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by

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EXTENSION PROPERTIES OF COMPLEX ANALYTIC OBJECTS

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Contents

Introduction 1

Chapter I. Around Theorems of Hartogs, Levi and Schwarz 2
1. Theorems of Hartogs and Levi and their immediate consequences 2
2. Non-standard versions of Levi's theorem 8
3. Schwarz reflection principle and its versions 16
4. Analytic objects and extensions along pseudoconvex exhaustions 18

Chapter II. Analytic Sets and Meromorphic Mappings 23
5. Analytic sets, ramified covers and currents 23
6. Extension properties of holomorphic mappings 28
7. Cycle space associated with a meromorphic mapping 32
8. Meromorphic mappings with values in Kähler spaces 43
9. Pluriclosed metrics and spherical shells 48

Chapter III. Applications 57
10. Coverings of compact complex manifolds 57
11. Sets of normality of families of meromorphic mappings 61
12. Separate analyticity and Rothstein-type theorems 67
13. Vanishing cycles in holomorphic foliations and foliated shells 70

Chapter IV. Holomorphic Bundles and Coherent Analytic Sheaves 79
14. Holomorphic bundles and coherent analytic sheaves 79
15. Levi flat hypersurfaces and roots of holomorphic line bundles 82
16. Thullen-type extension of bundles with constraints on curvature 86
17. Plateau problem, extension from the boundary and fillings 91
18. Appendix I. Placements of Hartogs figures 94
19. Appendix II. Historical notes 97

References 98

Introduction

In this survey we shall try to describe one of the remarkable features of complex analysis and geometry: the phenomenon of analytic continuation. The latter consists in the fact that the objects of study in this discipline, we shall call them *complex analytic objects*, often extend to a larger domain that there was their domain of definition at the beginning. The first and simplest example is the following Bochner-Hartogs extension theorem:

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Let K be a compact in a domain $D \subset \mathbb{C}^n$, $n \geqslant 2$, such that $D \setminus K$ is connected. Then every holomorphic function in $D \setminus K$ extends to a holomorphic function in D.

We shall explain in this text that this statement holds true for a great number of various analytic objects. The formal definition of the notion of a complex (and real) analytic object will be postponed to section 4. For the moment it will be sufficient to say that holomorphic and meromorphic functions, mappings, foliations, holomorphic bundles and coherent analytic sheaves are complex analytic objects.

The Table of Contents tells sufficiently about the content of this survey and therefore let us only very briefly outline the main goals of this text. We try to formulate in a possibly best and complete way the main results obtained in the subject since the very beginning at the end of 19th century. But as a rule the proofs of the principal statements will be sketched only in the case when they are not yet described in the monographic literature. The only exception is made for the beginning of Chapter I where the classical results are discussed. In the former case we send the interested reader to the corresponding surveys and books for more details. A specific attention will receive the developments which took place since the appearance of the book of Y.-T. Siu, [Si3], in 1974.

The extension phenomena, being one of the decisive features of complex analytic objects, makes its way accompanied from the very beginning with important applications and motivations. One of the first examples again belongs to F. Hartogs:

Let $f(z_1,...,z_n)$ be a function of $n \geqslant 2$ complex variables which is separately analytic, i.e., for every fixed $n-1$ variables f is holomorphic as a function of the remaining one. Then f is holomorphic as a function of n variables.

We pay a specific attention to such king of applications and motivations of extension results as well as formulate some open questions. A short historical note is added at the end of the text.

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Chapter I. Around Theorems of Hartogs, Levi and Schwarz

1. Theorems of Hartogs and Levi and their immediate consequences

For a positive real number r we denote by $\Delta_r(z_0) = \Delta(z_0,r)$ the disk of radius r in $\mathbb{C}$ centered at z_0, *i.e.*, $\Delta_r(z_0) = \{z \in \mathbb{C} : |z - z_0| < r\}$. Δ_r stands for the disk centered at the origin, Δ for the unit disk. By A_{r_1,r_2} we denote the open annulus of radii $r_1 < r_2$, *i.e.*, $A_{r_1,r_2} := \{z \in \mathbb{C} : r_1 < |z| < r_2\}$. The option $r_1 = 0$ is not excluded and $A_{0,r}$ will be denoted as Δ_r^*, the punctured disk. A ring domain in $\mathbb{C}^{n+1}$, $n \geqslant 1$, is defined as $R^{n+1}_{r_1,r_2} := A_{r_1,r_2} \times \Delta^n$, *i.e.*, $R^{n+1}_{r_1,r_2}$ is a product of an annulus and the unit polydisk Δ^n.

1.1. Theorems of Hurwitz and Levi.

Theorem 1.1. (A. Hurwitz, [Hw]). *Let f be a holomorphic function in $R^2_{1-r,1}$. Suppose that for some sequence $\{z_\nu\}$ of distinct complex numbers converging to zero restrictions $f_{z_\nu} := f(\cdot, z_\nu)$ holomorphically extend from $A_{1-r,1}$ to Δ. Then f holomorphically extends to the bidisk Δ^2 as a function of two variables.*

The proof goes as follows, see Fig.3(a). Write $f(\lambda,z) = \sum_{n=-\infty}^{\infty} a_n(z)\lambda^n$, where a_n are holomorphic in Δ. The fact that $f(\cdot,z_\nu)$ is holomorphic in Δ as a function of λ means that $a_n(z_\nu) = 0$ for all negative n. The set $A := \{z \in \Delta : a_n(z) = 0 \quad \forall n = -1,...\}$ is analytic. Since it contains a converging sequence it is the whole of Δ. I.e., $a_n \equiv 0$ for all $n < 0$. Therefore f is holomorphic in Δ^2.

Definition 1.1. A subset $A \subset \Delta^n$ is called *thick at origin* if for any neighborhood $V \ni 0$ the intersection $A \cap V$ is not contained in a proper analytic subset of V.

Theorem 1.2. *Let f be a holomorphic function in the ring domain $R^{n+1}_{1-r,1}$. Suppose that for z in some subset $A \subset \Delta^n$, thick at origin, restrictions $f_z := f(\cdot,z)$ holomorphically extend from $A_{1-r,1}$ to Δ. Then f holomorphically extends to Δ^{n+1} as a function of $n+1$ variables.*

Proof is the same as that of Theorem 1.1. Recall that a complex manifold/space X is called Stein if there exists a proper holomorphic imbedding $i : X \to \mathbb{C}^N$ for some N.

Corollary 1.1. *Let X be a Stein manifold (or, a reduced Stein space) and let $f : R^{n+1}_{1-r,1} \to X$ be a holomorphic mapping. Suppose that for z in some subset $A \subset \Delta^n$, thick at origin, restrictions $f_z := f(\cdot,z)$ holomorphically extend from $A_{1-r,1}$ to Δ. Then f holomorphically extends to Δ^{n+1}.*

Indeed, $i \circ f$ is defined by N holomorphic functions, say $f_1,...,f_N$. Extending every f_k to Δ^{n+1} by Theorem 1.2 we extend $i \circ f$ to a holomorphic mapping from Δ^{n+1} to $\mathbb{C}^N$. Its image is contained in $i(X)$ because the last is closed in $\mathbb{C}^N$. This gives us the extension of f itself.

Recall that a meromorphic function f on a complex manifold/normal space D is defined as locally being a quotient of two holomorphic functions. In more colloquial terms there should exist an open covering $\{D_\alpha\}$ of D and $h_\alpha \in \mathcal{O}(D_\alpha), g_\alpha \in \mathcal{O}^*(D_\alpha)$ such that $h_\alpha g_\beta = h_\beta g_\alpha$ on $D_\alpha \cap D_\beta$. Then $f = h_\alpha/g_\alpha$ on D_α. Taking a finer covering one can additionally suppose that for every $x \in D_\alpha$ germs of h_α and g_α are relatively prime in $\mathcal{O}_x$. Under this assumption we call a point x an *indeterminacy point* of f if $h_\alpha(x) = g_\alpha(x) = 0$. Observe that the set I_f of *indeterminacy points* of f is analytic of complex codimension $\geqslant 2$. By P_f denote the divisor of poles of f, by Z_f its divisor of zeroes. Then $I_f = P_f \cap Z_f$.

Before stating the Theorem of Levi about extension of meromorphic functions let us prove a lemma. Let $\mathcal{O}$ be an integral domain and $\mathcal{M}$ be its field of quotients (field of fractions). In our applications $\mathcal{O}$ will be the ring $\mathcal{O}(\Delta)$ of holomorphic functions in the unit disk and then $\mathcal{M} = \mathcal{M}(\Delta)$ will be the field of meromorphic functions in Δ.

Lemma 1.1. *A formal power series*

$$F(\lambda) = \sum_{n=-\infty}^{-1} a_n\lambda^n \in \mathcal{O}[[\lambda]] \tag{1.1}$$

represents a rational function $\frac{P(\lambda)}{Q(\lambda)}$ with $P,Q \in \mathcal{O}[\lambda]$ and $\mathsf{deg}\, Q \leqslant N$ *if and only if*

$$\begin{vmatrix} a_{-n_1} & a_{-n_2} & \dots & a_{-n_{N+1}} \\ . & . & \dots & . \\ a_{-n_1-N} & a_{-n_2-N} & \dots & a_{-n_{N+1}-N} \end{vmatrix} = 0 \tag{1.2}$$

for all $(N+1)$-tuples $n_1 < ... < n_{N+1}$.

Proof. Indeed, we look for a non-zero polynomial $Q(\lambda) = c_0 + c_1\lambda + ... + c_N\lambda^N$ with coefficients in $\mathcal{O}$ such that $F \cdot P \in \mathcal{O}[\lambda]$. But this condition means that for every $k \geqslant 1$ one should have

$$a_{-k}c_0 + ... + a_{-k-N}c_N = 0. \tag{1.3}$$

The last means that vectors $b_k := (a_{-k}, a_{-k-1},...,a_{-k-N})$, $k \in \mathbb{N}$ belong to the hyperplane with equation (1.3) in the $\mathcal{M}$-linear space $\mathcal{M}^{N+1}$. The latter means that every $N+1$ of them are linearly dependent, and this is precisely what tells the condition (1.2).

□

Theorem 1.3. *(E. Levi, [Lv]). Let f be a meromorphic function in $R^2_{1-r,1}$. Suppose that for some sequence $\{z_\nu\}$ of distinct complex numbers converging to zero restrictions $f_{z_\nu} := f(\cdot, z_\nu)$ meromorphically extend from $A_{1-r,1}$ to Δ and that the number of poles counted with multiplicities of these extensions is uniformly bounded. Then f meromorphically extends to the bidisk Δ^2 as a function of two variables.*

Proof. If $f \equiv \infty$ there is nothing to prove. If $f \not\equiv \infty$ but $f(\cdot,0) \equiv \infty$ we can multiply it by z^d with an appropriate d. This will not change the number of poles of $f(\cdot, z_\nu)$ for $z_\nu \neq 0$ and we can suppose that $f(\cdot,0) \not\equiv \infty$. Remark that the positive part $f^+(\lambda, z) := \sum_{n\geqslant 0} a_n(z)\lambda^n$ of the Laurent expansion of f is already holomorphic in the bidisk. Our task therefore is to extend $f^-(\lambda, z) := \sum_{n<0} a_n(z)\lambda^n$. By Lemma 1.1 applied to the ring $\mathbb{C}$ the extendability of $f^-(\lambda, z_\nu)$ to the disk together with the condition on poles means that for $a_n = a_n(z_\nu)$ the determinants (1.2) vanish. Therefore they vanish identically as functions of z. And therefore, again by Lemma 1.1 but this time applied to the ring $\mathcal{O}(\Delta)$, we have that $f^-(\lambda, z)$ is rational over the field $\mathcal{M}(\Delta)$. I.e., is meromorphic in Δ^2.

□

Example 1.1. The condition on uniform boundedness of poles in Levi's theorem cannot be removed. Take a sequence of polynomials $P_n(z) = \prod_{j=1}^{n-1}(z - 1/j)$ and consider the following function

$$f(\lambda, z) = \sum_{n=1}^{\infty} \frac{P_n(z)}{n!}\frac{1}{\lambda^n}.$$

f is holomorphic in $\mathbb{C}^* \times \mathbb{C}$ and for every $z_\nu = \frac{1}{\nu}$ its restriction to $\Delta \times \{z_\nu\}$ writes as

$$f(\lambda, z_\nu) = \sum_{n=1}^{\nu} \frac{P_n(z_\nu)}{n!}\frac{1}{\lambda^n}.$$

It is meromorphic and has a pole at origin of multiplicity ν. But for every $z \notin \{1, ..., \frac{1}{\nu}, ...\}$ the restriction $f(\cdot, z)$ has essential singularity at zero, *i.e.,* f is not extendable meromorphically to a neighborhood of the origin as a function of two variables.

Theorem of Levi extends with the same proof to the case of several variables. In the following theorem by saying that $f(\cdot, z)$ is well defined we mean that $A_{1-r,1} \times \{z\} \not\subset I_f$.

Theorem 1.4. *Let f be a meromorphic function in the ring domain $R^{n+1}_{1-r,1}$. Suppose that for z in some subset $A \subset \Delta^n$ thick at origin restrictions $f_z := f(\cdot, z)$ are well defined, meromorphically extend from $A_{1-r,1}$ to Δ and the number of poles counted with multiplicities of these extensions is uniformly bounded. Then f extends to Δ^{n+1} as a meromorphic function of $n+1$ variables.*

Remark 1.1. In applications one usually refers to theorems above in a less precise form asking, for example, that $f(\cdot, z)$ extends to Δ for z in some non-empty open subset U of Δ^n. Or, that this U is not contained in a countable union of locally closed proper analytic subsets of Δ^n. Under these (and analogous) assumptions it is straightforward to deduce the existence of such $N \in \mathbb{N}$ and $A \subset \Delta^n$ thick at some point that $f(\cdot, z)$ extends to Δ with the number of poles bounded by N for $z \in A$.

1.2. Theorem of Hartogs, globalizations. A typical example of the situation described in Remark 1.1 is the Hartogs-type extension statement, the so called *Hartogs' Lemma*. It is explained on the Fig.1(a). We call a Hartogs figure in $\mathbb{C}^{n+1}$, $n \geqslant 1$, the following domain

$$H^{n+1}_r := \left(A_{1-r,1} \times \Delta^n\right) \bigcup \left(\Delta \times \Delta^n_r\right) = R^{n+1}_{1-r,1} \cup \left(\Delta \times \Delta^n_r\right). \tag{1.4}$$

Here Δ^n_r stands for the polydisk of radius $r > 0$ in $\mathbb{C}^n$. As we just explained every holomorphic/meromorphic function in H^{n+1}_r extends to a holomorphic/meromorphic function in Δ^{n+1}. And one more variation. Let $\varphi : \bar{\Delta} \to \Delta^n$ be a holomorphic mapping continuous up to the boundary. Denote by C its graph in Δ^{n+1}. Let V^{n+1}_r be a domain in $\mathbb{C}^{n+1}$ which contains the ring domain $R^{n+1}_{1-r,1}$ plus a neighborhood of the graph C, a "curved" Hartogs figure, see Fig.

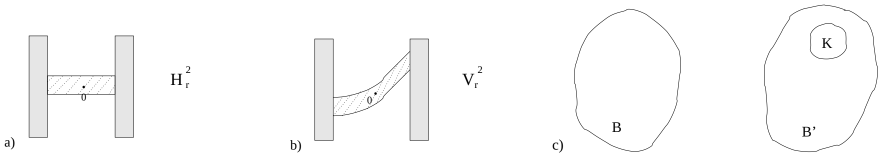

FIGURE 1. Fig.(a) shows the standard Hartogs figure, Fig.(b) a "curved" one. It should be underlined that both cases (a) and (b) were treated in [Ht2]. Fig.(c) explains how Hartogs thought about the Fig.(a) himself: function in question was supposed to be holomorphic in $B \times K$ plus in a neighborhood of ∂B times B'.

1(b). Then *every function holomorphic in* V_r^{n+1} *holomorphically extends to* Δ^{n+1}. The simplest way to prove this is to develop f into Taylor series with center at $(\lambda, \varphi(\lambda))$ and observe that for $|\lambda| \sim 1$ the radius of convergence (in z direction) is big. This gives an estimate for the coefficients of the Taylor series, and this estimate is preserved along the graph C by the maximum principle. To make the things easier remark that we can suppose f to be bounded. Let us state these results as a theorem.

Theorem 1.5. (Hartogs' Lemma, F. Hartogs, [Ht2]). *Every holomorphic function in* H_r^{n+1} *or* V_r^{n+1} *extends to a holomorphic function in the unit polydisk* Δ^{n+1}.

The same holds for meromorphic functions too, it follows from the theorem above via the Corollary 1.4 below. The following theorem in the case of holomorphic functions is also stated in [Ht2]. The first rigorous proof, using the Green formula, was given by S. Bochner in [Bo] both for holomorphic and meromorphic cases.

Theorem 1.6. (Bochner-Hartogs). *Let D be a domain in $\mathbb{C}^n$ and $K \Subset D$ a compact in D such that $D\setminus K$ is connected. Then every holomorphic (resp. meromorphic) function in $D\setminus K$ extends to a holomorphic (resp. meromorphic) function in D.*

Let $\rho(z) = \|z\|^2$ be the Euclidean distance function in $\mathbb{C}^n$. Fix $\delta > 0$ sufficiently small. Consider the set $T \subset \mathbb{R}^+$ of t-s such that the theorem holds for every compact $K \Subset D$ with connected complement satisfying

i) $K \subset \{\rho < t\}$;

ii) $K \subset D_\delta := \{z \in D : \mathsf{dist}(z, \partial D) \geqslant \delta\}$.

T is obviously closed and contains a neighborhood of zero. Indeed, if $0 \notin D$ there is nothing to prove. If $0 \in D$ then for $t > 0$ small enough K can be removed by just one Hartogs figure.

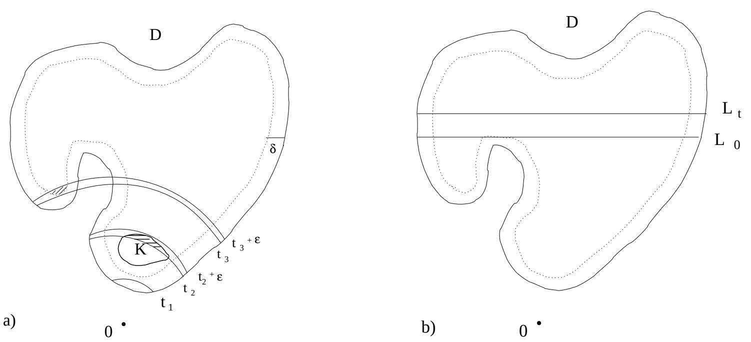

FIGURE 2. Fig.(a): if $t \leqslant t_1$ there is nothing to prove, if our compact K is contained in $\{\rho < t_2 + \varepsilon\} \cap D_\delta$ then the bump $K \setminus \{\rho < t_2\}$ can removed by appropriately placed Hartogs figures over the level set $\{\rho = t_2 + \varepsilon\}$. For $t = t_3$ no new problems appear, and for $t = t_3 + \varepsilon$ the newly appeared peace of $D_\delta \cap \{\rho < t_3 + \varepsilon\}$ on the left can be again easily removed by Hartogs figures. But if one tries to extend a function along a family of slices as on Fig.(b), there L_t approaches L_0, one might get in trouble with monodromy.

Let us see that T is also open. Let $t \in T$, set $\Sigma_t := D \cap \{\rho = t\}$. One obviously finds an $\varepsilon > 0$ such that for every $x \in \Sigma_t$, $\mathsf{dist}(x, \partial D) \geqslant \delta$, there exists an imbedding $i : \Delta^n \to D$ with the property that $i(H_r^n) \subset D \setminus \{\rho > t + \varepsilon\}$ and $i(\Delta^n) \ni x$. Now let K be a compact in $D \cap \{\rho < t + \varepsilon\}$. Using the bumping explained above one extends any holomorphic/meromorphic function from $D \setminus K$ to $(D \cap \{\rho < t\}) \setminus \tilde{K}$. Here $\tilde{K}$ is equal to K minus corresponding polydisks (a finite number of them, in fact). The complement to $\tilde{K}$ is the union of the complement to K and these polydisks, therefore it is also connected. Theorem follows by taking $\delta \to 0$.

Remark 1.2. A nice proof of the holomorphic case of this theorem, following ideas of L. Ehrenpreis from [Eh], can be found in [Ho1]. Many more other approaches are spread over the literature. We gave here the proof "by bumping" for the following two reasons. First: this way makes possible to get the needed statement reasonably simply and directly from the Hartogs' Lemma. Second: this method works not only for functions but also for other analytic objects and will be repeatedly used along this survey.

1.3. Relation to the Levi and Poincaré problems. Recall that a Riemann domain (D,p) over a complex manifold X is called *locally pseudoconvex over a point* $z \in X$ if there exists a Stein neighborhood $U \ni z$ such that all connected components of $p^{-1}(U)$ are Stein. (D,p) is called locally pseudoconvex over X if it is locally pseudoconvex over every point of X.

Theorem 1.7. *(Docquier-Grauert, [DG]). Let (D,p) be a Riemann domain over a Stein manifold X of dimension $n \geqslant 2$. If every holomorphic imbedding $h : H_r^n \to D$ extends to a holomorphic immersion $\hat{h} : \Delta^n \to D$ then D is a Stein manifold.*

As an obvious corollary from this criterion one gets one theorem of K. Stein: *a regular cover of a Stein manifold is Stein itself.* Recall that the locally pseudoconvex envelope of a Riemann domain (D,p) over a complex manifold X is the smallest locally pseudoconvex domain over X containing (D,p). Denote it as $(\hat{D},\hat{p})$.

Corollary 1.2. *Let (D,p) be a domain over a complex manifold X. Then every holomorphic/meromorphic function on D extends to a holomorphic/meromorphic function $\hat{f}$ on $\hat{D}$.*

For the proof of this corollary let us recall the construction of the pseudoconvex envelope first. Let $\mathcal{P}$ denote the set of pseudoconvex domains over X which contain (D,p). This $\mathcal{P}$ is non-empty, it contains (X, Id), and possesses a natural pre-order: $(R_1,p_1) \leqslant (R_2,p_2)$ if there exists a local homeomorphism $\varphi : R_1 \to R_2$ commuting with projections. In this case we say actually that (R_2,p_2) *contains* (R_1,p_1). This pre-order is *directed* in the sense that for any given $(R_1,p_1),(R_2,p_2) \in \mathcal{P}$ there exists (R_3,p_3) such that $(R_3,p_3) \leqslant (R_1,p_1)$ and $(R_3,p_3) \leqslant (R_2,p_2)$. Such (R_3,p_3) can be constructed as a *fiber product* of (R_1,p_1) with (R_2,p_2) over X: $R_1 \times_X R_2 := \{(x_1,x_2) \in R_1 \times R_2 : \mathsf{pr}_1(x_1) = p_2(x_2)\}$. This product with a natural projection to X is obviously locally pseudoconvex over X and is smaller than both of (R_1,p_1) and (R_2,p_2). The smallest element of $\mathcal{P}$ is our locally pseudoconvex envelope $(\hat{D},\hat{p})$ of (D,p). Now let a holomorphic/meromorphic function f on D be given. Its domain of existence, see the Cartan-Thullen construction in Theorem 4.1, must be necessarily locally pseudoconvex over X by Hartogs (resp. Levi) theorem and Docquier-Grauert criterium. Therefore it contains the pseudoconvex envelope $(\hat{D},\hat{p})$ of (D,p). Corollary follows. In the particular case when X is a Stein manifold the pseudoconvex envelope is actually the envelope of holomorphy.

Corollary 1.3. *Let (D,p) be a domain over a Stein manifold X and let $(\hat{D},\hat{p})$ be its envelope of holomorphy. Then every meromorphic function extends to a meromorphic function $\hat{f}$ on $(\hat{D},\hat{p})$.*

This gives the solution of the Poincaré problem for domains in Stein manifolds:

Corollary 1.4. *Let D be a domain in a Stein manifold, then every meromorphic function in D can be represented as a quotient of two holomorphic ones.*

Indeed, by Corollary 1.3 our meromorphic function f can be extended to a meromorphic function $\hat{f}$ on the envelope of holomorphy $\hat{D}$ of D. $\hat{D}$ is Stein and therefore we can apply

Theorem A of Cartan to the sheaf of ideals $\mathcal{J}_P$ of holomorphic functions vanishing along the divisor of poles P of $\hat{f}$. In particular we can find a global section g of this sheaf, *i.e.,* a holomorphic function on $\hat{D}$ vanishing along P with multiplicities not less than those of $\hat{f}$. Then $g\hat{f}$ is holomorphic. Corollary is proved.

1.4. Levi and Riemann theorems on complex spaces. All complex spaces in this text are supposed to be reduced, Hausdorff and countable at infinity.

Remark 1.3. Normality, if needed, will be required each time separately. One of the equivalent definitions of normality says that bounded holomorphic functions extend across proper analytic sets, *i.e.,* the Riemann extension theorem holds true on such spaces. This obviously implies that on normal space holomorphic functions extend across analytic sets of codimension two (without the assumption of boundedness). Let us state the Riemann theorem in a slightly more general form.

Theorem 1.8. *Let A be a proper analytic subset of a normal complex space X and f an L^2-bounded holomorphic function on $X\setminus A$. Then f extends to a holomorphic function on X.*

Extension of f across $A\setminus \mathsf{Sing}\,X$, after the application of the Fubini theorem, reduces to a simple one-dimensional argument. Extension across $\mathsf{Sing}\,X$ goes due to the supposed normality of X. Now let us state one more frequently used version of Levi's theorem.

Corollary 1.5. *Let A an analytic subset of codimension $\geqslant 2$ of a reduced, normal complex space X. Then every meromorphic function on $X\setminus A$ extends to a meromorphic function on X.*

Let f be our function. The problem is local, therefore we can suppose that there exists an analytic cover $\mathsf{c} : X \to \Delta^n$, see subsection 5.2, $n = \mathsf{dim}\,X$. Let d be the order of c. We can suppose that $\pi(A)$ is contained in the ramification divisor $\mathcal{R}$ of c. Indeed, $A\setminus\tilde{\mathcal{R}}$ (here $\tilde{\mathcal{R}} := \mathsf{c}^{-1}(\mathcal{R})$ stands for the branching divisor of c), is obviously removable by Theorem 1.4. Let $a_1(z),...,a_d(z)$ be the elementary symmetric functions of the branches of f at c-preimages of z. They are well defined and meromorphic on $\Delta^n\setminus\mathcal{R}$. Take any point $z\in\mathcal{R}\setminus\pi(A\cup I_f)$. Then for some neighborhood U of z either f or $1/f$ is holomorphic in $\pi^{-1}(U)$. In the former case symmetric functions $a_1(z),...,a_d(z)$ are obviously bounded in $U\setminus\mathcal{R}$ and by Riemann theorem holomorphically extend to U. In the latter case one proves meromorphicity of $a_1(z),...,a_d(z)$ using holomorphicity of $1/f$ plus obvious algebraic manipulation. Therefore our symmetric functions meromorphically extend to $\Delta^n\setminus\pi(A\cup I_f)$ and, since the last is of codimension at least two, again by Theorem 1.4 all a_j extend to Δ^n. Denote by $w=[w_0:w_1]$ homogeneous coordinates in $\mathbb{P}^1$ and consider the following analytic set in $\Delta^n\times\mathbb{P}^1$

$$\Gamma' := \{(z,w) : w_1^d + a_1(z)w_1^{d-1}w_0 + ... + a_d(z)w_0^d = 0\}. \tag{1.5}$$

Denote by Γ the irreducible component of the fiber product $X\times_{\Delta^n}\Gamma'$ which contains the graph of f over $X\setminus A$. Γ will be the graph of the desired extension of f to X.

1.5. Separate analyticity and domains of convergence. W. Osgood proved in [Os1] that a *bounded* separately holomorphic function in the bidisk is holomorphic. The proof was achieved by reducing the problem to the case of continuous functions by the means of Schwarz lemma. In his remarkable paper [Ht1] F. Hartogs removed the condition of boundedness in Osgood's theorem and obtained his famous *separate analyticity theorem*.

Theorem 1.9. *(F. Hartogs,* [Ht1]*). Let $f:\Delta^p\times\Delta^q\to\mathbb{C}$ be a function of two (vector) complex variables defined in the unit polydisk Δ^{p+q}, $p,q\geqslant 1$. Suppose that for every $z_1\in\Delta^p$ the function $f(z_1,\cdot)$ is holomorphic in Δ^q and the same for every $z_2\in\Delta^q$. Then f is holomorphic in Δ^{p+q}.*

A version stated in the Introduction is clearly equivalent to this one. The proof can be found in any text book on several complex variables and will be not reproduced here. Let us only say that in modern texts Theorem 1.9 is deduced from Hartogs' Lemma 1.5 and Osgood's theorem. This was not the way of Hartogs (in fact [Ht1] precedes [Ht2]). His proof is closer in spirit to the proof of the following highly non-trivial generalization of Theorem 1.9, which is due to J. Siciak. For the elements of pluripotential theory see subsection 5.3.

Theorem 1.10. *(J. Siciak, [Sc]). Let E and F be a non-pluripolar compacts in Δ^p and Δ^q respectively and let $f : E \times F \to \mathbb{C}$ be a function such that $f(z_1, \cdot)$ is holomorphic in Δ^q for every $z_1 \in E$, and the same for every $z_2 \in F$. Then f holomorphically extends to a neighborhood of $E^* \times F^*$, where E^* (resp. F^*) is the set of pluriregular points of E (resp. of F).*

We shall treat the separate analyticity properties of meromorphic mappings with values in general complex spaces in section 12 and Theorem 1.10 will play a key role there. The envelope of holomorphy of $E^* \times F^*$ (to which f from Theorem 1.10, in fact, extends) is explicitly described in [Sc], see Corollary 12.2 in subsection 12.2. For the proof of this result we refer to [JP].

Already in the paper of Cartan and Thullen, see [CT], questions about domains of existence of holomorphic/meromorphic functions were considered together with properties of domains of their convergence/normality. The following statement in the holomorphic case follows immediately from the maximum principle.

Proposition 1.1. *Let X be a domain in $\mathbb{C}^n$ and let $\{f_k\}$ be a sequence of holomorphic/meromorphic functions in X. Denote by D the maximal open subset of X such that $\{f_k\}$ converge on compacts of D. Then D is pseudoconvex.*

Remark 1.4. The case of meromorphic functions is much more delicate. It is, probably, sufficient to say that it is not obvious what does the convergence of meromorphic functions mean. This will be discussed in section 11, see Corollary 11.3 there.

1.6. Singularities. One more remarkable result of Hartogs is the following theorem.

Theorem 1.11. *(F. Hartogs, [Ht3]). Let S be a non-empty closed subset of the unit polydisk Δ^{n+1} such that $S \subset \Delta^n \times \Delta_{1-\varepsilon}$ for some $\varepsilon > 0$ and such that there exists a natural k such that for every $z' \in \Delta^n$ the set $S_{z'} := S \cap \Delta_{z'}$ consists of not more than k points. If there exists a holomorphic function f in $\Delta^{n+1} \setminus S$ such that f doesn't extend to a neighborhood of any point of S then S is an analytic subset of Δ^{n+1}.*

One also has the same statements with f being meromorphic. Indeed, by Corollary 1.4 we can represent $f = g/h$ where g, h are holomorphic in $\Delta^{n+1} \setminus S$. Now S must be singular for both g and h. This theorem of Hartogs was generalized by Oka. Recall that a closed subset $S \subset \Delta^{n+1}$ is called pseudoconcave if $\Delta^{n+1} \setminus S$ is a domain of holomorphy.

Theorem 1.12. *(K. Oka, [Ok]). Let S be a pseudoconcave subset of the unit polydisk Δ^{n+1} such that $S \subset \Delta^n \times \Delta_{1-\varepsilon}$ for some $\varepsilon > 0$. Set*

$$\mathcal{S} := \{z \in \Delta^n : S_z := (\{z\} \times \Delta) \cap S \text{ consists of a finite number of points } \}.$$

If $\mathcal{S}$ is not pluripolar then S is a complex hypersurface in Δ^{n+1}.

For the proof and more statements of such type we refer to the book of Nishino [Ni].

2. Non-standard versions of Levi's theorem

In this section we shall give a few non standard versions of the Levi's extension theorem.

2.1. A non-linear version: formulation and example. Let a sequence of holomorphic functions $\{\varphi_k : \Delta_{1+r} \to \Delta\}_{k=1}^{\infty}$ be given such that φ_k converge uniformly on Δ_{1+r} to some $\varphi_0 : \Delta_{1+r} \to \Delta$. We say that such sequence is a *test sequence* if $(\varphi_k - \varphi_0)|_{\partial\Delta}$ doesn't vanish for $k \gg 0$ and

$$\mathsf{VarArg}_{\partial\Delta}(\varphi_k - \varphi_0) \qquad \text{stays bounded when} \qquad k \to +\infty. \tag{2.1}$$

Denote by C_k the graph of φ_k in $\Delta_{1+r} \times \Delta$, by C_0 the graph of φ_0. Let f be a meromorphic function in the ring domain $R^2_{1-r,1+r}$. Suppose that for every k the restriction $f|_{C_k \cap R^2_{1-r,1+r}}$

extends to a meromorphic function on the curve C_k and that the number of poles counting with multiplicities of these extensions is uniformly bounded.

Theorem 2.1. *There exists an analytic family of holomorphic graphs $\{C_\alpha\}_{\alpha\in\mathcal{A}}$, parameterized by a Banach ball $\mathcal{A}$ of infinite dimension, such that:*

- *i) $f|_{C_\alpha\cap R^2_{1-r,1+r}}$ extends to a meromorphic function on C_α for every $\alpha\in\mathcal{A}$ and the number of poles counting with multiplicities of these extensions is uniformly bounded.*
- *ii) Moreover, f meromorphically extends as a function of two variables (λ,z) to the pinched domain $\mathcal{P} := \mathsf{Int}\left(\bigcup_{\alpha\in\mathcal{A}} C_\alpha\right)$ swept by C_α.*

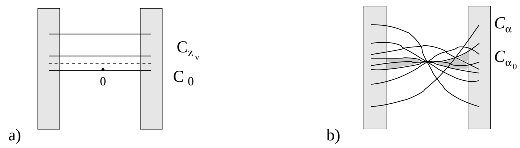

FIGURE 3. Fig. (a) illustrates horizontal disks in Theorems 1.1 and 1.3, and Fig. (b) the non-linear version of Theorem 2.1. The brighter dashed zone on this picture represents the ring domain $R^2_{1-r,1+r}$ and curves are the graphs C_α. Around C_{α_0}, the graph of $\varphi_0 = \varphi_{\alpha_0}$, the analytic family $\{C_\alpha\}_{\alpha\in\mathcal{A}}$ fills in an another (darker) dashed zone, a pinched domain $\mathcal{P}$. On this picture there is exactly one pinch, the point at which most of graphs intersect.

Let us discuss the notions of a *pinched domain* and *analytic family*, which appear in the context of Theorem 2.1. By an *analytic family* of holomorphic mappings from Δ to Δ we understand the quadruple $(\mathcal{X},\pi,\mathcal{A},\Phi)$ where:

- $\mathcal{X}$ is a complex manifold, which is either of finite dimension or a Banach one;
- a holomorphic submersion $\pi : \mathcal{X} \to \mathcal{A}$, where $\mathcal{A}$ is a positive dimensional complex (Banach) manifold such that for every $\alpha\in\mathcal{A}$ the preimage $\mathcal{X}_\alpha := \pi^{-1}(\alpha)$ is a disk;
- a holomorphic map $\Phi : \mathcal{X} \to \mathbb{C}^2$ of generic rank 2 such that for every $\alpha\in\mathcal{A}$ the image $\Phi(\mathcal{X}_\alpha) = C_\alpha$ is a graph of a holomorphic function $\varphi_\alpha : \Delta\to\Delta$. We write $\varphi(\lambda,\alpha) := \varphi_\alpha(\lambda)$.

In our applications $\mathcal{A}$ will be always a neighborhood of some α_0 and without loss of generally we may assume for convenience that $\varphi_{\alpha_0}\equiv 0$, *i.e.*, that $C_{\alpha_0} = \Delta\times\{0\}$. When $\mathcal{A}$ is a one-dimensional disk we say that our family is a *complex one-parameter* analytic family. Denote as $\bar{\mathcal{P}}_{\mathcal{X},\Phi}$ the image $\Phi(\mathcal{X})$, where $(\mathcal{X},\pi,\Delta,\Phi)$ is some complex one-parameter analytic family of complex disks in Δ^2. Point λ_0 such that $\varphi(\lambda_0,\alpha)\equiv 0$ as a function of α we call *a pinch* of $\bar{\mathcal{P}}_{\mathcal{X},\Phi}$ and say that $\bar{\mathcal{P}}_{\mathcal{X},\Phi}$ has a pinch at λ_0. Let us describe the shape of $\bar{\mathcal{P}}_{\mathcal{X},\Phi}$ near a pinch λ_0. Since $\varphi(\lambda_0,\alpha)\equiv 0$ we can divide it by $(\lambda-\lambda_0)^{l_0}$ with some (taken to be maximal) $l_0\geqslant 1$. I.e., in a neighborhood of $(\lambda_0,\alpha_0)\in\Delta\times\mathcal{A}$ we can write

$$\varphi(\lambda,\alpha) = (\lambda-\lambda_0)^{l_0}\varphi_1(\lambda,\alpha), \tag{2.2}$$

where $\varphi_1(\lambda_0,\alpha)\not\equiv 0$. Set

$$\Phi_1 : (\lambda,\alpha)\to(\lambda,\varphi_1(\lambda,\alpha)). \tag{2.3}$$

The image of Φ_1 contains a bidisk $\Delta^2_r(\lambda_0,0)$ of some radius $r>0$ centered at $(\lambda_0,0)$. Therefore with some constant $c>0$ one has

$$\bar{\mathcal{P}}_{\mathcal{X},\Phi} \supset \Delta^2_r(\lambda_0,0)\cap\{|z|<c|\lambda-\lambda_0|^{l_0}\}. \tag{2.4}$$

Definition 2.1. *A pinched domain is an open neighborhood $\mathcal{P}$ of $\bar{\Delta}\setminus\Lambda$, where Λ is a finite set of points in Δ, such that in a neighborhood of every $\lambda_0\in\Lambda$ domain $\mathcal{P}$ contains*

$$\Delta^2_r(\lambda_0,0)\cap\{|z|<c|\lambda-\lambda_0|^{l_0}\}\setminus\{(\lambda_0,0)\}. \tag{2.5}$$

We shall call l_0 the *order of the pinch* λ_0. After shrinking Δ (in λ-variable) if necessary, we can suppose that a domain $\mathcal{P}_{\mathcal{X},\Phi}$ which corresponds to a complex one-parameter analytic family $(\mathcal{X},\pi,\mathcal{A},\Phi)$ has only finite number of pinches, say at $\lambda_1,...,\lambda_N$ of orders $l_1,...,l_N$ respectively, and therefore $\mathcal{P}_{\mathcal{X},\Phi} := \bar{\mathcal{P}}_{\mathcal{X},\Phi}\setminus\{\lambda_1,...,\lambda_N\}$ is a pinched domain. Remark that $\bar{\mathcal{P}}_{\mathcal{X},\Phi}$ obviously contains every curve in a neighborhood $\mathcal{B}$ of $\varphi_0\equiv 0$ of the following subspace of finite codimension

$$\{\varphi\in \mathsf{Hol}\,(\Delta,\Delta) : \mathsf{ord}_0(\varphi,\lambda_j)\geqslant l_j\}\subset \mathsf{Hol}\,(\Delta,\Delta). \tag{2.6}$$

Remark 2.1. Therefore, let us make the following precisions: our pinched domains will be always supposed to have only finitely many pinches and moreover, these pinches *do not belong* to the corresponding pinched domain *by definition*. At that point it will be sufficient for us to remark that, as it is not difficult to see, the extension along one-parameter analytic families is equivalent to that of along of infinite dimensional ones, and both imply the extension to pinched domains. See [Iv12] for more details.

Example 2.1. *Without the condition (2.1) on the sequence $\{\varphi_k\}$ the theorem fails to be true. Let the function f be defined by the following series*

$$\sum_{n=1}^{\infty}3^{-4n^3}\prod_{j=1}^{n}\left[z-\left(\tfrac{2}{3}\lambda\right)^j\right]\lambda^{-n^2}z^n. \tag{2.7}$$

Then f is holomorphic in the ring domain $R:=\mathbb{C}^*\times\mathbb{C}$, holomorphically extends along every $C_k := \{z=\left(\frac{2}{3}\lambda\right)^k\}$, but there doesn't exist an analytic family $\{\varphi_\alpha\}_{\alpha\in\mathcal{A}}$ parameterized by a disk $\mathcal{A}\ni 0$, $\varphi_0\equiv 0$, such that $f|_{C_\alpha\cap(\mathbb{C}^*\times\mathbb{C})}$ meromorphically extends to C_α for all $\alpha\in\mathcal{A}$. The sequence $\varphi_k(\lambda)=\left(\frac{2}{3}\lambda\right)^k$ here converge to $\varphi_0\equiv 0$ and is not a test sequence. See [Iv12] *for more details.*

Definition 2.2. *Let's say that our functions $\{\varphi_k\}$ or, corresponding graphs $\{C_k\}$, are in* general position *if for every point $\lambda_0\in\Delta$ there exists a subsequence $\{\varphi_{k_p}\}$ such that zeroes of $\varphi_{k_p}-\varphi_0$ do not accumulate to λ_0.*

Theorem 2.1 implies the following *non-linear* Levi-type extension theorem:

Corollary 2.1. *If under the conditions of Theorem 2.1 curves $\{C_k\}$ are in general position then f extends to a meromorphic function in the bidisk $\Delta_{1+r}\times\Delta$.*

Remark 2.2. a) Let us explain the condition of the general position. Take the sequence $C_k=\{z=\frac{1}{k}\lambda\}$ in $\mathbb{C}^2$. Then the function $f(\lambda,z)=e^{\frac{z}{\lambda}}$ is holomorphic in $R:=\mathbb{C}^*\times\mathbb{C}$ and extends holomorphically onto every curve C_k. But it is not holomorphic (even not meromorphic) on $\mathbb{C}^2$. It is also holomorphic when restricted to any curve $C=\{z=\varphi(\lambda)\}$ provided $\varphi(0)=0$. Therefore the subspace H_0 of $\varphi\in\mathsf{Hol}\,(\Delta_{1+r},\Delta)$ such that f extends along the corresponding curve is of codimension one. In fact this is the general case: the Banach ball $\mathcal{A}$ in Theorem 2.1 appears as a neighborhood of the limit point α_0 in the subspace of finite codimension of a well chosen Banach space of holomorphic functions.

b) In order to prove Corollary 2.1 remark that pinches that appeared along the proof of Theorem 2.1 are limits of zeroes of φ_k. General condition assumption means that for every $\lambda_0\in\Delta$ we can take a subsequence such that the resulting pinched domain will not have a pinch in λ_0. The rest follows.

2.2. Finite dimensional families. Let us discuss the *finite dimensional* case in Theorem 2.1 first, *i.e.*, when φ_k *ad hoc* belong to some finite dimensional family $\mathcal{A}$, ex. family of lines or quadrics. In that case the proof is rather straightforward. To see this denote by $L^{1,2}(\mathbb{S}^1)$ the Sobolev space of complex valued functions on the unit circle having their first derivative in L^2. This is a complex Hilbert space with the scalar product $(h,g)=\int_0^{2\pi}[h(e^{i\theta})\bar{g}(e^{i\theta})+h'(e^{i\theta})\bar{g}'(e^{i\theta})]d\theta$. Recall that by Sobolev Imbedding Theorem $L^{1,2}(\mathbb{S}^1)\subset\mathcal{C}^{\frac{1}{2}}(\mathbb{S}^1)$, where $\mathcal{C}^{\frac{1}{2}}(\mathbb{S}^1)$ is the space of Hölder $\frac{1}{2}$ - continuous functions on $\mathbb{S}^1$. Denote by $H_+^{1,2}(\mathbb{S}^1)$ the subspace of $L^{1,2}(\mathbb{S}^1)$ which consists from functions holomorphically extendable to the unit disk Δ. By $H_-^{1,2}(\mathbb{S}^1)$ denote the subspace of functions holomorphically extendable to the complement of the unit disk in the Riemann sphere $\mathbb{P}^1$ and zero at infinity. Observe the following orthogonal decomposition

$$L^{1,2}(\mathbb{S}^1)=H_+^{1,2}(\mathbb{S}^1)\oplus H_-^{1,2}(\mathbb{S}^1). \tag{2.8}$$

We shall consider only the holomorphic case. After shrinking, if necessary, we can suppose that our function f is holomorphic in a neighborhood of $\bar{R}^2_{1-r,1}$. Consider the following analytic mapping $F : L^{1,2}(\mathbb{S}^1) \to L^{1,2}(\mathbb{S}^1)$

$$F : \varphi(\lambda) \to f(\lambda, \varphi(\lambda)), \tag{2.9}$$

and consider also the following integral operator $\mathcal{F} : H^{1,2}_+(\mathbb{S}^1) \to H^{1,2}_-(\mathbb{S}^1)$

$$\mathcal{F}(\varphi)(\lambda) = \frac{-1}{2\pi i}\int_{\mathbb{S}^1} \frac{f(\zeta,\varphi(\zeta)) - f(\lambda,\varphi(\lambda))}{\zeta-\lambda} d\zeta. \tag{2.10}$$

From the well known facts about the Hilbert transform, see §3 in [Iv12] for example, we get that $f(\lambda,\varphi(\lambda))$ extends to a holomorphic function in Δ_{1+r} if and only if $\mathcal{F}(\varphi) = 0$. Therefore we are interested in the zero set $\mathcal{A}^0$ of a holomorphic map $\mathcal{F}_{\mathcal{A}} : \mathcal{A} \to H^{1,2}_-(\mathbb{S}^1)$. But the zero set $\mathcal{A}^0$ of a holomorphic mapping from a finite dimensional manifold is an analytic set. Since this set contains a converging sequence $\{\alpha_k\}$ it has positive dimension and therefore contains a complex disk through α_0.

Remark 2.3. **a)** If C_k are intersections of $\Delta_{1+r} \times \Delta$ with algebraic curves of bounded degree, then they are included in a finite dimensional analytic (even algebraic in this case) family.

b) If $\varphi_k(\partial\Delta) \subset M$, where M is totally real in $\partial\Delta \times \bar{\Delta}$, and have bounded Maslov index then they are included in a finite dimensional analytic family.

c) If we do not suppose *ad hoc* that φ_k belong to some finite dimensional analytic family of holomorphic functions then the argument above is clearly not sufficient. The (well known) problem here is that a Banach (or, even Hilbert) analytic set has no *a priori* any analytic structure. The following simple example is very instructive. Take the following holomorphic map $\mathcal{F} : l^2 \to l^2 \oplus l^2$

$$\mathcal{F} : \{z_k\}_{k=1}^{\infty} \to \{\{z_k(z_k - 1/k)\} \oplus \{z_k z_j\}_{j>k}\}. \tag{2.11}$$

The zero set of $\mathcal{F}$ is a sequence $\{Z_k = (0,...,0,1/k,0,...)\}_{k\geqslant 1} \subset l^2$ together with zero. These Z_k-s might well be ours φ_k-s and therefore we cannot conclude the existence of families in the zero set of our $\mathcal{F}$ from (2.10) at this stage. Example 2.1 has precisely the feature just explained.

d) Let us remark that if $\{\varphi_k\}$ are taken from a finite dimensional family then they form a test sequence (for a generic choice of r), but in general a test sequence doesn't belong to any finite dimensional family. Take for example $\varphi_k(\lambda) = \frac{1}{k}\lambda^2 + e^{-k}\lambda^k$.

2.3. Sketch of the proof of Theorem 2.1. Let us explain the main lines of the proof of Theorem 2.1 in the general case. In the notations of the proof of Theorem 1.3 we can suppose that $\varphi_0 \equiv 0$, $f = f^-$ and the last is holomorphic in $A_{1-\varepsilon,1+\varepsilon} \times \Delta_{1+2\varepsilon}$. For $|\lambda| \sim 1$ the Taylor expansion of f writes as

$$f(\lambda, z) = \sum_{n=0}^{\infty} \tfrac{1}{n!}\tfrac{\partial^n f(\lambda,0)}{\partial z^n} z^n = \sum_{n=0}^{\infty} A_n(\lambda) z^n, \tag{2.12}$$

and we have the estimates

$$|A_n(\lambda)| = \tfrac{1}{n!}\left|\tfrac{\partial^n f(\lambda,0)}{\partial z^n}\right| \leqslant \tfrac{C}{(1+\varepsilon)^n}, \tag{2.13}$$

for some constant C, all $k \in \mathbb{N}$ and all $\lambda \in \mathbb{S}^1 := \partial\Delta$. Under the assumptions of the Theorem we see that meromorphic extensions $f_k(\lambda)$ of $f(\lambda, \varphi_k(\lambda))$ have uniformly bounded number of poles counted with multiplicities. As well as the numbers of zeroes of φ_k are uniformly bounded too. Up to taking a subsequence we can suppose that:

a) The number of poles of f_k-s, counted with multiplicities, is constant, say M, and these poles converge to the finite set $b_1,...,b_M \in \Delta_{1-\varepsilon}$ with corresponding multiplicities, *i.e.*, some of $b_1,...,b_M$ may coincide.

b) The number of zeroes of φ_k, counted with multiplicities, is also constant, say N and these zeroes converge to a finite set $a_1,...,a_N$ with corresponding multiplicities.

Step 1. For every k take a Blaschke product P_k having zeroes exactly at poles of f_k with corresponding multiplicities and subtract from $\{P_k\}$ a converging subsequence with the limit

$$P_0(\lambda) = \prod_{i=1}^{M} \frac{\lambda - b_i}{1-\bar{b}_i\lambda}. \tag{2.14}$$

Holomorphic functions $g_k := P_k f_k$ have uniformly bounded modulus on Δ and converge to some g_0, which is bounded with modulus by C (a constant from (2.13)). Therefore f_k converge on compacts of $\Delta \setminus \{b_1,...,b_M\}$ to a meromorphic function, which is nothing but A_0, and it satisfies the estimate

$$|A_0(\lambda)| \leqslant \frac{CC_1}{|\lambda - b_1|...|\lambda - b_M|}, \tag{2.15}$$

where $C_1 = \max\{\Pi_{i=1}^{M}|1-\bar{b}_i\lambda| : |\lambda| \leqslant 1\}$.

Step 2. Repeating the same argument one gets the estimate

$$|A_1(\lambda)| \leqslant \frac{1}{|\lambda - a_1|...|\lambda - a_N||\lambda - b_1||\lambda - b_N|} \cdot \frac{CC_1C_2}{1+\varepsilon} \tag{2.16}$$

for $\lambda \in \Delta \setminus \{a_1,...,a_N,b_1,...,b_M\}$. Here $C_2 = \max\{\Pi_{i=1}^{N}|1-\bar{a}_i\lambda| : |\lambda| \leqslant 1\}$.

Step 3. By induction one proves that A_n extends to a meromorphic function in Δ with the estimate

$$|A_n(\lambda)| \leqslant \frac{1}{\prod_{j=1}^{N}|\lambda - a_j|^n \prod_{j=1}^{M}|\lambda - b_j|} \cdot \frac{C'}{(1+\varepsilon)^n} \tag{2.17}$$

for $\lambda \in \Delta \setminus \{a_1,...,a_N,b_1,...,b_M\}$. Remark that (2.17) means, in particular, that $A_0,...,A_n$ have no other poles than $a_1,...,b_N$ with corresponding multiplicities.

Estimate (2.17) implies that (2.12) converges in the domain

$$\left\{(\lambda,z) \in \Delta^2 : |z| < c|\lambda - a_{j_1}|^{l_1}...|\lambda - a_{N_1}|^{l_{N_1}}\right\} \setminus \bigcup_{i=1}^{M}\{\lambda = b_i\}, \tag{2.18}$$

for an appropriately chosen $c > 0$. Here N_1 is the number of *different* a_j-s, which are denoted as $a_{j_1},...,a_{N_1}$ having corresponding multiplicities $l_1,...,l_{N_1}$. In particular we mean here that b_i are different from a_{j_1} for all i, j_1. Estimate (2.17) implies also that the extension of $f \cdot \prod_{j=1}^{M}(\lambda - b_j)$ to (2.18) is locally bounded near every vertical disk $\{\lambda = b_{i_1}\}$ and therefore extends across it by Riemann extension theorem. We conclude that f extends as a meromorphic function to the pinched domain

$$\mathcal{P} = \left\{(\lambda,z) \in \Delta^2 : |z| < c|\lambda - a_{j_1}|^{l_1}...|\lambda - a_{N_1}|^{l_{N_1}}\right\}, \tag{2.19}$$

and this proves the part (*ii*) of Theorem 2.1. For the rest we refer to [Iv12].

2.4. A "generalized Hartogs' Lemma". Let us give one more non-standard version of the Levi's extension theorem. It turns out that the statement of Hartogs theorem given on Fig. 1(b) remains true when φ is not necessarily holomorphic. Let $\varphi : \Delta_{1+r} \to \Delta$ be a continuous function with graph C. And let V be a domain which contains $R^2_{1-r,1+r}$ plus a neighborhood of C.

Theorem 2.2. *(E. Chirka, [Ch3]). Every function meromorphic in V meromorphically extends to Δ^2.*

This theorem follows from a more general one. Recall that a symplectic form on a real manifold X is a closed 2-form ω which is non-degenerate at any point on X. In this case X has even dimension, $\dim_{\mathbb{R}}(X) = 2n$, and ω^n is a volume form on X. Our principle example of symplectic forms are Kähler forms on complex manifolds. Let (X,ω) be a symplectic manifold. A real two-manifold $M \subset X$ is called *ω-positive* if $\omega|_M$ never vanish. By $c_1(X)$ denote the first Chern class of X, it is uniquely determined by the symplectic structure ω. By $c_1(X)[M]$ denote its value on M. Recall finally that a *rational curve* in a complex manifold X is an image of a non-constant holomorphic map $h : \mathbb{P}^1 \to X$.

Theorem 2.3. *Let M be a ω-positive immersed two-sphere in a disk-convex Kähler surface (X,ω) having only positive transversal self-intersections and such that $c_1(X)[M] > 0$. Then for any neighborhood U of M its envelope of meromorphy $(\hat{U},\pi)$ contains a rational curve C such that $c_1(X)[C] > 0$.*

For the notion of disk-convexity see subsection 6.3. For the moment it is sufficient to note that every compact complex manifold is disk-convex. In some cases like $X = \mathbb{P}^2$ this C is *symplectically* isotopic to M. More precisely we mean that there exists an isotopy $\{M_t\}_{t\in[0,1]}$ of ω-positive surfaces with transversal self-intersections such that $M_0 = M$ and $M_1 = C$. But in general this is not always the case.

Example 2.2. For $k \geq 1$ denote by H_k the projectivization of the bundle $E_k = \mathcal{O}\oplus\mathcal{O}(-k) \to \mathbb{P}^1$. H_k are called Hirzebruch surfaces. Denote by E_0 the exceptional curve in H_k and by F a fiber. Note that $E_0^2 = -k, F^2 = 0$ and $E_0 \cdot F = 1$. Let C be a smooth, irreducible complex curve in H_k which is not E_0. Then $C = nE_0 + mF$ in homology. We have that $0 \leq C\cdot E_0 = -kn+m$, and therefore $m \geq kn$. Therefore $C^2 = -n^2k + 2nm \geq 2n^2k - n^2k = n^2k$. Now take $k = 3$ and take $E_0 \cup F_{z_1} \cup F_{z_2}$, *i.e.,* the union of E_0 with two distinct fibers. Changing intersections by handles near points z_1 and z_2 we obtain an imbedded symplectic sphere M with $M^2 = 1$. This M is not isotopic to any smooth rational curve C, because for such C we should have either $C^2 \geq 3$ or $C^2 = 0$ and not $C^2 = M^2 = 1$.

Let us explain how Theorem 2.2 follows from Theorem 2.3. As it was explained at the beginning of the proof of Theorem 1.3 we can suppose that f is holomorphic in $A_{1-r,1+r}\times\Delta$ and $f = f^-$, *i.e.,* f is holomorphic on $(\mathbb{P}^1\setminus\bar{\Delta}_{1-r})\times\Delta$. Graph of φ can be obviously extended to a continuous graph M over $\mathbb{P}^1$, and, after perturbing M, we find that f is meromorphic in a neighborhood of a *smooth* sphere M, which is homologous to $\mathbb{P}^1\times\{pt\}$ in $\mathbb{P}^1\times\mathbb{P}^1 =: X$. Therefore $c_1(X)[M] = 2$. Making dilatations along second coordinate we can make M to be $\mathcal{C}^1$-close to $\mathbb{P}^1\times\{pt\}$ and therefore symplectic. Result follows now from Theorem 2.3.

Example 2.3. Let $(X,\omega) = (\mathbb{P}^2,\omega_{FS})$ be the complex projective plane with the Fubini-Study form and let $M\subset\mathbb{P}^2$ be and imbedded symplectic sphere, therefore $\delta = 0$. Recall that the first Chern class $c_1(\mathbb{P}^2)$ is represented by $3\omega_{FS}$. Therefore

$$c_1(\mathbb{P}^2)[M] = \int_M c_1(\mathbb{P}^2) = 3\int_M \omega_{FS} > 0,$$

i.e., we find ourselves under the assumptions of Theorem 2.3 and conclude that for every neighborhood $U\subset M$ its envelope of meromorphy $\hat{U}$ contains an imbedded rational curve C. Since $\mathbb{P}^2\setminus C$ is affine and, in particular, Stein we see that $\hat{U} = \mathbb{P}^2$. In particular every function meromorphic in a neighborhood of such M is rational.

Example 2.4. Let X be a ball in $\mathbb{C}^2$, ω the standard Euclidean form. Blow up the origin in $\mathbb{C}^2$ and denote by E the exceptional curve. By $\hat{X}$ denote the blown-up ball X. $\hat{X}$ is also Kähler, denote by ω_0 some Kähler form there. Consider a sufficiently small $\mathcal{C}^1$-perturbation of E. This will be a ω_0-symplectic sphere in $\hat{X}$, denote it by M. Chern class of the normal bundle to M is equal to that of for E and therefore is -1. So $c_1(\hat{X})[M] = 1$ and Theorem 2.3 applies. We got the following result.

Corollary 2.2. *The envelope of meromorphy of any neighborhood of M contains E.*

One can then blow down the picture to obtain downstairs a sphere M_1 the image of M under the blown-down map. This M_1 is homologous to zero, so cannot be symplectic, and for this M_1 our Theorem 2.3 cannot be applied.

Remark 2.4. Condition $c_1(X)[M] > 0$ in the Theorem 2.3 cannot be dropped. In [Nm1] Nemirovski, using results of Eliashberg-Harlamov and Forstneric, [F], showed that any imbedded complex curve C with $c_1(X)[M] \leq 0$ can be perturbed to an imbedded surface M which has a basis of Stein neighborhoods.

2.5. Sketch of the proof of Theorem 2.3. Now let us give a sketch of the proof of Theorem 2.3. It will crucially use the Gromov's theory of pseudoholomorphic curves. An almost complex structure on a real manifold X is a smooth section J of $\mathsf{End}(TX)$ such that $J^2 \equiv -\mathsf{Id}$. J is said to be tamed by a symplectic form ω if $\omega(\mathrm{v},J\mathrm{v}) > 0$ for every non-zero $\mathrm{v}\in TX$. Let (S,j) be a complex curve, here j stands for a complex structure on S. A J-holomorphic map is a $\mathcal{C}^1$-map $u:(S,j)\to(X,J)$ such that du commutes with the almost complex structures, $du\circ j = J\circ du$. Its image $M = u(S)$ is called a J-complex curve or a J-complex sphere if $(S,j) = \mathbb{P}^1$.

Step 1: Deformation of structures. Let U be a given neighborhood of M. Fix some other neighborhood U_1 of M which is relatively compact in U. Using ω-positivity of M we construct a smooth family $\{J_t\}_{t\in[0,1]}$ of almost complex structures on X satisfying the following properties:

a) J_0 is the given integrable structure on X.

b) For any $t \in [0,1]$ one has $\{x \in X : J_t(x) \neq J_0(x)\} \Subset U_1$.

c) M is J_1-holomorphic.

d) All $\{J_t\}$ are tamed by our Kähler form ω.

Denote by $(\hat{U},\pi)$ the envelope of meromorphy of U. We lift then structures J_t to $(\hat{U},\pi)$ in the following way. Having the natural imbedding $i : U \hookrightarrow \hat{U}$ we define the lift $\hat{J}_t$ by setting $\hat{J}_t|_{i(U)} := i_* J_t$ and extend $\hat{J}_t$ outside $i(U)$ as given integrable structure J_0 on $\hat{U}$. Again, $\hat{J}_t$ differs from J_0 only in $i(U_1)$. With some ambiguity of notations we denote $\hat{J}_t$ still as J_t. $i(U_1)$ and $i(U)$ will be identified with U_1 and U in the sequel.

