O}_X(L-2D) \longrightarrow \mathcal{O}_L(L-2D) \longrightarrow 0$$

leads to a short exact sequence in cohomology

$$0 \longrightarrow H^0(\mathcal{O}_L(L-2D)) \xrightarrow{d_L} H^1(\mathcal{O}_X(-2D)) \longrightarrow H^1(\mathcal{O}_X(L-2D)) \longrightarrow 0.$$

Proof: Let $L \sim lF + \sum_i l_i F_i$ and $2D \sim \delta F + \sum_i \delta_i F_i$ with the usual convention $0 \leq l_i, \delta_i < a_i$. With help of (1.1) we can compute the dimensions $h^1(\mathcal{O}_X(-2D)) = M + \delta - 1$ and $h^0(\mathcal{O}_L(L-2D)) = l + \#\{i \mid \delta_i \leq l_i\}$. Writing the divisor $L - 2D \sim fF + \sum_i f_i F_i$ in standard form with $0 \leq f_i < a_i$, we get an estimate

$$\begin{aligned} f &= l - \delta - M + \#\{i \mid \delta_i \leq l_i\} = -\delta - M + h^0(\mathcal{O}_L(L-2D)) \\ &\leq -\delta - M + h^1(\mathcal{O}_X(-2D)) = -1. \end{aligned}$$

As a consequence the cohomology group $H^0(\mathcal{O}_X(L-2D))$ vanishes. A further application of (1.1) shows

$$h^1(\mathcal{O}_X(L-2D)) = -f - 1 = h^1(\mathcal{O}_X(-2D)) - h^0(\mathcal{O}_L(L-2D)),$$

proving the surjectivity part of the claim. ♣

As a set $\mathbf{P}(D)\backslash U(D)$ is the union $\cup_L \mathbf{P}(ker(\cdot L))$ over all destabilizing curves associated to $\mathbf{P}(D)$. If a curve L is destabilizing for a bundle in $\mathbf{P}(D)$, then so are all curves in the complete linear system $|L|$. Of course not all these curves destabilize the same bundles in $\mathbf{P}(D)$. For a bundle $\mathcal{F}_Y$ over a variety Y we will denote the projective bundle of lines in $\mathcal{F}_Y$ by $\mathbf{P}(\mathcal{F}_Y)$. This definition coincides with the Grothendieck construction $\mathbf{P}roj(\mathcal{F}_Y^*)$ on the dual locally free sheaf $\mathcal{F}_Y^*$, as defined in Hartshorne [77], p.. 160.
The bilinear form

$$\alpha : H^1(\mathcal{O}_X(-2D)) \otimes H^0(\mathcal{O}_X(L)) \longrightarrow H^1(\mathcal{O}_X(-2D+L))$$

for a curve L associated to $\mathcal{O}_X(D)$ determines canonically a sheaf homomorphism

$$(*) \qquad \hat{\alpha} : H^1(\mathcal{O}_X(-2D)) \otimes \mathcal{O}_{|L|}(-1) \longrightarrow H^1(\mathcal{O}_X(-2D+L)) \otimes \mathcal{O}_{|L|}$$

$$(v \otimes u, \mathbf{C} \cdot u) \longmapsto (vu \otimes 1, \mathbf{C} \cdot u).$$

By the lemma above $\hat{\alpha}$ is fiberwise surjective, thus is a surjective bundle homomorphism with kernel $\mathcal{S}^*_{|L|}$, a bundle over the complete linear system $|L|$. Using the defining inclusion, we get a map

$$(**) \qquad \psi_{|L|} : \mathbf{P}(\mathcal{S}^*_{|L|}) \longrightarrow \mathbf{P}(H^1(\mathcal{O}_X(-2D)) \otimes \mathcal{O}_{|L|}(-1))$$
$$\cong \mathbf{P}(H^1(\mathcal{O}_X(-2D)) \otimes \mathcal{O}_{|L|}) = \mathbf{P}(D) \times |L| \xrightarrow{pr} \mathbf{P}(D).$$

The image of $\psi_{|L|}$ is an irreducible subvariety $Inst(L)$ of $\mathbf{P}(D)$. We will give a description of this image for a destabilizing vertical curve L. To do that we recall some geometric constructions.

A rational norm curve N in projective r–space $\mathbf{P}^r$ is a rational curve of degree r not contained in any linear subspace. The construction is straightforward: View $\mathbf{P}^r = \mathbf{P}(S_r V)$ as the space of lines in the r-th symmetric tensor power of a 2-dimensional vector space V. The curve N is the image of the map $\mathbf{P}^1 = \mathbf{P}(V) \longrightarrow \mathbf{P}^r$, given by $x \mapsto x^r$.

The join $W_1 * \ldots * W_l$ of subvarieties $W_1, \ldots, W_l$ of $\mathbf{P}^n$ is the smallest closed subvariety containing $span(w_1, \ldots, w_l)$ for all tuples $(w_1, \ldots, w_l) \in W_1 \times \ldots \times W_l$. The secant variety

$$Sec_l(N) = \underbrace{N * \ldots * N}_{\text{l-times}}$$

is a special case of this construction.

Let $\mathcal{O}_X(D)$ be an admissible line bundle with $2D \sim \delta F + \sum_i \delta_i F_i$, $0 \leq \delta_i < a_i$ and let M be the cardinality of the set $J = \{i \mid \delta_i \neq 0\}$.

