A NOTE ON FUNCTIONAL EQUATIONS OF L-ADIC POLYLOGARITHMS

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Contents

Abstract. We are studying l-adic analogues of the classical polylogarithms. Using an explicit arithmetic formula we show that they satisfy the same functional equations as classical polylogarithms. We also identify l-adic polylogarithms evaluated at roots of unity with Soulé classes.

1. Introduction

In [5] we defined l-adic polylogarithms and we showed that they satisfy functional equations analogous to functional equations of classical polylogarithms. In [1], motivated by an old result of O.Gabber, we have given in a commun work with H.Nakamura an explicit formula for l-adic polylogarithms. In this note we shall use this explicit formula to get functional equations of l-adic polylogarithms. We use them in [5] to study the action of Galois groups on $\pi_1(P^1_{\bar K}\setminus\{0,\mu_n,\infty\},\vec{01})$.

Let K be a number field. We choose in $\bar K$ a compatible family $(\xi_{l^n})_{n\in N}$ of l^n - roots of 1 such that $(\xi_{l^{n+1}})^l=\xi_{l^n}$.

For any $z\in K\setminus\{0\}$ we choose in $\bar K$ a compatible family $(z^{\frac{1}{l^n}})_{n\in N}$ of l^n- roots of z such that $(z^{\frac{1}{l^{n+1}}})^l=z^{\frac{1}{l^n}}$.

For any $z\in K\setminus\{0\}$ we define a Kummer character

$$\kappa_z:G_K\to Z_l$$

setting $\sigma(z^{\frac{1}{l^n}})=(\xi_{l^n})^{\kappa_z(\sigma)}\cdot z^{\frac{1}{l^n}}$ for all n.

Let us set $G_K:=Gal(\bar K/K)$. For any natural numbers n and m and any i such that $0\le i<l^n$ we define functions

$$\kappa_i^{n,m}(z):G_K\to Z/l^mZ$$

by the following equality

$$\frac{\sigma((1-\xi_{l^n}^{\chi(\sigma^{-1})(i-\kappa_z(\sigma))}\cdot z^{\frac{1}{l^n}})^{\frac{1}{l^m}})}{(1-\xi_{l^n}^{i}\cdot z^{\frac{1}{l^n}})^{\frac{1}{l^m}}}=\xi_{l^m}^{\kappa_i^{n,m}(z)(\sigma)}.$$

The family of functions $(\kappa_i^{n,m}(z)(\sigma))_{m\in N}$ is compatible, hence we get a function

$$\kappa_i^n(z):G_K\to Z_l.$$

Let us fix $\sigma\in G_K$. Then for each n we have a function

$$Z/l^n\to Z_l,\quad i\to\kappa_i^n(z)(\sigma).$$

Let $n > m$. The identities

$$\prod_{i\equiv i_0(l^m)} (1-\xi_{l^n}^{\chi(\sigma^{-1})(i-\kappa_z(\sigma))}\cdot z^{\frac{1}{l^n}}) = (1-\xi_{l^m}^{\chi(\sigma^{-1})(i_0-\kappa_z(\sigma))}\cdot z^{\frac{1}{l^m}})$$

and

$$\prod_{i\equiv i_0(l^m)} (1-\xi_{l^n}^{i}\cdot z^{\frac{1}{l^n}}) = (1-\xi_{l^m}^{i_0}\cdot z^{\frac{1}{l^m}})$$

imply that

$$\sum_{i\equiv i_0(l^m)} \kappa_i^n(z)(\sigma) = \kappa_{i_0}^m(z)(\sigma). \tag{1.1}$$

Hence we get a measure

$$\kappa(z)(\sigma) := (\kappa_i^n(z)(\sigma))_{n\in N, 0\le i<l^n}$$

on $Z_l = \varprojlim Z/l^n$. Let $n > m$ and let p be a natural number. The equality (1.1) implies that

$$\sum_{i=0}^{l^n-1} i^p\kappa_i^n(z)(\sigma) \equiv \sum_{i=0}^{l^m-1} i^p\kappa_i^m(z)(\sigma) \mod l^m. \tag{1.2}$$

Hence by the definition of the integration on Z_l we get

$$\int_{Z_l} x^p d\kappa(z)(\sigma) \equiv \sum_{i=0}^{l^n-1} i^p\kappa_i^n(z)(\sigma) \mod l^n. \tag{1.3}$$