Step 2: Deformation of M as a pseudoholomorphic curve. We construct a "semi-continuous" family of reduced J_t-complex curves $M_t \subset \hat{U}$ such that:

a) $M_1 = M$.

b) Each M_t is a union of its components, $M_t = \cup_i M_{t,i}$, and each component $M_{t,i}$ is a J_t-complex sphere eventually with singularities. Further, each component $M_{t,i}$ is defined with its multiplicity $m_{t,i} \in \mathbb{N}$ such that for each t one has $[M_t] = \sum_i m_{t,i} \cdot [M_{t,i}]$ in homology. In particular, for each t there exists a component M_{t,i_0} satisfying

$$c_1(X)[M_{t,i_0}] > 0. \tag{2.20}$$

c) There are finitely many "critical values" $t_1^* = 1 > t_2^* > \cdots > t_m^* = 0$ such in each subinterval $(t_{j+1}^*, t_j^*]$ the number of the components $M_{t,i}$, their homology classes $[M_{t,i}]$ and their multiplicities $m_{t,i}$ remain constant.

Remark 2.5. Semi-continuity of the family $\{M_t\}_{t\in[0,1]}$ means more precisely the following. M_t converges to $M_{t_k^*}$ when $t \searrow t_k^*$ in the sense described below. But then only those components of $M_{t_k^*}$ are subject to further deformation which satisfy (2.20).

Step 3: Kontinuitätssatz. Note that $\Sigma_t = M_t \backslash U_1$ is a complex curve in $\hat{U}$ with boundary on ∂U_1, and $\Sigma_1 = \varnothing$. Therefore by an appropriate version of the "continuity principle" Σ_t and therefore M_t stay in $\hat{U}$. But M_0 is holomorphic and satisfies $c_1(X)[X][M_0] > 0$ by *Step 2*. This finishes the proof.

Let us make few extended remarks with some more indications for the proofs of these steps. The first step of the proof is a simple topological fact. Steps 2 and 3 turn to be quite technical. Step 2 has two ingredients: a piece of Fredholm theory in moduli space of pseudoholomorphic curves and Gromov compactness theorem.

Remark 2.6. *Fredholm theory.* This part is quite far from the main lines of this survey and we shall not stop on it, see however §2 in [IS3] for detailed proof. Let us just formulate the final statement. For a given symplectic manifold (X,ω) we denote by $\mathcal{J}_\omega$ the space of ω-tamed $\mathcal{C}^{k,\alpha}$-smooth almost complex structures on X. Further, in the case when (X,ω) is a Kähler manifold and $U \subset X$ an open set we denote by J_0 the (integrable) complex structure on X and by $\mathcal{J}_\omega(U)$ the subspace of those $J \in \mathcal{J}_\omega$ which coincide with J_0 outside U. It is well-known that both $\mathcal{J}_\omega$ and $\mathcal{J}_\omega(U)$ are contractible Banach manifolds.

Theorem 2.4. *a) Let (X,ω) be a complex Kähler surface, $U \subset X$ an open set, $[M] \in \mathsf{H}_2(X,\mathbb{Z})$ a homology class representable by a sphere. Then for a generic path $h : [0,1] \to \mathcal{J}_\omega(U)$ for every $t \in [0,1]$ the structure $J_t := h(t)$ is regular and the moduli space $\mathscr{M}_{J_t}$ of J_t-complex spheres in $[M]$ is a smooth manifold of expected dimension.*

b) If for such a path $h(t)$ and some value $t_0 \in [0,1]$ the space $\mathscr{M}_{J_{t_0}}$ is non-empty and contains a J_{t_0}-complex curve $M_0 = u_0(S)$, then for every t sufficiently close to t_0 there exists a J_t-complex curve $M = u(S)$ parameterized by the map $u : S \to X$ close to $u_0 : S \to X$.

Let us notice that the both assumptions $\mathsf{dim}_{\mathbb{C}}X = 2$ and $S \cong \mathbb{S}^2$ in this theorem are essential: the assertion is wrong if either $\mathsf{dim}_{\mathbb{C}}X > 2$ or if S is a Riemann surface of higher genus. Further, there is another, hidden condition needed to make Theorem 2.4 meaningful. Namely, we need that one of the spaces $\mathscr{M}_{J_t}$ is non-empty. Since this space has expected dimension $2\cdot(c_1(X)[M]-1)$, it must be non-negative. This explains the origin of the condition $c_1(X)[M] > 0$ in Theorem 2.3.

The meaning of Theorem 2.4 is that the set of t-s in $[0,1]$ for which we can construct spheres M_t is open. Now we need to prove that it is also closed. To perform this second part of Step 2 and then Step 3 we need to recall the language, which was used in [IS2] to state the Gromov compactness theorem. Let us make this very briefly.

Remark 2.7. *Stable topology in Gromov compactness theorem.*

Definition 2.3. A stable curve over an almost-complex manifold (X,J) is a pair (C,u), where C is a connected complex nodal curve, possibly with boundary $\partial C = \bigcup_{i=1}^d \gamma_i$, and $u : C \to X$ is a J-holomorphic map which satisfies the following conditions:

1) if C_j is an irreducible component of C biholomorphic to $\mathbb{P}^1$ and $u(C_j) = \{point\}$, then C_j contains at least three nodes of C;
2) if C_j is a torus and again $u(C_j) = \{point\}$ then C_j contains at least one node of C.

Definition 2.4. A connected, oriented real surface with boundary $(\Sigma,\partial\Sigma)$ parameterizes a complex nodal curve C if there is a continuous map $\sigma : \Sigma \to C$ such that:

1) if $a \in C$ is a nodal point then $\gamma_a = \sigma^{-1}(a)$ is a smooth imbedded circle in $\Sigma\backslash\partial\Sigma$, and if $a \neq b$, then $\gamma_a \cap \gamma_b = \varnothing$;
2) $\sigma : \Sigma\backslash\bigcup_{i=1}^N \gamma_{a_i} \to C\backslash\{a_1,\dots,a_N\}$ is a diffeomorphism. Here $a_1,\dots,a_N$ are all the nodes of C.

Let a sequence $\{J_n\}$ of continuous almost complex structures on X be given which uniformly converge to a continuous structure J_∞.

Definition 2.5. We say that stable J_n-complex curves (C_n,u_n) converge to a stable J_∞-complex curve (C_∞,u_∞) in *Gromov topology*, if all C_n and C_∞ are parameterized by the same real surface Σ and there exist parameterizations $\sigma_n : \Sigma \to C_n$ and $\sigma_\infty : \Sigma \to C_\infty$ such that

i) $u_n\circ\sigma_n$ converges to $u_\infty\circ\sigma_\infty$ in $C^0(\overline{\Sigma},X)$-topology;
ii) if $\{a_i\}$ is the set of nodes of C_∞ and $\gamma_i := g_\infty^{-1}(a_i)$ are the corresponding circles in Σ, then on any compact $K \Subset \Sigma\backslash\cup_i\gamma_i$ the convergence $u_n\circ\sigma_n \longrightarrow u_\infty\circ\sigma_\infty$ is $L^{2,p}$-smooth;
iii) on any compact $K \Subset \Sigma\backslash\cup_i\gamma_i$ the complex structures $\sigma_n^*j_n$ converge to the complex structure $\sigma_\infty^*j_\infty$ in C^∞-topology.

Here j_n and j_∞ are complex structures on C_n and C_∞ respectively. The following result is the Gromov compactness theorem, see [Gro].

Theorem 2.5. *Let (C_n,u_n) be a sequence of stable over X J_n-holomorphic curves such that:*

a) J_n are of class $\mathcal{C}^0$ and are uniformly converging to $J \in \mathcal{C}^0$;
b) $\mathsf{area}[u_n(C_n)] \leqslant M$ for all n;
c) u_n converge near the boundary.

Then there is a subsequence (C_{n_k},u_{n_k}) which converges to a J-holomorphic stable over X curve (C_∞,u_∞). Moreover, for each boundary component γ there is an imbedding $\varphi_\infty : A_r \to C_\infty$ such that $u_n\circ\varphi_n \to u_\infty\circ\varphi_\infty = h_\infty$ on A_r.

For more details and proof we refer to [IS4]. This theorem comes out in deformation of curves in Step 2 as follows. Since our J_t-complex curves M_t are compact and symplectic and since they are isotopic on each interval $[t_k^*, t_{k+1}^*)$ we have that

$$\mathsf{area}_\omega M_t = \int\limits_{M_t} \omega$$

doesn't change when $t \to t_{k+1}^*$. Therefore Gromov compactness theorem gives us that M_t converges to some $M_{t_{k+1}^*}$ which is $J_{t_{k+1}^*}$-complex but may be reducible.

Discussion of the details on the continuity principle, *i.e.,* the step 3, we shall postpone till section 8, where a more general statement will naturally come out.

3. Schwarz reflection principle and its versions

3.1. Reflection of functions and varieties from totally real submanifolds. The formulation of the classical Reflection principle of H. A. Schwarz [Sw1] can be found in any text book on complex analysis. It obtained further developments along different lines. Let us start with a slight variation of it. Consider a real analytic totally real submanifold $W\subset\mathbb{C}^n$. Without loss of generality we may suppose that $\mathsf{dim}_{\mathbb{R}}W=n$. One can find local holomorphic coordinates in a neighborhood of a given point $p\in W$ such that $W=\mathbb{R}^n$ in these coordinates. Denote by Δ the unit disc in $\mathbb{C}$, by $\mathbf{S}$ the unit circle. Let $\beta\subset\mathbf{S}$ be a non-empty open subarc of $\mathbf{S}$. Let $u:\Delta\to\mathbb{C}^n$ be a holomorphic mapping continuous up to β and such that $u(\beta)\subset W$. Now the Schwarz Reflection Principle applies and we get that u holomorphically extends to some neighborhood V of β by reflection:

$$u(\zeta):=\begin{cases}u(\zeta)\text{ for }\zeta\in\bar\Delta\cap V,\\ \bar u(1/\bar\zeta)\text{ for }\zeta\in V\setminus\Delta.\end{cases}\tag{3.1}$$

It turns out that an analogous statements holds true also in non-integrable case.

Theorem 3.1. *Let (X,J) be a real analytic almost complex manifold and W a real analytic J-totally real submanifold of X. Let $u:\Delta\to X$ be a J-holomorphic map continuous up to β and such that $u(\beta)\subset W$. Then u extends to a neighborhood of β as a (real analytic) J-holomorphic map.*

In this case there is no reflection like (3.1), since a general almost complex structure doesn't admits any local (anti)-holomorphic maps. But the extension result still holds. For the proof of our Reflection Principle of Theorem 3.1, see [IS7], we need to study not only real analytic boundary values but also the smooth ones (with finite smoothness). For the method to work we need the precise regularity and a certain kind of uniqueness of smooth J-complex discs attached to a J-totally real submanifold. The result obtained is the following

Theorem 3.2. *Let $u:(\Delta,\beta)\to(X,W)$ be a J-holomorphic map of class $L^{1,2}\cap\mathcal{C}^0(\Delta\cup\beta)$, where W is J-totally real. Then:*

(i) *for any integer $k\geqslant 0$ and real $0<\alpha<1$ if $J\in\mathcal{C}^{k,\alpha}$ and $W\in\mathcal{C}^{k+1,\alpha}$ then u is of class $\mathcal{C}^{k+1,\alpha}$ on $\Delta\cup\beta$;*

(ii) *for $k\geqslant 1$ the condition $u\in L^{1,2}\cap\mathcal{C}^0(\Delta\cup\beta)$ and $u(\beta)\subset W$ can be replaced by the assumption that $\overline{u(\Delta)}$ is compact and the cluster set $\mathsf{cl}(u,\beta)$ is contained in W.*

Remark 3.1. If J is of class $\mathcal{C}^0$ and W of $\mathcal{C}^1$ then $u\in\mathcal{C}^\alpha$ up to β for all $0<\alpha<1$. This is proved in [IS5], Lemma 3.1. The rest is done in [IS7].

Let D be a domain in $\mathbb{C}^n$ stable under the conjugation map: $z\to\bar z$, and let A be an analytic set in $D\setminus\mathbb{R}^n$ of pure dimension 1. Denote by $\bar A$ the image of A under the conjugation.

Theorem 3.3. *(H. Alexander, [Ad]). Then intersection of the closure $\mathsf{cl}(A\cup\bar A)$ with D is an analytic set in D.*

For the proof we refer to [Ad] and [Si3]. The case $\mathsf{dim}\,A\geqslant 2$ is simpler and will be explained in section 5. Let us mention few open questions in this concern.

1. Let (X,J) be a real analytic almost complex manifold and W a real analytic J-totally real submanifold of X. Let C^+ be J-complex curve in $X\setminus W$. Does there exists a neighborhood V of W and a J-complex curve C^- in $V\setminus W$ (reflection of C^+) such that $\overline{(C^+\cup C^-)}\cap V$ is a J-complex curve in V?

2. The following question is a particular case of the previous one. Let C be a J-complex curve in the complement of a point. Will its closure $\bar C$ be a J-complex curve?

3.2. Segre varieties and mappings between real analytic hypersurfaces. Another line of developments can be described as follows. The central problem here is the following. Given bounded domains D and D' in $\mathbb{C}^n$ with smooth, real analytic boundaries and let $f : D \to D'$ be a proper holomorphic map (or, even a biholomorphic map). Does f extends to a holomorphic map between open neighborhoods of $\bar{D}$ and $\bar{D}'$? Dimension one case was solved already by Schwarz, it is explained on the Fig.4. If $n \geqslant 2$ and D, D' are strictly pseudoconvex C. Fefferman in [Fe] proved that a biholomorphic map $f : D \to D'$ smoothly extends to the boundary (for that it is sufficient that ∂D and $\partial D'$ are of class $\mathcal{C}^\infty$), and then S. Pinchuk in [Pn] proved, by inventing the reflection in higher dimensions, that the extended f is actually real analytic. The weakly pseudoconvex case for general proper holomorphic mappings was resolved affirmatively in [BR] and [DF]. The general case in dimensions $\geqslant 3$ in the following theorem is still open.

Theorem 3.4. *(Diederich-Pinchuk,* [DP1].[DP2]*). Let $D, D' \Subset \mathbb{C}^n$ be relatively compact domains in $\mathbb{C}^n$, $n \geqslant 2$, with real analytic boundaries and let $f : D \to D'$ be a proper holomorphic mapping.*

i) If $n = 2$ then f holomorphically extends to a neighborhood of $\bar{D}$.

ii) If $n \geqslant 3$ then the same as in (i) holds under an additional assumption that f is continuous up to the boundary.

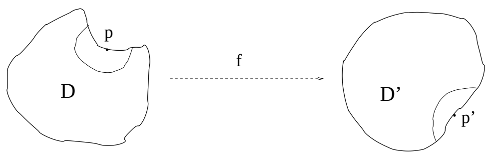

FIGURE 4. H.A. Schwarz proved in [Sw2, Sw3] that a conformal mapping f between simply connected regions D and D' with real analytic boundaries continuously extends to the boundary and then his reflection principle from [Sw1], applied locally near $p \in \partial D$ and $p' = f(p) \in \partial D'$, gives the affirmative answer to the main problem in dimension one. Continuous extension to the boundary for Jordan boundaries was conjectured by W. Osgood and proved by C. Carathéodory in [Ca].

A *real analytic* hypersurface M in a neighborhood of the origin of $\mathbb{C}^n$ is a level set of a real analytic function ρ in a neighborhood U of zero with nowhere vanishing gradient. I.e., $M = \{z \in U : \rho(z) = 0\}$ and $\nabla\rho(z) \neq 0\ \forall z \in U$. We shall usually suppose that $0 \in M$ and consider a germ $(M,0)$. Sometimes it is convenient to consider ρ as a convergent power series in $(z,\bar{z})$. A germ $(M,0)$ is called *real algebraic* if the defining function ρ can be taken to be a polynomial. A germ $(M,0)$ is called Levi non-degenerate at zero if the Levi form

$$L_\rho(0) = \left(\frac{\partial^2\rho(0)}{\partial z_k \partial \bar{z}_j}\right)_{k,j=1,\ldots,n} \tag{3.2}$$

is non-degenerate. If (M,p) and (M',p') are germs of real hypersurfaces in $\mathbb{C}^n$ and $\mathbb{C}^{n'}$ respectively then by a germ of a holomorphic mapping $f : (M,p) \to (M',p')$ we understand a germ of a holomorphic mapping $f : (\mathbb{C}^n,p) \to (\mathbb{C}^{n'},p')$ such that $f(M) \subset M'$. Recall that a proper holomorphic correspondence between D and D' is an irreducible analytic subset $\Gamma \subset D \times D'$ such that both projections $\mathsf{pr}_1|_\Gamma : \Gamma \to D$ and $\mathsf{pr}_2|_\Gamma : \Gamma \to D'$ are finite proper mappings. Now let us state two closely related results.

Theorem 3.5. *(S. Webster,* [We]*). Let $f : (M,0) \to (M',0)$ be a germ of a biholomorphic mapping between the germs of a Levi non-degenerate real algebraic hypersurfaces in $\mathbb{C}^n$, $n \geqslant 2$. Then the graph Γ_f of f is an open subset of a complex affine subvariety of $\mathbb{C}^n \times \mathbb{C}^n$.*

Theorem 3.6. *(R. Shafikov,* [Sf]*). Let D and D' two relatively compact, smoothly bounded domains in $\mathbb{C}^n$ with real algebraic boundaries. There exists a proper holomorphic correspondence* *between D and D' if and only if there exist points $p \in \partial D$ and $p' \in \partial D'$ and a germ of a biholomorphism $f:(M,p) \to (M',p')$.*

Remark 3.2. **a)** These results reduce the problem of global classification of bounded domains with real algebraic boundaries to a local one, and this was solved in [CM], also the real analytic case.
b) Correspondences naturally appear in the context of both theorems. Take, for example,

$$D = \{|z|^2 + |w|^2 < 1\} \quad \text{and} \quad D' = \{|z'|^2 + |w'|^4 < 1\}.$$

Mapping $f(z',w') = (z',w'^2)$ is a proper holomorphic mapping from D' to D which is locally biholomorphic on $D' \setminus \{w' = 0\}$. Therefore for any point $p \in D \setminus \{w = 0\}$ a germ of the inverse f^{-1} will be a germ of a biholomorphism between ∂D and $\partial D'$ which extends only to a correspondence. Finally:

Theorem 3.7. *(Pinchuk-Sukhov, [PS]). Let $(M,0)$ and $(M',0')$ be the germs of real analytic strictly pseudoconvex hypersurfaces, $n' \leqslant 2n$, and let $f:(M,0) \to (M',0')$ be a germ of a smooth CR-mapping. Then f holomorphically extends to a neighborhood of the origin.*

Recall that a smooth function/mapping $f: M \to \mathbb{C}$ is called CR (*i.e.,* Cauchy-Riemann), if it satisfies the Cauchy-Riemann equation in every complex tangential direction. Wether this statement holds without the assumption $n' \leqslant 2n$ is still open.

Remark 3.3. The principal tool in the proof of latter three theorems are Segre varieties, introduced by B. Segre in [Sg1] and reintroduced to the subject by S. Webster in [We]. These are germs of complex hypersurfaces

$$Q_\zeta := \{z : \rho(z,\bar{\zeta}) = 0\}, \tag{3.3}$$

where ρ is a defining function of the germ $(M,0)$. One observes that holomorphic germ $f:(M,0) \to (M',0')$ sends Q_ζ to $Q_{f(\zeta)}$. After that one extends f along Segre varieties first and then using "reflection" $\zeta \to Q_\zeta$ both in the source and in the target extends the map. We send an interested reader to surveys [BER], [DP3] and [Vit] for much more results on this subject.

4. Analytic objects and extensions along pseudoconvex exhaustions

It will convenient to formalize the notion of a *"complex analytic object"*, as well as state some general properties of these objects.

4.1. Analytic objects and maximal extensions. Let us denote by $\mathcal{T}_{lc}$ the category of *locally connected, Hausdorff* topological spaces and let $\mathcal{T}_0$ be some subcategory of $\mathcal{T}_{lc}$. About subcategory $\mathcal{T}_0$ we shall assume the following: if $X \in \mathcal{T}_0$ and if (D,φ) is a domain *over* X then $D \in \mathcal{T}_0$. In our applications $\mathcal{T}_0$ will be:

- the category of smooth real analytic manifolds $\mathcal{R}$.
- the category of reduced complex spaces $\mathcal{C}$.

Recall that a sheaf of sets over $X \in \mathcal{T}_{lc}$ is a triple $(\mathcal{S},\pi,X)$, where $\mathcal{S}$ is a topological space and $\pi : \mathcal{S} \to X$ is a surjective local homeomorphism. By $\mathcal{S}_x$ we denote the stalk of $\mathcal{S}$ at $x \in X$.

Definition 4.1. *A category of analytic objects $\mathcal{A}$ over $\mathcal{T}_0$ is defined by the following data:*

AO1) for every $X \in \mathcal{T}_0$ a sheaf of sets $\mathcal{A}^X$ over X is specified, we write it as $(\mathcal{A}^X,\pi_X,X)$ in more details;

AO2) the family $\{\mathcal{A}^X : X \in \mathcal{T}_0\}$ is coherent in the sense that for every morphism $f: X \to Y$ in $\mathcal{T}_0$ the inverse image $f^{-1}\mathcal{A}^Y$ is a subsheaf of $\mathcal{A}^X$.

Recall that the inverse image of a sheaf $(\mathcal{A}^Y,\pi_Y,Y)$ under a continuous map $f: X \to Y$ is the fiber product

$$X \times_Y \mathcal{A}^Y := \{(x,s) \in X \times \mathcal{A}^Y : f(x) = \pi_Y(s)\}$$

with a natural projection to X which is obviously a surjective local homeomorphism, *i.e.,* $f^{-1}\mathcal{A}^Y$ is a sheaf. By saying that $f^{-1}\mathcal{A}^Y$ is a subsheaf of $\mathcal{A}^X$ we mean as usually that an inclusion $\iota_f : f^{-1}\mathcal{A}^Y \to \mathcal{A}^X$ is specified. We shall ask from ι the following

AO3) if $f : X \to Y$ is a locally homeomorphic morphism in $\mathcal{T}_0$ then ι_f is bijective.

With some abuse of notations we shall write simply that $f^{-1}\sigma \in \Gamma(f^{-1}(U), \mathcal{A}^X)$ for $\sigma \in \Gamma(U, \mathcal{A}^Y)$. A good example for the beginning is the category $\mathcal{C}_T$ of continuous mappings with values in some fixed topological space T. I.e., $\mathcal{C}_T^X$ is the sheaf of sets of continuous mappings from X to T. In practice one rather rarely needs to work the whole category $\mathcal{A}$, but only with the sheaf $\mathcal{A}^X$ for some given X. The last will be called the *sheaf of analytic objects on* X. Sections of $\mathcal{A}^X$ on some open $D \subset X$ will be called simply an *analytic object* on D. We are interested in maximal domains of existence of a given analytic object or, a given family of analytic objects. As it is well known this task requires, in general, to consider analytic objects *over* a given space X. If $\mathcal{A}^X$ is a sheaf of analytic objects on X then via the axioms (AO2) and (AO3) for every domain (D, φ) over X we have the sheaf $\mathcal{A}^D \cong \varphi^{-1}\mathcal{A}^X$ and its global sections will be called analytic objects *over* X, *i.e.*, an analytic object over X is a triple (D, φ, σ) where (D, φ) is a domain over X and σ is a section of $\varphi^{-1}\mathcal{A}^X$. Recall that a mapping $\psi : (D, \varphi) \to (D', \varphi')$ is called a mapping *over* X if it commutes with projections, *i.e.*, $\varphi' \circ \psi = \varphi$. Such ψ is necessarily a local homeomorphism.

Remark 4.1. *Cartan-Thullen construction,* [CT]. Denote by I some non-empty set of indices. A family of analytic objects $(D', \varphi', \{\sigma'\}_{i\in I})$ is an *extension* of the family $(D, \varphi, \{\sigma_i\}_{i\in I})$ if there exists a mapping $\psi : D \to D'$ over X such that for every $i \in I$ one has $\psi^{-1}\sigma_i' = \sigma_i$. Denote by $\mathcal{E}(D, \varphi, \{\sigma_i\}_{i\in I})$ the category of all extensions of $(D, \varphi, \{\sigma_i\}_{i\in I})$ with obvious morphisms between extensions. The final object of this category is called the *maximal extension* of $(D, \varphi, \{\sigma_i\}_{i\in I})$ and is denotes as $(\hat{D}, \hat{\varphi}, \{\hat{\sigma}_i\}_{i\in I})$. It is easy to see that the maximal extension, if exists, is unique. Indeed, let $(D', \varphi', \{\sigma_i'\}_{i\in I})$ be another maximal extension and let $\hat{\psi} : D \to \hat{D}$, $\psi' : D \to D'$ be corresponding mappings over X. By maximality there exist $f' : D' \to \hat{D}$ and $\hat{f} : \hat{D} \to D'$ such that $\hat{f} \circ \hat{\psi} = \psi'$ and $f' \circ \psi' = \hat{\psi}$. Take some $x \in D$ and let $y = \psi'(x) \in D'$. Then $f'(y) = (f' \circ \psi')(x) = \hat{\psi}(x)$ and therefore $(\hat{f} \circ f')(y) = (\hat{f} \circ \hat{\psi})(x) = \psi'(x) = y$. I.e., the set of $y \in D'$ such that $(\hat{f} \circ f')(y) = y$ is not empty. This set is obviously open, because both $\hat{f}$ and f' are mappings over X. Finally it is obviously closed. Therefore $f' : D' \to \hat{D}$ is a homeomorphism.

Definition 4.2. *We shall say that a category $\mathcal{A}$ of analytic objects possesses the uniqueness property (or, satisfies the uniqueness theorem) if for every connected $X \in \mathcal{T}_0$ and any two sections σ_1, σ_2 of $\mathcal{A}^X$ such that $\sigma_1|_V = \sigma_2|_V$ for some non-empty open $V \subset X$ one has $\sigma_1 = \sigma_2$ on X.*

Let us point out that if $\sigma_1(x) = \sigma_2(x)$ for some $x \in X$ then they are equal in some neighborhood of x by the very definition of a section of a sheaf. Here one requires more: σ_1 and σ_2 should be equal everywhere. Category $\mathcal{C}_T$ of continuous mappings with values in T doesn't satisfy the uniqueness theorem (=uniqueness property). Category of real analytic mappings $\mathcal{M}_T$ with values in a real analytic manifold T is a category over $\mathcal{R}$ which satisfies the uniqueness theorem.

Theorem 4.1. *(Cartan-Thullen). Let $\mathcal{A}$ be a category of analytic objects over a category $\mathcal{T}_0 \subset \mathcal{T}_{lc}$. Then the following assertions are equivalent:*

i) for every $X \in \mathcal{T}_0$ and every family $(D, \varphi, \{\sigma_i\}_{i\in I})$ of sections of $\mathcal{A}^X$ over X there exists a maximal extension;

ii) for every $X \in \mathcal{T}_0$ every section of $\mathcal{A}^X$ over X admits a maximal extension;

iii) category $\mathcal{A}$ possesses the uniqueness property;

iv) for every $X \in \mathcal{T}_0$ the total space of $\mathcal{A}^X$ is Hausdorff.

Proof. (i) obviously implies *(ii)* . Let us prove that *(ii)* implies the uniqueness property, *i.e.*, *(iii)* . Let a connected $X \in \mathcal{T}_0$, non-empty open $V \subset X$ and sections $\sigma_1, \sigma_2 \in \Gamma(X, \mathcal{A}^X)$ be such that $\sigma_1|_V = \sigma_2|_V$. By assumption there exists a maximal extension (D, φ, σ) of $(V, \mathsf{Id}, \sigma_1|_V) = (V, \mathsf{Id}, \sigma_2|_V)$. Because of maximality of (D, φ, σ) there exist for $i = 1,2$ locally homeomorphic mappings $\psi_i : X \to D$, commuting with projections, such that $\psi_i^{-1}\sigma = \sigma_i$. Connectivity of X obviously implies that $\psi_1 = \psi_2$. Therefore we get that $\sigma_1 = \psi_1^{-1}\sigma = \psi_2^{-1}\sigma = \sigma_2$. Uniqueness property is proved.

Now let us prove that *(iii)* implies *(i)* . Fix some domain (D,φ) over some $X\in\mathcal{T}_0$ and some family $\{\sigma_i\}_{i\in I}$ of sections of $\mathcal{A}^X$ on D. denote by $\mathcal{A}_I^X$ the direct product of I copies of $\mathcal{A}^X$. The total space of $\mathcal{A}_I^X$ we denote with same letter. The base $\mathcal{B}_I$ of the topology on $\mathcal{A}_I^X$ is defined as follows: for an open $U\subset X$ and a family $\{\sigma_i\}_{i\in I}$ of sections of $\mathcal{A}^X$ on U set $\tilde{U}=\bigcup_{x\in U}\prod_{i\in I}\sigma_i(x)$. To check that $\mathcal{B}_I$ is a base take another open $\tilde{V}=\bigcup_{x\in V}\prod_{i\in I}\tau_i(x)$ such that $\tilde{V}\cap\tilde{U}\neq\varnothing$. This means that there exists $z\in V\cap U$ such that $\sigma_i(z)=\tau_i(z)$ for all $i\in I$. Let $z\in W\subset V\cap U$ be some connected neighborhood of z. By uniqueness property we have that $\sigma_i|_W=\tau_i|_W$. Therefore $\tilde{V}\cap\tilde{U}\supset\tilde{W}:=\bigcup_{x\in W}\prod_{i\in I}\sigma_i(x)$.

The natural projection $\pi:\mathcal{A}_I^X\to X$ is locally homeomorphic and $(D,\varphi,\{\sigma_i\}_{i\in I}))$ is naturally mapped to $(\mathcal{A}_I^X,\pi)$, *i.e.*, there exists a locally homeomorphic map $i:D\to\mathcal{A}_I^X$ commuting with projections φ and π. Let $\hat{D}$ be the connected component of $i(D)$ in $\mathcal{A}_I^X$. It easy to check that $(\hat{D},\pi|_{\hat{D}})$ is the required maximal extension.

The last item to prove is that the uniqueness property *(iii)* is equivalent to the separability of the topology on $\mathcal{A}^X$, *i.e.*, to *(iv)* . Indeed, let $\sigma_1(x)\neq\sigma_2(x)$ for some $x\in X$. Take a connected neighborhood U of x. Then for every $y\in U$ we have that $\sigma_1(y)\neq\sigma_2(y)$ by uniqueness. Vice versa, suppose that uniqueness theorem fails for sections of $\mathcal{A}^X$, *i.e.*, there exist $\sigma_1\neq\sigma_2\in\Gamma(X,\mathcal{A}^X)$ such that $\sigma_1|_V=\sigma_2|_V$ for some open non-empty $V\subset X$. Therefore the set W of $x\in X$ such that $\sigma_1(x)=\sigma_2(x)$ is non-empty and is obviously open. But it is not the whole of X. Take some $y\in\partial W$, *i.e.*, $\sigma_1(y)\neq\sigma_2(y)$ but every neighborhood of y intersects W, where these sections coincide. I.e., $\mathcal{A}^X$ is not Hausdorff. □

4.2. Real and complex analytic objects: examples. Let us adopt the following

Definition 4.3. *A category of analytic objects over the category $\mathcal{R}$ will be called a category of real analytic objects. A category of analytic objects over the category $\mathcal{C}$ will be called a category complex analytic objects.*

Sections of $\mathcal{A}^X$ will be called *real* or *complex* analytic objects on X. For the time being the reader could think about the following couple of examples:

- $\mathcal{O}_Y$ is the category of holomorphic maps with values in a complex space Y, $\mathcal{M}_Y$ stays for the category of meromorphic mappings with values in Y with the usual pull-backs as inverse transforms in both cases. I.e., $\mathcal{O}_Y^X$ (resp. $\mathcal{M}_Y^X$) is the sheaf of germs of holomorphic (resp. meromorphic) mappings from open subsets of X to Y. These categories do satisfy the uniqueness theorem and therefore for them the maximal extensions do exist.
- The category $\mathcal{A}$ of analytic sets, *i.e.*, $\mathcal{A}^X$ is a sheaf of locally closed analytic sets in X without taking into account multiplicities. Inclusions ι in these examples are obvious.
- The category $\mathcal{C}oh$ of coherent analytic sheaves, *i.e.*, $\mathcal{C}oh^X$ (or, $\mathcal{C}oh(X)$ in standard notations) is the sheaf of sets which elements are coherent analytic sheaves on X. Two sheaves $\mathcal{F}_1$ and $\mathcal{F}_2$ on X are said to be equal, *i.e.*, represent the same element of $\mathcal{C}oh^X$ if they are isomorphic. In particular, for a morphism $f:X\to Y$ (*i.e.*, a holomorphic mapping) and a coherent analytic sheaf $\mathcal{F}$ on Y the inclusion is $\iota_f(f^{-1}\mathcal{F})=\mathcal{O}_X\otimes_{f^{-1}\mathcal{O}_Y}f^{-1}\mathcal{F}$, the so called analytic inverse image, which is coherent if $\mathcal{F}$ was such. Categories $\mathcal{A}$ and $\mathcal{C}oh$ do not satisfy the uniqueness theorem and therefore maximal extensions for them in general do not exist.

By a q-concave Hartogs figure in $\mathbb{C}^n$ we shall understand the following domain

$$H_r^{n,q}:=A_{1-r,1}^{n-q}\times\Delta^q\cup\Delta^{n-q}\times\Delta_r^q. \tag{4.1}$$

Here $A_{1-r,1}^q:=\Delta^q\backslash\overline{\Delta}_{1-r}^q$ denotes a "generalized annulus" of dimension q. Note that $H_r^{n,n-1}=H_r^n$ in our notations. Correspondingly $H_r^{n,n-q}$ will be called a q-convex Hartogs figure.

Definition 4.4. *We say that a category of complex analytic objects $\mathcal{A}$ is a q-Hartogs category (for some $q\geqslant 1$) if the q-Hartogs-type extension lemma is valid for objects of $\mathcal{A}$, i.e., if the natural transform, which corresponds to the canonical inclusion $H_r^{n,n-q}\subset\Delta^n$ is surjective for every*

$n > q$ and any $0 < r < 1$. In other words this means that every analytic object $\sigma \in \Gamma(H_r^{n,n-q}, \mathcal{A})$ has an extension $\tilde{\sigma} \in \Gamma(\Delta^n, \mathcal{A})$.

Remark 4.2. a) Category $\mathcal{M}_{\mathbb{P}^1}$ is 1-Hartogs (theorem of Levi), while is $\mathcal{C}oh$ not. But it has a subcategory of sheaves satisfying the gap-sheaf condition which is 2-Hartogs, see Theorem 14.1. By Theorem 8.1 the category $\mathcal{M}_Y$ for a disk-convex Kähler space Y is 1-Hartogs too.

b) A (singular) holomorphic foliation of codimension d on X is defined by a meromorphic section of the appropriate Grassmann bundle $Gr(TX)$. The products $U \times Gr(T_xX) \equiv Gr(TX)|_U$, where U is a local chart, are projective. Therefore the 1-Hartogs extension works. Involutibility, being an analytic condition, is preserved by extension. Uniqueness condition is also obvious. Therefore the category $\mathcal{F}_d$ of codimension d singular holomorphic foliations is 1-Hartogs and possesses maximal extensions.

c) Roots of holomorphic line bundles do satisfy the 1-Hartogs property but do not satisfy the uniqueness theorem, see section 15.

d) Solutions of a general (*i.e.*, non-holomorphic) elliptic systems do satisfy uniqueness theorem but are not necessarily Hartogs in general. We shall not speak about real analytic objects in this text any more.

e) The interested reader may look to the recent survey [McK] and find out an impressive number of examples of complex analytic objects.

4.3. Extension accross q-concave boundary points. Along our exposition we adopt the following strategy. First, we look to the model situation: whether a given analytic object can be extended from a Hartogs figure $H_r^{n,q}$ of a appropriate concavity to the associated polydisk Δ^n? Then we use various ways to deduce from such local statement the global ones. Two such methods were already explained: the Docquier-Grauert theorem and bumping. Now we want to discuss one more: extension of analytic objects along the levels of an appropriate exhaustion functions. Let U be a domain in $\mathbb{C}^n$ and ρ a real-valued function on U of class $\mathcal{C}^2$.

Definition 4.5. *Function ρ is called strongly q-convex at $z_0 \in U$ if the Levi form $L_\rho(z)$, see (3.2), has at least $n-q+1$ positive eigenvalues.*

We say that ρ is q-convex in U if it is q-convex at every $z \in U$. One is of course interested not with q-convex functions themselves but with their level sets. In the case when $\nabla\rho(z) \neq 0$ one can always take a composition of $h \circ \rho$ with a sufficiently convex growing function h in such a way that $L_{h(\rho)}(z)$ will be positive along $\nabla\rho(z)$. Therefore q-convexity means that there exists at least $(n-q)$ positive eigenvalues which correspond to eigenvectors *tangent* to $\Sigma_0 := \{\rho = 0\}$. Now it is easy to see that if ρ is strongly q-convex at z_0 then one can imbed a polydisk $i : \Delta^n \to U$ in such a way that $z_0 \in i(\Delta^n)$ and $i(H_r^{n,q}) \subset \{\rho > 0\}$. Here $H^{n,q}$ stands for a q-concave Hartogs figure, see (4.1). The same can be done also when z_0 is a *critical* point of ρ, see Appendix I for example. The property of being q-convex doesn't depend on the choice of a coordinate system and therefore can be translated to real functions on complex manifolds. Moreover it is given in such a fashion that strong q-convexity is preserved under the restrictions onto a complex submanifolds, *i.e.*, if ρ is strongly q-convex on X and Y is a complex submanifold of X then $\rho|_Y$ is strongly q-convex on Y. This leads to the following

Definition 4.6. *A real-valued function ρ on a complex space X is called strongly q-convex at $x_0 \in X$ if there exists an open neighborhood $U \ni x_0$ and a holomorphic imbedding $h : U \to \tilde{U}$ into an open subset of some $\mathbb{C}^n$, and a strongly q-convex at $h(x_0)$ function $\tilde{\rho}$ on $\tilde{U}$ such that $\rho = \tilde{\rho} \circ h$.*

Let $\rho : X \to \mathbb{R}$ be an continuous exhaustion function, *i.e.*, such that for every $t \in \mathbb{R}$ the lower level set $X_t^- := \{\rho < t\}$ is relatively compact. If such ρ can be taken to be strongly q-convex we call X *q-complete*. Let D be a domain in a complex space X and $x_0 \in \partial D$. D is said to be *q-concave* at x_0 if there is a neighborhood $U \supset x_0$ and a strongly q-convex at x_0 function $\rho : U \to \mathbb{R}$ such that $D \cap U = \{x \in U : \rho(x) > 0\}$. We are going to discuss few general principles.

GP 1. If the category $\mathcal{A}$ of complex analytic objects is an $(n-q)$-Hartogs category then analytic objects from $\mathcal{A}$ tend to extend across the q-concave boundary points.

By saying "tend to extend“ we mean simply that this is not always true. If X is smooth at x_0 and the analytic object in question satisfies the uniqueness theorem then *GP1* is certainly true. Therefore it holds for holomorphic/meromorphic functions and holomorphic mappings for example. But, as we shall see in section 15, it turns to be wrong for the roots of holomorphic line bundles (they do not satisfy the uniqueness theorem). If X is singular at x_0 then the situation becomes even worse. Let E be a curve in $\mathbb{P}^2$ of degree 3, *i.e.*, an elliptic curve. Its self-intersection is 9. Blow up 10 points on E to make its self-intersection negative. Then there exists a 1-convex (and therefore disk-convex) neighborhood Y of E which can be blown down to normal space X with one singular point x_0, the image of E. A holomorphic inclusion $i : X \setminus \{x_0\} \to Y$ doesn't extend to x_0 holomorphically despite of the fact that Y is Kähler and doesn't contain rational curves. The last follows from Lemma 6.1.

GP 2. If analytic objects from $\mathcal{A}$ extend across q-concave boundary points and, in addition, obey an appropriate uniqueness theorem then they extend along q-convex exhaustions.

By an "appropriate" uniqueness theorem we mean for example that if σ_1,σ_2 are two objects from $\mathcal{A}^U$ for some neighborhood U of a q-concave point $x_0 \in \{\rho = 0\}$ such that $\sigma_1|_{\rho>0} = \sigma_2|_{\rho>0}$ then $\sigma_1 = \sigma_2$. See subsection 5.1 about analytic set for a meaningful example of a such situation. Now let $\rho : X \to (a,b)$ is a proper strongly q-convex exhaustion function and suppose our object σ is extended to $X^+_{t_0} := \{\rho(x) > t_0 \in (a,b)\}$. Applying *GP1* we extend σ locally across every boundary point and then, using uniqueness, we can glue these extensions and obtain the extension of σ to $X^+_{t_1}$ for some $t_1 < t_0$.

GP 3. If analytic objects from $\mathcal{A}$ extend across 1-concave boundary points in dimension q then they extend across $(n-q)$-concave boundary points in dimension n.

This can be seen by placing Hartogs domains appropriately from the pseudoconcave side, see Appendix I.

4.4. Bumping. Let D be a relatively compact domain in a q-complete complex space X and let K be a compact in D with connected complement.

GP 4. If the category $\mathcal{A}$ of complex analytic objects is $(n-q)$-Hartogs and satisfies an appropriate uniqueness theorem then every analytic object $\sigma \in \Gamma(D \setminus K, \mathcal{A}^X)$ extends to D.

Remark 4.3. In [Be] it is proved that for every reduced, Hausdorff complex space with countable basis Morse functions are dense in the space of smooth functions. In particular, if X admits a strongly q-convex exhaustion function ρ then it can be always supposed to be Morse and, moreover, that for any $t \in (a,b)$ the level set $\Sigma_t := \{\rho(x) = t\}$ contains at most one critical point. The property to be Morse is local and therefore we can suppose that X is an analytic subset of some domain $\tilde{U}$ in $\mathbb{C}^n$. Let $X = X^1 \supset X^2 \supset ...$ be the usual stratification of X by the successive singular loci. Connected components S^α_j of $X^\alpha \setminus X^{\alpha+1}$ are called strata of X. $\{S^i_j\}$ form a locally finite system in $\tilde{U}$ and the boundary ∂S^i_j of each stratum is a disjoin union of lower dimensional strata. If $x \in S^\alpha_j \subset \partial S^\beta_i$ then $T_x(S^\alpha_j, S^\beta_i)$ is defined to be the set of $\mathsf{dim}\, S^\beta_i$-dimensional complex planes H through x which are limits of tangent planes $T_y S^\beta_i$ when $y \in S^\beta_i$ tends to x. A smooth function ρ on $\tilde{U}$ is called a Morse function on X if its restriction to the positive dimensional strata is Morse in the standard sense and if for every $x \in S^\alpha_j \subset \partial S^\beta_i$ the restriction $d\rho(x)|_H$ doesn't vanish for every $H \in T_x(S^\alpha_j, S^\beta_i)$. Remark that local minima of a Morse function on a complex space are isolated. This is important for bumping method to work. For example one has the following:

Corollary 4.1. *Let D be a domain in a $(n-1)$-complete, normal complex space X and let $K \Subset D$ be a compact with connected complement. Then every meromorphic function f in $D \setminus K$ extends to a meromorphic function $\tilde{f}$ in D.*

Let $p_1,...,p_M \in X$ be the minima of the exhaustion ρ, *i.e.*, $\rho(x) = 0$ only for $x = p_1,...,p_M$. Fix some $\delta > 0$ and let S be the set of $s \in \mathbb{R}^+$ such that the assertion holds for compacts in $X^-_s \cap D$ such that $\mathsf{dist}(K,\partial D) > \delta$, here $X^-_s := \{\rho < s\}$. S contains a neighborhood of the origin. Indeed, if $p_1,...,p_M \notin D$ or, $\mathsf{dist}(p_i,\partial D) < \delta$ there is nothing to prove. Otherwise fix some $p_i \in D$. For s small enough (D,p_i) imbeds to $(\mathbb{C}^N,0)$. Now extension can be performed along the level sets of $\|z\|^2$ as an exhaustion function. S is also obviously closed. To see that S is open fix some $s_0 \in S$. Take some $s > s_0$ and cover $\Sigma_s := \{\rho = s\}$ by open sets $\{U_j\}$ such that for every U_j there exists a cover $\pi_j : U_j \to \Delta^n$ as in the Projection Lemma 18.1. Every restriction $f|_{U_j \cap X_s^+}$ extends to a neighborhood of $\Sigma_s \cap U_j$. These extensions are univalent because the eventual ramification divisors do not intersect $\Sigma_s \cap U_j$ and therefore are empty. Moreover, these extensions and coincide one with another due to the uniqueness theorem. I.e., f extends itself to a neighborhood of $\Sigma_s \cap U_j$. For s close enough to s_0 we have that $\Sigma_{s_0} \cap D_\delta \subset \bigcup_j U_j$ for some finite collection of j-s, here $D_\delta := \{x \in D : \mathsf{dist}(,\partial D) \geqslant \delta\}$. Now let $K \Subset D_\delta \cap X_s$ be our compact. By local extension proved above and once more by uniqueness f uniquely extended to $(D \setminus K) \cup \bigcup_j U_j$. Therefore $s \in S$. Corollary is proved.

Remark 4.4. This Corollary was recently proved in [MP] by a different method. We shall extend this corollary to the case of meromorphic mappings and analytic sets, see Corollaries 5.2 and 9.7.

Chapter II. Analytic Sets and Meromorphic Mappings

5. Analytic sets, ramified covers and currents

5.1. Extension properties of analytic sets. An analytic subset in a complex space X is a closed subset $A \subset X$ such that for every point $a \in A$ there exists a neighborhood $U \ni a$ and a finite number of holomorphic functions $h_1,...,h_k \in \mathcal{O}(U)$ such that $A = \{z \in U : h_1(z) = ... = h_k(z) = 0\}$. A *locally closed* analytic subset of X is a subset $A \subset X$, which is an analytic subset of some open subset U of X. Let us start with an example of a non-extendable analytic set.

Example 5.1. Take a smooth, closed Jordan curve $\gamma : [0,1] \to \Delta$ which is not real analytic at any of its points. Denote by D^+ the interior and by $D^- = \mathbb{P}^1 \setminus \bar{D}^+$ the exterior of γ. Consider the function $r(t) = \sqrt{1-|\gamma(t)|^2}$. Let g be the solution of the Dirichlet problem on D^- with boundary condition $g|_\gamma = \ln r$. Denote by h the harmonic conjugate to g, h is also smooth on $\bar{D}^-$. Let A' be the graph $(z,f(z))$ of the holomorphic function $f = \exp(g+ih)$ in $\mathbb{C}^2$. It is not difficult to see that A' closes to a complex curve in $\mathbb{P}^2 \setminus \bar{\mathbb{B}}^4$. Indeed, let $[Z_0 : Z_1 : Z_2]$ be homogeneous coordinates in $\mathbb{P}^2$. In the affine chart $\mathbb{C}^2 = U_0 = \{Z_0 \neq 0\}$ take the standard affine coordinates $z_1 = Z_1/Z_0$ and $z_2 = Z_2/Z_0$. We have that $z_2 \to f(0)$ when $z_1 \to \infty$. Therefore in the affine chart $U_1 = \{Z_1 \neq 0\}$ we shall have: $Z_0/Z_1 \to 0$ and $Z_2/Z_1 \to 0$ on A', which means that $A = A' \cup \{0\}$ is an analytic set. We got an analytic subset (a complex curve, in fact) $A \subset \mathbb{P}^2 \setminus \bar{\mathbb{B}}^4$ which closes to a compact surface with boundary $\Gamma(t) = (\gamma(t),r(t))$ on the unit sphere. But since the curve Γ is not real analytic at any of its points our A cannot be extended as an analytic subset to a neighborhood of any point of Γ.

However, if the dimension of an analytic set $A \subset D$ under the extension is bigger then the degree of concavity of the domain D then A extends to a neighborhood of the corresponding concave point of ∂D. As a model domain consider a q-concave Hartogs figure in $\mathbb{C}^n$ defined by (4.1). One has the following

Theorem 5.1. *(W. Rothstein,* [Rt2]*). Let A be a purely $(q+1)$-dimensional analytic subset of the q-concave Hartogs figure $H_r^{n,q}$, $q \geqslant 1$. Then A extends uniquely to a purely $(q+1)$-dimensional analytic subset $\tilde{A}$ of the unit polydisk Δ^n.*

Example 5.1 shows that this result is precise. It shows that a complex hypersurface A in Hartogs figure $H_r^n = H_r^{n,n-1}$ doesn't necessary extend to Δ^n, but it does extend to Δ^n from $H_r^{n,n-2}$. The situation with hypersurfaces can be made fairly precise.

Theorem 5.2. *(G. Dloussky,* [Dl]*). Let D be a domain in a Stein manifold, $\tilde{D}$ its envelope of holomorphy and let A be a complex hypersurface D. Then the envelope of holomorphy of $D \setminus A$ is either $\tilde{D}$ or, $\tilde{D} \setminus \tilde{A}$, where $\tilde{A}$ is a complex hypersurface extending A.*

And more generally:

Theorem 5.3. *(E. Chirka, [Ch2]). Let D be a domain in a Stein manifold, $\tilde{D}$ its envelope of holomorphy and A a closed pluripolar subset of D. Then the envelope of holomorphy of $D \setminus A$ is of the form $\tilde{D} \setminus \tilde{A}$, where $\tilde{A}$ is a closed pluripolar (possibly empty) subset of $\tilde{D}$.*

Let us give a corollary from the Rothstein's theorem.

Corollary 5.1. *Let D be a domain in $\mathbb{C}^n$ and A an analytic subset of $D \setminus \mathbb{R}^n$ whose every branch has dimension $\geqslant 2$. Then $\bar{A} \cap D$ is an analytic subset of D.*

The proof goes as follows. For every point $x \in \mathbb{R}^n \cap D$ one can put an 1-concave Hartogs figure $H_r^{n,1}$ to $D \setminus \mathbb{R}^n$ in such a way that the associated polydisk will contain x, see Lemma 2.20 in [Si3]. The result follows now from Theorem 5.1. The case of $\mathsf{dim}\, A = 1$ was considered in section 3. One more result of that type is the following theorem of Thullen-Remmert-Stein.

Theorem 5.4. *Let S is an analytic subset in a complex space X of dimension $q \geqslant 0$ and let G be an open subset of X which contains $X \setminus S$ and intersects each branch of S of dimension q. Let A be a pure q-dimensional analytic subset of G. Then the closure $\bar{A}$ of A is an analytic subset of X.*

In the case of analytic sets GP 1 and 2 work due to the appropriate mentioned uniqueness property, see Theorem 8.3 in [ST2] and gives the following.

Theorem 5.5. *(W. Rothstein, [Rt2], H. Fujimoto, [Fu1]). Purely $(q+1)$-dimensional analytic sets extend across q-concave boundary points on reduced complex spaces. Moreover, they extend along q-convex exhaustions.*

Local extension follows from Theorem 5.1 together with Projection Lemma 18.1. An important point is that this local extension is unique. After that one can glue the local extensions to obtain the global one. This implies a result in the spirit of Theorem 1.6 for analytic sets.

Corollary 5.2. *Let D be a relatively compact domain in a q-complete, reduced complex space X and let K be a compact in D with connected complement. Then every purely $(q+1)$-dimensional analytic subset of $D \setminus K$ uniquely extends to a purely $(q+1)$-dimensional analytic subset of D.*

The proof goes by bumping similarly to that of Corollary 4.1. We end up with the following two related results.

Theorem 5.6. *(E. Bishop, [Bs1]). Let S be an analytic subset in an open set $\Omega \subset \mathbb{C}^n$ and let A be an analytic subset of $\Omega \setminus S$ of locally finite volume near S. Then $\bar{A}$ is an analytic subset of Ω.*

Denote by B_r the ball of radius r in $\mathbb{C}^n$ centered at the origin. Now one deduces the following.

Theorem 5.7. *(W. Stoll, [Stl]). A subvariety X in $\mathbb{C}^n$ of pure dimension $p \geqslant 0$ is algebraic if and only if there exists a constant $C > 0$ such that for every $r \gg 0$*

$$\mathsf{vol}_{2p}(X \cap B_r) \leqslant C r^{2p}. \tag{5.1}$$

Indeed, the condition (5.1) means exactly that X regarded as an analytic set in $\mathbb{C}^n = \mathbb{P}^n \setminus \mathbb{P}^{n-1}$ has bounded volume with respect to the Fubini-Study metric. For more details and proofs of the results mentioned in this subsection we refer to [Ch1] and [Si3].

5.2. Extension of analytic covers. Recall that a *regular cover* is a locally homeomorphic map $\mathsf{c} : \tilde{X} \to X$ between Hausdorff topological spaces such that for every $x_0 \in X$ there exists a neighborhood $U_0 \ni x_0$ such that its preimage $\mathsf{c}^{-1}(U_0)$ is at most countable disjoint union of its connected components U_i and for every i the restriction $\mathsf{c}|_{U_i} : U_i \to U_0$ is a homeomorphism. As it is well known (and obvious) if $\mathsf{c} : \tilde{X} \to X$ is a regular cover then for every path $\gamma : [0,1] \to X$ and every $a \in \tilde{X}$ such that $\mathsf{c}(a) = \gamma(0)$ there exists a unique lift $\tilde{\gamma}$ of γ starting at a, *i.e.*, a path $\tilde{\gamma} : [0,1] \to \tilde{X}$ such that $\tilde{\gamma}(0) = a$ and $(\mathsf{c} \circ \tilde{\gamma})(t) = \gamma(t)$ for all $t \in [0,1]$.

An *unramified analytic cover* of complex spaces is by definition a regular cover of normal complex spaces. One can consider two different natural notions of extension of unramified covers. Let $(\tilde{X}, \mathsf{c}_X, X)$ be an unramified cover over X. One says that it extends to an unramified cover $(\tilde{Y}, \mathsf{c}_Y, Y)$ over Y if there exists a holomorphic imbedding $\varphi : X \to Y$ and a holomorphic imbedding/immersion $\Phi : \tilde{X} \to \tilde{Y}$ such that the following diagram

$$\begin{array}{ccc} \tilde{X} & \stackrel{\Phi}{\to} & \tilde{Y} \\ \downarrow \mathsf{c}_X & & \downarrow \mathsf{c}_Y \\ X & \stackrel{\varphi}{\to} & Y \end{array} \tag{5.2}$$

is commutative. Depending on whereas Φ is an *imbedding* or an *immersion* one gets the different notions of the extension of covers.

Example 5.2. Take as $X = \mathbb{C}^2 \setminus \mathbb{R}^2$ and as $\tilde{X}$ a d-sheeted cover of X. It cannot be extended over any point of $\mathbb{R}^2$ in the first sense because holomorphic functions on this $\tilde{X}$ do not separate points. But it obviously extends to a trivial cover $(\mathbb{C}^2, \mathsf{Id}, \mathbb{C}^2)$ after "gluing the sheets".

Definition 5.1. *A ramified analytic cover (or, simply an analytic cover) is a triple $(\tilde{X}, \mathsf{c}, X)$, where*

i) $\tilde{X}$ is a locally compact, Hausdorff topological space, X is a normal complex space;

ii) $\mathsf{c} : \tilde{X} \to X$ a continuous, surjective, zero-dimensional, proper map for which there exists a negligible subset $\mathcal{R} \subset X$ such that $\mathsf{c}|_{\tilde{X}\setminus\tilde{\mathcal{R}}} : \tilde{X} \setminus \tilde{\mathcal{R}} \to X \setminus \mathcal{R}$ is a finite unramified cover, where $\tilde{\mathcal{R}} := \mathsf{c}^{-1}(\mathcal{R})$;

iii) and such that $\tilde{X} \setminus \tilde{\mathcal{R}}$ is dense in $\tilde{X}$ and $\tilde{\mathcal{R}}$ doesn't separate $\tilde{X}$.

Zero-dimensional means that for every $x \in X$ the preimage $\mathsf{c}^{-1}(x)$ is discrete. Since c is in addition proper it is a *finite* map. A subset $\mathcal{R}$ of a normal complex space X is called *negligible* if for any neighborhood V of any point of $\mathcal{R}$ every bounded holomorphic function on $V \setminus \mathcal{R}$ holomorphically extends to V. $\tilde{\mathcal{R}}$ doesn't separate $\tilde{X}$ if for every point $r \in \tilde{\mathcal{R}}$ and every connected neighborhood $V \ni r$ the difference $V \setminus \tilde{\mathcal{R}}$ is connected as well. $\tilde{\mathcal{R}}$ is called the *branching divisor* of the cover, and $\mathcal{R}$ the *ramification divisor*. By the well known theorem of Grauert and Remmert one knows that $\tilde{X}$ inherits a unique structure of a normal complex space such that c becomes holomorphic and $\tilde{\mathcal{R}}$ analytic. Therefore, $\mathcal{R}$ and $\tilde{\mathcal{R}}$ are indeed divisors. Different proofs of Grauert-Remmert theorem can be found in [GR1], [Ni], also in [NS] and [De]. One can remark that either of the proofs simultaneously proves the removability of codimension two singularities for analytic covers. Therefore we get the following statement, which was our goal.

Proposition 5.1. *Let A be a codimension $\geqslant 2$ analytic subset of a normal complex space Y. Then any analytic cover X over $Y \setminus A$ uniquely extends to an analytic cover $\tilde{X}$ over the whole of Y. Moreover X injects to $\tilde{X}$.*

More details on the proof can be found in section 3 of [Iv11], where we follow [De].

Example 5.3. Consider as $X = \mathbb{P}^2 \setminus \bar{\mathbb{B}}^4$ and let $A_1 \subset X$ be an unextendable complex curve constructed in Example 5.1. A_1 cuts the unit sphere by a curve which is a graph of a curve $\gamma_1 \subset \Delta$ on the first coordinate plane, see construction there. Take this γ_1 sufficiently close to $\partial\Delta$. Do the same over the second coordinate plane to get another curve A_2. $X \setminus (A_1 \cup A_2)$ has homotopy type of $\mathbb{C}^*$ and therefore we can construct a finite cover $\tilde{X}$ of X ramified over $A = A_1 \cup A_2$ of any given degree d. Let c be the corresponding projection. $(\tilde{X}, \mathsf{c}, X)$ cannot be extended to any neighborhood of any point of $A \cap \partial\mathbb{B}^4$ because, otherwise its ramification curve A would extend to this neighborhood.

5.3. Elements of pluripotential theory. In order to proceed further we need to recall some generalities on pluripotential theory, for the proofs we mostly refer to [Kl]. Recall that a subset $S \subset \mathbb{C}^n$ is called *locally pluripolar* if for every point $s \in S$ there exists a neighborhood $U \ni s$ and a plurisubharmonic (psh for short) function u in U, which is not identically equal to $-\infty$, such that

$$S \cap U \subset \{x \in U : u(x) = -\infty\}.$$

S is *locally complete pluripolar* if in the situation as above $S \cap U = \{z \in U : u(z) = -\infty\}$. By Josefson's theorem, see [Kl], every locally pluripolar set is globally pluripolar or, simply, pluripolar. That means that there exists a psh-function $u \not\equiv -\infty$ in $\mathbb{C}^n$ such that $S \subset \{z : u(z) = -\infty\}$. One has also that countable unions of pluripolar sets are pluripolar. Let S be a subset of an open set $\Omega \subset \mathbb{C}^n$. Consider the following class of functions

$$\mathcal{U}(S,\Omega) = \{u \in \mathcal{P}_+(\Omega) : u|_S \geqslant 1\},$$

where by $\mathcal{P}_+(\Omega)$ we denote the class of non-negative plurisuperharmonic functions in Ω.

Definition 5.2. *The lower regularization w_* of the function*

$$w(\zeta,S,\Omega) = \inf\{u(\zeta) : u \in \mathcal{U}(S,\Omega)\}$$

is called a $\mathcal{P}$-measure of S in Ω, i.e.,

$$w_*(z,S,\Omega) = \liminf_{\zeta\to z} w(\zeta,S,\Omega). \tag{5.3}$$

Note that w_* is plurisuperharmonic in Ω.

Definition 5.3. A point $s_0 \in \Omega$ is called a *locally regular* point of S if for all $\varepsilon > 0$ one has $w_*(s_0, S \cap \Delta^n(s_0,\varepsilon), \Delta^n(s_0,\varepsilon)) = 1$. We shall also say that S is locally regular at s_0.

Theorem 5.8. *(Two Constants Theorem). Let v be plurisubharmonic in Ω such that $v|_S \leqslant M_0$ and $v|_\Omega \leqslant M_1$. Then for $z \in \Omega$ one has*

$$v \leqslant M_0 \cdot w_*(z) + M_1 \cdot [1 - w_*(z)]. \tag{5.4}$$

Indeed, function $\frac{M_1 - v}{M_1 - M_0}$ obviously belongs to $\mathcal{U}(S,\Omega)$ and therefore it should be $\geqslant$ than ω_*. This gives (5.4). We shall repeatedly use the following statement.

Proposition 5.2. *Let Ω be a pseudoconvex domain in $\mathbb{C}^n$. If a subset $S \subset \Omega$ is not locally regular at all its points then S is pluripolar.*

In fact the set $\{s \in S : w_*(s) > w(s)\}$ is *negligible* and therefore pluripolar, see [Kl]. The following lemma can be proved by Taylor expansion using (5.4).

