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LOWER ESTIMATES FOR THE SUPREMUM OF SOME RANDOM PROCESSES, II

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In this paper we give some lower estimates for the supremum of random processes of the type

$$(1) \qquad \sum_{i=1}^{n} a_i \xi_i(t) \varphi_i(x),$$

where $\{a_i\}_{i=1}^n$ are real coefficients, $\{\xi_i\}_{i=1}^n$ is a system of independent random variables on a probability space $(T, \mathcal{T}, \tau)$ normalized in $L_2(T, \mathcal{T}, \tau)$, and $\{\varphi_i\}_{i=1}^n$ is a system of norm one functions in an $L_2(X, \Sigma, \mu)$ space with (X, Σ, μ) being another probability space.

This work continues the investigation done in [2], where the simpler case $a_i = 1,\ 1 \leq i \leq n$ was considered. The method of the proof of the theorem below is the same as in [2], and is based on the application of a sharper version of the central limit theorem for sequences of independent vectors in $\mathbb{R}^2$. Our theorem generalizes results from Salem and Zygmund [5] (see Theorems 4.5.1 and 5.4.1 there), where only the case $\varphi_i = cos(ix)$ was considered. Let us remark that the method used in the paper [5] can be, after corresponding modifications, also applied for studying the process (1), for more general orthogonal systems.

As an application of the main estimate, we study random d-dimensional trigonometric and more general polynomials, and give some lower L_∞-estimates that could be useful in harmonic analysis and approximation theory.

Theorem. *For every $M < \infty$ there exist constants $C_j = C_j(M) > 0,\ j = 1, 2, 3$ and $q = q(M) > 0$ such that, whenever $\{\varphi_i\}_{i=1}^n$ is a system of functions in an $L_2(\mu)$ -space satisfying*

$$(1^\circ) \qquad \|\varphi_i\|_{L_2(\mu)} = 1 \quad \textit{and} \quad \|\varphi_i\|_{L_3(\mu)} \leq M, \quad \textit{for all} \quad 1 \leq i \leq n,$$

$$(2^\circ) \qquad \|\sum_{i=1}^{n} a_i \varphi_i\|_{L_2(\mu)} \leq M (\sum_{i=1}^{n} |a_i|^2)^{1/2}, \quad \textit{for all} \quad \{a_i\}_{i=1}^n,$$

and $\{\xi_i\}_{i=1}^n$ are independent random variables over a probability space $(T, \mathcal{T}, \tau)$ with

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(3°) $\mathbb{E}(\xi_i)=0$, $\mathbb{E}|\xi_i|^2=1$ *and* $(\mathbb{E}|\xi_i|^3)^{1/3}\le M$, *for all* $1\le i\le n$,

then, for any choice of coefficients $\{a_i\}_{i=1}^n$,

(i) $\tau\{t\in T, \|\sum_{i=1}^n a_i\,\xi_i(t)\varphi_i(x)\|_\infty \le C_1(\sum_{i=1}^n |a_i|^2)^{1/2}(1+\log R)^{1/2}\}\le C_2R^{-q}$,

where $R=\frac{(\sum_{i=1}^n |a_i|^2)^2}{\sum_{i=1}^n |a_i|^4}$, *and hence*

(ii) $$\mathbb{E}\|\sum_{i=1}^n a_i\,\xi_i\,\varphi_i\|_{L_\infty(\mu)}\ge C_3(\sum_{i=1}^n |a_i|^2)^{1/2}(1+\log\frac{(\sum_{i=1}^n |a_i|^2)^2}{\sum_{i=1}^n |a_i|^4})^{1/2}$$

Remark. By using the so-called contraction principle (see Theorem 4.9 in [3]), one can prove the inequality (ii) under the weaker assumption

(3′) $\mathbf{E}(\xi_i)=0$, $\mathbb{E}|\xi_i|^2=1$ and $\mathbb{E}|\xi_i|\ge 1/M$, for $1\le i\le n$,

instead of (3) above. However, we are not aware of other reduction methods that would allow to prove (i) under similar weaker assumptions.

The proof consists of several steps.

Step I. Fix coefficients $\{a_i\}_{i=1}^n$ so that $\sum_{i=1}^n |a_i|^2=1$, take $\varepsilon=\varepsilon(M)=\frac{1}{4}(\frac{3}{4M^2})^3$ and consider the set

$$E_1=\{x;\sum_{i=1}^n |a_i|^3|\varphi_i(x)|^3<\frac{M^3}{\varepsilon}\sum_{i=1}^n |a_i|^3\}$$

Then, by our assumptions,

$$\mu(E_1^c)\frac{M^3}{\varepsilon}\sum_{i=1}^n |a_i|^3\le\int\sum_{i=1}^n |a_i|^3|\varphi_i(x)|^3\,d\mu\le M^3\sum_{i=1}^n |a_i|^3$$

from which it follows that

$$\mu(E_1^c)\le\varepsilon$$

i.e. that $\mu(E_1)\ge 1-\varepsilon$.

