

QUANTIZATION OF SYMPLECTIC DYNAMICAL r -MATRICES AND THE QUANTUM COMPOSITION FORMULA

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ABSTRACT. In this paper we quantize symplectic dynamical r -matrices over a possibly nonabelian base. The proof is based on the fact that the existence of a star-product with a nice property (called strong invariance) is sufficient for the existence of a quantization. We also classify such quantizations and prove a quantum analogue of the classical composition formula for coboundary dynamical r -matrices.

INTRODUCTION

Let $\mathfrak{h} \subset \mathfrak{g}$ be an inclusion of Lie algebras and $H \subset G$ the corresponding inclusion of Lie groups. Let $U \subset \mathfrak{h}^*$ be an invariant open subset and let $Z \in (\wedge^3 \mathfrak{g})^{\mathfrak{g}}$. A *(coboundary) dynamical r -matrix* is a \mathfrak{h} -equivariant map $r : U \rightarrow \wedge^2 \mathfrak{g}$ satisfying the *(modified) classical dynamical Yang-Baxter equation*

$$\frac{1}{2}[r(\lambda), r(\lambda)] - \sum_i h_i \wedge \frac{\partial r}{\partial \lambda^i}(\lambda) = Z \quad (\lambda \in U),$$

where $(h_i)_i$ and $(\lambda^i)_i$ are dual bases of \mathfrak{h} and \mathfrak{h}^* , respectively. Then (following [14, 6])

$$(0.1) \quad \pi_r := \pi_{lin} + \sum_i \frac{\partial}{\partial \lambda^i} \wedge \vec{h}_i + \overrightarrow{r(\lambda)},$$

together with Z , defines a H -invariant (\mathfrak{g}) -quasi-Poisson structure on $M = U \times G$. Here π_{lin} is the linear Poisson structure on $U \subset \mathfrak{h}^*$.

Such a dynamical r -matrix is called *symplectic* if the \mathfrak{g} -quasi-Poisson manifold (M, π_r, Z) is symplectic, i.e. if $\pi_r^\# : T^*M \rightarrow TM$ is invertible and $Z = 0$.

By a *dynamical twist quantization* of $r(\lambda)$ we mean a \mathfrak{h} -equivariant map $J = 1 \otimes 1 + O(\hbar) : U \rightarrow \otimes^2 U_{\mathfrak{g}}[[\hbar]]$ satisfying the *semi-classical limit condition*

$$J(\lambda) - J^{2,1}(\lambda) = \hbar r(\lambda) + O(\hbar^2) \quad (\lambda \in U)$$

and the *modified dynamical twist equation*

$$J^{12,3}(\lambda) *_{PBW} J^{1,2}(\lambda + \hbar h^{(3)}) = \Phi^{-1} J^{1,23}(\lambda) *_{PBW} J^{2,3}(\lambda) \quad (\lambda \in U).$$

where $\Phi \in (\otimes^3 U_{\mathfrak{g}})^{\mathfrak{g}}[[\hbar]]$ is an associator quantizing Z , of which we know the existence from [5, Proposition 3.10]. Recall that $*_{PBW}$ is the Poincaré-Birkhoff-Witt star-product given on polynomial functions as the pull-back of the usual product in $U(\mathfrak{h}_{\hbar})^1$ by the symmetrization map $\text{sym} : S(\mathfrak{h})[[\hbar]] \rightarrow U(\mathfrak{h}_{\hbar})$. We also made use of the following notations:

$$J^{12,3}(\lambda) := (\Delta \otimes \text{id})(J(\lambda)) \quad \text{and}$$

$$J^{1,2}(\lambda + \hbar h^{(3)}) := \sum_{k \geq 0} \frac{\hbar^k}{k!} \sum_{i_1 \dots i_k} \frac{\partial^k J}{\partial \lambda^{i_1} \dots \partial \lambda^{i_k}}(\lambda) \otimes h_{i_1} \dots h_{i_k}.$$

In this paper we prove the following generalization of [13, Theorem 5.3] to the case of a nonabelian base:

¹ $\mathfrak{h}_{\hbar} = \mathfrak{h}[[\hbar]]$ with bracket $[\cdot, \cdot]_{\hbar} := \hbar[\cdot, \cdot]_{\mathfrak{h}}$.

