On integral models of an algebraic torus

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Abstract

Given an algebraic torus T defined over a $\mathfrak{p}$-adic field or over a number field, an explicit construction of a natural integral model of T is described. The constructed model is a faithfully flat affine scheme of finite type over the ring of integers of the field of definition of T. If T splits over a tamely ramified extension, the identity component of our model coincides with the identity component of the Néron–Raynaud model of T.

Résumé

On présente une description explicite d'un modèle entier naturel d'un tore algébrique T défini sur un corps $\mathfrak{p}$-adique ou sur un corps de nombres. Ce modèle est un schéma affine fidèlement plat de type fini sur l'anneau des entiers du corps de définition de T, et si T se déploie sur une extension modérément ramifiée, il a la même composante neutre que le modèle de Néron–Raynaud.

1 Introduction

Let T be an algebraic group defined over a field k of characteristic 0, and suppose k be the field of fractions of an integral domain $\mathfrak{o}$. A group scheme $\mathcal{T}$ defined over $\mathfrak{o}$ is said to be an $\mathfrak{o}$-integral model of T if the generic fibre $\mathcal{T}_k = \mathcal{T} \times_{\mathfrak{o}} k$ of $\mathcal{T}$ is isomorphic to T. So far no attempt to classify all the integral models of an algebraic group has been made (see, however, [26] for a discussion of the one-dimensional case). Building on the classical work of Néron on integral models of an Abelian variety, Raynaud [19] has initiated a study of integral models of a commutative algebraic group; the arising Néron–Raynaud models have been described in [1, ch.10]. We refer to [3], [4], [27], [13] for some further studies, relating to the theory of the Néron–Raynaud models. Not being minimal in any sense, an integral model of an algebraic torus may nonetheless have beautiful applications, see for instance [21]. In this paper, we are concerned with integral models of algebraic tori defined over a number field, or over a $\mathfrak{p}$-adic field. Let k be such a field, and let $\mathfrak{o}$ denote its ring of integers. In what follows, we shall describe an explicit construction of an $\mathfrak{o}$-integral model $\mathcal{T}_0$ of an algebraic torus T defined over k. The integral model $\mathcal{T}_0$ is a reduced faithfully flat affine scheme of finite type over $\mathfrak{o}$. Moreover, if the torus T splits over a tamely ramified extension of k, then the identity component of the scheme $\mathcal{T}_0$ is proved to be isomorphic to the identity component of the Néron–Raynaud model of T (see Theorem 3 in Section 5). Let us recall that in contrast to the Néron model of an Abelian variety, the Néron–Raynaud model of an algebraic torus is not necessarily of finite type over $\mathfrak{o}$, although its identity component is a scheme of finite type over $\mathfrak{o}$.

In a series of papers [25], [23], [24], [11], [12], two of the authors and their collaborators have investigated the behaviour of the scheme $\mathcal{T}_0$, or similar integral models, under reduction modulo a non-archimedean prime; in [25], [5], [23], [24], integral models of finite type over $\mathfrak{o}$ have been used for calculations, relating to the class numbers and Tamagawa numbers of an algebraic torus. Given a toric variety X, containing the torus T as its dense open orbit, one can define [10] an $\mathfrak{o}$-integral model $\mathcal{X}$ of X in such a way that the action of $\mathcal{T}_0$ on $\mathcal{X}$ extends the natural action of the torus T on X; one of the authors has studied the distribution of integer points $\mathcal{X}(\mathfrak{o})$ in the real locus $X(\mathbb{R})$, [14]–[17]. The construction of the model $\mathcal{T}_0$ described in this paper has its origins in the authors' work cited in this paragraph; our goal here is to further investigate the properties of this model.

In the next section, we fix our notation, recall following [22], [23] the basic definitions about algebraic tori, and construct an integral model $\mathcal{T}_0$ of an algebraic torus T defined over the field k of fractions of an integral domain, assuming the torus T splits over a finite normal extension of k having an integral basis over k. Our approach, which emerged from a series of papers by Voskresenskii and Fomina [5], [25] (cf. also [23], [24]), is based on transferring the Hopf algebra structure from the coordinate ring of the torus to the coordinate ring of the integral model. Alternatively, following the original approach of Raynaud [19], one can embed the torus T in a quasi-split torus, and use this embedding to obtain an explicit description of $\mathcal{T}_0$ by a system of polynomial equations, cf. [14]–[17], [10], [12]. This construction leads to a functorial description of our model, enabling us to explicitly compute $\mathcal{T}_0(\mathcal{A})$ for any $\mathfrak{o}$-algebra $\mathcal{A}$ (Corollary 1). In Section 3, we investigate the properties of the constructed model of an algebraic torus T defined over a $\mathfrak{p}$-adic field. We define the notion of reduction of the torus T in terms of the model $\mathcal{T}_0$, cite the known results concerning the behaviour of reduction modulo a non-archimedean prime, and give a few examples. In Section 4, the number field case is investigated. A rather plausible conjecture stated at the beginning of that section would imply that one can always choose a splitting field of T having an integral basis over its field of definition. At present, we can not prove this conjecture however, and proceed without making any additional assumptions on a splitting field. One can either directly define a functor from the category of commutative $\mathfrak{o}$-algebras to the category of commutative groups and then prove that this functor is representable by an affine group scheme, or to "glue" a global model from the local ones constructed in Section 3. Either of the methods gives, up to an $\mathfrak{o}$-isomorphism, the same integral model. In Section 5, the constructed integral model $\mathcal{T}_0$ is compared to the Néron–Raynaud model. Appendix 1 contains a proof of a technical lemma from Section 5. In Appendix 2, we compare the scheme $\mathcal{T}_0$ to some other integral models appearing in the early work of Voskresenskiĭ and Fomina.

Notation 1. *As usual, $\mathbb{Q}, \mathbb{Q}_p, \mathbf{F}_p, \mathbb{Z}$, and $\mathbb{Z}_p$ stand for the field of rational numbers, the field of p-adic numbers, the prime field with p elements, the ring of rational integers, and the ring of p-adic integers respectively. Let A^* denote the group of invertible elements in a commutative ring A. In what follows, $G_{m,A}$ and $G_{a,A}$ stand, respectively, for the multiplicative group and the additive group defined over a commutative ring A.*

2 Generalities; the basic construction

Let T be an algebraic torus of dimension d defined over a field k of characteristic zero; it splits over the algebraic closure $\bar{k}$ of k :

$$T \times_k \bar{k} \cong G_{m,\bar{k}}^d.$$

The projections $\chi_i \colon T \times_k \bar{k} \to G_{m,\bar{k}}$ $(i = 1, \ldots, d)$ are defined over a finite normal extension L of k; we call any such field L a splitting field of T. These projections generate a free Abelian group $\hat{T}$, the group of rational characters of T. Let $\Gamma = \mathrm{Gal}(L \mid k)$ denote the Galois group of the extension $L \mid k$; the group $\hat{T}$ may be regarded as a Γ-module of rank d over $\mathbb{Z}$. Let

$$\rho\colon \Gamma \to \mathrm{GL}(d, \mathbb{Z}),\ \rho(g) = (r_{ji}(g))_{1 \le i,j \le d}$$

be the integral representation of the group Γ afforded by this Γ-module.

Notation 2. *Given a multiplicatively written Abelian group H, let $\vec{a} = (a_1, \ldots, a_d),\ a_i \in H,\ 1 \le i \le d,\ \vec{a} \in H^d$, and let*

$$\vec{a}^{\,g} = \vec{b},\ g\prod_{j=1}^{d} a_j = \prod_{i=1}^{d} b_i,\ b_i = \prod_{j=1}^{d} a_j^{r_{ji}(g)},\ 1 \le i \le d,\ g \in \Gamma. \tag{2.1}$$

With this notation, one can define a natural k-linear action of the group Γ on the group ring

$$L[\hat{T}] := L[\chi_1, \ldots, \chi_d, \chi_1^{-1}, \ldots, \chi_d^{-1}]$$

of the multiplicative group $\hat{T}$ as follows:

$$g(\sum_{\chi \in \hat{T}} \alpha(\chi)\chi) = \sum_{\chi \in \hat{T}} g\alpha(\chi)\ g\chi,\ g \in \Gamma, \tag{2.2}$$

where

$$\alpha \mapsto g\alpha,\ \alpha \in L,\ g \in \Gamma$$

stands for the Galois group action of Γ on L.