Next notice that the function

$$\varphi(x)=\sum_{i=1}^n |a_i|^2|\varphi_i(x)|^2$$

satisfies $\|\varphi\|_{L_1(\mu)} = 1$ and, since $L_3(\mu)$ in 2-convex,

$$\|\varphi\|_{L_{3/2}(\mu)} = \left(\int (\sum_{i=1}^{n} |a_i|^2 |\varphi_i(x)|^2)^{3/2} d\mu\right)^{2/3} = \|(\sum_{i=1}^{n} |a_i|^2 |\varphi_i|^2)^{1/2}\|_{L_3(\mu)}^2$$
$$\leq \sum_{i=1}^{n} |a_i|^2 \|\varphi_i\|_{L_3(\mu)}^2 \leq M^2.$$

Consider now the set

$$E_2 = \{x; \varphi(x) > \frac{1}{4}\}$$

and observe that

$$\int_{E_2^c} \varphi(x) d\mu \leq \frac{1}{4}$$

so that

$$\frac{3}{4} \leq \int_{E_2} \varphi(x) d\mu \leq \|\varphi\|_{L_{3/2}} \mu(E_2)^{1/3} \leq M^2 \mu(E_2)^{1/3}.$$

This yields that $\mu(E_2) \geq (\frac{3}{4M^2})^3$ so that, if we set

$$E_3 = \{x \in E_2; \varphi(x) \leq 2(\frac{4M^2}{3})^3\}$$

then

$$2(\frac{4M^2}{3})^3(\mu(E_2) - \mu(E_3)) < \int_{E_2 \sim E_3} \varphi(x) dx \leq \|\varphi\|_{L_1(\mu)} = 1$$

from which it follows that

$$\mu(E_3) \geq \frac{1}{2}(\frac{3}{4M^2})^3 = 2\varepsilon.$$

In order to get the final conclusion, put $E = E_1 \cap E_3$ and observe that:

(i) $\mu(E) \geq \varepsilon(M) > 0$

(ii) For $x \in E$, $\sum_{i=1}^{n} |a_i|^3 |\varphi_i(x)|^3 < \frac{256}{27} M^9 \sum_{i=1}^{n} |a_i|^3$,

(iii) For $x \in E$, the function $\varphi(x) = \sum_{i=1}^{n} |a_i|^2 |\varphi_i(x)|^2$ satisfies the inequalities

$$\frac{1}{4} < \varphi(x) < 2(\frac{4M^2}{3})^3 = \gamma(M).$$

Step II. The change of density. Define a new measure ν by

$$d\nu = \begin{cases} \chi_{E^c}(x)d\mu, & x \in E^c \\ \chi_E(x)(\frac{\varphi(x)\mu(E)}{\int_E \varphi(u)d\mu})d\mu, & x \in E \end{cases}$$

and notice that ν is a probability measure on the same measure space as μ. Moreover, if we put

$$\psi_i(x) = \begin{cases} \varphi_i(x); & x \in E^c \\ \varphi_i(x)(\frac{\int_E \varphi(u)d\mu}{\varphi(x)\mu(E)})^{1/2}; & x \in E \end{cases} \quad ; 1 \leq i \leq n$$

then

(i) $\|\psi_i\|_{L_2(\nu)} = 1, \quad \text{for all} \quad 1 \leq i \leq n,$

(ii) $\|\sum_{i=1}^n a_i\psi_i\|_{L_2(\nu)} = \|\sum_{i=1}^n a_i\varphi_i\|_{L_2(\mu)} \leq M(\sum_{i=1}^n |a_i|^2)^{1/2},$

for all $1 \leq i \leq n$,

(iii) $\psi(x) = \sum_{i=1}^n |a_i|^2 |\psi_i(x)|^2 = \frac{1}{\mu(E)} \int_E \varphi(u)d\mu = K^2,$

for $x \in E$, where

$$\frac{1}{4} \leq K^2 < \gamma(M)$$

(iv) $\sum_{i=1}^n |a_i|^3 |\psi_i(x)|^3 \leq \beta(M) \sum_{i=1}^n |a_i|^3,$

for $x \in E$ and $\beta(M) = 10^5 M^{18}$. Finally, notice that for $x \in E$ and $t \in T$,

$$|\sum_{i=1}^n a_i\xi_i(t)\psi_i(x)| \leq 5M^3 |\sum_{i=1}^n a_i\xi_i(t)\varphi_i(x)|$$

so that

$$\|\sum_{i=1}^n a_i\xi_i(t)\psi_i(x)\|_{L_\infty(\nu)} \leq 5M^3 \|\sum_{i=1}^n a_i\xi_i(t)\varphi_i(x)\|_{L_\infty(\mu)}.$$

Hence, it suffices to prove the assertion for the system $\{\psi_i\}_{i=1}^n$ restricted to the set E.