Theorem 0.1. *Any symplectic dynamical r -matrix admits a dynamical twist quantization (with $\Phi = 1$).*

Furthermore, two dynamical twist quantizations $J_1, J_2 : U \rightarrow \otimes^2 U \mathfrak{g}[[\hbar]]$ of r are said *gauge equivalent* if there exists a \hbar -equivariant map $T = 1 + O(\hbar) : U \rightarrow U \mathfrak{g}[[\hbar]]$ such that

$$T^{12}(\lambda) *_{PBW} J_1(\lambda) = J_2(\lambda) *_{PBW} T^1(\lambda + \hbar h^{(2)}) *_{PBW} T^2(\lambda).$$

In this context we also prove the following generalization of [13, Section 6] to the case of a nonabelian base, asserting that dynamical twist quantizations are classified by the *second dynamical r -matrix cohomology* (see Definition 3.2):

Theorem 0.2. *Let r be a symplectic dynamical r -matrix. Then the space of gauge equivalence classes of dynamical twist quantizations of r (with $\Phi = 1$) is an affine space modeled on $H_r^2(U, \mathfrak{g})[[\hbar]]$.*

A class of examples of symplectic dynamical r -matrices is given by *nondegenerate reductive splittings*. A reductive splitting $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ (with $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$) is called nondegenerate if there exists $\lambda \in \mathfrak{h}^*$ for which $\omega(\lambda) \in \wedge^2 \mathfrak{m}^*$ defined by $\omega(\lambda)(x, y) = \langle \lambda, [x, y]_{|\mathfrak{h}} \rangle$ is nondegenerate. Then $r_{\mathfrak{h}}^{\mathfrak{m}}(\lambda) := -\omega(\lambda)^{-1} \in \wedge^2 \mathfrak{m} \subset \wedge^2 \mathfrak{g}$ defines a symplectic dynamical r -matrix on the invariant open subset $\{\lambda | \det \omega(\lambda) \neq 0\} \subset \mathfrak{h}^*$ (see [11, Proposition 1] or [14, Theorem 2.3]; see also [7, Proposition 1.1] for a more algebraic proof). Moreover, one can use it to “compose” dynamical r -matrices (see [11, Proposition 1] and [7, Proposition 0.1]):

Proposition 0.3 (The composition formula). *Assume that $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{m}$ is a nondegenerate reductive splitting and let $r_{\mathfrak{t}}^{\mathfrak{m}} : \mathfrak{t}^* \supset V \rightarrow \wedge^2 \mathfrak{h}$ be the corresponding symplectic dynamical r -matrix. If $\rho : \mathfrak{h}^* \supset U \rightarrow \wedge^2 \mathfrak{g}$ is a dynamical r -matrix with $Z \in (\wedge^3 \mathfrak{g})^{\mathfrak{g}}$, then*

$$\theta_{\rho} := r_{\mathfrak{t}}^{\mathfrak{m}} + \rho|_{\mathfrak{t}^*} : \mathfrak{t}^* \supset U \cap V \longrightarrow \wedge^2 \mathfrak{g}$$

is a dynamical r -matrix with the same Z .

We prove that one can “quantize” the map $\rho \mapsto \theta_{\rho}$:

Theorem 0.4. *With the hypothesis of Proposition 0.3, there exists a map*

$$\Theta : \{\text{Dynamical twist quantizations of } \rho\} \longrightarrow \{\text{Dynamical twist quantizations of } \theta_{\rho}\}$$

which keeps the associator Φ fixed.

The paper is organized as follows.

In Section 1 we first recall basic facts about quasi-Poisson manifolds, compatible quantizations and related results. We then give a sufficient condition for the existence of dynamical twist quantizations.

In Section 2 we give a very short proof of Theorem 0.1, using a result stating the existence of a quantum momentum map (see also [10, 12]) which is based on Fedosov’s well-known globalization procedure [8, 9]. We start the section with a summary of basic ingredients of Fedosov’s construction.

In Section 3 we prove Theorem 0.2 using a variant of the well-established classification of star-products on a symplectic manifold by formal series with coefficients in the second De Rham cohomology group of the manifold.