It follows from (1.3) and the fact that $\kappa_i^n(z)(\sigma) \equiv \kappa_i^{n,n}(z)(\sigma) \mod l^n$ that

$$\int_{Z_l} x^p d\kappa(z)(\sigma) \equiv \sum_{i=0}^{l^n-1} i^p\kappa_i^{n,n}(z)(\sigma) \mod l^n. \tag{1.4}$$

Let us set

$$\ell_{p+1}(z)(\sigma) := \int_{Z_l} x^p d\kappa(z)(\sigma).$$

The function $\ell_n(z)$ appears naturally when studying the action of G_K on the $\pi_1(P^1_{\bar K}\setminus\{0,1,\infty\},\vec{01})$ - torsor of l-adic paths from $\vec{01}$ to z.

We embed the group $\pi_1(P^1_{\bar K}\setminus\{0,1,\infty\},\vec{01})$ into the Q_l -algebra $Q_l\{\{X,Y\}\}$ of formal power series in two non commuting variables X and Y. The image of the loop around 0 (resp. around 1) is the power series e^X (resp. e^Y). The embedding is continous and multiplicative and we denote it by k. Let γ be a path from $\vec{01}$ to z. Then $\gamma^{-1}\cdot\sigma(\gamma)$ is a loop in $\pi_1(P^1_{\bar K}\setminus\{0,1,\infty\},\vec{01})$. Let x be a loop around 0 - one of the standard generators of $\pi_1(P^1_{\bar K}\setminus\{0,1,\infty\},\vec{01})$. We can write

$$logk(\gamma^{-1}\cdot\sigma(\gamma)\cdot x^{-\kappa_z(\sigma)}) = l_1(z)_\gamma(\sigma)Y + \sum_{n=2}^{\infty} l_n(z)_\gamma(\sigma)[[X,Y]X^{n-2}] + ...$$

The main result in [1] is that the function $l_n(z)_\gamma$ coinsides with $\ell_n(z)$ for some choice of a path γ from $\vec{01}$ to z.

2. Functional equations

In this section we show that the function $\ell_n(z)$ satisfy certain functional equations known as distribution relations. These equations were already proved in [5] using geometric methods. They were used in [5] in the study of actions of Galois groups on $\pi_1(P^1_{\bar K}\setminus\{0,\mu_n,\infty\},\vec{01})$.

Theorem 2.1. *Let r be a positive integer prime to l and let ξ_r be a primitive r - root of 1. We assume that ξ_r and z are in K. For each $0 \leq a < r$ we choose a compatible family $(\xi_r^{\frac{a}{l^n}})_{n\in N}$ of l^n - roots of ξ_r^a such that each $\xi_r^{\frac{a}{l^n}} \in \mu_r$. Then we have*

$$\ell_m(z^r) = r^{m-1}\Big(\sum_{\xi^r=1} \ell_m(\xi z)\Big).$$

Proof. Observe that we have the identities

$$\prod_{a=0}^{r-1}\left(\prod_{i=0}^{l^n-1}\left(\frac{\sigma(1-\xi_{l^n}^{\chi(\sigma^{-1})(i-\kappa_z(\sigma))}\cdot\xi_r^{\frac{a}{l^n}}\cdot z^{\frac{1}{l^n}})}{1-\xi_{l^n}^{i}\cdot\xi_r^{\frac{a}{l^n}}\cdot z^{\frac{1}{l^n}}}\right)^{i^{m-1}}\right)^{r^{m-1}} =$$

$$\prod_{i=0}^{l^n-1}\left(\prod_{a=0}^{r-1}\left(\frac{\sigma(1-\xi_r^{\frac{a}{l^n}}\cdot\xi_{l^n}^{\chi(\sigma^{-1})(i-\kappa_z(\sigma))}\cdot z^{\frac{1}{l^n}})}{1-\xi_r^{\frac{a}{l^n}}\cdot\xi_{l^n}^{i}\cdot z^{\frac{1}{l^n}}}\right)\right)^{(ir)^{m-1}} =$$

$$\prod_{i=0}^{l^n-1}\left(\frac{\sigma(1-\xi_{l^n}^{\chi(\sigma^{-1})(ir-r\kappa_z(\sigma))}\cdot(z^r)^{\frac{1}{l^n}})}{1-\xi_{l^n}^{ir}\cdot(z^r)^{\frac{1}{l^n}}}\right)^{(ir)^{m-1}} =$$

$$\prod_{i=0}^{l^n-1}\left(\frac{\sigma(1-\xi_{l^n}^{\chi(\sigma^{-1})(i-\kappa_{z^r}(\sigma))}\cdot(z^r)^{\frac{1}{l^n}})}{1-\xi_{l^n}^{i}\cdot z^{r\frac{1}{l^n}}}\right)^{i^{m-1}} \cdot l^n\text{–th power.}$$