3.4. Proposition: Suppose dim $\mathbf{P}(D) \geq 1$. Then there is a rational norm curve N and M disjoint points $\{p_j\} \subset N$, such that the following statements hold:

i) If $L \sim lF + \sum_i l_i F_i$ with $0 \leq l_i < a_i$ is an associated destabilizing divisor, then

$$Inst(L) = Sec_l(N) * (\underset{I \subset J}{*} p_j) \quad \text{with} \quad I = \{i \mid l_i \neq 0\}.$$

ii) For any pair of destabilizing divisors L and L' the intersection $Inst(L) \cap Inst(L')$ is a finite union $\cup_k Inst(L_k)$. This union is over all subvarietes $Inst(L_k)$ for divisors L_k, which are destabilizing and associated to $\mathbf{P}(D)$, such that there are nontrivial maps $\mathcal{O}_X(L_k) \longrightarrow \mathcal{O}_X(L)$ and $\mathcal{O}_X(L_k) \longrightarrow \mathcal{O}_X(L')$.

iii) The subvariety $\mathbf{P}(D) \setminus U(D)$ of unstable bundles is a finite union $\cup Inst(L)$ for destabilizing curves L associated to $\mathbf{P}(D)$.

Proof: The dimension of $H^1(\mathcal{O}_X(-2D))$ is $M + \delta - 1 = k + 1$. We first show that the map

$$\alpha : H^1(\mathcal{O}_X(-2D)) \otimes H^0(\mathcal{O}_X(kF)) \longrightarrow H^1(\mathcal{O}_X(-2D + kF)) \cong \mathbf{C}$$

is nondegenerate. To prove this we take $(k+1)$ disjoint generic fibers $F^0, \ldots, F^k$. The differential

$$d_{F^0+\ldots+F^k} : H^0(\mathcal{O}_{F^0+\ldots+F^k}) \longrightarrow H^1(\mathcal{O}_X(-2D))$$

is an isomorphism. Since $d_{F^0+\ldots+F^k} = d_{F^0} + \ldots + d_{F^k}$, we have found a base $f^l, 0 \le l \le k$ of the latter space. Looking at the kernel of the multiplication of $H^1(\mathcal{O}_X(-2D))$ with the divisor $(\sum_0^k F^l - F^i)$, we notice that the set $\{f^1, \ldots, f^k\} \setminus \{f^i\}$ is a base of this kernel. Thus α is nondegenerate and $H^1(\mathcal{O}_X(-2D))$ can be identified with the dual of $H^0(\mathcal{O}_X(kF))$. Using the commuting diagram

$$\begin{array}{ccc} H^1(\mathcal{O}_X(-2D)) & \xrightarrow{\delta_i F_i} & H^1(\mathcal{O}_X(-2D + \delta_i F_i)) \\ \downarrow = & & (a_i - \delta_i)F_i \downarrow \\ H^1(\mathcal{O}_X(-2D)) & \xrightarrow{a_i F_i} & H^1(\mathcal{O}_X(-2D + F)) \end{array}$$

and counting dimensions, we see that multiplication with $\delta_i F_i$ gives the same kernel as does multiplication with $a_i F_i$. This shows that $p_i = \mathbf{P}(Im(d_{\delta_i F_i}))$ is a point on the subvariety $Im(\psi_{|F|})$. Moreover to study $\mathbf{P}(Im(d_{L'}))$ we can replace any divisor L' associated to $\mathbf{P}(D)$ by a divisor $L \ge L'$, which again is associated to $\mathbf{P}(D)$, such that $h^0(\mathcal{O}_X(L)) = h^0(\mathcal{O}_X(L'))$ and L is linear equivalent to a multiple of a generic fiber.
So let us for a moment fix such a divisor $L \sim lF$. Using the canonical isomorphism

$$H^0(\mathcal{O}_X(kF)) \cong S_k(H^0(\mathcal{O}_X(F))) = S_k V,$$

the map

$$H^1(\mathcal{O}_X(-2D)) \xrightarrow{\cdot L} H^1(\mathcal{O}_X(-2D + L))$$

is the dual of the map

$$\begin{array}{c} S_{k-l} V \longrightarrow S_k V \\ x \longmapsto L \cdot x. \end{array}$$

In particular $\hat{\alpha}$ in $(*)$ is the defining map for the dual of the l–secant bundle of the rational normal curve, also named Schwarzenberger bundle (cf. Schwarzenberger, [64] and Spindler–Trautmann [87], p. 5f and p. 16f). In particular $Inst(L)$ is the l–secant variety of a rational norm curve N in $\mathbf{P}(D)$. The first claim in the theorem is now immediate. Statement iii) is a consequence of proposition (3.1). To prove the second statement it is sufficient to show that for destabilizing effective divisors L and L' associated to $\mathbf{P}(D)$ there is a third such divisor L'', such that

$$Im(d_L) \cap Im(d_{L'}) = Im(d_{L''}).$$

Let C denote the union of the curves L and L', respecting of course the multiplicities, and let L'' be the intersection $L \cap L'$. We will construct a base of $Im(d_C)$, such that subsets of this base constitute bases of $Im(d_L)$ and $Im(d_{L'})$. Therefore the intersection of these two subsets gives a base for the intersection $Im(d_L) \cap Im(d_{L'})$, which will turn out to be $Im(d_{L''})$. The construction relies on the fact that L and L' are destabilizing. By definition this implies

$$(D - L) \cdot H > 0 \quad \text{and} \quad (D - L') \cdot H > 0$$

and consequently $h^0(\mathcal{O}_X(-2D + L + L')) = 0$. In particular $d_{L+L'}$ is injective. Because of the inequality of divisors $C \leq L + L'$, d_C is injective, too. As a divisor, L is a sum $\sum l_\alpha F_\alpha$ of reduced vertical divisors F_α, counted with multiplicities l_α Similarly $L' = \sum l'_\alpha F_\alpha$ and $C = \sum(max(l_\alpha, l'_\alpha))F_\alpha = \sum c_\alpha F_\alpha$. Note that $Im(d_C) = \oplus_\alpha Im(d_{c_\alpha F_\alpha})$. The sequence

$$Im(d_{F_\alpha}) \subseteq Im(d_{2F_\alpha}) \subseteq \ldots \subseteq Im(d_{c_\alpha F_\alpha})$$

constitutes a flag. So we may choose a base of $Im(d_C)$ respecting both the direct sum decomposition and the flag structure. It is immediate that such a base will do the job. ♣

In order to apply the Weil conjecture to the moduli space $\mathcal{M}_X^H(0,0)$, we need a nice decomposition of the secant varieties. Let $S_{|lF|}$ denote the l-secant bundle for the rational norm curve in $\mathbf{P}(D)$, which is defined over the complete linear system $|lF|$ with $2l < dim\mathbf{P}(D) = r$. Let Sec_l denote the l-secant variety

$$\underbrace{N * \ldots * N}_{l-times}$$

of the rational norm curve $N \subset \mathbf{P}(D)$.