In Section 4 we prove a quantum analogue of the composition formula for classical dynamical r -matrices. We start with a new proof of the classical composition formula (Proposition 0.3) using (quasi-)Poisson reduction. We then derive its quantum counterpart (Theorem 0.4) using quantum reduction.

Notations. We denote by $\mathcal{O}_{\mathfrak{h}^*} = S(\mathfrak{h})$ polynomial functions on \mathfrak{h}^* , $\mathcal{O}_U = C^{\infty}(U) \supset \mathcal{O}_{\mathfrak{h}^*}$ smooth functions on $U \subset \mathfrak{h}^*$, and \mathcal{O}_G the ring of smooth functions on the (eventually formal) Lie group G . For any element $x \in \mathfrak{g}$ we denote by \overrightarrow{x} (resp. \overleftarrow{x}) the corresponding left (resp.

right) invariant vector field on G .

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1. A SUFFICIENT CONDITION FOR THE EXISTENCE OF A DYNAMICAL TWIST QUANTIZATION

Let $r : \mathfrak{h}^* \supset U \rightarrow \wedge^2 \mathfrak{g}$ be a coboundary dynamical r -matrix. Denote by π_r the bivector field on $M = U \times G$ given by (0.1).

1.1. Quasi-Poisson manifolds and their quantizations. Recall from [1, 2] that a (\mathfrak{g}) -*quasi-Poisson manifold* is a manifold X together with a \mathfrak{g} -action $\rho : \mathfrak{g} \rightarrow \mathfrak{X}(X)$, an invariant bivector field $\pi \in \Gamma(X, \wedge^2 TX)^\mathfrak{g}$ and an element $Z \in (\wedge^3 \mathfrak{g})^\mathfrak{g}$ such that

$$(1.1) \quad [\pi, \pi] = \rho(Z).$$

Let $\{f, g\} := \langle \pi, df \wedge dg \rangle$ ($f, g \in \mathcal{O}_X$) be the corresponding *quasi-Poisson bracket*. Then equation (1.1) is equivalent to

$$\{\{f, g\}, h\} + \{\{h, f\}, g\} + \{\{g, h\}, f\} = \langle \rho(Z), df \wedge dg \wedge dh \rangle \quad (f, g, h \in \mathcal{O}_X).$$

One can define the *quasi-Poisson cochain complex* of (X, π, Z) as follows: k -cochains are $C_\pi^k(X) = \Gamma(X, \wedge^k TX)^\mathfrak{g}$ and the differential is $d_\pi = [\pi, -]$. The fact that $d_\pi \circ d_\pi = 0$ follows from an easy calculation:

$$d_\pi \circ d_\pi(x) = [\pi, [\pi, x]] = \frac{1}{2}[[\pi, \pi], x] = \frac{1}{2}[\rho(Z), x] = 0 \quad (x \in C_\pi^k(X)).$$

Let us now fix an associator $\Phi \in (\otimes^3 U\mathfrak{g})^\mathfrak{g}[[\hbar]]$ quantizing Z (we know it exists from [5, Proposition 3.10]). Following [6, Definition 4.4], by a *quantization* of a given quasi-Poisson manifold (X, π, Z) we mean a series $* \in \text{Bidiff}(X)^\mathfrak{g}[[\hbar]]$ of invariant bidifferential operators such that

- $f * g = fg + O(\hbar)$ for any $f, g \in \mathcal{O}_X$,
- $f * g - g * f = \hbar\{f, g\} + O(\hbar^2)$ for any $f, g \in \mathcal{O}_X$, and
- if we write $m_*(f \otimes g) := f * g$ for $f, g \in \mathcal{O}_X$, and $\tilde{\Phi} := S^{\otimes 3}(\Phi^{-1})$, then²

$$(1.2) \quad m_* \circ (m_* \otimes \text{id}) = m_* \circ (\text{id} \otimes m_*) \circ \rho^{\otimes 3}(\tilde{\Phi}).$$

Here, S denotes the antipode of $U\mathfrak{g}$.

One has a natural notion of *gauge transformation* for quantizations. It is given by an element $Q = \text{id} + O(\hbar) \in \text{Diff}(X)^\mathfrak{g}[[\hbar]]$ that act on $*$'s in the usual way:

$$f *^{(Q)} g := Q^{-1}(Q(f) * Q(g)) \quad (f, g \in \mathcal{O}_X).$$

More precisely, if $(\Phi, *)$ is a quantization of (Z, π) then $(\Phi, *^{(Q)})$ is also. In this case we say that $*$ and $*^{(Q)}$ are *gauge equivalent*.