Step III. Fix m, which will be determined later, and notice that

$$\frac{1}{\nu(E)^m}\int_E\cdots\int_E\frac{1}{m^2}\sum_{j,k=1}^{m}|\sum_{i=1}^{n}|a_i|^2\psi_i(x_j)\psi_i(x_k)|^2 d\nu_1(x_1)\dots d\nu(x_m)\leq$$

$$\leq\frac{1}{(\nu(E)m)^2}\sum_{j,k=1}^{m}\int_E\int_E|\sum_{i=1}^{n}|a_i|^2\psi_i(x_j)\psi_i(x_k)|^2 d\nu(x_j)d\nu(x_k)\leq$$

$$\leq\frac{M^2}{(\nu(E)m)^2}\sum_{j,k=1}^{m}\int_E\sum_{i=1}^{n}|a_i|^4|\psi_i(x_j)|^2 d\nu(x_j)\leq(\frac{M}{\varepsilon(M)})^2\sum_{i=1}^{n}|a_i|^4$$

Hence, one can find m points $\{x_j\}_{j=1}^m$ in the set E such that

$$\frac{1}{m^2}\sum_{j,k=1}^{m}|\sum_{i=1}^{n}|a_i|^2\psi_i(x_j)\psi_i(x_k)|^2<(\frac{M}{\varepsilon(M)})^2\sum_{i=1}^{n}|a_i|^4.$$

Step IV. For $x\in E$ and $\rho>0$, put

$$E_\rho(x)=\{t\in T;\sum_{i=1}^{n}a_i\,\xi_i(t)\psi_i(x)>\rho\}.$$

Our aim is to show that, for every M, there exist constants $1>\alpha(M)>0$, $C_2=C_2(M)$ and $q=q(M)>0$ such that if $\{x_j\}_{j=1}^m$ are the points selected in Step III and $\rho=\frac{\alpha K}{2}(1+\log R)^{1/2}$, with $\alpha\leq\alpha(M)$, then

$$\tau(T\sim\bigcup_{j=1}^{m}E_\rho(x_j))\leq C_2R^{-q}. \tag{2}$$

In fact, it suffices to prove (2) for $R>R_0(M)$ because the case $1\leq R\leq R_0(M)$ can be taken care by just increasing the constant C_2. Put

$$f(t)=\sum_{j=1}^{m}\chi_{E_\rho(x_j)}(t);\quad t\in T,$$

and observe that if $\tau(\bigcup_{j=1}^m E_\rho(x_j))<\kappa$, for some $\kappa>0$, then by the Cauchy-Schwartz inequality, we get that

$$\mathbb{E}|f|\leq(\mathbb{E}|f|^2)^{1/2}\tau(\bigcup_{j=1}^{m}E_\rho(x_j))^{1/2}<\kappa^{1/2}(E|f|^2)^{1/2},$$

which means that the inequality

$$\mathbb{E}|f|\geq(1-C_2R^{-q})^{1/2}(\mathbb{E}|f|^2)^{1/2} \tag{3}$$

implies (2), and thus our assertion with $C_1=\frac{\alpha\kappa}{2}$.

Step V. In order to estimate $\mathbb{E}|f|$ and $(\mathbb{E}|f|^2)^{1/2}$, we shall use a sharper version of the one and two-dimensional central limit theorem. More precisely, we shall use Proposition 1 from [2], which is due to V. Rotar' [4] (see also Corollary 17.2 in [1]).

The first application of the above result is done in the one-dimensional case, when, for fixed $1 \leq j \leq m$, we put

$$X_i(t) = a_i \xi_i(t) \psi_i(x_j); 1 \leq i \leq n, t \in T.$$

Then

$$\begin{aligned}
\rho_3 &= \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}|X_i|^3 = \frac{1}{n} \sum_{i=1}^{n} |a_i|^3 \mathbb{E}|\xi_i|^3 |\psi_i(x_j)|^3 \leq \\
&\leq \frac{M^3}{n} \sum_{i=1}^{n} |a_i|^3 |\psi_i(x_j)|^3 \leq \frac{M^3 \beta(M)}{n} \sum_{i=1}^{n} |a_i|^3, \\
V &= \frac{1}{n} \sum_{i=1}^{n} cov(X_i) = \frac{1}{n} \sum_{i=1}^{n} |a_i|^2 \mathbb{E}|\xi_i|^2 |\psi_i(x_j)|^2 = \frac{K^2}{n} \quad \text{and} \quad \lambda = \frac{K^2}{n},
\end{aligned}$$

which, by the afore mentioned version of the central limit theorem, yields that

$$|\tau(E_\rho(x_j)) - \frac{n^{1/2}}{(2\pi)^{1/2} K} \int_{\frac{\rho}{n^{1/2}}}^{\infty} e^{-\frac{y^2 n}{2K^2}} dy| < C_1' \frac{M^3 \beta(M)}{K^3} \sum_{i=1}^{n} |a_i|^3$$

or, by a change of variable, that

$$|\tau(E_\rho(x_j)) - \frac{1}{(2\pi)^{1/2} K} \int_{\rho}^{\infty} e^{-\frac{u^2}{2K^2}} du| < C_2' \sum_{i=1}^{n} |a_i|^3 \leq C_2' (\sum_{i=1}^{n} |a_i^4|)^{1/2}$$

for some constant $1 < C_2^1 = C_2^1(M) < \infty$.