3.5. Lemma: The projective bundle $\mathbf{P}(S^*_{|lF|})$ over $|lF|$ admits a stratification

$$\mathbf{P}(S^*_{|lF|}) = \coprod_{i=0}^{l-1} A_i$$

with locally closed smooth strata $A_i \cong (Sec_{(l-i)} \setminus Sec_{(l-i-1)}) \times \mathbf{P}^i$.

Proof: We use the same notation as in the proof of (3.4) and abbreviate $S_i = S_iV$ and $\mathbf{P}^i = \mathbf{P}S_i$. Multiplication $S_{l-i} \otimes S_i \longrightarrow S_l$ induces a $\binom{l}{i}$-fold covering

$$\phi_i : X_i = \mathbf{P}^{l-i} \times \mathbf{P}^i \longrightarrow \mathbf{P}^l = |lF|.$$

Consider the tautological diagram

$$
\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & 0 & & & & \mathcal{F} & & \\
 & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & S_{(r-l)} \otimes \mathcal{O}_{X_i}(-1,-1) & \longrightarrow & S_r \otimes \mathcal{O}_{X_i} & \longrightarrow & \phi_i^* \mathcal{S}_{\mathbf{P}^l} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow = & & \downarrow & & \\
0 & \longrightarrow & S_{(r-l+i)} \otimes \mathcal{O}_{X_i}(-1,0) & \longrightarrow & S_r \otimes \mathcal{O}_{X_i} & \longrightarrow & \mathcal{S}_{\mathbf{P}^{l-i}} \times \mathcal{O}_{\mathbf{P}^i} & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \\
 & & \mathcal{F} & & & & 0 & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & & .
\end{array}
$$

The epimorphism in in the last row defines an inclusion j fitting into the commuting diagram:

$$
\begin{array}{ccccccc}
\mathbf{P}(\mathcal{S}^*_{\mathbf{P}^{l-i}}) \times \mathbf{P}^i & = & \mathbf{P}(\mathcal{S}^*_{\mathbf{P}^{l-i}} \times \mathcal{O}_{\mathbf{P}^i}) & \stackrel{j}{\hookrightarrow} & \mathbf{P}(\mathcal{S}^*_{\mathbf{P}^l}) \times_{\mathbf{P}^l} \mathbf{P}^i & \stackrel{pr_1}{\longrightarrow} & \mathbf{P}(\mathcal{S}^*_{\mathbf{P}^l}) \\
\downarrow pr_1 & & \downarrow \psi'_{l-i} & & \downarrow & & \downarrow \psi_l \\
\mathbf{P}(\mathcal{S}^*_{\mathbf{P}^{l-i}}) & \stackrel{\psi_{l-i}}{\longrightarrow} & Sec_{l-i} & \hookrightarrow & Sec_l & = & Sec_l
\end{array}
$$

Here the maps ψ_l are just the maps $\psi_{|lF|}$ defined earlier in (**). The leftmost square implies:

$$(\psi'_{l-i})^{-1}(Sec_{(l-i)} \setminus Sec_{(l-i-1)}) \cong (\psi_{l-i})^{-1}(Sec_{(l-i)} \setminus Sec_{(l-i-1)}) \times \mathbf{P}^i$$

The proposition is a consequence of the following
Claim: The map $\beta_i = pr_1 \circ j$ induces an isomorphism

$$(\psi'_{l-i})^{-1}(Sec_{(l-i)} \setminus Sec_{(l-i-1)}) \cong (\psi_l)^{-1}(Sec_{(l-i)} \setminus Sec_{(l-i-1)}).$$

For the proof consider the diagram

$$
\begin{array}{ccccc}
\mathbf{P}(\mathcal{S}^*_{\mathbf{P}^{l-i}}) \times \mathbf{P}^i & \stackrel{j}{\hookrightarrow} & \mathbf{P}(\mathcal{S}^*_{\mathbf{P}^l}) \times_{\mathbf{P}^l} X_i & \stackrel{\overline{\pi}}{\longrightarrow} & X_i \\
\downarrow \beta_i & & \downarrow & & \downarrow \phi_i \\
\psi_l^{-1}(Sec_{(l-i)}) & \hookrightarrow & \mathbf{P}(\mathcal{S}^*_{\mathbf{P}^l}) & \stackrel{\pi}{\longrightarrow} & \mathbf{P}^l \\
\downarrow \psi_l & & \downarrow \psi_l & & \\
Sec_{(l-i)} & \hookrightarrow & Sec_l & &
\end{array}
$$

View $\mathbf{P}^l$ as a symmetric product $S^l(\mathbf{P}^1)$, where $\mathbf{P}^1 = \mathbf{P}S_1$. Then the projective bundle $\mathbf{P}(\mathcal{S}^*_{\mathbf{P}^l})$ can be written as a subspace

$$\mathbf{P}(\mathcal{S}^*_{\mathbf{P}^l}) = \{(a; x_1, \ldots, x_l) \in \mathbf{P}^r \times \mathbf{P}^l \text{ with } a \in span(x_1^r, \ldots, x_l^r)\} \subset \mathbf{P}^r \times \mathbf{P}^l,$$

where $\mathbf{P}^r = \mathbf{P}(S_r)$. Here we use the convention to denote by

$$span(\underbrace{y, \ldots, y}_{i})$$

the i–th osculating space. The subspace $\psi_l^{-1}(Sec_{(l-i)})$ then is:

$$\{(a; x_1, \ldots, x_l) \in \mathbf{P}^r \times \mathbf{P}^l \mid a \in span(x_{j_1}^r, \ldots, x_{j_{l-i}}^r) \text{ for } \{x_{j_1}, \ldots, x_{j_{l-i}}\} \subset \{x_1, \ldots, x_l\}\}.$$