²We thank Pavel Etingof for pointing to us that one has to use $\tilde{\Phi}$ instead of Φ in this definition.

1.2. Classical and quantum momentum maps. Let (X, π, Z) be \mathfrak{g} -quasi-Poisson manifold and let \mathcal{G} be a Lie algebra with Lie group \mathbf{G} . A *momentum map* is a smooth \mathfrak{g} -invariant map $\mu : M \rightarrow \mathcal{G}^*$ such that $\mu_*\pi = \pi_{lin}$ and for which the corresponding infinitesimal action $\mathcal{G} \rightarrow \mathfrak{X}(X); x \mapsto \{\mu^*x, -\}$ integrates to a right action of \mathbf{G} .

Let us describe the reduction procedure with respect to a given momentum map μ . First of all \mathbf{G} acts on $\mu^{-1}(0)$ and hence one can define the reduced space $X_{red} := \mu^{-1}(0)/\mathbf{G}$. Let us assume that it is smooth (this is the case when 0 is a regular value and \mathbf{G} acts freely); its function algebra is $\mathcal{O}_{X_{red}} = \mathcal{O}_X^{\mathcal{G}}/(\mathcal{O}_X^{\mathcal{G}} \cap \mathcal{I}_0)$, where \mathcal{I}_0 is the ideal generated by $\text{im}(\mu^*)$. Since μ is \mathfrak{g} -invariant then \mathfrak{g} acts on $\mu^{-1}(0)$. Moreover the \mathfrak{g} -action and the \mathcal{G} -action commute (because π and μ are \mathfrak{g} -invariant), consequently \mathfrak{g} also acts on X_{red} . Now observe that $\mathcal{O}_X^{\mathcal{G}} = \{f \in \mathcal{O}_X \mid \{f, \mathcal{I}_0\} \subset \mathcal{I}_0\}$, therefore the quasi-Poisson bracket $\{, \}$ naturally induces a quasi-Poisson bracket (with the same Z) on $\mathcal{O}_{X_{red}}$. In other words, X_{red} inherits a structure of a quasi-Poisson manifold from the one of X .

Now assume that we are given a quantization $(*, \Phi)$ of the quasi-Poisson manifold (X, π, Z) . By a *quantum momentum map* quantizing μ we mean a map of algebras

$$\mathbf{M} = \mu^* + O(\hbar) : (\mathcal{O}_{\mathcal{G}^*}[[\hbar]], *_PBW) \longrightarrow (\mathcal{O}_X[[\hbar]], *)$$

taking its values in \mathfrak{g} -invariant functions, and such that for any $f \in \mathcal{O}_X$ and any $x \in \mathcal{G}$ one has $[\mathbf{M}(x), f]_* = \hbar\{\mu^*x, f\}$.

Remark 1.1. One only needs to know \mathbf{M} on linear functions $x \in \mathcal{G}$.

Let us describe the quantum reduction with respect to a given quantum momentum map \mathbf{M} . First denote by \mathcal{I} the right ideal generated by $\text{im}(\mathbf{M})$ in $(\mathcal{O}_X[[\hbar]], *)$ and observe that its normalizer is $\mathcal{O}_X^{\mathfrak{g}}[[\hbar]]$. Therefore $\mathcal{O}_X^{\mathfrak{g}}[[\hbar]] \cap \mathcal{I}$ is a two-sided ideal in $\mathcal{O}_X^{\mathfrak{g}}[[\hbar]]$, and we can define the reduced algebra $\mathcal{A} := \mathcal{O}_X^{\mathfrak{g}}[[\hbar]]/(\mathcal{O}_X^{\mathfrak{g}}[[\hbar]] \cap \mathcal{I})$. Since $\mathcal{I} \cong \mathcal{I}_0[[\hbar]]$ then $\mathcal{A} \cong \mathcal{O}_{X_{red}}[[\hbar]]$ (as $\mathbb{R}[[\hbar]]$ -modules). It is easy to see that the induced product on $\mathcal{O}_{X_{red}}[[\hbar]]$, together with Φ , gives a quantization of the quasi-Poisson structure on X_{red} .