We shall use the notation $g_1 \asymp g_2$ whenever there is a universal constant $0 < C < \infty$ so that $C^{-1} < g_2(\xi)/g_1(\xi) \leq C$ for the relevant values of the parameter ξ. With this notation, we recall that

$$\int_{\rho}^{\infty} e^{-\xi^2/2} d\xi \asymp \frac{1}{\rho e^{\frac{\rho^2}{2}}}; \rho > 1.$$

Hence if with $\alpha(M)$ and $0 < \alpha \leq \alpha(M)$ fixed, we impose on R a condition of the type $R > R_0(\alpha(M)) = R_0(M)$ then the error term in the application of the central limit theorem is much less than the main term and also $\rho > K$ so we can deduce easily that

$$\tau(E_\rho(x_j)) \asymp K^2 \rho^{-1} e^{-\frac{\rho^2}{2K^2}}; 1 \leq j \leq m$$

and thus

$$\mathbb{E}|f| = \sum_{j=1}^{m} \tau(E_\rho(x_j)) \asymp mK^2\rho^{-1}e^{-\frac{\rho^2}{2K^2}}.$$

Step VI. Since

$$\mathbb{E}|f|^2 = E|f| + \sum_{\substack{j,k=1 \\ j\neq k}}^{m} \tau(E_\rho(x_j) \cap E_\rho(x_k))$$

the proof of (3) requires that we estimate each of the terms appearing in the right hand side of the above identity.

Put

$$\sigma_1 = \{(j,k);\ 1 \leq j \neq k \leq m,\ |\sum_{i=1}^{n} |a_i|^2 \psi_i(x_j)\psi_i(x_k)| < \frac{1}{10}\},$$

and notice that

$$\frac{|\sigma_1^c|}{(10m)^2} \leq \frac{1}{m^2} |\sum_{i=1}^{n} |a_i|^2 \psi_i(x_j)\psi_i(x_k)|^2 < (\frac{M}{\varepsilon(M)})^2 \sum_{i=1}^{n} |a_i|^4$$

from which it follows that

$$|\sigma_1^c| \leq (\frac{10Mm}{\varepsilon(M)})^2 \sum_{i=1}^{n} |a_i|^4.$$

Hence,

$$\sum_{\substack{(j,h)\in\sigma_1^c \\ j\neq k}} \tau(E_\rho(x_j) \cap E_\rho(x_k)) \leq (\frac{10KMm}{\varepsilon(M)})^2 \sum_{i=1}^{n} |a_i|^4 \rho^{-1} e^{-\frac{\rho^2}{2K^2}} \leq$$

$$\leq (\frac{10KMm}{\varepsilon(M)})^2 \frac{\rho^{-1}e^{-\frac{\rho^2}{2K^2}}}{R}.$$

On the other hand,

$$\frac{1}{R^{1/2}}(E|f|)^2 \asymp \frac{(mK\rho^{-1}e^{-\frac{\rho^2}{2K^2}})^2}{R^{1/2}}$$

so if $\alpha(M)$ is sufficiently small then

$$\sum_{\substack{(j,h)\in\sigma_1^c \\ j\neq k}} \tau(E_\rho(x_j) \cap E_\rho(x_k)) < \frac{1}{R^{1/2}}(E|f|)^2.$$

Assume now that

$$m = [4\rho e^{\frac{\rho^2}{2K^2}} R^{1/2}] + 1$$

and notice that this condition implies that

$$\mathbb{E}|f| \leq \frac{W}{R^{1/2}}(E|f|)^2,$$

for some absolute constant W.

At this time, we want to point out that the value of m above depends on that of α which will be made precise only later.

The main part of the argument of the proof will be devoted to evaluate the expression

$$\sum_{\substack{(j,k)\in\sigma_1 \\ j\neq k}} \tau(E_\rho(x_j) \cap E_\rho(x_k)).$$

Step VII. Fix a pair $s = (j,k) \in \sigma_1$ and consider the random vectors in $\mathbf{R}^2$ defined by

$$X_i^s(t) = (a_i\xi_i(t)\psi_i(x_j),\ a_i\xi_i(t)\psi_i(x_k));\ 1 \leq i \leq n,\ t \in T.$$

In order to apply the central limit theorem for these random vectors, we notice that

$$\rho_3^s \equiv \frac{1}{n}\sum_{i=1}^n |a_i|^3 \mathbb{E}|\xi_i|^3 (|\psi_i(x_j)|^2 + |\psi_i(x_k)|^2)^{3/2} \leq \frac{8M^3\beta(M)}{n}\sum_{i=1}^n |a_i|^3$$

while

$$V^s \equiv \frac{1}{n}\begin{pmatrix} \sum_{i=1}^n |a_i|^2|\psi_i(x_j)|^2 & \sum_{i=1}^n |a_i|^2\psi_i(x_j)\psi_i(x_k) \\ \sum_{i=1}^n |a_i|^2\psi_i(x_j)\psi_i(x_k) & \sum_{i=1}^n |a_i|^2|\psi_i(x_k)|^2 \end{pmatrix} =$$

$$= \frac{1}{n}\begin{pmatrix} K^2 & \sum_{i=1}^n |a_i|^2\psi_i(x_j)\psi_i(x_k) \\ \sum_{i=1}^n |a_i|^2\psi_i(x_j)\psi_i(x_k) & K^2 \end{pmatrix}.$$