Lemma 5.1. *Suppose that function f is holomorphic and bounded with modulus by M in $\Delta^n \times \Delta^q_\varepsilon$ for some $0 < \varepsilon < 1$. Suppose that for $s \in S \subset \Delta^n$ the restriction $f_s = f(s,\cdot)$ extends holomorphically to Δ^q and that all these extensions are also bounded with modulus by M. If $s_0 \in \Delta^n$ is a locally regular point of S then for every $0 < R < 1$ there exists $r > 0$ such that f holomorphically extends to $\Delta^n(s_0,r) \times \Delta^q(R)$ and is bounded with modulus by $\frac{2M}{(1-R)^q}$ there.*

Denote by (z_1,z_2) the coordinates in $\mathbb{C}^n \times \mathbb{C}^q$ and write

$$f(z_1,z_2) = \textstyle\sum_{m\in\mathbb{N}^q} a_m(z_1) z_2^m.$$

Functions $|a_m|$ are plurisubharmonic in Δ^n and satisfy

$$|a_m| \leqslant \tfrac{M}{\varepsilon^{|m|}} \text{ on } \Delta^n \text{ and } |a_m| \leqslant M \text{ on } S.$$

Let w_* be the $\mathcal{P}$-measure of S in Δ^n. For a given $c < 1$ there exists $r > 0$ such that $\omega_*(z_1) \geqslant c$ when $z_1 \in \Delta^n(s_0,r)$. Theorem 5.8 tells now that for $z_1 \in \Delta^n(s_0,r)$ one has

$$|a_m(z_1)| \leqslant M + \tfrac{M}{\varepsilon^{|m|}}(1-c) \leqslant 2M,$$

if c was taken close to 1. The statement of Lemma follows.

5.4. Positive currents. Denote by $\mathcal{D}^{k,k}(\Omega)$ the space of $\mathcal{C}^\infty$-forms of bidegree (k,k) with compact support on a complex manifold Ω. $\varphi \in \mathcal{D}^{k,k}(\Omega)$ is real if $\bar\varphi = \varphi$. The dual space $\mathcal{D}_{k,k}(\Omega)$ is the space of currents of bidimension (k,k) or, bidegree $(n-k,n-k)$, $n = \mathsf{dim}_{\mathbb{C}}\Omega$. $T \in \mathcal{D}_{k,k}(\Omega)$ is real if $\langle T, \bar\varphi\rangle = \overline{\langle T,\varphi\rangle}$ for all $\varphi \in \mathcal{D}^{k,k}(\Omega)$.

Definition 5.4. *A current $T \in \mathcal{D}_{k,k}(\Omega)$ is called positive if for all $\varphi_1,...,\varphi_k \in \mathcal{D}^{1,0}(\Omega)$*

$$\left\langle T, \frac{i}{2}\varphi_1\wedge\bar\varphi_1\wedge\ldots\wedge\frac{i}{2}\varphi_k\wedge\bar\varphi_k\right\rangle \geqslant 0. \tag{5.5}$$

T is negative if $-T$ is positive.

Positive currents have coefficients measures. Current T is called *closed* if $dT = 0$, *pluriclosed* if $dd^cT = 0$, *plurinegative* if $dd^cT \leqslant 0$. We shall have mostly the following two examples in mind.

- Let A be a pure k-dimensional analytic subset in Ω, then the *current of integration* $[A]$ on A, which is a closed positive current in Ω of bidimension (k,k), see [Lg], is defined as

$$\langle [A], \varphi\rangle := \int_A \varphi. \tag{5.6}$$

- Let $f : \Omega \to X$ be a holomorphic mapping to a complex manifold/space X. Suppose X is possesses a closed (or, pluriclosed/plurinegative) metric form ω, then $f^*\omega$ is a positive closed (resp. pluriclosed/plurinegative) form on Ω. Moreover, if f has some mild "singularities" in Ω, ex. points of indeterminacy, then $f^*\omega$ often extends across these singularities as a closed (resp. pluriclosed/plurinegative) current.

Let T be a current of dimension k with coefficients measures. Its mass on an open $U \subset \Omega$ is

$$\|T\|(U) = \mathsf{sup}\{|\langle T,u\rangle| : u \in \mathcal{D}^k(U), |u(x)| \leqslant 1 \text{ for all } x \in U\}. \tag{5.7}$$

Here $|u(x)|$ is the Euclidean norm of the k-covector $u(x)$. If T has locally integrable coefficients then its mass norm coincides with L^1-norm. If $T = [A]$ is a current of integration on analytic set A then its mass is equal to the volume of A. Let S be a closed subset in Ω. We say that a current T, defined on $\Omega\setminus S$, has locally finite mass near S if for any open, relatively compact $U \Subset \Omega$ one has $\|T\|(U\setminus S) < \infty$. For a current T, which has locally finite mass near a closed set $S \subset \Omega$, one denotes by $\tilde T$ its trivial extension to Ω. It is defined as follows. Take a sequence of functions $\chi_n \in \mathcal{C}^\infty(\Omega)$ such that

- $\chi_n \equiv 0$ in a neighborhood of S;
- $\chi \nearrow 1$ uniformly on compacts of $\Omega\setminus S$.

Then $\tilde T$ is defined as the limit of $\chi_n T$. This limit exists and doesn't depend on the choice of χ_n, see [Kl, Lg] for more details. If T was positive on $\Omega\setminus S$ its trivial extension stays to be positive. When S is, in addition, complete pluripolar one can chose χ_n to be plurisubharmonic, see [El1].

Theorem 5.9. *(H. El Mir, [El1]). Let S be a closed, complete pluripolar subset of an open set $\Omega \subset \mathbb{C}^n$ and let T be a positive, closed current on $\Omega\setminus S$ such that T has a locally finite mass near S. Then the trivial extension $\tilde T$ of T is a closed positive current on Ω.*

An important case when S is an analytic subset of Ω was proved previously by H. Skoda in [Sk1]. Via the analyticity of upper level sets of Lelong numbers for positive closed currents, proved by Siu in [Si4], this theorem implies the Theorem 5.6 of Bishop.

Theorem 5.10. *(H. El Mir, [El2]). Let S be a closed, complete pluripolar subset of an open set $\Omega \subset \mathbb{C}^n$ and let T be a positive, plurinegative current on $\Omega\setminus S$ such that T has a locally finite mass near S. Then dd^cT has a locally finite mass near S and the residual current R, defined as*

$$R := \widetilde{dd^cT} - dd^c\tilde T, \tag{5.8}$$

is closed, positive and supported on S.

In particular $\tilde T$ is again plurinegative. If S is compact then the condition on T can be relaxed.

Theorem 5.11. *(K. Dabbek, F. Elkhadra, H. El Mir,* [DEM]*). Let K be a complete pluripolar compact in an open set $\Omega\subset\mathbb{C}^n$ and let T be a positive, plurinegative current on $\Omega\setminus K$. Then T has a finite mass near K and therefore the conclusion of Theorem 5.10 holds.*

Theorem 5.12. *For every $0<r<1$ there exists a constant C_r such that for every pure q-dimensional variety A in Δ^n one has*

$$\mathsf{vol}(A\cap\Delta^n_{1-r})\leqslant C_r\mathsf{vol}(A\cap H^{n,q}_r). \tag{5.9}$$

When $n=2,q=1$ this was proved by Oka in [Ok], the general case is due to Riemenschneider, see [Rn]. For positive, plurinegative, bidimension (q,q)-currents an analogous Oka-type inequality was proved by Fornaess and Sibony in [FS]. Namely, if T is such then one has

$$\|T\|(\Delta^n_{1-r})+\|dd^cT\|(\Delta^n_{1-r})\leqslant C_r\|T\|(H^{n,q}_r). \tag{5.10}$$

We refer to the survey of H. Skoda [Sk2] and lectures of H. El Mir [El2] for much more results on extension of currents, as well as to the paper [FS] for more details about the Oka-type inequalities.

6. Extension properties of holomorphic mappings

Along the following sections we shall generalize results on extension of holomorphic and meromorphic functions to the case of mappings with values in complex spaces. In this section we would like to describe complex spaces X such that holomorphic mappings with values in X extend as well as holomorphic functions do.

6.1. Mappings with values in q-complete complex spaces. As it was remarked in Corollary 1.1 holomorphic mappings with values in Stein spaces extend from the Hartogs figure to the associated polydisk. Remark that by theorem of Narasimhan [Nr] Stein spaces are precisely 1-complete ones. In [IS1] the following generalization of Corollary 1.1 was proved.

Theorem 6.1. *Let X be a q-complete complex space. Then every holomorphic mapping $f: H^{n,n-q}_r\to X$ extends to a holomorphic mapping $\hat f:\Delta^n\to X$.*

The proof uses the result of Barlet, [Ba2], about the existence on q-complete complex spaces certain dd^c-exact strictly positive (q,q)-forms. It should be said that in [IS1] it was proved that every *meromorphic* $f:H^{n,n-q}_r\to X$ *meromorphically* extends to Δ^n. But going through the proof one can observe that if f was supposed to be holomorphic on $H^{n,n-q}_r$ the extension will stay holomorphic. In fact the proof becomes much more easier, one only need to use the Rouché Principle of Theorem 11.2 from section 11.

6.2. Stein neighborhoods of Stein subvarieties and rational curves. One more observation which can be made from Corollary 1.1 is that finding "big" Stein subsets in X might be useful for our task of extension of holomorphic mappings with values in X.

Theorem 6.2. *(Y.-T. Siu,* [Si6]*) Let Γ be a Stein subspace of a complex space X. Then there exists an open $U\supset\Gamma$ which is Stein. If, moreover, Γ and X are smooth then there exists a neighborhood $U\supset\Gamma$ which is biholomorphic to a neighborhood of the zero section in the normal bundle $\mathcal{N}_\Gamma$ to Γ in X.*

For the proof we refer to [Si6] or to [Co] and [Dm1]. Remark that a biholomorphism in Theorem 6.2, which is proved to be identity on Γ, provides a holomorphic retraction $\mathsf{r}:U\to\Gamma$. Now let us turn to holomorphic mappings with values in a given complex space X. Remark that if X contains a rational curve then Hartogs-type extension theorem for holomorphic maps with values in X fails. Indeed, let

$$\pi:\mathbb{C}^2\setminus\{0\}\to\mathbb{P}^1\quad \pi:(z_1,z_2)\to[z_1:z_2] \tag{6.1}$$

be the standard projection which sends a point $z=(z_1,z_2)$ to the line l_z passing through z and the origin. The limit set at zero of this map is the whole $\mathbb{P}^1$. Therefore π doesn't extend to zero. Let now $C=h(\mathbb{P}^1)$ be a rational curve in X. Then the composition $f=h\circ\pi:\mathbb{C}^2\setminus\{0\}\to X$ doesn't extend to the origin for the same reasons as π. Our first result, which will be proved in the next subsection, says that for Kähler X rational curves are the *only* obstructions for holomorphic extension. Let us derive a lemma which explains this. Let S is a compact complex curve in a complex manifold X. We suppose that X is equipped with some distance d compatible with its topology. Denote by $S_\varepsilon^{'}$ a domain in S which is the interior of the complement to the ε-neighborhood of a finite number of points $p_1,...,p_N$. We shall always suppose that $N\geqslant 1$ (if not we just mark some $p_1\in S$) and that all singular points of S are among $p_1,...,p_N$. Recall that an *analytic disk* in X is a holomorphic map φ of a neighborhood of $\bar{\Delta}$ to X.

Lemma 6.1. *Suppose that for every $\delta>0$ there exists an analytic disk $\varphi:\bar{\Delta}\to X$ such that:*

i) $S_\varepsilon^{'}$ is contained in a δ-neighborhood of $\varphi(\bar{\Delta})$;

ii) $\min\{d(x,y):x\in\varphi(\partial\Delta),y\in S_\varepsilon^{'}\}>2\delta$;

iii) the intersection of $\varphi(\bar{\Delta})$ with the boundary of the δ-neighborhood of $S_\varepsilon^{'}$ is contained in the δ-neighborhood of the boundary $\partial S_\varepsilon^{'}$.

Then S is a rational curve.

Indeed, since $S_\varepsilon^{'}$ is Stein we can apply Theorem 6.2 and find a Stein neighborhood $U\supset\bar{S}_\varepsilon^{'}$ with a holomorphic retraction $\mathsf{r}:U\to S_\varepsilon^{'}$. We can suppose that U is a δ-neighborhood of $S_\varepsilon^{'}$ for some $\delta>0$. Let $\varphi:\bar{\Delta}\to X$ be an analytic disk satisfying (*i*) - (*iii*) . Set $D:=\varphi^{-1}(U)$. Conditions (*i*) - (*iii*) imply that $\mathsf{r}\circ\varphi:D\to S_\varepsilon^{'}$ is proper and surjective. I.e., is a ramified analytic cover. Let γ be one of the generators of $H_1(S,\mathbb{Z})$ and let $\tilde{\gamma}$ be its lift to D. Then $\tilde{\gamma}$ is homologous to some linear combination of boundary components of D. Consequently γ must be homologous to a cycle in $\partial S_\varepsilon^{'}$. This shows that $H_1(S,\mathbb{Z})=0$, *i.e.*, S is a Riemann sphere.

6.3. Holomorphic mappings with values in Kähler spaces. Recall that a complex space X is called *disk-convex* if for every compact $K\subset X$ there exists another compact $\hat{K}\subset X$ such that for every analytic disk $\varphi:\bar{\Delta}\to X$ with $\varphi(\partial\Delta)\subset K$ one has $\varphi(\bar{\Delta})\subset\hat{K}$. Every compact space is obviously disk-convex. As well as every Stein or, holomorphically convex one. We are prepared to state and prove the following

Theorem 6.3. *For a disk-convex Kähler space X the following conditions are equivalent:*

i) every holomorphic mapping $f:H_r^{n+1}\to X$ holomorphically extends to Δ^{n+1};

ii) X doesn't contain rational curves.

Proof. We need to prove that (*ii*) $\Rightarrow$ (*i*) only. The proof will be done in four steps. For $z=(z_1,...,z_n)\in\Delta^n$ denote $\Delta_z:=\{z\}\times\Delta$. Let U be the maximal open subset of Δ^n such that f holomorphically extends to the Hartogs figure over U

$$H_U^{n+1}(r):=A_{1-r,1}\times\Delta^n\cup\Delta\times U. \tag{6.2}$$

Step 1. Vanishing of the homology of Hartogs domains. Let H^c stand for the homology with compact supports.

Lemma 6.2. *For any domain $U\subset\Delta^n$ one has*

$$H_2^c\big(H_U^{n+1}(r)\big)=0. \tag{6.3}$$

For the proof write the Mayer-Vietoris sequence for the homology groups of the pair of domains $\Delta\times U$ and $A_{1-r,1}\times\Delta^n$:

$$\begin{aligned}...\to H_2^c(A_{1-r,1}\times U)\xrightarrow{\beta}H_2^c(\Delta\times U)\to H_2^c(H_U^{n+1}(r))\to H_1^c(A_{1-r,1}\times U)\xrightarrow{\alpha}\\ H_1^c(\Delta\times U)\oplus H_1^c(A_{1-r,1}\times\Delta^n)\to ....\end{aligned} \tag{6.4}$$

Here we used the obvious fact that $H_2^c(A_{1-r,1}\times\Delta^n)=0$. Denote by π_1 and π_2 the natural projections of $\mathbb{C}^{n+1}=\mathbb{C}^n\times\mathbb{C}$ onto the first and second factors respectively. If $\gamma\in H_1^c(A_{1-r,1}\times U)$ is such that $\gamma\sim 0$ in $\Delta\times U$ then $\pi_1(\gamma)\sim 0$ in U. And if $\gamma\sim 0$ in $A_{1-r,1}\times\Delta^n$ then $\pi_2(\gamma)\sim 0$ in $A_{1-r,1}$. Therefore from Künneth formula for the product $A_{1-r,1}\times U$ one has for such γ that $\gamma\sim 0$ in $A_{1-r,1}\times U$. This proves the injectivity of the homomorphism α. The surjectivity of β is obvious. This gives the result.

For a holomorphic map $f: H_U^{n+1}(r)\to(X,\omega)$ define the following *area function*

$$a(z)=\mathsf{area}_{\omega}f(\Delta_z)=\int_{\Delta}f|_{\Delta_z}^{*}\omega, \tag{6.5}$$

i.e., $a(z)$ is the area of the image of the disk Δ_z. In what follows without any additional explanations we shall suppose, after shrinking $A_{1-r,1}$ if necessary, that f is defined in a neighborhood of $\overline{\Delta^n\times A_{1-r,1}}$ and therefore the formula (6.5) has perfectly sense.

Step 2. Estimate of the area. Suppose we extended f to $H_U^{n+1}(r)$ for some, taken to be maximal domain $U\subset\Delta$. If $U\neq\Delta^n$ take some point $s_0\in\partial U\cap\Delta^n$ and take a sequence of analytic disks $\Delta_k=\Delta\times\{s_k\}$, $s_k\in U$, approaching the disc $\Delta_0:=\Delta\times\{s_0\}$. Let $\tilde\omega:=f^*\omega$ be the pull-back of ω by f. This is a closed (1,1)-form on $H_U^{n+1}(r)$. Due to Lemma 6.2 and usual duality $\tilde\omega=d\gamma$ for some smooth 1-form γ on $H_U^{n+1}(r)$. This gives the following estimate of the areas of the analytic discs $\Gamma_{f_k}:=$ the graph of $f|_{\Delta_k}$ in $\Delta^{n+1}\times X$:

$$\mathsf{area}[\Gamma_{f_k}]=\pi+\int_{\Delta_k}\tilde\omega=\pi+\int_{\partial\Delta_k}\gamma.$$

The last quantity obviously stays bounded when $k\to\infty$. Bishop compactness theorem tells us now that some subsequence of graphs $\{\Gamma_{f_k}\}$ (we suppose that it is the sequence itself) converges to a complex analytic variety Γ of pure dimension one in $\Delta_0\times X$.

Step 3. The structure of the limit. Γ decomposes as $\Gamma=\hat\Gamma\cup\Gamma_0$, where Γ_0 is the graph of $f|_{\Delta_0}$ and all irreducible components of $\hat\Gamma$ are rational curves. This follows from the description of convergence in Gromov compactness theorem, see Theorem 2.5 and definitions involved there. But it can be also seen straightforwardly from Lemma 6.1. Therefore all compact components of Γ are rational curves. Since we do not have them in X we see that Γ_{f_k} converge to Γ_0.

Step 4. Extension to a neighborhood of s_0. Take a Stein neighborhood U of Γ_0 in $\Delta^{n+1}\times X$. Then all Γ_{f_k} will belong to U for $k>>1$. Also some neighborhood of the boundary $\partial\Gamma_{f_0}$ is in U (after shrinking if necessary). We are exactly in the conditions of Hurwitz Theorem 1.1 as it was explained at the beginning of this section. Therefore we can extend f to a neighborhood of s_0. Theorem is proved.

6.4. Holomorphic mappings to hyperbolic spaces. For generalities on Kobayashi pseudo-distance we refer to [Ko]. A complex space X is called Kobayashi hyperbolic (simply hyperbolic along this text) if the Kobayashi pseudo-distance is actually a distance on X. It that case this distance is compatible with the given topology of X, denote it as $d_k(\cdot,\cdot)$. In the case $X=\Delta$ the Kobayashi distance coincides with the Poincaré distance. It is also obvious from the very definition of d_k that holomorphic mappings are distance decreasing. This implies the following

Proposition 6.1. *The family $\mathcal{O}(\Delta,X)$ of holomorphic mappings form the unit disk to a Kobayashi hyperbolic complex space is equicontinuous. If (X,d_k) is complete then every bounded subfamily $\mathcal{F}\subset\mathcal{O}(\Delta,X)$ is relatively compact.*

Here by saying that $\mathcal{F}$ is *bounded* we mean that there exists a compact $K\subset X$ such that $f(\Delta)\cap K\neq\varnothing$ $\forall f\in\mathcal{F}$. Mappings with values in *complete* hyperbolic spaces satisfy the following

Definition 6.1. *We say that holomorphic mappings from Δ to a complex space X satisfy the disk condition if any sequence $\{\varphi_k:\Delta\to X\}$ of holomorphic mappings with values in X which for some $0<r<1$ uniformly converge on $A_{1-r,1}$ converge uniformly on the whole of Δ.*

The proof follows from the uniform equicontinuity of $\{\varphi_k\}$. This implies the following

Theorem 6.4. *Every holomorphic mapping $f : H_r^{n+1} \to X$, where X is a complete Kobayashi hyperbolic complex space, extends to a holomorphic mapping $\hat{f} : \Delta^{n+1} \to X$.*

The proof of Theorem 6.3 goes through. Remark that Step 2 is not needed, Step 3 is straightforward from the disk-condition.

Remark 6.1. In fact for *compact* hyperbolic spaces a much stronger extension statement holds true: every holomorphic mapping $f : \Delta^* \to X$ extends to the origin, see [Kw]. This can be viewed as a characterization of compact Kobayashi hyperbolic manifolds. For *complete* hyperbolic manifold this is not true: Δ^* is complete.

A close class to hyperbolic spaces are spaces with non-positive holomorphic sectional curvature. Let $h = hdz \otimes d\bar{z}$ be a Hermitian pseudo-metric on the unit disk. “Pseudo“ means that $h \geqslant 0$ but may vanish somewhere. Metric h has non-positive curvature if

$$K_h(z) = -\tfrac{2}{h(z)} \Delta \ln h(z) \leqslant 0 \tag{6.6}$$

at every $z \in \Delta$ such that $h(z) > 0$. Let furthermore (X,h) be a Hermitian space, *i.e.*, a complex space with a Hermitian metric h. For a holomorphic mapping $f : \Delta \to X$ we get canonically the induced metric $h_f = h_f dz \wedge d\bar{z}$ on Δ. h_f is a non-negative real function, uniquely determined by f (and h). We say that h has a *non-positive holomorphic sectional curvature* if for any holomorphic mapping $f : \Delta \to X$ the induced metric $h_f = h_f dz \wedge d\bar{z}$ has non-positive curvature. Corresponding K_{h_f} is called the holomorphic curvature of h along the holomorphic (or, complex) disk $f(\Delta)$.

Theorem 6.5. *(B. Shiffman,* [Sh1], *P. Griffiths,* [Gr]*). Let (X,h) be a complete Hermitian space with non-positive holomorphic sectional curvature. Then every holomorphic mapping $f : H_r^{n+1} \to X$ extends to a holomorphic mapping $\hat{f} : \Delta^{n+1} \to X$.*

Denote by $[z,w]$ the closed interval in $\mathbb{C}$ from z to w, by $S_r = \partial\Delta_r$ the circle of radius r. In what follows $L_h(\gamma)$ stands for the length of the curve γ with respect to the pseudo-metric h. The key point is the following

Lemma 6.3. *Let $0 < a < b < 1$ and $c = (b+a)[2\pi b(b-a)]^{-1}$. Let h be a Hermitian pseudo-metric on Δ with non-positive curvature. Then for all $z,w \in \Delta_a$ one has*

$$L_h([z,w]) \leqslant c|z-w| L_h(S_b). \tag{6.7}$$

Proof. Remark that non-positivity of curvature (6.6) means that $\ln h$ is subharmonic. Therefore $\sqrt{h}$ is subharmonic too. By the definition of arc length

$$L_h(S_b) = b\int_0^{2\pi} \sqrt{h(be^{i\theta})}d\theta \quad \text{and} \quad L_h([z,w]) = |z-w|\int_0^1 \sqrt{h(tz+(1-t)w)}dt.$$

For $s \in \bar{\Delta}_b$ by Poisson's formula we have

$$\sqrt{h(s)} \leqslant \frac{1}{2\pi}\int_0^{2\pi} \frac{(b^2-|s|^2)}{|be^{i\theta}-s|^2}\sqrt{h(be^{i\theta})}d\theta.$$

Since obviously $(b^2-|s|^2)|be^{i\theta}-s|^{-2} \leqslant (b+a)/(b-a)$ for $|s| < a$ we see that

$$L_h([z,w]) \leqslant \frac{|z-w|}{2\pi}\frac{b+a}{b-a}\int_0^{2\pi} \sqrt{h(be^{i\theta})}d\theta = cL_h(S_b),$$

as stated. □

From Lemma 6.3 we immediately see that holomorphic mappings with values in a complete Hermitian manifold of non-positive holomorphic sectional curvature satisfy the disk condition of Definition 6.1. Theorem 6.5 follows.

Remark 6.2. As it was explained in section 1 from Docquier-Grauert theorem it follows that holomorphic mappings with values in manifolds as in Theorems 6.3, 6.5, 6.4 extend from domains over Stein manifolds to their envelopes of holomorphy (and moreover to locally pseudoconvex envelopes for the case of domains over arbitrary manifolds).

7. Cycle space associated with a meromorphic mapping

The remaining part of this chapter will be devoted to the extension of meromorphic mappings.

7.1. Meromorphic mappings. Let D and X be reduced complex spaces. D will be also always supposed to be normal. By pr_1 and pr_2 denote the natural projections of the product $D \times X$ onto D and X respectively. Let A be a proper analytic subset of D and $f : D \setminus A \to X$ be a holomorphic mapping.

Definition 7.1. *We say that f defines a meromorphic map from D to X if the closure of the graph of f is an analytic set Γ_f in $D \times X$ and the restriction $\mathsf{pr}_1|_{\Gamma_f} : \Gamma_f \to D$ is proper.*

This notion defines the meromorphicity in the sense of Remmert, [R], (other types of meromorphicity will be not considered in this text). We shall denote this meromorphic mapping also as f (*i.e.,* in the same way as its holomorphic part), and by Γ_f the graph of either of these mappings. Projection $\mathsf{pr}_1|_{\Gamma_f} : \Gamma_f \to D$ is surjective and maps irreducible components of Γ_f to irreducible components of D. The minimal analytic set A such that f is holomorphic on $D \setminus A$ is called the *indeterminacy set* of f and will be denoted as I_f. Since D is *normal* the set I_f is of codimension $\geqslant 2$ and $x \in I_f$ if and only if $\mathsf{dim}\,(\mathsf{pr}_1|_{\Gamma_f})^{-1}(x) > 0$.

Remark 7.1. 1. If V is an irreducible complex subvariety of D, which doesn't belong entirely to I_f, then the restriction of f to V is the meromorphic map having as the graph the *irreducible component* of $\Gamma_f \cap (V \times X)$ which projects to V generically one to one. In particular, this graph is not necessarily the whole intersection $\Gamma_f \cap (V \times X)$.

2. One more important remark is that if $\mathsf{dim}\, D = 1$ then every meromorphic map $f : D \to X$ is *holomorphic* (whatever X is). This follows from the fact that $\mathsf{codim}\, I_f \geqslant 2$ and therefore $I_f = \varnothing$ in this case. As well as if $\mathsf{dim}\, D \geqslant 2$ and $\mathsf{dim}\, V = 1$ with $V \not\subset I_f$, then the restriction of f to V is holomorphic.

3. The *full image* of a set $L \subset D$ under f is defined as $f[L] := \mathsf{pr}_2\,(\Gamma_f \cap [L \times X])$. It is probably worth to notice once more that since D is normal $x \in I_f$ if and only if $\mathsf{dim}\, f[x] \geqslant 1$. This follows from the obvious observation that $I_f = \mathsf{pr}_1\left(\{(x_1,x_2) \in \Gamma_f : \mathsf{dim}_{(x_1,x_2)} \mathsf{pr}_1|_{\Gamma_f}^{-1}(x_1) \geqslant 1\}\right)$.

In order to better understand the notion of a meromorphic map let us give the description of meromorphic maps in the case when X is projective. Recall that a complex projective manifold/algebraic space is a compact complex manifold/space X which admits a holomorphic imbedding $i : X \to \mathbb{P}^N$ into a complex projective space for some N. Let us see that a meromorphic mapping from a domain D with values in $\mathbb{P}^N$ can be naturally represented by N meromorphic functions on D. This will be explained in the process of the proof of the following related fact.

Proposition 7.1. *Let $f : D \to \mathbb{P}^N$ be a meromorphic mapping. Then for every point $x_0 \in D$ there exists a neighborhood $V \ni x_0$ and holomorphic functions $\varphi^0, ..., \varphi^N$ in V such that*

$$f(z) = [\varphi^0(z) : ... : \varphi^N(z)]. \tag{7.1}$$

Proof. Denote by $[w_0 : w_1 : ... : w_N]$ the homogeneous coordinates of $\mathbb{P}^N$. Let $U_j = \{w \in \mathbb{P}^N : w_j \neq 0\}$ and let $w_0/w_j, ..., w_N/w_j$ be the affine coordinates in U_j. Set $D_j = f^{-1}(U_j)$. Since U_0 is isomorphic to $\mathbb{C}^N$ the restriction $f|_{D_0} : D_0 \longrightarrow U_0$ is given by holomorphic functions $w_1/w_0 = f^1(z), ..., w_N/w_0 = f^N(z)$. The coordinate change in $\mathbb{P}^N$ shows that $f|_{D_0 \cap D_j} : D_0 \cap D_j \longrightarrow \mathbb{P}^N$ is given by functions $w_1/w_0 = 1/f^j(z), ..., w_N/w_0 = f^N(z)/f^j(z)$ which are holomorphic in D_j. Therefore functions $f^1, ..., f^N$ are meromorphic on $D_0 \cup D_j$. This proves that $f^1, ..., f^N$ are meromorphic on $\bigcup_{j=0}^N D_j = D$.

Remark 7.2. I.e., we proved on this stage that a meromorphic mapping $f : D \to \mathbb{P}^N$ can be *globally* represented by N meromorphic functions $f^1, ..., f^N$, *i.e.,* they are meromorphic *on the whole of* D. In particular meromorphic functions on D are precisely the meromorphic mappings from D to $\mathbb{P}^1$.

If $f^1 \equiv ... \equiv f^N \equiv 0$ then $f(D) \equiv 0 \in U_0$. If not, let $f_1 \not\equiv 0$. One finds holomorphic functions h_j et g_j $0 \le j \le N$ in a neighborhood V of a given point $x_0 \in D$, $g_j \not\equiv 0$ for $j = 1, ..., N$, such that

$$f^1 = \frac{h_1}{g_1}, ..., f^N = \frac{h_N}{g_N}.$$

And therefore gets

$$f := \left[1 : \frac{h_1}{g_1} : ... : \frac{h_N}{g_N}\right] = \left[\prod_{j=1}^N g_j : h_1 \prod_{j=2}^N g_j : ... : h_N \prod_{j=1}^{N-1} g_j\right].$$

This proves that f can be locally written in the form (7.1) as claimed. □

Remark 7.3. a) If the zero sets of φ^j contain a common divisor then we can divide all φ^j by its equation and get a representation such that $GCD(\varphi^1, ..., \varphi^N) = 1$ in every $\mathcal{O}_x$, $x \in V$. In that case the indeterminacy set of f is

$$I_f \cap V = \{z \in V : \varphi^0(z) = ... = \varphi^N(z) = 0\}. \tag{7.2}$$

Representation (7.1) satisfying (7.2) will be called *reduced*.

b) In section 6 we considered an example of a holomorphic mapping (6.1) $\pi : \mathbb{C}^2 \setminus \{0\} \to \mathbb{P}^1$ which doesn't extend to the origin holomorphically. But it is easy to see that π extends to zero meromorphically. Indeed, if (z_1, z_2) and $[w_0 : w_1]$ denote the standard coordinates in $\mathbb{C}^2$ and homogeneous coordinates in $\mathbb{P}^1$ then the graph of π in $\mathbb{C}^1 \times \mathbb{P}^1$ closes to an analytic set which is defined by the equation $z_2 w_0 - z_1 w_1 = 0$.

Now it is clear that Theorem 1.4 gives the following statement.

Corollary 7.1. *Let $f : R^{n+1}_{1-r,1} \to X$, $n \geqslant 1$, be a meromorphic mapping to a complex algebraic space. Suppose that for z in some subset A, which is not contained in a countable union of locally closed proper analytic subsets, the restriction f_z of f to $A_{1-r,1} \times \{z\}$ is well defined and extends to Δ. Then f extends to a meromorphic mapping $\hat{f} : \Delta^{n+1} \to X$.*

7.2. Boundedness of area and volume, normality of the image space. As one can suggest from the very definition of a meromorphic mapping f the most interesting information about it is concentrated in that part of its graph Γ_f which lies over the indeterminacy set I_f. Let us make few remarks on this issue. Given a holomorphic mapping $f : D \setminus A \to X$ between complex manifolds/spaces, where $\operatorname{codim} A \geqslant 2$. By Bishop's Theorem 5.6 its graph Γ_f extends to an analytic subset of $D \times X$ (or, equivalently, its closure is analytic) if and only if its volume is locally bounded near A, *i.e.,* for every compact $K \subset D$ one has

$$\operatorname{vol}[\Gamma_f \cap (K \times X)] < \infty. \tag{7.3}$$

Example 7.1. Consider a Hopf surface $X^2 := \mathbb{C}^2 \setminus \{0\}/z \sim 2z$. The natural projection $\pi : \mathbb{C}^2 \setminus \{0\} \to X^2$ covers X by infinitely many times. In fact every shell $\frac{1}{2^{n+1}} \leqslant \|z\| \leqslant \frac{1}{2^n}$ is mapped by π *onto* X^2. Therefore the volume of the graph of π is not locally bounded near $A = \{0\}$ and π doesn't extend meromorphically to zero.

The remark about local boundedness of volume of a meromorphic map near its indeterminacy set suggests one possible way to extend them: try to estimate the volume of Γ_f near the eventual singularity S. This approach in principle can work only if S is "reasonably small“. But even in this case it looks to be problematic as the following example of Shiffman and Taylor shows.

Example 7.2. Let D be the unit ball in $\mathbb{C}^n$, $n \geqslant 2$ and $S = \{z_2 = ... = z_n = 0\}$. Then there exists a plurisubharmonic function in D, real analytic on $D \setminus S$ such that

$$\int_{D(r)\setminus S} (dd^c u)^n = +\infty$$

for every $0 < r < 1$.

Remark that if X is Kähler and ω is a Kähler form on X and if $f: D\setminus S \to X$ is holomorphic then $\tilde{\omega} := f^*\omega$ is a closed positive $(1,1)$-current on $D\setminus S$. If f extends trough S then $\tilde{\omega}$ extends too. In particular, locally there should exist in this case a plurisubharmonic function u such that $\tilde{\omega} = dd^c u$. And now the example above shows what type of difficulties could occur: the unboundedness of the integral shows that the volume of Γ_f is not locally finite near S. Details about this example can be found in [Si5]. In [Si5] this difficulty has been overcome in the Kähler case by using the analyticity of superlevel sets of Lelong numbers of $\tilde{\omega}$ and its powers. Our approach will be different and will work not only in the case of Kähler targets. Namely, it is not difficult to show that for Kähler ω the areas of disks

$$\mathsf{area}(f(\Delta_{z'})) = \int_{f(\Delta_{z'})} \omega = \int_{\Delta_{z'}} \tilde{\omega}$$

$z' = (z_1, ..., z_{n-1}) \in \Delta^{n-1}$ are uniformly bounded. Also in Example 7.2 this is true, if one understands $\int_{\Delta_{z'}} \tilde{\omega}$ as $\int_{\Delta_{z'}} dd^c u$. Our approach will be based on this observation. The key point in this approach is Theorem 7.3, see the following section.

Remark 7.4. Now let us briefly discuss our basic assumptions on the source and target spaces of a meromorphic mapping. The source space, say D, will be always supposed to be normal (and reduced). The reason can be seen from Corollary 1.5, *i.e.,* this is needed already to extend functions. The image space will be supposed to be just reduced. But let us note that in the proofs we can always suppose that X is, in addition, also normal. Indeed, let $\mathrm{n}: \tilde{X} \to X$ be the normalization of X. n is a holomorphic map which is bimeromorphic. In addition it is biholomorphic outside of the preimage of the proper analytic subset $\mathcal{N}$ of X of non-normal points of X. Now, let $f: D \to X$ be our meromorphic mapping, which we want to extend to some $\tilde{D} \supset D$. If $f(D) \not\subset \mathcal{N}$ we can lift it to $\tilde{f}: D \to \tilde{X}$, extend (if possible), and then pull down the extension obtained. If $f(D) \subset \mathcal{N}$ then do the same for $X = \mathcal{N}$. Therefore we can suppose in the process of proof, if needed, that the target space X is normal.

7.3. Sequences and families of meromorphic disks. Let X be a complex space, equipped with some Hermitian metric h. By ω we denote, as usually, $(1,1)$-form canonically associated with h. Let ω_0 be some metric form on D (usually clear from the context). Set $\omega = \omega_0 + \omega_h$, this is a metric form on $D \times X$. Let $q \geqslant 1$ be the dimension of D. The volume of the graph Γ_φ of a meromorphic mapping $\varphi: D \to X$ is given by

$$\mathsf{vol}(\Gamma_\varphi) = \textstyle\int_{\Gamma_\varphi} \omega^q = \int_D (\varphi^*\omega_h + \omega_0)^q. \tag{7.4}$$

Here by $\varphi^*\omega_h$ we denote the preimage of ω_h under φ, *i.e.,* $\varphi^*\omega_h = (\mathsf{pr}_1)_*\mathsf{pr}_2^*\omega_h$.

Remark 7.5. Let us give the sense to the formula (7.4). The first integral there has perfectly sense since we are integrating a smooth form over a complex variety. Denote by I_φ^ε the ε-neighborhood of the indeterminacy set I_φ of φ. Then (7.4) shows that the limit

$$\lim_{\varepsilon\to 0} \int_{D\setminus \bar{I}_\varphi^\varepsilon} \left(\omega_0 + \varphi^*\omega_h\right)^n = \lim_{\varepsilon\to 0} \int_{D\setminus \bar{I}_\varphi^\varepsilon} \sum_{p=0}^{n} C_n^p \omega_0^{n-p} \wedge \varphi^*\omega_h^p \tag{7.5}$$

exists. Therefore all $\varphi^*\omega_h^p$ are well defined on D as positive currents.

Definition 7.2. *By a* meromorphic q-disc *in a complex space X we shall understand a meromorphic mapping $\varphi: D \to X$, where D is a relatively compact domain in some irreducible, normal complex space of pure dimension $q \geqslant 1$.*

We shall mostly suppose that φ is defined in a neighborhood of $\bar{D}$. The case $q = 1$ is quite special. In that case meromorphic disk is actually holomorphic, see Remark 7.1. When $D = \Delta$ we called such disk an analytic disk. Recall furthermore that the Hausdorff distance between two subsets A and B of a metric space (Y, ρ) is a number $\rho(A,B) = \inf\{\varepsilon : A^\varepsilon \supset B, B^\varepsilon \supset A\}$. Here by A^ε we denote the ε-neighborhood of the set A, i.e. $A^\varepsilon = \{y \in Y : \rho(y, A) < \varepsilon\}$. Now let $\{\varphi_r : D \to X\}$ be a sequence of meromorphic mappings.

Definition 7.3. *We shall say that $\{\varphi_r\}$ strongly converge on compacts in D to a meromorphic map $\varphi : D \to X$, if for every relatively compact open $D_1 \Subset D$ the graphs $\Gamma_{\varphi_r} \cap (D_1 \times X)$ converge in the Hausdorff metric on $D_1 \times X$ to the graph $\Gamma_\varphi \cap (D_1 \times X)$.*

We shall write $\varphi_r \to \varphi$ to denote the strong convergence. Later on in section 11 we shall see that a strongly converging sequence converges in (even a stronger) *topology of cycles*. In particular the volumes of Γ_{φ_r} over compacts in D stay uniformly bounded, see Theorem 11.1. For the proof of the following lemma we refer to [Iv6], Lemma 2.3.1.

Lemma 7.1. *Let $\{\varphi_r : D \to X\}$ be a sequence of meromorphic q-disks in a complex space X. Suppose that there exists a compact $K \subset X$ and a constant $C < \infty$ such that:*
a) $\varphi_r(D) \subset K$ for all r;
b) $\mathrm{vol}(\Gamma_{\varphi_r}) \leqslant C$ for all r.
Then there exists a subsequence $\{\varphi_{r_j}\}$ such that:
1) the sequence $\{\Gamma_{\varphi_{r_j}}\}$ converges in Hausdorff metric to an analytic subset Γ of $D \times X$ of pure dimension q;
2) $\Gamma = \Gamma_\varphi \cup \hat{\Gamma}$, where Γ_φ is the graph of some meromorphic mapping $\varphi : D \to X$ and $\hat{\Gamma}$ is a pure q-dimensional analytic subset of $D \times X$ such that $A := \mathrm{pr}_1(\hat{\Gamma})$ is a proper analytic subset of D;
3) $\varphi_{r_j} \to \varphi$ on compacts in $D \setminus A$;
4) one has

$$\lim_{j\to\infty} \mathrm{vol}(\Gamma_{\varphi_{r_j}}) \geqslant \mathrm{vol}(\Gamma_\varphi) + \mathrm{vol}(\hat{\Gamma}). \tag{7.6}$$

5) For every $1 \leqslant p \leqslant \dim X - 1$ there exists a positive constant $\nu_p = \nu_p(K,h)$ such that the volume of every pure p-dimensional compact analytic subset of X which is contained in K is not less then ν_p.
6) Put $\hat{\Gamma} = \bigcup_{p=0}^{q-1} \Gamma_p$, where Γ_p is a union of all irreducible components of $\hat{\Gamma}$ such that $\dim[\mathrm{pr}_1(\Gamma_p)] = p$. Then

$$\mathrm{vol}_{2q}(\hat{\Gamma}) \geqslant \textstyle\sum_{p=0}^{q-1} \mathrm{vol}_{2p}(A_p) \cdot \nu_{q-p}, \tag{7.7}$$

where $A_p = \mathrm{pr}_1(\Gamma_p)$.

Now let us turn to the families of meromorphic disks. Let S be a set, and $W \Subset \mathbb{C}^q$ an open subset equipped with the usual Euclidean metric from $\mathbb{C}^q$.

Definition 7.4. *By a family of meromorphic q-disks in a complex space X we shall understand a subset $\mathcal{F} \subset S \times W \times X$ such that, for every $s \in S$ the set $\mathcal{F}_s = \mathcal{F} \cap \{s\} \times W \times X$ is a graph of a meromorphic mapping of W into X. If S is equipped with topology we shall say that the family $\mathcal{F}$ is continuous at point $s_0 \in S$ if $\mathcal{H} - \lim_{s\to s_0} \mathcal{F}_s = \mathcal{F}_{s_0}$.*

Here by $\mathcal{H} - \lim_{s\to s_0} \mathcal{F}_s$ we denote the limit of closed subsets of $\mathcal{F}_s$ in the Hausdorff metric on $W \times X$. As we already mentioned after Definition 7.3 such continuity is equivalent to the continuity in the cycle topology of Definition 7.7. $\mathcal{F}$ is continuous if it is continuous at each point of S. If W_0 is open in W then the restriction $\mathcal{F}_{W_0}$ is naturally defined as $\mathcal{F} \cap (S \times W_0 \times X)$. Now let S be a complex space itself.

Definition 7.5. *We call a family $\mathcal{F}$ meromorphic if the closure $\hat{\mathcal{F}}$ of the set $\mathcal{F}$ is an analytic subset of $S \times W \times X$ and the natural projection $\hat{\mathcal{F}} \to S \times W$ is proper.*

Our main statement about meromorphic families Theorem 7.1 concerns with the interaction of notions of continuity and meromorphicity of families of meromorphic q-disks. Consider a meromorphic mapping $f : V \times W_0 \longrightarrow X$ to a reduced complex space X, where V is a domain in $\mathbb{C}^p$ and $W_0 \Subset W \Subset \mathbb{C}^q$ are domains in $\mathbb{C}^q$, $p,q \geqslant 1$. Let S be some subset of V and $s_0 \in S$ some accumulation point of S. Suppose that for each $s \in S$ the restriction $f_s = f|_{\{s\}\times W_0}$ is well defined, *i.e.*, $W_s := \{s\} \times W_0 \not\subset I_f$, and meromorphically extends to W. We suppose additionally that there is a compact $K \subset X$ such that for all $s \in S$ one has $f_s(W) \subset K$. Let, as in Lemma 7.1 denote by $\nu_j = \nu_j(K)$ the minima of volumes of j-dimensional compact analytic subsets contained in our compact $K \subset X$. Fix some $W_0 \Subset W_1 \Subset W$ and put

$$\nu = \mathsf{min}\{\mathsf{vol}(A_{q-j}) \cdot \nu_j : j = 1, \ldots, q\}, \tag{7.8}$$

where A_{q-j} are running over all $(q-j)$-dimensional analytic subsets of W, intersecting $\bar{W}_1$. Clearly $\nu > 0$. In the following theorem the volumes of graphs are taken over W.

Theorem 7.1. *Suppose that there exists a neighborhood $U \ni s_0$ such that, for all $s_1, s_2 \in S \cap U$*

$$|\mathsf{vol}(\Gamma_{f_{s_1}}) - \mathsf{vol}(\Gamma_{f_{s_2}})| < \nu/2. \tag{7.9}$$

If $s_0 \in S$ is a locally regular point of S then there exists a neighborhood $V_1 \ni s_0$ in V such, that f meromorphically extends to $V_1 \times W_1$.

Proof. The proof will be done in three steps.

Step 1. $\{\Gamma_{f_s}\}$ is continuous at s_0. Indeed, let $s_n \in S, s_n \to s_0$ as $n \to \infty$. Then from (7.9) we see that $\mathsf{vol}(\Gamma_{f_{s_n}})$ are uniformly bounded and thus by Lemma 7.1 $\Gamma_{f_{s_0}} \subset (W_0 \times X)$ extends to a graph of meromorphic mapping over W. The graph of this extension be also denoted as $\Gamma_{f_{s_0}}$. Now if one could find a sequence $s_n \in S, s_n \longrightarrow s_0$ as $n \longrightarrow \infty$ such that $\Gamma_{f_{s_n}} \not\longrightarrow \Gamma_{f_{s_0}}$ in Hausdorff metric, then by Lemma 7.1, using the boundedness of volumes of Γ_{f_s}, one finds a subsequence, still denoted as s_n such that $\Gamma_{f_{s_n}} \longrightarrow \Gamma \supset \Gamma_{f_{s_0}}$, but not equal $\Gamma_{f_{s_0}}$. But then, by the relations (7.6) and (7.7) of Lemma 7.1 one has that

$$\lim_{n \longrightarrow \infty} \mathsf{vol}(\Gamma_{f_{s_n}}) \geqslant \nu + \mathsf{vol}(\Gamma_{f_{s_0}}),$$

which contradicts (7.9).

Our aim is to prove now that the family $\mathcal{F} = \bigcup_{s \in S} \Gamma_{f_s} \subset S \times W_0 \times X$ extends to a meromorphic family on $V_1 \times W_1 \times X$ for some neighborhood $V_1 \ni s_0$.

Step 2. $\mathcal{F}$ extends to a neighborhood of $\{s_0\} \times \mathsf{Reg}\,\Gamma_{f_{s_0}}$. Fix a point $z_0 \in \mathsf{Reg}\,\Gamma_{f_{s_0}} \cap (W_0 \times X)$. Here we consider $\Gamma_{f_{s_0}}$ as analytic space itself. So $\mathsf{Reg}\,\Gamma_{f_{s_0}}$ is connected dense subset in $\Gamma_{f_{s_0}}$ and $\mathsf{Sing}\,\Gamma_{f_{s_0}} := \Gamma_{f_{s_0}} \backslash \mathsf{Reg}\,\Gamma_{f_{s_0}}$ is a proper analytic subset of $\Gamma_{f_{s_0}}$. Take a point $z_1 \in \mathsf{Reg}\,\Gamma_{f_{s_0}} \cap (W \times X)$ and take a path $\gamma : [0,1] \longrightarrow \mathsf{Reg}\,\Gamma_{f_{s_0}}$ from z_0 to z_1. We shall prove that there is a neighborhood Ω of $\gamma([0,1])$ in $W \times X$ and a neighborhood $V \ni s_0$ such that $\mathcal{F} \cap (V \times \Omega)$ extends to an analytic set in $V \times \Omega$.

By T denote the set of those $t \in [0,1]$ that there exists a neighborhoods $\Omega_t \supset \gamma([0,t])$ and $V_t \ni s_0$ such that $\mathcal{F} \cap V_t \times \Omega_t$ extends to an analytic set in $V_t \times \Omega_t$. Note that T is open and contains the origin. Now let t_0 be the cluster point of T. Find the chart $\Sigma \cong \Delta^q \times \Delta^n$ for the space $W \times X$ in the neighborhood of $\gamma(t_0)$ with coordinates $u_1, \ldots, u_q, v_1, \ldots, v_n$ in such a way that $\gamma(t_0) = 0$ and $\Gamma_{f_{s_0}} \cap \Sigma = \{(u,v) : v = F_0(u)\}$ for some holomorphic map $F_0 : \Delta^q \longrightarrow \Delta^n$. By the Hausdorff continuity of our family $\{\Gamma_{f_s}\}$ in s_0 $\Gamma_{f_s} \cap \Omega = \{(u,v) : v = F_s(u)\}$ for s close to s_0, F_s holomorphic and continuously depending on s.

Take $t_1 \in T$ close to t_0, such that $\gamma([0,t_1]) \subset \Sigma$. We have some neighborhoods $V_{t_1} \ni s_0, \Omega_{t_1} \ni \gamma(t_1)$ such that $\mathcal{F}$ extends analytically to $V_{t_1} \times \Omega_{t_1}$. Let $u_1 \in \Delta^q$ be such that $\gamma(t_1) = (u_1, F_0(u_1))$. Then there is a neighborhood, say $\Delta^q_r(u_1)$, such that $\Gamma_{f_s} \cap (V_{t_1} \times \Delta^q_r(u_1) \times \Delta^n)$ is defined by the equation $v = F(u,s)$, where $F(u,s) = F_s(u) : V_{t_1} \times \Delta^q_r(u_1) \to \Delta^n$ as above. From the condition of the Lemma we see that for $s \in S$ close to s_0 $F(u,s)$ extends onto Δ^q. So by Lemma 5.1 $F(u,s)$ extends holomorphically to $V_{t_0,\varepsilon} \times \Delta^q_{1-\varepsilon}$, where ε is arbitrarily small ($V_{t_0,\varepsilon}$ depending on ε). But this means that $\mathcal{F}$ extends analytically onto $V_{t_0} \times \Delta^q_{1-\varepsilon} \times \Delta^n$. Thus T is closed and coincides with $[0,1]$.

We proved in fact that for any compact subset $R \subset \mathsf{Reg}\,\Gamma_{f_{s_0}} \cap (W_1 \times X)$ there are neighborhoods $V_R \ni s_0, \Omega_R \supset R$ such that Γ_f analytically extend to $V_R \times \Omega_R$.

Step 3. $\mathcal{F}$ is analytic near s_0. Cover the set $\mathsf{Sing}\,\Gamma_{f_{s_0}} \cap [\{s_0\} \times \bar{W}_1 \times X]$ with a finite number of open charts of the form $1/2V_\alpha \times \Omega_\alpha$, where $V_\alpha \cong \Delta^q$ and $\Omega_\alpha \cong \Delta^n$, and such that $\Gamma_{f_{s_0}} \cap (V_\alpha \times \Omega_\alpha)$ is analytic cover of V_α. By Step 1 we can find an open neighborhoods $V_R \ni s_0$ and $\Omega_R \supset R = \Gamma_{f_{s_0}} \setminus [\bigcup_\alpha 1/2V_\alpha \times \Omega_\alpha]$ such that $\mathcal{F}$ analytically extends to $V_R \times \Omega_R$.

Fix now some α. All that remained to prove is that Γ_f analytically extends to $V'_\alpha \times V_\alpha \times \Omega_\alpha$ for some neighborhood of $V'_\alpha \ni s_0$. But this again follows from Lemma 5.1 applied to the coefficients of polynomials which define the cover $\Gamma_{f_s} \cap (V_\alpha \times \Omega_\alpha) \to V_\alpha$. Theorem is proved. □

7.4. Cycle space associated with a meromorphic mapping. We shall freely use the results from the theory of cycle spaces developed by D. Barlet in [Ba1]. Recall that an analytic q-cycle in a reduced, normal complex space Y is a formal sum $Z = \sum_j n_j Z_j$, where $\{Z_j\}$ is a locally finite sequence of analytic subsets of pure dimension q and n_j are positive integers called multiplicities of Z_j. By $|Z| := \bigcup_j Z_j$ we denote the support of Z. Set $A^q(r,1) = \Delta^q \setminus \bar{\Delta}^q(r)$, *i.e.*, $A^q(r,1) = A^q_{r,1}$ in our previous notations. Let X be a reduced, normal complex space equipped with some Hermitian metric and let a holomorphic mapping $f : \bar{\Delta}^n \times \bar{A}^q(r,1) \to X$ be given. We shall start with the following space of cycles related to f. Fix some positive constant C and consider the set $\mathcal{C}_{f,C}$ of all analytic q-cycles Z in $Y := \Delta^{n+q} \times X$ such that:

(a) $Z \cap [\Delta^n \times \bar{A}^q(r,1) \times X] = \Gamma_{f_z} \cap [\bar{A}^q_z(r,1) \times X]$ for some $z \in \Delta^n$, where Γ_{f_z} is the graph of the restriction $f_z := f\,|_{A^q_z(r,1)}$. Here $A^q_z(r,1) := \{z\} \times A^q(r,1)$. This means, in particular, that for this z the mapping f_z extends meromorphically from $\bar{A}^q_z(r,1)$ to $\bar{\Delta}^q_z := \{z\} \times \bar{\Delta}^q$.

(b) $\mathsf{vol}(Z) < C$ and the support $|Z|$ of Z is connected.

We put $\mathcal{C}_f := \bigcup_{C>0} \mathcal{C}_{f,C}$ and we are going to show that $\mathcal{C}_f$ is an analytic space of finite dimension in a neighborhood of each of its points.

Definition 7.6. *By a coordinate chart adapted to Z we shall understand an open set V in Y such that $V \cap |Z| \neq \varnothing$ together with an isomorphism j of V onto a closed subvariety $\tilde{V}$ in the neighborhood of $\bar{\Delta}^q \times \bar{\Delta}^k$ such that $j^{-1}(\bar{\Delta}^q \times \partial\Delta^k) \cap |Z| = \varnothing$.*

We shall denote such a chart as (V, j). The image $j(Z)$ of cycle Z under the isomorphism j is the image of the underlying analytic set together with multiplicities. Sometimes we shall, following Barlet, denote: $\Delta^q = U, \Delta^k = B$ and call the quadruple $E = (V, j, U, B)$ a *scale* adapted to Z. If $\mathsf{pr} : \mathbb{C}^q \times \mathbb{C}^k \to \mathbb{C}^q$ is the natural projection, then the restriction $\mathsf{pr}\,|_{j(Z)}: j(Z) \to \Delta^q$ is a branched cover of degree say d. The number k depends on the imbedding dimension of Y (or X in our case). Sometimes we shall skip j in our notations and consider $Z \cap V$ as an analytic subset readily in a neighborhood of $\bar{\Delta}^q \times \bar{\Delta}^k$ such that $Z \cap (\bar{\Delta}^q \times \partial\Delta^k) = \varnothing$. The branched cover $\mathsf{pr}\,|_Z: Z \cap (\Delta^q \times \Delta^k) \to \Delta^q$ defines in a natural way a mapping $\varphi_Z : \Delta^q \to \mathsf{Sym}^d(\Delta^k)$, the d-th symmetric power of Δ^k, by setting $\varphi_Z(z) = (\mathsf{pr}\,|_Z)^{-1}(z)$. This allows to represent a cycle $Z \cap \Delta^{q+k}$ with $|Z| \cap (\bar{\Delta}^q \times \partial\Delta^k) = \varnothing$ as the graph of a d-valued holomorphic map.

Definition 7.7. *The topology defined by parameterizations φ_Z is called the cycle topology.*

This defines a metrizable topology on the space of q-cycles in Y, and this topology is equivalent to the *topology of currents*: $Z_r \to Z$ if for any continuous (q,q)-form χ with compact support one has

$$\int_{Z_r} \chi \to \int_Z \chi,$$

see [Fj]. It is also equivalent to the *Hausdorff* topology under an additional condition of *boundedness of volumes*, *i.e.*, $Z_r \to Z$ if and only if for every compact $K \Subset Y$ there exists $C_K > 0$ such that $\mathsf{vol}_{2q}(Z_r \cap K) \leqslant C_K$ and $Z_r \cap K \to Z \cap K$ with respect to the Hausdorff distance. We denote the space of q-cycles on Y endowed with the topology described as above by $\mathcal{C}^{\mathsf{loc}}_q(Y)$. By $\mathcal{B}_q(X)$ we denote the Barlet space of *compact* q-cycles in X, *i.e.*, cycles with compact support.

Without loss of generality we suppose that our holomorphic mapping f is defined on $\Delta^n(a) \times A^q(r_1, b)$ with $a, b > 1, r_1 < r$. Now, each $Z \in \mathcal{C}_f$ can be covered by a finite number of adapted neighborhoods (V_α, j_α). Such covering we be called an *adapted covering*. Denote the union $\bigcup_\alpha V_\alpha$ by W_Z. Taking this covering to be small enough, we can furthermore suppose the following.

(c) If $V_{\alpha_1} \cap V_{\alpha_2} \neq \varnothing$, then on every irreducible component of the intersection $Z \cap V_{\alpha_1} \cap V_{\alpha_2}$ a point x_1 is fixed so that:

- (c_1) *either there exists a polycylindric neighborhood $\Delta_1^q \subset \Delta^q$ of $\mathsf{pr}_{\alpha_1}(j_{\alpha_1}(x_1))$ such that the chart $V_{12} = j_{\alpha_1}^{-1}(\Delta_1^q \times \Delta^q)$ is adapted to Z and is contained in V_{α_2}, where V_{12} is given the same imbedding j_{α_1}, here pr_α is the projection pr which corresponds to the chart V_α;*
- (c_2) *or, this is fulfilled for V_{α_2} instead of V_{α_1}.*

(d) For some fixed $1 < c < a$ if $y \in V_\alpha$ is such that $\mathsf{pr}_1(y) \in \bar{\Delta}^n(c) \times A^q(\frac{r+1}{2}, 1)$, then $\mathsf{pr}_1(\bar{V}_\alpha) \subset \bar{\Delta}^n(\frac{a+c}{2}) \times A^q(r,1)$. Here by $\mathsf{pr}_1 : \Delta^{n+q} \times X \to \Delta^{n+q}$ we denote the natural projection.

Case (c_1) can be realized when the imbedding dimension of V_{α_1} is smaller or equal to that of V_{α_2}, and (c_2) in the opposite case, see [Ba1], pp. 91-92. Let $E = (V, j, U, B)$ be a scale on the complex space Y. Denote by $H_Y(\bar{U}, \mathsf{Sym}^d(B)) := \mathsf{Hol}_Y(\bar{U}, \mathsf{Sym}^d(B))$ the Banach analytic set of all d-sheeted analytic subsets on $\bar{U} \times B$, contained in $j(Y)$. As it was told the subsets W_Z together with the topology of uniform convergence on $H_Y(\bar{U}, \mathsf{Sym}^d(B))$ define a metrizable topology on our cycle space $\mathcal{C}_f$, which is equivalent to the topology of currents.

We refer the reader to [Ba1] for the definition of the isotropicity of the family of elements from $H_Y(\bar{U}, \mathsf{Sym}^d(B))$ parameterized by some Banach analytic set $\mathcal{S}$. Space $H_Y(\bar{U}, \mathsf{Sym}^d(B))$ can be endowed by another (more rich) analytic structure. This new analytic space will be denoted by $\hat{H}_Y(\bar{U}, \mathsf{Sym}^d(B))$. The crucial property of this new structure is that the tautological family $\hat{H}_Y(\bar{U}, \mathsf{Sym}^d(B)) \times U' \to \mathsf{Sym}^d(B)$ is isotropic in $H_Y(\bar{U}', \mathsf{Sym}^d(B))$ for any relatively compact polydisk $U' \Subset U$, see [Ba1]. The key point is that for isotropic families $\{Z_s : s \in \mathcal{S}\}$ parameterized by Banach analytic sets the following projection changing theorem of Barlet holds.

Theorem 7.2. *(D. Barlet, [Ba1]). If the family $\{Z_s : s \in \mathcal{S}\} \subset H_Y(\bar{U}, \mathsf{Sym}^d(B))$ is isotropic, then for any scale $E_1 = (V_1, j_1, U_1, B_1)$ in $U \times B$ adapted to some Z_{s_0}, there exists a neighborhood U_{s_0} of s_0 in $\mathcal{S}$ such that $\{Z_s : s \in U_{s_0}\}$ is again isotropic in V_1.*

This means, in particular, that the mapping $s \to Z_s \cap V_1 \subset H_Y(\bar{U}_1, \mathsf{Sym}^d(B_1))$ is analytic, i.e., can be extended to a neighborhood of any $s \in U_{s_0}$. Neighborhood means here a neighborhood in some complex Banach space where $\mathcal{S}$ is defined as an analytic subset. This leads naturally to the following

Definition 7.8. *A family $\mathcal{Z}$ of analytic cycles in an open set $W \subset Y$, parameterized by a Banach analytic set $\mathcal{S}$, is called analytic in a neighborhood of $s_0 \in \mathcal{S}$ if for any scale E adapted to Z_{s_0} there exists a neighborhood $U \ni s_0$ such that the family $\{\mathcal{Z}_s : s \in U\}$ is isotropic.*

7.5. Analyticity of $\mathcal{C}_f$ and construction of $\mathcal{G}_f$. Let $f : \bar{\Delta}^n \times \bar{A}^q(r,1) \to X$ be our map. Take a cycle $Z \in \mathcal{C}_f$ and a finite covering (V_α, j_α) satisfying conditions (c) and (d). As above, put $W_Z = \bigcup V_\alpha$. We divide V_α's into two types.