This implies

$$r^{m-1}\sum_{a=0}^{r-1}\Big(\sum_{i=0}^{l^n-1} i^{m-1}\kappa_i^{n,n}(\xi_r^a z)\Big) = \sum_{i=0}^{l^n-1} i^{m-1}\kappa_i^{n,n}(z^r).$$

The theorem follows from the congruences (1.2) and (1.4). □

To consider the case when l divides r we need the following lemma.

Lemma 2.2. *Let $\xi_{l^\nu} \in K$ and let $z \in K$. We have*

$$\kappa_{\xi_{l^\nu}^j z}(\sigma) = \frac{(\chi(\sigma)-1)j}{l^\nu} + \kappa_z(\sigma).$$

Proof. We have $\frac{\sigma(\xi_{l^{\nu+n}}^j\cdot z^{\frac{1}{l^n}})}{\xi_{l^{\nu+n}}^j\cdot z^{\frac{1}{l^n}}} = \frac{\sigma(\xi_{l^{\nu+n}}^j)}{\xi_{l^{\nu+n}}}\cdot\xi_{l^n}^{\kappa_z(\sigma)} = \xi_{l^{\nu+n}}^{(\chi(\sigma)-1)j}\cdot\xi_{l^n}^{\kappa_z(\sigma)} = \xi_{l^n}^{\frac{(\chi(\sigma)-1)j}{l^\nu}+\kappa_z(\sigma)}$.

□

Theorem 2.3. *Let $\xi_{l^\nu} \in K$ and let $z \in K$. We have*

$$\ell_m(z^{l^\nu}) = l^{\nu(m-1)}\sum_{j=0}^{l^\nu-1}\ell_m(\xi_{l^\nu}^j\cdot z) + \sum_{a=1}^{m-1}\binom{m-1}{a} l^{\nu(m-1-a)}\sum_{j=0}^{l^\nu-1} j^a\ell_{m-a}(\xi_{l^\nu}^j\cdot z).$$

Proof. We have

$$\prod_{i=0}^{l^n-1}\left(\frac{\sigma(1-\xi_{l^n}^{\chi(\sigma^{-1})(i-\kappa_{z^{l^\nu}}(\sigma))}\cdot(z^{l^\nu})^{\frac{1}{l^n}})}{(1-\xi_{l^n}^{i}\cdot(z^{l^\nu})^{\frac{1}{l^n}})}\right)^{i^{m-1}} =$$

$$\prod_{0\leq i<l^n,\, i=j+kl^\nu}\left(\frac{\sigma(1-\xi_{l^n}^{\chi(\sigma^{-1})j}\cdot\xi_{l^n}^{\chi(\sigma^{-1})(kl^\nu-l^\nu\kappa_z(\sigma))}\cdot z^{\frac{1}{l^{n-\nu}}})}{(1-\xi_{l^n}^{j}\cdot\xi_{l^n}^{kl^\nu}\cdot z^{\frac{1}{l^{n-\nu}}})}\right)^{(j+kl^\nu)^{m-1}} =$$

$$\prod_{j=0}^{l^\nu-1}\prod_{k=0}^{l^{n-\nu}-1}\left(\frac{\sigma(1-\xi_{l^{n-\nu}}^{\chi(\sigma^{-1})(k-\kappa_{\xi_{l^\nu}^j\cdot z}(\sigma))}\cdot(\xi_{l^\nu}^j\cdot z)^{\frac{1}{l^{n-\nu}}})}{(1-\xi_{l^{n-\nu}}^{k}\cdot(\xi_{l^\nu}^j\cdot z)^{\frac{1}{l^{n-\nu}}})}\right)^{(j+kl^\nu)^{m-1}} =$$

$$\prod_{a=0}^{m-1}\prod_{j=0}^{l^\nu-1}\prod_{k=0}^{l^{n-\nu}-1}\left(\left(\frac{\sigma(1-\xi_{l^{n-\nu}}^{\chi(\sigma^{-1})(k-\kappa_{\xi^j_{l^\nu}\cdot z}(\sigma))}\cdot(\xi^j_{l^\nu}\cdot z)^{\frac{1}{l^{n-\nu}}})}{(1-\xi^k_{l^{n-\nu}}\cdot(\xi^j_{l^\nu}\cdot z)^{\frac{1}{l^{n-\nu}}})}\right)^{k^{m-1-a}}\right)^{j^a\binom{m-1}{a}l^{\nu(m-1-a)}}.$$