Hence,

$$\det V^s = \frac{1}{n^2}(K^4 - |\sum_{i=1}^n |a_i|^2\psi_i(x_j)\psi_i(x_k)|^2) > \frac{1}{n^2}(\frac{1}{4} - \frac{1}{100}) = \frac{1}{5n^2}$$

and

$$\operatorname{trace} V^s = \frac{2K^2}{n}.$$

Denote the eigenvalues of V^s by λ_1 and λ_2, and observe that they should be positive. Suppose that $0 < \lambda_1 \leq \lambda_2$, and notice that $\lambda_1 + \lambda_2 = \operatorname{trace} V^s$, which yields that $\lambda_2 \leq \frac{2K^2}{n}$. Therefore, since det $V^s = \lambda_1 \lambda_2$ it follows that

$$\frac{1}{5n^2} \leq \lambda_1 \lambda_2 \leq \frac{2K^2}{n} \lambda_1,$$

i.e.

$$\lambda_1 \geq \frac{1}{10K^2 n}.$$

By using again the central limit theorem, in the version of [4] which contains an error term, as we did before, we get that

$$|\tau(E_\rho(x_j) \cap E_\rho(x_k)) - \frac{1}{2\pi(\det V^s)^{1/2}} \int_{\frac{\rho}{\sqrt{n}}}^{\infty} \int_{\frac{\rho}{\sqrt{n}}}^{\infty} e^{-\frac{1}{2}\langle Y,(V^s)^{-1}Y\rangle} dY \leq$$

$$\leq C_1^1 n^{-1/2} \rho_3 \lambda^{-3/2} \leq D_1 \sum_{i=1}^{n} |a_i|^3,$$

for some constant $D_1 = D_1(M) < \infty$. Hence,

$$\tau(E_\rho(x_j) \cap E_\rho(x_k)) \leq D_1 \sum_{i=1}^{n} |a_i|^3 + \frac{1}{2\pi(\det(nV^s))^{1/2}} \int_\rho^\infty \int_\rho^\infty e^{-\frac{1}{2}\langle U,(nV^s)^{-1}U\rangle} dU$$

which yields that

$$\sum_{s=(j,k)\in\sigma_1} \tau(E_\rho(x_j) \cap E_\rho(x_k)) \leq D_1 m^2 \sum_{i=1}^{n} |a_i|^3 +$$

$$+ \int_\rho^\infty \int_\rho^\infty \sum_{s\in\sigma_1} \frac{1}{2\pi(\det(nV^s))^{1/2}} e^{-\frac{1}{2}\langle U,(nV^s)^{-1}U\rangle} dU.$$

Notice that

$$D_1 m^2 \sum_{i=1}^{n} |a_i|^3 \leq D_1 m^2 (\sum_{i=1}^{n} |a_i|^4)^{1/2} = \frac{D_1 m^2}{R^{1/2}} \leq \frac{(\mathbb{E}|f|)^2}{R^{1/9}},$$

provided that $R \geq R_0(M)$. As we have already mentioned, the case when R is relatively small can be taken care by increasing the constant C_2 in the statement of the Theorem. It remains to compare the expression

$$\int_\rho^\infty \int_\rho^\infty \sum_{s\in\sigma_1} \frac{1}{2\pi(\det(nV^s))^{1/2}} e^{-\frac{1}{2}\langle U,(nV_s)^{-1}U\rangle} dU$$

with the expression

$$\sum_{s\in\sigma_1} \tau(E_\rho(x_j))\cdot\tau(E_\rho(x_k)) = \int_\rho^\infty\int_\rho^\infty \sum_{s\in\sigma_1} \frac{1}{2\pi K^2} e^{-\frac{1}{2K^2}(u_1^2+u_2^2)} du_1 du_2 + \delta$$
$$= \frac{|\sigma_1|}{2\pi K^2}\int_\rho^\infty\int_\rho^\infty e^{-\frac{1}{2K^2}(u_1^2+u_2^2)} du_1 du_2 + \delta,$$

where

$$\delta \le m^2 (C_2^1)^2 \sum_{i=1}^n |a_i|^4.$$

As before, we can ensure that δ is dominated by $\frac{(E|f|)^2}{R^{1/9}}$. In order to compare these two integral expressions, notice that

$$(nV^s)^{-1} = \frac{1}{\det(nV^s)}\begin{pmatrix} K^2 & -\sum_{i=1}^n |a_i|^2\psi_i(x_j)\psi_i(x_k) \\ -\sum_{i=1}^n |a_i|^2\psi_i(x_j)\psi_i(x_k) & K^2 \end{pmatrix}$$

so if we introduce the notation

$$c_s = \sum_{i=1}^n |a_i|^2\psi_i(x_j)\psi_i(x_k);\ s\in\sigma_1,$$

then

$$(nV^s)^{-1} = \begin{pmatrix} \frac{1}{K^2-\frac{|c_s|^2}{K^2}} & -\frac{c_s}{K^4-|c_s|^2} \\ -\frac{c_s}{K^4-|c_s|^2} & \frac{1}{K^2-\frac{|c_s|^2}{K^2}} \end{pmatrix}.$$