The subspace $\psi_l^{-1}(Sec_{(l-i)} \setminus Sec_{(l-i-1)})$ consists of the elements, for which the span containing a is unique. The invers for β_i is given by:

$$(a; x_1, \ldots, x_l) \longmapsto (a; x_{j_1}, \ldots, x_{j_{l-i}}; x_1, \ldots, \hat{x}_{j_1}, \ldots, \hat{x}_{j_2}, \ldots, \hat{x}_{j_{l-i}}, \ldots x_l).$$

♣

For the next lemma let N_r denote a rational norm curve in $\mathbf{P}^r$ and let $p_1, \ldots p_s$ be disjoint closed points on N_r

3.6. Lemma: There is an isomorphism

$$(Sec_l(N_r) * \{p_1\} * \ldots * \{p_s\}) \setminus \{p_s\} \cong \mathbf{P}(\mathcal{O}_A(1) \oplus \mathcal{O}_A) \setminus \mathbf{P}(\mathcal{O}_A),$$

where A is the join $A = Sec_l(N_{r-1}) * \{p_1\} * \ldots * \{p_{s-1}\}$.

Proof: Let $\mathbf{P}^{r-1} \subset \mathbf{P}^r$ be a hyperplane intersecting N_r in $r-1$ disjoint points such that the points $p_1, \ldots, p_{s-1}$ are contained in that hyperplane, but not p_s. We will show that for

$$B = (Sec_l(N_r) * \{p_1\} * \ldots * \{p_s\}) \setminus \{p_s\}$$

the intersection $B \cap \mathbf{P}^{r-1}$ is isomorphic to A. Furthermore we will show that B is isomorphic to the join $A * \{p_s\}$. The claim then follows from the well known isomorphism

$$\mathbf{P}^r \setminus \{p_s\} \cong \mathbf{P}(\mathcal{O}_{\mathbf{P}^{r-1}}(1) \oplus \mathcal{O}_{\mathbf{P}^{r-1}}) \setminus \mathbf{P}(\mathcal{O}_{\mathbf{P}^{r-1}}).$$

As a first step we will show that the intersection $N_{r-1} = \mathbf{P}^{r-1} \cap (N_r * \{p_s\})$ is a rational norm curve in the hyperplane. For this we take a generic hyperplane H in $\mathbf{P}^{r-1}$ and count the number of intersections with N_{r-1}. The hyperplane in $\mathbf{P}^r$ generated by H and p_s intersects N_r in r disjoint points, one of them being p_s. The line connecting one of these points with p_s is contained in $N_r * \{p_s\}$ and intersects H exactly once. This gives $r-1$ points of intersection of H with N_{r-1}. On the other hand any point of intersection of H with N_{r-1} lies on a line connecting p_s with a point on N_r. This point on N_r then is an intersection point with the hyperplane in $\mathbf{P}^r$ generated by H and p_s. This implies that the curve N_{r-1} has degree $(r-1)$ and therefor has to be a rational norm curve in $\mathbf{P}^{r-1}$. Furthermore $N_r * \{p_s\} = N_{r-1} * \{p_s\}$. Associativity of the join construction gives the result. ♣

Let X denote a scheme of finite type, defined over the complex numbers.

3.7. Proposition: There is a subring $A \subset \mathbf{C}$ of finite type over $\mathbf{Z}$ and an extension

$$\begin{array}{ccc} X & \hookrightarrow & \mathcal{X} \\ f\downarrow & & \downarrow f^e \\ Spec(\mathbf{C}) & \hookrightarrow & Spec(A) \end{array}$$

such that f^e is flat. This extension can be constructed simultaneously for a finite number of schemes $X_1, \ldots, X_n$ and a finite number of maps $g_{ij} : X_i \longrightarrow X_j$ such that there exist maps g_{ij}^e over $spec(A)$ extending g_{ij}. Moreover these extensions can be chosen such that
 i) if f is smooth (resp. proper, irreducible, integer), then so is f^e.
 ii) if $g_{ij}.X_i \longrightarrow X_j$ is a locally closed immersion, then so is $g_{ij}^e : \mathcal{X}_i \longrightarrow \mathcal{X}_j$.

A proof can be found in EGA IV,2,3,4. References are assembled e.g. in I. Bauer–Kosarew [88], 2.1 and 2.2. ♣

Let X be a smooth projective variety and $\mathcal{X}$ an extension over the ring A as above. We may assume a given prime to be invertible in A.

3.8. Proposition: There exists a finite residue field $\mathbf{F}_q$ of A with separable closure $\overline{\mathbf{F}}_q$ such that the étale cohomology $H^i(X_{\overline{\mathbf{F}}_q}; \mathbf{Q}_l)$ of the reduction $X_{\overline{\mathbf{F}}_q}$ of $\mathcal{X}$ is isomorphic to the singular cohomology $H^i(X; \mathbf{Q}_l)$ for all i.

Proof: The construction of $\mathbf{Q}_l$–cohomology is by taking limits over $\mathbf{Z}/l^n$–cohomology and then moding out torsion, compare Freitag–Kiehl [87], I,12. For finite $\mathbf{Z}/l^n$–cohomology the isomorphisms are a consequence of the proper base change theorem and the fact that the higher direct image sheaves $R^i f_*(\mathbf{Z}/l^n)$ are all locally constant, cf. Freitag–Kiehl [87], p. 61 and p. 94.