Let $B = (L[\hat{T}])^{\Gamma}$ be the ring of invariants under this action; it follows then that $T = \mathrm{Spec}\, B$, and $B = k[T]$. Thus given a finite normal extension $L \mid k$ with $\Gamma = \mathrm{Gal}(L \mid k)$ and a Γ-module M, there is a unique algebraic torus T defined over k with $M \cong \hat{T}$. The ring $L[\hat{T}]$ is a Hopf L-algebra, the operations of co-multiplication m^*, co-inversion i^*, and co-unit e^* being defined as follows:

$$m^*(\chi_j) = \chi_j \otimes \chi_j,\ i^*(\chi_j) = \chi_j^{-1},\ e^*(\chi_j) = 1,\ 1 \le j \le d. \tag{2.3}$$

Since the k-subalgebras B and L of the ring $L[\hat{T}]$ are linearly disjoint, it follows that

$$L[\hat{T}] = LB \cong L \otimes_k B. \tag{2.4}$$

To compute the Hopf k-algebra structure on B, inherited from $L[\hat{T}]$, let us fix a basis $\{\omega_1, \ldots, \omega_n\}$ of the extension $L \mid k$, and let

$$\omega_i \omega_j = \sum_{l=1}^{n} c_{ij}^{(l)} \, \omega_l, \ c_{ij}^{(l)} \in k, \ 1 \le i, j, l \le n.$$

In view of (2.4), it follows that

$$L[\hat{T}] = B\omega_1 \oplus \ldots \oplus B\omega_n. \tag{2.5}$$

In particular,

$$\chi = f_1(\chi) \, \omega_1 + \ldots + f_n(\chi) \, \omega_n, \ f_j(\chi) \in B, \ 1 \le j \le n$$

for $\chi \in \hat{T}$. Let B_0 stand for the k-subalgebra of B generated by the set of functions

$$\mathfrak{b} = \{f_i(\chi) \ \mid \ \chi \in \hat{T}, 1 \le i \le n\}.$$

Since $LB_0 = L[\hat{T}]$, it follows from the relation (2.2) that $B = (LB_0)^\Gamma = B_0$. Consequently, the $2nd$ functions

$$\{f_i(\chi_j), \ f_i(\chi_j^{-1}) \mid \ 1 \le i \le n, \ 1 \le j \le d\}$$

generate the k-algebra B. We shall use, however, the more convenient set of generators defined as follows. Write, in accordance with (2.5),

$$\chi_m = x_1^{(m)} \omega_1 + \cdots + x_n^{(m)} \omega_n, \ x_i^{(m)} \in B, \ 1 \le i \le n,$$

and let

$$y_m = \prod_{g \in \Gamma} g \cdot \chi_m, \ 1 \le m \le d;$$

clearly, $y_m \in B$ for every m. Let

$$\mathfrak{c} = \{x_l^{(m)}, y_m^{-1} \ \mid \ 1 \le m \le d, 1 \le l \le n\}. \tag{2.6}$$

The set $\mathfrak{c}$ generates the k-algebra B. Moreover, let

$$y_m \chi_m^{-1} = F_1^{(m)} \omega_1 + \ldots + F_n^{(m)} \omega_n, \ 1 \le m \le d,$$

and

$$1 = a_1\omega_1 + \ldots + a_n\omega_n,$$

then

$$\{F_j^{(m)},\ a_j \mid 1 \leq m \leq d,\ 1 \leq j \leq n\} \subseteq B_0.$$

The formulae (2.3) may be rewritten in the coordinates $\mathfrak{c}$ as follows:

$$\begin{aligned} m^*(x_l^{(m)}) &= \sum_{1\leq i,j\leq n} c_{ij}^{(l)} x_i^{(m)} \otimes x_j^{(m)},\ i^*(x_l^{(m)}) = \frac{F_l^{(m)}}{y_m}, \\ e^*(x_l^{(m)}) &= a_l,\ 1 \leq m \leq d,\ 1 \leq l \leq n. \end{aligned} \tag{2.7}$$

Equations (2.7) define a Hopf k-algebra structure on B; it is, of course, consistent with the Hopf L-algebra structure on $L[\hat{T}]$ defined by the relations (2.3), so that (2.4) is in fact an L-isomorphism of Hopf algebras.

From now on, the field k is assumed to be the field of fractions of an integral domain denoted by $\mathfrak{o}$. Suppose, moreover, that the integral closure $\mathfrak{O}$ of $\mathfrak{o}$ in L is an $\mathfrak{o}$-free module, and call an $\mathfrak{o}$-basis of $\mathfrak{O}$ an integral basis of the extension $L \mid k$. Suppose $\{\omega_1, \ldots, \omega_n\}$ be such a basis. Let A be the $\mathfrak{o}$-subalgebra of B generated by the set $\mathfrak{c}$, and let $\mathcal{T}_0 = \text{Spec } A$.

Lemma 1. *We have $A \subseteq k[T]$, $A \cap k = \mathfrak{o}$, $A \otimes_{\mathfrak{o}} k \cong kA = k[T]$, and $\mathfrak{O}[\hat{T}] \subseteq \mathfrak{O}A$.*

Proof. It follows from the definition of the $\mathfrak{o}$-algebra A.

Proposition 1. *$\mathcal{T}_0$ is a reduced $\mathfrak{o}$-group scheme. The definition of this group scheme does not depend, up to an $\mathfrak{o}$-isomorphism, on the choice of a splitting field L of the torus T, on the choice of the integral basis $\{\omega_1, \ldots, \omega_n\}$ of the extension $L \mid k$, and on the choice of the basis $\{\chi_1, \ldots, \chi_d\}$ in the group of characters $\hat{T}$. The group scheme $\mathcal{T}_0$ is an integral model of the torus T.*

Proof. It follows from the definition (2.7) that A is an $\mathfrak{o}$-Hopf algebra because

$$c_{ij}^{(l)} \in \mathfrak{o},\ a_l \in \mathfrak{o},\ 1 \leq i, j, l \leq n,$$

therefore $\mathcal{T}_0$ is an $\mathfrak{o}$-group scheme. Since the ring A contains no nilponent elements, the group scheme $\mathcal{T}_0$ is reduced. Moreover, a change of an integral basis is a linear $\mathfrak{o}$-invertible transformation; it unduces an automorphism of the $\mathfrak{o}$-algebra A, and it does not change Spec A. Since $\mathcal{T}_0 \times_{\mathfrak{o}} k = T$, the scheme $\mathcal{T}_0$ is an integral model of the torus T. Finally, let L_0 be the minimal splitting field of T; then $L_0 \subseteq L$, and the Galois group of the extension $L \mid L_0$ is contained in the kernel of the integral representation ρ of the group Γ. Hence the definition of $\mathcal{T}_0$ does not depend on the choice of L, as asserted.

Definition 1. *The group $\mathfrak{o}$-scheme $\mathcal{T}_0$ is called the standard model of T.*

Remark 1. Given an arbitrary morphism of schemes $h\colon S' \to S$, one can define the Weil restriction functor $R_{S'|S}(X)$ with respect to h as follows, [1, 7.6]. For any S'-scheme X, let $R_{S'|S}(X)$ be the scheme representing the functor $Y \mapsto \mathrm{Hom}_{S'}(Y \times_S S', X)$, with Y ranging over the S-schemes. This is a generalisation of the Weil restriction of scalars functor $R_{F|k}$ for a finite extension of fields $F \mid k$.