Type 1. These are V_α as in (d). For them put

$$H_\alpha := \bigcup_z \{[\Gamma_{f_z} \cap \bar{A}_z^q(r,1) \times X] \cap V_\alpha\} \subset H_Y(\bar{U}_\alpha, \mathsf{Sym}^{d_\alpha}(B_\alpha)). \tag{1.2.1}$$

The union is taken over all $z \in \Delta^n$ such that V_α is adapted to Γ_{f_z}.

Type 2. These are all others. For V_α of this type we put $H_\alpha := \hat{H}_Y(\bar{U}_\alpha, \mathsf{Sym}^{d_\alpha}(B_\alpha))$.

All H_α are open sets in complex Banach analytic subsets and for V_α of the first type they are of dimension n and smooth. The latter follows from Barlet-Mazet theorem, which tells that if $h : A \to \mathcal{S}$ is a holomorphic injection of a finite dimensional analytic set A into a Banach analytic set $\mathcal{S}$, then $h(A)$ is also an Banach analytic set of finite dimension, see [Mz].

For every irreducible component of $V_\alpha \cap V_\beta \cap Z_l$ we fix a point $x_{\alpha\beta l}$ on this component (the subscript l indicates the component), and a chart $V_\alpha \cap V_\beta \supset (V_{\alpha\beta l}, \varphi_{\alpha\beta l}) \ni x_{\alpha\beta l}$ adapted to this component as in (c). Put $H_{\alpha\beta l} := \hat{H}(\Delta^q, \mathsf{Sym}^{d_{\alpha\beta l}}(\Delta^p))$. In the sequel it will be convenient to introduce an order on our finite covering $\{V_\alpha\}$ and write $\{V_\alpha\}_{\alpha=1}^N$.

Consider finite products $\Pi_{(\alpha)}H_\alpha$ and $\Pi_{(\alpha\beta l)}H_{\alpha\beta l}$. In the second product we take only triples with $\alpha < \beta$. These are Banach analytic spaces and by the Projection Changing Theorem of Barlet, for each pair $\alpha < \beta$ we have two holomorphic mappings $\Phi_{\alpha\beta} : H_\alpha \to \Pi_{(l)}H_{(\alpha\beta l)}$ and $\Psi_{\alpha\beta} : H_\beta \to \Pi_{(l)}H_{\alpha\beta l}$. This defines two holomorphic maps $\Phi, \Psi : \Pi_{(\alpha)}H_\alpha \to \Pi_{\alpha<\beta,l}H_{\alpha\beta l}$. The kernel $\mathcal{A}$ of this pair, i.e., the set of $h = \{h_\alpha\}$ with $\Phi(h) = \Psi(h)$, consists exactly from analytic cycles in the neighborhood W_Z of Z. This kernel is a Banach analytic set, and moreover the family $\mathcal{A}$ is an analytic family in W_Z in the sense of Definition 7.8.

Lemma 7.2. *$\mathcal{A}$ is an analytic set of finite dimension.*

Proof. Take a smaller covering $\{V_\alpha^{'}, j_\alpha\}$ of Z. Namely, $V_\alpha^{'} = V_\alpha$ for V_α of the first type and $V_\alpha^{'} = j_\alpha^{-1}(\Delta_{1-\varepsilon}^q \times \Delta^k)$ for the second. In the same manner define $H_\alpha^{'}$ and $H^{'} := \Pi_\alpha H_\alpha^{'}$. Repeating the same construction as above we obtain a Banach analytic set $\mathcal{A}^{'}$. We have a holomorphic mapping $K : \mathcal{A} \to \mathcal{A}^{'}$ defined by the restrictions. The differential $dK \equiv K$ of this map is a compact operator by Montel's theorem.

Let us show that we also have an inverse *analytic* map $F : \mathcal{A}^{'} \to \mathcal{A}$. The analyticity of F means more precisely that it should be defined in some neighborhood of $\mathcal{A}^{'}$ in $H^{'}$. For scales $E_\alpha = (V_\alpha, U_\alpha, B_\alpha, j_\alpha)$ of the second type the mapping $F_\alpha : \mathcal{A}^{'} \to H_Y(\bar{U}_\alpha, \mathsf{Sym}^{d_\alpha}B_\alpha)$ is defined by the isotropicity of the family $\mathcal{A}^{'}$ as in [Ba1]. In particular, this F_α extends analytically to a neighborhood in $H^{'}(!)$ of each point of $\mathcal{A}^{'}$.

For scales $E_\alpha = (V_\alpha, U_\alpha = U_\alpha^{'}, B_\alpha, j_\alpha)$ of the first type define F_α as follows. Let $Y = (Y_\alpha)$ be some point from $H^{'}$. Using the fact that $H_\alpha = H_\alpha^{'}$ in this case, we can correctly define $F_\alpha(Y) := Y_\alpha$ viewed as an element of H_α. This directly defines F_α on the whole $H^{'}$. Analyticity is also obvious.

Put $F := \Pi_\alpha F_\alpha : \mathcal{A}^{'} \to \mathcal{A}$. F is defined and analytic in a neighborhood of each point of $\mathcal{A}^{'}$. Observe further that $\mathsf{Id} - dK \circ dF$ is Fredholm. Since $\mathcal{A}^{'} \subset \{h \in \Pi_{(i)}H_i^{'} : (\mathsf{Id} - K \circ F)(h) = 0\}$, we obtain that $\mathcal{A}^{'}$ is an analytic subset in a complex manifold of finite dimension. □

Remark 7.6. **a)** Isotropicity condition was crucial to get the inverse map F in the proof.
b) If Z was compact cycle then this proof gives the proof of the Douadi's theorem: the space of compact analytic cycles is a finite dimensional analytic space in a neighborhood of each of its points. Remark that in this case one needs to consider only scales of type 2.

Therefore $\mathcal{C}_f$ is an analytic space of finite dimension in a neighborhood of each of its points. $\mathcal{C}_{f,C}$ are open subsets of $\mathcal{C}_f$. Note further that for $C_1 < C_2$ the set $\mathcal{C}_{f,C_1}$ is an open subset of $\mathcal{C}_{f,C_2}$. This implies that for each irreducible component $\mathcal{K}_C$ of $\mathcal{C}_{f,C}$ there is a unique irreducible component $\mathcal{K}$ of $\mathcal{C}_f$ containing $\mathcal{K}_C$ and moreover $\mathcal{K}_C$ is an open subset of $\mathcal{K}$. Of course, in general the dimension of irreducible components of $\mathcal{C}_f$ is not bounded, and in fact the space $\mathcal{C}_f$ is to big. Let us denote by $\mathcal{G}_f$ the union of irreducible components of $\mathcal{C}_f$ that contain at least one irreducible cycle or, in other words, a cycle of the form Γ_{f_z} for some $z \in \Delta^n$. Denote by $\mathcal{Z}_f := \{Z_a : a \in \mathcal{C}_f\}$ the universal family.

Lemma 7.3. *1. Irreducible cycles form an open dense subset $\mathcal{G}_f^0$ in $\mathcal{G}_f$.*

2. The dimension of $\mathcal{G}_f$ is not greater than n.

Proof. *1.* $\mathcal{G}_f^0$ is clearly open, this follows immediately from (4) and (6) of Lemma 7.1. Denote by $\hat{\mathcal{C}}_f$ the normalization of $\mathcal{C}_f$ and denote by $\hat{\mathcal{Z}}_f$ the pull-back of the universal family under the normalization map $\mathcal{N} : \hat{\mathcal{C}}_f \to \mathcal{C}_f$. Consider the following "forgetting of extra compact components" mapping $\Pi : \hat{\mathcal{C}}_f \to \hat{\mathcal{C}}_f$. Note that each cycle $Z \in \hat{\mathcal{C}}_f$ can be uniquely represented as $Z = \Gamma_{f_s} + \Sigma_{j=1}^N B_s^j$, where each B_s^j is a compact analytic q-cycle in $\Delta_s^q(r) \times X$ with connected support. Mark those B_s^j, which possess the following property: there is a neighborhood in V of Z in $\hat{\mathcal{C}}_f$ such that every cycle $Z_1 \in V$ decomposes as $Z_1 = \hat{Z}_1 + B_1$, where B_1 is a compact cycle in a neighborhood of B_s^j in the Barlet space $\mathcal{B}_q(X)$. Our mapping $\Pi : \hat{\mathcal{C}}_f \to \hat{\mathcal{C}}_f$ sends each cycle Z to the cycle obtained from this Z by deleting all the marked components. This is clearly an analytic map. Every irreducible cycle is clearly a fixed point of Π. Thus the set of fixed points is open in $\hat{\mathcal{G}}_f \subset \hat{\mathcal{C}}_f$ and so contains the whole $\hat{\mathcal{G}}_f$.

Now we shall prove that every fixed point Z of Π is a limit of irreducible cycles. For the sequel remark that the compositions $\psi := p \circ \mathbf{ev} : \mathcal{Z}_f \to \Delta^{n+q}$ and $\varphi := \mathsf{pr}_1 \circ \mathbf{ev} \circ \pi^{-1} : \mathcal{C}_f \to \Delta^n$ are well defined. Here $\mathsf{pr}_1 : \Delta^{n+q} \times X \to \Delta^n$ is one more natural projection and $\mathbf{ev} : \mathcal{Z}_f \to \Delta^{n+q} \times X$ is the natural evaluation map. Let $\varphi(Z) = s \in \Delta^n$ and $Z = \Gamma_{f_s} + \Sigma_{j=1}^N B_s^j$. Z being a fixed point of Π means that in any neighborhood of Z one can find a cycle Z_1 such that $Z_1 = \Gamma_{f_{s_1}} + \Sigma_{j=2}^N B_{s_1}^j$, where $B_{s_1}^j$ are compact cycles close to B_s^j. Observe that every cycle in a neighborhood of Z_1 has the same form, *i.e.,* in its decomposition one has $j \geqslant 2$. This follows from Lemma 7.1. Since Z_1 is also a fixed point for Π, we can repeat this procedure N times to obtain finally an irreducible cycle in a given neighborhood of Z. We conclude that $\mathcal{G}_f^0$ is dense in $\mathcal{G}_f$.

2. Take an irreducible $Z \in \mathcal{G}_f^0 \cap \mathsf{Reg}(\mathcal{G}_f)$. Take a neighborhood $Z \in V \subset \mathsf{Reg}(\mathcal{G}_f)$ that consists from irreducible cycles only. Then $\varphi|_V : V \to \Delta^n$ is injective and holomorphic. Therefore $\dim \mathcal{G}_f \leqslant n$. □

Definition 7.9. *We shall call the space $\mathcal{G}_f$ the cycle space associated to a meromorphic map f. By $\mathcal{G}_{f,C}$ we shall denote the open subset of $\mathcal{G}_f$ consisting of Z with $\mathsf{vol}(Z) < C$.*

7.6. Proof of the Main Statement. . Now we are ready to state and prove the main result of this section, namely Theorem 7.3. From now on we restrict our universal family $\mathcal{Z}_f$ onto $\mathcal{G}_f$ without changing notations. I.e., now $\mathcal{Z}_{f,C} := \{Z_a : a \in \mathcal{G}_{f,C}\}$, $\mathcal{Z}_f := \bigcup_{C>0} \mathcal{Z}_{f,C}$ and $\pi : \mathcal{Z}_f \to \mathcal{G}_f$ is the natural projection. $\mathcal{Z}_f$ is a complex space of finite dimension. We have an evaluation map

$$\mathbf{ev} : \mathcal{Z}_f \to \Delta^{n+q} \times X, \tag{7.10}$$

defined by $Z_a \in \mathcal{Z}_f \to Z_a \subset \Delta^{n+q} \times X$. Evaluation map (7.10) will be used in the proof of the theorem below. Recall that we suppose that our complex space X is equipped with some Hermitian metric h.

Theorem 7.3. *Let a holomorphic map $f : \bar{\Delta}^n \times \bar{A}^q(r,1) \to X$ into a complex space X be given. Suppose that:*

1) for every $z \in \bar{\Delta}^n$ the restriction f_z extends meromorphically to the q-disk $\bar{\Delta}_z^q$;

2) the volumes of graphs of these extensions are uniformly bounded;

3) there exists a compact $K \Subset X$ which contains $f(\bar{\Delta}^n \times \bar{A}^q(r,1))$ and $f(\bar{\Delta}_z^q)$ for all $z \in \bar{\Delta}^n$.

Then f extends meromorphically to Δ^{n+q}.

Proof. Denote by $\nu = \nu_q(K)$ the minimal volume of a compact q-dimensional analytic subsets in K, $\nu > 0$ by Lemma 7.1. Denote by W the maximal open subset of Δ^n such that f extends meromorphically onto $\Delta^n \times A^q(r,1) \cup W \times \Delta^q$. Set $S = \Delta^n \setminus W$. Let

$$S_l = \{z \in S : \mathsf{vol}(\Gamma_{f_z}) \leqslant l \cdot \frac{\nu}{2}\}. \tag{7.11}$$

The maximality of W (and thus the minimality of S) and Theorem 7.1 imply that $S_{l+1} \setminus S_l$ are pluripolar and by the Josefson theorem so is S. In particular, $W \neq \varnothing$. Consider the analytic space

$$\mathcal{G}_{f,2C_0,c} := \{Z \in \mathcal{G}_{f,2C_0} : \|\varphi(Z)\| < c\}, \tag{7.12}$$

where $0 < c \leqslant 1$ is fixed. C_0 is taken here such that $\mathsf{vol}(\Gamma_{f_z}) \leqslant C_0$ for all $z \in \bar{\Delta}^n$. Since, by Lemma 7.3 cycles of the form Γ_{f_z} are dense in $\mathcal{G}_{f,2C_0,1}$, we have that for every $Z \in \mathcal{G}_{f,2C_0,1}$ in fact $\mathsf{vol}(\mathbf{ev}(Z)) \leqslant C_0$. Therefore we see that $\bar{\mathcal{G}}_{f,C_0,1} \cap \varphi^{-1}(\Delta^n(1))$ is closed and open in $\mathcal{G}_{f,2C_0,1}$ and in fact coincides with $\mathcal{G}_{f,2C_0,1}$. Closures we take in the cycle space $\mathcal{G}_f$.

For any $c<1$ the set $\bar{\mathcal{G}}_{f,C_0,c}=\varphi^{-1}(\bar{\Delta}^n(c))$ is compact by the Harvey-Shiffman generalization of Bishop's theorem. Therefore $\varphi:\mathcal{G}_{f,2C_0,1}\to\Delta^n$ is proper. Therefore $\mathbf{ev}:\mathcal{Z}_f\to\Delta^{n+q}\times X$ is also proper and by the Remmert proper mapping theorem its image is an analytic set extending the graph of f. The latter follows from the fact that $\varphi(\mathcal{G}_{f,2C_0,1})\supset W$ and therefore in fact $\varphi(\mathcal{G}_{f,2C_0,1})=\Delta^n(1)$. □

7.7. Versions of Levi's theorem. It should be said that Theorem 7.3 will be not always sufficient for us in this text. Moreover, if one looks on it from the point of view of Levi's Theorem 1.4 one naturally becomes interested wether it is possible to give a more precise statement. Two points should be explained here.

First is about boundedness of volumes condition. Of course it is a necessary condition. But also it is not difficult to see that in the case $X=\mathbb{P}^1$ this is exactly the boundedness of poles counted with multiplicities (remark that winding numbers of $f(z_1,\cdot)|_{\partial\Delta}$ are fixed). Therefore for general X it is the boundedness of volumes condition which replaces the boundedness of number of poles in Levi's theorem.

Second, in applications one often deals with the situation where a holomorphic map $f:\Delta^n\times A^q(r,1)\to X$ extends from $A^q_z(r,1)$ to Δ^q_z not for all $z\in\Delta^n$ but only for z in some "thick" set S. Recall that a subset $S\subset\Delta^n$ we call thick at s_0 if for any neighborhood U of s_0 $U\cap S$ is not contained in a proper analytic subset of U. In the case of dimension one, *i.e.*, $n=1$ the set S is thick at s_0 if and only if S contains a sequence $\{s_n\}$ which converges to s_0.

Theorem 7.4. *Let $f:\Delta\times A^q_{1-r,1}\to X$ be a holomorphic map to a reduced complex space X. Suppose that for a sequence $\{s_n\}$ of points in Δ, converging to the origin the restrictions $f_{s_n}:=f|_{A^q_{s_n}}$ extend meromorphically to $\Delta^q_{s_n}$. Suppose in addition that:*

1) there exists a compact $K\Subset X$ such that $\left[\bigcup_{n=1}^{\infty}f(\Delta^q_{s_n})\right]\cup f(\Delta\times A^q_{1-r,1})\subset K$;

2) volumes of graphs $\Gamma_{f|_{\Delta_{s_n}}}$ are uniformly bounded.

Then there exists an $\varepsilon>0$ such that f extends as a meromorphic map to $\Delta(\varepsilon)\times\Delta^q$.

In dimensions starting from two the situation becomes more complicated, the same statement for $f:\Delta^n\times A_{1-r,1}\to X$ with $n\geqslant 2$ fails to be true as the following example shows, see [Iv5].

Example 7.3. *There exists a compact complex 4-fold X^4 and holomorphic mapping $f:\Delta\times\Delta_{\frac{1}{2}}\times A(\frac{1}{2},1)\to X^4$ such that:*

(1) for any $s\in S=\{(z_0,z_2)\in\Delta\times\Delta_{\frac{1}{2}}:|z_0|^2>|z_2|^2\}$ the restriction $f_s=f|_{A_s(r,1)}$ extends holomorphically to Δ_s;

(2) for any $t>1$ there is a constant $C_t<\infty$ such that for all $s\in S_t=\{(z_0,z_2)\in\Delta\times\Delta_{\frac{1}{2}}:|z_0|^2>t\cdot|z_2|^2\}$ one has $\mathsf{area}(\Gamma_{f_s})\leqslant C_t$;

(3) but for all $z\in\Delta^2\setminus\bar{S}=\{(z_0,z_2)\in\Delta\times\Delta_{\frac{1}{2}}:|z_0|^2<|z_2|^2\}$ the inner circle of the annulus $A_z(r,1):=\{z_1\in\Delta_z:1>|z_1|^2>r^2\}$ consists from essentially singular points of $f_z:A_z(r,1)\to X^4$, here $r^2=|z_2|^2-|z_0|^2$.

Remark that S_t in this example is thick at origin. Let us give a condition on X sufficient to maintain the conclusion of Theorem 7.4 also for $n\geqslant 2$. Denote by $\mathbf{ev}:\mathcal{Z}\to X$ the natural evaluation map from the universal space $\mathcal{Z}$ over $\mathcal{B}_q(X)$ to X.

Definition 7.10. *Let us say that X has unbounded cycle geometry in dimension q if there exists a path $\gamma:[0,1[\to\mathcal{B}_q(X)$ with $\mathsf{vol}_{2q}(\mathbf{ev}(Z_{\gamma(t)}))\to\infty$ as $t\to\infty$ and $\mathbf{ev}(Z_{\gamma(t)})\subset K$ for all t, where K is some compact in X.*

Theorem 7.5. *Let $f:\Delta^n\times A^q(r,1)\to X$ be a holomorphic mapping into a normal, reduced complex space X. Suppose that there is a constant $C_0<\infty$ and a compact $K\Subset X$ such that for s in some subset $S\subset\Delta^n$, which is thick at origin the following holds:*

(a) the restrictions $f_s:=f|_{A^q_s(r,1)}$ extend meromorphically onto the polydisk Δ^q_s, and

$\mathsf{vol}(\Gamma_{f_s}) \leqslant C_0$ *for all* $s \in S$;

(b) $f(\Delta^n \times A^q(r,1)) \subset K$ *and* $f_s(\Delta^q) \subset K$ *for all* $s \in S$.

If X *has bounded cycle geometry in dimension* q*, then there exists a neighborhood* $W \ni 0$ *in* Δ^n *and a meromorphic extension of* f *onto* $W \times \Delta^q$.

We shall use the Theorem 7.5 also in the case when $q = 1$ (but not only). In this case it admits a nice refinement. A 1-cycle $Z = \Sigma_j n_j Z_j$ is called rational if all Z_j are rational curves, *i.e.,* an images of the Riemann sphere $\mathbb{P}^1$ in X under a non constant holomorphic mappings. Considering the space of rational cycles $\mathcal{R}(X)$ instead of Barlet space $\mathcal{B}_1(X)$ we can define as in Definition 7.10 the notion of bounded rational cycle geometry.

Corollary 7.2. *Suppose that in the conditions of Theorem 7.5 one has additionally that* $q = 1$. *Then the conclusion of this theorem holds provided* X *has bounded rational cycle geometry.*

Let us give the proofs of Theorems 7.4, 7.5 and of Corollary 7.2.

Case $n = 1$. Define $\mathcal{G}_0$ as the set of all limits $\{\Gamma_{f_{s_n}}, s_n \in S, s_n \to 0\}$. Consider the union $\hat{\mathcal{G}}_0$ of those components of $\mathcal{G}_{f,2C_0}$ that intersect $\mathcal{G}_0$. At least one of these components, say $\mathcal{K}$, contains two points a_1 and a_2 such that Z_{a_1} projects onto Δ^q_0 and Z_{a_2} projects onto Δ^q_s with $s \neq 0$. This is so because S contains a sequence converging to zero. Consider the restriction $\mathcal{Z}_f\mid_{\mathcal{K}}$ of the universal family to $\mathcal{K}$. This is a complex space of finite dimension. Join the points a_1 and a_2 by an analytic disk $h : \Delta \to \mathcal{K}$, $h(0) = a_1, h(1/2) = a_2$. Then the composition $\psi = \varphi \circ h : \Delta \to \Delta$ is not degenerate because $\psi(0) = 0 \neq s = \psi(1/2)$. Here $\varphi := \mathsf{pr}_1 \circ \mathbf{ev} \circ \pi^{-1} : \mathcal{C}_f \to \Delta^n$ was defined in the proof of Lemma 7.3. Map φ restricted to $\mathcal{G}_f$ will be denoted also as φ. Therefore ψ is proper and obviously such is the map $\mathbf{ev} : \mathcal{Z}\mid_{\psi(\Delta)} \to F(\mathcal{Z}\mid_{\psi(\Delta)}) \subset \Delta^{1+q} \times X$. Therefore $\mathbf{ev}(\mathcal{Z}\mid_{\psi(\Delta)})$ is an analytic set in $W \times \Delta^q \times X$ for small enough W extending Γ_f by the reason of dimension. This proves Theorem 7.4.

Case $n \geqslant 2$. We shall treat this case in two steps.

Step 1. Fix a point $z \in \Delta^n$ *such that* $\varphi(\mathcal{G}_f) \ni z$. *Then there exists a relatively compact open* $V \subset \mathcal{G}_f$, *which contains* $\mathcal{G}_{f,C_0}$ *such that* $\varphi(V)$ *is an analytic variety in some neighborhood* W *of* z. Indeed, consider the analytic subset $\varphi^{-1}(z)$ in $\mathcal{G}_f$. Every Z_a with $a \in \varphi^{-1}(z)$ has the form $B_a + \Gamma_{f_z}$, where B is a compact cycle in $\Delta^q_z \times X$. Therefore connected components of $\varphi^{-1}(z)$ parameterize connected and closed subvarieties in $\mathcal{B}_q(\Delta^q \times X)$. Holomorphicity of f on $\Delta^n \times A^q(r,1)$ and condition (b) of the Theorem 7.5 imply that $B_a \subset \bar{\Delta}^q_z \times K$. So, if $\varphi^{-1}(z)$ had non compact connected components, this would imply the unboundedness of cycle geometry of X.

Therefore all connected components of $\varphi^{-1}(z)$ should be compact. Let $\mathcal{K}$ denote the union of connected components of $\varphi^{-1}(z)$ intersecting $\mathcal{G}_{f,C_0}$. Since $\mathcal{K}$ is compact, there obviously exist a relatively compact open $V \Subset \mathcal{G}_f$ containing $\mathcal{G}_{f,C_0}$ and $\mathcal{K}$, and a neighborhood $W \ni z$ such that $\varphi\mid_V : V \to W$ is proper. By the Remmert's proper mapping theorem. $\varphi(V) \subset W$ is an analytic subset of W.

Step 2. If S *is thick at* z *then there exists a neighborhood* $W \ni z$ *such that* f *meromorphically extends onto* $W \times \Delta^q$. Indeed, since $\varphi(V) \supset S \cap W$ and S is thick at the origin, the first step implies that $\varphi(V) \cap W = W$. Since $V \Subset \mathcal{G}_f$ there exist a constant C s.t. $\mathsf{vol}\{Z_s : s \in V\} \leqslant C$. This allows to apply the Theorem 7.3 and obtain the extension of f onto $W \times \Delta^q$. This proves the Theorem 7.5.

Case $q = 1$. The limit of a sequence of analytic disks of bounded area is an analytic disk plus a rational cycle, see Lemma 6.1. Therefore we need to consider only the space of rational cycles in this case. The rest is obvious. This gives Corollary 7.2. □

7.8. A remark about spaces with bounded cycle geometry. To apply the Theorem 7.5 one needs to check the boundedness of cycle geometry of the space X. We shall do that for a wide class of complex spaces in Proposition 7.2 below.

Remark 7.7. We start from the following simple observation:
Every compact complex manifold of dimension $q+1$ carries a strictly positive (q,q)-form $\Omega^{q,q}$ with $dd^c\Omega^{q,q}=0$.

Indeed, either a compact complex manifold carries a dd^c-closed strictly positive (q,q)-form or it carries a bidimension $(q+1,q+1)$-current T with $dd^cT\geqslant 0$ but $\not\equiv 0$. In the case of $\mathsf{dim}\, X=q+1$ such current is nothing but a nonconstant plurisubharmonic function, which doesn't exists on compact X.

a) Let us introduce the class $\mathcal{G}_q$ of normal complex spaces, carrying a nondegenerate positive dd^c-closed strictly positive (q,q)-forms. Note that the sequence $\{\mathcal{G}_q\}$ is rather exhaustive: $\mathcal{G}_q$ contains all compact complex manifolds of dimension $q+1$.

b) Introduce furthermore the class of normal complex spaces $\mathcal{P}_q^-$ which carry a strictly positive (q,q)-form $\Omega^{q,q}$ with $dd^c\Omega^{q,q}\leqslant 0$. Note that $\mathcal{P}_q^-\supset\mathcal{G}_q$.

Proposition 7.2. *Let $X\in\mathcal{P}_q^-$ and let $\mathcal{K}$ be an irreducible component of $\mathcal{B}_k(X)$ such that* $\mathbf{ev}(\mathcal{Z}\mid_{\mathcal{K}})$ *is relatively compact in X. Then:*

1) $\mathcal{K}$ is compact.

2) If $\Omega^{q,q}$ is a dd^c-negative (q,q)-form on X, then $\int_{Z_s}\Omega^{q,q}\equiv const$ for $s\in\mathcal{K}$.

3) X has bounded cycle geometry in dimension k.

Proof. 1) Let $\mathbf{ev}:\mathcal{Z}\mid_{\mathcal{K}}\to X$ be the evaluation map, and let $\Omega^{q,q}$ be a strictly positive dd^c-negative (q,q)-form on X. Then $\int_{Z_s}\Omega^{q,q}$ measures the volume of Z_s. Let us prove that the function $v(s)=\int_{Z_s}\Omega^{q,q}$ is plurisuperharmonic on $\mathcal{K}$. Take an analytic disk $\varphi:\Delta\to\mathcal{K}$. Then for any nonnegative test function ψ on Δ by Stokes theorem and reasons of bidegree we have

$$<\psi,\Delta\varphi^*(v)>=\int_{\Delta}\Delta\psi\cdot\int_{Z_{\varphi(s)}}\Omega^{q,q}=\int_{\mathcal{Z}|_{\varphi(\Delta)}}dd^c(\pi^*\psi)\wedge\Omega^{q,q}=$$

$$=\int_{\mathcal{Z}|_{\varphi(\Delta)}}\pi^*\psi\wedge dd^c\Omega^{q,q}\leqslant 0.$$

Here $\pi:\mathcal{Z}\mid_{\mathcal{K}}\to\mathcal{K}$ is the natural projection. So $\Delta\varphi^*(v)\leqslant 0$ for any analytic disk in $\mathcal{K}$ in the sense of distributions. Therefore v is plurisuperharmonic.

Note that by Harvey-Shiffman generalization of Bishop's theorem $v(s)\to\infty$ as $s\to\partial\mathcal{K}$. So by the minimum principle $v\equiv\mathsf{const}$ and $\mathcal{K}$ is compact again by Bishop's theorem.
2) The same computation shows that $\int_{Z_s}\Omega^{q,q}$ is plurisuperharmonic for any dd^c-negative (q,q)-form. Since $\mathcal{K}$ is proved to be compact, we obtain the statement.
3) Let $\mathcal{R}$ be any connected component of $\mathcal{B}_q(X)$. Write $\mathcal{R}=\bigcup_j\mathcal{K}_j$, where $\mathcal{K}_j$ are irreducible components. From (1) we have that v is constant on $\mathcal{R}$. So if $\{\mathcal{K}_j\}$ is not finite then $\mathcal{R}$ has an accumulation point $s=\mathsf{lim}\, s_j$ by Bishop's theorem, where all s_j belong to different components $\mathcal{K}_j$ of $\mathcal{R}$. This contradicts the fact that $\mathcal{B}_q(X)$ is a complex space. □

Remark 7.8. Remark that a Kähler spaces obviously have bounded cycle geometry in all dimensions. Indeed, if ω is Kähler then $d\omega^q=0$ for any q.

8. Meromorphic mappings with values in Kähler spaces

8.1. Hartogs-type extension theorem for Kähler spaces. Let h be some Hermitian metric on a complex manifold X and let ω_h be the associated $(1,1)$-form. Recall that ω_h (and h itself) is called Kähler if $d\omega_h=0$. When X is a (reduced) complex space a metric form ω on X is defined as follows. Let $\{U_\alpha\}$ be a locally finite, open covering of X such that for every α there exists a holomorphic imbedding $i_\alpha:U_\alpha\to V_\alpha$, where V_α is open in some $\mathbb{C}^N$. Then $\omega|_{U_\alpha}$ should be the restriction of a metric form $\tilde\omega_\alpha$ in V_α to $i_\alpha(U_\alpha)$ *i.e.*, $\omega|_{U_\alpha}=i_\alpha^*\tilde\omega_\alpha$. ω is called Kähler if every $\tilde\omega_\alpha$ is Kähler, *i.e.*, closed. Complex projective space $\mathbb{P}^n$ carries the so called Fubini-Study form, which in homogeneous coordinates $Z = [z_0 : ... : z_n]$ writes as $dd^c \ln \|Z\|^2$ and is therefore Kähler. As a result every projective manifold/algebraic space is Kähler. As it was remarked in Corollary 7.1 every algebraic space possesses the Hartogs extension property for meromorphic mappings. This can be generalized to the case of Kähler spaces.

Theorem 8.1. *Let X be a disk-convex, reduced Kähler space. Then every meromorphic mapping $f : H_r^{n+1} \to X$, $n \geqslant 1$, extends to a meromorphic mapping $\hat{f} : \Delta^{n+1} \to X$.*

Proof. As it was explained in Remark 7.4 we can suppose that X is normal.

Step 1. Area function. For $z = (z_1, ..., z_n) \in \Delta^n$ set $\Delta_z := \{z\} \times \Delta$. Let U be the maximal open subset of Δ^n such that f meromorphically extends to a Hartogs figure over U, *i.e.*, to $H_U^{n+1}(r)$, see (6.2). Remark that the area function (6.5) is well defined for $z \in U$ such that Δ_z is not entirely contained in the *indeterminacy set* I_f of f. As it was explained in Remark 7.1 for such z the restriction $f|_{\Delta_z}$ is well defined and holomorphic. Set

$$A_{\mathrm{v}}' := \{z \in \Delta^n : \Delta_z \cap H_U^{n+1}(r) \subset I_f\} \tag{8.1}$$

and call $A_{\mathrm{v}} := A_{\mathrm{v}}' \times \Delta$ the *vertical subset* of I_f. It is clear that A_{v} is an analytic subset of $H_U^{n+1}(r)$ of codimension $\geqslant 2$ and that A_{v}' itself is a proper analytic subset (of codimension $\geqslant 2$) of Δ^n. Therefore the formula (6.5) has perfectly sense and the integral in the right hand side of it is finite for every $z \in U \setminus A_{\mathrm{v}}'$. We need to prove more: that $a(z)$ is *locally bounded* in $(\bar{U} \cap \Delta^n) \setminus A_{\mathrm{v}}'$.

Step 2. Vanishing of the cohomology of Hartogs domains. Let us improve Lemma 6.2 and prove that for any domain $U \subset \Delta^n$ and any analytic set A in $H_U^{n+1}(r)$ of codimension $\geqslant 2$ one has

$$H_{DR}^2\left(H_U^{n+1}(r) \setminus A\right) = 0, \tag{8.2}$$

where H_{DR} stands for the de Rham cohomology. Indeed, duality (see [Rh], Théorème 17, Chapitre IV) it follows from (6.3) that

$$H_{DR}^2\left(H_U^{n+1}(r)\right) = 0. \tag{8.3}$$

Consider the exact sequence

$$H^2\left(H_U^{n+1}(r)\right) \to H^2\left(H_U^{n+1}(r) \setminus A\right) \to H_A^3\left(H_U^{n+1}(r)\right) \tag{8.4}$$

Since A has real codimension at least four we have that $\mathcal{H}_A^j(\mathbb{C}) = 0$ for $j = 1,2,3$ and therefore the result follows.

Let $\tilde{\omega} := f^*\omega$ be the pull-back of the Kähler form. It is a smooth closed 2-form on $H_U^{n+1}(r) \setminus I_f$. By (8.2) this form is exact, *i.e.*, there exists a smooth 1-form γ on $H_U^{n+1}(r) \setminus I_f$ such that

$$d\gamma = \tilde{\omega}. \tag{8.5}$$

Step 3. Extension outside of the vertical analytic set. We shall prove that f meromorphically extends to $\Delta^{n+1} \setminus A_{\mathrm{v}}$. Take a point $s_0 \in (\partial U \cap \Delta^n) \setminus A_{\mathrm{v}}'$ and find a neighborhood V of it such that $V \Subset \Delta^n \setminus A_{\mathrm{v}}$. Then for $z \in U \cap V$, if V was taken small enough, one has for some constant C independent of $z \in V$

$$a(z) = \int_{\Delta_z} \tilde{\omega} = \int_{\partial \Delta_z} \gamma \leqslant C. \tag{8.6}$$

Boundedness of the cycle geometry of a Kähler space permits us to apply Theorem 7.5 to this case. Indeed, U is obviously thick at s_0. Therefore f extends to $W \times \Delta$ for some neighborhood and the step is proved.

Step 4. Extension across an analytic sets. In the previous step we proved that f holomorphically extends to $\Delta^{n+1} \setminus A_{\mathrm{v}}$. The last step of the proof used the following Thullen-type meromorphic extension statement.

Lemma 8.1. *Let $f : \Delta^{n+1} \setminus A \to X$ be a meromorphic mapping into a disk-convex Kähler space, where A is analytic subset in Δ^{n+1} of codimension $\geqslant 2$. Then f extends to a meromorphic mapping $\hat{f} : \Delta^{n+1} \to X$.*

Take a point $s_0 \in A$ and choose coordinates in a neighborhood $U = \Delta^{n-1} \times \Delta^2$ of it such that the restriction $\pi_A : A \to \Delta^{n-2}$ of the natural projection $\pi : \Delta^{n-1} \times \Delta^2 \to \Delta^{n-1}$ is proper (we suppose that A is of pure codimension two for simplicity). Then the area function $a(z)$ is well defined for $z \in \Delta^{n-1}$ because $A_{\rm v} = \varnothing$ now. Repeating Step 3 we extend f to Δ^{n+1}. Lemma and theorem are proved. □

Let us give a non-linear version of Theorem 8.1 in the spirit of Theorem 2.1. Let's call a family $\{\varphi_t \in \mathsf{Hol}(\Delta_{1+\varepsilon}, \Delta) : t \in T\}$ a *test family* if the exists $N \in \mathbb{N}$ such that for every pair $s \neq t \in T$ there exists a radius $1 - \varepsilon/2 < r < 1 + \varepsilon/2$ such that $(\varphi_s - \varphi_t)|_{\partial\Delta_r}$ doesn't vanish and has winding number $\leqslant N$. As usual by C_t we denote the graph of φ_t.

Corollary 8.1. *Let X be a reduced, disk-convex complex space and $f : R^2_{1-r,1+r} \to X$ a meromorphic mapping. Suppose that there exists an uncountable test family of holomorphic maps $\{\varphi_t : \Delta_{1+\varepsilon} \to \Delta : t \in T\}$ such that $f|_{C_t \cap R^2_{1-r,1+r}}$ holomorphically extends to C_t for every $t \in T$. Then f extends to a meromorphic mapping from a pinched domain $\mathcal{P}$ to X.*

For the proof we refer to [Iv12].

8.2. Banach neighborhoods of stable curves. To finish the sketch of the proof of Theorem 2.3 we need to explain a necessary version of a continuity principle, *i.e.*, the step 3. But in order to do so we shall need first to describe neighborhoods of *non-compact* curves in stable topology. This description will allow us to draw analytic families through two sufficiently close curves, see Proposition 8.1 below.

Let U be an open set in some complex Banach space L.

Definition 8.1. *We say that a closed subset $\mathcal{M} \subset U$ is a Banach analytic set of finite codimension if for every point $m \in \mathcal{M}$ there exists a neighborhood B of m and an analytic map $F : B \to \mathbb{C}^N$, for some N, such that $\mathcal{M} \cap B = \{x \in B : F(x) = 0\}$.*

As we already know nothing good can be said about Banach analytic sets in general. A converging sequence of points in l^2, see example in Remark 2.3 (c), can be realized as a Hilbert analytic set. Moreover, every metric compact, *e.g.*, interval $[0,1]$, can be realized as a Banach analytic set in an appropriate Banach space, see [Rm] p. 33. However, the structure of Banach analytic sets of *finite codimension* is pretty nice according to the following theorem, which is due to Ramis.

Theorem 8.2. *Let $\mathcal{M} \subset U$ be a Banach analytic set of finite codimension, $0 \in \mathcal{M}$. Then there is a neighborhood $B \ni 0$ such that $\mathcal{M} \cap B$ is a finite union of irreducible components $\mathcal{M}_j$, each of them being a finite ramified cover of a neighborhood of zero in the subspace L_j of L of finite codimension.*

Fix now a complex space X and a stable curve (C_0, u_0) over X parameterized by a real surface Σ, see Definition 2.4. The key result we need is the following.

Theorem 8.3. *There exist a Banach analytic sets of finite codimension $\mathcal{M}$ and $\mathcal{C}$ and holomorphic maps $\mathcal{U} : \mathcal{C} \to X$ and $\pi : \mathcal{C} \to \mathcal{M}$, such that:*

a) for any $\lambda \in \mathcal{M}$ fiber $C_\lambda := \pi^{-1}(\lambda)$ is a nodal curve parameterized by Σ and $C_{\lambda_0} = C_0$ for some $\lambda_0 \in \mathcal{M}$;

b) (C_λ, u_λ) with $u_\lambda := \mathcal{U}|_{C_\lambda}$ is a stable curve over X and $u_{\lambda_0} = u_0$;

c) if (C', u') is sufficiently close to (C_0, u_0) in Gromov topology, then there exists $\lambda' \in \mathcal{M}$ such that $(C', u') = (C_{\lambda'}, \mathcal{U}|_{\lambda'})$.

Proof. Let us sketch the proof of this theorem assuming for simplicity that X is a manifold. Let (C_0, u_0) be our stable curve over X. Denote by E the pull-back $u_0^* TX$ of the tangent bundle to X together with natural holomorphic structure on it. We suppose that u_0 extends $L^{1,p}$-smoothly onto the boundary of C_0 to be able to consider the $L^{1,p}$-sections of E, here some $p > 2$ is fixed.

Cover C_0 by a finite family of open sets U_i with boundaries in such a way that intersections $U_{ij} := U_i \cap U_j$ have piecewise smooth boundaries. In the case when U_i contains a nodal point it should be a union of two disks. We take U_i sufficiently small to find a coordinate chart V_i containing $u_0(U_i)$.

If U_i is smooth, consider a complex Banach manifold of holomorphic maps $\mathcal{H}^{1,p}(U_i,V_i)$ with a tangent space at $u_i^0 = u_0\mid_{U_i}$ equal to $\mathcal{H}^{1,p}(U_i,E)$, the space of holomorphic $L^{1,p}$-sections of E over U_i. If U_i is a neighborhood of a node, we consider a complex Banach manifold of holomorphic $L^{1,p}$-maps from $\mathcal{A}$ to V_i with tangent space at u_i equal to $\mathcal{H}^{1,p}(U_i,E)$, the space of pairs of holomorphic $L^{1,p}$-sections f_1 and f_2 over the components $\mathcal{A}_0^1$ and $\mathcal{A}_0^2$ of the standard node $\mathcal{A}_0$ respectively, such that $f_1(0) = f_2(0)$.

Denote by B_i open neighborhoods of u_i^0 in these Banach manifolds. Repeat the same construction for $U_{ij}, i<j$, using as a coordinate chart V_i, and get the Banach manifolds $\mathcal{H}^{1,p}(U_{ij},V_i)$ with tangent spaces $\mathcal{H}^{1,p}(U_{ij},E)$.

Denote by $\varphi_{ij} : V_j \to V_i$ the coordinate change. We can consider the following analytic map between complex Banach manifolds

$$\Phi : \Pi_i B_i \to \Pi_{i<j}\mathcal{H}^{1,p}(U_{i,j},V_i), \qquad (\Phi(\{h_i\}))_{ij} := \varphi_{ij}(h_j) - h_i. \tag{8.7}$$

Zero level set of this map is a Banach analytic set and is by construction some neighborhood in Gromov topology of (C_0,u_0) in the space of stable complex curves over X. Denote this neighborhood by $\mathcal{M}$. Differential $d\Phi_{u_0}$ of this map at u_0 coincides with the differential of Čech complex

$$\begin{array}{rccc} \delta: & \sum_{i=1}^{l}\mathcal{H}^{1,p}(U_i,E) & \longrightarrow & \sum_{i<j}\mathcal{H}^{1,p}(U_{ij},E) \\ \delta: & (v_i)_{i=1}^{l} & \longmapsto & (v_i - v_j) \end{array} \tag{8.8}$$

This differential has the following properties:

i) the image $\mathsf{Im}\,(\delta)$ is of finite codimension and closed; more over, $\mathsf{Coker}\,(\delta) = H^1(C_0,E) = H^1(C_{comp},E)$, where C_{comp} denotes the union of compact irreducible components of C_0;

ii) the kernel $\mathsf{Ker}\,(\delta)$ admits a closed complementing.

Denote by $T = \mathsf{Im}\, d\Phi_{u_0}$ and by S its finite dimensional complement. Let π_T be a projection onto T parallel to S. Implicit function theorem applied to $\pi_T\circ\Phi$ tells us that $\mathcal{M}$ is contained in complex Banach manifold $\mathcal{N}$ with tangent space at u_0 equal to $\mathsf{Ker}\, d\Phi = \mathsf{Ker}\,\delta = \mathcal{H}^{1,p}(C_0,E)$. Now our $\mathcal{M}$ is a Banach analytic subset of $\mathcal{N}$ defined by the equation $\pi_S\circ\Phi = 0$, where π_S is a projection onto finite dimensional vector space S parallel to T. Therefore $\mathcal{M}$ is of finite codimension. □

For more details on the proof we refer to [IS3]. The following proposition is an immediate corollary of Theorem 8.3 and will be used in the proof of the Continuity principle. Consider a sequence (C_n,u_n) of stable curves over a complex manifold X, which converges in Gromov topology to (C_0,u_0). We suppose that C_n are smooth, except C_0.

Proposition 8.1. *There exist a natural N and smooth complex surface $\mathcal{C}$ together with a surjective holomorphic map $\pi_{\mathcal{C}} : \mathcal{C} \to \Delta$ and holomorphic map $\mathcal{U} : \mathcal{C} \to X$ such that the family $\{(C_\lambda,u_\lambda)\}$ with $C_\lambda := \pi_{\mathcal{C}}^{-1}(\lambda)$ and $u_\lambda := \mathcal{U}|_{C_\lambda}$ is a holomorphic family of stable curves over X joining (C_N,u_N) with (C_0,u_0). More precisely:*

1) For every $\lambda \in \Delta$ $C_\lambda = \pi_{\mathcal{C}}^{-1}(\lambda)$ is a connected nodal curve with boundary ∂C_λ and the pair $(C_\lambda, u_\lambda := \mathcal{U}\mid_{C_\lambda})$ is a stable curve over X.

2) For λ outside of zero C_λ is connected and smooth.

3) (C_0,u_0) is our limit and there is $\lambda_N \in \Delta$ such that $(C_{\lambda_N}, u_{\lambda_N}) = (C_N,u_N)$.

4) There are open subsets $W_1,...,W_m$ of $\mathcal{C}$ such that every W_j is biholomorphic to $\Delta\times A_j$, where A_j is an annulus in $\mathbb{C}$ and the following diagrams are commutative

$$\begin{array}{ccc} W_j & \sim & \Delta\times A_j \\ \pi_{\mathcal{C}}\downarrow & & \downarrow\pi_\Delta \\ \Delta & = & \Delta \end{array} \tag{8.9}$$

Each annulus $C_\lambda \cap W_j \cong \{\lambda\} \times A_j$ is adjacent to one boundary component of C_λ, and the number m of domains W_j is equal to the number of boundary components of every C_λ.

Proof. Let a sequence of stable curves over X converges to (C_0, u_0). Take a neighborhood $\mathcal{M}$ of (C_0, u_0) in Gromov topology, which is realized as a Banach analytic set of finite codimension in Banach ball $\mathbf{B}$. Denote by λ_0 the point on $\mathcal{M}$ which corresponds to (C_0, u_0). By the theorem of Ramis $\mathcal{M}$ in the neighborhood of λ_0 has finite number of irreducible components. One of them, denote it as $\mathcal{M}_1$ should contain a point λ_N which corresponds to (C_N, u_N). Further, there is a closed linear subspace L of $\mathbf{B}$ of finite codimension such that $\mathcal{M}_1$ is a finite covering of $L \cap \mathbf{B}$. Now one can easily find an analytic disk passing through λ_0 and λ_N. $\square$

8.3. Continnuity Principle. Let U be a domain in a complex manifold X and let Y be a reduced complex space.

Definition 8.2. *An envelope of meromorphy of U relative to Y is a maximal domain $(\hat{U}_Y, \pi)$ over X, which contains U (i.e., there exists an imbedding $i : U \to \hat{U}_Y$ with $\pi \circ i = \mathsf{Id}$), such that every meromorphic mapping $f : U \to Y$ extends to a meromorphic mapping $\hat{f} : \hat{U}_Y \to Y$.*

Theorem 8.4. (Continuity Principle-I) *Let U be a domain in a complex Hermitian surface (X, ω) and let $(\hat{U}_Y, \hat{\pi})$ be its envelope of meromorphy relative to a disk-convex Kähler space Y. Let (C_n, u_n) be a sequence of smooth curves over $\hat{U}_Y$ parameterized by the same surface Σ, such that*

1) $\mathsf{area}(u_n(C_n))$ with respect to $\pi^*\omega$ are uniformly bounded;

2) u_n $\mathcal{C}^1$-converges in the neighborhood of ∂C_n;

3) $(\pi \circ u_n)(C_n)$ are contained in some compact of U.

Then $u_n(C_n)$ are contained in some compact of $\hat{U}_Y$.

This result can be reformulated in more familiar terms as follows. Let $\{(C_t, u_t)\}_{t\in[0,1]}$ be a continuous (in the Gromov topology) family of complex curves over X with boundaries, parameterized by a unit interval. More precisely, for each $t \in [0,1[$ a smooth Riemann surface with boundary $(C_t, \partial C_t)$ is given together with the holomorphic mapping $u_t : C_t \longrightarrow X$, which is $\mathcal{C}^1$-smooth up to the boundary. Note that C_1 is not supposed to be smooth, *i.e.*, it can be a nodal curve. As well as there is no assumption on hove parameterizations depend on t. Suppose that in the neighborhood V of $u_0(C_0)$ a meromorphic map f into complex space Y is given.

Definition 8.3. *We shall say that mapping f meromorphically extends along the family (C_t, u_t) if for every $t \in [0,1]$ a neighborhood V_t of $u_t(C_t)$ is given, and given a meromorphic map $f_t : V_t \longrightarrow Y$ such that*

a) $V_0 = V$ and $f_0 = f$;

b) if $V_{t_1} \cap V_{t_2} \neq \varnothing$ then $f_{t_1}|_{V_{t_1}\cap V_{t_2}} = f_{t_2}|_{V_{t_1}\cap V_{t_2}}$.

Theorem 8.5. (Continuity principle-II). *Let U be a domain in a complex Hermitian surface (X, ω). Let $\{(C_t, u_t)\}_{t\in[0,1]}$ be a continuous family of complex curves over X with boundaries in a relatively compact subdomain $U_1 \Subset U$. Suppose also that $u_0(C_0) \subset U$ and that C_t for $t \in [0,1[$ are smooth. Then every meromorphic mapping $f : U \to Y$ to a disk-convex Kähler space Y extends meromorphically along the family (C_t, u_t).*

The discussion made above leads to the following

Corollary 8.2. *If U is a domain in a complex Hermitian surface (X, ω) and $\{(C_t, u_t)\}$ a family satisfying the conditions of the continuity principle, then this family can be lifted to the envelope $\hat{U}_Y$, i.e., there exists a continuous family $\{(C_t, \hat{u}_t)\}$ of complex curves in $\hat{U}_Y$ such that $\pi \circ \hat{u}_t = u_t$ for every t.*

Of course, the point here is that the map can be extended to the neighborhood of $u_1(C_1)$, which is a reducible curve having in general compact components. For further details on the proof of the continuity principle we refer to [IS2]. This finishes the proof of Theorem 2.3 and gives, in fact, the following more general statement.

Theorem 8.6. *Let M be a ω-positive immersed two-sphere in a compact Kähler surface (X,ω) having only positive transversal self-intersections and such that $c_1(X)\cdot[M]>0$. Then for any neighborhood U of M its envelope of meromorphy $(\hat{U}_Y,\pi)$ of U relative to any disk-convex Kähler space Y contains a rational curve C such that $c_1(X)\cdot[C]>0$.*

Example 8.1. This example was explained to us by E. Chirka, and it shows that our continuity principle is not valid when the complex dimension of the manifold X is more then two. Take as X the total space of the bundle $\mathcal{O}(-1)\oplus\mathcal{O}(-1)$ over $\mathbb{P}^1$. Denote an affine coordinate on $\mathbb{P}^1$ by z, coordinates on the fibers by ξ_1, ξ_2 and η_1, η_2, such that $\eta_1=z\xi_1$ and $\eta_2=z\xi_2$. Identify $\mathbb{P}^1$ with the zero section of the bundle. Consider meromorphic function $f=e^{\xi_2/\xi_1}$. The set of essential singularity of f is $\{\xi_1=0\}$, which contains the zero section $\mathbb{P}^1$. Consider the following sequence of analytic disks C_n in $U:=X\backslash\{\xi_1=0\}$, $C_n:=\{\xi_2=0,|z|\leqslant n,\xi_1=\frac{z}{n}\}$. The limit of this sequence is $C_0=\mathbb{P}^1\cup\Delta_\infty$, where $\Delta_\infty:=\{\eta_2=0,z=\infty,|\eta_1|\leqslant 1\}$ and f doesn't extend to a neighborhood of this C_0.

Let us give one corollary of the continuity principle as an illustration.

Corollary 8.3. *Let X be a complex surface with one singular normal point p. Let D be a domain in X, $\partial D\ni p$. Suppose there is a sequence (C_n,u_n) of stable curves over $D\subset X$ converging to (C_0,u_0) in Gromov topology and such that*

a) there is a compact $K\subset D$ with $u_n(\partial C_n)\subset K$ for all n;

b) $p\in u_0(C_0)$.

Then every meromorphic function from D extends to the neighborhood of p.

9. Pluriclosed metrics and spherical shells

Unlike the Kähler case meromorphic mappings with values in general complex manifolds can have even point singularities. We already saw such example in the case when X is a Hopf surface, see Example 7.1. The singularity set of the projection π there is a point. For more examples see examples 7.3, 9.2 and 10.1 in this text, as well as §3 in [Iv8].

9.1. Pluriclosed and plurinegative metric forms. It was proved by P. Gauduchon in [Ga] that every Hermitian metric on every compact complex surface is conformal to such metric h that its associated (1,1)-form ω_h is dd^c-closed, i.e., $dd^c\omega_h=0$.

Definition 9.1. *Let h be a Hermitian metric on a complex manifold X and let ω_h be the corresponding $(1,1)$-form. We call ω_h (and h itself) pluriclosed or, dd^c-closed, if $dd^c\omega_h=0$. The form ω_h (and the metric h) is called plurinegative or, dd^c-negative, if $dd^c\omega_h\leqslant 0$.*

Recall that $d^c:=\frac{i}{4\pi}(\bar\partial-\partial)$ and therefore $dd^c=\frac{i}{2\pi}\partial\bar\partial$, in particular $dd^c\ln|z|^2=\delta_0$. If X is a complex space we say that a metric form ω is dd^c-closed (resp. dd^c-negative) if it is locally a restriction of such a form under the local imbedding of X to $\mathbb{C}^N$. Let Ω be a domain in $\mathbb{C}^n$.

Definition 9.2. *A subset $K\subset\Omega$ is called (complete) p-polar if for any $a\in\Omega$ there exist a neighborhood $V\ni a$ and coordinates $z_1,...,z_n$ in V such that the sets $K_{z_I^0}=K\cap\{z_{i_1}=z_{i_1}^0,...,z_{i_p}=z_{i_p}^0\}$ are (complete) pluripolar in the subspaces $V_{z_i^0}:=\{z\in V: z_{i_1}=z_{i_1}^0,...,z_{i_p}=z_{i_p}^0\}$ for almost all $z_I^0=(z_{i_1}^0,...,z_{i_p}^0)\in\pi^I(V)$, where I runs over a finite set of multi-indices with $|I|=p$, such that $\{(\pi^I)^*\omega_e\}_I$ generates the space of (p,p)-forms. Here $\pi^I(z_1,...,z_n)=(z_{i_1},...,z_{i_p})$ denotes the projection onto the space of variables $(z_{i_1},...,z_{i_p})$.*

Suppose that K is, in addition, of Hausdorff codimension four. Take a point $a\in K$ and a complex two-dimensional plane $P\ni a$ such that $P\cap K$ is of dimension zero. A sphere $\mathbb{S}^3=\{x\in P:\|x-a\|=\varepsilon\}$ with $\varepsilon>0$ small enough will be called a *transverse sphere* if $\mathbb{S}^3\cap K=\varnothing$.

Theorem 9.1. *Let $f : H_r^{n+1} \to X$ be a meromorphic map to a reduced disk-convex complex space X which admits a plurinegative Hermitian metric form. Then:*

i) f extends to a meromorphic map $\hat{f} : \Delta^{n+1} \setminus S \to X$, where S is a closed $(n-1)$-polar subset of Δ^{n+1} of Hausdorff dimension $2n-2$;

ii) if, in addition, ω is pluriclosed and $S \neq \varnothing$, then for every transverse sphere $\mathbb{S}^3 \subset \Delta^{n+1} \setminus S$ its image $f(\mathbb{S}^3)$ is not homologous to zero in X.

Remark 9.1. A (two-dimensional) *spherical shell* in a complex space X is the image Σ of the standard sphere $\mathbb{S}^3 \subset \mathbb{C}^2$ under a holomorphic map of some neighborhood of $\mathbb{S}^3$ into X such that Σ is not homologous to zero in X. Theorem states that if the singularity set S of our map f is non-empty and the metric form ω is pluriclosed then X contains spherical shells.

Example 9.1. 1. Let X be the Hopf surface $X = (\mathbb{C}^2 \setminus \{0\})/(z \sim 2z)$ and $f : \mathbb{C}^2 \setminus \{0\} \to X$ be the canonical projection. The $(1,1)$-form $\omega = \frac{i}{2}\frac{(dz,dz)}{\|z\|^2} := \frac{i}{2}\frac{dz_1\wedge d\bar{z}_1 + dz_2\wedge d\bar{z}_2}{\|z\|^2}$ is well defined on X and $dd^c\omega = 0$. In this example one easily sees that f is not extendable to zero and that the image of the unit sphere from $\mathbb{C}^2$ is not homologous to zero in X. Note also that $dd^c f^*\omega = dd^c\omega = -c_4\delta_{\{0\}} dz \wedge d\bar{z}$, where c_4 is the volume of the unit ball in $\mathbb{C}^2$ and $\delta_{\{0\}}$ is the delta-function.

2. In §3.6 of [Iv8] an Example 3.7 of a 4-dimensional compact complex manifold X and a holomorphic mapping $f : \mathbb{B}^2 \setminus \{s_k\} \to X$ is constructed, where $\{s_k\}$ is a sequence of points converging to zero, such that f cannot be meromorphically extended to the neighborhood of any s_k. There one finds also Example 3.6 where the singularity set S is of Cantor-type and pluripolar.

Example 9.2. On a Hopf three-fold $X = (\mathbb{C}^3 \setminus \{0\})/(z \sim 2z)$ the analogous metric form $\omega = \frac{i}{2}\frac{(dz,dz)}{\|z\|^2}$ is no longer pluriclosed but only plurinegative (i.e. $dd^c\omega \leqslant 0$). Moreover, if we consider ω as a bidimension $(2,2)$ current, then it will provide us a natural obstruction for the existence of a pluriclosed metric form on X. For this X the natural projection $\pi : \mathbb{C}^3 \setminus \{0\} \to X$ has singularity of codimension three and X doesn't contains spherical shells of dimension two (but it contains a spherical shell of dimension three).

All compact complex surfaces admit pluriclosed Hermitian metric forms, therefore we obtain:

Corollary 9.1. *If X is a compact complex surface, then:*

(a) every meromorphic map $f : H_r^{n+1} \to X$ extends to $\Delta^{n+1} \setminus A$, where A is an analytic set of pure codimension two;

(b) if Ω is a domain on a Stein surface and $K \Subset \Omega$ is a compact with connected complement, then every meromorphic map $f : \Omega \setminus K \to X$ extends to $\Omega \setminus \{$finite set$\}$. If this set is not empty (respectively, if A from (a) is non-empty), then X contains a spherical shell.

Remark 9.2. The fact that in the case of surfaces the singularity set A is a genuine analytic set requires some additional (not complicated) considerations. They are given in §3.4 of [Iv8], where also some other cases when A can be proved to be analytic are discussed.

There is a hope that the surfaces with spherical shells could be classified, as well as surfaces of class VII_0 containing a rational curve. Therefore the following somewhat surprising speculation, which immediately follows from Corollary 9.1, could be of some interest:

Corollary 9.2. *If a compact complex surface X is not "among the known ones" then for every domain D in a Stein surface every meromorphic mapping $f : D \to X$ is in fact holomorphic and extends as a holomorphic mapping $\hat{f} : \hat{D} \to X$ of the envelope of holomorphy $\hat{D}$ of D into X.*

At this point let us note that the notion of a spherical shell, as we understand it here, is different from the notion of *global spherical shell* from [Ka1] and therefore Corollary 9.2 is indeed not more than a "speculation". A real two-form ω on a complex manifold X is said to "tame" the complex structure J if for any non-zero tangent vector $v \in TX$ we have $\omega(v, Jv) > 0$. This is equivalent to the property that the $(1,1)$-component $\omega^{1,1}$ of ω is strictly positive. Complex manifolds admitting a *closed* form, which tames the complex structure, are of special interest. The class of such manifolds contains all Kähler manifolds. On the other hand, such metric forms are dd^c-closed. Indeed, if $\omega = \omega^{2,0} + \omega^{1,1} + \bar{\omega}^{2,0}$ and $d\omega = 0$, then $\partial\omega^{1,1} = -\bar{\partial}\omega^{2,0}$. Therefore $dd^c\omega^{1,1} = 2i\partial\bar{\partial}\omega^{1,1} = 0$. So the Main Theorem applies to meromorphic mappings into such manifolds. In fact, the technique of the proof gives more:

Corollary 9.3. *Suppose that a compact complex manifold X admits a strictly positive $(1,1)$-form, which is the $(1,1)$-component of a closed form. Then every meromorphic map $f : H_U^{n+1}(r) \to X$ extends to Δ^{n+1}.*

9.2. Proof in dimension two. Let us outline the proof of the Theorem 9.1, first in dimension two, *i.e.*, when $n = 1$.

Step 1. Estimate of the Laplacian. Recall that for an open $U \subset \Delta$ we denote by $H_U^2(r)$ the Hartogs figure over U, see (6.2). Let $f : H_r^2 \to X$ be our mapping. Performing dilatations in the vertical direction we can without loss of generality suppose that f is defined and holomorphic in a neighborhood of $\Delta \times \partial\Delta$.