For a maximal ideal in $A \otimes \mathbf{Q}$ the quotient field is a number field by Deligne–Illusie [87], p. 257. If $\wp$ is the inverse image in A of such a maximal ideal in $A \otimes \mathbf{Q}$, then a maximal ideal s of A containing $\wp$ will give a finite quotient field $\mathbf{F}_q$. Let R denote the strict henselization of A. Then the inclusion $A \longrightarrow \mathbf{C}$ admits an extension to R. Using this extension we get a specialization isomorphism

$$R^i f_*(\mathbf{Z}/l^n)_s \longrightarrow R^i f_*(\mathbf{Z}/l^n)_0$$

of the stalk over the special point to the stalk over the generic point, cf Freitag–Kiehl [87], p. 96. Taking inverse limits over n gives the claimed isomorphisms of étale cohomology groups. But for the generic fiber X the étale and the singular cohomology coincide. ♣

Let $\mathcal{M}_X^H(0,0)$ be the moduli space of rank–2 bundles with vanishing Chern classes over the Dolgachev surface $X = X(a_1, \ldots, a_n)$ for pairwise coprime a_i.

3.9. Theorem: The rational cohomology groups $H^i(\mathcal{M}_X^H(0,0); \mathbf{Q})$ vanish in odd dimensions i.

Proof: Using 3.8. and 3.7. we can take the reduction $\mathcal{M}_{\overline{\mathbf{F}}_q}$ of the moduli space to the separate closure $\overline{\mathbf{F}}_q$ of a finite field $\mathbf{F}_q$ and compute the Betti numbers of the étale cohomology groups $H^i(\mathcal{M}_{\overline{\mathbf{F}}_q};\mathbf{Q}_l)$. To do that we compute the zeta–function of $\mathcal{M}_{\overline{\mathbf{F}}_q}$. Let $\nu_k(\mathcal{M}_{\overline{\mathbf{F}}_q})$ denote the number of closed points in the reduction $\mathcal{M}_{\mathbf{F}_{q^k}}$ of the moduli space to the finite field $\mathbf{F}_{q^k}$. The zeta–function then is defined by

$$Z(\mathcal{M}_{\overline{\mathbf{F}}_q};t) = exp\left(\sum_{k=1}^{\infty}\frac{\nu_k(\mathcal{M}_{\overline{\mathbf{F}}_q})\cdot t^k}{k}\right).$$

Claim: The space $\mathcal{M}_{\overline{\mathbf{F}}_q}$ satisfies the following property:

(C) There are integers $b_2, b_4, \ldots, b_{2n}$ such that for any power q^k of q the number $\nu_k(\mathcal{M}_{\overline{\mathbf{F}}_q})$ is given by

$$\nu_k(\mathcal{M}_{\overline{\mathbf{F}}_q}) = \sum_{i=0}^{n} b_{2i}(q^k)^i.$$

Assuming this claim we can prove the theorem. Using the formula

$$log\left(\frac{1}{1-x}\right) = \sum_{k=1}^{\infty}\frac{x^k}{k},$$

the zeta–function takes the form

$$(***) \qquad Z(\mathcal{M}_{\overline{\mathbf{F}}_q},t) = \prod_{i=0}^{n}(1-q^it)^{-b_{2i}}.$$

The theorem of Deligne [74] states that the zeta–function is a rational function

$$Z(\mathcal{M}_{\overline{\mathbf{F}}_q},t) = \frac{P_1\cdot P_3\ldots P_{2n-1}}{P_0\cdot P_2\ldots P_{2n}}.$$

The degree of the polynomial P_i is the dimension of $H^i(\mathcal{M}_{\overline{\mathbf{F}}_q};\mathbf{Q}_l)$ and the zeroes of P_i are of absolute value $q^{i/2}$.
Application of this theorem to $(***)$ gives the result.

It remains to prove the claim. The condition (C) has some nice properties:

i) Projective space $\mathbf{P}^n_{\overline{\mathbf{F}}_q}$ satisfies (C). More generally every projective bundle $\mathbf{P}^n(\mathcal{E}_V)$ over a variety V defined over $Spec(\mathbf{F}_q)$ satisfies (C), if V does satisfy (C).

ii) Condition (C) is additive in the following sense:
Suppose a variety V_0 admits a stratification $\amalg V_i = V_0$ with locally closed subvarieties V_i, all defined over $\mathbf{F}_q$. If each V_i with $i \neq i_0$ satisfies (C), then so does V_{i_0}.
As an immediate consequence of ii) one has

iii) Let $V_1, \ldots, V_n$ be irreducible closed subvarieties in V such that $V_i \cap V_j$ is a union $\cup V_l$ and any V_i satisfies (C). Then the union $\cup_i V_i$ also satisfies (C).

Because of 3.7. a nice stratification, i.e. with locally closed strata, of the moduli space $\mathcal{M}_X^H(0,0)$ will give a nice stratification of the reduced moduli space $\mathcal{M}_{\overline{\mathbf{F}}_q}$. We thus only have to show that the strata $U(D)$ of 2.6. have reductions satisfying (C). Using ii) above this is equivalent to $\mathbf{P}(D) \setminus U(D)$ having reductions satisfying (C). By 3.4. and iii) above it is sufficient to show this for $Inst(L)$. By lemma 3.6. and i) this is a consequence, if the reduction of the secant varieties Sec_l satisfy (C). This, however, follows from easy induction over l using 3.5. and i). ♣

The torsion subgroup of $H^3(Y;\mathbf{Z})$ for a smooth variety Y, in algebraic geometry known as topological Brauer group, is a birational invariant, cf. Grothendieck [68], p. 50 and p. 138. Since it vanishes for $Y = \mathbf{P}^2$ we obtain

3.10. Corollary: The conjecture of Fintushel–Stern holds for all components of $\mathcal{R}(\Sigma)$ of real dimension ≤ 6.

4. The Betti numbers

In this chapter we apply the results of the paper to compute the Betti numbers of each component of the representation spaces. By explicitely computing some complex 2–dimensional components of the representation space for a particular Seifert sphere with five multiple fibers we illustrate the general situation. These low dimensional examples already reveal some of the characteristic features of the representation spaces.