Example 1. Let $F \mid k$ be a finite field extension of degree d. The quasi-split torus $S = R_{F|k}G_{m,F}$ splits over any normal extension $L \mid k$ with $F \subseteq L$; let $\Gamma_1 = \mathrm{Gal}(L/F)$ be the Galois group of the extension $L \mid F$. The module of characters $\hat{S} = \mathbb{Z}[\Gamma/\Gamma_1]$ is a permutation module. One can choose a basis $\{\chi_1, \dots, \chi_d\}$ of $\hat{S}$ in such a way that Γ_1 stabilises χ_1 and

$$\chi_i = \sigma_i\chi_1,\ \sigma_i \in \Gamma/\Gamma_1,\ 1 \le i \le d.$$

Assuming the integral closure $\mathfrak{O}$ of $\mathfrak{o}$ in L is an $\mathfrak{o}$-free module, let $\{\omega_1, \dots, \omega_n\}$ be an integral basis of $L \mid k$, let $\chi_1 = x_1\omega_1 + \cdots + x_n\omega_n$, and let $\mathfrak{O}_F := \mathfrak{O} \cap F$. Let $A = \mathfrak{o}[x_1, \dots, x_n, y^{-1}]$, where $y := \chi_1 \cdots \chi_d$ is a form of degree d in n variables, then $\mathcal{S}_0 := \mathrm{Spec}\, A$ is the standard model of S. In this case, $\mathcal{S}_0 \cong R_{\mathfrak{O}_F|\mathfrak{o}}G_{m,\mathfrak{O}_F}$, and $\mathcal{S}_0(\mathcal{A}) = (\mathcal{A} \otimes_{\mathfrak{o}} \mathfrak{O}_F)^*$ for any $\mathfrak{o}$-algebra $\mathcal{A}$; in particular, $\mathcal{S}_0(\mathfrak{o}) = \mathfrak{O}_F^*$.

Let $\mathfrak{G}_k = \mathrm{Gal}(\bar{k} \mid k)$ be the absolute Galois group of k. The correspondence $T \mapsto \hat{T}$ is an exact (anti)-duality between the category of algebraic k-tori and the category of $\mathbb{Z}(\mathfrak{G}_k)$-modules, of finite rank and free over $\mathbb{Z}$ (cf. for instance [22], [23]). In this sense, the functorial description of the torus (and of its standard integral model) to be given now is dual to the Hopf algebra approach. There is a canonical embedding

$$i\colon T \hookrightarrow S \tag{2.8}$$

of the torus T into the quasi-split torus $S = R_{L|k}G^d_{m,L}$, corresponding to the canonical projection of the modules of characters

$$\hat{i}\colon \hat{T} \otimes_{\mathbb{Z}} \mathbb{Z}[\Gamma] \to \hat{T}.$$

Let $\mathcal{S}_0 = R_{\mathfrak{O}|\mathfrak{o}}G^d_{m,\mathfrak{O}}$ be the standard model of the torus S, and let $j\colon S \to \mathcal{S}_0$ be the canonical morphism of the generic fibre S of $\mathcal{S}_0$ into the $\mathfrak{o}$-group scheme $\mathcal{S}_0$. Let us recall, [8, Def. I.6.10.1–2], [1, p. 55], that *the scheme-theoretic closure* of a scheme X in a scheme Y with respect to a morphism $f\colon X \to Y$ may be regarded as the smallest closed subscheme of Y containing $f(X)$.

Proposition 2. *The standard model $\mathcal{T}_0$ of the torus T is isomorphic to the scheme-theoretic closure of T in $\mathcal{S}_0$ with respect to the morphism $j \circ i$.*

Proof. By definition, $k[S] = k[\mathfrak{d}]$ with

$$\mathfrak{d} := \{s_j^{(i)},\ z_i^{-1} \mid 1 \leq i \leq d,\ 1 \leq j \leq n\},$$

where

$$z_i = \prod_{g\in\Gamma} \sum_{j=1}^{n} s_j^{(i)} g\omega_j,\ 1 \leq i \leq d.$$

The embedding $i\colon T \hookrightarrow S$ induces a homomorphism $i^*\colon k[\mathfrak{d}] \to k[T]$ of k-algebras defined by

$$i^*\colon s_j^{(i)} \mapsto x_j^{(i)},\ z_i^{-1} \mapsto y_i^{-1},\ 1 \leq i \leq d,\ 1 \leq j \leq n.$$

The kernel of i^* coincides with the ideal of the relations satisfied by the variables in the set $\mathfrak{c}$ defined by (2.6). Let $\mathcal{D} = \mathfrak{o}[\mathfrak{d}]$; the scheme $\mathcal{S}$ is clearly isomorphic to Spec $\mathcal{D}$. Let $\varphi = i^* \circ j^*$, where $j^*\colon \mathfrak{o}[\mathfrak{d}] \hookrightarrow k[\mathfrak{d}]$ is the canonical inclusion induced by j. Since $A = \varphi(\mathcal{D})$, it follows that the scheme $\mathcal{T}_0$ is isomorphic to the scheme-theoretic closure of the torus T in $\mathcal{S}_0$, as asserted.

Let us consider the following $d+1$ linear forms

$$r(x) = x_1^{(0)}\omega_1 + \cdots + x_n^{(0)}\omega_n,\ t_i(x) = x_1^{(i)}\omega_1 + \cdots + x_n^{(i)}\omega_n,\ 1 \leq i \leq d,$$

in $\mathfrak{O}[x]$, $x := (x_j^{(i)})_{1\leq j\leq n,\, 0\leq i\leq d}$. Let J_0 be the ideal in $\mathfrak{o}[x]$ defined by the relations

$$g\vec{t} = \vec{t}^{\,g},\ g \in \Gamma,\ r(x)\prod_{i=1}^{d} t_i(x) = 1, \tag{2.9}$$

where, in notation (2.1), $\vec{t} := (t_1(x), \ldots, t_d(x))$, and let $J = J_0 k[x]$ be the corresponding ideal in $k[x]$.

Corollary 1. *The schemes T and $\mathcal{T}_0$ can be alternatively described as follows:*

$$T = \text{Spec } k[x]/J$$

and

$$\mathcal{T}_0 = \text{Spec } \mathfrak{o}[x]/J_0. \tag{2.10}$$

Proof. Clearly, $B \cong k[x]/J$; the proof of Proposition 2 shows that $A \cong \mathfrak{o}[x]/J_0$.

Notation 3. *Given a commutative* $\mathfrak{o}$*-algebra* $\mathcal{A}$, *let*

$$\mathcal{A}_d := [(\mathcal{A} \otimes_{\mathfrak{o}} \mathfrak{O})^*]^d. \tag{2.11}$$

Let, in notation of (2.1),

$$g(\vec{\alpha}) = (g\vec{\alpha})(\vec{\alpha}^g)^{-1}, \ g\vec{\alpha} := (g\alpha_1, \ldots, g\alpha_d), \ g \in \Gamma,$$

for $\vec{\alpha} \in \mathcal{A}_d$.

Corollary 2. *We have*

$$\mathcal{T}_0(\mathcal{A}) = (\mathcal{A}_d)^\Gamma \tag{2.12}$$

for any commutative $\mathfrak{o}$*-algebra* $\mathcal{A}$, *where* $(\mathcal{A}_d)^\Gamma := \{\vec{\alpha} \mid \vec{\alpha} \in \mathcal{A}_d, \ g(\vec{\alpha}) = \vec{\alpha}\}$ *is the subgroup of* Γ*-invariants of the group* $\mathcal{A}_d$. *In particular,*

$$\mathcal{T}_0(\mathfrak{o}) = [(\mathfrak{O}^*)^d]^\Gamma. \tag{2.13}$$

Proof. Since $\mathcal{T}_0(\mathcal{A}) = \mathrm{Hom}_{\mathfrak{o}}(A, \mathcal{A})$, equation (2.12) follows from the equation (2.10) and the definition (2.11) of the group $\mathcal{A}_d$.

3 Arithmetic applications (local fields)

Let k be a $\mathfrak{p}$-adic field, that is a finite extension of $\mathbb{Q}_p$; here $\mathfrak{p}$ stands for the maximal ideal of the ring $\mathfrak{o}$ of integers of k, so that $\mathfrak{O}$ is the ring of integers of L. The extension $L \mid k$ is known to have an integral basis. Given a finite extension of fields $F \mid k$, let $\mathfrak{O}_F$ and U_F denote the ring of integers of F and the maximal compact subgroup of $T(F)$ respectively. Some of the properties of the standard model $\mathcal{T}_0$ are summarized in the following theorem.

Theorem 1. *Let* $\mathcal{T}_0$ *be the standard model of* T. *Then*

(i) $A \subset k[T]$, $A \cap k = \mathfrak{o}$, $A \otimes_{\mathfrak{o}} k \cong kA = k[T]$, *and* $\mathfrak{O}[\hat{T}] \subseteq \mathfrak{O}A$;

(ii) *the* $\mathfrak{o}$*-scheme* $\mathcal{T}_0$ *is reduced, faithfully flat and of finite type;*

(iii) $\mathcal{T}_0(\mathfrak{o}) = \mathrm{Hom}_\Gamma(\hat{T}, \mathfrak{O}^*) = U_k$;

(iv) $\mathcal{T}_0(\mathfrak{O}_F) \subseteq U_F$ *for any finite extension* $F \mid k$;

(v) $f(u) \in \mathfrak{o}$ *for any* $f \in A$ *and any* $u \in U_k$.