Lemma 9.1. *If the metric form ω on a disk-convex complex space X is plurinegative and U is maximal open subset of Δ such that f extends to $H_U^2(r)$ then $\partial U \cap \Delta$ is complete polar in Δ.*

Take a point $z_0 \in \partial U \cap \Delta$. Choose a relatively compact neighborhood V of z_0 in Δ. Denote by $T = \frac{i}{2} t^{\alpha\bar{\beta}} dz_\alpha \wedge d\bar{z}_\beta$ the plurinegative current $f^*\omega + dd^c\|z\|^2$. Consider the area function

$$a(z_1) = \frac{i}{2} \cdot \int_{|z_2| \leqslant 1} t^{2\bar{2}}(z_1, z_2) dz_2 \wedge d\bar{z}_2. \tag{9.1}$$

The condition that dd^cT is negative means that

$$\frac{\partial^2 t^{1\bar{1}}}{\partial z_2 \partial \bar{z}_2} + \frac{\partial^2 t^{2\bar{2}}}{\partial z_1 \partial \bar{z}_1} - \frac{\partial^2 t^{1\bar{2}}}{\partial z_2 \partial \bar{z}_1} - \frac{\partial^2 t^{2\bar{1}}}{\partial z_1 \partial \bar{z}_2} \leqslant 0 \tag{9.2}$$

on $H_U^2(r)$. Now we can estimate the Laplacian of a:

$$\Delta a(z_1) = 2i \int_{|z_2| \leqslant 1} \frac{\partial^2 t^{2\bar{2}}}{\partial z_1 \partial \bar{z}_1} dz_2 \wedge d\bar{z}_2 \leqslant 2i \int_{|z_2| \leqslant 1} \left(-\frac{\partial^2 t^{1\bar{1}}}{\partial z_2 \partial \bar{z}_2} + \frac{\partial^2 t^{1\bar{2}}}{\partial z_2 \partial \bar{z}_1} + \frac{\partial^2 t^{2\bar{1}}}{\partial z_1 \partial \bar{z}_2} \right) dz_2 \wedge d\bar{z}_2 =$$

$$= 2i \int_{|z_2|=1} \frac{\partial t^{1\bar{1}}}{\partial z_2} dz_2 + 2i \int_{|z_2|=1} \frac{\partial t^{1\bar{2}}}{\partial \bar{z}_1} d\bar{z}_2 - 2i \int_{|z_2|=1} \frac{\partial t^{2\bar{1}}}{\partial z_1} dz_2 = \psi(z_1). \tag{9.3}$$

Inequality (9.3) holds for $z_1 \in V \cap U$, but the right hand side ψ is smooth in the whole of V. This means that the area behaves roughly as a superharmonic function. Let φ be a smooth solution of

$$\Delta\varphi = \psi. \tag{9.4}$$

Set

$$h := a - \varphi. \tag{9.5}$$

Denote by E the set of points $z_0 \in \partial U \cap V$ such that $a(z) \to +\infty$ as $z \in V, z \to z_0$. Note that $h(z)$ also tends to $+\infty$ in this case and note that h is harmonic in $V \setminus E$. For any point $z_\infty \in [\partial U \cap V] \setminus E$ we can find a sequence $\{z_n\} \subset V, z_n \to z_\infty$ such that $a(z_n) \leqslant C$. By Theorem 7.4 f extends to $\Delta(\varepsilon, z_\infty) \times \Delta$ for some $\varepsilon > 0$. Therefore $a(z)$ must tend to infinity when $z \to \partial U$ as well as h. By standard argument, see the proof of Lemma 2.4 in [Iv8] for details, this implies that h is superharmonic on the whole of V and therefore its polar set E has zero Hausdorff dimension, see [Gl]. Repeating this argument for $V = \Delta$ we get that E is *complete* polar, $h(z) \to +\infty$ when $z \to E$ and that h is harmonic on $\Delta \setminus E$. Lemma is proved.

Remark 9.3. We can add to E the discrete in $\Delta \setminus E$ set of points s_1 such that $\Delta_{s_1} \cap I_f \neq \varnothing$. Adding to h terms $-\ln \frac{|z_1 - s_1|}{2}$ with appropriate coefficients we can insure that the enlarged E is still complete polar, $h = +\infty$ exactly on E and harmonic elsewhere. Set $S_1 = E$ and remark that f is *holomorphic* on $\Delta^2 \setminus (S_1 \times \Delta)$. Finally remark that for pluriclosed ω (9.3) is an *equality*.

Step 2. Extension of the current $T = f^*\omega + dd^c||z||^2$. Interchanging coordinates in $\mathbb{C}^2$ and repeating the Step 1 we see that f holomorphically extends to $\Delta^2 \setminus (S_1 \times S_2)$, where S_1 and S_2 are complete polar compacts (after shrinking). Set $S = S_1 \times S_2$, this is a complete pluripolar compact in Δ^2 of Hausdorff dimension zero and f is holomorphic on $\Delta^2 \setminus S$.

Lemma 9.2. *T has locally summable coefficients on the whole of Δ^2.*

Indeed, consider the area function $a(z_1)$ as in (9.1). We remarked that $h = a - \varphi$ is superharmonic and obviously $\not\equiv +\infty$. Therefore $h \in L^1_{loc}(\Delta)$. Since φ is smooth we get that $a \in L^1_{loc}(\Delta)$. This proves that $t^{2,\bar 2} \in L^1_{loc}(\Delta^2)$. Analogously $t^{1\bar 1} \in L^1_{loc}(\Delta^2)$. Positivity of T means that $t^{1\bar 1}t^{2\bar 2} - |t^{1\bar 2}|^2 \geqslant 0$ and therefore

$$\int_{\Delta^2} |t^{1\bar 2}| \leqslant \int_{\Delta^2} \sqrt{t^{1\bar 1}}\sqrt{t^{2\bar 2}} \leqslant \left(\int_{\Delta^2} t^{1\bar 1}\right)^{1/2}\left(\int_{\Delta^2} t^{2\bar 2}\right)^{1/2}. \tag{9.6}$$

Which means that $t^{1\bar 2} \in L^1_{loc}(\Delta^2)$. Lemma is proved.

$T := f^*\omega + dd^c||z||^2$ has coefficients in $L^1_{loc}(\Delta^2)$ and therefore has trivial extension $\tilde T$ to Δ^2. Set $\mu_T := dd^c\tilde T - \widetilde{dd^cT}$. By Theorem 5.11 μ_T is a non-positive measure supported on S.

Step 3. Appearance of shells. Suppose that the metric form ω on X is pluriclosed. Take a relatively compact disc $D \Subset \Delta$ such that $\partial D \cap S_1 = \varnothing$ and set $W_D := D \times \Delta$. Set furthermore $\partial_0 W_D := \bar D \times \partial\Delta$ and $W_{\partial D} = \partial D \times \bar\Delta$. Therefore $\partial W_D = \partial_0 W_D \cup W_{\partial D}$. Denote by $\widetilde{dd^ca}$ the smooth by (9.3) extension of dd^ca from $D \setminus S_1$ to D. From (9.3) we see that

$$\widetilde{dd^ca} = \frac{i}{8\pi}\Delta a dz_1 \wedge d\bar z_1 = \frac{-1}{4\pi}\left(\int_{\partial\Delta} \frac{\partial t^{1\bar 1}}{\partial z_2}dz_2 + \frac{\partial t^{1\bar 2}}{\partial \bar z_1}d\bar z_2 - \frac{\partial t^{2\bar 1}}{\partial z_1}dz_2\right)dz_1 \wedge d\bar z_1. \tag{9.7}$$

Lemma 9.3. *If ω is pluriclosed then for a relatively compact disk $D \Subset \Delta$ such that $\partial D \cap S_1 = \varnothing$ one has the following two relations*

$$\int_{\partial W_D} d^cT = \int_{\partial D} d^ca - \int_D \widetilde{dd^ca} \tag{9.8}$$

and

$$\int_{\partial W_D} d^cT = \int_{\partial D} d^ch. \tag{9.9}$$

Proof. Write $8\pi d^cT = 2i(\bar\partial - \partial)T = (\partial - \bar\partial)\left[t^{\alpha\bar\beta}dz_\alpha \wedge d\bar z_\beta\right] =$

$$= \frac{\partial t^{1\bar 1}}{\partial z_2}dz_2 \wedge dz_1 \wedge d\bar z_1 - \frac{\partial t^{1\bar 1}}{\partial \bar z_2}d\bar z_2 \wedge dz_1 \wedge d\bar z_1 + \frac{\partial t^{1\bar 2}}{\partial z_2}dz_2 \wedge dz_1 \wedge d\bar z_2 - \frac{\partial t^{1\bar 2}}{\partial \bar z_1}d\bar z_1 \wedge dz_1 \wedge d\bar z_2 +$$
$$+ \frac{\partial t^{2\bar 1}}{\partial z_1}dz_1 \wedge dz_2 \wedge d\bar z_1 - \frac{\partial t^{2\bar 1}}{\partial \bar z_2}d\bar z_2 \wedge dz_2 \wedge d\bar z_1 + \frac{\partial t^{2\bar 2}}{\partial z_1}dz_1 \wedge dz_2 \wedge d\bar z_2 - \frac{\partial t^{2\bar 2}}{\partial \bar z_1}d\bar z_1 \wedge dz_2 \wedge d\bar z_2. \tag{9.10}$$

Therefore we get

$$\int_{\partial_0 W_D} d^cT = \frac{1}{8\pi}\int_D\left(\int_{\partial\Delta} \frac{\partial t^{1\bar 1}}{\partial z_2}dz_2 - \frac{\partial t^{1\bar 1}}{\partial \bar z_2}d\bar z_2 + \frac{\partial t^{1\bar 2}}{\partial \bar z_1}d\bar z_2 - \frac{\partial t^{2\bar 1}}{\partial z_1}dz_2\right)dz_1 \wedge d\bar z_1 = -\frac{1}{2}\int_D \widetilde{dd^ca} -$$
$$- \frac{1}{8\pi}\int_D\int_{\partial\Delta} \frac{\partial t^{1\bar 1}}{\partial \bar z_2}d\bar z_2 \wedge dz_1 \wedge d\bar z_1 = -\frac{1}{2}\int_D \widetilde{dd^ca} + \frac{1}{8\pi}\int_D\int_{\partial\Delta} \frac{\partial t^{1\bar 1}}{\partial z_2}dz_2 \wedge dz_1 \wedge d\bar z_1. \tag{9.11}$$

At the same time again from (9.10) we get

$$\int_{W_{\partial D}} d^cT = \frac{1}{8\pi}\int_{\partial D}\int_\Delta \frac{\partial t^{1\bar 2}}{\partial z_2}dz_2 \wedge dz_1 \wedge d\bar z_2 - \frac{\partial t^{2\bar 1}}{\partial \bar z_2}d\bar z_2 \wedge dz_2 \wedge d\bar z_1 + \frac{\partial t^{2\bar 2}}{\partial z_1}dz_1 \wedge dz_2 \wedge d\bar z_2 -$$
$$- \frac{\partial t^{2\bar 2}}{\partial \bar z_1}d\bar z_1 \wedge dz_2 \wedge d\bar z_2 = \frac{1}{8\pi}\int_{\partial D}\int_\Delta d_{z_2}\left(t^{1\bar 2}dz_1 \wedge \partial\bar z_2 - t^{2\bar 1}dz_2 \wedge \partial\bar z_1\right) + \frac{1}{8\pi}\frac{2}{i}\int_{\partial D}(\partial - \bar\partial)a =$$
$$= \frac{1}{8\pi}\int_{W_{\partial D}} d\left(t^{1\bar 2}dz_1 \wedge \partial\bar z_2 - t^{2\bar 1}dz_2 \wedge \partial\bar z_1\right) + \frac{i}{4\pi}\int_{\partial D}(\bar\partial - \partial)a = \frac{1}{8\pi}\int_{\partial(W_{\partial\Delta})}\left(t^{1\bar 2}dz_1 \wedge \partial\bar z_2 - t^{2\bar 1}dz_2 \wedge \partial\bar z_1\right) +$$
$$+ \int_{\partial D} d^ca = -\frac{1}{8\pi}\int_{\partial(\partial_0 W)}\left(t^{1\bar 2}dz_1 \wedge \partial\bar z_2 - t^{2\bar 1}dz_2 \wedge \partial\bar z_1\right) + \int_{\partial D} d^ca = \int_{\partial D} d^ca -$$

$$-\frac{1}{8\pi}\int_{\partial_0 W} d\left(t^{1\bar 2}dz_1\wedge d\bar z_2 - t^{2\bar 1}dz_2\wedge \partial\bar z_1\right) = \int_{\partial D} d^c a + \frac{1}{8\pi}\int_D\int_{\partial\Delta}\left(\frac{\partial t^{1\bar 2}}{\partial \bar z_1}d\bar z_2 - \frac{\partial t^{2\bar 1}}{\partial z_1}dz_2\right)\wedge dz_1\wedge d\bar z_1 \tag{9.12}$$

From (9.11) and (9.12) we get

$$\int_{W_{\partial D}} d^cT + 2\int_{\partial_0 W_D} d^cT = \int_{\partial D} d^c a - \int_D \widetilde{dd^c a} + \frac{1}{8\pi}\int_D\int_{\partial\Delta}\left(\frac{\partial t^{1\bar 1}}{\partial z_2}dz_2 - \frac{\partial t^{1\bar 1}}{\partial \bar z_2}d\bar z_2\right)\wedge dz_1\wedge d\bar z_1 +$$

$$+\frac{1}{8\pi}\int_D\int_{\partial\Delta}\left(\frac{\partial t^{1\bar 2}}{\partial \bar z_1}d\bar z_2 - \frac{\partial t^{2\bar 1}}{\partial z_1}dz_2\right)\wedge dz_1\wedge d\bar z_1 = \int_{\partial D} d^c a - \int_D \widetilde{dd^c a} + \int_{\partial_0 W_D} d^cT,$$

and this gives (9.8). As for (9.9) remark that since $h = a - \varphi$ with

$$dd^c\varphi = \frac{i}{2\pi}\partial\bar\partial\varphi = \frac{i}{8\pi}\Delta\varphi dz_1\wedge d\bar z_1 = \frac{i}{8\pi}\Delta a dz_1\wedge d\bar z_1 = \widetilde{dd^c a},$$

we get

$$\int_{\partial D} d^c h = \int_{\partial D} d^c a - \int_{\partial D} d^c\varphi = \int_{\partial D} d^c a - \int_D dd^c\varphi = \int_{\partial D} d^c a - \int_D \widetilde{dd^c a} = \int_{\partial W_D} d^cT.$$

□

Corollary 9.4. *Suppose that ω is pluriclosed and that for a relatively compact disk $D \Subset \Delta$ such that $\partial D\cap S_1 = \varnothing$ one has*

$$\int_{\partial W_D} d^cT = 0. \tag{9.13}$$

Then f meromorphically extends to W_D.

Indeed, denote by $\mu_h := dd^c h$ the negative measure supported on S_1. Using smoothing by convolution and (9.8) we obtain

$$\mu_h(D\cap S_1) = \int_D dd^c h = \int_{\partial D} d^c h = \int_{\partial W_D} d^cT. \tag{9.14}$$

If the latter is zero, as assumed, we get that h is smooth in D. That means that a is smooth, *i.e.*, that the area function is bounded near S_1. Theorem 7.3 with $n = q = 1$ gives the extension of f across S_1. Corollary is proved.

Remark 9.4. Now let us explain how do shells appear.

- By Stokes' formula $\int_{\partial W_D} d^cT = \int_{W_D} dd^cT = \mu_T(W_D\cap S)$, the latter is a negative measure supported on the singular set S. Therefore if this integral is non-zero for some W_D we can find a ball $B\subset W_D$ with $\partial B\cap S = \varnothing$ such that $\int_B d^cT \neq 0$.
- At the same time $\int_{\partial B} d^cT = \int_{f(\partial B)} d^c\omega$. If the latter is non-zero this means that $f(\partial B) \not\sim 0$, *i.e.*, is a spherical shell in X.

9.3. Proof in all dimensions. We pass to the case $n \geqslant 2$. To extend f from H_r^{n+1} to Δ^{n+1} it is obviously sufficient to extend f from $\Delta^{n-1}\times H_r^2$ to Δ^{n+1}. If, in addition, some singularities appear they will be of the same nature in both cases. We denote the coordinates in $\mathbb{C}^{n+1}$ as $z = (z', z_n, z_{n+1})$, where $z' := (z_1, ..., z_{n-1})$ and set $\Delta^2_{z'} := \{z'\}\times\Delta^2$. Our problem is local and it is not difficult to see that after a shrinking and choosing the slope of coordinates (z_n, z_{n+1}) appropriately we can suppose that our f is holomorphic on $\Delta^{n-1}\times A_{1-r,1}\times\Delta$. This implies that the indeterminacy set $I_{f_{z'}}$ of every restriction $f_{z'} := f|_{\{z'\}\times H_r^2}$ is discrete. Remark also that $I_f\cap(\{z'\}\times H_r^2) = I_{f_{z'}}$.

Step 4. Let us state this step in the form of a lemma.

Lemma 9.4. *Suppose that the metric form ω is plurinegative. Then there exists a closed $(n-1)$-polar subset $S \subset \Delta^{n+1}$ of Hausdorff dimension $2n-2$ and a holomorphic extension of f to $(\Delta^{n-1}\times\Delta^2)\setminus S$ such that the current $T := f^*\omega$ has locally summable coefficients in a neighborhood of S. Moreover, $dd^c\tilde{T}$ is negative.*

Proof. Remark that by Steps 1,2 for every $z' \in \Delta^{n-1}$ the restriction $f_{z'} := f|_{\{z'\}\times H_r^2}$ holomorphically extends to $\Delta^2_{z'} \setminus S_{z'}$, where $S_{z'}$ are closed complete pluripolar subsets in $\Delta^2_{z'}$ of Hausdorff dimension zero. In addition $S_{z'} = S_{z',1} \times S_{z',2}$ for some complete polar closed subsets $S_{z',1}$ and $S_{z',2}$ in Δ. We want to prove that f extends to a full-dimensional neighborhood of every point of $\Delta^2_{z'} \setminus S_{z'}$. Denote by $\pi_n : \Delta^{n+1} \to \Delta^n$ (resp. $\pi_{n+1} : \Delta^{n+1} \to \Delta^n$) the natural projection $\pi_n : (z', z_n, z_{n+1}) \to (z', z_n)$ (resp. $\pi_{n+1} : (z', z_n, z_{n+1}) \to (z', z_{n+1})$). Take a point $v = (z', z_n, z_{n+1})$ such that $z_{n+1} \notin S_{z',2}$. Find a point $w = (z', z_n, w_{n+1}) \in \{z'\} \times H_r^2$ such that $w_{n+1} \notin S_{z',2}$ as well. After that find a domain $U \Subset \{(z', z_n)\} \times \Delta$ containing both z_{n+1} and w_{n+1} such that $\pi_{n+1}(U) \cap S_{z',2} = \varnothing$. Remark that $f|_{\{z'\}\times\Delta\times U}$ is holomorphic. By Theorem 15.1 there exists a Stein neighborhood V of the graph $\Gamma_{f|_{\{z'\}\times\Delta\times U}}$ in $\Delta^{n+1} \times X$. Consider the mapping $\hat{f}(z) = (z, f(z))$ to the graph. Remark that for $\varepsilon > 0$ small enough we have that $\hat{f}(\Delta^{n-1}_\varepsilon \times \partial\Delta \times U) \subset V$. This follows from holomorphicity of f on $\Delta^{n-1} \times A_{1-r,1} \times \Delta$. As well as for some neighborhood $U_0 \Subset U$ of w_{n+1} we have that $\hat{f}(\Delta^{n-1}_\varepsilon \times \Delta \times U_0) \subset V$. This follows from the fact that $\{(z', w_{n+1})\} \times \Delta$ doesn't intersect I_f. The standard Hartogs theorem for holomorphic mappings into Stein spaces, see Corollary 1.1, implies that $\hat{f}$, and therefore f itself, holomorphically extends to a neighborhood $\Delta^{n-1}_\varepsilon \times \Delta \times U$ of v. By this we extended f to a neighborhood of $\Delta^{n+1} \setminus \bigcup_{z'\in\Delta^{n-1}} \Delta \times S_{z',2}$.

Now let us repeat the same along coordinate z_n. Namely let $v = (z', z_n, z_{n+1})$ be such that $z_{n+1} \in S_{z',2}$ but $z_n \notin S_{z',1}$. Take $w = (z', w_n, z_{n+1})$ such that $w_n \in A_{1-r,1}\setminus S_{z',1}$. Find $U \Subset \Delta\setminus S_{z',1}$ biholomorphic to the disk and such that $U \ni z_n, w_n$. Take ρ such that $\partial\Delta_\rho \cap S_{z',2} = \varnothing$ but $\Delta_\rho \ni z_{n+1}$. f is up to now holomorphically extended to $\Delta^{n-1}_\varepsilon \times U \times \partial\Delta_\rho$. As well as for some neighborhood $U_0 \Subset U$ of w_n f is holomorphic in $\Delta^{n-1}_\varepsilon \times U_0 \times \Delta_\rho$. As above it follows that f holomorphically extends to a neighborhood $\Delta^{n-1}_\varepsilon \times U \times \Delta_\rho$ of v. I.e., f is holomorphically extended to a neighborhood of $\Delta^{n+1} \setminus \bigcup_{z'\in\Delta^{n-1}} S_{z',1} \times \Delta$.

This gives the extension of f to a neighborhood, say U, of

$$\Delta^{n+1} \setminus \bigcup_{z'\in\Delta^{n-1}} S_{z'}. \tag{9.15}$$

Set $S := \Delta^{n+1} \setminus U$. This is a closed subset of Δ^{n+1} such that for every $z' \in \Delta^{n-1}$ one has $S \cap \Delta^2_{z'} \subset S_{z'}$, *i.e.*, S is $(n-1)$-polar. Taking coordinates (z_n, z_{n+1}) with different slopes and then intersecting the sets S thus obtained we obtain a closed $(n-1)$-polar set, call it again S, which for and open set of complex 2-directions has the property that its intersection with every 2-plain in this direction is of dimension zero. By standard geometric measure theory, see ex. [Mt], this implies that S is of Hausdorff dimension $2n-2$.

Consider now the current $T = f^*\omega$ defined on U. Note that T is smooth, positive and $dd^cT \leqslant 0$ there. By Lemma 9.2 for every $z' \in \Delta^{n-1}$ one has that $T_{z'} := T|_{\Delta^2_{z'}} \in L^1_{loc}(\Delta^2_{z'})$ and consequently every $T_{z'}$ extends to a plurinegative current $\tilde{T}_{z'}$ on $\Delta_{z'}$. Apply (5.10) our trivial extensions $\tilde{T}_{z'}$ of $T_{z'}$ to obtain that the masses $\|\tilde{T}_{z'}\|(\Delta^2)$ are uniformly bounded on z' on compacts in Δ^{n-1}. On L^1 the mass norm coincides with the L^1-norm. So taking the second factor in $\Delta^{n-1}\times\Delta^2$ with different slopes and using Fubini theorem we obtain that $T \in L^1_{loc}(\Delta^{n-1}\times\Delta^2)$. Denote by $\tilde{T}$ the trivial extension of T. All that is left to prove is that $dd^c\tilde{T}$ is negative. It is enough to show that for any collection L of $(n-1)$ linear functions $\{l_1, ..., l_{n-1}\}$ the measure $dd^c\tilde{T} \wedge \frac{i}{2}\partial l_1 \wedge \overline{\partial l_1} \wedge ... \wedge \frac{i}{2}\partial l_{n-1} \wedge \overline{\partial l_{n-1}}$ is nonpositive, see [Ho2]. Complete these functions to a coordinate system $\{z_1 = l_1, ..., z_{n-1} = l_{n-1}, z_n, z_{n+1}\}$ and note that for almost all collections L the set $\Delta^2_{z'} \cap U \supset \Delta^2_{z'} \setminus S_{z'}$ for all $z' \in \Delta^{n-1}$. Therefore $\tilde{T}|_{z'}$ are plurinegative for all such z'.

Take a nonnegative function $\varphi \in \mathcal{D}(\Delta^{n+1})$. We have

$$(n-1)! < dd^c\tilde{T} \wedge \frac{i}{2}\partial l_1 \wedge \overline{\partial l_1} \wedge \ldots \wedge \frac{i}{2}\partial l_{n-1} \wedge \overline{\partial l_{n-1}}, \varphi >= \int_{\Delta^{n+1}} \tilde{T} \wedge (dd^c\|z'\|^2)^{n-1} \wedge dd^c\varphi =$$

$$= \int_{\Delta^{n-1}} (dd^c\|z'\|^2)^{n-1} \int_{\Delta^2} (\tilde{T})_{z'} \wedge dd^c\varphi = \int_{\Delta^{n-1}} (dd^c\|z'\|^2)^{n-1} \int_{\Delta^2} \tilde{T}_{z'} \wedge dd^c\varphi =$$

$$= \int_{\Delta^{n-1}} (dd^c\|z'\|^2)^{n-1} \int_{\Delta^2} dd^c(\tilde{T})_{z'} \wedge \varphi \leqslant 0.$$

We used here Fubini theorem for L^1-functions, the fact that $(\tilde{T})_{z'} = \tilde{T}_{z'}$ for currents from L^1_{loc} that are smooth outside of a suitably situated set S, and finally the plurinegativity of $\tilde{T}_{z'}$. Therefore $\tilde{T}$ is plurinegative. Lemma and Step are proved, as well as part *(i)* of the theorem. □

Step 5. The case of a pluriclosed metric form. Let U denotes now the maximal open subset of Δ^{n+1} such that f meromorph8ically extends to U. Set this time $S = \Delta^{n+1} \setminus V$. We have that for every $z' \in \Delta^{n-1}$ one has $S \cap \Delta^2_{z'} \subset S_{z'}$, and the latter is closed, complete polar of Hausdorff dimension zero for every $z' \in \Delta^{n-1}$. Fix a point $a \in S$ and suppose that there is a transversal sphere $\mathbb{S}^3 = \{x \in P : \|x - a\| = \varepsilon\}$ on some two-plane P through a such that $f(\mathbb{S}^3)$ is homologous to zero in X. We shall prove that in this case f meromorphically extends to a neighborhood of a. Write $W = B^{n-1} \times B^2$ for some neighborhood of this point such that $(\bar{B}^{n+1} \times \partial B^2) \cap S = \varnothing$ and for every $z^{'} \in B^{n-1}$ one has $f(\partial B^2_{z'}) \sim 0$.

Lemma 9.5. *Suppose that the metric form w on X is pluriclosed and for all $z' \in B^{n-1}$ $f(\partial B^2_{z'}) \sim 0$ in X. Then:*

i) $dd^c\tilde{T} = 0$ in the sense of distributions.

ii) There exists a $(1,0)$-current γ in W, smooth in $W \setminus S$, such that $\tilde{T} = i(\partial\bar{\gamma} - \bar{\partial}\gamma)$.

The proof will be omitted, see however Lemma 2.8 in [Iv8]. Finally we have the following

Lemma 9.6. *If $\tilde{T}$ is pluriclosed, then the volumes $\Gamma_{f'_z} \cap B^2_z \times X$ are uniformly bounded for $z \in B^{n-1}_r$ and f extends meromorphically onto W.*

Proof. Find $\gamma^{1,0}$ for $\tilde{T}$ as in Lemma 9.5. Smoothing by convolutions we still have $\tilde{T}_\varepsilon = i(\partial\bar{\gamma}^{1,0}_\varepsilon - \bar{\partial}\gamma^{1,0}_\varepsilon)$. Then for $z' \in B^{n-1}$ and $R_{z'} := R \cap B^2_{z'}$ we have:

$$\mathsf{vol}(\Gamma_{f_{z'}}) = \int_{B^2_{z'} \setminus R_{z'}} S^2 = \lim_{\varepsilon \searrow 0} \int_{B^2_{z'} \setminus R_{z'}} \tilde{T}^2_\varepsilon \leqslant \lim_{\varepsilon \searrow 0} \int_{B^2_{z'}} \tilde{R}^2_\varepsilon = \lim_{\varepsilon \searrow 0} \int_{B^2_{z'}} i^2(\partial\bar{\gamma}^{1,0}_\varepsilon - \bar{\partial}\gamma^{1,0}_\varepsilon)^2 \leqslant$$

$$\leqslant \lim_{\varepsilon \searrow 0} \int_{B^2_{z'}} i^2 d(\bar{\gamma}^{1,0}_\varepsilon - \gamma^{1,0}_\varepsilon) \wedge d(\bar{\gamma}^{1,0}_\varepsilon - \gamma^{1,0}_\varepsilon) = \lim_{\varepsilon \searrow 0} \int_{\partial B^2_{z'}} i^2(\bar{\gamma}^{1,0}_\varepsilon - \gamma^{1,0}_\varepsilon) \wedge d(\bar{\gamma}^{1,0}_\varepsilon - \gamma^{1,0}_\varepsilon) =$$

$$= \int_{\partial B^2_{z'}} i^2(\bar{\gamma}^{1,0} - \gamma^{1,0}) \wedge d(\bar{\gamma}^{1,0} - \gamma^{1,0}) \leqslant \mathsf{const}.$$

In the first inequality we used the positivity of T. In the second – the fact that $-i^2\bar{\partial}\bar{\gamma}^{1,0}_\varepsilon \wedge \partial\gamma^{1,0}_\varepsilon$ is positive and $\bar{\partial}\bar{\gamma}^{1,0}_\varepsilon \wedge \bar{\partial}\bar{\gamma}^{1,0}_\varepsilon = 0$. Finally $\gamma^{1,0}_\varepsilon \to \gamma^{1,0}$ on $\bar{B}^{n-1} \times \partial B^2$, since $\gamma^{1,0}$ is smooth there. This gives the required bound for $\mathsf{vol}(\Gamma_{f_{z'}}) = \int_{B^2_{z'} \setminus A_{z'}} S^2$.

Theorem 7.3 (with $q = 2$) gives us now the extension of f onto $W \cong B^{n-1} \times B^2$. We proved that if the singularity set S of f is non-empty then X contains spherical shells. Theorem 9.1 is proved. □

9.4. Proof of Corollary 9.3, Kähler fibrations. Let us now prove Corollary 9.3. Namely, we suppose that the metric form ω on our space X is the $(1,1)$-component of some closed real two-form ω_0, *i.e.*, that there is a $(2,0)$-form $\omega^{2,0}$ such that $\omega_0 = \omega^{2,0} + \omega + \bar{\omega}^{2,0}$ and $d\omega_0 = 0$. As we remarked before stating Corollary 9.3 such ω is obviously dd^c-closed. Therefore the machinery of the proof of Theorem 9.1 applies to this case. Therefore our mapping f can be extended meromorphically to $\Delta^{n+1} \setminus A$, where A is either empty or is $(n-1)$-polar of transverse Hausdorff dimension zero.

Suppose $A \neq \varnothing$. Take a point $a \in A$ with a neighborhood $W \ni a$ biholomorphic to $B^{n-1} \times B^2$ and such that $\pi\mid_{\hat{A}\cap W}: B^{n-1} \times B^2 \to B^{n-1}$ is proper. Here $\hat{A} = A \cup I(f)$ is the union of A with the set of points of indeterminacy of f. Let us prove that $dd^c\tilde{T} = 0$ in W, where $T = f^*\omega$ on $\Delta^{n+1} \setminus \hat{A}$. From Lemma 9.5 we see that all we must prove is that $\int_{\partial B^2_{z'}} d^c\tilde{T}_\varepsilon = 0$ for all $z' \in B^{n-1}$. Indeed, let $T^0 = f^*\omega_0$ and $T^{2,0} = f^*\omega^{2,0}$ on $\Delta^{n+1} \setminus \hat{A}$. Then, since $dT^0 = d^cT^0 = 0$, one has:

$$\int_{\partial B^2_{z'}} d^c\tilde{T}_\varepsilon = \int_{\partial B^2_{z'}} d^c(\tilde{T}_\varepsilon - T^0_\varepsilon) = \int_{\partial B^2_{z'}} d^c(-\tilde{T}^{2,0}_\varepsilon - \bar{T}^{2,0}_\varepsilon).$$

Take a cut-off function η with support in a neighborhood of $B^2_{z'}$. Then

$$\int_{\partial B^2_{z'}} d^c\tilde{T}^{2,0}_\varepsilon = \int_{\partial B^2_{z'}} d^c(\eta\tilde{T}^{2,0})_\varepsilon = \int_{B^2_{z'}} dd^c(\eta\tilde{T}^{2,0})_\varepsilon = 0 \tag{2.5.2}$$

by the reasons of bidegree. So $\tilde{T}$ is pluriclosed on W and we can extend f onto the whole W using Lemma 9.6. Corollary 9.3 is proved.

Let us give a corollary from this result. Recall that a complex deformation of a compact complex manifold X is a complex manifold $\mathcal{X}$ together with a proper surjective holomorphic map $\mathsf{pr} : \mathcal{X} \to \Delta$ od rank one with connected fibers and such that the fiber $\mathcal{X}_0$ over zero is biholomorphic to X. From [Hi] one knows that if $\mathcal{X}_0$ is Kähler this doesn't implies that the neighboring fibers X_t are Kähler.

Corollary 9.5. *Let $\mathsf{pr} : \mathcal{X} \to Y$ be a complex deformation over a compact complex manifold of dimension $n \geqslant 2$ with compact Kähler fibers. Let $S \subset Y$ be a closed subset such that $Y \setminus S$ is Stein/or admits an $(n-1)$-convex exhaustion. Then any meromorphic section of $\mathsf{pr} : \mathcal{X} \to Y$, defined in a neighborhood of S extends to a meromorphic section over the whole of Y.*

Remark 9.5. There is no assumption on how the Kähler metrics on fibers depend on the point on the base. Of course the total space $\mathcal{X}$ don't need to be Kähler and even locally Kähler.

Proof. *Step 1. Every point $y \in Y$ has a neighborhood U such that $\mathcal{X}_U = \mathsf{pr}^{-1}(U)$ possesses a Hermitian metric such that its Kähler form ω_U is a $(1,1)$-component of a closed form.*

To see this take a coordinate neighborhood $U \ni y$ such that $\mathcal{X}_U$ is diffeomorphic to $U \times \mathcal{X}_y$. Let $\mathsf{pr}_1 : U \times \mathcal{X}_y : \mathcal{X}_y$ be the projection onto the second factor. Let ω_y be a Kähler form on $\mathcal{X}_y$. Consider the following 1-form on $\mathcal{X}_U$: $\omega_U = \mathsf{pr}^* dd^c|z|^2 + \mathsf{pr}_1^*\omega_y$, where z is the vector of local coordinates on U. ω_U is d-closed. Its $(1,1)$-component is positive for U small enough, since ω_y is positive on $\mathcal{X}_y$.

Let ρ be a strictly plurisubharmonic Morse exhaustion function on the Stein manifold $W := Y \setminus S$. Set $W_t = \{y \in W : \rho(y) > t\}$. Given a meromorphic section v on the neighborhood of S. Then v is defined on some W_t. The set T of t such that v meromorphically extends onto W_t is non-empty and close.

Step 2. T is open. Let $t \in T$, then v is well defined and meromorphic on W_t. Set $S_t = \{y \in W : \rho(y) = t\}$. Fix a point $y_0 \in S_t$. Take a neighborhood U of y_0 and form ω_U as in the *Step 1.* If y_0 is a regular point of S_t then there exists a Hartogs figure $H \subset W_t$ such that the corresponding polydisk $D \ni y_0$. By Corollary 9.3 the meromorphic mapping $\mathrm{v} : H \to D \times \mathcal{X}_{y_0}$ can be meromorphically extended to D and we are done.

If y_0 is a critical point of S_t we can still apply Corollary 9.3 appropriately placing a Hartogs figure near the critical point of a strictly plurisubharmonic Morse function. Therefore v extends to W_t for all t and Corollary is proved. $\square$

Definition 9.3. *We say that a complex space X possesses a holomorphic (resp. meromorphic) extension property if every holomorphic (resp. meromorphic) mapping $f : H_r^n \to X$ holomorphically (resp. meromorphically) extends to Δ^n.*

By Docquier-Grauert Theorem 1.7 for such X every holomorphic/meromorphic mapping from a domain D in Stein manifold with values in X holomorphically/meromorphically extends to the envelope of holomorphy $\hat{D}$ of D.

Corollary 9.6. *Let X_t be a complex deformation of a compact Kähler manifold X_0. Then for $t \sim 0$ X_t possesses a meromorphic extension property.*

Indeed, Step 1 in the proof of the Corollary 9.5 tells us that for $t \sim 0$ the fiber X_t admits a Hermitian metric such that its associated form is a $(1,1)$-component of a closed form. Therefore Corollary 9.3 applies to X_t.

In concern with the material of this chapter let us ask a few questions.

Question 1. Suppose all X_t for $t \neq 0$ possed meromorphic extension property. Does X_0 possesses it as well? And in other direction: if X_0 possesses a mer. ext. prop. does X_t possesses it for t close to zero?

Conjecture 1. Let $f : \Delta_*^k \to X$ be a meromorphic mapping from a punctured polydisk, $k \geqslant 2$, to a compact complex space X. Suppose that $\mathsf{vol} f(\Delta_*^k) < \infty$. Prove that f meromorphically extends to zero.

Conjecture 2. *Let $f : \Delta_*^{k+1} \to X$ be a meromorphic map from punctured $(k+1)$-disk into a compact complex space of dimension $k+1$, $k \geqslant 1$. Prove that $\mathsf{vol} f(A_{r,1}^k) = O(\mathsf{log}^{\frac{k+1}{k}}(\frac{1}{r}))$.*

It is likely that one can say more about the singularity set A of the extended mapping in Theorems 9.1 and 10.2.

Question 2. Let X is a compact complex manifold carrying a pluriclosed metric form, and let $f : H_r^3 \to X$ is a meromorphic mapping. Let S is a minimal closed subset of Δ^3 such that f extends onto $\Delta^3 \setminus S$. If $S \neq \varnothing$ then each connected component of S should be a complex curve.

For a general complex manifold X without special metrics the answer to the last question can be negative, see examples in the last section of [Iv8].

9.5. Extension of meromorphic correspondences along $(n-1)$-convex exhaustions. Let a meromorphic map $f : D \to Y$ be given, where Y is a reduced complex space and D is a domain in a reduced, normal complex space X which is $(n-1)$-concave at some point $x_0 \in \partial D$. Let $\pi : (U, x_0) \to (\Delta^n, 0)$ is a projection as in the Projection Lemma 18.1. Denote by d the branching number of π. The composition $f \circ \pi^{-1}$ defines in a natural way a d-valued meromorphic correspondence between $H_r^{n,n-1}$ and X.

Definition 9.4. *A d-valued meromorphic correspondence between complex spaces H and X is an irreducible analytic subset $Z \subset H \times X$ such that the restriction $\mathsf{pr}_1|_Z$ of the natural projection onto the first factor to Z is proper, surjective and generically d - to - one. More generally a k-dimensional, $k \geqslant 0$, meromorphic correspondence is a $\mathsf{dim}\, D + k$-dimensional analytic set Γ in $D \times X$ such that the restriction $\mathsf{pr}_1|_\Gamma : \Gamma \to D$ is proper.*

Therefore the extension of f to a neighborhood of x_0 is equivalent to the extension of Z from $H_r^{n,n-1}$ to the associated polydisk. Indeed, suppose that Z extends to Δ^n. Then $Z \circ \pi$ extends *to a correspondence* $\tilde{Z}$ on U. Denote by $\tilde{f}$ the irreducible component of $\tilde{Z}$ which contains the graph of f. If $\tilde{f}$ is not singlevalued then it should have a non-empty branching divisor B. But $B \cap D = \varnothing$, contradiction. It is clear that if f was also a correspondence it will produce no additional complications in the question of extending it to a neighborhood of x_0. So our task is to understand how far the problem of extending of correspondences goes from the problem of extension of mappings. Let Z be a d-valued meromorphic correspondence between the Hartogs figure H and X. Z defines in a natural way a mapping $f_Z : H \to \mathsf{Sym}^d(X)$, the symmetric power of X of degree d. Clearly the extension of Z to Δ^n is equivalent to the extension of f_Z to Δ^n. If X was, for example, a Kähler space, then $\mathsf{Sym}^d(X)$ is a Kähler space by [V]. So, meromorphic correspondences with values in Kähler manifolds are extendable through pseudoconcave boundary points. In fact in [V] it is proved that the Barlet space $B_k(X)$ of k-dimensional compact cycles of a Kähler space X is Kähler again. This implies the extendability also of meromorphic correspondences with values in Kähler spaces.

Corollary 9.7. *Let $f : D \to Y$ be a meromorphic correspondence (of any dimension) from a domain D in a normal complex space X to a disk-convex reduced Kähler space Y. Suppose that D is $(n-1)$-concave at $x_0 \in \partial D$, $n = \mathsf{dim}\, D$. Then f extends as a meromorphic correspondence to a neighborhood of x_0 in X.*

This implies also a Bochner-Hartogs-type statement in the spirit of Theorem 1.6.

Corollary 9.8. *Let X be a normal, $(n-1)$-complete complex space ($n = \mathsf{dim}\, X$), D a relatively compact domain in X and $K \Subset D$ a compact in D such that $D \setminus K$ is connected. Then every meromorphic correspondence between $D \setminus K$ and a disk-convex Kähler space Y can be extended as a meromorphic correspondence to D.*

The proof is literally the same as for meromorphic functions, see Corollary 4.1. For the manifolds and spaces carrying a pluriclosed metric form, this is no longer the case, even if they do not contain spherical shells. The following example is constructed in [Iv8], see Example 3.5 there.

Example 9.3. There exists a compact complex (elliptic) surface Y such that:
(a) every meromorphic map $f : H_r^2 \to Y$ extends meromorphically to Δ^2;
(b) but there exists a two-valued meromorphic correspondence Z between $\mathbb{C}_*^2$ and Y that cannot be extended to the origin.

The point here is that $\mathsf{Sym}^2 Y$ may contain a spherical shell even if Y contains none.

Chapter III. Applications

10. Coverings of compact complex manifolds

Let us apply the extension results of preceding sections to the case when our mapping $f : D \to X$ is a regular cover, see subsection 5.2. To give an immediate idea of the type of applications we have in mind let's state the following well known fact.

Corollary 10.1. *Let D be a bounded domain in $\mathbb{C}^n$ which regularly covers a compact complex manifold X. Then D is a domain of holomorphy.*

Proof follows immediately from Theorem 6.4 and Docquier-Grauert criterium of Theorem 1.7. Indeed, since D is Kobayashi hyperbolic so is also the manifold X. Remark that by theorem of Siegel, see [Sg2], such X is projective, in particular Kähler. Our first question will be if one can remove the assumption on D to be bounded?

10.1. Coverings of Kähler manifolds.

Corollary 10.2. *Let D be a domain over a Stein manifold which regularly covers a compact Kähler manifold X. Then D is Stein.*

Denote as $\mathsf{c} : D \to X$ the covering map. Let us remark that our X cannot contain rational curves. Indeed, let $h : \mathbb{P}^1 \to C \subset X$ be a rational curve. Then $\mathsf{c}^{-1} \circ h$ is well defined, because of simple connectivity of $\mathbb{P}^1$. This gives a rational curve in D, contradiction. To prove that D is Hartogs convex take any locally biholomorphic map $h : H^n_r \to D$. Then $\mathsf{c} \circ h$ extends to a locally biholomorphic map $\hat{h} : \Delta^n \to X$ by Theorem 6.3. Now we can lift $\hat{h}$ back and obtain an extension $\tilde{h}$ of h to Δ^n. Corollary is proved. More generally one has the following.

Corollary 10.3. *Let (D, π) be a domain over a complex manifold Y which regularly covers a compact Kähler manifold X. Then (D, π) is locally pseudoconvex over Y.*

Proof. Let again $\mathsf{c} : D \to X$ be the covering map. Take a point $y_0 \in Y$ and let B_0 be a coordinate ball around y_0. Denote by $\{B_i\}_{i \geqslant 1}$ the set of all connected components of $\pi^{-1}(B_0)$. We need to establish the Hartogs convexity of every B_i. In order to do this take a holomorphic imbedding $h : H^n_r \to B_1$ and, applying Theorem 8.1, extend the composition $\mathsf{c} \circ h$ of h with the cover map $\mathsf{c} : D \to X$ to the polydisk Δ^n, *i.e.,* to a meromorphic map $\tilde{h} : \Delta^n \to X$. The simple connectivity of $\Delta^n \setminus I_{\tilde{h}}$ together with the fact that $\mathsf{c} : D \to X$ is a regular covering gives us a locally biholomorphic extension of h to $\Delta^n \setminus I_{\tilde{h}}$ as a mapping with values in B_1. We want to prove that $I_{\tilde{h}}$ is empty. If this is not the case take a smooth point a on $I_{\tilde{h}}$ and a small polydisk $U = \Delta^n$ with center at a and reduce our situation to the following. There exists a locally biholomorphic map $h : \Delta^n \setminus \Delta^{n-k} \times \{0\} \to B_1$, $k \geqslant 2$, such that:

i) $h_0 := \pi \circ h$ extends to a holomorphic imbedding of Δ^n to B_0;

ii) $\tilde{h} := \mathsf{c} \circ h$ extends meromorphically to Δ^n with $I_{\tilde{h}} = \Delta^{n-k} \times \{0\}$;

iii) a fortiori h is an imbedding on $\Delta^n \setminus \Delta^{n-k} \times \{0\}$.

Items (*i)* and (*iii)* hold true because π and h are locally biholomorphic, therefore one just needs to take U small enough. Take a resolution of $\mathsf{c} \circ h$, *i.e.,* a complex manifold E together with proper holomorphic surjection $p : E \to \Delta^n$ which is biholomorphic over $\Delta^n \setminus \Delta^{n-k} \times \{0\}$ and such that the lift $\hat{h} := \tilde{h} \circ p$ is holomorphic. Using h and (*iii)* we can attach E to B_1, *i.e.,* $Z := E \cup D$ is what is called a local modification of D along a locally closed center $h(\Delta^{n-k} \times \{0\})$. Denote by $\tilde{\mathsf{c}} : Z \to X$ the holomorphic map obtained this way. Take a point $b \in E$ and find a point $c \in D$ such that $\tilde{\mathsf{c}}(b) = \tilde{\mathsf{c}}(c) =: p$. Take a path $\gamma : [0, 1] \to Z$ such that $\gamma(0) = c$ and $\gamma(1) = b$. Then the path $\beta := \tilde{\mathsf{c}}(\gamma)$ is closed. It can be lifted by c^{-1} to D with initial value c. But then it cannot reach b, contradiction. I.e., $I_{\tilde{h}}$ is empty. This means that $\tilde{h}$ is holomorphic on Δ^n and therefore h takes values in B_1 on the whole of Δ^n. The Hartogs convexity of B_1 is proved. Therefore D is locally pseudoconvex. □

Remark 10.1. a) In Corollary 10.3 the condition on X to be compact Kähler can be weakened to that of being of class $\mathcal{C}$, *i.e.,* bimeromorphic to a compact Kähler manifold. Indeed, all that we need for the proof is the extendability of meromorphic mappings, and this property is bimeromorphically invariant.

b) In these corollaries we do exploit the fact that a regular cover is locally biholomorphic, but not in a point of extending it. Only in some additional (easy) speculations. Now let us indicate that in some cases the fact that a covering map is not an arbitrary holomorphic map but it is *locally biholomorphic* can be used also in the difficult part of *extending* it.

Definition 10.1. *Recall that a complex manifold X is called infinitesimally homogeneous if the global sections of its tangent bundle generate the tangent space at each point.*

All parallelizable manifolds are infinitesimally homogeneous, as well as all Stein manifolds and all complex homogeneous spaces under an action of a real Lee group. Every Riemann domain (D, π) over an infinitesimally homogeneous manifold is infinitesimally homogeneous itself. Let (D, π) be such a domain over an infinitesimally homogeneous X.

Theorem 10.1. *Let X be a compact, infinitesimally homogeneous, Kähler manifold. Then every locally biholomorphic mapping $f : D \to X$ from a domain D over a complex manifold into*

X extends to a locally biholomorphic mapping $\hat{f} : \hat{D} \to X$ of the pseudoconvex envelope $\hat{D}$ of D into X.

For the proof we refer to [McK] or [Iv9], the last survey contains more results about the extension of equidimensional mappings. In particular every locally biholomorphic mapping from a domain D in complex manifold Y to $\mathbb{P}^n$ extends to a locally biholomorphic mapping of the pseudoconvex envelope $\hat{D}$ of D to $\mathbb{P}^n$. This was earlier proved in [Iv1] together with some more results on extension of locally biholomorphic mappings in [Iv2].

10.2. Weak convexity of covers of non-Kähler manifolds. Consider the case when X is a compact complex surface regularly covered by some domain $D \subset M$. We can apply then Corollary 9.1 and get the following

Corollary 10.4. *D is either locally pseudoconvex or, equal to a locally pseudoconvex domain minus a discrete set. In the latter case X contains a spherical shell.*

When dimension is $\geqslant 3$ the following example of Kato 10.1 shows that one cannot expect that the covering domain of a compact complex manifold is pseudoconvex minus some "small" set.

Example 10.1. *(M. Kato, [Ka2]).* Namely Kato had constructed a compact three-fold X, which is a quotient of $D = \{[z_0 : \ldots : z_3] \in \mathbb{P}^3 : |z_0|^2 + |z_1|^2 < |z_2|^2 + |z_3|^2\}$ by a co-compact properly discontinuous subgroup $G \subset PGL(4,\mathbb{C})$. Denote by $\pi : D \to X$ the natural projection. Consider the hyperplane $P = \{z \in \mathbb{P}^3 : z_0 = 0\} \cong \mathbb{P}^2$. Then $D \cap P = \mathbb{P}^2 \setminus \bar{\mathbb{B}}^2$, here $\mathbb{B}^2$ is a ball $\{|z_1|^2 > |z_2|^2 + |z_3|^2\}$ in $\mathbb{P}^2$. Therefore $\pi|_{D\cap P} : \mathbb{P}^2 \setminus \bar{\mathbb{B}}^2 \to X$ cannot be extended to a neighborhood of any point on $\partial\mathbb{B}^2$.

However one can get some minimal convexity. Following the approach of [IS5] let us sketch the proof of the following:

Theorem 10.2. *Let X be a compact complex manifold of dimension $n \geqslant 2$. Then every holomorphic map $f : H_r^{n,1} \to X$ with zero-dimensional fibers extends meromorphically to $\Delta^n \setminus S$, where S is a zero-dimensional complete pluripolar set. If S is non-empty then for every ball B with center $s \in S$ such that $\partial B \cap S = \varnothing$ its image $f(\partial B)$ is not homologous to zero in X, i.e., $f(\partial B)$ is a spherical shell (of dimension n) in X.*

Remark 10.2. a) More generally in this theorem one can suppose that X is of any dimension but carries a strictly positive dd^c-closed $(n-1,n-1)$-form.

b) A spherical shell of dimension n in complex manifold/space X is an image Σ of the unit sphere $\mathbb{S}^{2n-1} \subset \mathbb{C}^n$ under a meromorphic map h from a neighborhood of $\mathbb{S}^{2n-1}$ into X such that $\Sigma = h(\mathbb{S}^{2n-1})$ is not homologous to zero in X.

Proof. The proof can be achieved by induction on the dimension n. Case $n+1 = 2$ is a particular case of Theorem 9.1. Let $f : H_r^{n+1,1} \to X$ be given.

Step 1. Reductions. For $s \in \Delta$ set $A_s^n := A_{1-r,1}^n \times \{s\} \subset H_r^{n+1,1}$. Changing coordinates, as it s explained in [IS5] Lemma 5, we can without loss of generality assume that:

a) f is non-degenerate and holomorphic on a neighborhood of $\overline{A_{1-r,1}^n \times \Delta}$;

b) for all $s \in \Delta$ A_s^n contains no curves contracted by f to a point;

c) for all $s \in \Delta$ there do not exist non-empty disjoint open $V_1, V_2 \subset A_s^n$ with $f(V_1) = f(V_2)$.

Denote by U the biggest open subset of Δ such that f can be meromorphically extended onto the Hartogs domain

$$H_U(r) := \left[A_{1-r,1}^n \times \Delta\right] \cup \left[\Delta^n \times U\right].$$

Let ω be a strictly positive (n,n)-form on X with $dd^c\omega = 0$. For $s \in U$ set

$$\mu(s) = \int\limits_{\Delta_s^n} f^*\omega. \tag{10.1}$$

Step 2. The function μ is positive and smooth on U. Moreover, it s Laplacian smoothly extends onto the whole disk Δ. Proof is the same as in Step 1 of the proof of Theorem 9.1. For the case $n=2$ see also Lemma 6 from [IS5].

Step 3. We give it in the form of a lemma.

Lemma 10.1. *Suppose that f is as above (in particular non-degenerate) and that there exists a sequence $\{s_k\}\in U$ converging to $s_0\in\Delta$ such that $\mu(s_k)$ is bounded. Then:*
1) *$f_0 := f|_{A^n_{s_0}}$ meromorphically extends to $\Delta^n_{s_0}$;*
2) *the volumes of the graphs $\Gamma_{f_{s_k}}$ are uniformly bounded in k;*
3) *f meromorphically extends to $\Delta^n\times U_0$ for some neighborhood U_0 of s_0.*

The proof of this lemma is based on Theorem 7.5 with $q=n$. The main point is to bound the volumes of graphs. The latter can be achieved similarly to the proof of an analogous Lemma 7 from [IS5].

Perturbing slightly the slope of coordinates $(z_2,...,z_{n+1})$ we extend f as a holomorphic, zero-dimensional map to $\Delta^{n+1}\setminus S$, where S is an 1-polar closed subset of Δ^{n+1} of zero Hausdorff dimension.

Step 4. $\partial U\cap\Delta$ is complete polar. The proof of this statement is quite similar to the proof of steps 2-3 of Theorem 9.1, see however pp. 705-706 in [IS5] for more details.

$\square$

Corollary 10.5. *The universal cover $\tilde V$ of a compact complex n-fold V is weakly $(n-1)$-convex in the following sense: every meromorphic mapping $f : H^{n,1}_r \to \tilde V$ extends to Δ^n unless V contains an n-dimensional spherical shell.*

Theorem 10.2 just given is stated in [IS5] as Proposition 12 and its proof follows the lines of the proof of the main result of that paper:

Theorem 10.3. *Every meromorphic map $f : H^{3,1}(r)\to X$, where X is a three-dimensional compact complex manifold, extends to a meromorphic map from $\Delta^3\setminus S$ to X, where S is a closed complete pluripolar subset of Hausdorff dimension zero. Moreover, if $S\neq\varnothing$, then for every transversal sphere $\mathbb{S}^5$ in $\Delta^3\setminus S$ its image $f(\mathbb{S}^5)$ is not homologous to zero in X. I.e. if $S\neq\varnothing$ then X should contain a 3-dimensional spherical shell.*

Theorems 9.1, 10.2 and 10.3 suggest the following conjecture. In subsection 7.8 we introduced the class $\mathcal{G}_q$ of reduced complex spaces possessing a strictly positive dd^c-closed (q,q)-form.

Conjecture 3. *We conjecture that every meromorphic map $f : H^{n,n-q}(r)\to X$, where $X\in\mathcal{G}_q$ and is disk-convex in dimension q (e. compact), extends to a meromorphic map from $\Delta^n\setminus S$ to X, where S is a closed $(n-q-1)$-polar subset of transverse Hausdorff dimension zero. Moreover, if $S\neq\varnothing$, then for every transversal sphere $\mathbb{S}^{2q+1}$ in $\Delta^n\setminus S$ its image $f(\mathbb{S}^{2q+1})$ is not homologous to zero in X. I.e. if $S\neq\varnothing$ then X should contain a $(q+1)$-dimensional spherical shell. Moreover, we think that S, if non-empty, should be an analytic set of pure codimension $q+1$ or, at worst, an at most countable union of analytic sets of pure codimension $q+1$.*

Theorem 9.1 proves this conjecture in the case $q=1$, Theorem 10.3 for the case $q=2$, both except conjectured analyticity of the singular set S. General case looks to be quite technical. The main difficulty lies in the fact that it is impossible in general to make the reductions (a)-(c) as above (or, as in §1 from [IS5]). Note that reductions (d)-(e) from §1 of [IS5]can be achieved in all dimensions. A part of these reductions is the following theorem proved in [Cha].

Theorem 10.4. *Let $f : H^n_r\to X$ be a degenerate meromorphic mapping to a compact complex space. Then f extends to Δ^n.*

10.3. Coverings by "large" domains in complex projective space. To stay within a reasonable generality we shall restrict ourselves in this subsection with subdomains of $\mathbb{P}^n$ which cover compact complex manifolds (this includes also subdomains of $\mathbb{C}^n \subset \mathbb{P}^n$). However many statements have an obvious meaning (reformulation) in the case of domains in general complex manifolds. Locally pseudoconvex domains over both $\mathbb{C}^n$ and $\mathbb{P}^n$ are Stein (with one exception: $\mathbb{P}^n$ itself). They can cover both Kähler and non-Kähler manifolds. An example of Stein domain covering a non-Kähler compact manifold is any Inoue surface with $b_2 = 0$. Their universal cover is $\mathbb{C} \times H$, where H is the upper half-plane of $\mathbb{C}$. A domain $D \subset \mathbb{P}^n$ is said to be "large" if its complement $\Lambda := \mathbb{P}^n \setminus D$ is "small" in some sense. Different authors give different sense to the notion of being "small", see [Ka3, La], and therefore we shall reserve ourselves from giving a precise definition.

Let's start from the remark that if $\Lambda \neq \varnothing$ then its Hausdorff n-dimensional (resp. $(n-1)$-dimensional) measure is non-zero if n is even (resp. odd), see [La]. In $\mathbb{P}^2$ or $\mathbb{P}^3$ this means the same: $h_2(\Lambda) > 0$. Both cases is easy to realize by examples. We have the following statement.

Proposition 10.1. *Suppose a domain $D \subset \mathbb{P}^3$ covers a compact complex threefold X.*
Case 1. If the complement $\Lambda = \mathbb{P}^3 \setminus D$ is locally a finite union of two-dimensional submanifolds, then Λ is a union of finitely many complex lines.
Case 2. If the complement $\Lambda = \mathbb{P}^3 \setminus D$ is locally a finite union of three-dimensional submanifolds, then Λ is foliated by complex lines.

Proof. Take a point p on the limit set Λ and find a point $q \in D$ and a sequence of automorphisms $\gamma_n \subset \Gamma$ such that $\gamma_n(q) \to p$. Here Γ is a subgroup of $Aut(D)$ such that $D/\Gamma = X$. Due to the Hausdorff dimension condition on Λ there exists a line $l \ni q$ such that $l \cap \Lambda = \varnothing$. Then $\gamma_n(l)$ will converge to a line in Λ passing through p. $\square$

Remark 10.3. In [Ka3] an example of Λ of dimension 3 in $\mathbb{P}^3$ is constructed.

11. Sets of normality of families of meromorphic mappings

Questions of extension of meromorphic mappings come closely together with questions about their convergence and separate analyticity. Along the following two sections shall say more about these issues.

11.1. Strong convergence of meromorphic mappings. When working with sequences of meromorphic functions and, more generally, mappings one finds himself bounded to consider several notions of their convergence. It occurs that pseudoconvexity or not of domains of convergence/normality in the case of meromorphic mappings crucially depends on the type of convergence one is looking for. Let us describe the ways one can define what does it means that a sequence $\{f_k\}$ of meromorphic mappings between complex manifolds/spaces D and X converge to a meromorphic mapping $f : D \to X$. The only condition that one is supposed to respect is that for holomorphic mappings our notion of convergence should coincide with the uniform convergence on compacts in D. The latter is denoted as $f_k \rightrightarrows f$. The most obvious notion of convergence was already used and called *strong convergence*, see Definition 7.3. It turns that strong convergence is even stronger that it is postulated in its definition.

Theorem 11.1. *If f_k strongly converge to f then for every compact $K \Subset D$ the volumes $\Gamma_{f_k} \cap (K \times X)$ are uniformly bounded and therefore Γ_{f_k} converge to Γ_f in the topology of cycles.*

For the proof we refer to [IN]. Here is one more nice feature of the strong convergence.

Theorem 11.2. *(Rouché's principle). Let a sequence of meromorphic mappings $\{f_k\}$ between normal complex spaces D and X strongly converge on compacts in D to a meromorphic map f. Then:*

(a) If f is holomorphic then for any relatively compact open subset $D_1 \subset D$ all

restrictions $f_k|_{D_1}$ are holomorphic for k big enough, and $f_k \rightrightarrows f$.
(b) If $\{f_k\}$ are holomorphic then f is also holomorphic and $f_k \rightrightarrows f$.

For the proof we refer to [Iv7]. Strong convergence has also some disadvantages. For example domains of strong convergence and strong normality are quite arbitrary. We shall explain this in more details. Let $\mathcal{F}$ be a family of meromorphic mappings from a normal complex space D to a disk-convex reduced complex space X.

Definition 11.1. *The* set of normality *of $\mathcal{F}$ is the maximal open subset of D, we shall denote it as $\mathcal{N}_{\mathcal{F}}$, such that $\mathcal{F}$ is relatively compact on $\mathcal{N}_{\mathcal{F}}$. If $\mathcal{F} = \{f_k\}$ is a sequence then the* set of convergence *of $\mathcal{F}$, denote as $\mathcal{C}_{\mathcal{F}}$, is the maximal open subset of D such that f_k converge on compacts of this subset.*

To be relatively compact in this definition means that from every sequence of elements of $\mathcal{F}$ one can extract a converging on compacts subsequence. The sense of convergence (strong or other, see below) should be each time specified.