Let $D \sim dF + \sum_i d_i F_i$ denote an admissible divisor. It will be convenient to use the divisor $2D \sim \delta F + \sum_i \delta_i F_i$. In case a_1 is even this corresponds to looking at flat $SO(3)$–bundles over $X(a_1/2, a_2 \ldots, a_n)$ instead of flat $SU(2)$–bundles over the Dolgachev surface $X(a_1, \ldots, a_n)$. In the first chapter we have shown that these are equivalent points of view.

Let $(\pm d_1, \ldots, \pm d_n)$ be an n–tuple with $d_i \in \mathbf{Z}/a_i$. The first question to ask is whether there exists a component in the moduli space $\mathcal{M}_X^H(0,0)$ realizing this tuple as rotation numbers. Here X denotes a Dolgachev surface $X(a_1, \ldots, a_n)$. Without any loss of generality we may and will assume that none of the d_i vanishes. To fix an integer representing $\pm d_i$ we impose the condition

$$(1) \qquad \frac{a_i}{2} \leq d_i < a_i \quad \text{with the only exception} \quad 0 < d_n < \frac{a_n}{2}, \quad \text{if } n \text{ is even.}$$

The integers δ_i then satisfy

$$0 \leq \delta_i < a_i \quad \text{and} \quad \delta_i = 2d_i mod\ a_i.$$

4.1. Proposition: There exists a component in the moduli space $\mathcal{M}_X^H(0,0)$ realizing the tuple $(\pm d_1, \ldots, \pm d_n)$ as rotation numbers, if and only if the following numerical conditions are satisfied:

i) $\sum_{i=1}^n (\delta_i/a_i) > 1$.

ii) For any combination of integers m_i subject to the conditions $0 \leq m_i \leq 2$ and $\sum_{i=1}^n m_i = n-2$ the estimate

$$\sum_{i=1}^{n}(-1)^{m_i}\left(\frac{\delta_i}{a_i}\right) < 1 + 2\#\{i \mid m_i = 2\}$$

must hold. If $\delta_1 = 0$, then m_1 has to be set $m_1 = 1$.

The special case of $n = 3$ was shown by Fintushel and Stern [88].

Proof: Let D denote a divisor corresponding to a stratum of maximal dimension. In the final part of the proof of theorem (2.6) we saw that the number of d_i with $a_i/2 \leq d_i$ is either $n-1$ or n, depending on whether n is even or odd. In case n even the definition of admissible bundles then forces $d_n < a_n/2$. Furthermore the proof of (2.6) showed that $2D$ is linearly equivalent to $-F + \sum_{i=1}^n \delta_i F_i$ with $0 \leq \delta_i < a_i$. A multiple of the divisor $2D$ is effective. Hence the inequality

$$2D \cdot H = \left(-1 + \sum_{i=1}^{n} \frac{\delta_i}{a_i}\right) \cdot (F \cdot H) > 0$$

must hold. For the ample divisor H the product $F \cdot H$ is positive. This shows the first condition, together with the choice of the d_i, being equivalent to the existence of an admissible divisor realizing the given rotation numbers.

It remains to check, whether or not the whole component $\mathbf{P}(D)$ is unstable. By proposition (3.1) this amounts to search for a destabilizing curve $L \sim lF + \sum_i l_i F_i$, for which the corresponding subvariety $Inst(L)$ has maximal dimension $n - 3 - \#\{i \mid \delta_i = 0\}$. The dimension of $Inst(L)$ is $\#\{l_i \neq 0\} + 2l - 1$. So the candidates for destabilizing the whole of $\mathbf{P}(D)$ can be described the following way: Let m_i be a combination as in the statement and set

$$l = \#\{i \mid m_i = 2\} \quad \text{and} \quad l_i = \begin{cases} \delta_i & \text{if } m_i = 1 \\ 0 & \text{else.} \end{cases}$$

By proposition (3.1) the crucial condition on L for being destabilizing is that the divisor $D - L$ has to be admissible, in particular the inequality $2(D-L)\cdot H > 0$ has to be satisfied, which reduces to:

$$\left(-1 + \sum_{i=1}^{n} \frac{\delta_i}{a_i}\right) - \left(2l + 2\sum_{i\epsilon 1}^{n} \frac{l_i}{a_i}\right) > 0.$$

But this is just condition ii) of the statement. ♣

In the third chapter we described the subvarieties $Inst(L) \subset \mathbf{P}(D)$ representing bundles destabilized by a curve $L \sim lF + \sum_i l_i F_i$. These subvarieties are joins

$$\underbrace{N * \ldots * N}_{l} * p_{i_1} * \ldots * p_{i_k},$$

where N is a rational norm curve in $\mathbf{P}(D)$ and the p_i's are points on this curve. There are only two possibile choices for the l_i: zero or δ_i. The points p_i in the join above correspond to the nonvanishing l_i's.

Again we look at a divisor D corresponding to a stratum of maximal dimension. We already saw in the proof of (4.1) how to check whether a given join

$$\underbrace{N * \ldots * N}_{l} * p_{i_1} * \ldots * p_{i_k} \subset \mathbf{P}(D)$$

represents unstable bundles.

4.2. Remark: A curve $L \sim lF + \sum_i l_i F_i$ associated to $\mathbf{P}(D)$ is destabilizing if and only if the following estimate holds:

$$\sum_{i=1}^{n} (-1)^{m_i} \left(\frac{\delta_i}{a_i}\right) > 1 + 2l, \quad \text{where} \quad m_i = \begin{cases} 0 & , \text{if } l_i = 0 \\ 1 & , \text{if } l_i = \delta_i. \end{cases}$$

♣

We now turn to the computation of the Betti numbers of a given component of the moduli space. Let $D \sim dF + \sum_i d_i F_i$ be a divisor satisfying (1) and $L \sim lF + \sum_i l_i F_i$ a destabilizing curve associated to $\mathbf{P}(D)$. The subvariety $Inst(L) \subset \mathbf{P}(D)$ has dimension $2l - 1 + \#\{i \mid l_i = \delta_i\}$. Using (4.1) we can check whether $\mathbf{P}(D)$ contains stable bundles and if so, we may use (4.2) to count the number e_j of subvarieties $Inst(L) \subset \mathbf{P}(D)$ of (complex) dimension $\leq (j-1)$ and the number f_j of subvarieties $Inst(L) \subset \mathbf{P}(D)$ of codimension $\leq j$. We denote the dimension of $\mathbf{P}(D)$ by r.