Proof. Lemma 1 gives (i); it has been already remarked that the scheme $\mathcal{T}_0$ is reduced (Proposition 1), and it is clearly of finite type. It follows from Corollary 2 of Proposition 5 in [2, chap. II, § 3] that the $\mathfrak{o}$-module A is flat, because the only maximal ideal $\mathfrak{m}$ of the local ring $\mathfrak{o}$ is principal and the module A is torsion free. Moreover, since $\mathfrak{m}A \neq A$ the module A is faithfully flat [2, chap. I, § 3, Proposition 1]. This proves (ii). Assertion (iii) follows from the equation (2.13). To prove (iv), it suffices to observe that the group $\mathcal{T}_0(\mathfrak{O}_F)$ is compact; therefore it is contained in the maximal compact subgroup U_F of $T(F)$, as asserted. Finally, (v) follows from (iii) and (2.10).

Let $\kappa = \mathfrak{o}/\mathfrak{p}$ be the residue field of k, and let $T_\mathfrak{p} = \mathcal{T}_0 \times_\mathfrak{o} \kappa$ be the special fibre of the group scheme $\mathcal{T}_0$.

Definition 2. *We say that the torus T has a good reduction if $T_\mathfrak{p}$ is a κ-algebraic torus; otherwise, the torus T is said to have a bad reduction. The torus T has a smooth reduction if $T_\mathfrak{p}$ is a smooth scheme.*

Notation 4. *Let e be the ramification index of the extension $L \mid k$, let E be the maximal unramified extension of k in L, and let ε and λ denote the residue fields of the fields E and L respectively. Moreover, let μ_l stand for the group of l-th roots of unity. Let us recall that, given a finite extension $F \mid k$, a norm torus $R^1_{F|k}G_m$ is defined as the kernel of the norm map $R_{F|k}G_{m,F} \to G_{m,k}$.*

It can be easily proved that the torus T has a good reduction if the extension $L \mid k$ is unramified. This result is contained in the following proposition.

Proposition 3. *Suppose that a k-torus T splits over a normal unramified extension L of the field k. Then $\mathcal{T}_0 = \mathrm{Spec}\ (\mathfrak{O}[\hat{T}])^\Gamma$, and the torus T has a good reduction.*

Proof. Since the extension $L \mid k$ is unramified and $g\chi_m \in \hat{T}$, equations

$$g\chi_m = x_1^{(m)} g\omega_1 + \cdots + x_n^{(m)} g\omega_n,\ g \in \Gamma,\ 1 \leq i \leq n,\ 1 \leq m \leq d,$$

show that the set $\mathfrak{c}$, generating the $\mathfrak{o}$-algebra A, is contained in $\mathfrak{O}[\hat{T}]$; therefore $\mathfrak{O}A \subseteq \mathfrak{O}[\hat{T}]$. On the other hand, $\mathfrak{O}[\hat{T}] \subseteq \mathfrak{O}A$ by Lemma 1; thus $\mathfrak{O}[\hat{T}] = \mathfrak{O}A$. Since $A = (\mathfrak{O}A)^\Gamma$, it follows that $\mathcal{T}_0 = \mathrm{Spec}\ (\mathfrak{O}[\hat{T}])^\Gamma$. We have

$$A \otimes_\mathfrak{o} \kappa = (\mathfrak{O}[\hat{T}])^\Gamma \otimes_\mathfrak{o} \kappa = (\lambda[\hat{T}])^\gamma,$$

where γ denotes the Galois group of the extension $\lambda \mid \kappa$ (of course, $\gamma \cong \Gamma$ in our case). Therefore

$$T_{\mathfrak{p}} = \text{Spec } A \otimes_{\mathfrak{o}} \kappa = \text{Spec } (\lambda[\hat{T}])^{\gamma},$$

and it follows that $T_{\mathfrak{p}}$ is a κ-torus. This proves the proposition.

The argument used in the proof of Proposition 3 leads to the following observation.

Corollary 3. *Let $T_F = T \times_k F$, and let $\mathcal{T}_F$ be the standard integral model of T_F. If the extension $F \mid k$ is unramified, then $\mathcal{T}_F \cong \mathcal{T}_0 \times_{\mathfrak{o}} \mathfrak{O}_F$ and $\mathcal{T}_0(\mathfrak{O}_F) = U_F$.*

Proof. Without loss of generality, it may be assumed that $F \subseteq L$. Write $\mathcal{T}_F = \text{Spec } A_F$; by Corollary 1, $A \cong \mathfrak{o}[x]/J_0$ and $A_F \cong \mathfrak{O}_F[u]/J_0^{(F)}$, where $u := (u_j^{(i)})_{1 \le j \le m,\, 0 \le i \le d}$ with $m := [L : F]$, and $J_0^{(F)}$ is the ideal in $\mathfrak{O}_F[u]$ generated by the relations analogous to the relations (2.9). Since the extension $F \mid k$ is unramified, the natural map $A \otimes_{\mathfrak{o}} \mathfrak{O}_F \to A_F$ is easily seen to be an isomorphism. Hence $\mathcal{T}_0(\mathfrak{O}_F) = \mathcal{T}_F(\mathfrak{O}_F)$, and therefore $\mathcal{T}_0(\mathfrak{O}_F) = U_F$ in view of Theorem 1 (iv).

Suppose the extension $L \mid k$ be tamely ramified, then the torus T has a smooth reduction; however, if $e > 1$, the scheme $T_{\mathfrak{p}}$ is not necessarily a torus and may contain a non-trivial unipotent κ-subgroup. If the extension $L \mid k$ is wildly ramified, that is if $p|e$, then $T_{\mathfrak{p}}$ does not need to be a reduced scheme.

Example 2. Let $k = \mathbb{Q}_p$, let $L = k(\sqrt{p})$, and let $T = R^1_{L|k} G_m$ be the one-dimensional norm torus, that is the kernel of the norm map $R_{L|k} G_{m,L} \to G_{m,k}$. We have $\mathcal{T}_0 = \text{Spec } A$ with $A := \mathbb{Z}_p[x, y]/(x^2 - py^2 - 1)$, so that

$$T_p \cong \text{Spec } \mathbf{F}_p[x, y]/(x^2 - 1).$$

If $p \neq 2$, the group scheme T_p is isomorphic to $\mu_2 \times G_{a,\mathbf{F}_p}$; hence the torus T has a smooth reduction at $p \neq 2$. If $p = 2$, then the scheme T_p is not reduced and therefore $\mathcal{T}_0$ is not a smooth scheme. In accordance with the general theory, we have $\mathcal{T}_0(\mathfrak{o}) = \mathcal{T}_0(\mathbb{Z}_p) = U_k(= T(k))$. The integral model $\mathcal{T}_0$ described in this example is minimal among all the reduced integral models of T, cf. [26].

Let $L \mid k$ be a tamely ramified extension of degree n. The following proposition describes the behaviour of the norm torus $T = R^1_{L|k} G_m$ under reduction modulo the maximal ideal $\mathfrak{p}$ of $\mathfrak{o}$. We state it without proof, making use of Notation 4.

Proposition 4. *Let $\mathcal{T}_0$ be the standard integral model of T, let $Y = T_{\mathfrak{p}}$ denote the special fibre of T with respect to this model, and let Y^0 be the identity component of Y.*

(i) *If the extension $L \mid k$ is unramified, then $Y = R^1_{\lambda|\kappa} G_m$ is a κ-torus.*

(ii) *If $L \mid k$ is a tamely ramified extension, then $Y^0 = R^1_{\varepsilon|\kappa} G_m \times N$, where N is an unipotent group, and $Y/Y^0 \cong \mu_e$.*

(iii) *Suppose that the extension $L \mid k$ is totally and tamely ramified. Then $Y = \mu_e \times N$, where the unipotent subgroup N is isomorphic to a direct product of groups W_i of Witt vectors of length l_i, $1 \le i \le m$, and the numbers l_i, m can be explicitly computed; in particular, if $n < p$, then N is isomorphic to a direct product of $n-1$ copies of the additive group $G_{a,\kappa}$.*

Let us refer to the original papers [25, §4], [12], [18], [23, 10.4] for the proof of this proposition and further details. It should be mentioned that one can reduce the calculation of reduction of an arbitrary torus with a tamely ramified splitting field to the special case of norm tori [12].