Example 11.1. a) Let X be $\mathbb{P}^3$ blown up in one point. Then for every open subset D of $\mathbb{C}^2$ one can find a sequence of holomorphic mappings of $\mathbb{C}^2$ to X with D as its set of strong normality. To see this, let (w_1, w_2, w_3) be coordinates of the affine part of $\mathbb{P}^3$. We suppose that the blown-up point is zero in these coordinates. For $a = (a_1, a_2) \neq (0,0)$ and $n \in \mathbb{N}$ define a mapping $f_{n,a} : \mathbb{C}^2 \to X$ by $(w_1, w_2, w_3) = (z_1 - a_1, z_2 - a_2, 1/n)$. If one takes A to be the set of all points in $\mathbb{C}^2 \setminus \bar{D}$ with rational coordinates, then $\mathcal{F} = \{f_{n,a} : n \in \mathbb{N}, a \in A\}$ will be the family with $\mathcal{N}_{\mathcal{F}} = D$ in the strong sense.
b) Let X be a Hopf three-fold $X := \mathbb{C}^3 \setminus \{0\}/z \sim 2z$. Denote by $\pi : \mathbb{C}^3 \setminus \{0\} \to X$ the canonical projection. Let $D \Subset \mathbb{C}^2$ be any bounded domain. Take a sequence $\{a_n\} \subset D$ accumulating to every point on ∂D. Let $g_n : \mathbb{C}^2 \to \mathbb{C}^3$ be defined as $g_n(z) = (z - a_n, 1/n)$. Set $f_n := \pi \circ g_n$. Then the set of strong normality of $\mathcal{F} = \{f_n\}$ has D as one of its connected components.

11.2. Weak convergence. Let D and X be reduced complex spaces, D normal, and let $\{f_k\} \subset \mathcal{M}(D, X)$ be a sequence of meromorphic mappings.

Definition 11.2. *We say that f_k converge* weakly *to $f \in \mathcal{M}(D, X)$ (w-converge) if there exists an analytic subset A in D of codimension at least two such that f_k converge strongly to f on compacts in $D \setminus A$.*

If in Example 11.1 (b) one takes a converging to zero sequence $\{a_n\}$ then corresponding f_n will converge on compacts of $\mathbb{C}^2 \setminus \{0\}$ but the limit will not extend to zero meromorphically. I.e., f_n will not converge weakly in any neighborhood of the origin. If one does the same in Example 11.1 (a) then f_n will converge (but only weakly).

Remark 11.1. f_k converge weakly to f if and only if for every compact of $D \setminus I_f$ all f_k are holomorphic in a neighborhood of this compact for k big enough and uniformly converge there to f as holomorphic mappings. Indeed, let A be the minimal analytic set of codimension $\geqslant 2$ such that f_k converge strongly to f on $D \setminus A$. Then A must be contained in I_f because if there exists a point $a \in A \setminus I_f$ then f is holomorphic in some neighborhood $V \ni a$ and then, by the Rouché Principle f_k for $k \gg 1$ are holomorphic on compacts in $V \setminus A$ and converge uniformly (on compacts) to f there. From here and the fact that $\operatorname{codim} A \geqslant 2$ one easily gets that f_k are holomorphic on compacts on the whole of V and converge to f uniformly.

Now let us turn to the sets of weak convergence/normality. Sets of strong normality obviously are well defined, *i.e.,* they do exist. The existence of sets of weak normality follows from Remark 11.1, see however the proof of Corollary 1.2.1(a) in [Iv7]. Domains of weak convergence/normality of meromorphic mappings turn to be pseudoconvex for a large class of target manifolds (unlike to the case of strong convergence). This follows from the "mutual propagation principle" of the following theorem.

Theorem 11.3. *Let D be a domain in a Stein manifold $\hat{D}$ such that $\hat{D}$ is an envelope of holomorphy of D and let $f_k : \hat{D} \to X$ be a weakly converging on D sequence of meromorphic mappings with values in a disk-convex complex space X. Then:*

(a) If the weak limit f on f_k meromorphically extends from D to $\hat{D}$ then f_k weakly converge to f on the whole of $\hat{D}$.

(b) If, in addition, the space X carries a pluriclosed metric form then the weak limit f of f_k meromorphically extends to $\hat{D}$ and then the part (a) applies.

For the proof we refer to [Iv7]. As a result the sets of weak normality are locally pseudoconvex provided the target is disk-convex and carries dd^c-closed metric form. Namely we have the following

Corollary 11.1. *Let $\mathcal{F} \subset \mathcal{M}(D,X)$ be a family of meromorphic mappings from a complex manifold D to a disk-convex space X which carries dd^c-closed metric form. Then the set of weak normality $\mathcal{N}_{\mathcal{F}}$ of $\mathcal{F}$ is pseudoconvex. If $\mathcal{F} = \{f_k\}$ is a sequence then the set of its weak convergence is pseudoconvex.*

The proof of this corollary clearly follows from Theorem 11.3.

11.3. Gamma convergence. Let again $\{f_k\}$ be a sequence of meromorphic mappings between complex spaces D and X. Let $f \in \mathcal{M}(D,X)$ be a meromorphic map.

Definition 11.3. *We say that f_k Γ-converge to f if:*

i) there exists an analytic subset $A \subset D$ such that f_k strongly converge to f on compacts of $D \setminus A$;

ii) for every divisor H in X, such that $f(D) \not\subset H$ and every compact $K \Subset D$ the volumes of $f_k^*H \cap K$ counted with multiplicities are uniformly bounded for $k \gg 1$.

Remark 11.2. This notion is weaker than weak convergence because A can have components of codimension one. It might be convenient to add to A the indeterminacy set of f and then, see Remark 11.1, f_k will converge to f uniformly on compacts of $D \setminus A$ as holomorphic mappings. Condition (*ii*) is also satisfied for a weakly converging sequence, because divisors f^*H extend from $D \setminus A$ to D and if they have bounded volume on compacts of $D \setminus A$ then the same is true on compacts of D. All this obviously follows from the ingredients involved in the proof of Bishop's compactness theorem, see [St].

Example 11.2. *a)* Consider the following sequence of holomorphic mappings $f_k : \Delta \to \mathbb{P}^1$:

$$f_k : z \to \left[1 : 1 + \frac{1}{z} + \ldots + \frac{1}{z^k k!}\right] = \left[z^k : z^k + z^{k-1} + \ldots + \frac{1}{k!}\right]. \tag{11.1}$$

It is clear that f_k converge on compacts of $\Delta \setminus \{0\}$ to $f(z) = [1 : e^{\frac{1}{z}}]$ but, as it is clear from the second expression in (11.1), the preimage counting with multiplicities of the divisor $H = \{Z_0 = 0\}$ is $k[0]$ (here $[Z_0 : Z_1]$ are homogeneous coordinates in $\mathbb{P}^1$), *i.e.,* has unbounded volume. And indeed, this sequence should not be considered as converging one, because its limit is not holomorphic on Δ.

b) Set $f_k(z) = [z : z - \frac{1}{k}] : \Delta \to \mathbb{P}^1$. This sequence clearly converges to the constant map $f(z) = [z : z] = [1 : 1]$ on compacts of $\Delta \setminus \{0\}$. Moreover, the preimage of any divisor $H = \{P(z_0, z_1) = 0\} \not\supset f_k(\Delta)$ is $\{z \in \Delta : P(z, z - \frac{1}{k}) = 0\}$, *i.e.,* is a set of points, uniformly bounded in number counting with multiplicities. Therefore this sequence Γ-converge (but doesn't converge weakly).

Example 11.3. Consider the following sequence of meromorphic functions on Δ^2 (*i.e.,* meromorphic mappings to $\mathbb{P}^1$):

$$f_k(z_1, z_2) = [z_1 : 2^{-k} z_2^k].$$

The limit map is constant $f(z) = [1 : 0]$. f_k converge to f strongly (uniformly in fact) on compacts of $\Delta^2 \setminus \{z_1 = 0\}$. If $[Z_0 : Z_1]$ are homogeneous coordinates in $\mathbb{P}^1$ then the preimage of the divisor $[Z_1 = 0]$ is $k[z_2 = 0]$, *i.e.,* this sequence doesn't converge even in Γ-sense.

Remark 11.3. Examples 11.2 and 11.3 are examples of sequences converging outside of an analytic set of codimension one, which are not Γ-converging. In the first case the limit doesn't extend to the whole source, in the second it does. Convergence of meromorphic mappings of this type was introduced and studied by Rutishauser in [Ru].

If in Definition 11.1 the underlying convergence is the Γ-convergence we get the corresponding notions of a convergence/normality set. Let us conclude this general discussion with the following

Proposition 11.1. *Let X be a disk-convex complex space carrying a pluriclosed metric form. Then the sets of Γ-convergence/normality of meromorphic mappings with values in X are locally pseudoconvex.*

For the proof we refer to [IN].

Remark 11.4. Strong convergence (or *s*-convergence) will be denoted by $f_k \to f$, weak one (or *w*-convergence) as $f_k \rightharpoonup f$, and Γ-convergence as $f_k \xrightarrow{\Gamma} f$. Note that in the second and third definitions we *impose* that the limit f is defined and meromorphic on the whole of D if, even, the convergence takes place only on some part of D. In the first case the limit exists on the whole of D automatically, see once more Example 11.1 in concern with this.

11.4. Convergence of meromorphic functions. For the better understanding of these notions of convergence let us give their description in the case when X is projective, *i.e.,* imbeds into $\mathbb{P}^N$ for some N. In that special case the notions of convergence listed above permit an explicit analytic description as follows. Every meromorphic mapping f with values in $\mathbb{P}^N$ can be locally represented as (7.1) and its indeterminacy set I_f is then given as in (7.2). One has the following.

Theorem 11.4. *Let $\{f_k\}$ be a sequence of meromorphic mappings from a complex manifold D to $\mathbb{P}^N$. Then:*

i) $f_k \xrightarrow{\Gamma} f$ if and only if for any point $x_0 \in D$ there exists a neighborhood $V \ni x_0$, reduced representations $f_k = [f_k^0 : ... : f_k^N]$ and not necessarily reduced representation $f = [f^0 : ... : f^N]$ such that for every $0 \leqslant j \leqslant N$ the sequence f_k^j converge to f^j uniformly on V;

ii) $f_k \rightharpoonup f$ if and only if $f_k \xrightarrow{\Gamma} f$ and the limit representation $f = [f^0 : ... : f^N]$ is reduced;

iii) $f_k \to f$ if and only if $f_k \rightharpoonup f$ and corresponding non-pluripolar Monge-Ampère masses converge, i.e., for every $1 \leqslant p \leqslant n = \dim D$ one has

$$\left(dd^c \|z\|^2\right)^{n-p} \wedge \left(dd^c \ln \|f_k\|^2\right)^p \to \left(dd^c \|z\|^2\right)^{n-p} \wedge \left(dd^c \ln \|f\|^2\right)^p \tag{11.2}$$

weakly on compacts in D.

Here in (11.2) we suppose that $V = \Delta^n$, $z_, ..., z_n$ are standard coordinates and $\|f\|^2 = |f^0|^2 + ... + |f^N|^2$, *i.e.,* $dd^c \ln \|f\|^2$ is the pullback of the Fubini-Study form by f. Non-pluripolar MA mass of $\ln \|f\|^2$ of order p here means

$$\int_{D\setminus Z_f} \left(dd^c \|z\|^2\right)^{n-p} \wedge \left(dd^c \ln \|f\|^2\right)^p, \tag{11.3}$$

where Z_f is the analytic sets of common zeroes of $f^0, ..., f^N$. If this couple has ho common divisors then $Z_f = I_f$. For the proof of this theorem we refer to [IN].

Remark 11.5. a) Reducibility or not of the limit representation $f = [f^0 : ... : f^N]$ in this theorem doesn't depend on the choice of *converging* representations $f_k = [f_k^0 : ... : f_k^N]$, provided they are taken to be reduced (the last can be assumed always). Indeed, any other reduced representation of f_k has the form $f_k = [g_k f_k^0 : ... : g_k f_k^N]$, where g_k is holomorphic and nowhere zero. If the newly chosen representations converge to some representation of f then g_k must converge, say to g and this g is nowhere zero by Rouché's theorem. Therefore the obtained representation of the limit is $f = [gf^0 : ... : gf^N]$ and it is reduced if and only if $f = [f^0 : ... : f^N]$ was reduced.

Let us make the following remark. Let ω_{FS} be the Fubini-Study form on $\mathbb{P}^N$. For a holomorphic map $f : \bar{\Delta} \to \mathbb{C}^N$ (we suppose f to be defined in a neighborhood of the closure $\bar{\Delta}$), the area of $f(\Delta)$ with respect to the Fubini-Study form is

$$\mathsf{area}_{FS}f(\Delta)=\int_{\Delta}f^*\omega_{FS}. \tag{11.4}$$

Denote by $Z=(Z_0,...,Z_N)$ coordinates in $\mathbb{C}^{N+1}$ and let $\pi:\mathbb{C}^{N+1}\setminus\{0\}\to\mathbb{P}^N$ be the standard projection. Consider the following singular $(1,1)$-form on $\mathbb{C}^{N+1}$

$$\omega_0=dd^c\ln\|Z\|^2. \tag{11.5}$$

Lemma 11.1. *For a holomorphic lift $F=(f^0,...,f^N):\bar{\Delta}\to\mathbb{C}^{N+1}$ of $f:\bar{\Delta}\to\mathbb{P}^N$ (i.e., $f=\pi\circ F$) such that $F|_{\partial\Delta}$ doesn't vanishes one has*

$$\mathsf{area}_{FS}f(\Delta)=\int_{\partial\Delta}d^c\ln\|F\|^2-N_F. \tag{11.6}$$

Here N_F is the number of zeroes of F counted with multiplicities.

This readily follows from the King's residue formula, see [Kg], but we shall give a simple direct proof. By the very definition of the Fubini-Study form one has $\pi^*\omega_{FS}=\omega_0$. And therefore in a neighborhood of a point $a\in\Delta$ such that $F(a)\neq 0$ one has that $f^*\omega_{FS}=F^*\omega_0$. As the result

$$\mathsf{area}_{FS}f(\Delta)=\int_{\Delta}f^*\omega_{FS}=\int_{\Delta\setminus Z_F}F^*\omega_0, \tag{11.7}$$

where $Z_F:=\{z_1,...,z_k\}$ is the set of zeroes of F, i.e., such z_l that $f^j(z_l)=0$ for all $j=0,...,N$. Let n_l be the multiplicity of zero z_l. Then $F(z)=(z-z_i)^{n_l}(g^0(z),...,g^N(z))$, where at least one of g^j-s is not zero at z_l. We have that

$$dd^c\ln\|F\|^2=n_l\delta_{z_l}+dd^c\ln\|G\|^2,$$

where $G(z)=(g^0(z),...,g^N(z))$. Therefore $dd^c\ln\|G\|^2$ is an extension of $F^*\omega_0$ to z_l. The rest obviously follows from the Stokes formula.

Let us observe the following immediate corollary from this lemma.

Corollary 11.2. *Let $f_k:D\to\mathbb{P}^N$ be a Γ-converging sequence of meromorphic mappings and let L be a divisor in D such that f_k converge uniformly on compacts of $D\setminus L$. Let $V\cong\Delta^{n-1}\times\Delta$ be a chart adapted to L and to the limit M of $f_k^*H_0$, where $H_0=[Z_0=0]$. Then the areas of the analytic discs $f_k(\Delta_{z'})$ are uniformly bounded in $z'\in\Delta^{n-1}$ and $k\in\mathbb{N}$.*

Indeed, let (z',z_n) be coordinates in $\Delta^{n-1}\times\Delta$. Denote by $F_k=(f_k^0,...,f_k^N)$ lifts of f_k to $\mathbb{C}^{N+1}$. Consider restrictions $f_k|_{\Delta_{z'}}$. Due to the fact that our chart is adapted to $M=\mathsf{lim}\, f_k^*H_0$ we have that f_k^0 doesn't vanishes on $\partial\Delta_{z'}$ for $k\gg 1$ and, since it is also adapted to L the lifts $F_k=(f_k^0,...,f_k^N)$ converge in a neighborhood of $\partial\Delta_{z'}$. By (11.6) we have

$$\mathsf{area}_{FS}f_k(\Delta_{z'})\leqslant\int_{\partial\Delta_{z'}}d^c\ln\|F_k\|^2\leqslant c, \tag{11.8}$$

i.e., the areas are uniformly bounded for $z'\in\Delta^{n-1}$ and all k.

Now let us discuss the convergence of meromorphic functions. Meromorphic functions on a complex manifold D are exactly the meromorphic mappings from D to $\mathbb{P}^1$. I.e., all our previous results and notions are applicable to this case. Therefore we can summarize as follows.

Corollary 11.3. *Volumes of graphs of Γ-converging sequence of meromorphic functions are uniformly bounded on compacts. If a sequence $\{f_k\}$ of meromorphic functions converges weakly then it converges strongly. Domains of convergence/normality of meromorphic functions are pseudoconvex for all types of convergence: weak=strong and gamma.*

Indeed, let f be the gamma limit of f_k. Inequality (11.8) implies that in an appropriately chosen local coordinates (z', z_n) one has

$$\mathsf{vol}(\Gamma_{f_k|_{\Delta^n}}) = \int_{\Delta^n}(dd^c||z||^2)^n + \int_{\Delta^n}(dd^c||z||^2)^{n-1}\wedge f_k^*\omega_{FS} \leqslant \int_{\Delta^n}(dd^c||z||^2)^n + \\ + \int_{\Delta^{n-1}}(dd^c||z||^2)^{n-1}\int_{\partial\Delta_{z'}} d^c\ln\|F_k\|^2 \leqslant \mathsf{const}. \tag{11.9}$$

Therefore after going to a subsequence we get that the Hausdorff limit $\hat{\Gamma} := \lim\Gamma_{f_k}$ is a purely n-dimensional analytic subset of $D\times\mathbb{P}^1$. We claim that if f_k converge to f weakly then, in fact, $\lim\Gamma_{f_k} = \Gamma_f$. If not take any irreducible component Γ of this limit different from Γ_f. Denote by γ its projection to D. γ is a proper analytic set of codimension at least two D. But then Γ should be contained in $\gamma\times\mathbb{P}^1$ and the last analytic set is of dimension $\dim D - 1$. This is impossible, because all components of $\lim\Gamma_{f_k}$ are of pure dimension $\dim D$. Therefore $\gamma = \varnothing$ and $\lim\Gamma_{f_k} = \Gamma_f$. The rest follows from Theorem 11.3 and Proposition 11.1.

11.5. Behavior of volumes of graphs under weak and gamma convergence. Let us discuss the following question: suppose meromorphic mappings $f_k : D \to X$ converge in some sense to a meromorphic map f, what can be said about the behavior of volumes of graphs of f_k over compacts in D? If f_k converge to f strongly then, as it was remarked in Theorem 11.1, for every relative compact $V \Subset D$ we have that

$$\mathsf{vol}(\Gamma_{f_k|_V}) \to \mathsf{vol}(\Gamma_{f|_V}). \tag{11.10}$$

When f_k converges only weakly one cannot, of course expect anything like (11.10). At most what one can expect is that volumes of Γ_{f_k} stay bounded over compacts in D and converge to the volume of Γ_f plus volumes of exceptional components. I.e., the question is if for a weakly converging sequence $\{f_k\}$ one has that for every relatively compact open $V \Subset D$ there exists a constant C_V such that

$$\mathsf{vol}(\Gamma_{f_k|_V}) \leqslant C_V \qquad \text{for all} \quad k. \tag{11.11}$$

This turns to be wrong in general, the following example was communicated to us by A. Rashkovski, see more details in [IN].

Example 11.4. There exists a sequence $\varepsilon_k \searrow 0$ such that *holomorphic* mappings $f_k : \mathbb{B}^3 \to \mathbb{P}^3$ defined as

$$f_k : (z_1, z_2, z_3) \to [z_1 : z_1 - \varepsilon_k : z_2 : z_3^k] \tag{11.12}$$

converge weakly to $f(z) = [z_1 : z_1 : z_2 : 0]$ on compacts of the unit ball $\mathbb{B}^3 \subset \mathbb{C}^3$, but the volumes of graphs of f_k over the ball $\mathbb{B}^3(1/2)$ of radius $1/2$ diverge. In fact

$$\mathsf{vol}(\Gamma_{f_k}) \cap (\mathbb{B}^3(1/2)\times\mathbb{P}^3) \geqslant k. \tag{11.13}$$

Remark 11.6. Remark that this example has the source and image manifolds of dimension three.

1. A one dimensional manifold X either properly imbeds to $\mathbb{C}^n$ (when X is noncompact) or is projective and therefore imbeds to $\mathbb{P}^n$. In both cases by Theorem 11.4 we have convergence of reduced representations to a, may be non-reduced representation of the limit. And then the estimate (11.9) applies and gives the uniform bound on volumes.

2. If the dimension n of the source D is 2 the boundedness of volumes of graphs of a weakly converging sequence is automatic. This can be seen at least in two ways.

First, in projective case this readily follows from the following formula of King, see [Kg]:

$$d\left[d^c\ln(\|f\|^2)\wedge dd^c\ln(\|f\|^2)\right] = \chi_{D\setminus I_f}\left[\left(dd^c\ln(\|f\|^2)\right)^2\right] - \sum_j n_j\,[Z_j], \tag{11.14}$$

provided I_f has pure codimension two. Z_j are irreducible components (branches) of the indeterminacy set I_f of f. If it has branches of higher codimension then around these branches a higher order non-pluripolar masses can be expressed in a similar way. Now if f_k weakly converge to f formula (11.14) immediately gives a uniform bound of corresponding MA masses (even together with that concentrated on pluripolar sets I_{f_k}). If $n=2$ then that's all we need.

3. Second, using Skoda potentials, or Green functions, as it was done in [Iv7] Theorem 2, one can bound non-pluripolar Monge-Ampère masses of order two also in the case of weakly converging sequence with values in disk-convex Kähler X. This observation implies that if X is disk-convex Kähler and $\mathsf{dim}\, D=2$ then the volumes of graphs of weakly converging sequences of meromorphic mappings $D\to X$ are uniformly bounded over compacts in D.

Moreover, it was proved in [Ne] that volumes of weakly converging sequence are bounded also in the case when X is any compact complex surface. The proof uses Kaähler case separately and then the fact that a non-Kähler surface has only finitely many rational curves.

Remark 11.7. Let us remark that there is one more important case when the volumes of graphs of weakly (even Γ) converging sequence necessarily stay bounded: namely when $\{f_k\}$ is a Γ-converging sequence of meromorphic mappings between projective manifolds X and Y. Indeed the volumes of graphs Γ_{f_k} are uniformly bounded as it is straightforward from Besout theorem.

11.6. Rational connectivity of the exceptional components of the limit. Strong convergence obviously implies the weak one:

$$\mathsf{s}\text{-convergence} \Longrightarrow \mathsf{w}\text{-convergence}. \tag{11.15}$$

We want to understand what obstructs a weakly converging sequence to converge strongly. The problem is that by Theorem 11.1 the volumes of graphs of a strongly converging sequence are uniformly bounded over compacts in the source. As we saw starting from dimension three the volumes of graphs of a weakly converging sequence can diverge to infinity over compacts of D. Nevertheless for a sequence Γ_{f_k} of weakly converging meromorphic graphs we can consider the Hausdorff limit $\hat{\Gamma}$ (its always exists after taking a subsequence). Set $\Gamma := \overline{\hat{\Gamma}\setminus\Gamma_f}$, where Γ_f is the graph of the limit map f, and call Γ *a bubble*. For $a\in\mathsf{pr}_1(\Gamma)$ set $\Gamma_a := \mathsf{pr}_2(\mathsf{pr}_1^{-1}(a)\cap\Gamma)$. The following theorem describes the structure of the bubble.

Theorem 11.5. *Let X be a disk-convex manifold carrying a dd^c-closed metric form and let $f_k : D\to X$ be a weakly converging sequence of meromorphic mappings. Then for every point $a\in\mathsf{pr}_1(\Gamma)$ the fiber Γ_a is rationally connected.*

Here by saying that a closed subset Γ_a of a complex manifold is *rationally connected* we mean that every two distinct points $p,q\in\Gamma_a$ can be connected by a chain of rational curves in Γ_a, *i.e.*, by a *connected* union $C=\bigcup_j C_j$ of finitely many rational curves, each entirely contained in Γ_a. For the proof of Theorem 11.5 we refer to [IN].

12. Separate analyticity and Rothstein-type theorems

12.1. A Rothstein-type extension theorem. Let us start with the following Rothstein-type statement.

Theorem 12.1. *Let $V\subset\mathbb{C}^p$ and $W_0\Subset W\Subset\mathbb{C}^q$ be domains, $p,q\geqslant 1$, and let E be a non-pluripolar subset of V. Let furthermore a meromorphic mapping $f:V\times W_0\to X$ with values in a reduced complex space X be given. Suppose that for every $z\in E$ the restriction $f_z := f\mid_{\{z\}\times W_0}$ is well defined and extends meromorphically to $W_z := \{z\}\times W$. Then for every domain W' such that $W_0\Subset W'\Subset W$ there exists a pluripolar subset $E'\subset E$ such that f meromorphically extends to a neighborhood of $V\times W_0\cup(E\setminus E')\times W'$.*

Proof. Fix some $\tilde{W}$ such that $W'\Subset\tilde{W}\Subset W$. For every $z\in E$ the restriction f_z is well defined and extends meromorphically to $\overline{\tilde{W}}$. Fix some compact exhaustion $K_1\Subset ...\Subset K_n\Subset ...$ of X. Denote by E_n the set of those $z\in E$ that $f_z(\tilde{W})\subset K_n$. Since $\bigcup_n E_n = E$ we see that starting from some n_0 all E_n are not pluripolar. If we shall prove that for $n\geqslant n_0$ there exists a pluripolar E_n' and an extension of f to a neighborhood of $V\times W_0\cup(E_n\setminus E_n')\times W'$, then f will be extended to a neighborhood of $V \times W_0 \cup (E \setminus E') \times W'$, where $E' = \cup_n E'_n$. The latter is pluripolar by Josefson's theorem. Therefore we may additionally suppose that there exists a compact $K \Subset X$ such that $f_z(\tilde{W}) \subset K$ for all $z \in E$. Take ν as in (7.8) before Theorem 7.1 for ours $W' \Subset \tilde{W}$ and K. Let R be the maximal open subset of E such that f extends to a neighborhood of $V \times W_0 \cup R \times W'$. Set $S = E \setminus R$. Define furthermore

$$S_k = \{z \in S : \mathsf{vol}(\Gamma_{f_z}) \leqslant k\frac{\nu}{2}\},$$

where graphs are taken over $\tilde{W}$. Note that by Lemma 7.1 S_k are closed and they are increasing. Also $\bigcup_{k \geqslant 1} S_k = S$. By Theorem 7.1 and maximality of $R = E \setminus S$ the sets $S_{k+1} \setminus S_k$ are not locally regular at any of their points. Therefore the sets $S_1, S_2 \setminus S_1, ...$ are pluripolar and so is S. Theorem is proved. □

Remark 12.1. This theorem is proved in [Iv6], Corollary 2.5.1. We reproduce it here because in [Iv6] it was mistakenly stated that one can take $W' = W$. This is not true in general.

We have the following obvious corollary from Theorem 12.1.

Corollary 12.1. *If in the conditions of the Theorem 12.1 the complex space X possess the meromorphic extension property then:*

i) one can take $E' = E \setminus E^*$, where E^* is a set of pluriregular points of E; also $W' = W$ in this case, i.e., f extends meromorphically to a neighborhood of $V \times W_0 \cup E^* \times W$;

ii) if, in addition, if $E = V$ then f extends meromorphically to $V \times W$.

Indeed, f in this case extends to the envelope of holomorphy of the neighborhood of $V \times W_0 \cup (E \setminus E') \times W'$, which is obviously a neighborhood of $V \times W_0 \cup E^* \times W'$. Taking an increasing exhaustion $W_0 \Subset W_1 \Subset ...$ of W and applying the result consecutively for every W_n on the place of W' we get the result.

12.2. Separate analyticity of meromorphic mappings and the radius of meromorphy. Let us turn now to the separate meromorphicity. The following statement for holomorphic functions is due to J. Siciak, see [Sc].

Theorem 12.2. *Let E and F be a non-pluripolar subsets in domains $V \Subset \mathbb{C}^p$ and $W \Subset \mathbb{C}^q$ respectively, $p, q \geqslant 1$, and let G be some pluripolar subset of $V \times W$. Let further some mapping $f : E \times F \setminus G \to X$ to a complex space X be given. Suppose that:*

i) for every $z \in E$, such that $\{z\} \times F \not\subset G$ the restriction $f_z := f|_{\{z\} \times F}$ is well defined and meromorphically extends to $\{z\} \times W$;

ii) for every $w \in F$, such that $E \times \{w\} \not\subset G$ the restriction $f^w := f|_{E \times \{w\}}$ is well defined and meromorphically extends to $V \times \{w\}$.

Then for every $V' \Subset V$, $W' \Subset W$ there exist pluripolar subsets $E' \subset E$, $F' \subset F$ and a meromorphic extension $\tilde{f}$ of f to some neighborhood of $(E \setminus E') \times W' \cup V' \times (F \setminus F')$.

Proof. As in the proof of Theorem 12.1 without loss of generality we can suppose that f_z extends to $\overline{\tilde{W}}$ for some $W \Subset \tilde{W}$ and for all $z \in E$ such that $\{z\} \times F \not\subset G$. Moreover, we can suppose that there exists a compact $K \subset X$ such that $f_z(\tilde{W}) \subset K$ for all $z \in E$. The same for f^w-s: they can supposed to be meromorphic on $\tilde{V}$ for some $V \Subset \tilde{V}$, the same for all $w \in F$, such that $E \times \{w\} \not\subset G$. Let us prove first that there exists a point $Z_0 = (z_0, w_0) \in E \times F$ such that f holomorphically extends to a neighborhood of Z_0. Set

$$E_k = \{z \in E : \mathsf{vol}(\Gamma_{f_z}) \leqslant k\frac{\nu}{2}\},$$

where ν is defined for $W \Subset \tilde{W}$ and K as in (7.8). Since E is not pluripolar, there exists k and $z_1 \in E_{k+1} \setminus E_k$ such that $E_{k+1} \setminus E_k$ is locally regular at z_1. The same reasoning as at the beginning of the proof of Theorem 7.1 shows that the family $\{\Gamma_{f_z} : z \in E_{k+1} \setminus E_k\}$ is continuous in a neighborhood of z_1. Take a point $w_0 \in W$ such that f_{z_1} is *holomorphic* in a neighborhood of w_0. Remark that this w_0 can be taken to be a locally regular point of $F_{l+1} \setminus F_l$ for some l, where

$$F_l = \{w \in F : \mathsf{vol}(\Gamma_{f^w}) \leqslant k\frac{\nu}{2}.$$

ν should be taken to satisfy (7.8) also for $V' \Subset V$ and K. From Hausdorff continuity of the family Γ_{f_z} in a neighborhood of z_1 we get immediately that all f_z are holomorphic in a neighborhood of w_0 for $z \in E_{k+1} \setminus E_k$ close to z_1, see the Rouché Principle of Theorem 11.2. Find a point $z_0 \in E_{k+1} \setminus E_k$ close to z_1 where f^{w_0} is holomorphic. Now the separate analyticity Theorem 1.10 for functions tells us that the point $Z_0 = (z_0, w_0)$ is such as needed. To end the proof apply two times coordinatewise the Rothstein-type Theorem 12.1. □

Corollary 12.2. *Suppose that under the conditions of Theorem 12.2 the space X possesses the meromorphic extension property. Then:*

i) one can take $E \setminus E' = E^*$, $F \setminus F' = F^*$ and $W' = V$, $W' = W$, and moreover, a neighborhood $\Omega_{E,F}$ to which every such f extends depends only on E and F and is equal to

$$\Omega_{E,F} = \{(z_1, z_2) \in V \times W : w_*(z_1, E^*, V) + w_*(z_2, F^*, W) > 1\}. \tag{12.1}$$

ii) if, in addition, $E = V$ and $F = W$ then f extends meromorphically to $V \times W$.

The reason is that the envelope of holomorphy of any neighborhood of $E^* \times F^*$ contains this $\Omega_{E,F}$, see [Sc], Theorem 7.1. It should be said however that if the target space X doesn't possess a mer. ext. prop. then the "excluded" pluripolar set E' (and F') in both Theorems 12.1 and 12.2 can be bigger than $E \setminus E^*$ (resp. than $F \setminus F^*$). The most striking in this respect is the following example.

Example 12.1. *(A. Hirschowitz, [Hr]).* There exists a compact complex surface X (of class VII) and a holomorphic mapping $f : \Delta^2 \setminus \{0\} \to X$ such that for any complex curve $C \ni 0$ the restriction $f|_{C\setminus\{0\}}$ holomorphically extends to 0, but f doesn't extend meromorphically to a neighborhood of the origin. Here by saying that $f|_{C\setminus\{0\}}$ holomorphically extends to 0 one means that for a local injective parametrization $\mathsf{j} : (\Delta, 0) \to (C, 0)$ the composition $f \circ \mathsf{j}$ extends to 0. This example shows that one can have $E = E^* = \Delta$ in our theorems but with non-empty E', (namely $E' = \{0\}$ in this case). This example also shows (and this was the primary reason for constructing it) that the meromorphicity in the sense of Stoll (not considered in this text) is different from the meromorphicity in the sense of Remmert (which we adapt here). For Kähler targets these notions coincide, Siu, [Si5].

We end up with the two following statements.

Theorem 12.3. *(B. Shiffman, [Sh3]). Let E be a set of full measure in Δ^n and let $f : E \to X$ be a mapping with values in a reduced complex space X such that for almost every coordinate disk Δ the restriction of f to $\Delta \cap E$ extends to Δ as a holomorphic map with values in X. If the space X possesses the meromorphic extension property then f meromorphically extends to Δ^n.*

The proof can be achieved along the similar arguments as presented in this section. Let Ω be an open subset of $\mathbb{C}$ and D a connected open neighborhood of the origin in $\mathbb{C}$. Let f be a meromorphic mapping from $\Omega \times D$ to a reduced complex space X. For $z \in \Omega$ define the *radius of meromorphy* $r_f(z)$ of f to be the supremum of all r such that f meromorphically extends to a neighborhood of $\{z\} \times \Delta_r$.

Corollary 12.3. *If the space X possesses the meromorphic extension property (ex. X is disk-convex Kähler) then function $\ln \frac{1}{r_f(z)}$ is subharmonic in Ω.*

The proof is the same as that of Proposition 1.4 from [Si3] and will be omitted. Remark that both Theorem 12.3 and Corollary 12.3 apply when X is disk-convex Kähler due to Theorem 8.1.

Remark 12.2. Our treatment of separate analyticity is rather short. For the case of functions much more results can be found in the book [JP]. The case of mappings deserves a more detailed description which, we hope, will be given elsewhere. For the time being let us underline that our approach to the separate analyticity is based on the following three main ingredients: Theorem 1.10 of Siciak, Theorem of Josefson and Theorem 7.1. The main point is that the mer. ext. property on the image space X occurs to be "almost not needed". What is lost is just a pluripolar set.

12.3. Increasing the dimension of the source manifold. Let us discuss one interesting purely meromorphic phenomena.

Definition 12.1. *We say that holomorphic (resp. meromorphic) mappings with values in reduced complex space X are:*

i) Hartogs (n,q)-extendable if every holomorphic (resp. meromorphic) map from $H^{n,q}(r)$ to X extends holomorphically (resp. meromorphically) to Δ^n;

ii) Thullen (n,q)-extendable if for any closed pluripolar subset S of Δ^q every holomorphic (resp. meromorphic) map from

$$T_r^{n,q}(S) := \Delta^{n-q} \times (\Delta^q \setminus S) \cup (\Delta_r^{n-q} \times \Delta^q) \tag{12.2}$$

to X extends holomorphically (resp. meromorphically) to Δ^n.

Note that Hartogs (n,q)-extendibility obviously implies the Thullen-type one. Vice versa is not true. Take the example of Kato 10.1 and let $f : T_r^{2,1}(S) \to X$ be some meromorphic map. While $f : T_r^{2,1}(S)$ is simply-connected we can consider the lift $F = \pi^{-1} \circ f : f : T_r^{2,1}(S) \to D \subset \mathbb{P}^3$. By a Thullen-type extension theorem for meromorphic functions F extends to Δ^2. But $F(\Delta^2) \cap \partial D = \varnothing$ because one cannot touch ∂D by a bidisk. Thus $\pi \circ F$ gives an extension of f to Δ^2.

1. Remark that an appropriate Thullen-type extendibility is sufficient for Corollary 12.1.
2. If holomorphic mappings with values in X are Hartogs (n,q)-extendable then they are Hartogs (n',q')-extendable for all (n',q') with either $n' \geqslant n$ or $q' \geqslant q$. The proof is straightforward and is given in Lemmas 2.2.1 and 2.2.2 of [Iv6]. In particular the category $\mathcal{O}_X$ for such X is $(n-q)$-Hartogs.
3. Meromorphic case is surprisingly different. We refer to [Iv6] for the following example which shows that meromorphic Hartogs $(2,1)$-extendability doesn't imply Hartogs $(3,1)$-extendability!

Example 12.2. There exists a compact complex three-fold X such that:

(a) For every domain D in $\mathbb{C}^2$ every meromorphic mapping $f : D \longrightarrow X$ extends to a meromorphic mapping $\hat{f} : \hat{D} \longrightarrow X$. Here $\hat{D}$ stands for the envelope of holomorphy of D.

(b) But there exists a meromorphic mapping $F : \mathbb{B}^3 \setminus \{0\} \longrightarrow X$ from punctured 3-ball to X which does not extend to the origin.

13. Vanishing cycles in holomorphic foliations and foliated shells

In the present section we shall explain a certain "non-parametric" variation of previous results, a certain mélange of techniques from section 9 and subsections 8.2, 8.3 of section 8. The outcome will be then applied to holomorphic foliations by curves.

13.1. Holomorphic fibrations and generalized Hartogs figures. Let us start from a particular holomorphic foliations by curves, the so called *fibrations* by curves, *i.e.*, triples (W,π,V) where W and V are complex manifolds of dimensions $\dim W = \dim V + 1$ and $\pi : W \to V$ is a surjective holomorphic submersion with connected fibers. For $z \in V$ denote by $W_z = \pi^{-1}(z)$ the corresponding fiber. A holomorphic mapping $f : (W,\pi,V) \to (W',\pi',V')$ is said to be a *foliated immersion* if it is an immersion and sends leaves to leaves. By saying this we mean that there exists a holomorphic map $f_{\mathrm{v}} : V \to V'$ such that for all $z \in V$ one has $f(W_z) \subset W'_{f_{\mathrm{v}}(z)}$.

Definition 13.1. *A* generalized Hartogs figure *is a quadruple* (W,π,U,V)*, where* W *and* V *are complex manifolds,* U *an open subset of* V *and* $\pi : W \to V$ *is a holomorphic submersion such that:*

i) for all $z \in V\setminus U$ *the fiber* W_z *is diffeomorphic to an annulus;*
ii) for $z \in U$ *the fiber* W_z *is diffeomorphic to a disk.*

Remark 13.1. a) Generalized Hartogs figures are fibrations of a special type: they are concave in the most naïve and clear sense. Manifold W has a distinguished part of the boundary formed by the outer boundaries $\partial_0 W_z$ of annuli W_z. We shall suppose that W is smooth up to this part of its boundary and denote it by $\partial_0 W$, *i.e.*, $\partial_0 W = \cup_{z\in V}\partial_0 W_z$. Projection π is also supposed to be smooth up to $\partial_0 W$ and therefore $\pi : \partial W_0 \to V$ is a circle fibration. For $z \in U$ the outer boundary $\partial_0 W_z$ is actually the boundary of the disc W_z.

b) The standard Hartogs figure $H_r^{n+1} = (\Delta^n \times A_{1-r,1}) \cup (\Delta_r^n \times \Delta)$ carries a natural *vertical fibration* $\mathcal{L}^{\mathrm{v}}$, (or horizontal in one presents it as in (1.4)). Leaves $\mathcal{L}^{\mathrm{v}}_{z'}$ are discs Δ if $||z'|| < r$ and annuli for $r \leqslant ||z'|| < 1+r$. Here $z' = (z_1,...,z_n)$ and $||\cdot||$ is the polydisk-norm in $\mathbb{C}^n$. Remark now that $(H_r^{n+1},\mathcal{L}^{\mathrm{v}})$ fits, of course, into the Definition 13.1 with $V = \Delta^n$, $U = \Delta_r^n$ and π being the restriction of the canonical "vertical" projection $\mathbb{C}^{n+1} \to \mathbb{C}^n$ to H_r^{n+1}.

c) Let $P := \Delta^2$ be the unit bicylinder in $\mathbb{C}^2$ and $B = \partial P$ its boundary. For some $0 < \varepsilon < 1$ let $B^r = \{z \in \mathbb{C}^2 : 1-r < \max\{|z_1|,|z_2|\} < 1+r\}$ be a shell around B. Denote by $\pi : \mathbb{C}^2 \to \mathbb{C}$ the canonical projection $\pi(z) = z_1$ onto the first coordinate of $\mathbb{C}^2$. Note that B^r is foliated by π over the disc Δ_{1+r} of radius $1+r$. Denote this foliation by $\mathcal{L}^{\mathrm{v}}$ and call it *a vertical foliation*. Its leaves $\mathcal{L}^{\mathrm{v}}_{z_1} := \pi^{-1}(z_1)$ are discs Δ_{1+r} if $1-r < |z_1| < 1+r$ and are annuli $A_{1-r,1+r} := \Delta_{1+r}\setminus\bar{\Delta}_{1-r}$ if $|z_1| \leqslant 1-r$.

Definition 13.2. *The pair* $(B^r,\mathcal{L}^{\mathrm{v}})$ *will be called the* standard foliated shell.

Note that the standard foliated shell is also a generalized Hartogs figure. Namely it can be viewed as $(B^r,\pi,A_{1-r,1+r},\Delta_{1+r})$.

d) The difference between the generalized and the standard Hartogs figures can be understood from the example explained on the Figure 5. Namely the following can happen.

Example 13.1. There exists a generalized Hartogs figure W over a disc (*i.e.*, both $\varnothing \neq U \subset V$ are discs in $\mathbb{C}$) with the following property: whenever a holomorphic foliated imbedding $h : (z_1,z_2) \to (h_1(z_1),h_2(z_1,z_2))$ of $H = H_r^2$ into W is given such that $h_1(0) = z^0 \in U$ then necessarily $h_1(\Delta) \subset U$ (whatever $r > 0$ is).

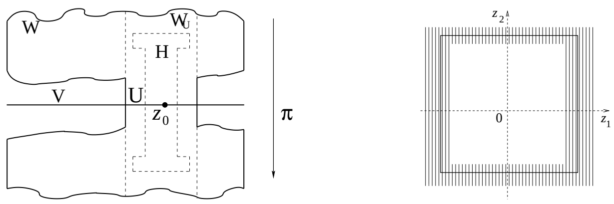

FIGURE 5. On the left is the standard Hartogs figure imbedded into a generalized Hartogs figure W constructed in [CI]. Every attempt to imbed H_r^2 into this W will look like this picture: if the fiber over the origin in H_r^2 is mapped to a fiber over some point $z_0 \in U$ then the image of H_r^2 will newer leave $W|_U$. The *standard foliated shell* on the right is foliated by discs and annuli over the disc Δ_{1+r}. In particular, $(B^r,\mathcal{L}^{\mathrm{v}})$ is a generalized Hartogs figure.

Definition 13.3. *If* $U = \varnothing$ *we call* $(W,\pi,\varnothing,V)$ *trivial, if* $U = V$ *we call* (W,π,V,V) *complete and in the latter case often denote it as* (W,π,V).

The standard Hartogs figure is newer trivial by definition, *i.e.*, it is commonly accepted that always $\varepsilon > 0$. Let D be a non-empty open subset of V. Set $W|_D = \pi^{-1}(D)$ and consider it also as a generalized Hartogs figure $(W|_D,\pi|_D,D\cap U,D)$, a subfigure of (W,π,U,V).

13.2. An unparametrized version of Levi's theorem. The following notion comes back to [Ti], see also [Bl]. Let X be a complex space and $f : A_{1-r,1} \to X$ be a holomorphic immersion.

Definition 13.4. *We say that f extends to Δ after a reparametrization if for some $\delta > 0$ there exists an imbedding $h : A_{1-\delta,1} \to A_{1-r,1}$ sending $\partial\Delta$ to $\partial\Delta$ and preserving the canonical orientation of $\partial\Delta$, such that $f \circ h$ holomorphically extends to Δ.*

It is clear that such h, if it exists, should be holomorphic. We shall use also the following form of this notion. Let γ be a simple oriented loop on a bordered Riemann surface W. The latter should be viewed simply as a collar adjacent to γ. Let $f : W \to X$ be a holomorphic immersion. Suppose that there exist a Riemann surface $\widetilde{W}$, which is a bordered disc with boundary $\tilde{\gamma}$ (canonically oriented), and a biholomorphic mapping h from a collar adjacent to $\tilde{\gamma}$ to W (smooth up to the boundaries) and sending $\tilde{\gamma}$ onto γ, preserving orientations, such that the composition $f \circ h$ holomorphically extends to the disc $\widetilde{W}$. Then we shall say that f extends to the disc $\widetilde{W}$ after a reparametrization. If such $\widetilde{W}, \tilde{\gamma}$ and h do exist but are not specified we shall say simply that f holomorphically extends to a disk *after a reparametrization*. In the sequel we shall consider only the case when f is a *generic injection* (i.e. injective outside of a finite set). Then its extension after reparametrization, which we also require to be a generic injection, is unique (if exists). Uniqueness means here up to a biholomorphic automorphism of the disc. Now let's turn to the families of immersions.

Definition 13.5. *A holomorphic mapping $f : (W, \pi, V) \to X$ of a fibration (W, π, V) into a complex space X is called generically injective if for all $z \in V$ outside of a proper analytic subset $A \subset V$ the restriction $f_z := f|_{W_z}$ is a generic injection.*

Note that we do not ask f to be generically injective itself but only its restrictions onto generic fibers. Actually f may not even be an immersion. However in most cases mappings appearing in this text will be both immersions and generic injections. We shall also need a corresponding notion for the meromorphic case.

Definition 13.6. *A meromorphic mapping $f : W \to X$ between complex spaces is a meromorphic immersion if it is an immersion outside of its indeterminacy set I_f. If, moreover, (W, π, V) is a holomorphic fibration then a meromorphic mapping f is called generically injective if $f|_{W_z}$ is a generic injection for z outside of a proper analytic subset of V.*

Let a holomorphic generic injection $f : (W, \pi, U, V) \to X$ of a generalized Hartogs figure into a complex space X be given and let $\hat{U}$ be some open subset of V containing U.

Definition 13.7. *We say that f extends to the Hartogs figure $(\widetilde{W}, \pi, \hat{U}, V)$ after a respirometriction if there exists a foliated biholomorphism of trivial figures $h : (\partial_0\widetilde{W}, \pi, \varnothing, V) \to (\partial_0 W, \pi, \varnothing, V)$ such that $f \circ h$ extends to a generically injective meromorphic map $\tilde{f} : (\widetilde{W}, \pi, \hat{U}, V) \to X$.*

Remark that if f extends as a meromorphic map being a generic injection on (W, π, U, V) with $U \neq \varnothing$ then its extension will be automatically a generic injection. However in the definition above we do not exclude the case when $U = \varnothing$.

Theorem 13.1. *Let $f : (W, \pi, \varnothing, V) \to X$ be a generically injective holomorphic map of a trivial Hartogs figure into a complex space X. Suppose that $\dim V = 1$ and that for some sequence $z_n \to z_0 \in V$ restrictions $f|_{W_{z_n}} : W_{z_n} \to X$ holomorphically extend as generic injections to a discs $\widetilde{W}_{z_n}$ after a reparametrization. Suppose additionally that:*

i) $\tilde{f}|_{\widetilde{W}_{z_n}}(\widetilde{W}_{z_n})$ stay in some compact of X;

ii) $\operatorname{area}\left(\tilde{f}|_{\widetilde{W}_{z_n}}(\widetilde{W}_{z_n})\right)$ are uniformly bounded.

Then there exists a neighborhood $D \ni z_0$ such that f extends meromorphically onto a figure $(\widetilde{W}, \pi, D, V)$ after a reparametrization. Moreover, the extension $\tilde{f}$ is a generically injective meromorphic map.

Proof. Remark that the statement *(ii)* in this theorem doesn't depend on a particular choice of metric. Writing $\tilde{f}|_{\widetilde{W}_{z_n}}$ here we mean that for every n a reparametrization map $h_{z_n} : \partial\widetilde{W}_{z_n} \to \partial W_{z_n}$ is given such that $\tilde{f}|_{\partial\widetilde{W}_{z_n}} := f|_{\partial W_{z_n}} \circ h_{z_n}$ extends generically injectively and holomorphically to the disc $\widetilde{W}_{z_n}$.

Set $\tilde{f}_n = \tilde{f}|_{\widetilde{W}_{z_n}}$ and consider $(\widetilde{W}_{z_n}, \tilde{f}_n)$ as complex discs over X in the sense of Definition 2.3, parameterized by a fixed disc Σ. Applying Theorem 2.5 we can find a subsequence from $(\widetilde{W}_{z_n}, \tilde{f}_n)$ that converges in the sense of Definition 2.5 to a stable curve $(\widetilde{W}_0, \tilde{f}_0)$ over X, parameterized again by Σ. Be careful, this $\widetilde{W}_0$ may have compact components. Denote by $\mathcal{C}$ the space of discs over X which are close to $(\widetilde{W}_0, \tilde{f}_0)$ and which are reparametrizations of restrictions $f|_{W_z}$ near the boundary, here z are close to z_0.

Cover $\widetilde{W}_0$ as in the proof of Theorem 8.3 by open sets U_j in such a way that:

1) All U_j are either discs, annuli or pants. The boundary annulus is one of them, denote it as U_{j_0}). This covering has the property that each intersecting pair U_i, U_j intersects by an annulus denoted as $U_{i,j}$.

2) For each j, except j_0, consider a Banach analytic space (manifold if X was a manifold) H_j of holomorphic maps from U_j to X. For j_0 take as H_{j_0} the space $\{f|_{\partial W_z} : z$ in a neighborhood of $z_0\}$. This is a one dimensional space of holomorphic maps from U_{j_0} to X. ∂W_z stays here for an annulus adjacent to $\partial_0 W_z$, which is naturally identified with U_{j_0}.

3) The same type Banach analytic spaces $H_{i,j}$ of holomorphic maps $U_{i,j} \to X$ for intersecting U_i and U_j are considered.

Denote by $\mathcal{C}$ the Banach analytic set $\Phi^{-1}(0)$, where Φ is the "gluing" holomorphic map constructed in the same way as in (8.7). We can repeat now the argument of Douady, *i.e.*, Lemma 7.2, and get that $\mathcal{C}$ is finite dimensional analytic set. In fact it is clearly of dimension not more than one. But since it contains the sequence $(\widetilde{W}_{z_n}, \tilde{f}_n)$ its dimension is actually one. Therefore $\mathcal{C}$ is a usual analytic set by Barlet-Mazet theorem, [Mz], *i.e.*, is a complex curve in our case. Restriction $\mathcal{C} \to \mathcal{W}$ is an analytic map and it is *proper* (!), because nondegenerate analytic maps between complex curves are always proper. Therefore its image is the whole of $\mathcal{W}$. We get an extension $\tilde{f}_z$ for all z close to z_0 as a family by a tautological map $\tilde{f} : \tilde{\mathcal{W}} \to X$. Here $\tilde{\mathcal{W}}$ is a tautological family of discs over $\mathcal{W}$. □

13.3. Vanishing ends in singular holomorphic foliations by curves. One of the ways to define a *singular* holomorphic foliation by curves $\mathcal{L}$ on a complex manifold X is the following. Take a sufficiently fine open covering $\{\Omega_\alpha\}$ of X. Then $\mathcal{L}$ will be defined by the nonvanishing identically holomorphic vector fields $\mathrm{v}_\alpha \in \mathcal{O}(\Omega_\alpha, TX)$ which are related on a non-empty intersections $\Omega_{\alpha,\beta} := \Omega_\alpha \cap \Omega_\beta$ as $\mathrm{v}_\alpha = h_{\alpha,\beta}\mathrm{v}_\beta$. Here $h_{\alpha,\beta} \in \mathcal{O}^*(\Omega_{\alpha,\beta})$. After dividing by common factors one immediately sees that the singular set of $\mathcal{L}$, which is defined as $\mathsf{Sing}\,\mathcal{L} := \{z : \mathrm{v}_\alpha(z) = 0\}$, is an analytic subset of X of codimension at least two. Set $X^{\mathsf{reg}} := X \setminus \mathsf{Sing}\,\mathcal{L}$. For a point $z \notin \mathsf{Sing}\,\mathcal{L}$ the leaf $\mathcal{L}_z^0$ through z is, by definition, the leaf of the smooth foliation $\mathcal{L}^{\mathsf{reg}} := \mathcal{L}|_{X^{\mathsf{reg}}}$. If $z \in \mathsf{Sing}\,\mathcal{L}$ then leaves through z are not defined, *i.e.*, a stationary point is not considered as being a trajectory. The pair $(X, \mathcal{L})$ is called a *foliated manifold* or a *foliated pair*.

Remark 13.2. This notion requires a certain justification. Let A be an analytic set in a complex manifold X of codimension $\geqslant 2$ and let a *smooth* holomorphic foliation by curves $\mathcal{L}$ on $X \setminus A$ be given. The latter is understood classically as being defined by "flowboxes": every point in $X \setminus A$ possesses a "foliated" neighborhood (a flowbox) where leaves of $\mathcal{L}$ are plaques $\{z_1 = c_1, ..., z_{n-1} = c_{n-1}\}$, $n = \mathsf{dim}_{\mathbb{C}} X$. Then for every point $a \in A$ there exists a neighborhood $\Omega \ni a$ and a holomorphic vector field v on Ω whose trajectories on $\Omega \setminus A$ are the leaves of $\mathcal{L}|_{\Omega\setminus A}$. Indeed, $\mathcal{L}$ naturally indices a holomorphic mapping $\tau : X \setminus A \to P(TX)$ which sends z to the tangent $T_z\mathcal{L}_z \in P(T_zX)$. By Corollary 7.1 τ extends to a meromorphic mapping $\hat{\tau} : X \to P(TX)$. Take a neighborhood $\Omega \ni a$ such that $TX|_\Omega$ is generated by $\partial_{z_1},...,\partial_{z_n}$ for some local coordinates $(z_1,...,z_n)$. Let $w = [w_1 : ... : w_n]$ be the induced homogeneous coordinates on fibers $P(T_zX)$, $z \in \Omega$. Write $\hat{\tau}$ in these coordinates as in Proposition 7.1, *i.e.,* $\hat{\tau}(z) = (z,[\tau_1(z) : ... : \tau_n(z)]$, where τ_j are holomorphic in Ω and without common factors. Now $\mathrm{v} := \tau_1\partial_{z_1} + ... + \tau_n\partial_{z_n}$ is our vector field.

Example 13.2. Let $\mathrm{v} = P(x,y)\partial/\partial x + Q(x,y)\partial/\partial y$ be a complex vector field on $\mathbb{C}^2$ with polynomial coefficients. Trajectories of v define a holomorphic foliation $\mathcal{L}$, which naturally extends to the complex projective plane $\mathbb{P}^2$. Vice versa, every holomorphic foliation on $\mathbb{P}^2$ is defined as the set of trajectories of a polynomial vector field starting from an appropriately chosen affine chart.

Let $(X,\mathcal{L})$ be a foliated manifold. Take a point $z^0 \in X^{\mathrm{reg}}$ and denote by $\mathcal{L}^0_{z^0}$ the leaf of $\mathcal{L}^{\mathrm{reg}}$ passing through z^0. Take a local transverse to $\mathcal{L}^{\mathrm{reg}}$ hypersurface D through z^0, *i.e.,* a complex hypersurface in some neighborhood of z^0 in X^{reg} which is everywhere transverse to the leaves of $\mathcal{L}^{\mathrm{reg}}$. The domain

$$\mathcal{L}^0_D := \bigcup_{z\in D} \mathcal{L}_z \tag{13.1}$$

we shall call a *Poincaré domain* of $\mathcal{L}$ over the transversalis D. Furthermore recall that a parabolic end of $\mathcal{L}^0_{z^0}$ is a closed subset $E \subset \mathcal{L}^0_{z^0}$ which is biholomorphic to the closed punctured disc $\bar{\Delta}^* = \{\zeta \in \mathbb{C} : 0 < |\zeta| \leq 1\}$. By ∂E we shall denote the biholomorphic image of the circle $\{|\zeta| = 1\}$, the outer boundary of the end E. Foliation $\mathcal{L}$ may have a nontrivial holonomy along ∂E, which can be finite or infinite. Consider the case when holonomy is finite. Recall what does that mean. Take a local transversalis D through some point in ∂E, call it again z^0. If one takes a point $z \in D$ close to z^0 and travels on $\mathcal{L}^0_z$ along the path γ_z close to $\gamma_{z^0} = \partial E$ then one certainly hits D in a neighborhood of z^0 by a point $g(z)$. This defines a local biholomorphism $g : (D,z^0) \to (D,z^0)$ which generates a subgroup $< g >$ of the group $Bihol(D,z^0)$ of local biholomorphisms of D fixing z^0. We suppose that $< g >$ is finite, *i.e.,* $g^d = \mathsf{Id}$ for some $d \geq 1$ and this d is always taken to be the minimal satisfying this property. This d is called the order of the holonomy of $\mathcal{L}$ along ∂E.

Lemma 13.1. *Let $E \subset \mathcal{L}^0_{z^0}$ be a parabolic end with holonomy of order d. Then for a sufficiently small $r > 0$ there exists a foliated holomorphic immersion $f : \Delta^n \times A_{1-r,1+r} \to \mathcal{L}^0_D$ such that:*

i) $f(\{0\}\times A_{1-r,1+r}) \subset \mathcal{L}^0_{z^0}$ and the restriction $f|_{\{0\}\times A_{1-r,1+r}} : \{0\}\times A_{1-r,1+r} \to \mathcal{L}^0_{z^0}$ is a regular cover of order d, i.e., covers d-times some imbedded annulus in $\mathcal{L}^0_{z^0}$ and $f(\{0\}\times\partial\Delta) = d\cdot\partial E$.

ii) For all $z \in \Delta^n$ outside a proper analytic subset $A \subset \Delta^n$ the restriction $f|_{\{z\}\times A_{1-\varepsilon,1+\varepsilon}}$: $\{z\} \times A_{1-r,1+r} \to \mathcal{L}_z$ is an imbedding.

For the proof we refer to Lemma 3.1 in [Iv10]. As we see from item (*ii*) our f is a *generic injection* of the trivial Hartogs figure $\Delta^n \times A_{1-\varepsilon,1+\varepsilon}$ over a polydisk in the sense of Definition 13.5 and results of the previous subsection are applicable to such f.

Definition 13.8. *A parabolic end E is called a* vanishing end *of order d if:*

i) the holonomy of $\mathcal{L}$ along ∂E is finite of order $d \geq 1$;

ii) the generic injection $f : \Delta^n \times A_{1-r,1+r} \to \mathcal{L}^0_D$, constructed above, extends as a foliated meromorphic immersion $\tilde{f} : \widetilde{W} \to X$ from a complete Hartogs figure $(\widetilde{W},\pi,\Delta^n)$ over Δ^n to X after a reparametrization;

iii) the intersection of $\widetilde{W}_0 := \pi^{-1}(0)$ with the set of points of indeterminacy $I_{\tilde{f}}$ of $\tilde{f}$ consists of a single point $a \in \widetilde{W}_0$.

The point $q = \tilde{f}|_{\widetilde{W}_0}(a)$ will be called the *endpoint* of the vanishing end E (or of the leaf $\mathcal{L}^0_z$). Following Brunella, see [Br3], we add all vanishing endpoints to the leaf $\mathcal{L}^0_{z^0}$ and call the curve obtained a *completed leaf* through z^0. Completed leaf will be denoted as $\mathcal{L}_{z^0}$. For each $z \in D$ take a holonomy cover $\hat{\mathcal{L}}^0_z$ of the leaf $\mathcal{L}^0_z$. Recall that a holonomy cover of $\mathcal{L}^0_z$ is a cover with respect to the holonomy subgroup $\mathsf{Hol}(z,\mathcal{L}^0_z)$ of the fundamental group $\pi(z,\mathcal{L}^0_z)$. That means that in the construction of $\hat{\mathcal{L}}^0_z$ two pathes γ_1, γ_2 from z to some $w \in \mathcal{L}^0_z$ define the same point of $\hat{\mathcal{L}}^0_z$ if and only if $\gamma_1 \circ \gamma_2^{-1} \in \mathsf{Hol}(z, \mathcal{L}^0_z)$, *i.e.*, if the holonomy along $\gamma_1 \circ \gamma_2^{-1}$ is trivial. Set

$$\hat{\mathcal{L}}^0_D = \bigcup_{z \in D} \hat{\mathcal{L}}^0_z. \tag{13.2}$$

This set has the natural structure of a complex manifold together with the natural projection $\pi : \hat{\mathcal{L}}^0_D \to D$ which sends $\hat{\mathcal{L}}^0_z$ to z. It admits also the natural locally biholomorphic foliated map $p : \hat{\mathcal{L}}^0_D \to \mathcal{L}^0_D \subset X^0$ which sends $\hat{\mathcal{L}}^0_z$ to $\mathcal{L}^0_z$ with $p|_{\hat{\mathcal{L}}^0_z} : \hat{\mathcal{L}}^0_z \to \mathcal{L}^0_z$ being the canonical holonomy covering map. Call $\hat{\mathcal{L}}^0_D$ the *holonomy covering Poincaré domain* of $\mathcal{L}$ over D or, shorter, a holonomy Poincaré domain.