4.3 Theorem: The Betti numbers of a component of the moduli space $\mathcal{M}_X^H(0,0)$ with rotation numbers $(\pm d_1, \ldots, \pm d_n)$ are given by the formulae $b_{2j} = b_{2r-2j} = 1 + e_j - f_j$ and $b_{2j+1} = 0$.

Proof: We use the counting method of the proof of theorem (3.9) and consider the reduction $\mathcal{M}_{\mathbf{F}_q}$ of the moduli space to a suitable finite field $\mathbf{F}_q$. All spaces in the sequel will be defined over $\mathbf{F}_q$. We will therefore omit the index $\mathbf{F}_q$.

Let $\mathcal{C}$ denote the component of $\mathcal{M}_X^H(0,0)$ with rotation numbers $(\pm d_1, \ldots, \pm d_n)$. We will prove the following formula:

$$\nu_k(\mathcal{C}) = \nu_k(\mathbf{P}^r) + \sum (\nu_k(\mathbf{P}^{s_L}) - \nu_k(\mathbf{P}^{t_L})), \tag{2}$$

where the sum is over all linear equivalence classes of destabilizing curves L, t_L is the dimension of $Inst(L)$ and $\mathbf{P}^{s_L} = \mathbf{P}(D - L)$. Applying the remarks we made in the proof of (3.9) to (2), combined with the equality

$$s_L + t_L = r - 1 \tag{3}$$

gives the claim. So it remains to show (2) and (3). For the latter we have to check

$$r = h^1(\mathcal{O}_X(-2D-2L)) + dim\,(Inst(L)).$$

This, however, is clear, since

$$\begin{aligned} h^1(\mathcal{O}_X(-2D-2L)) &= h^1(\mathcal{O}_X(-2D)) - 2l - \#\{i \mid 2(d_i - l_i) \leq 0\} \\ &= r + 1 - 2l - \#\{i \mid l_i \neq 0\} \end{aligned}$$

and $dim\,Inst(L) = 2l + \#\{i \mid l_i \neq 0\} - 1$. The proof of (2) is a bit more involved. We start with the equation

$$\nu_k(C) = \sum_{\mathcal{O}_X(C)\in I} \nu_k(U(C))$$

where $I \subset Z$ is the subset of all admissible bundles with $C \sim cF + \sum_i c_iF_i$ such that $(\pm c_1, \ldots, \pm c_n) = (\pm d_1, \ldots, \pm d_n)$. The linear equivalence $C \sim D - L$ gives a 1–1 correspondence of admissible bundles $\mathcal{O}_X(C) \in I \setminus \mathcal{O}_X(D) = I^*$ and complete linear systems $|L|$ of destabilizing curves associated to $\mathbf{P}(D)$. So in order to prove (2) it suffices to show

$$\sum_{I^*} \nu_k(\mathbf{P}^{t_L}) = \sum_{I} \nu_k(\mathbf{P}(C) \setminus U(C)). \tag{4}$$

For a destabilizing curve L associated to $\mathbf{P}(C)$ we denote by $J_C(L)$ the subspace

$$J_C(L) = (Inst(L) \setminus \coprod_{L'<L} Inst(L')) \subset \mathbf{P}(C).$$

With this notation we can give a stratification

$$\mathbf{P}(C) \setminus U(C) = \coprod_{I_C^*} J_C(L),$$

where the sum is over all destabilizing curves L associated to $\mathbf{P}(C)$.

Let L be a destabilizing curve associated to $\mathbf{P}(D)$. If L' is another curve associated to $\mathbf{P}(D)$, then it is destabilizing and moreover $(L - L')$ is a destabilizing curve associated to $\mathbf{P}(D - L')$. This yields

$$\begin{aligned} \sum_I \nu_k(\mathbf{P}(C) \setminus U(C)) &= \sum_{\mathcal{O}_X(D-L)\in I} \left(\sum_{I^*_{D-L}} \nu_k(J_{D-L'}(L - L')) \right) \\ &= \sum_{\mathcal{O}_X(D-L)\in I^*} \left(\sum_{L'-L} \nu_k(J_{D-L'}(L - L')) \right). \end{aligned} \tag{5}$$

Note that $J_C(L)$ is of the form

$$(N * \ldots * N * p_{i_1} * \ldots * p_{i_m}) \setminus (\cup Inst(L')),$$

where each of the $Inst(L')$ is a similar join with either one factor missing or one of the N's replaced by a p_{i_j}. We claim

(6) $$\nu_k(Inst(L)) = \nu_k(\mathbf{P}^{t_L}).$$

Before proving this equation we look at its consequences: First it would follow $\nu_k(J_C(L)) = \nu_k(J_D(\tilde{L}))$, where $\tilde{L} \sim \tilde{l}F + \sum_i \tilde{l}_i F_i$ is a destabilizing curve associated to $\mathbf{P}(D)$ with $\tilde{l} = l$ and $(\tilde{l}_i = 0 \iff l_i = 0)$ for $L \sim lF + \sum_i l_i F_i$. Furthermore (5) could be reformulated

$$\begin{aligned}\sum_I \nu_k(\mathbf{P}(C) \setminus U(C)) &= \sum_{\mathcal{O}_X(D-L) \in I^\bullet} \left(\sum_{\tilde{L}<L} \nu_k(J_D(\tilde{L})) \right) \\ &= \sum_{\mathcal{O}_X(D-L) \in I^\bullet} \nu_k(Inst(L)) \\ &= \sum_{\mathcal{O}_X(D-L) \in I^\bullet} \nu_k(\mathbf{P}^{t_L}),\end{aligned}$$

proving (2). So it remains to show (6). Using (3.6) we see

$$\nu_k(N * \ldots * N * p_{i_1} * \ldots * p_{i_m}) = q^k \cdot \nu_k(N * \ldots * N * p_{i_1} * \ldots * p_{i_{m-1}}) + 1.$$