4 Arithmetic applications (number fields)

Let now k be a number field, that is a finite extension of $\mathbb{Q}$, and let $\mathfrak{o}$ be the ring of integers of k, so that $\mathfrak{O}$ stands for the ring of integers of L. If the extension $L \mid k$ has an integral basis, that is if $\mathfrak{O}$ is an $\mathfrak{o}$-free module, then the standard model $\mathcal{T}_0$ of T can be defined as in Section 2. The definition of the $\mathfrak{o}$-group scheme $\mathcal{T}_0$ does not depend on the choice of a splitting field L of the k-torus T, in view of Proposition 1. One may ask whether a splitting field of an algebraic k-torus can be always chosen to have an integral basis over k. If the following conjecture holds true, it is indeed the case.

Conjecture 1. *Given two number fields F and k, there is a finite normal extension $L \mid k$ such that L has a k-integral basis, and $F \subseteq L$.*

By a theorem of Fröhlich's [6], this conjecture is certainly true if the 2-class group of k is either trivial, or non-cyclic; we have not succeeded, however, to either prove or disprove this assertion in general. Therefore we proceed to construct an integral model of the k-torus T, without assuming $\mathfrak{O}$ to be an $\mathfrak{o}$-free module.

Definition 3. *Consider a functor*

$$\mathfrak{F} : \mathcal{A} \mapsto (\mathcal{A}_d)^{\Gamma} \tag{4.1}$$

from the category of commutative $\mathfrak{o}$*-algebras to the category of Abelian groups. An* $\mathfrak{o}$*-group scheme* $\mathcal{T}_0$ *representing the functor* $\mathfrak{F}$ *is said to be the standard model of the* k*-torus* T*.*

Remark 2. By Corollary 2, this definition agrees with Definition 1.

Notation 5. *Let* $\mathcal{P}$ *be the set of all the (non-zero) prime ideals of* $\mathfrak{o}$*, and let* $\mathfrak{p} \in \mathcal{P}$*. In what follows,* $\nu_{\mathfrak{p}}$*,* $k_{\mathfrak{p}}$*,* $\mathfrak{o}_{\mathfrak{p}}$*, and* $\mathfrak{o}_{(\mathfrak{p})} := \{\alpha \mid \alpha \in k,\ \nu_{\mathfrak{p}}(\alpha) \geq 0\}$ *stand for the* $\mathfrak{p}$*-adic valuation, the completion of* k *at* $\mathfrak{p}$*, the ring of integers of* $k_{\mathfrak{p}}$*, and the localisation of* $\mathfrak{o}$ *at* $\mathfrak{p}$ *respectively; let*

$$\mathfrak{o}^{(\mathfrak{p})} := \{\alpha \mid \alpha \in k,\ \nu_{\mathfrak{q}}(\alpha) \geq 0 \ for\ \ \mathfrak{q} \neq \mathfrak{p},\ \mathfrak{q} \in \mathcal{P}\}.$$

Further, let $\mathfrak{O}_{(\mathfrak{p})} = \mathfrak{o}_{(\mathfrak{p})}\mathfrak{O} = \mathfrak{o}_{(\mathfrak{p})} \otimes_{\mathfrak{o}} \mathfrak{O}$ *and* $\mathfrak{O}^{(\mathfrak{p})} = \mathfrak{o}^{(\mathfrak{p})}\mathfrak{O} = \mathfrak{o}^{(\mathfrak{p})} \otimes_{\mathfrak{o}} \mathfrak{O}$*.*

The ring $\mathfrak{O}_{(\mathfrak{p})}$ (resp. $\mathfrak{O}^{(\mathfrak{p})}$) coincides with the integral closure of $\mathfrak{o}_{(\mathfrak{p})}$ (resp. $\mathfrak{o}^{(\mathfrak{p})}$) in L; moreover, the ring $\mathfrak{O}_{(\mathfrak{p})}$ is an $\mathfrak{o}_{(\mathfrak{p})}$-free module. Let $\mathcal{T}_{(\mathfrak{p})} = \operatorname{Spec} A_{(\mathfrak{p})}$ be the standard $\mathfrak{o}_{(\mathfrak{p})}$-model of a k-torus T; by construction, $A_{(\mathfrak{p})} \subseteq k[T]$. Let $A = \bigcap_{(\mathfrak{p}) \in \mathcal{P}} A_{(\mathfrak{p})}$. Making use of the well-known results in the approximation theory of lattices, see for instance [20, §2], one can show that $\operatorname{Spec} A$ is an $\mathfrak{o}$-model of the torus T of finite type over $\mathfrak{o}$. We shall describe the $\mathfrak{o}$-algebra A explicitly, and then show that $\mathcal{T}_0 := \operatorname{Spec} A$ is indeed the standard model of T in the sense of Definition 3.

Let $\mathfrak{C}$ be the Steinitz invariant of the $\mathfrak{o}$-module $\mathfrak{O}$; it is an ideal class in k, and $\mathfrak{O} \cong \mathfrak{a} \oplus M$ for $\mathfrak{a} \in \mathfrak{C}$, with a free $\mathfrak{o}$-module M of rank $n-1$, $n := [L : k]$, cf. [7, Theorem 13 on p. 95]. Let us choose two distinct prime ideals $\mathfrak{p}$ and $\mathfrak{p}_1$ in $\mathfrak{C}$, and let $\mathfrak{p}\mathfrak{p}_1^{-1} = (\pi)$ with $\pi \in k^*$. Since $\mathfrak{p} \in \mathfrak{C}$, one can choose a basis $\{\omega_1, \ldots, \omega_n\}$ of the extension $L \mid k$ satisfying the following condition

$$\mathfrak{O} = \mathfrak{p}\omega_1 \oplus \mathfrak{o}\omega_2 \oplus \ldots \oplus \mathfrak{o}\omega_n. \tag{4.2}$$

Clearly, $\omega_i \in \mathfrak{O}$ for $2 \leq i \leq n$, and $(\omega_1)\mathfrak{p} \subseteq \mathfrak{O}$. Furthermore, it follows that $\{\pi\omega_1, \omega_2, \ldots, \omega_n\}$ is an $\mathfrak{o}_{(\mathfrak{p})}$-basis of $\mathfrak{O}_{(\mathfrak{p})}$, and that $\{\omega_1, \ldots, \omega_n\}$ is an $\mathfrak{o}^{(\mathfrak{p})}$-basis of an $\mathfrak{o}^{(\mathfrak{p})}$-free module $\mathfrak{O}^{(\mathfrak{p})}$. As at the end of Section 2, let

$$x := (x_j^{(i)}),\ y := (y_j^{(i)}),\ 1 \leq j \leq n,\ 0 \leq i \leq d,$$

stand for two arrays of $n(d+1)$ independent variables, and let $J^{(\mathfrak{p})}$ (respectively, $J_{(\mathfrak{p})}$) denote the corresponding ideal of relations (2.9) in $\mathfrak{o}^{(\mathfrak{p})}[x]$ (respectively, in $\mathfrak{o}_{(\mathfrak{p})}[y]$). Clearly,

$$A_{(\mathfrak{p})} = \mathfrak{o}_{(\mathfrak{p})}[y]/J_{(\mathfrak{p})};$$

let

$$A^{(\mathfrak{p})} = \mathfrak{o}^{(\mathfrak{p})}[x]/J^{(\mathfrak{p})},$$

then $A^{(\mathfrak{p})} \subseteq k[T]$. Let $\mathcal{T}^{(\mathfrak{p})} = \operatorname{Spec} A^{(\mathfrak{p})}$ be the standard $\mathfrak{o}^{(\mathfrak{p})}$-model of the torus T. Let $I_{(\mathfrak{p})} = J_{(\mathfrak{p})} \cap \mathfrak{o}[x,y]$, $I^{(\mathfrak{p})} = J^{(\mathfrak{p})} \cap \mathfrak{o}[x,y]$, and let I_0 stand for the ideal in $\mathfrak{o}[x,y]$ generated by the relations

$$\pi y_1^{(i)} = x_1^{(i)},\ y_j^{(i)} = x_j^{(i)},\ 2 \le j \le n,\ 0 \le i \le d. \tag{4.3}$$

Let $I = (I_{(\mathfrak{p})}, I^{(\mathfrak{p})}, I_0)$ and let $A = \mathfrak{o}[x,y]/I$. By construction, A is a finitely generated $\mathfrak{o}$-algebra. The following theorem is the main result of this section.