Hence for any N we have

$$\sum_{i=0}^{l^n-1} i^{m-1}\kappa_i^{n,N}(z^{l^\nu})(\sigma)=l^{\nu(m-1)}\sum_{0\le j<l^\nu}(\sum_{k=0}^{l^{n-\nu}-1}k^{m-1}\kappa_k^{n-\nu,N}(\xi^j_{l^\nu}z)(\sigma))+$$

$$\sum_{a=1}^{m-1}\binom{m-1}{a}l^{\nu(m-1-a)}(\sum_{j=0}^{l^\nu-1}j^a(\sum_{k=0}^{l^{n-\nu}-1}k^{m-1-a}\kappa_k^{n-\nu,N}(\xi^j_{l^\nu}\cdot z)(\sigma))).$$

This implies that

$$\ell_m(z^{l^\nu})=l^{\nu(m-1)}\sum_{j=0}^{l^\nu-1}\ell_m(\xi^j_\nu z)+\sum_{a=1}^{m-1}\binom{m-1}{a}l^{\nu(m-1-a)}\sum_{j=0}^{l^\nu-1}j^a\ell_{m-a}(\xi^j_\nu z).$$

$\square$

Now we consider a general case.

Theorem 2.4. *Let p be a positive integer. Assume that $p=r\cdot l^\nu$ where r is a prime to l. Assume that ξ_r and ξ_{l^ν} are in K. Then we have*

$$\ell_m(z^p)=p^{m-1}\sum_{\xi^p=1}\ell_m(\xi z)+r^{m-1}\sum_{a=1}^{m-1}\binom{m-1}{a}l^{\nu(m-1-a)}\sum_{k=0}^{r-1}\sum_{0\le j<l^\nu}j^a\ell_{m-a}(\xi^j_{l^\nu}\xi^k_r z).$$

Proof. The theorem follows from Theorems 2.1 and 2.3. $\square$

Corollary 2.5. *Let $G_n\subset G_K$ be a subgroup on which functions $\ell_k(z)$ vanish for all $z\in K$ and all $k<n$. Then*

$$\ell_m(z^p)=p^{m-1}\sum_{\xi^p=1}\ell_m(\xi z)$$

on G_n.

3. l-adic polylogarithms and Soulé classes

3.1. We recall briefly the construction of the Soulé classes. Let K be a number field. Let us set $K_n:=K(\mu_{l^n})$ and $K_\infty:=K(\mu_{l^\infty})$. The extension $K\subset K_n$ is unramified outside l. Let M be a maximal Galois extension of K unramified outside l.

Let us set $G:=Gal(M/K)$, $G_n:=Gal(M/K_n)$, $G_\infty:=Gal(M/K_\infty)$, $\Gamma_n:=Gal(K_n/K)$ and $\Gamma_\infty:=Gal(K_\infty/K)$. The groups G_n and G_∞ are normal subgroups of G and we have $G/G_n\simeq\Gamma_n$ and $G/G_\infty\simeq\Gamma_\infty$. Let us set $\mathcal{R}_n:=\mathcal{O}_{K_n}[\frac{1}{l}]$ and $\mathcal{R}:=\mathcal{O}_K[\frac{1}{l}]$. Let us set $X_n:=\mathrm{Spec}\mathcal{R}_n$, $X_\infty:=\varprojlim_n X_n$ and $X:=\mathrm{Spec}\mathcal{R}$.