It follows that

$$\int_\rho\int_\rho \sum_{s\in\sigma_1} \frac{1}{2\pi(\det(nV^s))^{1/2}} e^{-\frac{1}{2}\langle U,(uV^s)^{-1}U\rangle} dU =$$
$$= \int_\rho^\infty\int_\rho^\infty \sum_{s\in\sigma_1} \frac{1}{2\pi(\det(nV^s))^{1/2}} e^{\{-\frac{u_1^2+u_2^2}{2(K^2-\frac{|c_s|^2}{K^2})}+\frac{c_s u_1 u_2}{K^4-|c_s|^2}\}} du_1 du_2.$$

Now, for $s\in\sigma_1$, put

$$a_s = \frac{1}{K^2-\frac{|c_s|^2}{K^2}},\quad b_s = \frac{c_s}{K^4-|c_s|^2}$$

and notice that, for any value of $L>1$,

$$\int_{L_\rho}^{\infty}\int_{\rho}^{\infty}\sum_{s\in\sigma_1}\frac{1}{2\pi(\det(nV^s))^{1/2}}e^{\{-\frac{a_s}{2}(u_1^2+u_2^2)+b_s u_1u_2\}}du_1du_2 \le$$
$$\le 32\int_{L_\rho}^{\infty}\int_{\rho}^{\infty}\sum_{s\in\sigma_1}e^{-\frac{a_s-b_s}{2}(u_1^2+u_2^2)}du_1du_2 \asymp \frac{1}{L\rho^2}\sum_{s\in\sigma_1}e^{-\frac{a_s-b_s}{2}(L^2+1)\rho^2}.$$

But, for any $s\in\sigma_1$,

$$a_s-b_s=\frac{K^2-c_s}{K^4-|c_s|^2}=\frac{1}{K^2+c_s}\ge\frac{1}{\gamma(M)+\frac{1}{10}}=\gamma'(M)>0$$

which implies that

$$\int_{L_\rho}^{\infty}\int_{\rho}^{\infty}\sum_{s\in\sigma_1}\frac{1}{2\pi(det(nV^s))^{1/2}}e^{\{-\frac{a_s}{2}(u_1^2+u_2^2)+b_s u_1u_2\}}\le\frac{m^2}{L\rho^2}e^{\frac{-\gamma'(M)(L^2+1)\rho^2}{2}}.$$

If $L^2+1=max(\frac{2}{\gamma'(M)K^2},2)$ then

$$\frac{m^2}{L\rho^2}e^{\frac{-\gamma'(M)(L^2+1)\rho^2}{2}}\le\frac{D_2(E|f|)^2}{R^q},$$

for suitable $D_2=D_2(M)<\infty$ and $q=q(M)>0$. Hence, for another constant $D_3=D_3(M)<\infty$,

$$\sum_{s\in\sigma_1}\tau(E_\rho(x_j)\cap E_\rho(x_k))\le\frac{D_3(\mathbb{E}|f|)^2}{R^q}+$$
$$+\int_{\rho}^{L_\rho}\int_{\rho}^{L_\rho}\sum_{s\in\sigma_1}\frac{1}{2\pi(\det(nV^s))^{1/2}}e^{\{-\frac{a_s}{2}(u_1^2+u_2^2)+b_s u_1u_2\}}du_1du_2.$$

Step VIII. In order to complete the proof, we have to compare the expression

$$A=\sum_{s\in\sigma_1}\frac{1}{2\pi(\det(nV^s))^{1/2}}e^{\{-\frac{a_s}{2}(u_1^2+u_2^2)+b_s u_1u_2\}},$$

pointwise in the range $\rho\le u_1,u_2\le L\rho$, with the expression

$$B=\frac{1}{2\pi K^2}|\sigma_1|e^{-\frac{1}{2K^2}}(u_1^2+u_2^2),$$

which appears in the calculation of the expression $\sum_{s\in\sigma_1}\tau(E_\rho(x_j))\tau(E_\rho(x_k))$. We need to show that $A\le B(1+D_4R^{-q'})$, for some choice of $D_4=D_4(M)<\infty$ and $q'=q'(M)>0$. To this end, set

$$\delta_r=\{s\in\sigma_1;\frac{1}{2^r}\le|c_s|<\frac{1}{2^{r-1}}\}$$

and observe that, by Step III,

$$\frac{|\delta_r|}{2^{2r}} \le \sum_{s\in\sigma_1} |c_s|^2 \le (\frac{M}{\varepsilon(M)})^2 m^2 \sum_{i=1}^{n} |a_i|^4$$

i.e.