A quantization $*$ of a quasi-Poisson manifold (M, π, Z) with a momentum map $\mu : M \rightarrow \mathcal{G}^*$ for which μ^* itself defines a quantum momentum map (it will be $\mathbf{M} = U(\mu^*) \circ \text{sym}$) is called *strongly (\mathcal{G} -)invariant*.

1.3. Compatible quantizations. Let us first observe that π_r defines a quasi-Poisson structure on $M = U \times G$. Here the action is $\mathfrak{g} \ni x \mapsto \overleftarrow{x} \in \mathfrak{X}(M)$ (it generates left translations). Remark that since $Z \in (\wedge^3 \mathfrak{g})^{\mathfrak{g}}$ then $\overleftarrow{Z} = \overrightarrow{Z}$. Moreover, the natural map $M \rightarrow \mathfrak{h}^*, (\lambda, g) \mapsto \lambda$ is a momentum map and the corresponding right H -action is given by $(\lambda, g) \cdot h := (\text{Ad}_h^* \lambda, gh)$. Conversely,

Proposition 1.2 ([14], Proposition 2.1). *A function $\rho \in \mathcal{O}(M, \wedge^2 \mathfrak{g})$ is a coboundary classical dynamical r -matrix if and only if*

$$\pi = \pi_{lin} + \sum \frac{\partial}{\partial \lambda^i} \wedge \overrightarrow{h_i} + \overrightarrow{\rho(\lambda)}$$

defines a \mathfrak{g} -quasi-Poisson structure on $U \times G$.

Proof. The proof given in [14] is for the case when $Z = 0$, but it admits a straightforward generalization. \square

Following Ping Xu ([14]), by a *compatible quantization* of π_r we mean a quantization $*'$ which is such that for any $u, v \in \mathcal{O}_{\mathfrak{h}^*}$ and any $f \in \mathcal{O}_G$, $u *' v = u *_PBW v$, $f *' u = fu$ and

$$(1.3) \quad u *' f = \sum_{k \geq 0} \frac{\hbar^k}{k!} \sum_{i_1, \dots, i_k} \frac{\partial^k u}{\partial \lambda^{i_1} \dots \partial \lambda^{i_k}} \overrightarrow{h_{i_1}} \cdots \overrightarrow{h_{i_k}} \cdot f.$$

Proposition 1.3. *There is a bijective correspondance between compatible quantizations of π_r and dynamical twist quantizations of r .*

Proof. Let \ast' be a compatible quantization of π_r . Since \ast' is G -invariant then for all $f, g \in \mathcal{O}_G$ one has

$$(f \ast' g)(\lambda) = \overrightarrow{J(\lambda)}(f, g) \quad (\lambda \in \mathfrak{h}^*)$$

with $J : U \rightarrow \otimes^2 U\mathfrak{g}[[\hbar]]$. Moreover

Lemma 1.4. *\ast' is strongly \mathfrak{h} -invariant³.*

Proof of the lemma. Let $f = gu$ ($g \in \mathcal{O}_G$ and $u \in \mathcal{O}_{\mathfrak{h}^*}$) on $U \times G$. Then for any $h \in \mathfrak{h}$ one has

$$\begin{aligned} h \ast' f - f \ast' h &= h \ast' (gu) - (gu) \ast' h = h \ast' (g \ast' u) - (g \ast' u) \ast' h \\ &= (h \ast' g) \ast' u - g \ast' (u \ast' h) \quad (\Phi \text{ acts trivially}) \\ &= (g \ast' h + \hbar(\overrightarrow{h} \cdot g)) \ast' u - g \ast' (u \ast' h) \\ &= g \ast' ([h, u]_{\ast'}) + \hbar(\chi_h \cdot g) \ast' u = g([h, u]_{\ast'PBW}) + \hbar(\chi_h \cdot g)u \\ &= \hbar(g(\chi_h \cdot u) + (\chi_h \cdot g)u) = \hbar(\chi_h \cdot f) \end{aligned}$$

Hence for any $f \in \mathcal{O}_M$, $h \ast' f - f \ast' h = \hbar(\chi_h \cdot f)$. \square

Therefore using [14, Proposition 3.2] one obtains that J is H -equivariant. The following lemma ends the first part of the proof:

Lemma 1.5. *J satisfies the dynamical twist equation.*

Proof of the lemma. Let us define $\mathbf{L} : \mathfrak{g} \ni x \mapsto \overrightarrow{x}$ and $\mathbf{R} : \mathfrak{g} \ni x \mapsto \overleftarrow{x}$, and denote by $m^{(n)} : \mathcal{O}_M^{\otimes n} \rightarrow \mathcal{O}_M$; $f_1 \otimes \cdots \otimes f_n \mapsto f_1 \cdots f_n$ the standard n -fold product of functions. A computation in [14] emphases the fact that for all $f, g, h \in \mathcal{O}_G$, one has⁴

$$m_{\ast'} \circ (m_{\ast'} \otimes \text{id})(f \otimes g \otimes h) = \overrightarrow{J^{12,3}(\lambda) \ast'_{PBW} J^{1,2}(\lambda + \hbar h^{(3)})}(f \otimes g \otimes h)$$

and

$$m_{\ast'} \circ (\text{id} \otimes m_{\ast'})(f \otimes g \otimes h) = \overrightarrow{J^{1,23}(\lambda) \ast'_{PBW} J^{2,3}(\lambda)}(f \otimes g \otimes h).$$

Therefore,

$$\begin{aligned} m_{\ast'} \circ (\text{id} \otimes m_{\ast'}) \circ \mathbf{R}^{\otimes 3}(\tilde{\Phi})(f \otimes g \otimes h) &= m^{(3)}(\mathbf{L}^{\otimes 3}(J^{1,23}(\lambda) \ast'_{PBW} J^{2,3}(\lambda))\mathbf{R}^{\otimes 3}(\tilde{\Phi})(f \otimes g \otimes h)) \\ &= m^{(3)}(\mathbf{R}^{\otimes 3}(\tilde{\Phi})\mathbf{L}^{\otimes 3}(J^{1,23}(\lambda) \ast'_{PBW} J^{2,3}(\lambda))(f \otimes g \otimes h)) \\ &= \overleftarrow{S^{\otimes 3}(\Phi^{-1})}(\mathbf{L}^{\otimes 3}(J^{1,23}(\lambda) \ast'_{PBW} J^{2,3}(\lambda))(f \otimes g \otimes h)) \\ &= \overrightarrow{\Phi^{-1}}(\mathbf{L}^{\otimes 3}(J^{1,23}(\lambda) \ast'_{PBW} J^{2,3}(\lambda))(f \otimes g \otimes h)) \\ &= \overrightarrow{\Phi^{-1} J^{1,23}(\lambda) \ast'_{PBW} J^{2,3}(\lambda)}(f \otimes g \otimes h), \end{aligned}$$

where the equality before the last one follows from the invariance of Φ . This ends the proof of the lemma. \square

³In particular \ast' is H -invariant. It was not noticed in [14], where the definition of compatible star-products includes this H -equivariance property. The lemma claims that it comes for free (like in the classical situation).

⁴The reader must pay attention to the following important remark: for any $P \in \otimes^n U\mathfrak{g}$ we denote by \overrightarrow{P} (resp. \overleftarrow{P}) the corresponding left (resp. right) invariant multidifferential operator, while $\mathbf{L}^{\otimes n}(P)$ (resp. $\mathbf{R}^{\otimes n}(P)$) is an element in $\otimes^n \text{Diff}(G)^{G_{left}}$ (resp. $\otimes^n \text{Diff}(G)^{G_{right}}$). Namely, $\overrightarrow{P} = m^{(n)} \circ (\mathbf{L}^{\otimes n}(P))$.