Vanishing ends of $\hat{\mathcal{L}}^0_z$ are defined similarly to that of $\mathcal{L}^0_z$. Let E be a parabolic end of $\hat{\mathcal{L}}^0_{z^0}$ Take $f : \Delta^n \times A_{1-\varepsilon, 1+\varepsilon} \to \hat{\mathcal{L}}^0_D$ such that:

i) $f : \Delta^n \times A_{1-r, 1+r} \to \hat{\mathcal{L}}^0_D$ is an imbedding;
ii) $f(\{0\} \times \partial\Delta) = \partial E$ (note that $d = 1$ in this case).

The only difference that now f takes values in $\hat{\mathcal{L}}^0_D$ and f is an imbedding. The last is because the holonomy of the foliation $\hat{\mathcal{L}}^0$ on $\hat{\mathcal{L}}^0_D$ is trivial.

Definition 13.9. *E is called a vanishing end of $\hat{\mathcal{L}}^0_{z^0}$ if $h = p \circ f$ extends to a meromorphic foliated immersion $\tilde{h} : \widetilde{W} \to X$ after a reparametrization (not f itself as in Definition 13.8) and $\widetilde{W}_0$ intersects the indeterminacy set $I_{\tilde{h}}$ of $\tilde{h}$ by exactly one point.*

The union of $\hat{\mathcal{L}}^0_z$ with all its vanishing endpoints equipped with an obvious complex structure will be denoted as $\hat{\mathcal{L}}_z$. We shall call it also *a completed holonomy covering leaf* of the leaf $\mathcal{L}^0_z$. Set $\hat{\mathcal{L}}_D := \bigcup_{z \in D} \hat{\mathcal{L}}_z$ and call it the *completed* holonomy Poincaré domain over D.

Lemma 13.2. *i) The completed holonomy Poincaré domain possesses the natural structure of a foliated complex manifold with foliation given by the natural projection $\pi : \hat{\mathcal{L}}_D \to D$ defined as above by $\pi(\hat{\mathcal{L}}_z) = z$.*

ii) The natural foliated holomorphic immersion $p : \hat{\mathcal{L}}^0_D \to \mathcal{L}^0_D$ extends to a meromorphic foliated immersion $p : \hat{\mathcal{L}}_D \to X$ and its restrictions $p|_{\hat{\mathcal{L}}_z} : \hat{\mathcal{L}}_z \to \mathcal{L}_z$ are ramified at vanishing ends.

For the proof we refer to Lemma 3.2 in [Iv10].

Remark 13.3. Cover $p_{z^0} : \hat{\mathcal{L}}_{z^0} \to \mathcal{L}_{z^0}$ is an *orbifold cover* in the sense that its ramification index at point a depends only on $b := p_{z^0}(a)$. This is also an unbounded cover in the sense that for every a there exists a disc-neighborhood $V \ni b$ such that $p_{z^0}^{-1}(V)$ is a disjoint union of discs W_j with centers a_j, preimages of b, such that every restriction $p_{z^0}|_{W_j} : W_j \to V$ is a proper cover ramified over b.

13.4. Vanishing cycles and simultaneous uniformization. Now let us give the definition of a vanishing cycle in a singular holomorphic foliation by curves. A cycle in $\mathcal{L}^0_z$ is by definition a closed path (a loop) $\gamma : [0,1] \to \mathcal{L}^0_z$. Let $\hat{\gamma} : [0,1] \to \hat{\mathcal{L}}^0_z$ be a cycle in $\hat{\mathcal{L}}^0_z$ which is not homotopic to zero in $\hat{\mathcal{L}}^0_z$.

Definition 13.10. *We say that $\hat{\gamma}$ is a vanishing cycle if for some sequence $z_n \to z$ there exist loops $\hat{\gamma}_n$ in $\hat{\mathcal{L}}^0_{z_n}$ uniformly converging to $\hat{\gamma}$ which are homotopic to zero in the corresponding leaves $\hat{\mathcal{L}}_{z_n}$.*

(a) We say that $\hat{\gamma}$ is an algebraic vanishing cycle if γ is not homotopic to zero in $\hat{\mathcal{L}}^0_z$ but is homotopic to zero in the completed leaf $\hat{\mathcal{L}}_z$.

(b) If $\hat{\gamma}$ is not homotopic to zero also in $\hat{\mathcal{L}}_z$ we call it an essential vanishing cycle.

Remark 13.4. There is an analogy (rather deep in fact) between *algebraic/essential* vanishing cycles and poles/essential singularities of meromorphic functions. Really, pole of a meromorphic function f becomes a regular point if one completes $\mathbb{C}$ to $\mathbb{P}^1$ and considers f as a holomorphic mapping into the latter manifold. However, an essential singular point remains a singularity of f also after this operation. The same happens with cycles. For the moment let us say the following.

a) Suppose $\mathcal{L}^{\text{sing}} = \varnothing$, *i.e.,* if $\mathcal{L}$ has no singularities. In that case classically a cycle $\gamma \subset \mathcal{L}_z$ is called *a vanishing cycle* if the following two conditions hold:

- γ is not homotopic to zero in $\mathcal{L}_z$;
- there exist a sequence of points $z_n \to z$ and a sequence of loops $\gamma_n : [0,1] \to \mathcal{L}_{z_n}$ such that γ_n uniformly converge to γ and each γ_n is homotopic to zero in $\mathcal{L}_{z_n}$.

In the smooth case every vanishing cycle is an essential vanishing cycle, more precisely projects to a vanishing cycle under the holonomy covering map $\hat{\mathcal{L}}_z \to \mathcal{L}_z$. Let us explain this in more details. In that case vanishing ends do not exist and, in particular, $p_{z^0} : \hat{\mathcal{L}}_{z^0} \to \mathcal{L}_{z^0}$ is an unramified cover. Let $\gamma_0 \subset \mathcal{L}_{z^0}$ be a vanishing cycle and $\gamma_n \subset \mathcal{L}_{z_n}$ be cycles homotopic to zero and converging to γ_0. All γ_n lift to cycles $\hat{\gamma}_n \subset \hat{\mathcal{L}}_{z_n}$ converging to the lift $\hat{\gamma}_0 \subset \hat{\mathcal{L}}_{z^0}$ of γ_0. All $\hat{\gamma}_n$ are homotopic to zero. But $\hat{\gamma}_0$ cannot be homotopic to zero. Therefore we get a vanishing cycle $\hat{\gamma}_0$ in $\hat{\mathcal{L}}_{z^0}$. Vice verse, let $\hat{\gamma}_0$ and $\hat{\gamma}_n$ be as above in the holonomy covering leaves. Then $\hat{\gamma}_n$ project to cycles homotopic to zero in corresponding leaves. But $\hat{\gamma}_0$ project to some γ_0 which cannot be homotopic to zero because in the latter case its lift $\hat{\gamma}_0$ (as lift of any curve homotopic to zero) should be homotopic to zero itself. Therefore γ_0 is a vanishing cycle in $\mathcal{L}_{z^0}$.

b) Algebraic vanishing cycles in the leaf $\hat{\mathcal{L}}_z^0$ can be removed (*i.e.,* one can make these cycles homotopic to zero) by adding to $\hat{\mathcal{L}}_z^0$ *vanishing ends*.

c) It is known also (it follows from [Br3]) that if X is Kähler, then all vanishing cycles (of any $\mathcal{L}$) are algebraic. It follows also from a more general statement of Corollary 13.1.

d) Now the necessity of considering generalized Hartogs figures comes (once more) from the simple observation that: *every vanishing cycle produces a natural generalized (or topological) Hartogs figure around it*, see the sketch of the proof of Theorem 13.2 below.

e) Classically vanishing cycles became the object of study in foliation theory since the seminal paper of Novikov [N], where he used them to produce a compact leaf in every smooth foliation by surfaces on $\mathbb{S}^3$, see also [H].

Further, for $z \in D$ denote by $\tilde{\mathcal{L}}_z$ the universal cover of the completed holonomy leaf $\hat{\mathcal{L}}_z$. I.e., we take the orbifold universal cover of $\mathcal{L}_z$, see Remark 13.3. On the union

$$\tilde{\mathcal{L}}_D = \bigcup_{z \in D} \tilde{\mathcal{L}}_z \tag{13.3}$$

one defines a natural topology in the following way. An element of $\tilde{\mathcal{L}}_D$ is a path γ in some leaf $\hat{\mathcal{L}}_z$ starting from z and ending at some point $w \in \hat{\mathcal{L}}_z$. γ and $\gamma^{'}$ define the same point if their ends coincide and they are homotopic (inside $\hat{\mathcal{L}}_z$) with ends fixed. A neighborhood of $\gamma \subset \hat{\mathcal{L}}_z$ in $\tilde{\mathcal{L}}_D$ is the set of pathes γ'-s in the leaves $\hat{\mathcal{L}}_{z'}$ with z' close to z which are themselves close to γ. γ' "close" to γ is understood here as closed in the topology of uniform convergence in the space $\mathcal{C}([0,1], X)$ of continuous mappings from $[0,1]$ to X.

Definition 13.11. *$\tilde{\mathcal{L}}_D$ with the topology just described is called the universal covering Poincaré domain of $\mathcal{L}$ over D.*

The natural projection $\pi : \hat{\mathcal{L}}_D \to D$ lifts to $\pi : \tilde{\mathcal{L}}_D \to D$ (and will be denoted with the same letter). There is a distinguished section $\sigma : D \to \tilde{\mathcal{L}}_D$ sending z to z. The mapping $p : \hat{\mathcal{L}}_D \to X$ lifts to $\tilde{\mathcal{L}}_D$ and stays to be a *meromorphic foliated immersion* $\tilde{p} : \tilde{\mathcal{L}}_D \to X$ in the sense that it is a foliated immersion outside of its indeterminacy set.

Due to the eventual presence of essential vanishing cycles the natural topology on the covering cylinder might be not Hausdorff. Let us explain this in more details. Non-separability of the natural topology on $\tilde{\mathcal{L}}_D$ means that:

- there exist $z \in D$ and $w \in \hat{\mathcal{L}}_z$ and two pathes γ_1, γ_2 from z to w such that $\gamma_1 \circ \gamma_2^{-1}$ is not homotopic to zero in $\hat{\mathcal{L}}_z$;
- there exist some sequence $z_n \to z$ in D, some sequence $w_n \in \hat{\mathcal{L}}_{z_n}$ converging to w, some sequences of pathes γ_1^n and γ_2^n from z_n to w_n each converging uniformly to γ_1 and γ_2 such that $\gamma_1^n \circ (\gamma_2^n)^{-1}$ are homotopic to zero in $\mathcal{L}_{z_n}$.

And that exactly means that $\gamma_1 \circ \gamma_2^{-1}$ is an essential vanishing cycle. Vice verse, if $\gamma : [0,1] \to \hat{\mathcal{L}}_z$ is an essential vanishing cycle starting and ending at z, then γ and the trivial path $\beta \equiv z$ represent two non-separable points in $\tilde{\mathcal{L}}_D$.

Remark 13.5. This explains that apart from the question of existence of compact leaves vanishing cycles come into a play as obstructions to the simultaneous uniformization of leaves. Indeed, take a transversalis D to the leaves of $\mathcal{L}$. As we just explained a leaf $\mathcal{L}_z \subset \mathcal{L}_D$ containing a vanishing cycle exists if and only if the natural topology of $\tilde{\mathcal{L}}_D$ is not separable (*i.e.,* is not Hausdorff). Separability of $\tilde{\mathcal{L}}_D$ means that the leaves of $\mathcal{L}$ which cut D can be simultaneously uniformized. Therefore a vanishing cycle in some leaf $\mathcal{L}_z \subset \mathcal{L}_D$ is an obstruction to such simultaneous uniformization. $\mathcal{L}$ is called *uniformizable* if for any transversalis D the Poincaré domain $\mathcal{L}_D$ can be uniformized. Therefore $\mathcal{L}$ is uniformizable if and only if it doesn't contain a vanishing cycle in any of its leaves. This explains one more reason for the interest in studying vanishing cycles.

13.5. Vanishing cycles and foliated shells. We approach the main goal in this section, which consists in showing that a vanishing cycle in a *holomorphic* foliation by curves generates a very rich complex geometric object: a *foliated shell.* Let $(X,\mathcal{L})$ be a foliated manifold and let $h : (B^r, \mathcal{L}^{\mathrm{v}}) \to (X,\mathcal{L})$ be a *foliated holomorphic immersion* of the standard foliated shell into $(X,\mathcal{L})$. We suppose in addition that h takes its values in X^{reg}. An immersion between two foliated manifolds is called *foliated* if it sends leaves to leaves. Denote by Σ the image of the boundary B under h.

Definition 13.12. *The image $h(B^r)$ is called a foliated shell in $(X,\mathcal{L})$ if:*

i) immersion h is a generic injection, i.e., is such that for all $z_1 \in \Delta_{1+r}$ except of a finite set the restriction $h|_{\mathcal{L}^{\mathrm{v}}_{z_1}} : \{z_1\} \times A_{1-r,1+r} \to X$ is an imbedding;

ii) Σ is not homologous to zero in X.

Roughly speaking condition (*i*) means that h is (much) better than simply an immersion. The main point is of course the condition (*ii*) , see Fig. 5.

Example 13.3. The reader should think about the Hopf surface $H^2 = \mathbb{C}^2 \setminus \{0\}/z \sim 2z$. The same vertical foliation $\mathcal{L}^{\mathrm{v}}$ is invariant under the action $z \sim 2z$ and therefore projects to a foliation $\mathcal{L}$ on H^2. Let $h : \mathbb{C}^2 \setminus \{0\} \to H^2$ be the canonical projection. It obviously induces a "foliated inclusion" $h : (B^r, \mathcal{L}^{\mathrm{v}}) \to (H^2, \mathcal{L})$. $\Sigma = h(B)$ is of course not homologous to zero in H^2.

We call a $(1,1)$-form ω on X *a taming form* for $\mathcal{L}$ if $\omega|_{\mathcal{L}} > 0$. Foliations admitting a pluriclosed taming form we shall call *pluritamed*. Our result is the following.

Theorem 13.2. *Let $(X,\mathcal{L})$ be a disc-convex foliated manifold which admits a dd^c-closed taming form and let $z^0 \in X^{\mathrm{reg}}$ be a point. Then the following statements are equivalent:*

i) The leaf $\hat{\mathcal{L}}_{z^0}$ contains an essential vanishing cycle.

ii) For every transversal $D \ni z^0$ there exists an imbedded disc $z^0 \in \Delta \subset D$ such that

$$\mathcal{L}^0_{\Delta} := \bigcup_{z\in\Delta} \mathcal{L}^0_z$$

contains a foliated shell.

Remark 13.6. a) Statement (ii) means that the mapping $h : B^r \to X$, which "supports" the foliated shell in X, takes values in the cylinder $\mathcal{L}_D$, but $\Sigma = h(B)$ is not homologous to zero in the whole of X!

b) A transversalis D is irrelevant in this theorem: if $\mathcal{L}_z$ contains a vanishing cycle then (*ii*) is true for *every* transversalis $D \ni z$.

c) Let us stress here that X may contain a two-dimensional shell, but it may not be a *foliated* shell for the given foliation $\mathcal{L}$. A simple example is the elliptic fibration on the same Hopf surface H^2. This fibration doesn't admit a foliated shell, while H^2 itself does contain a two-dimensional shell.

d) In the process of the proof of Theorem 13.2 one can establish the following useful characterization of shells:

Proposition 13.1. *Let w be a dd^c-closed taming form for $\mathcal{L}$. A holomorphic foliated immersion $h : B^r \to X$ represents a foliated shell if and only if it is a generic injection and*

$$\int_B d^c(h^*\omega) \neq 0. \tag{13.4}$$

I.e., not only $h(B)$ is not homologous to zero in X but, moreover, the distinguished closed 3-form $d^c\omega$ doesn't vanish on $h(B)$. From Proposition 13.1 we immediately obtain the following:

Corollary 13.1. *If the taming form ω of the foliation $\mathcal{L}$ is d-closed then $\mathcal{L}$ has no vanishing cycles.*

The meaning of Theorem 13.2 is that a topological property of $(X, \mathcal{L})$ to contain a vanishing cycle is equivalent to a complex geometric (even analytic) property to contain a foliated shell. The last is very restrictive as the following corollary shows.

Corollary 13.2. *Let X be a compact complex surface and $\mathcal{L}$ a (singular) holomorphic foliation by curves such that some leaf $\mathcal{L}_z$ of $\mathcal{L}$ contains a vanishing cycle γ. Then:*

i) either X is a modification of a Hopf surface and $\mathcal{L}_z$ is an elliptic curve;

ii) or, X is a modification of a Kato surface and the closure of $\mathcal{L}_z$ is a rational curve.

13.6. Sketch of the proof of Theorem 13.2. Let us briefly outline the main ingredients of the proof. We start with *(i)* $\Rightarrow$ *(ii)* .

Step 1. This step is purely topological. One proves that if the leaf $\hat{\mathcal{L}}_{z^0}$ contains an essential vanishing cycle then it also contains an *imbedded* essential vanishing cycle $\hat{\gamma}_0$, see Lemma 3.4 in [Iv10]. This means simply that $\hat{\gamma} : \mathbb{S}^1 \to \hat{\mathcal{L}}_{z^0}$ is an imbedding. Fix a transversalis $D \ni z^0$. Using the fact that for some $z \in D$ close to z^0 there exists an imbedded loop $\hat{\gamma}_z \subset \hat{\mathcal{L}}_z^0$ which bounds a disk in $\hat{\mathcal{L}}_z$, one easily constructs a generalized Hartogs figure (W, π, U, V) where V is a domain in D which contains both z_0 and z, U a neighborhood of z in D, W is an appropriate open subset of $\hat{\mathcal{L}}_D$ and π is the restriction to W of the natural projection $\hat{\mathcal{L}}_D \to D$.

Step 2. Restrict the holonomy covering projection $p : \hat{\mathcal{L}}_D \to X$ to W. Remark that on W projection p is a *holomorphic* foliated immersion by construction. Indeed, W can be taken away from vanishing ends. Now using Theorem 13.1 on the place of Theorem 7.4 one can prove the following non-parametric version of Theorem 9.1.

Theorem 13.3. *Let $(X, \mathcal{L})$ be a disk-convex foliated manifold admitting a pluriclosed taming form ω and let $f : (W, \pi, U, V) \to (X, \mathcal{L})$ be a generically injective foliated holomorphic map from a non-trivial generalized Hartogs figure (i.e., $U \neq \varnothing$) to X. Then f extends after a reparametrization to a foliated meromorphic map $\tilde{f} : (\tilde{W} \setminus S, \pi, V) \to (X, \mathcal{L})$ of the complete generalized Hartogs figure minus a proper closed subset S to X. Moreover this S, if non-empty, has the following structure. For every point $s \in S$ there exists a coordinate neighborhood $D = \Delta^{n-1} \times \Delta \times \Delta$ of s with coordinates $z = (z', z_n, z_{n+1})$ such that $s = 0$ in these coordinates and:*

i) the restriction to $S \cap D$ of the natural projection $\pi_D : (z', z_n, z_{n+1}) \to z'$ is proper and for every $z' \in \Delta^{n-1}$ the intersection $S_{z'} := \Delta^2_{z'} \cap S$ is non-empty;

ii) the restriction to D of the projection $\pi : \tilde{W} \to V$ coincides with the natural projection $\pi_n : (z', z_n, z_{n+1}) \to (z', z_n)$ and for every $z' \in \Delta^{n-1}$ the set $S_{z',1} := \pi_n(S) \cap (\{z'\} \times \Delta)$ is complete polar of Hausdorff dimension zero;

iii) moreover, $f(\partial\Delta^2_{z'})$ is not homologous to zero in X, i.e., is a foliated shell.

By this theorem p extends after a reparametrization to $\tilde{W}\setminus S$, where $\tilde{W}$ is a complete Hartogs figure over V and S is locally of the form $S=\cup_{z'\in\Delta^{n-1}}S_{z'}$, where $S_{z'}=\bigcup_{z_n\in S_{z',1}}S_{z',z_n}$ with $S_{z',1}$ being closed complete polar compact in Δ for every z' and S_{z',z_n} compacts in Δ for every $z_n\in S_{z',1}$.
Step 3. $z^0=(z_0',z_n',z_{n+1}')$ must belong to $S_1:=\bigcup_{z'}S_{z',1}$, otherwise $\hat{\gamma}^0$ would bound a disc. I.e., S_1 and therefore S is non-empty. By item *(iii)* of Theorem 13.3 this gives a shell $f(\partial\Delta^2_{z_0'})$ in X and this shell is naturally foliated.
Step 4. The implication *(ii)* $\Rightarrow$ *(i)* is slightly simpler. The mapping $h:(B^r,\mathcal{L}^{\mathrm{v}})\to(X^0,\mathcal{L})$, which defines a foliated shell in a pluritamed foliated manifold extends to $P^r\setminus\bigcup_{z_1\in S_1}S_{z_1}$, where P^r is the r-neighborhood of the polydisk, and S_1 is as above. S_1 must be non-empty because h gives a shell. Let $0\in S_1$. Then one proves that h sends $\{0\}\times(\Delta\setminus S_0)$ to the leaf which contains an essential vanishing cycle. For more details we send the interested reader to [Iv10].

Chapter IV. Holomorphic Bundles and Coherent Analytic Sheaves

14. Holomorphic bundles and coherent analytic sheaves

14.1. Generalities on extensions of bundles and sheaves. Let us discuss now the extension properties of holomorphic bundles and, more generally, coherent analytic sheaves.

1. Let $D\subset\widetilde{D}$ be domains in a complex manifold (or, a normal complex space) X and let $\mathcal{N}$ be a holomorphic bundle (resp. a coherent analytic sheaf) on D. One says that $\mathcal{N}$ extends from D to $\widetilde{D}$ if there exists a holomorphic bundle $\widetilde{\mathcal{N}}$ (resp. a coherent analytic sheaf) on $\widetilde{D}$ and an isomorphism $\varphi:\widetilde{\mathcal{N}}|_D\to\mathcal{N}$ of bundles (resp. of sheaves).

2. If $\mathcal{N}$ is a line bundle then it extends as a bundle if and only if it extends as a coherent analytic sheaf. The way to see this is to pass to the second dual $\widetilde{\mathcal{N}}^{**}$ of the extended sheaf, which is reflexive. And then apply Lemma 26 from [Fri] to conclude that $\widetilde{\mathcal{N}}^{**}$ is locally free, *i.e.,* is a bundle extension of $\mathcal{N}$.

3. The feature mentioned in the previous item is specific for line bundles. The rank two subbundle $\mathcal{N}\subset\mathcal{O}^3$ over $\mathbb{C}^3\setminus\{0\}$ with the stalk $\mathcal{N}_z=\{w:w_1z_1+w_2z_2+w_3z_3=0\}$ extends to the origin as a coherent sheaf but not as a bundle.

4a. Let a holomorphic bundle $\mathcal{F}$ be defined on a domain $X^+_{t^*}$ in a complex manifold X (ex. $X^+_{t^*}=\{\rho>t^*\}$ the upper level set of some exhaustion function), and let U be another domain such that $U\cap X^+_{t^*}$ is non-empty and connected. If $\mathcal{F}|_{U\cap X^+_{t^*}}$ is trivial then $\mathcal{F}$ extends to $X^+_{t^*}\cup U$. Indeed, one can use the trivialization, say $\varphi:\mathcal{F}|_{U\cap X^+_{t^*}}\to\mathbb{C}^r\times(U\cap X^+_{t^*})$, $r=\mathsf{rk}\mathcal{F}$, to glue $\mathcal{F}$ with the trivial bundle $\mathbb{C}^r\times U$ on U.

4b. At the same time if $\mathcal{F}$ is trivializable on a subdomain $U_1\subset U\cap X^+_{t^*}$ then it might be not sufficient to make such a gluing. Simply because the trivialization φ need not to extend to $U\cap X^+_{t^*}$ in general.

4c. If there are two such domains U_1 and U_2 and, in addition, $U_1\cap U_2$ and $U_1\cap U_2\cap X^+_{t^*}$ are connected and if $\mathcal{F}^i$ denotes the extensions of $\mathcal{F}$ onto $U_i\cup X^+_{t^*}$ as on the Fig. 6 left, then we get a transition function $\varphi_{12}:U_1\cap U_2\cap X^+_{t^*}\to Gl(r,\mathbb{C})$. In order that $\mathcal{F}^i$ glue together to a holomorphic bundle on $U_1\cup U_2\cup X^+_{t^*}$ it is necessary and sufficient that φ_{12} extends to a non-degenerate $Gl(r,\mathbb{C})$-function on $U_1\cap U_2$. This last condition need not to be satisfied in general.

5. Finally let us consider a very instructive example.

Example 14.1. Consider the line bundle $\mathcal{F}$ on the punctured bidisk $\check{\Delta}^2$ given by the transition function $e^{\frac{1}{z_1z_2}}$. Blow up the origin and denote by $\mathbb{P}^1$ the exceptional divisor. Cover $\mathbb{P}^1$ by two bidisks U_1 and U_2 as on the Figure 6 right. Then $\mathcal{F}|_{U_j\setminus\mathbb{P}^1}$ is trivial for $j=1,2$ (every holomorphic bundle on $\Delta\times\check{\Delta}$ is trivial) and therefore extends to $\check{\Delta}^2\cup U_j$ as a holomorphic line bundle $\mathcal{F}^j$. But the transition function

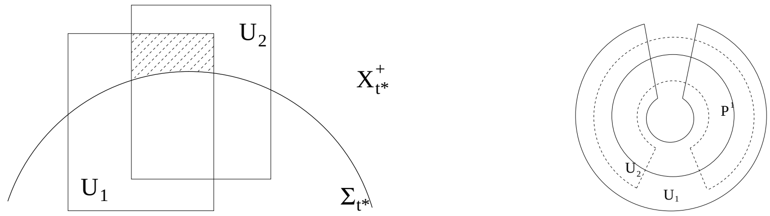

FIGURE 6. The transition function φ_{12} between $\mathcal{F}^1_{t^*}$ and $\mathcal{F}^2_{t^*}$ is defined only in $U_1 \cap U_2 \cap U^+_{t^*}$ - dashed zone on the left of picture. But it should be defined on $U_1 \cap U_2$. Since the envelope of holomorphy of $U_1 \cap U_2 \cap U^+_{t^*}$ is much smaller then $U_1 \cap U_2$ the extension of φ_{12} to $U_1 \cap U_2$ is not automatic, *i.e.,* cannot be achieved via the classical Hartogs theorem. The transition function φ_{12} between $\mathcal{F}^1$ and $\mathcal{F}^2$ on the right is defined only in $(U_1 \cap U_2) \setminus \mathbb{P}^1$. But it should be defined on $U_1 \cap U_2$ in order for $\mathcal{F}$ to extend.

between $\mathcal{F}^1$ and $\mathcal{F}^2$ doesn't extend though $\mathbb{P}^1 \cap U_1 \cap U_2$ and therefore $\mathcal{F}$ doesn't extend to the blown up bidisk. And it shouldn't, because otherwise its direct image would extend $\mathcal{F}$ downstairs, and this is not the case. In particular, we see that a Hartogs extension property fails for holomorphic bundles. But this is not the only problem with them: uniqueness property also fails to be true, because all bundles of the same rank are locally isomorphic to the trivial one.

14.2. Gap-sheaves and extension of coherent analytic sheaves. Let $\mathcal{F}$ be a coherent analytic sheaf on a complex space X.

Definition 14.1. *For a non-negative integer d the d-th absolute gap-sheaf $\mathcal{F}^{[d]}$ of $\mathcal{F}$ is defined by the presheaf*

$$U \to \operatorname*{ind.lim}_{A \in \mathcal{A}_d(U)} \Gamma(U \setminus A, \mathcal{F}). \tag{14.1}$$

Here A runs over the directed set $\mathcal{A}_d(U)$ analytic subsets of U of dimension $\leqslant d$.

In other words a section $\sigma \in \Gamma(U, \mathcal{F}^{[d]})$ is a collection of sections $\sigma_i \in \Gamma(U_i \setminus A_i, \mathcal{F})$ for some locally finite open covering $\{U_i\}$ of U and some analytic subsets $A_i \subset U_i$ of dimensions $\leqslant d$ such that $\sigma_i|_{(U_i \setminus A_i) \cap (U_j \setminus A_j)} = \sigma_j|_{(U_i \setminus A_i) \cap (U_j \setminus A_j)}$ for all i, j.

Theorem 14.1. *(Y.-T.Siu, [Si1]). Let $\mathcal{F}$ be a coherent analytic sheaf on the ring domain $R^{n,2}_r := \Delta^{n-2} \times A^2_{r,1}$, $n \geqslant 3$, such that $\mathcal{F}^{[n-2]} = \mathcal{F}$. Suppose that for every t in some thick subset $T \subset \Delta^{n-2}$ the restriction $\mathcal{F}(t)$ of $\mathcal{F}$ to $\{t\} \times A^2_{r,1}$ extends to a coherent analytic sheaf on $\{t\} \times \Delta^2$. Then $\mathcal{F}$ extends to a coherent analytic sheaf $\tilde{\mathcal{F}}$ on Δ^n such that $\tilde{\mathcal{F}}^{[n-2]} = \tilde{\mathcal{F}}$.*

Remark 14.1. a) Remark that a locally free sheaf obviously satisfies the gap-sheaf condition of this theorem and therefore extends (but not necessarily as a free sheaf) to the associated polydisk. A line bundle of course extends as a bundle. To be thick in this theorem means that T is not contained in a countable union of proper locally closed analytic subsets.

b) Theorem 14.1 contains as a special case the statement of the extendability of bundles/sheaves with gap conditions from the $(n-2)$-concave Hartogs domain $H^{n,n-2}_r$ to the associated polydisk.

Let us give the proof of this theorem for a very particular an easy case of *line* bundles. First of all one can easily classify the line bundles on the two-dimensional annulus $A^2_{r,R} := \Delta^2_R \setminus \bar{\Delta}^2_r$, here $0 < r < R$ are fixed. For this cover $A^2_{r,R}$ by two Stein subsets

$$U_1 := \Delta_R \times A_{r,R} \quad \text{and} \quad U_2 := A_{r,R} \times \Delta_R.$$

Let L be a holomorphic line bundle on $A^2_{r,R}$. Since restrictions $L|_{U_i}$ are holomorphically trivial our bundle L is completely determined by its transition function $f : U_{1,2} \to \mathbb{C}^*$. Here $U_{12} := U_1 \cap U_2 = A_{r,R} \times A_{r,R}$. Furthermore, since $H^1(A^2_{r,R}, \mathbb{Z}) = H^2(A^2_{r,R}, \mathbb{Z}) = 0$ we get in the exponential sequence the following exact part

$$0 \to H^1(A^2_{r,R}, \mathcal{O}) \to H^1(A^2_{r,R}, \mathcal{O}^*) \to 0.$$

This means that holomorphic line bundles on $A^2_{r,R}$ are precisely the exponents of classes from $H^1(A^2_{r,R})$. The last group can be easily computed if one takes into account that our covering $\{U_1, U_2\}$ is acyclic. Let g be a holomorphic function in $A_{r,R} \times A_{r,R}$. One obviously finds $g_i \in \mathcal{O}(U_i)$ such that

$$g(z) - g_1(z) + g_2(z) = \sum_{k,l>0} \frac{a_{kl}}{z_1^k z_2^l} =: g_L(z).$$

Therefore our bundle L is *uniquely* defined by the transition function

$$f_L(z) = e^{g_L(z)} = e^{\sum_{k,l>0} \frac{a_{kl}}{z_1^k z_2^l}}, \tag{14.2}$$

where series $\sum_{k,l>0} \frac{a_{kl}}{z_1^k z_2^l}$ converge in the annulus $(\mathbb{P} \setminus \bar{\Delta}_r) \times (\mathbb{P} \setminus \bar{\Delta}_r)$.

Now let L be a holomorphic line bundle on $R_r^{n,2}$. It is uniquely determined by its transition function $f_L \in \mathcal{O}^*(\Delta^{n-2} \times A_{r,1} \times A_{r,1})$. This function writes as

$$f(z'', z_{n-1}, z_n) = e^{\sum_{k,l>0} \frac{a_{kl}(z'')}{z_{n-1}^k z_n^l}},$$

with a_{kl} holomorphically depending on $z'' = (z_1, ..., z_{n-2})$. $\mathcal{F}(t)$ extends to $\{t\} \times \Delta^2$ if and only if $a_{kl}(t) = 0$ for all $k, l > 0$. Since the set T of such t-s is not analytic by the assumption of the theorem we conclude that all $a_{kl} \equiv 0$. I.e. L is trivial and therefore extends to Δ^n.

Remark 14.2. *The role of the gap condition.* In general, when $\mathcal{F}$ is a sheaf the conclusion of the Theorem 14.1 fails to be true without the condition on the gap-sheaf. Take a curve A constructed in Example 5.1 and imbed it together with $\mathbb{P}^2$ to $\mathbb{P}^3$ in a canonical way as $\mathbb{P}^2 = \{z_3 = 0\} \subset \mathbb{P}^3$. Then A will be attached to the boundary of the unit ball $\mathbb{B}^6 \subset \mathbb{C}^3 \subset \mathbb{P}^3$ and will be not extendable to a neighborhood of any point on $A \cap \partial\mathbb{B}^6$. Let $\mathcal{J}_A$ be the ideal sheaf of A and let $\mathcal{F}_A := \mathcal{O}/\mathcal{J}_A$ be the quotient sheaf. Then $\mathcal{F}_A$ obviously doesn't extend to a neighborhood of any point on $A \cap \partial\mathbb{B}^6$ as a coherent analytic sheaf. Remark that $\mathcal{F}_A$ fails to satisfy the gap-condition of Theorem 14.1 because $\mathcal{F}_A^{[1]} = 0 \neq \mathcal{F}_A$.

Now let us turn to subsheaves.

Definition 14.2. *Let X be a complex space, $\mathcal{F}$ a coherent sheaf on X, and $\mathcal{G} \subset \mathcal{F}$ a coherent subsheaf. The d-th relative gap-sheaf of $\mathcal{G}$ in $\mathcal{F}$ is the sheaf $\mathcal{G}_{[d],\mathcal{F}}$ associated with the presheaf*

$$U \mapsto \big\{ s \in \Gamma(U, \mathcal{F}) \, : \, s|_{U \setminus A} \in \Gamma(U \setminus A, \mathcal{G}) \text{ for some analytic subset } A \subset U \text{ of } \mathsf{dim}\, A \leq d \big\}.$$

Therefore local sections of $\mathcal{G}_{[d],\mathcal{F}}$ are local sections of $\mathcal{G}$ defined outside analytic sets of dimension $\leq d$ which extend as sections of $\mathcal{F}$.

Theorem 14.2. *(Siu-Trautmann, [ST1]). Let $\mathcal{F}$ be a coherent sheaf on Δ^n and $\mathcal{G}$ a coherent subsheaf of $\mathcal{F}|_{R_r^{n,2}}$ such that $\mathcal{G}_{[n-2],\mathcal{F}} = \mathcal{G}$. Assume that there exists a thick subset $T \subset \Delta^{n-2}$ such that for every $t \in T$ the sheaf $\mathsf{Im}\,(\mathcal{G}|_{\{t\}\times A^2_{r,1}} \to \mathcal{F}|_{\{t\}\times A^2_{r,1}})$ extends to a coherent subsheaf of $\mathcal{F}|_{\{s\}\times\Delta^2}$. Then $\mathcal{G}$ extends to Δ^n as a coherent analytic subsheaf $\widetilde{\mathcal{G}}$ of $\mathcal{F}$, such that $\widetilde{\mathcal{G}}_{[n-2],\mathcal{F}} = \widetilde{\mathcal{G}}$.*

For the proof of Theorems 14.1 and 14.2 we refer to the book [Si3]. For the proof of the following result we refer to the book [ST2], see Theorem 10.4.3. Let X be a reduced complex space and let $\rho : X \to \mathbb{R}^+$ be a proper strongly $(n-2)$-convex function. Set $X_c^+ = \{\rho > c\}$.

Theorem 14.3. *Let $\mathcal{F}$ be a coherent analytic sheaf on X and let $\mathcal{G}$ be a coherent analytic subsheaf of $\mathcal{F}|_{X_c^+}$ which satisfies the relative $(n-2)$-gap condition: $\mathcal{G}_{[n-2]\mathcal{F}} = \mathcal{G}$. Then $\mathcal{G}$ extends to X as a coherent analytic subsheaf of $\mathcal{F}$ satisfying $\mathcal{G}_{[n-2]\mathcal{F}} = \mathcal{G}$.*

Let us sketch the proof of the analogous statement in absolute case.

Theorem 14.4. *Let $\mathcal{F}$ be a coherent analytic sheaf on X_c^+ satisfying $\mathcal{F}^{[n-2]} = \mathcal{F}$. Then $\mathcal{F}$ uniquely extends to a coherent analytic sheaf $\tilde{\mathcal{F}}$ on X satisfying $\tilde{\mathcal{F}}^{[n-2]} = \tilde{\mathcal{F}}$.*

Proof. Let x_0 be an $(n-2)$-concave boundary point and $\pi : (U,x_0) \to (V,0)$ as in the Projection Lemma 18.1. The direct image $\pi_*\mathcal{F}$ is a coherent analytic sheaf on $i(H_r^{n,n-2})$ by Grauert's theorem, and it obviously satisfies the gap sheaf condition. Therefore by Theorem 14.1 extends to a coherent analytic sheaf $\widetilde{\pi_*\mathcal{F}}$ on $V = i(\Delta^n)$. Denote by $\mathcal{G} := \pi^*\widetilde{\pi_*\mathcal{F}}$ the *analytic* inverse image of $\widetilde{\pi_*\mathcal{F}}$. This is a coherent analytic sheaf, see §9.6 in [Dm2], which admits a natural epimorphism $m : \mathcal{G}|_{\pi^{-1}(i(H_r^{n,n-2}))} \to \mathcal{F}$. This epimorphism extends to $\{\rho > 0\} \cap U$, where ρ is the strongly $(n-2)$-convex function in question. Denote by $\mathcal{K}$ the kernel of m. $\mathcal{K}$ is a coherent analytic subsheaf of $\mathcal{G}$ satisfying the relative gap-sheaf condition and therefore it extends to a neighborhood of x_0 by Theorem 14.3 as a coherent analytic sheaf $\tilde{\mathcal{K}}$. The quotient sheaf $\mathcal{G}/\tilde{\mathcal{K}}$ will be the desired extension of $\mathcal{F}$. □

Remark 14.3. Theorems 14.1, 14.2, 14.3 and 14.4 hold for (sub)-sheaves satisfying the d-th (relative)-gap condition across d-concave boundaries for $0 \leqslant d \leqslant n-2$. See the quoted sources. It is worth noting that when $\mathcal{F}$ is the locally free sheaf of rank one then its extension will be the line bundle as well.

14.3. Slicing and separate extension of holomorphic vector bundles. Let us give one separate analyticity result for holomorphic vector bundles. Denote coordinates in the polydisk $\Delta^n = \Delta^{n-1} \times \Delta$ as $(w_1,\dots,w_{n-1},z)$, where $w = (w_1,\dots,w_{n-1})$ are coordinates in Δ^{n-1}. By a $(n-k)$-index we understand a multi-index $I = (i_1,\dots,i_{n-k})$ such that $1 \le i_1 < \dots < i_{n-k} \le n-1$. For such I we denote by $\pi_I : \Delta^{n-1} \to \Delta^{n-k}$ the projection $(w_1,\dots,w_{n-1}) \mapsto (w_{i_1},\dots,w_{i_{n-k}})$, and by $\Delta_{I,v}^{k-1}$ the slice $\pi_I^{-1}(v)$ for $v \in \Delta^{n-k}$. Let $A = A_{\rho,1} \subset \Delta$ be an annulus, E a holomorphic vector bundle over A, and E_1, E_2 two extensions of E over Δ. The latter means that we have fixed isomorphisms $\varphi_i : E_i|_A \xrightarrow{\cong} E$, $i = 1,2$. We say that extensions E_1, E_2 *coincide* if the spaces of sections $\varphi_1[\mathcal{O}(\Delta,E_1)]$ and $\varphi_2[\mathcal{O}(\Delta,E_2)]$ coincide, considered as the subspaces of $\mathcal{O}(A,E)$.

Example 14.2. Let $A = A_{\frac{1}{2},1}$. Take $E = A \times \mathbb{C}$, this is the total space of the trivial bundle $\mathcal{O}$ on A. For every $k \geqslant 1$ and every $p \in \Delta_{\frac{1}{2}}$ consider the bundle $E_{k,p}$ whose sheaf of sections is $\mathcal{O}(k[p])$, *i.e.*, the sheaf of holomorphic in $\Delta \setminus \{p\}$ functions having at p a pole of order at most k. Every $E_{k,p}$ is an extension of E, but they are all distinct.

Theorem 14.5. *Let E be a holomorphic vector bundle over $\Delta^{n-1} \times A$, $n \ge 3$ and let for every $(n-k)$-index $I \subset \{1,...,n-1\}$ and every $v \in \Delta^{n-k}$ a holomorphic extension $\widetilde{E}_{I,v}$ of $E|_{\Delta_{I,v}^{k-1} \times A}$ to $\Delta_{I,v}^{k-1} \times \Delta$ be given. Suppose that there exists a set $W^* \subset \Delta^{n-1}$ of full measure such that for every $w \in W^*$ and any two $(n-k)$-indices $I_1 \neq I_2$ we have that for $v_1 := \pi_{I_1}(w)$ and $v_2 := \pi_{I_2}(w)$ extensions $\widetilde{E}_{I_1,v_1}$ and $\widetilde{E}_{I_2,v_2}$ restricted to $\{w\} \times \Delta$ coincide. Then there exists an extension of E to the polydisk as a coherent analytic sheaf $\widetilde{\mathscr{E}}$, such that for every $(n-k)$-index I and almost every $v \in \Delta^{n-k}$ the restriction $\widetilde{\mathscr{E}}|_{\Delta_{I,v}^{k-1} \times \Delta}$ is the sheaf of holomorphic sections of $\widetilde{E}_{I,v}$. Moreover, the singular set of $\widetilde{\mathscr{E}}$ is an analytic subset of Δ^n of codimension $> k$.*

A special case of this statement can be found in §3 of [She2]. The proof of the general case will appear in [She3].

15. Levi flat hypersurfaces and roots of holomorphic line bundles

In this section we shall discuss a quite specific question about extension of *roots* of holomorphic line bundles. As a motivation for this question we refer to the paper [Oh] of T. Ohsawa, where extension of roots is related to the existence of real analytic Levi flat hypersurfaces in some (like $\mathbb{P}^n$, $n \geqslant 2$) compact complex manifolds. In particular, Corollary 15.1 below implies the non-existence of real analytic Levi flat hypersurfaces in $\mathbb{P}^n$ for $n \geqslant 3$ and shows that for $n = 2$ there is a specific obstruction for this method to work. It should be said that the case $n \geqslant 3$ is not new and is contained in a more general result of [LN].

Let $\mathcal{N}$ be a holomorphic line bundle on a (non compact) complex manifold X. Suppose that X admits a strongly $(n-1)$-convex exhaustion function ρ. Suppose that for some $t^* \in \rho(X)$ and some $k\in\mathbb{N}$ our $\mathcal{N}$ admits a k-th root $\mathcal{F}$ on $X^+_{t^*}:=\{\rho>t^*\}$. I.e., $\mathcal{F}$ is a holomorphic line bundle on $X^+_{t^*}$ such that $\mathcal{F}^{\otimes k}$ is isomorphic to $\mathcal{N}|_{X^+_{t^*}}$. Denote by $r_{t^*}:\mathcal{F}^{\otimes k}\to\mathcal{N}|_{X^+_{t^*}}$ some isomorphism. We want to extend $\mathcal{F}$ to a neighborhood of Σ_{t^*} and then to the whole of X together with r_{t^*}.

15.1. Extension of roots. Following [Iv11] let us describe the obstructions to the extension of roots. Fix a point $x_0\in\Sigma_{t^*}$ and take a polydisk neighborhood $V\ni x_0$. Set $V^+:=V\cap X^+_{t^*}$. Since $\mathcal{N}|_V$ is trivial its restriction to V^+ is trivial too. Therefore $c_1(\mathcal{N}|_{V^+})=0$. Since $kc_1(\mathcal{F})=c_1(\mathcal{N}|_{V^+})$ we get that $c_1(\mathcal{F})=0$. Holomorphic line bundles with vanishing Chern class on an (arbitrary) complex manifold V^+ are described by the following exact sequence

$$0\to\mathbb{Z}\to H^0(V^+,\mathcal{O})\xrightarrow{\mathsf{exp}}H^0(V^+,\mathcal{O}^*)\xrightarrow{\delta}H^1(V^+,\mathbb{Z})\xrightarrow{i}H^{0,1}(V^+)\to Pic^0(V^+)\to 0. \quad (15.1)$$

Here $i:H^1(V^+,\mathbb{Z})\to H^{0,1}(V^+)$ is the composition of $H^1(V^+,\mathbb{Z})\to H^1(V^+,\mathcal{O})$ with the Dolbeault isomorphism $D:H^1(V^+,\mathcal{O})\to H^{0,1}(V^+)$. $Pic^0(V^+)$ appears here as the kernel of the map c_1 from the group $H^1(V^+,\mathcal{O}^*)$ of all holomorphic line bundles on X to $H^2(V^+,\mathbb{Z})$. I.e., $Pic^0(V^+)$ is exactly the group of holomorphic line bundles on V^+ with vanishing first Chern class, that is, topologically trivial ones. The arrow $H^{0,1}(V^+)\to Pic^0(V^+)$ is the composition of the inverse to the Dolbeault isomorphism $D^{-1}:H^{0,1}(V^+)\to H^1(V^+,\mathcal{O})$ and the exponential map $\mathsf{exp}:H^1(V^+,\mathcal{O})\to H^1(V^+,\mathcal{O}^*)$. If the map

$$H^0(V^+,\mathcal{O})\xrightarrow{\mathsf{exp}}H^0(V^+,\mathcal{O}^*)$$

is surjective then (15.1) writes as

$$0\to H^1(V^+,\mathbb{Z})\xrightarrow{i}H^{0,1}(V^+)\to Pic^0(V^+)\to 0, \quad (15.2)$$

which means that

$$Pic^0(V^+)\cong H^{0,1}(V^+)/H^1(V^+,\mathbb{Z}). \quad (15.3)$$

This is the case for example if all holomorphic functions on V^+ are constant or, all holomorphic functions from V^+ extend to some simply connected $V\supset V^+$ and this is exactly our case: all holomorphic functions from $V^+=V\cap X^+_{t^*}$ extend to a neighborhood of $V^+\cap\Sigma_{t^*}$ by Hartogs' Lemma. Remark furthermore that in our concrete situation the only case when V^+ is not simply connected is when ρ is strictly $(n-1)$-convex with index at x_0 equal to $2n-2$, the maximal possible value. In that case $H_1(V^+,\mathbb{Z})=\mathbb{Z}$. Fix a generator C of $H_1(V^+,\mathbb{Z})$ and a cohomology class $A\in H^1(V^+,\mathbb{Z})$ such that $\langle A,C\rangle=1$. Set $B:=i(A)$, where $i:H^1(V^+,\mathbb{Z})\to H^{0,1}(V^+)$ is from (15.2) as above. For $0\leqslant l<k$ denote by $\mathcal{F}_k(l)$ the holomorphic line bundle which corresponds to $\frac{l}{k}B$ under the isomorphism (15.3), $\mathcal{F}_k(0)$ is trivial.

Theorem 15.1. *Let $\rho:X\to\mathbb{R}$ be a strongly $(n-1)$-convex exhaustion function on a complex manifold X and let $\mathcal{N}$ be a holomorphic line bundle on X. Suppose that for some $t^*\in\rho(X)$ the restriction $\mathcal{N}|_{X^+_{t^*}}$ admits a k-th root $\mathcal{F}$. Then:*

i) $\mathcal{F}$ extends to a k-th root of $\mathcal{N}$ to a neighborhood of Σ_{t^*} provided all points on Σ_t are either smooth, or strongly q-convex with $q\leqslant n-2$, or strictly $(n-1)$-convex with index of the critical point less then $2n-2$;

ii) if Σ_{t^*} is strictly $(n-1)$-convex at some critical point $\mathsf{c}\in U_j\cap\Sigma_{t^*}$ of index $2n-2$ then either $\mathcal{F}$ extends to a neighborhood of c to a k-th root of $\mathcal{N}$ or there exist a neighborhood V of c such that $\mathcal{F}|_{V^+}$ is isomorphic to some $\mathcal{F}_k(l)$ with $1\leqslant l<k$ there.

Proof. *i)* In all cases collected in the first part $V^+:=V\cap X^+_{t^*}$ is connected and simply connected as well (by Morse Lemma). Therefore $Pic(V^+)=H^{0,1}(V^+)$. If ω represents $\mathcal{F}$ in $H^{0,1}(V^+)$ then $k\omega$ represents $\mathcal{N}|_{V^+}$. Since $k\omega=0$ we obtain that $\omega=0$. Let us glue the local extensions obtained. Cover Σ_{t^*} by a finite number of such coordinate neighborhoods U_j that:

i) Each U_j is biholomorphic to a polydisk Δ^n and $U_j\cap X^+_{t^*}$ are connected;

ii) the associated Hartogs figures H^n_ε are contained in $U_j\cap X^+_{t^*}$;

iii) the double $U_{ij} := U_i \cap U_j$ and triple $U_{ijk} := U_i \cap U_j \cap U_k$ intersections are connected, simply connected and moreover, $U_{ij} \cap X^+_{t^*}$ and $U_{ijk} \cap X^+_{t^*}$ are connected and simply connected as well.

Extend $\mathcal{F}_{t^*}$ through all of them, *i.e.*, get line bundles $\mathcal{F}^j_{t^*}$ over $V_j := X^+_{t^*} \cup U_j$ - extensions of $\mathcal{F}_{t^*}$. We need to prove that these extensions match together, *i.e.*, that transition functions extend from $U_i \cap U_j \cap X^+_{t^*}$ to $U_i \cap U_j$ for all $U_i \cap U_j \neq 0$. Denote by $\varphi_j : \mathcal{F}^j_{t^*}|_{X^+_{t^*}} \to \mathcal{F}_{t^*}$ the corresponding isomorphisms. Then we get the transition functions $\varphi_{ij} = \varphi_i^{-1} \circ \varphi_j : \mathcal{F}^j_{t^*}|_{U_i \cap U_j \cap X^+_{t^*}} \to \mathcal{F}^i_{t^*}|_{U_i \cap U_j \cap X^+_{t^*}}$. As it was said already, we need to extend φ_{ij} onto $U_i \cap U_j$ and this is not automatic as it was explained on the Fig. 6. Denote by $\psi_j : \mathcal{N}|_{V_j} \to (\mathcal{F}^j_{t^*})^{\otimes k}$ the isomorphisms, obtained as extensions of the isomorphism $\psi_{t^*} : \mathcal{N}|_{U^+_{t^*}} \to \mathcal{F}^{\otimes k}_{t^*}$. From the diagram

$$\begin{array}{ccc} \mathcal{N}|_{U_i \cap U_j \cap X^+_{t^*}} & \xrightarrow{\mathsf{tr}} & \mathcal{N}|_{U_i \cap U_j \cap X^+_{t^*}} \\ \big\downarrow \psi_j & & \big\downarrow \psi_i \\ (\mathcal{F}^j_{t^*}|_{U_i \cap U_j \cap X^+_{t^*}})^{\otimes k} & \xrightarrow{\varphi^{\otimes k}_{ij}} & (\mathcal{F}^i_{t^*}|_{U_i \cap U_j \cap X^+_{t^*}})^{\otimes k} \end{array} \tag{15.4}$$

we see that $\varphi^{\otimes k}_{ij} = \psi_i \circ \psi_j^{-1}$ on $U_i \cap U_j \cap X^+_{t^*}$. But ψ_j (resp. ψ_i) is defined over U_j (resp. U_i) and tr is a transition map of a globally existing bundle $\mathcal{N}$, therefore $\psi_i \circ \mathsf{tr} \circ \psi_j^{-1}$ is defined over $U_i \cap U_j$ extending $\varphi^{\otimes k}_{ij}$. But then φ_{ij} also extends onto $U_i \cap U_j$ as a k-th root on an extendable non-vanishing function on a simply connected domain. It is easy to see that the cocycle condition for the extended transition maps will be preserved. Indeed, it is satisfied on $U_{ijk} \cap X^+_{t^*}$, so it will be satisfied on U_{ijk} too. The case is proved.

ii) Let the index be $2n-2$. By considerations as above for a cohomology class $\omega \in H^{0,1}(V^+)$ representing $\mathcal{F}$ we have that $k\omega \in H^1(V^+, \mathbb{Z})$, *i.e.*, it is nothing but $\frac{l}{k}B$ for some integer $1 \leqslant l < k$. Bundle which corresponds to $\frac{l}{k}B$ we denoted as $\mathcal{F}_k(l)$. This finishes the proof.

□

Remark 15.1. Case $q \leqslant n-2$ in the theorem above is also served by the much more general Theorem 14.1 of Siu. But in the particular case of the roots of line bundles we prefer for the readers convenience to give an uniform approach which serves all cases.

Corollary 15.1. *Let $\rho : X \to \mathbb{R}$ be a strongly $(n-1)$-convex exhaustion function on a complex manifold X without critical points of index $2n-2$ and let $\mathcal{N}$ be a holomorphic line bundle on X. Suppose that for some $t_0 \in \rho(X)$ the restriction $\mathcal{N}|_{X^+_{t_0}}$ admits a k-th root $\mathcal{F}$. Then $\mathcal{F}$ extends to the whole of X as a k-th root of $\mathcal{N}$.*

Proof. Let $r : \mathcal{F}^{\otimes k} \to \mathcal{N}|_{X^+_{t_0}}$ be some isomorphism. Denote by T the set of points t on the interval $\rho(X)$ such that the pair $(\mathcal{F}, r)$ extends to X^+_t. T is non-empty, it contains t_0.

T *is open*. Indeed, let $(\mathcal{F}, r)$ be extended to $X^+_{t^*}$. By Theorem 15.1 our pair extends to a neighborhood of Σ_{t^*} because we forbid the case (*ii*) . Therefore our "analytic object" $(\mathcal{F}, r)$ extends to $X^+_{t'}$ for some $t' < t$.

T *is closed*. By saying that $\mathcal{F}$ extends to X^+_t as a k-th root $\mathcal{F}_t$ of $\mathcal{N}$, we mean the following:

a) Holomorphic line bundles $\mathcal{F}_{t_1}$ on $X^+_{t_1}$ are defined for all $t_1 \geqslant t$ together with isomorphisms $\psi_{t_1} : \mathcal{N}|_{X^+_{t_1}} \to \mathcal{F}^{\otimes k}_{t_1}$.

b) For all pairs $t_1 \geqslant t_2 \geqslant t$ the bundle $\mathcal{F}_{t_2}$ is an extension of $\mathcal{F}_{t_1}$, *i.e.*, an isomorphism $\varphi_{t_1 t_2} : \mathcal{F}_{t_2}|_{X^+_{t_1}} \to \mathcal{F}_{t_1}$ is given, and these isomorphisms satisfy:

b)$_1$ $\varphi_{t_1 t_1} = \mathsf{Id}$;

b)$_2$ $\varphi_{t_1 t_2} \circ \varphi_{t_2 t_3} = \varphi_{t_1 t_3}$ for $t_1 \geqslant t_2 \geqslant t_3$.

c) Isomorphisms ψ_{t_1} and $\varphi_{t_1t_2}$ are natural in the sense that for every pair $t_1 \geqslant t_2 \geqslant t$ the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{N}|_{X_{t_2}^+} & \xrightarrow{r_{t_1t_2}} & \mathcal{N}|_{X_{t_1}^+} \\ \Big\downarrow \psi_{t_2} & & \Big\downarrow \psi_{t_1} \\ \mathcal{F}_{t_2}^{\otimes k} & \xrightarrow{\varphi_{t_1t_2}^{\otimes k}} & \mathcal{F}_{t_1}^{\otimes k} \end{array} . \tag{15.5}$$

Here $r_{t_1t_2} : \mathcal{N}|_{X_{t_2}^+} \to \mathcal{N}|_{X_{t_1}^+}$ is the natural restriction operator of the globally existing bundle $\mathcal{N}$.

Now suppose that for every $t > t^*$ the bundle $\mathcal{F}$ extends onto X_t^+ as a k-th root $\mathcal{F}_t$ of $\mathcal{N}$. We define the presheaf $\mathcal{F}_{t^*}$ on $X_{t^*}^+$ as the projective limit of $\mathcal{F}_t$-s for $t > t^*$:

$$\mathcal{F}_{t^*} := \varprojlim \mathcal{F}_t. \tag{15.6}$$

A section σ of $\mathcal{F}_{t^*}$ over an open set $V \subset X_{t^*}^+$ is a product $\prod_{t>t^*} \sigma_t$ of sections σ_t of $\mathcal{F}_t$ over $V \cap X_t^+$ such that $\varphi_{t_1t_2}(\sigma_{t_2}) = \sigma_{t_1}$ for every pair $t_1 \geqslant t_2 > t^*$. Restriction map for $W \subset V$ in $\mathcal{F}_{t^*}$ is $\prod_{t>t^*} \sigma_t \to \prod_{t>t^*} \sigma_t|_{W\cap U_t^+}$, which is correctly defined.

It is easy to see that the presheaf, so defined, is actually a sheaf and that this sheaf is locally free of rank one. Isomorphisms $\varphi_{tt^*} : \mathcal{F}_{t^*}|_{X_t} \to \mathcal{F}_t$ for $t > t^*$ are naturally defined as $\varphi_{tt^*}(\prod_{t>t^*} \sigma_t) = \sigma_t$. For $t = t^*$ we set $\varphi_{t^*t^*} = \mathsf{Id}$.

Take some $t_1 > t^*$. It is not difficult now to see that $\psi_{t_1} : \mathcal{N}|_{X_{t_1}^+} \to \mathcal{F}_{t_1}^{\otimes k}$ extends to an isomorphism $\psi_{t^*} : \mathcal{N}|_{X_{t^*}^+} \to \mathcal{F}_{t^*}^{\otimes k}$ and the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{N}|_{X_{t^*}^+} & \xrightarrow{r_{t_1t^*}} & \mathcal{N}|_{X_{t_1}^+} \\ \Big\downarrow \psi_{t^*} & & \Big\downarrow \psi_{t_1} \\ \mathcal{F}_{t^*}^{\otimes k} & \xrightarrow{\varphi_{t_1t^*}^{\otimes k}} & \mathcal{F}_{t_1}^{\otimes k}. \end{array} \tag{15.7}$$

The proof goes by continuity: for, if ψ_{t_1} is already extended to ψ_t, then take the composition $(\varphi_{tt^*}^{-1})^{\otimes k} \circ \psi_t : \mathcal{N}|_{X_t} \to \mathcal{F}_{t^*}^{\otimes k}|_{X_t}$. It is an analytic morphism of sheaves and in local chats it is given by holomorphic functions. Therefore it can be extended through a pseudoconcave boundary.

15.2. Extension of roots across the contractible analytic sets. The complement to a Levi flat hypersurface could be not exactly Stein but only 1-convex. In this case one theorem of Grauert tells us that the only reason for such component of the complement not to be exhausted by a strictly psh-function is that it can contain a *contractible* analytic set. It turns not to be a problem in our case. Recall that a compact analytic set E in a normal complex space X is called *contractible* if there exists a normal complex space Y, a compact analytic set A in Y of codimension at least two and a holomorphic map (a contraction) $\mathsf{c} : X \to Y$ which is a biholomorphism between $X \setminus E$ and $Y \setminus A$. If A is zero dimensional one calls E *exceptional*.

Theorem 15.2. *Let E be a contractible analytic set in a normal complex space X and let $\mathcal{N}$ be a holomorphic line bundle on X. Suppose $\mathcal{N}$ admits a k-th root $\mathcal{F}$ on $X \setminus E$. Then $\mathcal{F}$ extends to a holomorphic line bundle $\widetilde{\mathcal{F}}$ on the whole of X. Moreover, $\widetilde{\mathcal{F}}^{\otimes k}$ doesn't depend on k.*

Remark 15.2. a) The extension $\widetilde{\mathcal{F}}$ of $\mathcal{F}$ can fail to be a k-th root of $\mathcal{N}$ on E. Let X be the blown-up $\mathbb{C}^2$ at origin and E be the exceptional curve. Set $\mathcal{N} := [E]$. By the adjunction formula $\mathcal{N}|_E$ is the normal bundle of the imbedding $E \subset X$ *i.e.*, is $\mathcal{O}_E(-1)$. At the same time $\mathcal{N}|_{X\setminus E}$ is trivial and as such admits a trivial root $\mathcal{F} = \mathcal{O}$ on $X \setminus E = \mathbb{C}^2 \setminus \{0\}$, and this $\mathcal{F}$ is a root of $\mathcal{N}|_{X\setminus E}$ of any given degree $k \in \mathbb{N}$. But $\mathcal{N}$ doesn't have roots of any degree $k > 1$ on E.

b) At the same time Theorem 15.2 means that $\mathcal{N}|_{X\setminus E}$ can be (differently) extended to a line bundle, say $\widetilde{\mathcal{N}}$ on X, having the extension $\widetilde{\mathcal{F}}$ of $\mathcal{F}$ as its k-th root. And, moreover, $\widetilde{\mathcal{N}}$ is the same for all k. I.e., we can once forever modify $\mathcal{N}$ on the contractible set and make all roots of $\mathcal{N}$, which one can take on $X \setminus E$, to extend onto the whole of X as roots of the modified bundle.

c) Note that according to our definition of a contractible set all compact analytic sets of codimension at least two are contractible. For them, in fact, more is true, see Lemma 15.1: $\mathcal{F}$ extends to a k-th root of $\mathcal{N}$ (and not of some other $\widetilde{\mathcal{N}}$).