$$|\delta_r| \le \min(m^2, (\frac{M}{\varepsilon(M)})^2 2^{2r} m^2 \sum_{i=1}^{n} |a_i|^4)$$

and

$$|\sigma_1| = \bigcup_{r=3}^{\infty} \delta_r.$$

For $s \in \delta_r$, we have that

$$|(a_s - \frac{1}{K^2})(u_1^2+u_2^2) - 2b_s u_1 u_2| \le |\frac{(c_s/K)^2(u_1^2+u_2^2) - 2c_s u_1 u_2}{K^4 - c_s^2}| \le$$
$$\le 32(2^{-2r+4}(u_1^2+u_2^2) + 2^{-r+1}u_1u_2) \le V_1 2^{-r} L^2 \rho^2,$$

provided of course that $\rho \le u_1,\ u_2 \le L\rho$, and V_1 is a suitably chosen universal constant. Moreover, $s \in \delta_r$ also yields that

$$\det(nV^s) = K^4 - |c_s|^2 \ge K^4 - \frac{1}{2^{2r-2}} \ge \frac{K^4}{(1+V_2 2^{-r})^2},$$

for some universal constant V_2. It follows that

$$A \le \frac{B\cdot S}{|\sigma_1|},$$

where

$$S = \sum_{r=3}^{\infty} |\delta_r|(1+\frac{V_2}{2^r}) e^{\frac{V_1 L^2 \rho^2}{2^r}}$$
$$= \sum_{r=3}^{[\frac{1}{3}\log_2 R]} + \sum_{r=1+[\frac{1}{3}\log_2 R]}^{\infty} = S_1 + S_2.$$

However,

$$S_1 \le \frac{1}{3}\log_2 R (\frac{M}{\varepsilon(M)})^2 R^{2/3} m^2 R^{-1} (1+\frac{V_2}{8}) e^{\frac{V_1 L^2 \rho^2}{8}}$$
$$\le D_4 \frac{\log_2 R}{R^{1/3}} m^2 e^{\frac{V_1 L^2 \rho^2}{8}} \le \frac{D_5 m^2}{R^{1/4}} \le D_6 \frac{|\sigma_1|}{R^{1/4}},$$

for suitable constants $D_4 = D_4(M)$, $D_5 = D_5(M)$ and $D_6 = D_6(M)$, provided $\alpha(M)$ satisfies a condition of type

$$\frac{\alpha(M)K^2V_1L^2}{4} < \frac{1}{20}.$$

Also

$$S_2 \leq \sum_{r=1+[\frac{1}{3}\log_2 R]}^{\infty} |\delta_r|(1+\frac{V_2}{R^{1/3}})(1+\frac{V_4}{R^{1/4}}) \leq |\sigma_1|(1+\frac{V_5}{R^{1/4}}),$$

for suitable universal constants V_4 and V_5, if $\alpha(M)^2L^2V_1 < 1$. The final conclusion is that

$$|S| \leq |\sigma_1|(1+\frac{D_7}{R^{1/4}}),$$

for some constant $D_7 = D_7(M)$, and this completes the proof.

□

Remarks. 1. It is useful to understand the order in which various constants appearing in the proof are selected. Once M is fixed, the change of density argument together with the selection of the new system $\{\psi_i\}_{i=1}^n$ give rise to the constants $\varepsilon = \varepsilon(M) > 0$, $\gamma = \gamma(M) < \infty$, $\beta = \beta(M) = 10^5M^{18}$ and $K = K(M)$. With $\gamma'(M) = \frac{1}{\gamma(M)+\frac{1}{16}}$ we then choose $L(M)$ so that

$$L^2 + 1 = \max(\frac{2}{\gamma'(M)K^2}, 2).$$

Once the universal constants V_1 and V_2 are determined we select $\alpha = \alpha(M)$ subject to various conditions which are spread all over the proof. Then we fix $\rho = \frac{\alpha K}{2}(1+\log R)^{1/2}$ provided $R = \frac{(\sum_{i=1}^m |a_i|^2)^2}{\sum |a_i|^4}$ is large relative to a constant $R_0 = R_0(M)$; otherwise, we complete the proof immediately by taking C_2 sufficiently large. Finally, we select the number of points $m = [4\rho R^{1/2}e^{\frac{\rho^2}{2K^2}}] + 1$ for which we proceed with the random selection argument described in Step III.

2. As in the previous paper [2], the theorem above remains true of we replace $p = 3$ by any other value of $2 < p$.

The main theorem can be used, for instance, in order to study random processes of the type

$$F_{N,\alpha,d}(t,x) = \sum_{\substack{n=(n_1,\dots,n_d)\in \mathbb{Z}_+^d \\ 1\leq |n_1\dots n_d| \leq N}} \frac{\xi_n(t)\varphi_n(x)}{(n_1\dots n_d)^\alpha},$$

where the dimension $d \geq 1$, $0 < \alpha < \frac{1}{2}$ and the random variables $\{\xi_n\}$ and the system $\{\varphi_n\}$, which both are indexed by $n \in \mathbb{Z}_+^d$, satisfy the conditions of the theorem, for some choice of M.

In this case, we conclude the existence of constants $K_1 = K_1(M,\alpha)$, $K_2 = K_2(M,\alpha)$ and $q = q(M,\alpha) > 0$ so that

$$\tau\{t \in T, \|F_{N,\alpha,d}(t,x)\|_{L_\infty(\mu,x)} \leq K_1 N^{\frac{1}{2}-\alpha}(1+\log N)^{\frac{d}{2}}\} \leq K_2 N^{-q}, \tag{4}$$

for all N.