Theorem 2. *The affine $\mathfrak{o}$-scheme $\mathcal{T}_0 := \operatorname{Spec} A$ is isomorphic to the standard model of T, in the sense of Definition 3.*

Proof. Since $\mathcal{T}_{(\mathfrak{p})}$ (respectively, $\mathcal{T}^{(\mathfrak{p})}$) is isomorphic to the standard $\mathfrak{o}_{(\mathfrak{p})}$- (respectively, $\mathfrak{o}^{(\mathfrak{p})}$-) model of T, the scheme $\mathcal{T}_0$ is an $\mathfrak{o}$-model of the torus T. Let us consider an $\mathfrak{o}$-homomorphism $\varphi : A \to \mathcal{A}$ of A in a commutative $\mathfrak{o}$-algebra $\mathcal{A}$; let

$$\varphi(x_j^{(i)}) = a_j^{(i)},\ 1 \le j \le n,\ 0 \le i \le d,$$

then $\varphi(y_j^{(i)}) = b_j^{(i)}$, with

$$\pi b_1^{(i)} = a_1^{(i)},\ b_j^{(i)} = a_j^{(i)},\ 2 \le j \le n,\ 0 \le i \le d, \tag{4.4}$$

in view of (4.3). Since

$$\{a_j^{(i)},\ b_j^{(i)}\ \mid\ 1 \le j \le n,\ 0 \le i \le d\} \subseteq \mathfrak{o},$$

it follows from (4.4) that the elements $a_1^{(i)}$, $0 \le i \le d$, of the ring $\mathfrak{o}$ range over the ideal $\mathfrak{p}$; therefore the elements

$$\alpha^{(i)} = a_1^{(i)}\omega_1 + a_2^{(i)}\omega_2 + \ldots + a_n^{(i)}\omega_n,\ 0 \le i \le d,$$

range over the $\mathfrak{O}$-algebra $\mathcal{A} \otimes_{\mathfrak{o}} \mathfrak{O}$, in view of (4.2). Now, it follows from the definition of the ideals $I_{(\mathfrak{p})}, I^{(\mathfrak{p})}$ that the homomorphisms φ are in one-to-one correspondence with the elements $\alpha = (\alpha^{(1)}, \ldots, \alpha^{(d)})$ of the group $(\mathcal{A}_d)^{\Gamma}$, as asserted.

Remark 3. The ideal I of the Noetherian ring $\mathfrak{o}[x,y]$ is finitely generated; consequently, the $\mathfrak{o}$-algebra A can be described explicitly in terms of generators and relations.

Remark 4. Two of the authors have constructed [10] a natural integral model of an affine toric variety defined over a number field. A suitable adaptation of the argument used in the proof of Theorem 2 shows that this model is isomorphic to an affine group scheme, although it has been originally defined in [10] by "gluing" together two affine schemes.

Notation 6. *For each $\mathfrak{p}$ in $\mathcal{P}$, we fix a prime ideal $\mathfrak{P}$ of $\mathfrak{O}$ with $\mathfrak{P}|\mathfrak{p}$; let $L_{\mathfrak{P}}$ denote the $\mathfrak{P}$-adic completion of L, and let $\Gamma_{\mathfrak{p}}$ be the Galois group of the field extension $L_{\mathfrak{P}} \mid k_{\mathfrak{p}}$.*

Proposition 5. *For every $\mathfrak{p}$ in $\mathcal{P}$, the $\mathfrak{o}_{\mathfrak{p}}$-scheme $\mathcal{T}_0 \times_{\mathfrak{o}} \mathfrak{o}_{\mathfrak{p}}$ is isomorphic to $\mathcal{T}_{0\mathfrak{p}}$, the standard integral $\mathfrak{o}_{\mathfrak{p}}$-model of the $k_{\mathfrak{p}}$-torus $T \times_k k_{\mathfrak{p}}$. The group scheme $\mathcal{T}_0$ is a reduced faithfully flat $\mathfrak{o}$-scheme.*

Proof. Let us recall the well-known identity

$$\mathfrak{O} \otimes_{\mathfrak{o}} \mathfrak{o}_{\mathfrak{p}} = \sum_{g \in \Gamma/\Gamma_{\mathfrak{p}}} \oplus\, \mathfrak{O}_{g\mathfrak{P}}, \tag{4.5}$$

see for instance [7, Ch. III, eq.(1.8)]; in view of Theorem 2, decomposition (4.5) leads to the asserted relation $\mathcal{T}_0 \times_{\mathfrak{o}} \mathfrak{o}_{\mathfrak{p}} \cong \mathcal{T}_{0\mathfrak{p}}$. Since "being a reduced faithfully flat scheme" is a local property, the second assertion of the proposition follows from the first one and Theorem 1, (ii) (cf. also Corollary to Proposition 15 in [2, Ch. II, § 3]).

Remark 5. Let $\mathbf{A}$ stand for the ring of adeles of k, and let

$$\mathbf{A}^{(\infty)} = \prod_{v|\infty} k_v \prod_{\mathfrak{p}} \mathfrak{o}_{\mathfrak{p}}$$

denote the subring of integral adeles. We have $\mathcal{T}_0(\mathbf{A}^{(\infty)}) \subseteq \mathcal{T}_0(\mathbf{A}) = T(\mathbf{A})$. The finite group $H(\mathcal{T}_0) := T(\mathbf{A})/T(k)\mathcal{T}_0(\mathbf{A}^{(\infty)})$ is called the class group of T. Let $\mathcal{T}_1$ be another integral model of T of finite type over $\mathfrak{o}$, then $\mathcal{T}_1(\mathbf{A}^{(\infty)}) \subseteq \mathcal{T}_0(\mathbf{A}^{(\infty)})$ in view of Proposition 5 and Theorem 1, (iii). Therefore there is a surjective homomorphism of finite groups $H(\mathcal{T}_1) \to H(\mathcal{T}_0)$; in particular, the order $h(T)$ of the group $H(\mathcal{T}_0)$ divides the order of the group $H(\mathcal{T}_1)$ for any other integral model $\mathcal{T}_1$ of T of finite type. The number $h(T)$ is called the class number of T. The reader is referred to [23, § 20] for further details concerning the class groups of algebraic tori.

5 Comparison with the Néron–Raynaud model

In what follows, k stands for either a number field, or a $\mathfrak{p}$-adic field, and $\mathfrak{o}$ denotes the ring of integers of k. The goal of this section is to compare the standard $\mathfrak{o}$-model $\mathcal{T}_0$ of an algebraic k-torus T to the Néron–Raynaud model of this torus.

Definition 4. *A separated group $\mathfrak{o}$-scheme $\mathcal{N}$ is said to be the Néron–Raynaud model of T, if its generic fibre $\mathcal{N}_k = \mathcal{N} \times_{\mathfrak{o}} k$ is isomorphic to T and, for any smooth $\mathfrak{o}$-scheme Y and any k-morphism $u_k \colon Y \times_{\mathfrak{o}} k \to T$, there is an $\mathfrak{o}$-morphism $u \colon Y \to \mathcal{N}$ extending the morphism u_k.*

Every algebraic k-torus has a (unique) Néron–Raynaud model. The Néron–Raynaud model of an algebraic k-torus is not necessarily a scheme of finite type over $\mathfrak{o}$; it is, however, locally of finite type (=:lft) [1, Ch.10, Proposition 1.6].

Definition 5. (cf. [1, p.154]) *The identity component $\mathcal{G}^{(0)}$ of a group $\mathfrak{o}$-scheme $\mathcal{G}$ is defined to be the open subscheme of $\mathcal{G}$ which is the union of all the identity components $\mathcal{G}_{\mathfrak{p}}^{(0)}$ of the fibres $\mathcal{G}_{\mathfrak{p}}$ over the points $\mathfrak{p}$ of Spec $\mathfrak{o}$.*

The identity component of the Néron–Raynaud model of an algebraic k-torus is always a scheme of finite type over $\mathfrak{o}$ [1, p. 290].