The exact sequence of sheaves on X_n

$$1\to\mu_{l^n}\to\mathbf{G_m}\stackrel{l^n}{\to}\mathbf{G_m}\to 1$$

induces a long exact sequence of cohomology

$$0\to H^0(X_n,\mu_{l^n})\to H^0(X_n,\mathbf{G_m})\to H^0(X_n,\mathbf{G_m})\stackrel{\delta}{\to}H^1(X_n,\mu_{l^n})\dots.$$

Let $(\xi_{l^n})_{n\in N}$ be a compatible family of l^n - roots of 1 i.e. $(\xi_{l^{n+1}})^l=\xi_{l^n}$ for all n. The element ξ_{l^n} defines an element in $H^0(X_n,\mu_{l^n})$, which we denote by ξ_{l^n}. Let

$\alpha_n := \xi_{l^n} \cup \ldots \cup \xi_{l^n}$ ($i-1$ - times) and let $u_n \in \mathcal{R}_n^* = H^0(X_n; \mathbf{G_m})$. Observe that $\alpha_n \in H^0(X_n, \mu_{l^n}^{\otimes(i-1)})$. Let

$$N_n : H^1(X_n, \mu_{l^n}^{\otimes i}) \to H^1(X, \mu_{l^n}^{\otimes i})$$

be a transfer map associated to an etale covering $p_n : X_n \to X$. Let us set

$$x_n^i := N_n(\alpha_n \cup \delta(u_n)).$$

The next lemma one finds in [3].

Lemma 3.1. *Let $(u_n)_{n\in N}$, $u_n \in \mathcal{R}_n^*$ be a family of l - units such that $N_{n+1,n}(u_{n+1}) = u_n$, where $N_{n+1,n} : \mathcal{R}_{n+1}^* \to \mathcal{R}_n^*$ is a norm. Then $r_{n+1,n}(x_{n+1}^i) = x_n^i$, where $r_{n+1,n} : H^1(X, \mu_{l^{n+1}}^{\otimes i}) \to H^1(X, \mu_{l^n}^{\otimes i})$ is induced by the projection $\mu_{l^{n+1}} \to \mu_{l^n}$. Hence the family $(x_n^i)_{n\in N}$ defines an element $x^i = x^i((u_n)_{n\in N}) = (x_n^i)_{n\in N} \in H^1(X, Z_l(i))$.*

The covering $p_n : X_n \to X$ is Galois. It follows from [2] Lemma 6 that we have a commutative diagram

$$\begin{array}{ccc}
(\mathcal{R}_n^*/(\mathcal{R}_n^*)^{l^n})\otimes\mu_{l^n}^{\otimes(i-1)} & \xrightarrow[\delta\otimes id_{\mu_{l^n}^{\otimes(i-1)}}]{} & H^1(X_n; \mu_{l^n}^{\otimes i}) \\
{\scriptstyle N_{\Gamma_n}}\downarrow & & \downarrow{\scriptstyle p_n^*\circ N_n} \\
(\mathcal{R}_n^*/(\mathcal{R}_n^*)^{l^n})\otimes\mu_{l^n}^{\otimes(i-1)} & \xrightarrow[\delta\otimes id_{\mu_{l^n}^{\otimes(i-1)}}]{} & H^1(X_n; \mu_{l^n}^{\otimes i})
\end{array}$$

where $N_{\Gamma_n}(c) = \prod_{\sigma\in\Gamma_n} \sigma(c)$.

Observe that

$$N_{\Gamma_n}\big(u_n \otimes \xi_{l^n}^{\otimes(i-1)}\big) = \prod_{\sigma\in\Gamma_n} (\sigma(u_n))\otimes(\xi_{l^n}^{\chi(\sigma)})^{\otimes(i-1)} =$$

$$\prod_{\sigma\in\Gamma_n} (\sigma(u_n))^{\bar{\chi(\sigma)}^{i-1}} \otimes \xi_{l^n}^{\otimes(i-1)},$$

where $\bar{\chi(\sigma)}$ is a positive integer such that $0 \leq \bar{\chi(\sigma)} < |\Gamma_n|$ and $\bar{\chi(\sigma)} \equiv \chi(\sigma)$ mod $|\Gamma_n|$. Hence we have proved the following result.

Lemma 3.2. *The element*

$$p_n^*(x_n^i) \in H^1(X_n; \mu_{l^n}^{\otimes i}) \approx Hom(G_n; \mu_{l^n})\otimes\mu_{l^n}^{\otimes(i-1)}$$

is given by the Kummer character associated to $\prod_{\sigma\in\Gamma_n}(\sigma(u_n))^{\bar{\chi(\sigma)}^{i-1}}$.

The element $x^i = x^i((u_n)_{n\in N}) \in H^1(X, Z_l(i))$ is called a Soulé class. Further we shall identify Soulé classes with ℓ-adic polylogarithms.