This estimate is obtained by recalling that the cardinality of the set Γ_N, of all d-tuples $n = (n_1, \dots, n_d) \in \mathbb{Z}_d^+$ so that $1 \leq |n_1 \dots n_d| \leq N$, satisfies

$$|\Gamma_N| \asymp N(1+\log N)^{d-1}.$$

Hence, the computation of the expression

$$\Big(\sum_{\substack{n \in \mathbb{Z}_+^d \\ 1 \leq |n_1 \dots n_d| \leq N}} \frac{1}{(n_1 \dots n_d)^{2\alpha}}\Big)^{1/2}; 0 < \alpha \leq \frac{1}{2},$$

shows that it is of order of magnitude

$$\Big(\frac{1}{N^{2\alpha}} N(1+\log N)^{d-1}\Big)^{1/2} = N^{\frac{1}{2}-\alpha}(1+\log N)^{\frac{d-1}{2}}.$$

Then (4) follows immediately from the main theorem since R is polynomial in N.

This estimate is sharp in the case when $\{\varphi_n\}$ is the d-dimensional trigonometric system and $\{\xi_n\}$ the usual Rademacher functions, since in this case

$$\mathbb{E}\|F_{N,\alpha,d}(t,x)\|_{L_\infty(\mu,x)} \leq K_3 N^{\frac{1}{2}-\alpha}(1+\log N)^{\frac{d}{2}},$$

for some constant K_3.

In the case $\alpha = \frac{1}{2}$, the direct application of the main theorem does not yield a sharp estimate since R is now of logarithmic order. In order to overcome this difficulty, we consider instead the random process

$$F'_{N,\frac{1}{2},d}(t,x) = \sum_{\substack{n=(n_1,\dots,n_d) \in \mathbb{Z}_+^d \\ \sqrt{N} < |n_1 \dots n_d| \leq N}} \frac{\xi_n(t)\varphi_n(x)}{(n_1 \dots n_d)^{\frac{1}{2}}}$$

for which the theorem gives that

$$\mathbb{E}\|F'_{N,\frac{1}{2},d}(t,x)\|_{L_\infty(\mu,x)} \geq c(1+\log N)^{\frac{d}{2}},$$

for some constant $c > 0$.Since $\{\xi_n\}$ are independent random variables of mean zero, it follows that also

$$\mathbf{E}\|F_{N,\frac{1}{2},d}(t,x)\|_{L_\infty(\mu,x)} \geq \mathbb{E}\|F'_{N,\frac{1}{2},d}(t,x)\|_{L_\infty(\mu,x)} \geq c(1+\log N)^{\frac{d}{2}}.$$

This estimate is sharp in the classical case discussed above.

In a similar manner, we can study also random processes of the form

$$Q_{N,\beta,d}(t,x) = \sum_{\substack{n=(n_1,\dots,n_d)\in \mathbb{Z}_+^d \\ 1<|n_1\dots n_d|\le N}} \frac{\xi_n(t)\varphi_n(x)}{(n_1\dots n_d)^{\frac{1}{2}}(1+\log^\beta(n_1\dots n_d))},$$

where $\frac{d}{2} < \beta < \frac{d+1}{2}$ and the systems $\{\xi_n\}$ and $\{\varphi_n\}$ are as before.

For $\beta > \frac{d}{2}$, a direct calculation shows that

$$\mathbb{E}\|Q_{N,\beta,d}(t,x)\|_{L_2(\mu,x)} \le K_4,$$

for some $K_4 = K_4(d) < \infty$ and all N. On the other hand, by applying the main theorem again to the auxiliary process

$$Q'_{N,\beta,d}(t,x) = \sum_{\substack{n\in \mathbb{Z}_+^d \\ \sqrt{N}<|n_1\dots n_d|\le N}} \frac{\xi_n(t)\varphi_n(x)}{(n_1\dots n_d)^{\frac{1}{2}}(1+\log^\beta(n_1\dots n_d))}$$

we obtain, as before, that

$$\mathbb{E}\|Q_{N,\beta,d}(t,x)\|_{L_\infty(\mu,x)} \ge c(1+\log N)^{\frac{d+1}{2}-\beta},$$

which is of interest only if $\beta < \frac{d+1}{2}$. Again, this estimate is sharp in the case of the trigonometric functions and that of the Rademacher functions. This fact can be checked as in [5] p. 284.

Remark. As we have seen in the above examples, it is sometimes useful to replace the assertion (ii) of the theorem by

$$\text{(ii}'\text{)}\quad \mathbb{E}\|\sum_{i=1}^n a_i\,\xi_i\,\varphi_i\|_{L_\infty(\mu)} \ge C_3 \max_{\Lambda\subset\{1,\dots,n\}} (\sum_{i\in\Lambda}|a_i|^2)^{1/2}(1+\log\frac{(\sum_{i\in\Lambda}|a_i|^2)^2}{\sum_{i\in\Lambda}|a_i|^4})^{1/2}.$$

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