Conversely, let $J = \sum_{\alpha} f_{\alpha} A_{\alpha} \otimes B_{\alpha}$ be a dynamical twist quantization of r ($f_{\alpha} \in \mathcal{O}_U[[\hbar]]$ and $A_{\alpha}, B_{\alpha} \in U\mathfrak{g}$). Following [14] we define a G -invariant product $*'$ on $\mathcal{O}_M[[\hbar]]$ by

$$g_1 *' g_2 := \sum_{k \geq 0, \alpha} \frac{\hbar^k}{k!} \sum_{i_1, \dots, i_k} f_{\alpha} *_{PBW} (\overrightarrow{A}_{\alpha} \cdot \frac{\partial^k g_1}{\partial \lambda^{i_1} \dots \partial \lambda^{i_k}}) *_{PBW} (\overrightarrow{B}_{\alpha} \overrightarrow{h_{i_1}} \dots \overrightarrow{h_{i_k}} \cdot g_2).$$

One can check by direct computations that \mathfrak{h} -equivariance of J implies strong \mathfrak{h} -invariance of $*'$, and that the dynamical twist equation implies equation (1.2). \square

Remark 1.6. Since $\mathcal{O}_{\mathfrak{h}^*} = S(\mathfrak{h})$ is generated as a vector space by h^n , $h \in \mathfrak{h}$ and $n \in \mathbb{N}$, then one can rewrite condition (1.3) as

$$h^n *' f = \sum_{k=0}^n \hbar^k C_n^k(\overrightarrow{h}^k \cdot f) h^{n-k}.$$

We saw in Lemma 1.4 that a compatible quantization always satisfies the strongly \mathfrak{h} -invariance condition. In what follows we show that this condition is actually sufficient for the existence of a compatible quantization.

1.4. A sufficient condition for the existence of a compatible quantization.

Proposition 1.7. *Assume that we are given a strongly \mathfrak{h} -invariant quantization $*$ of π_r on M . Then there exists a gauge equivalent compatible quantization $*'$ of π_r . Therefore there exists a dynamical twist quantization J of r .*

Proof. First observe that $h * h' - h' * h = \hbar[h, h']_{\mathfrak{h}} = [h, h']_{\mathfrak{h}\hbar}$. Therefore we have an algebra morphism

$$a : U(\mathfrak{h}\hbar) \longrightarrow (\mathcal{O}_M, *).$$

Then define the algebra morphism $Q : \mathcal{O}_{\mathfrak{h}^* \times G} = S(\mathfrak{h}) \otimes \mathcal{O}_G \longrightarrow \mathcal{O}_M$ as follows:

$$Q(fu) = f * a(\text{sym}(u)) \quad (u \in S(\mathfrak{h}), f \in \mathcal{O}_G),$$

where $\text{sym} : S(\mathfrak{h})[[\hbar]] \longrightarrow U(\mathfrak{h}\hbar)$ is the isomorphism sending h^n to h^n for any $h \in \mathfrak{h}$. Thus $Q(h^n \otimes f) = f * \underbrace{h * \dots * h}_{n \text{ times}}$, and since $*$ can be expressed as a series $m_0 + O(\hbar)$ of bidifferential

operators on M then Q can be expressed as a series $\text{id} + O(\hbar)$ of differential operators on M . Moreover it is obviously \mathfrak{g} -invariant (since $*$ is), consequently we have a new quantization $*'$ of π_r , gauge equivalent to $*$, defined as follows: for any $f, g \in \mathcal{O}_M$,

$$f *' g = Q^{-1}(Q(f) * Q(g)).$$

Let us now check that $*'$ satisfies all Xu's properties for compatible quantizations.

- for any $u, v \in S(\mathfrak{h})$,

$$\begin{aligned} u *' v &= Q^{-1}(a(\text{sym}(u)) * a(\text{sym}(v))) \\ &= Q^{-1}(a(\text{sym}(u)\text{sym}(v))) \\ &= Q^{-1}(a(\text{sym}(u *_{PBW} v))) = u *_{PBW} v \end{aligned}$$

- let $u \in S(\mathfrak{h})$ and $f \in \mathcal{O}_G$, then $f *' u = Q^{-1}(f * a(\text{sym}(u))) = fu$. Let us now compute $u *' f$; we can assume that $u = h^n$, $h \in \mathfrak{h}$, and then

$$\begin{aligned} u *' f &= Q^{-1}(a(\text{sym}(u)) * f) = Q^{-1}(\underbrace{h * \dots * h}_{n \text{ times}} * f) \\ &= Q^{-1}\left(\sum_{k=0}^n C_n^k \hbar^k (\overrightarrow{h}^k \cdot f) * \underbrace{h * \dots * h}_{n-k \text{ times}}\right) \\ &= \sum_{k=0}^n \hbar^k C_n^k (\overrightarrow{h}^k \cdot f) h^{n-k} \end{aligned}$$

- Since $*$ is a H -invariant star-product, then Q is a H -invariant gauge equivalence. Therefore $*'$ is also H -invariant.