So we are done, if we can show

$$\nu_k(Sec_l(N)) = \nu_k(\mathbf{P}^{2l-1}) = \frac{x^{2l}-1}{x-1},$$

where $x = q^k$. The inductive claim is

(7) $$\nu_k(Sec_l(N) \setminus Sec_{l-1}(N)) = \frac{x^{2l} - x^{l-2}}{x-1}.$$

We invoke (3.5) to the effect

$$\begin{aligned}&\nu_k(Sec_l(N) \setminus Sec_{l-1}(N)) = \\ &= \nu_k(\mathbf{P}^l) \cdot \nu_k(\mathbf{P}^{l-1}) - \sum_{i=1} l - 1 \nu(Sec_{l-i} \setminus Sec_{l-i-1}) \cot \nu_k(\mathbf{P}^i) = \\ &= (x-1)^{-2}((x^{l+1}-1)(x^l-1) - \sum_{i=1}^{l-2} x^{2l-2i-2}(x^2-1)(x^{i+1}-1)) = \\ &= (x-1)^{-1}((\sum_{i=0}^{l} x^i)(x^l-1) - \sum_{i=0}^{l-2} x^{2i}(x+1)(x^{l-i}-1)) \\ &= (x-1)^{-1}((x^{2l} + \sum_{i=0}^{l-2} x^{l+i+1} - \sum_{i=0}^{l-1} x^i) - (\sum_{i=0}^{l-3} x^{l+i+1} + \sum_{i=0}^{l-1} x^{l+i} - \sum_{i=0}^{2l-3} x^i)) = \\ &= \frac{x^{2l} - x^{2l-2}}{x-1}.\end{aligned}$$

That finishes the proof. ♣

We will conclude our considerations with complex 2–dimensional examples. Any component of the moduli space $\mathcal{M}_X^H(0,0)$ of this dimension has to be the projective plane $\mathbf{P}^2$ or a blow–up of a Hizebruch surface.

Let $(a_1,\dots,a_n) = (256, 103, 257, 259, 303)$ be our chosen tuple. We will consider the following five combinations of rotation numbers:

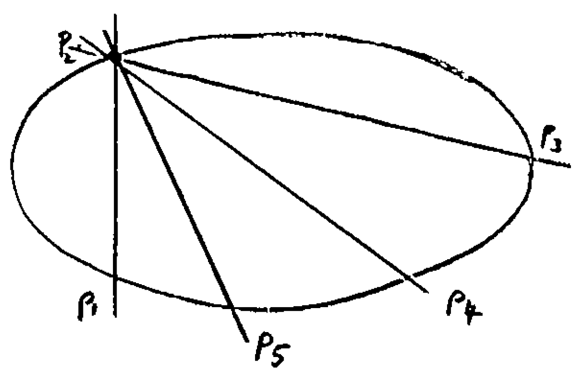
	d_{256}	d_{103}	d_{257}	d_{259}	d_{303}
case 1	154	62	155	156	182
case 2	255	102	256	258	302
case 3	154	67	168	169	197
case 4	180	72	180	180	212
case 5	129	52	180	180	212

In case 1 the ratio δ_i/a_i is slightly greater than 0.2 for all i. The conditions of (4.1) are satisfied and by (4.2) there are no destabilizing curves. Thus $\mathbf{P}(D) \cong \mathbf{P}^2$ is stable.

In the next case the ratio δ_i/a_i is slightly less than 1 for all i. Again there is a component in the moduli space realizing the chosen rotation numbers. In this case, however, there is a whole bunch of destabilizing curves $L \sim lF + \sum_i l_i F_i$, which we list below:

geometry of $Inst(L)$	l	l_{256}	l_{103}	l_{257}	l_{259}	l_{303}
N a quadric in $\mathbf{P}^2$	1	0	0	0	0	0
p_1, point on N	0	254	0	0	0	0
p_2, point on N	0	0	101	0	0	0
p_3, point on N	0	0	0	255	0	0
p_4, point on N	0	0	0	0	257	0
p_5, point on N	0	0	0	0	0	301
$\overline{p_1p_2}$, connecting line	0	254	101	0	0	0
$\overline{p_2p_3}$, connecting line	0	0	101	255	0	0
$\overline{p_2p_4}$, connecting line	0	0	101	257	0	0
$\overline{p_2p_5}$, connecting line	0	0	101	0	0	301

Using (4.3) we see that the resulting space has the Betti numbers of $\mathbf{P}^2$, hence has to be the projective plane. The figure on the right visualizes the subspace of unstable bundles in the projective plane. To get the moduli space one has to replace the points p_i by lines. Then the strict transforms of the curves in the picture are to be blown down to points. One can easily check that the resulting space actually is $\mathbf{P}^2$. In general any combination of these five points, the connecting lines and the quadric may occur, as long as the sequence of blow–ups and blow–downs will give a smooth surface. Checking the cases



one can see that the only possibilities for the resulting space are: $\mathbf{P}^2$, $\mathbf{P}^1 \times \mathbf{P}^1$, and $\mathbf{P}^2$ blown up in up to five points.

Computing the cases 3, 4, 5 reveals in case 3 one single destabilizing curve. The resulting space thus is $\mathbf{P}^2$ blown up in this point. In case 4 we get five unstable points in the plane. The resulting space consequently is $\mathbf{P}^2$ blown up in these five points.

Case 5 finally gives an example of $\mathbf{P}^1 \times \mathbf{P}^1$: There are two unstable points in $\mathbf{P}^2$ and the connecting line is unstable, too. Let's summarize:

4.4. Corollary: It is exactly the following complex surfaces which arise as 2–dimensional components in the moduli spaces $\mathcal{M}_X^H(0,0)$: $\mathbf{P}^2$, blown up in 0 to 5 points, and $\mathbf{P}^1 \times \mathbf{P}^1$.

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