3.2. Let r be a positive integer prime to l. Let $K = Q(\mu_r)$. Let $0 \leq q < r$. Observe that

$$\sum_{i=0}^{l^n-1} i^{m-1}\kappa_i^{n,M}(\xi_r^q) = \sum_{i=0,(i,l)=1}^{l^n-1} i^{m-1}\kappa_i^{n,M}(\xi_r^q) + \sum_{i=0,(i,l)\geq l}^{l^n-1} i^{m-1}\kappa_i^{n,M}(\xi_r^q) \tag{3.1}$$

and

$$\kappa_{li}^{n,M}(\xi_r^q) = \kappa_j^{n-1,M}(\xi_r^{ql^{-1}}). \tag{3.2}$$

It follows from 3.1. and 3.2 and from section 1 that

$$\int_{Z_l} x^{m-1} d\kappa(\xi_r^q)(\sigma) = \int_{Z_l\setminus lZ_l} x^{m-1} d\kappa(\xi_r^q)(\sigma) + l^{m-1}\int_{Z_l} x^{m-1} d\kappa(\xi_r^{ql^{-1}})(\sigma).$$

Hence by the definition of l-adic polylogarithms we have

$$\ell_m(\xi_r^q)(\sigma) = \int_{Z_l \setminus lZ_l} x^{m-1} d\kappa(\xi_r^q)(\sigma) + l^{m-1}\ell_m(\xi_r^{ql^{-1}})(\sigma). \tag{3.3}$$

The function

$$G_K \ni \sigma \to \int_{Z_l \setminus lZ_l} x^{m-1} d\kappa(\xi_r^q)(\sigma) \in Z_l$$

we shall identify with a Soulé class.

The family of l-units $u_n^{(q)} := (1-\xi_r^{\frac{q}{l^n}} \cdot \xi_{l^n})$ $(n \in N)$ is compatible i.e. $N_{n+1,n}(1-\xi_r^{\frac{q}{l^{n+1}}} \cdot \xi_{l^{n+1}}) = (1-\xi_r^{\frac{q}{l^n}} \cdot \xi_{l^n})$ for all $n \in N$. It follows from Lemma 3.2 that the corresponding Soulé classe $x^m((1-\xi_r^{\frac{q}{l^n}} \cdot \xi_{l^n})_{n\in N})$ restricted to $G_{Q(\mu_{r\cdot l^\infty})} := Gal(\bar{Q}/Q(\mu_{r\cdot l^\infty}))$ is given by the Kummer character associated to $\prod_{i=0,\,(i,l)=1}(1-\xi_r^{\frac{q}{l^n}} \cdot \xi_{l^n})^{i^{m-1}}$. This implies that

$$x^m((1-\xi_r^{\frac{q}{l^n}} \cdot \xi_{l^n})_{n\in N})(\sigma) = \int_{Z_l \setminus lZ_l} x^{m-1} d\kappa(\xi_r^q)(\sigma) \tag{3.4}$$

for any $\sigma \in G_{Q(\mu_{r\cdot l^\infty})}$.

Proposition 3.3. (see also [5]) *We have*

$$x^m((1-\xi_r^{\frac{q}{l^n}} \cdot \xi_{l^n})_{n\in N}) = \ell_m(\xi_r^q) - l^{m-1}\ell_m(\xi_r^{ql^{-1}})$$

on $G_{Q(\mu_{r\cdot l^\infty})}$.

Proof. The proposition follows from (3.3) and (3.4). □

Corollary 3.4. *Let p be the order of l in $(Z/r)^*$. Then*

$$(1-l^{p(m-1)})\ell_m(\xi_r^q) = \sum_{i=0}^{p-1} l^{i(m-1)} x^m((1-\xi_r^{\frac{q\cdot l^{-i}}{l^n}} \cdot \xi_{l^n})_{n\in N})$$

on $G_{Q(\mu_{r\cdot l^\infty})}$.