Example 3. Let $T = G_{m,\mathbb{Q}}$, then

$$\mathcal{N} = \bigcup_{a \in \mathbb{Z}, a \geq 1} \text{Spec } \mathbb{Z}[at, \frac{1}{at}]$$

is the Néron–Raynaud model of T over $\mathbb{Z}$, where, by definition,

$$\text{Spec } \mathfrak{o}[at, \frac{1}{at}] \cap \text{Spec } \mathfrak{o}[bt, \frac{1}{bt}] = \text{Spec } \mathfrak{o}[at, bt, \frac{1}{at}, \frac{1}{bt}] (= \text{Spec } \mathfrak{o}[at, \frac{1}{at}, \frac{a}{b}, \frac{b}{a}]).$$

We have $\mathcal{N}^{(0)} \cong G_{m,\mathbb{Z}}$, so that the identity component of $\mathcal{N}$ is isomorphic to the standard model of T in this case, cf. [1, Example 10.1.5].

Let L be a splitting field of the torus T, so that $T_L \cong G_{m,L}^d$, $T_L := T \times_k L$. Let $\mathcal{N}_L$ be the Néron–Raynaud model of the L-torus $G_{m,L}^d$; the group scheme $\mathcal{N}_L$ is defined over $\mathfrak{O}$, the ring of integers of L. The Néron–Raynaud model $\mathcal{S}$ of the quasi-split torus $R_{L|k}T_L$ is isomorphic to the group scheme $R_{\mathfrak{O}|\mathfrak{o}}\mathcal{N}_L$ [1, Ch.7, Proposition 6.6 and Ch.10, Proposition 1.4], whose identity component coincides with $\mathcal{S}_0 = R_{\mathfrak{O}|\mathfrak{o}} G_{m,\mathfrak{O}}^d$ [18, Lemma 3.1]. Let $i \colon T \hookrightarrow S$ be the embedding (2.8) defined in Section 2. Let $\mathcal{T}$ be the scheme-theoretic closure of T in $\mathcal{S}$, then the Néron–Raynaud model $\mathcal{N}$ of the torus T can be constructed as a "group smoothening" of the scheme $\mathcal{T}$ [1, Ch.7, Proposition 1.6 and Ch.10, Proposition 1.4].

Example 4. Let us return to the norm torus of Example 2; we use notation introduced there. If p is odd, T_p is a smooth, unipotent group scheme; the identity component of the scheme T_p is isomorphic to the additive group $G_{a,\mathbf{F}_p}$. The same can be said about the identity component of the Néron–Raynaud model of T [4, Ex. 4.3]. Note, in particular, that the identity component $\mathcal{T}_0^{(0)}$ of the scheme $\mathcal{T}_0$ coincides with the open subscheme $\mathcal{T}_0 \setminus \mathcal{T}_0^{(1)}$, where $\mathcal{T}_0^{(1)} :=$ Spec $\mathbb{Z}_p[x,y]/(x+1,p)$. If $p=2$, then the scheme T_p is not reduced, and therefore $\mathcal{T}_0$ is not a smooth scheme; in this case, the Néron–Raynaud model of T can be obtained from $\mathcal{T}_0$ by blowing up two closed points, *loc.cit.* More generally, in a recent paper [13], the Néron–Raynaud model of the norm torus associated to a degree p cyclic extension of $\mathfrak{p}$-adic fields has been explicitly described and its reduction has been calculated.

The following theorem is the main result of this section.

Theorem 3. *Suppose that the torus T splits over a tamely ramified extension $L \mid k$. Then the identity component $\mathcal{T}_0^{(0)}$ of the standard integral model $\mathcal{T}_0$ of T over $\mathfrak{o}$ is isomorphic to the identity component $\mathcal{N}^{(0)}$ of the Néron–Raynaud model $\mathcal{N}$ of T over $\mathfrak{o}$.*

We start with the following three lemmata.

Lemma 2. *The standard model $\mathcal{T}_0$ of the torus T is isomorphic to the scheme-theoretic closure of T in the group $\mathfrak{o}$-scheme $\mathcal{S}_0$.*

Proof. If one can find a splitting field of the torus T having an integral basis over k, the assertion of the lemma follows from Proposition 2. Alternatively, one can argue as follows. Let $\mathcal{A}$ be a commutative $\mathfrak{o}$-algebra. The definition of the restriction of scalars (cf. Remark 1) shows that $\mathcal{S}_0(\mathcal{A}) = \mathcal{A}_d$, in notation of (2.11); therefore the assertion of the lemma follows from Definition 3.

Lemma 3. *Suppose that $\mathcal{T}$, the scheme-theoretic closure of T in $\mathcal{S}$, is a smooth group $\mathfrak{o}$-scheme. Then the identity component $\mathcal{T}_0^{(0)}$ of the standard model $\mathcal{T}_0$ of T is isomorphic to the identity component $\mathcal{N}^{(0)}$ of the Néron–Raynaud model $\mathcal{N}$ of T over $\mathfrak{o}$.*

Proof. It follows that $\mathcal{T} = \mathcal{N}$, for the process of "group smoothening" is not needed in this case since the scheme $\mathcal{T}$ is smooth. Therefore

$$\mathcal{T}^{(0)} = (\mathcal{T} \cap \mathcal{S}^{(0)})^{(0)} = (\mathcal{T} \cap \mathcal{S}_0)^{(0)} = \mathcal{T}_0^{(0)}$$

in view of Lemma 2, as asserted.

Lemma 4. *Let us consider an algebraic torus T defined over a $\mathfrak{p}$-adic field k. Suppose that the torus T splits over a tamely ramified extension $L \mid k$. Then the group $\mathfrak{o}$-scheme $\mathcal{T}$ is smooth.*

Proof. One may repeat the proof of [4, Theorem 4.2], applying the argument used by Edixhoven to the Néron–Raynaud model of a torus, rather than to the Néron model of an Abelian variety studied in [4]. The reader would find a complete proof of the lemma along these lines in Appendix 1.

Corollary 4. *If the torus T splits over a tamely ramified extension $L \mid k$, then the group $\mathfrak{o}$-scheme $\mathcal{T}$ is smooth.*

Proof. The assertion follows from Lemma 4 and Proposition 5 since being smooth is a local property.

Proof of Theorem 3. It is a consequence of Lemma 3 and Corollary 4.

Remark 6. As it has been already noted, the integral model $\mathcal{T}_0$ is minimal in the sense of [23, § 20], cf. Remark 5 at the end of Section 4. The one-dimensional scheme considered in Example 2 is minimal in the sense of [26]. If the torus T splits over a tamely ramified extension of its field of definition, then the integral model $\mathcal{T}_0$ is minimal in the stronger sense of Theorem 3.

6 Concluding remarks

Let T be an algebraic torus defined over a number field, or over a $\mathfrak{p}$-adic field. It seems to be an interesting and difficult problem to obtain a reasonable classification of the integral models of the torus T. The open questions posed in Appendix 2 may be regarded as but a rather modest step in this direction.

Open problem. Describe explicitly (the connected component of) the Néron–Raynaud model of the torus T. Is it true that for any torus T as above, the connected component of the Néron–Raynaud model of T is an affine scheme?

In this paper, we have solved that problem for those tori T which split over a tamely ramified extension of their field of definition. The reader may consult [13] for some calculations, relating to a norm torus with the wildly ramified minimal splitting field.

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Appendix 1

The goal of this appendix is to prove Lemma 4; our argument follows the proof of Theorem 4.2 in [4] rather closely.

One first remarks that the property of a scheme being smooth descends under a quasi-compact and faithfully flat base change. To be more precise, consider a morphism $f\colon X \to Y$ of two Z-schemes X and Y, let $Z' \to Z$ be a quasi-compact and faithfully flat morphism of schemes Z' and Z, and let $f'\colon X' \to Y'$ denote the morphism of the Z'-schemes $X' = X \times_Z Z'$ and $Y' = Y \times_Z Z'$ induced by f. Suppose that f' is a smooth morphism, then the morphism f is also smooth [9, 17.7.3(ii)], [1, p.54]. Our next observation is that for any local ring R, its strict Henselization R^{sh} is a faithfully flat extension of R [1, Cor. 2.4.9], and it is also Noetherian provided R be Noetherian [1, Prop. 2.3.10]; hence the natural morphism Spec $R^{sh} \to$ Spec R is quasi-compact [8, I.6.1]. Therefore one may assume that the ring $\mathfrak{o}$ is strictly Henselian; it follows then that the extension $L \mid k$ may be assumed to be totally (and tamely) ramified, so that p does not divide n, the degree of $L \mid k$, the Galois group $\Gamma = \mathrm{Gal}(L \mid k)$ is isomorphic to μ_n, the group of n-th roots of unity, and $\mu_n \subseteq L^*$.