The key point in the proof is the following statement.

Lemma 15.1. *Let X be a normal complex space and A a codimension two analytic subset of X. Let a holomorphic line bundle $\mathcal{N}$ over X be given. Suppose that $\mathcal{N}|_{X\setminus A}$ admits a k-th root $\mathcal{F}$. Then $\mathcal{F}$ extends as a k-th root of $\mathcal{N}$ onto the whole of X.*

For the proof we refer to [Iv11]. Let now $\mathsf{c} : X \to Y$ be the contraction. Lemma 15.1 is now applicable to the direct images of $\mathsf{c}_*\mathcal{F}|_{X\setminus E}$ and $\mathsf{c}_*\mathcal{N}|_{X\setminus E}$. Both are holomorphic line bundles on $Y \setminus A$. The second one extends onto Y as a coherent analytic sheaf by the Theorem about direct image sheaves of Grauert. Therefore it extends onto Y as a holomorphic line bundle. Denote this extension (with some abuse of notation) as $\mathsf{c}_*\mathcal{N}$. By Lemma 15.1 $\mathsf{c}_*\mathcal{F}$ extends as a holomorphic line bundle onto Y and stays there to be a k-th root of $\mathsf{c}_*\mathcal{N}$. Denote by $\mathsf{c}_*\mathcal{F}$ this extension. Now $\mathsf{c}^*\mathsf{c}_*\mathcal{F}$ will be an extension of $\mathcal{F}$ to X, which is a k-th root of $\mathsf{c}^*\mathsf{c}_*\mathcal{N}$, and the last is an extension of $\mathcal{N}|_{X\setminus E}$ onto X. Theorem is proved.

Let us remark that the Thullen-type extension theorem for roots of holomorphic line bundles also holds true. For the proof of the following statement we refer to [Iv11].

Theorem 15.3. *Let Y be an analytic set in a connected complex manifold X and let G be a domain in X, which contains $X \setminus Y$ and which intersects every irreducible component of Y of codimension one. Let $\mathcal{N}$ be a holomorphic line bundle on X such that it admits a k-th root $\mathcal{F}$ on G. Then $\mathcal{F}$ extends to a holomorphic line bundle onto the whole of X and stays there to be a k-th root of $\mathcal{N}$.*

16. Thullen-type extension of bundles with constraints on curvature

Certain analytic objects which do not possess a Hartogs-type extension property nevertheless do possess a weaker one, the so called Thullen-type, see Definition 12.1. In this section we shall describe few sufficient conditions on holomorphic vector bundle which imply a Thullen-type extension for them.

16.1. Limit holonomy and Sobolev extension of unitary connections. Let E be a complex vector bundle over a manifold X, ∇ a connection in E, *i.e.*, a complex linear mapping $\nabla : \Gamma(E) \to \Lambda^1 \otimes \Gamma(E)$ which satisfies the Leibnitz rule: $\nabla(fe) = df \otimes e + f\nabla e$ for every smooth function f and every section e. Let $\gamma : [0,1] \to X$ be a piece wisely smooth path and $x_0 := \gamma(0)$, $x_1 := \gamma(1)$ be its ends. Then the ∇-parallel transport along γ is a linear operator $\tau_\gamma : E_{x_0} \to E_{x_1}$ between the fibers over the end points of γ. If γ is closed, $\gamma(0) = \gamma(1)$, then τ_γ is an endomorphism of the fiber E_{x_0}.

Definition 16.1. *The conjugacy class of the ∇-parallel transport along a closed path γ is called the* holonomy *of the connection ∇ along γ and is denoted by η_γ.*

It is known that η_γ is independent of the choice of the point on the closed path γ. If E is equipped with a Hermitian metric $\langle\cdot,\cdot\rangle$ and ∇ is a unitary connection, *i.e.*, $\nabla\langle e,h\rangle = \langle\nabla e,h\rangle + \langle e,\nabla h\rangle$ for any $e,h \in \Gamma(E)$, then the parallel transport $\tau_\gamma : E_{x_0} \to E_{x_1}$ is a unitary operator. Therefore in the unitary case we consider the holonomy η_γ as a conjugacy class of unitary matrices, *i.e.*, a conjugacy class in the unitary group $\mathbf{U}(k)$ where k is the rank of E. Such a conjugacy class is determined by its eigenvalues. The convergence $\eta_\nu \longrightarrow \eta_\infty$ of conjugacy classes of unitary matrices is understood as the convergence of the sets of eigenvalues counted with multiplicities. Or, equivalently, as existence of representatives $A_\nu \in \mathbf{U}(k)$ of the classes η_ν such that the A_ν converge. In this case the limit matrix $A_\infty = \lim A_\nu$ is a representative of the limit class $\eta_\infty = \lim \eta_\nu$.

The following result was proved in [SS] in the case of $\mathrm{dim}_{\mathbb{R}}X = 4$ and $\mathrm{rank}_{\mathbb{C}}E = 2$, and in [She1] in the general case. Let Y be the unit ball in $\mathbb{R}^{m-2}$, $m \geq 2$, and let B be the unit ball in $\mathbb{R}^m$. Set $\check{B} := B\backslash Y$ and identify Y with $Y \times \{0\} \subset B$. Suppose that E is a smooth Hermitian vector bundle of rank m over $\check{B}$ equipped with a unitary connection ∇. Denote by F_∇ the curvature of ∇, by $\gamma_{y,r}$ the circle of radius r in the disc $\{y\} \times \Delta$, and by $\eta_{y,r}$ the holonomy of ∇ along $\gamma_{y,r}$.

Theorem 16.1. a) *(Existence of the limit holonomy). Assume that the curvature F_∇ is locally integrable near Y. Then there exists a subset $Y^* \subset Y$ of full measure such that the limit $\lim\limits_{r\to 0} \eta_{y,r} =: \eta_{y,0}$ exists for every y from Y^*.*

b) *(Constancy of the limit holonomy). Assume that the curvature F_∇ is locally square integrable near Y. Then there exists a subset $Y^* \subset Y$ of full measure and the conjugacy class η^* such that for every y from Y^* the limit $\lim\limits_{r\to 0} \eta_{y,r}$ exists and equals η^*.*

Remark 16.1. Moreover, under the conditions of part **(b)** of this theorem the following is proved in [She1]. Let $U,\widetilde{U} \subset B$ be open subsets containing Y and $\Phi : U \to \widetilde{U}$ a $\mathcal{C}^1$-diffeomorphism mapping Y onto Y, and $\tilde{\eta}_{y,r}$ the holonomy of the connection $\tilde{\nabla} := \Phi^*\nabla$ along $\gamma_{y,r}$. Then exists a subset $\widetilde{Y}^* \subset Y$ of full measure such that for every y from $\widetilde{Y}^*$ the limit $\lim\limits_{r\to 0} \tilde{\eta}_{y,r}$ exists and equals the same class η^*. This means that the limit holonomy stays invariant under the coordinate changes and is therefore well defined when Y is a real codimension two submanifold of a real manifold X.

Definition 16.2. *In the situation* **(b)** *of the theorem above the conjugacy class η^* is called the limit holonomy of the connection ∇ around the submanifold Y. We say that the limit holonomy is trivial if the conjugacy class η^* consists of the identity matrix.*

Let B_t be the ball of radius t, set $\check{B}_t := B_t\backslash Y$. We have the following result on local structure of connections with a given limit holonomy, for its proof we refer to [She1].

Theorem 16.2. *(V. Shevchishin, [She1]). There exists a constant $\varepsilon > 0$, depending on $n \geq 4$, $m = \mathrm{rank}_{\mathbb{C}}E$, and a constant $C_p < \infty$ depending on n, m and $n/2 \leq p < \infty$ such that the following holds. Let (E,h,∇) be a smooth Hermitian vector bundle with a unitary connection over $\check{B}$. Assume that $\|F_\nabla\|_{L^{m/2}(B)} < \varepsilon$ that the limit holonomy of ∇ around Y is trivial. Then there exists a flat unitary connection $\nabla^\flat$ in the bundle E over $\check{B}_{\frac{1}{2}}$ such that the $\mathrm{End}(E)$-valued 1-form $A := \nabla - \nabla^\flat$ satisfies the following a priori estimate*

$$\|A\|_{L^p(B_{\frac{1}{2}})} + \left\|\nabla^\flat A\right\|_{L^p(B_{\frac{1}{2}})} \leq C_p \cdot \|F_\nabla\|_{L^p(B)}. \tag{16.1}$$

In addition the holonomy of $\nabla^\flat$ around Y is also trivial.

Let us make some remarks about this theorem. It means in other words that one can find a unitary frame ξ of E in $\check{B}_{\frac{1}{2}}$ such that $\nabla^\flat = d$ in this frame, $\nabla = d + A$, and (16.1) reads as

$$\|A\|_{L^{1,p}(B_{\frac{1}{2}})} \leq C_p \cdot \|F_\nabla\|_{L^p(B)}. \tag{16.2}$$

Furthermore, the L^p-norm of a two-form $F_{\alpha,\beta}dx^\alpha \wedge dx^\beta$ is defined as

$$\|F\|^p_{L^p(B)} = \int_B \left(\sum_{\alpha,\beta} |F_{\alpha,\beta}|^2\right)^{p/2}. \tag{16.3}$$

$L^{m/2}$-norm is conformally invariant in the following sense. Let $\tau_t : B \to B$ be a contraction, *i.e.*, $\tau_t x = tx$, $0 < t < 1$. The one easily checks that

$$\|F\|_{L^{m/2}(B_t)} = \|\tau_t^* F\|_{L^{m/2}(B)}. \tag{16.4}$$

Now the theorem implies that the unitary connection extends in the Sobolev sense to the whole of B.

16.2. Thullen-type extension theorem for holomorphic bundles with L^2-bounds on curvature. Let X be a complex manifold of $\mathsf{dim}_{\mathbb{C}}X := n \geq 2$, $Y \subset X$ either an analytic subset of X or, a $\mathcal{C}^1$-submanifold of real codimension 2, and $G \subset X$ an open set which contains $X \backslash Y$ and intersects each irreducible component of Y of pure codimension 1 or resp. each connected component of Y. Set $Y' := Y \backslash G$. Recall that a Chern connection on a holomorphic Hermitian bundle E on a complex manifold is the unique connection which preserves both holomorphic and Hermitian structures of E. The former means that the $(0,1)$-component of the connection form A in holomorphic frame satisfies the following integrability condition

$$\bar{\partial}A^{0,1} + A^{0,1} \wedge A^{0,1} = 0. \tag{16.5}$$

Theorem 16.3. *(V. Shevchishin,* [She2]*). Let (E,h) a holomorphic Hermitian vector bundle over G. Assume that the curvature F_∇ of the Chern connection ∇ is locally L^2-integrable in a neighborhood of each point y on Y. Then E extends to X as a reflexive sheaf $\mathscr{E}$. Moreover, if the curvature F_∇ is locally L^p-integrable for some $p \geq 2$, then the singular set* $\mathsf{Sing}(\mathscr{E})$ *has complex codimension at least $[p]+1$ in X.*

In particular, in complex dimension two E extends as a bundle. And more generally, if $F_\nabla \in L^n_{loc}(X)$, then E again extends as a bundle. We describe main steps of the proof indicating the techniques lying behind, referring to [She2] for the complete proof. It follows from the "Thullen-type" structure of the singular set $Y' = Y \backslash G$ that the limit holonomy of the Chern connection ∇ in E is trivial.

Step 1. Consider first the special case when the curvature is L^n_{loc}-integrable. In this situation Theorem 16.2 provides that for every point y on Y there exists a neighborhood $V \subset X$ of y biholomorphic to the unit ball $B \subset \mathbb{C}^n$ and frame ξ of E in $V \backslash Y$ such that the connection form A of ∇ in the frame ξ is $L^{1,n}$-regular and its $(0,1)$-component $A^{0,1}$ fulfills the integrability condition (16.5). Remark that Sobolev imbedding, which reads in $\mathbb{C}^n = \mathbb{R}^{2n}$ as $L^{1,p}_{loc} \subset L^{\frac{2np}{2n-p}}$, gives $L^{1,n}_{loc} \subset L^{2n}_{loc}$, and therefore if $A \in L^{1,n}_{loc}$ then $A \wedge A \in L^n_{loc}$.

Step 2. Non-linear matrix-valued Dolbeault lemma. Let $V \subset X$ be an open set and ξ the unitary frame of E in $V \backslash Y$ constructed in the previous step. Set $k := \mathsf{rank}(E)$. Then every local frame η of E in an open set $W \subset V \backslash Y$ has the form $\eta = \xi \cdot g$ for some unique $\mathsf{Mat}(k,\mathbb{C})$-valued function g in W. Such a frame η is holomorphic if and only if g satisfies the equation $\bar{\partial}g + A^{0,1}g = 0$. The latter equation can be written as $g^{-1} \cdot \bar{\partial}g = -A^{0,1}$.

Theorem 16.4. *Let Γ be a L^{2n}-integrable* $\mathsf{Mat}(k,\mathbb{C})$*-valued $(0,1)$-form in the unit ball B in $\mathbb{C}^n$ which satisfies the condition*

$$\bar{\partial}\Gamma + \Gamma \wedge \Gamma = 0$$

in the sense of distributions. Then there exists $\mathsf{Mat}(k,\mathbb{C})$*-valued function g in B such that both g and g^{-1} are $L^{1,p}_{loc}$-regular for every $p < 2n$, and such that g is a solution of the equation*

$$\bar{\partial}g + \Gamma \cdot g = 0.$$

Moreover, if g_1, g_2 are two matrix-valued functions with this property, then $g_2 = g_1 \cdot h$ for some holomorphic $\mathsf{Mat}(k,\mathbb{C})$*-valued function in B.*

Applying this result to $\Gamma = A^{0,1}$ we obtain the assertion of Theorem 16.3 in the case when the curvature F is L^n_{loc}-integrable for $n = \mathsf{dim}_{\mathbb{C}}X$. Moreover, E in this case extends as a bundle, and the sheaf of holomorphic sections of the extended bundle $\widetilde{E}$ admits the following characterization: for every $2 \leq p < \infty$

$$\mathscr{O}(U,\widetilde{E}) = \{s \in \mathscr{O}(U \backslash Y, E) : s \in L^p_{loc}(U,E)\}.$$

Or, in another words, let $\eta = \xi \cdot g$ be the holomorphic frame obtained in this theorem. Write $\eta = (\eta_1, ..., \eta_k)$ and define an extension of E through Y as $(\mathcal{O}^k, \varphi)$, where $\varphi : \mathcal{O}^k|_{\check{B}} \to E$ is defined as

$$\varphi : (f_1, ..., f_k) \to (f_1\eta_1, ..., f_k\eta_k).$$

Step 3. Slicing and separate extension. Assume that the curvature F is locally L^p-integrable for some $2 \le p < n = \dim_{\mathbb{C}} X$. Take maximal integer $k \le p$. Consider a polydisk Δ^n in X and represent it as the product $\Delta^k \times \Delta^{n-k}$. For v in the second factor Δ^{n-k} we set $\Delta^k_v := \Delta^k \times \{v\} \subset \Delta^k \times \Delta^{n-k}$. We imagine Δ^n to be "sliced" in the polydisks Δ^k_v. This induces a slicing of the singular set with slices $Y_v := Y \cap \Delta^k_v$. On each Δ^k_v we have the slice of the bundle $E_v := E|_{\Delta^k_v \setminus Y_v}$ equipped with the induced Hermitian metric and Chern connection whose curvature is the restriction of the total curvature F. By Fubini's theorem, for almost all $v \in \Delta^{n-k}$ the curvature is L^k-integrable on Δ^k_v. The limit holonomy of the restricted bundles $E_v = E|_{\Delta^k_v \setminus Y_v}$ around the corresponding singular set Y_v is trivial for almost all v. By the previous step, for almost all $v \in \Delta^{n-k}$ the bundle E_v extends holomorphically from $\Delta^k_v \setminus Y_v$ to Δ^k_v. One can show that holomorphic sections of those extensions are those holomorphic sections of E_v over $\Delta^k_v \setminus Y_v$ which are locally L^q-integrable for some fixed $2 \le q < \infty$. In this situation Theorem 3.1 in [She2] or, Theorem 14.3 of this survey, ensures the assertion of Theorem 16.3 with the following description of the extension $\widetilde{\mathscr{E}}$: for every $2 \le p < \infty$ and every open set $U \subset X$ as in Theorem 16.3 one has

$$\widetilde{\mathscr{E}}(U) = \left\{ s \in \mathscr{O}(U \setminus Y, E) : \ s \text{ is } L^p_{loc}\text{-integrable on almost all } k\text{-discs in } U \right\}.$$

16.3. Holomorphic bundles with semipositive curvature. Let X be a complex manifold of dimension $n \ge 2$, $Y \subset X$ an analytic subset, and $G \subset X$ an open set containing $X \setminus Y$ meeting each irreducible component of Y of codimension 1. Let E be a holomorphic vector bundle over a complex manifold X. E is called *Nakano positive* if it carries a Hermitian metric with positive semidefinite curvature form, see definition below.

Theorem 16.5. *Let (E, h) a holomorphic Hermitian vector bundle over G with Nakano positive curvature. Then E admits an extension to X as a reflexive coherent sheaf $\widetilde{\mathscr{E}}$. Moreover, every local section of $\widetilde{\mathscr{E}}$ over an open set $U \Subset X$ is L^2_{loc}-integrable with respect to h in U.*

Proof. We follow main ideas of the paper [Si2]. Since the original proof in this paper is very brief we shall try to present a more detailed demonstration.

Step 1. Bochner-Nakano identity. Let X be a complex manifold of dimension n equipped with a Kähler metric with the Kähler form ω. In local coordinates, if ω has a representation $\omega = Im(h_\omega) = \frac{i}{2}\sum h_{jk} dz_j \wedge d\bar{z}_k$, then the metric is given by $h_\omega = \sum h_{jk} dz_j \otimes d\bar{z}_k$. We use the metric h_ω to define the volume form $dV = \frac{\omega^n}{n!}$ on X and the Hodge operator $*$. As usually, $[A, B]$ denotes the commutator $A \circ B - B \circ A$ of linear operators. Further, for an operator S between Hermitian vector spaces or bundles we denote by S^* its adjoint operator and by $\overline{S}$ its complex conjugate if the latter is well-defined. The *Hodge-Lefschetz operators* $L = L_\omega$ and $\Lambda = \Lambda_\omega$ on the spaces of (p,q)-forms are defined by $L_\omega(\alpha) := \alpha \wedge \omega$ and $\Lambda_\omega(\alpha) := *(L_\omega(*\alpha))$, so that $\Lambda_\omega = L^*_\omega$.

Now let (E, h_E) be a holomorphic Hermitian vector bundle on X. We denote by ∇_E the operator of the Chern connection in E and by $D_E : \mathcal{A}^k(X, E) \to \mathcal{A}^{k+1}(X, E)$ its extension to E-valued differential forms. Set $D^{1,0}_E =: \partial_E$, $D^{0,1}_E =: \overline{\partial}_E$, and let ∂^*_E, $\overline{\partial}^*_E$ be the adjoint operators. Notice that the curvature of the Chern connection in (E, h_E) can be computed by $F_E := D^2_E = \partial_E \overline{\partial}_E + \overline{\partial}_E \partial_E$. Besides, we introduce the Laplace operators $\Delta'_E := \partial_E \partial^*_E + \partial^*_E \partial_E$, and $\Delta''_E := \overline{\partial}_E \overline{\partial}^*_E + \overline{\partial}^*_E \overline{\partial}_E$. The following formula is known as the *Bochner-Nakano* identity.

Lemma 16.1. *Assume that the metric given by ω is Kähler. Then*

$$\Delta''_E = \Delta'_E + [iF_E, \Lambda] \tag{16.6}$$

The proof, see [Na], is the same as the classical equality $\Delta' = \Delta''$ of the ∂- and $\overline{\partial}$-Laplacians on the (p,q)-forms on Kähler manifolds which is a special case of 16.6 with (E,h) being the trivial bundle $\mathcal{O}$ equipped with the standard metric. First, one proves the Hodge identities

$$[\overline{\partial}_E^*, L] = i\partial_E, \qquad [\partial_E^*, L] = -i\overline{\partial}_E,$$
$$[\overline{\partial}_E, \Lambda] = i\partial_E^*, \qquad [\partial_E^*, \Lambda] = -i\overline{\partial}_E^*,$$

and then substitute them in the definitions of Δ'_E and Δ''_E. The rest of calculation is the same as in the classical case, with the difference that now we have $\partial_E\overline{\partial}_E + \overline{\partial}_E\partial_E = F_E$ instead of $\partial\overline{\partial} + \overline{\partial}\partial = 0$. We refer to [Dm2], Ch. VII for the generalizations to the non-Kähler case.

Step 2. Positivity of the curvature of holomorphic Hermitian bundles. Recall that a holomorphic Hermitian bundle (E,h) with the curvature F_E on a complex manifold X is *Nakano positive* if for every section v of the bundle $T^{1,0}X \otimes E$ one has $\langle iF_E(v), v\rangle \geq 0$.

For a Kähler manifold (X,ω) and a holomorphic Hermitian vector bundle (E,h_E) denote by $L^2_{(p,q)}(X,E)$ the space of L^2-integrable E-valued (p,q)-form on X with the usual norm

$$\|u\|^2_{L^2_{(p,q)}(X,E,h)} = \|u\|^2_{L^2_{(p,q)}} := \int_X \langle u,u\rangle dV.$$

Lemma 16.2. *Let (X,ω) be a Kähler complex manifold of dimension n, (E,h) a holomorphic Hermitian vector bundle on X, and F its curvature. Then the Hermitian form*

$$u, v \mapsto \int_X i\big\langle [F,\Lambda]u, v\big\rangle dV \tag{16.7}$$

on the space of smooth $(n,1)$-forms with compact support is semi-positive if and only if the bundle (E,h) is Nakano positive. Furthermore, let φ be a strictly psh function such that $i\partial\overline{\partial}\varphi \geq c\cdot\omega$ with some constant $c>0$. Define the new metric $h_\varphi := e^{-\varphi}h$ on E and let F_φ be the curvature of (E,h_φ). Then

$$\int_X \big\langle [iF_\varphi,\Lambda]u, u\big\rangle dV \geq c\cdot\|u\|^2_{L^2(X)} \tag{16.8}$$

for every $u \in L^2_{n,q}(X,E,h)$ and every $1 \leq q \leq n$.

Step 3. Solving with estimates the $\overline{\partial}$-equation with values in a positive bundle.

Theorem 16.6. *Let X be a Stein manifold of dimension n, ω a Kähler form on X, (E,h) a Nakano-positive holomorphic Hermitian vector bundle, and φ a strictly PSH function such that $i\partial\overline{\partial}\varphi \geq c\cdot\omega$ with some constant $c>0$. Set $h_\varphi := e^{-\varphi}h$. Then for every $1\leq q\leq n$ and every $u \in L^2_{n,q}(X,E,h_\varphi)$ such that $\overline{\partial}u = 0$ in the weak sense there exists $v \in L^2_{n,q-1}(X,E,h_\varphi)$ such that $\overline{\partial}_E v = u$ and $\overline{\partial}_E^* v = 0$ in the weak sense and*

$$c\cdot\|v\|^2_{L^2_{n,q-1}(X,E,h_\varphi)} \leq \|u\|^2_{L^2_{n,q}(X,E,h_\varphi)}.$$

In the case when the metric on X is complete and E is the trivial bundle $\mathcal{O}$ this result is the core of Hörmander's L^2-theory, see [Ho1]. The generalization to the case of vector bundle was obtained by Nakano [Na]. An important improvement of the L^2-method, allowing to treat the case of non-complete metric, was obtained by Demailly [Dm3]. The idea of the proof is as follows. Take a new Kähler form $\hat{\omega}$ on X such that the corresponding metric is complete. Then for every $\varepsilon>0$ the metric defined by the Kähler form $\omega_\varepsilon := \omega + \varepsilon\cdot\hat{\omega}$ is also complete. Denote by $\|\cdot\|_{L^2_{n,q}(X,\omega_\varepsilon,E,h)}$ the L^2-norm with respect to the metric given by the Kähler form ω_ε. Then for every $\varepsilon>0$ one has

$$\|u\|_{L^2_{n,q}(X,\omega_\varepsilon,E,h)} \leq \|u\|_{L^2_{n,q}(X,\omega,E,h)}.$$

So by the "usual" Hörmander's L^2-theory, for every $\varepsilon>0$ there exists $v_\varepsilon \in L^2_{n,q-1}(X,\omega_\varepsilon,E,h)$ with

$$c\cdot\|v_\varepsilon\|^2_{L^2_{n,q-1}(X,\omega_\varepsilon,E,h)} \leq \|u\|^2_{L^2_{n,q}(X,\omega_\varepsilon,E,h)} \leq \|u\|^2_{L^2_{n,q}(X,E,h)}$$

which satisfies the equations $\bar\partial_E v_\varepsilon = u$ and $(\bar\partial_E)^*_\varepsilon v_\varepsilon = 0$ (where $(\bar\partial_E)^*_\varepsilon$ is the operator adjoint to $\bar\partial_E$ with respect to the metric given by ω_ε). Finally, take a sequence ε_ν decreasing to 0. Then some subsequence of the sequence v_{ε_ν} converges to a desired solution $v \in L^2_{n,q-1}(X,E,h)$ with the properties stated in the theorem.

Step 4. Finding local sections L^2-bounded near singularity. Consider the following situation. Let X be a Stein manifold of dimension n with a Kähler form ω, $Y \subset X$ an analytic set of pure codimension 1, $G \subset X$ an open set which contains $X \backslash Y$, (E,h) a holomorphic Hermitian bundle of rank m over G with Nakano positive curvature. Finally. let $V \Subset X$ be a relatively compact Stein domain, denote by $\overline{V}$ the closure of V in X.

Theorem 16.7. *Let $p \in V \cap G$ be a point. Assume that there exists a non-constant holomorphic function f on X which is vanishing at p and non-zero on $Y \backslash G$. Then there exist holomorphic sections $s_1, \dots, s_m \in \mathcal{O}(V \cap G, E)$ which are L^2-integrable in V and which generate the basis of the fiber of E at the point p.*

Proof. The proof follows the ideas of [Si2]. Let $\mathsf{det}(TX) = \Lambda^n(TX)$ be the anti-canonical bundle, $h_{\mathsf{det},\omega}$ the metric on $\mathsf{det}(TX)$ induced by Kähler metric on X, and $F_{\mathsf{det},\omega}$ the corresponding curvature. Find a smooth real function ψ such that $i\partial\bar\partial\psi \ge iF_{\mathsf{det},\omega}$ and set $h_{\mathsf{det},\psi} := h_{\mathsf{det},\omega} \cdot e^{-\psi}$. Then the holomorphic Hermitian line bundle $(\mathsf{det}(TX), h_{\mathsf{det},\psi})$ is positive. Now define $(E',h') := (E,h) \otimes (\mathsf{det}(TX), h_{\mathsf{det},\psi})$. Then (E',h') is a Nakano-positive bundle over G such that $E' \otimes \Lambda^{n,0}X \cong E$. In particular, holomorphic sections of $E' \otimes \Lambda^{n,0}X$ are holomorphic sections of E, and we can use *Step 3* to construct such sections.

Let $Z \subset X$ be the zero set of the function f. Then Z is Stein and lies in G. So by the theorem of Siu [Si6] Z admits a Stein neighborhood in U. Fix a smooth cut-off function χ on X which is identically 1 in some smaller neighborhood of Z and whose support $\mathsf{supp}(\chi)$ lies in U.

Since U is Stein, every fiber of the bundle E is generated by global holomorphic sections. Take any vector v in the fiber E_p of this bundle at the point p and let $s^* \in \mathcal{O}(U,E)$ be a holomorphic section in U such that $s^*(p) = v$. We claim that $\alpha := f^{-1} \cdot \bar\partial\chi \cdot s^*$ is L^2-integrable in V. To show this let us observe that the set $\overline{V} \cap \mathsf{supp}(\chi)$ is a compact set lies in $U \subset G$. Consequently, the vector valued form $\bar\partial\chi \cdot s^*$ is L^2-integrable in V. Further, the set $\overline{V} \cap \mathsf{supp}(\bar\partial\chi)$ is also compact, and hence the function $|f|$ achieves its minimum at some point q. This minimum can not be 0, since otherwise f would vanish at q and q would lie on the set Z, in contradiction to the fact that χ is identically 1 in some neighborhood of Z and hence $\mathsf{supp}(\bar\partial\chi) \cap Z = \varnothing$. Consequently, f^{-1} is uniformly bounded on $\overline{V} \cap \mathsf{supp}(\bar\partial\chi)$ and so $\alpha = f^{-1} \cdot \bar\partial\chi \cdot s^*$ is L^2-integrable in V as asserted. Due to the isomorphism $E' \otimes \Lambda^{n,0}X \cong E$, we can write the L^2-integrability as $\alpha \in L^2_{0,1}(V \backslash Y, E, \omega) = L^2_{n,1}(V \backslash Y, E', \omega)$. Further, since f is non-vanishing in $V \cap \mathsf{supp}(\bar\partial\chi)$ $\alpha = f^{-1} \cdot \bar\partial\chi \cdot s^*$ is $\bar\partial$-closed, $\bar\partial\alpha = 0$. Now by Theorem 16.6, $\alpha = \bar\partial\psi$ for some $\psi \in L^2_{n,0}(V \backslash Y, E', \omega) = L^2(V \backslash Y, E, \omega)$. Consequently, $\bar\partial(\chi \cdot s^* - f \cdot \psi) = 0$. This means that $\chi \cdot s^* - f \cdot \psi =: s$ is a holomorphic section of E over $V \backslash Y$ which is L^2-integrable. By Riemann's extension theorem, s extends holomorphically to the set $V \cap G$.

Next, recall that χ is identically 1 in a neighborhood of the point p. It follows that in a neighborhood of p the form $\alpha = f^{-1} \cdot \bar\partial\chi \cdot s^* = \bar\partial\psi$ vanishes identically, and so ψ is holomorphic near p. Thus $s(p) = \chi(p) \cdot s^*(p) - f(p) \cdot \psi(p) = 1 \cdot s^*(p) - 0 \cdot \psi(p) = s^*(p) = v$. This means that $s(z)$ is a L^2-integrable holomorphic section of E in $V \cap G$ which takes the prescribed value v at the point p. This implies Theorem 16.7 and subsequently Theorem 16.5. □

17. Plateau problem, extension from the boundary and fillings

17.1. Complex Plateau problem. We follow [Dh1]. Recall that a subset $\Gamma \subset \mathbb{R}^n$ is called m-rectifiable if Γ is an image of a bounded set $B \subset \mathbb{R}^m$ under a Lipschitz continuous map $f : B \to \mathbb{R}^n$. Γ is called $(\mathcal{H}^m, m)$-rectifiable if $\mathcal{H}^m(\Gamma) < \infty$ and there exists at most countable collection of m-rectifiable compacts $K_j \subset \mathbb{R}^n$ such that $\mathcal{H}^m(\Gamma \setminus \cup_i K_i) = 0$. Here $\mathcal{H}^m$ stands for the m-Hausdorff measure in $\mathbb{R}^n$. In particular, the current $[\Gamma]$ of integration over such Γ is well defined. Such currents are called $(\mathcal{H}^m, m)$-rectifiable. We say that a compact $\Gamma \subset \mathbb{R}^n$ is of class A_m if Γ is $(\mathcal{H}^m, m)$-rectifiable and its tangent cone at $\mathcal{H}^m$-almost all points is an m-dimensional subspace of $\mathbb{R}^n$. Recall that the tangent cone to Γ at x is

$$\mathsf{Tan}\,(\Gamma, x) := \{\mathrm{v} \in \mathbb{R}^n : \forall \varepsilon > 0, \exists y \in \Gamma, \exists c > 0 \text{ such that } |y - x| < \varepsilon \text{ and } |\mathrm{v} - c(y - x)| < \varepsilon\}.$$

Note that a connected, compact in $\mathbb{R}^n$ of finite length is of class A_1, see Lemma 1.3 in [Dh1].

An $(\mathcal{H}^{2p-1}, 2p-1)$-rectifiable current Γ in $\mathbb{C}^n$ is called maximally complex if it is a sum of currents of bidimensions $(p, p-1)$ and $(p-1, p)$. In other words if Γ annihilates all $(k, 2p-1-k)$-forms except for $k = p, p-1$. If Γ is a $\mathcal{C}^1$-manifold this means that its tangents at all points are maximally complex subspaces of $\mathbb{C}^n$. A holomorphic p-chain in $\mathbb{C}^n \setminus \Gamma$ is by definition a locally finite linear combination A with integer coefficients of pure p-dimensional analytic subsets of $\mathbb{C}^n \setminus \Gamma$. If this chain has locally bounded $2p$-dimensional volume in $\mathbb{C}^n$ it defines a (p,p)-current $[A]$ in $\mathbb{C}^n$. If, in addition $\partial[A] = [\Gamma]$ in the sense of currents then we say that $[\Gamma]$ bounds $[A]$.

Theorem 17.1. *(T.-C. Dinh,* [Dh1]*.) Let $[\Gamma]$ be a closed current in $\mathbb{C}^n$ with compact support Γ of class A_{2p-1}. Then the following holds.*

i) If $p = 1$ then $[\Gamma]$ bounds a holomorphic 1-chain if and only if it satisfies the moment condition, i.e.,

$$\int_\Gamma \omega = 0 \tag{17.1}$$

for every polynomial 1-form in $\mathbb{C}^n$.

ii) If $p \geqslant 2$ then $[\Gamma]$ bounds a holomorphic p-chain if and only if $[\Gamma]$ is maximally complex.

The proof of this theorem goes roughly as follows. First one proves the part *(i)* and then cuts Γ by subspaces of dimension $n - p + 1$. From maximal complexity of $[\Gamma]$ one deduces that these slices satisfy the moment condition (17.1). This permits to prove that they bound holomorphic 1-chains. Their union form then the needed holomorphic p-chain which solves the boundary problem. Let us give more details.

Step 1. The statement of this step is contained in the following lemma, see the corresponding Slicing Lemma 1.4 in [Dh1].

Lemma 17.1. *Let Γ be a compact of class A_{2p-1} in $\mathbb{C}^n$. Then for almost all orthogonal projections $\Pi : \mathbb{C}^n \to \mathbb{C}^{p-1}$ and almost all $x \in \mathbb{C}^{p-1}$ the tangent cone $\mathsf{Tan}\left(\Gamma \cap \Pi^{-1}(x), y\right)$ is a real line for $\mathcal{H}^1$-almost all $y \in \Gamma \cap \Pi^{-1}(x)$ and therefore $\Gamma \cap \Pi^{-1}(x)$ is of class A_1.*

Step 2. Let Π be a projection satisfying the conclusion of the previous step. By $< [\Gamma], \Pi, x >$ denote the corresponding slice current.

Lemma 17.2. *If $[\Gamma]$ is maximally complex then for almost all x the slices $< [\Gamma], \Pi, x >$ satisfy the moment condition (17.1).*

The case when Γ is a maximally complex, compact $\mathcal{C}^1$-submanifold of $\mathbb{C}^n$ is due to Harvey and Lawson, [HL]. We send the interested reader to the lectures [Ha] for the detailed exposition of this case and to [Dh1] for the proof of Theorem 17.1. Using the techniques developed in [Dh1] the following statement was proved in [KS2].

Theorem 17.2. *Let X be a q-complete complex manifold and $[\Gamma]$ a closed, maximally complex current in X of dimension $2p-1$ with compact support of class A_{2p-1}. If $p \geqslant q + 1$ then $[\Gamma]$ bounds a unique holomorphic p-chain.*

17.2. Extension from the boundary. In [Bo] apart of Theorem 1.6 also the following statement was proved.

Theorem 17.3. *Let D be a smoothly bounded relatively compact domain in $\mathbb{C}^n$, $n \geqslant 2$. Then every CR-function on ∂D extends to a holomorphic function in D.*

Theorem 17.2 implies the following corollary in the spirit of Bochner's result.

Corollary 17.1. *Let D be a smoothly bounded, relatively compact domain with connected boundary in an $(n-1)$-complete complex space X, $n = \dim X \geqslant 2$. Them every CR-function extends to a holomorphic function in D.*

Indeed, it is sufficient to solve the Plato problem for the graph of the function in question. Let D be a smoothly bounded domain in a Stein manifold X of dimension $n \geqslant 2$. A compact $K \Subset \partial D$ is called *removable* if every CR-function on $\partial D \setminus K$ extends to a holomorphic function in D.

Theorem 17.4. *(G. Lupacciolu [Lu], J.-P. Rosay - E.L. Stout [RS]). In the situation as above the following conditions are equivalent:*
i) K is removable;
ii) K is $\mathcal{O}(\bar{D})$-convex if $n=2$ or $H^{0,1}(X\setminus K)=0$ if $n \geqslant 3$;
iii) $H^{0,1}_{\Phi}(X\setminus K)=0$, if $n\geqslant 3$.

Here $\mathcal{O}(\bar{D}) := \mathcal{C}(\bar{D}) \cap \mathcal{O}(D)$ and $H^{0,1}_{\Phi}$ stands for the cohomology group with compact supports. We refer to [CS2] for details and much more results in this direction.

17.3. Filling by disks a neighborhood of a CR-submanifold. All considerations in this subsection will be purely local. I.e., we shall consider the germs of smooth real submanifolds in $\mathbb{C}^n$. A submanifold of $\mathbb{C}^n$ is called a Cauchy-Riemann submanifold, CR-submanifold for short, if the dimension of its complex tangent $T^c_xM := T_xM \cap i(T_xM)$ is independent of $x \in M$. An analytic disk $\varphi : \bar{\Delta} \to \mathbb{C}^n$ (we suppose here that φ is smooth up to the boundary) is said to be *attached* to M if $\varphi(\partial\Delta) \subset M$. The pioneering result in this direction was obtained by E. Bishop. Let $(M,0)$ be a germ of a CR-submanifold in $(\mathbb{C}^n,0)$. Choose coordinates in such a way that $T^c_0M = \mathbb{C}^m \times \{0^{n-m}\}$.

Theorem 17.5. *(E. Bishop, [Bs2]). For every sufficiently small analytic disk $\varphi : \bar{\Delta} \to \mathbb{C}^m$ such that $\varphi(1) = 0^m$ there exists an analytic disk $\Phi = (\varphi,\psi)$ attached to M and such that $\Phi(1)=0$.*

These disks are commonly called Bishop's disks. Using Bishop's disks several remarkable results were obtained since. Let us mention only one of them.

Theorem 17.6. *(J.-M. Trépreau, [Tr]). Let M be a germ of a real hypersurface in $\mathbb{C}^n$, $n \geqslant 2$. Assume that there doesn't exist a germ of complex hypersurface in M through 0. Then all CR-functions on M holomorphically extend to the same one-sided neighborhood of M.*

This result is obtained by filling this one-sided neighborhood by Bishop's disks. We send the interested reader to the survey [Tu] of A. Tumanov for the questions of one sided extensions of CR-functions and extensions to the wedges from hypersurfaces and also from submanifolds of higher codimension.

17.4. Fillings holes in complex manifolds. A 1-corona is a pair (X,ρ), where X is a complex manifold and $\rho : X \to (t_1,t_2)$ is a proper strictly plurisubharmonic function. Possibilities $t_1 = -\infty$ and $t_2 = +\infty$ are not excluded. One also says that X has a concave end or a hole, here $n \geqslant 2$ always. If $t_1 = -\infty$ one says that X has a *hyperconcave* end. A completion of X is a normal Stein space $\hat{X}$ with finitely many singular points and an imbedding $i : X \to \hat{X}$ such that for $t_1 < t^* < t_2$ the set

$$\left(\hat{X} \setminus i(X)\right) \cup i\left(\{\rho \leqslant c\}\right) \tag{17.2}$$

is a compact. One also says that $\hat{X}$ fills in a hole in X.

Theorem 17.7. *(H. Rossi,* [Ro]*). If the dimension of the* 1*-corona is* $\geqslant 3$ *then it always admits a Stein completion.*

For $n=2$ this is not longer true. Take a double covering of D^- from Example 5.1 to get a counter example. However the following statement was proved in [DM].

Theorem 17.8. *(T.-C. Dinh, G. Marinescu,* [DM]*). A hyperconcave end can be always completed also in dimension two.*

It was proved in [BL] that a 1-corona can be completed if and only if the vector space $H^1(X,\mathcal{O})$ is Hausdorff, only if statement actually was proved in [AV]. In other words the space of $\bar\partial$-exact $\mathcal{C}^{\infty}_{0,1}$-forms should be closed in $\mathcal{C}^{\infty}$-topology. If dimension is $\geqslant 3$ then $H^1(X,\mathcal{O})$ is even finite dimensional by Andreotti-Grauert theorem.

Conjecture 4. *One can introduce the object called q-corona requesting $\rho : X \to (t_1,t_2)$ to be proper and strongly q-convex. It is conjectured in* [Si7] *that q-corona can be also completed provided* $\mathsf{dim}\, X \geqslant q+2$.

18. Appendix I. Placements of Hartogs figures

Along this survey we used several times in certain way imbedded Hartogs figures. Let us give more details on this issue.

18.1. Normal form of a strictly q-convex function. Let $\rho : X \to \mathbb{R}$ be a real valued function on a complex manifold. It will be supposed in the sequel to be Morse and all perturbations will be supposed to be small enough in $\mathcal{C}^2$-norm in order for ρ to stay Morse. Function ρ is called *strictly q-convex* at x_0 if the Levi form L_{ρ,x_0} of ρ at x_0 has exactly $n-q+1$ positive eigenvalues, here $1 \leqslant q \leqslant n$. Our considerations will be local and therefore we suppose that ρ is defined in a coordinate chart centered at zero with $\rho(0)=0$ and 0 is a non-degenerate critical point of ρ. Let $z=(z_1,...,z_n)$ be the corresponding local coordinates. Then ρ writes as

$$\rho(z) = 2\mathsf{Re}\, H_{\rho,0}(z) + L_{\rho,0}(z) + o(\|z\|^2) \tag{18.1}$$

in a neighborhood of the origin. Here

$$H^z_{\rho,0}[z] := \sum_{k,l=1}^{n} \frac{\partial^2\rho(0)}{\partial z_k \partial z_l} z_k z_l \quad \text{and} \quad L^z_{\rho,0}[z] := \sum_{k,l=1}^{n} \frac{\partial^2\rho(0)}{\partial z_k \partial \bar z_l} z_k \bar z_l$$

the complex Hessian and the Levi form of ρ at the origin respectively. Under a complex linear change of coordinates $z = U\zeta$ they undergo the change as follows

$$H^{\zeta}_{\rho,0} = U^t H^z_{\rho,0} U \quad \text{and} \quad L^{\zeta}_{\rho,0} = U^t L^z_{\rho,0} \bar U.$$

In the $\mathbb{C}$-vector space $T_{x_0}X$ consider the non-degenerate Hermitian form g, which in the basis $\frac{\partial}{\partial z} := \{\frac{\partial}{\partial z_1},...,\frac{\partial}{\partial z_n}\}$ is given by $g(\mathrm{v},\mathrm{w}) = \mathrm{v}^t L^z_{\rho,0}\bar{\mathrm{w}}$. By the well known theorem there exists a basis $\eta_1,...,\eta_n$ in which g is diagonal of the form $I_{n,q} := [\underbrace{-1,...,-1}_{q-1},\underbrace{1,...,1}_{n-q+1}]$. Let $\zeta = (\zeta_1,...,\zeta_n)$ be coordinates centered at x_0 such that $\frac{\partial}{\partial \zeta_k} = \eta_k$ for $k=1,...,n$. In these coordinated L^{ζ}_{ρ,x_0} will have the diagonal form as above.

Remark 18.1. In general, if a strongly q-convex function ρ is not $(q-1)$-convex the diagonalized Levi form will look as follows $\{\underbrace{0,...,0}_{1...s},\underbrace{-1,...,-1}_{s+1...q-1},\underbrace{1,...,1}_{q...n}\}$. Assuming that our coordinate chart is the unit ball B_1 take a cut-off function φ, which is $\mathbb{1}$ on $[0,1/3]$ and zero on $[2/3,1]$, and consider $\tilde\rho(z) := \rho(z) - \varepsilon\varphi(\|z\|)(|z_1|^2 + ... + |z_s|^2)$. For $\varepsilon > 0$ sufficiently small we shall have that $\tilde\rho \leqslant \rho$, $\tilde\rho$ close to ρ in $\mathcal{C}^2$-norm on B_1, and equal to ρ on $B_1 \setminus B_{2/3}$. Therefore we can replace ρ by $\tilde\rho$ globally on X and extend our analytic object along the level sets of $\tilde{\rho}$. Remark that this $\tilde{\rho}$ is strictly q-convex at x_0 and that x_0 is the only its critical point in B_1. Rescailing variables $z_1,...,z_s$ we rewrite our function as follows

$$\tilde{\rho}(z) = 2H_{\rho,0}(z) - \sum_{k=1}^{q-1}|z_k|^2 + \sum_{k=q}^{n}|z_k|^2 + o(\|z\|^2). \tag{18.2}$$

By a separate unitary changes in $z^{'} := (z_1,...,z_{n-q})$ and $z^{''} := (z_q,...,z_n)$ we bring our strictly q-convex function to the *normal form*

$$\rho(z) = -\sum_{k=1}^{q-1}a_k(z_k^2+\bar{z}_k^2) + \sum_{l=q}^{n}a_l(z_l^2+\bar{z}_l^2) + 2\mathsf{Re}\,H(z^{'},z^{''}) - ||z^{'}||^2 + ||z^{''}||^2 + o(\|z\|^2), \tag{18.3}$$

where one can make $a_k, a_l \geqslant 0$ and the “mixed part” has the form

$$H(z^{'},z^{''}) = \sum_{k=1,l=q}^{q-1,n} a_{k,l}z_kz_l.$$

Remark that further simplification of the quadratic part is not possible, *i.e.*, one cannot get reed from the mixed terms, see Theorem 4.5.15 in [HJ]. After that, setting $\hat{\rho}(z) := \tilde{\rho}(z) - \varphi(\frac{\|z\|}{\varepsilon})o(\|z\|^2)$ with $\varepsilon > 0$ small enough, we obtain a new function close to $\hat{\rho}$ in $\mathcal{C}^2$-norm, which coincides with $\hat{\rho}$ in $B_\varepsilon \setminus B_{2\varepsilon/3}$ and is a quadratic polynomial in $B_{\varepsilon/3}$. Therefore one often can suppose that ρ has only the *nice* critical points, which means that ρ near such x_0 is equal to its quadratic part. We shall assume this from now on, *i.e.*, we shall write

$$\rho(z) = -\sum_{k=1}^{q-1}a_k(z_k^2+\bar{z}_k^2) + \sum_{l=q}^{n}a_l(z_l^2+\bar{z}_l^2) + 2\mathsf{Re}\,H(z^{'},z^{''}) - ||z^{'}||^2 + ||z^{''}||^2. \tag{18.4}$$

18.2. Hartogs figures near q-concave boundaries: smooth case. If $q = 1$, *i.e.*, our ρ is strictly plurisubharmonic we can obviously transform (18.4) to

$$\rho(z) = \sum_{k=1}^{s}(x_k^2 - \varepsilon_k y_k^2) + \sum_{k=s+1}^{n}(x_k^2 + \varepsilon_k y_k^2), \tag{18.5}$$

where $0 < \varepsilon_k < 1$ for $k = 1,...,s$ and $0 < \varepsilon_k \leqslant 1$ for $k = s+1,...,n$. Here s is he Morse index of ρ at x_0. Consider the case of maximal index $s = n$. The slope of the cone $\sum_{k=1}^{n}(x_k^2 - \varepsilon_k y_k^2) = 0$ is bigger than 1 (this reflects strict plurisubharmonicity), and can be made arbitrarily big as it is explained on the Fig.7. There $U^+ := \{\rho > 0\}$ and $\Sigma_0 := \{\rho = 0\}$. For positive parameters t and

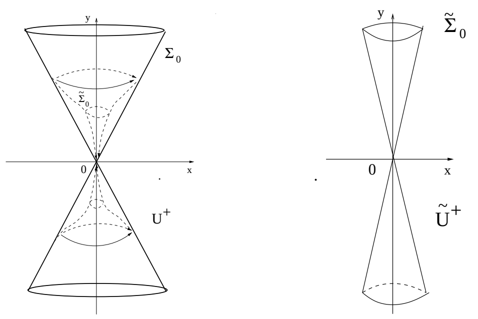

FIGURE 7. Σ_0 on the left is the zero level of ρ. It can be deformed to $\tilde{\Sigma}_0 := \{\tilde{\rho} = 0\}$ by perturbing ρ to $\tilde{\rho}$ appropriately in such a way that the slope becomes bigger. Then, after rescailing and cutting off the term $o(\|z\|^2)$ one gets the cone with bigger slope. If the analytic object in question extends across *smooth* 1-convex boundaries one can using this procedure extend it to $U \cap (\mathbb{C}^n \setminus \mathbb{R}^n)$ for some neighborhood of the origin U.

τ consider the quadric

$$R_{t,\tau} := \{z \in K^2 : (z_1+t)^2 + z_2^2 + ... + z_n^2 = \tau^2\}. \tag{18.6}$$

Remark that $R_{0,\tau}$ lies in U^+ and moves to the left when t increases, see the Fig.8. Cutting each $R_{t,\tau}$ along an ε-neighborhood of the real demi-hyperplane $\{x_2 = 0, x_1 \leqslant -t\}$ one gets the discs $D_{t,\tau}$. From them it is not difficult to construct a Hartogs figure in U^+ such that the associated polydisk contains the origin, see Appendix in [Iv11] for more details.

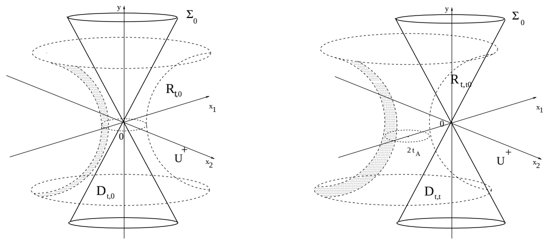

FIGURE 8. When t runs along an appropriate interval in $\mathbb{R}$ discs $D_{t,\tau}$ move as on this picture. One can complexify t to get a one-parameter holomorphic family of disks for t in a complex neighborhood of that interval. Remark that boundaries of $D_{t,\tau}$ stay in U^+, while interiors sweep a neighborhood of the origin.

If x_0 is a strictly q-convex point then the picture as on Fig.8 depends on the parameter $z^{'}$ ($\mathbb{R}$-linearly) and one can imbed $H^{n-q+1,1}$ to $U^+ \cap \mathbb{C}^{n-q+1}_{z''}$ as in strictly pseudoconvex case and then multiply it by Δ^q_ε with $\varepsilon > 0$ small enough. This product contains a (biholomorphic image of a) Hartogs figure $H^{n,q}_r$ such that the associated polydiks contains the origin.

18.3. Hartogs figures near $\mathbb{R}^n$. To put a polydisk near $\mathbb{R}^n$ in $\mathbb{C}^n$, $n \geqslant 2$, one realizes $\mathbb{R}^n$ as the Shilov boundary of the polydisk by exponential map

$$\mathsf{exp} : (z_1, ..., z_n) \to (e^{iz_1}, ..., e^{iz_n}),$$

and then touches the convex set $\bar{\Delta}^n$ at $\mathbb{1} = (1, ..., 1) = e(0)$ by the hyperplane L with the equation

$$z_1 + ... + z_n = n.$$

Remark that $L \cap \bar{\Delta}^n = \{\mathbb{1}\}$. Remark also that exp is biholomorphic on the cube $K = (-\pi, \pi)^{2n}$ and set $V = \mathsf{exp}(K)$. Take an $(n-1)$-disk Δ^{n-1} on L centered at $\mathbb{1}$ and an 1-disk Δ in the orthogonal complement $L^\perp$ such that $\Delta^{n-1} \times \Delta \subset V$. All what is left is to take a sufficiently small subdisk δ in $\Delta \cap \{\mathsf{Re}(z_1 + ... + z_n) > 0\}$. Then for $r > 0$ small enough the Hartogs figure

$$H^{n,1}_r := \left(\Delta^{n-1} \setminus \bar{\Delta}^{n-1}_{1-r}\right) \times \Delta \cup \left(\Delta^{n-1} \times \delta\right)$$

will be the needed one. See [Si3], Lemma 2.20 for more details.

18.4. Hartogs figures near q-concave boundaries: singular case. In general we have the following powerful technical statement.

Lemma 18.1. *(Projection Lemma). If x_0 is a q-concave point of ∂D, $q \leqslant n-1$, then one can find a neighborhood $U \ni x_0$ and a finite, proper holomorphic map $\pi : (U, x_0) \to (\Delta^n, 0)$ such that $\pi(U \cap D)$ contains the q-concave Hartogs figure $H^{n,q}_r$ for some $r > 0$.*

For $q = n-1$ this was proved in [Fu1], see Lemma 5.2, for the case of general any $1 \leqslant q \leqslant n-1$ we refer to Theorem 8.4 in [ST2].

19. Appendix II. Historical notes

These notes are not exhaustive, they only give a complementary information to that what is contained in the corresponding chapters and sections of the main body of this survey.

Chapter I.

1. Theorem 1.1 should be in my opinion attributed to A. Hurwitz, who in his talk on the 1st ICM remarked, see [Hw] p.104, that an analytic function of several complex variables cannot possess isolated singularities "as one can easily prove with the help of generalized Laurent theorem". And this is exactly the proof we gave. Hartogs figures with corresponding statement known as Hartogs' Lemma appeared in [Ht2], where the Theorem 1.6 is also stated. The proof however goes as on Fig. 2 (b). Another attempt to derive this theorem from Hartogs' Lemma was made in the lectures of W. Osgood [Os2], but, in my opinion his arguments are even less convincing than that of Hartogs. Therefore it is generally recognized that Hartogs proved Theorem 1.6 for the case when D and K are round balls and the first proof of the general statement belongs to S. Bochner. At the same time the proof we gave in this text shows that nevertheless Theorem 1.6 directly and reasonably obviously follows from the Hartogs' Lemma.

Theorem 1.3 was proved by E. Levi in [Lv] and the proof we gave is that of Levi. Holomorphicity of *bounded* separately analytic functions was proved by W. Osgood in [Os1]. F. Hartogs in [Ht1] removed the boundedness condition. Non-trivial and far reaching "cross" version of separate analyticity of Theorem 1.10 is due to J. Siciak. Poincaré problem (more precisely the weak Poincaré problem) of Corollary 1.4 was proved for domains in Stein manifolds in [KS1]. For $D = \mathbb{C}^2$ the statement was obtained in [P2] in a stronger form: the corresponding entire functions can be chosen to have relatively prime germs at every point. In this stronger form theorem doesn't hold already for domains in $\mathbb{C}^2$, an example can be found in [Ni].

2. A non-linear version of Levi's theorem, *i.e.,* Theorem 2.1 was obtained in [Iv12]. In the case when $\{C_t\}_{t\in T}$ are non-horizontal straight disks, *i.e.,* intersections of lines with Δ^2, Corollary 2.1 is due to T.-C. Dinh, see [Dh2] Corollaire 1. Again for straight disks and Kähler manifolds in the image Corollary 2.1 was obtained by F. Sarkis in [Sr]. Theorem 2.3 was proved in [IS1, IS2] answering a question of A. Vitushkin. The method of construction of envelopes, which is used here is based on the Gromov's theory of pseudoholomorphic curves, [Gro]. The formulation of the "local version" of Theorem 2.3, *i.e.,* Theorem 2.2 was proposed by A. Domrin (see Introduction in [Ch3]) as a sort of test question for Theorem 2.3 and was proved by E. Chirka in [Ch3] following the methods of [IS1] but in a more simple way using the Vekua theory. He called it "a generalized Hartogs' Lemma". Answering the question, posed by Chirka, J.-P. Rosay in [Rs] constructed an example showing that a "generalized Hartogs' Lemma" doesn't hold in $\mathbb{C}^3$. Another approach to envelopes, based on Seiberg-Witten theory, was proposed by S. Nemirovski in [Nm2].

3. Reflection principle is due to H.A. Schwarz, see [Sw1]. For non-integrable structures the statements of Theorems 3.1 and 3.2 were obtained in [IS7]. Segre varieties were introduced by B. Segre in [Sg1] and first used for the task of extension by S. Webster in [We].

4-5. Construction used in the proof of Theorem 4.1 is due to H. Cartan and P. Thullen, [CT]. The notion of q-convexity on complex spaces was introduced by W. Rothstein in [Rt2].

6. Theorem 6.3 was proved in [Iv3]. Theorem 6.5 was conjectured by S.-S. Chern in [Che] and proved independently by B. Shiffman in [Sh1] and P. Griffiths in [Gr].

Chapter II.

7-8-9-10. Theorem 7.1 was proved in [Iv6]. Cycle space theory was developed by D. Barlet in [Ba1]. The necessary adaption for meromorphic mappings was done in [Iv8]. Theorem 8.1 was proved in [Iv4] answering the conjecture of Griffiths from [Gr]. The Thullen-type case, *i.e.,* Lemma 8.1 was proved earlier by Y.-T. Siu in [Si5] by a different method. Nonlinear version, *i.e.,* Corollary 8.1 was proved in [Iv12]. Continuity Principle of Theorem 8.2 was proved in [IS1, IS2]. Theorem 9.1 was proved in [Iv8].

Chapter III.

11. Corollary 10.2 was conjectured in [CH] and proved in [Iv4] (in fact it follows already from the result of [Iv3], *i.e.,* from Theorem 6.3 of this text).

12. Domains of convergence of holomorphic/meromorphic functions where, probably, for the first time considered by G. Julia in [J] and then by Cartan and Thullen in [CT] and W. Saxer in [Sa] simultaneously with their domains of existence. In particular in [J] and [CT] it was proved that these domains are (in some sense) pseudoconvex. Later domains of convergence of meromorphic functions were studied in [Ru]. In these early papers convergence was understood as convergence of holomorphic functions/mappings, *i.e.,* outside of the union of the indeterminacy sets. The same notion is used in many of recent papers. Three types of convergence which were discussed in this text were introduced in [Iv7], were the Rouché Principle of Theorem 11.2 and Propagation principle of Theorem 11.3 where proved. Convergence of meromorphic mappings with values in $\mathbb{P}^n$ as in the part *(i)* of Theorem 11.4 was introduced and studied by Fujimoto in [Fu2], who called it *meromorphic*, or *m*-convergence. Theorems 11.4 and 11.5 were proved in [IN]. Example 11.4 was proposed to us by A. Rashkovski and published in [IN].

13. In the case of $X = \mathbb{C}$, *i.e.,* for holomorphic functions Corollary 12.1 *(ii)* is due to F. Hartogs, [Ht1], in the case of $X = \mathbb{P}^1$, *i.e.,* for meromorphic functions to W. Rothstein, see [Rt1]. Separate analyticity in the form of Corollary 12.2 *(ii)* is due to the same authors for holomorphic (resp. meromorphic) cases. Theorem 12.2 for meromorphic functions is due to M. Kazaryan, see [Kz]. Rothstein-type Theorem 12.1 for holomorphic mappings is due to B. Shiffman, see [Sh2]. Theorem 12.2 for holomorphic mappings to manifolds with hol. ext. prop was proved in [Al], using approach of B. Shiffman. Theorems 12.1 and 12.2 as they are stated here were proved in [Iv6], see Corollaries 2.5.1 and 2.5.3 respectively.

14. Theorem 13.1 was proved in [Iv9], Example 13.1 was constructed in [CI] and it shows the necessity of modifying the notion of a vanishing end, which belongs to M. Brunella, see discussion before the Theorem 3.1 in [Br3]. The necessary modification was undertaken in [Br1], where for this task a certain version of unparametrized Hartogs-type extension lemma was proved. Corollary 13.2 was independently obtained in [Br2] and [Iv9].

Chapter IV

15-16-17-18. For the historical notes on extensions of sheaves we refer to the Historical Notes in [ST2]. The idea to use extensions of roots of holomorphic line bundles was proposed by T. Ohsawa in [Oh]. Results of section 15 were obtained in [Iv11]. A special case of Theorem 14.5 was proved in [She1], a general case will appear in [She3]. Results of subsection 16.1 were obtained in [She2], Theorem 16.3 was proved in [She1]. The idea of the proof of Theorem 16.5 belongs to Y.-T. Siu, see [Si2], the necessary L^2-estimate of Theorem 16.6 was obtained using Hörmander theory by J.-P. Demailly in [Dm3]. A complete proof, sketched here, will appear in [She3]. Conjecture 4 can be found in [Si7].

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