3.3. Let r be an integer prime to l and let ν be a positive integer. Let $K = Q(\mu_{l^\nu \cdot r})$. Then $K_{n+\nu} = Q(\mu_{l^{n+\nu}\cdot r})$ and $\Gamma_{n+\nu} = Z/l^n$. Let $\alpha \in \Gamma_{n+\nu} = Z/l^n$. Then $\alpha(\xi_{l^{n+\nu}}) = \xi_{l^{n+\nu}}^{1+l^\nu\alpha}$. Let q be an integer prime to l and let $0 \le a < r$. The family of l-units $(1-\xi_r^{\frac{a}{l^{n-\nu}}} \cdot \xi_{l^n}^q)_{n\ge\nu}$ is compatible i.e. $N_{n+1,n}(1-\xi_r^{\frac{a}{l^{n-\nu+1}}} \cdot \xi_{l^{n+1}}^q) = (1-\xi_r^{\frac{a}{l^{n-\nu}}} \cdot \xi_{l^n}^q)$ for all $n \ge \nu$. We have

$$N_{\Gamma_{n+\nu}}((1-\xi_r^{\frac{a}{l^n}} \cdot \xi_{l^{n+\nu}}^q) \otimes \xi_{l^{n+\nu}}^{\otimes(m-1)}) = \prod_{\alpha\in Z/l^n} (1-\xi_r^{\frac{a}{l^n}} \cdot \xi_{l^{n+\nu}}^{(1+l^\nu\alpha)q}) \otimes (\xi_{l^{n+\nu}}^{1+l^\nu\alpha})^{\otimes(m-1)} =$$

$$\prod_{\alpha=0}^{l^n-1} \left(1-\xi_r^{\frac{a}{l^n}} \cdot \xi_{l^{n+\nu}}^{(1+l^\nu\alpha)q}\right)^{(1+l^\nu\alpha)^{(m-1)}} \otimes \xi_{l^{n+\nu}}^{\otimes(m-1)} = \prod_{k=0}^{m-1}\prod_{\alpha=0}^{l^n-1} \left(1-\xi_r^{\frac{a}{l^n}} \cdot \xi_{l^n}^{\alpha q} \cdot \xi_{l^{n+\nu}}^q\right)^{\binom{m-1}{k} l^{\nu k}\alpha^k} \otimes \xi_{l^{n+\nu}}^{\otimes(m-1)}$$

$$= \prod_{k=0}^{m-1} \left(\prod_{\alpha=0}^{l^n-1} (1-\xi_{l^n}^{\alpha q} \cdot (\xi_r^a \cdot \xi_{l^\nu}^q)^{\frac{1}{l^n}})^{(\alpha q)^k}\right)^{\binom{m-1}{k} l^{\nu k} q^{-k}} \otimes \xi_{l^{n+\nu}}^{\otimes(m-1)}.$$

The Kummer character associated to the last term is

$$\sum_{k=0}^{m-1} \binom{m-1}{k} l^{\nu k} \cdot q^{-k} (\sum_{j=0}^{l^n-1} j^k \kappa_j^{n,n}(\xi_{l^\nu}^q \cdot \xi_r^a)).$$

Hence we get the following result.

Proposition 3.5. *We have*

$$x^m((1-\xi_r^{\overline{\frac{a}{l^{n-\nu}}}}\cdot\xi_{l^n}^q)_{n\geq\nu}) = l^{\nu(m-1)}q^{-(m-1)}\ell_m(\xi_r^a\cdot\xi_{l^\nu}^q)+\sum_{k=0}^{m-2}\binom{m-1}{k}\cdot l^{\nu k}\cdot q^{-k}\cdot\ell_{k+1}(\xi_r^a\cdot\xi_{l^\nu}^q)$$

on $Gal(\bar{Q}/Q(\mu_{l^\infty})$.

In the next proposition we generalize Corollary 8.4.2 from [5].

Theorem 3.6. *The functions* $\ell_m(\xi_r^q\cdot\xi_{l^\nu}^q)$ *for* $0 < a \leq \frac{r-1}{2}$, $(a,r)=1$ *and* $0 < q \leq \frac{l^\nu}{2}$, $(q,l)=1$ *are linearly independent over* Z_l.

Proof. It is sufficient to show that the Soulé classes $x^m((1-\xi_r^{\overline{\frac{a}{l^{n-\nu}}}}\cdot\xi_{l^n}^q)_{n\geq\nu})$ for $0 < a \leq \frac{r-1}{2}$, $(a,r)=1$ and $0 < q \leq \frac{l^\nu}{2}$, $(q,l)=1$ are linearly independent over Z_l. This follows from the fact that l-units $(1-\xi_r^a\cdot\xi_{l^\nu}^q)$ for $0 < a \leq \frac{r-1}{2}$, $(a,r)=1$ and $0 < q \leq \frac{l^\nu}{2}$, $(q,l)=1$ are linearly independent in the group of cyclotomic l-units of $Q(\mu_{rl^\nu})$ tensored by $\mu_{l^n}^{\otimes(m-1)}$. □

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