Since the group $\mathfrak{o}$-scheme $\mathcal{T}$ satisfies the Néron property, the natural right action of the group Γ on $T \times_k L$ extends uniquely to the right action on $\mathcal{T}_{\mathfrak{O}} = \mathcal{T} \times_{\mathfrak{o}} \mathfrak{O}$, and the natural morphism $\mathcal{T}_{\mathfrak{O}} \to$ Spec $\mathfrak{O}$ is equivariant under this action. As in [4, 2.4], this leads to a right action of Γ on the $\mathfrak{o}$-scheme $\mathcal{S}$, the morphism $\mathcal{T} \to \mathcal{S}$ being Γ-equivariant. This action can be described as follows. Let $\mathcal{S}'$ denote the Néron–Raynaud model, defined over $\mathfrak{O}$, of the torus $G^d_{m,L}$; the right action of the group Γ on $\mathfrak{O}$ and on $\mathcal{S}'$ is equivariant with respect to the morphisms Spec $\mathfrak{O} \to$ Spec $\mathfrak{o}$ and $\mathcal{S}' \to$ Spec $\mathfrak{O}$. To define an equivariant right action of Γ on the group $\mathfrak{o}$-scheme $\mathcal{S}$, let us consider an $\mathfrak{o}$-scheme Z and let $P \in \mathcal{S}(Z)$. We have to define P^g for every g in Γ. Let $Z' = Z \times_{\mathfrak{o}} \mathfrak{O}$, let $\rho_{\mathfrak{O}}(g)$ (resp. $\rho_{\mathcal{S}'}(g)$) be the automorphism of Spec $\mathfrak{O}$ (resp. of $\mathcal{S}'$) induced by g, and let $\rho_{Z'}(g) = 1_Z \times_{\mathfrak{o}} \rho_{\mathfrak{O}}(g)$. Since

$$\mathcal{S}(Z) = \mathrm{Hom}_{\mathfrak{o}}(Z, \mathcal{S}) = \mathrm{Hom}_{\mathfrak{O}}(Z', \mathcal{S}'),$$

one may let

$$P^g = \rho_{\mathcal{S}'}(g) \circ P \circ \rho_{Z'}(g)^{-1}.$$

Since the group Γ acts trivially on T, the group scheme $\mathcal{T}$ is a closed subscheme of the scheme of fixed points $\mathcal{S}^\Gamma$. The generic fibre of the scheme $\mathcal{S}^\Gamma$ is isomorphic to T because

$$S(A)^\Gamma = [R_{L|k}(T_L)(A)]^\Gamma = T(A \otimes_k L)^\Gamma = T(A)$$

for any k-algebra A. The order of the group Γ is prime to the residue characteristic, therefore it follows from Proposition 3.4 of [4] that the scheme $\mathcal{S}^\Gamma$ is smooth. Moreover, the scheme $\mathcal{T}$ is a closed subscheme of $\mathcal{S}$, its generic fibre is isomorphic to T by construction, and it is flat (see [1, proof of Prop. 10.1.4 on p.291]). The scheme $\mathcal{S}^\Gamma$ is also a closed and flat subscheme of $\mathcal{S}$ (see [4, Prop. 3.1]). Thus the schemes $\mathcal{T}$ and $\mathcal{S}^\Gamma$ have the same generic fibre and are both flat, therefore it follows from [9, Prop. 2.8.5] that $\mathcal{T} = \mathcal{S}^\Gamma$. Hence $\mathcal{T}$ is smooth, as asserted.

Appendix 2

Let k be a $\mathfrak{p}$-adic field. In this appendix, we briefly discuss three integral models of an algebraic k-torus T. These models have been introduced and studied in [25], [23], [24] in an attempt to define an integral model invariantly, in terms of the maximal compact subgroups U_F of the groups $T(F)$ as F varies over the finite extensions of k. The first model is an affine $\mathfrak{o}$-group scheme $\mathcal{T}_1 = \text{Spec } A_1$, where

$$A_1 = \{f \in k[T] \mid f(U_k) \subseteq \mathfrak{o}\}. \tag{A.1}$$

It turns out that the scheme $\mathcal{T}_1$ is not necessarily of finite type over $\mathfrak{o}$. For let $k = \mathbb{Q}_p$ and $T = G_m$, then (A.1) becomes

$$A_1 = \{f \in \mathbb{Q}_p[t, t^{-1}] \mid f(\mathbb{Z}_p^*) \subseteq \mathbb{Z}_p\};$$

therefore $A_1 \supseteq A_{11}$, where

$$A_{11} := \mathbb{Z}_p[t, t^{-1}, u_1, \ldots, u_n, \ldots],\ u_n := \frac{1}{p^n}(t^{p^{n-1}(p-1)} - 1).$$

One can easily calculate the behaviour of the the scheme $\mathcal{T}_{11} :=$ Spec A_{11} under reduction modulo p. We have $A_{11} \otimes_{\mathbb{Z}_p} \mathbf{F}_p = \mathbf{F}_p[t, t^{-1}, u_1]/(t^{p-1} - 1)$, for $(t^{p-1} - 1) \otimes 1 = p(u_1 \otimes 1) = 0$ and

$$u_n = u_1^{p^{n-1}} p^{p^{n-1}-n} = 0 \text{ for } n \geq 3;\ u_2 = \begin{cases} u_1^2, & \text{if } p = 2, \\ 0, & \text{if } p \geq 3. \end{cases}$$

In particular, the special fibre $\mathcal{T}_1 \times_{\mathbb{Z}_p} \mathbf{F}_p$ of the scheme $\mathcal{T}_1$ is not isomorphic to $G_{m,\mathbf{F}_p}$.

Open question 1. It may be interesting to further investigate the properties of the group scheme $\mathcal{T}_1$. Is it true, for instance, that $A_1 = A_{11}$ when we let $k = \mathbb{Q}_p$ and $T = G_m$?

Let now

$$A_2 = \{f \in k[T] \mid f(U_F) \subseteq \mathfrak{O}_F \text{ for every finite unramified extension } F \mid k\},$$

let

$$A_3 = \{f \in k[T] \mid f(U_F) \subseteq \mathfrak{O}_F \text{ for every finite extension } F \mid k\},$$

and let $\mathcal{T}_i = \operatorname{Spec} A_i$, $i = 2, 3$. Clearly, $A_3 \subseteq A_2 \subseteq A_1$, and $A \subseteq A_2$ by Corollary 3.

Open question 2. Is $\mathcal{T}_i$, $i = 2, 3$, a scheme of finite type over $\mathfrak{o}$? Is it true that $A_2 = A$?

Let now $k = \mathbb{Q}_p$, $L = k(\sqrt{p})$, $T = R^1_{L|k} G_m$ be chosen as in Example 2, and suppose that $p > 3$. We show that $\mathcal{T}_0 \neq \mathcal{T}_3$ in this case. By definition,

$$\mathcal{T}_0(\mathfrak{O}) = \{(x, y) \mid (x, y) \in \mathfrak{O}^2,\ x^2 - py^2 = 1\},$$

or

$$\mathcal{T}_0(\mathfrak{O}) = \{\varepsilon \mid \varepsilon = x + y\pi,\ \varepsilon^{-1} = x - y\pi,\ (x, y) \in \mathfrak{O}^2\},$$

where $\pi := \sqrt{p}$. Therefore

$$\mathcal{T}_0(\mathfrak{O}) \subseteq \{\varepsilon \mid \varepsilon \in \mathfrak{O}^*,\ \varepsilon = \pm 1 \bmod \pi\}.$$

If $p > 3$, it follows that $\mathcal{T}_0(\mathfrak{O})$ is not even a subgroup of finite index in $\mathfrak{O}^*$ and, in particular, $\mathcal{T}_0(\mathfrak{O}) \neq U_L$ since $U_L = \mathfrak{O}^*$. Let $f(x, y) = y$, then $f \in A$ but $f \notin A_3$.

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