The proposition is proved. \square

Remark 1.8. The gauge transformation Q constructed above obviously satisfies $Q(h) = h$ for any $h \in \mathfrak{h}$.

2. QUANTIZATION OF SYMPLECTIC DYNAMICAL r -MATRICES

In this section we prove Theorem 0.1. We start by recalling Fedosov's construction of star-products on a symplectic manifold (for more details we refer to [8, 9]).

2.1. Fedosov's star-products. Let (M, ω) a symplectic manifold and denote by $\pi = \omega^{-1}$ the corresponding Poisson bivector. Then its tangent bundle TM inherits a Poisson structure $\tilde{\pi}$ expressed locally as

$$\tilde{\pi} = \pi^{ij}(x) \frac{\partial}{\partial y^i} \wedge \frac{\partial}{\partial y^j},$$

where y^i 's are coordinates in the fibers. This Poisson structure is regular and constant on the symplectic leaves which are the fibers $T_x M$ of the bundle. Therefore it is quantized by the series of fiberwise bidifferential operators $\exp(\hbar \tilde{\pi})$. It defines an associative product \circ on sections of $W = \hat{S}(T^*M)[[\hbar]]$ that naturally extends to $\Omega^*(M, W)$. The center of $(\Omega^*(M, W), \circ)$ consists of forms that are constant in the fibers, i.e. lying in $\Omega^*(M)[[\hbar]]$.

By assigning the degree $2k + l$ to sections of $\hbar^k S^m(T^*M)$ there is a natural decreasing filtration

$$W = W_0 \supset W_1 \supset \cdots \supset W_i \supset W_{i+1} \supset \cdots \supset \mathcal{O}_M.$$

Now fix (once and for all) a torsion free connection ∇ on M with Christoffel's symbols Γ_{ij}^k . One can assume without loss of generality that it is symplectic (see [9, Section 2.5]). Then consider

$$\partial : \Omega^*(M, W) \rightarrow \Omega^{*+1}(M, W)$$

its induced covariant derivative. In Darboux local coordinates we have

$$\partial = d + \frac{1}{\hbar}[\Gamma, -]_{\circ},$$

where $\Gamma = -\frac{1}{2}\Gamma_{ijk}y^i y^j dx^k$ is a local 1-form with values in W ($\Gamma_{ijk} = \omega_{il}\Gamma_{jk}^l$). One has

$$\partial^2 = -\frac{1}{\hbar}[R, -]_{\circ}$$

where $R = \frac{1}{4}R_{ijkl}y^i y^j dx^k \wedge dx^l$, and $R_{ijkl} = \omega_{im}R_{jkl}^m$ being the curvature tensor of the symplectic connection ∇ .

Let us consider more general derivations of $(\Omega^*(M, W), \circ)$ of the form

$$D = \partial - \delta + \frac{1}{\hbar}[r, -]_{\circ}$$

where $r \in \Omega^1(M, W)$ and $\delta = \frac{1}{\hbar}[\omega_{ij}y^i dx^j, -]_{\circ}$. A simple calculation yields that

$$D^2 = -\frac{1}{\hbar}[\Omega, -]_{\circ}$$

where $\Omega = \omega + R + \delta r - \partial r - \frac{1}{\hbar}r^2 \in \Omega^2(M, W)$ is called the *Weyl curvature* of D . In particular D is flat (i.e. $D^2 = 0$) if and only if $\Omega \in \Omega^2(M)[[\hbar]]$ (i.e. is a central 2-form), and in this case Bianchi identity for ∇ implies that $d\Omega = D\Omega = 0$.

In computing D^2 one sees that $\delta : \Omega^*(M, W_k) \rightarrow \Omega^{*+1}(M, W_{k-1})$ has square zero and that the torsion freeness of ∇ implies $\delta\partial + \partial\delta = 0$. Then we define a homotopy operator $\kappa : \Omega^*(M, W_k) \rightarrow \Omega^{*-1}(M, W_{k+1})$ on monomials $a \in \Omega^p(M, S^q(T^*M))$: if $p + q \neq 0$ then

$$\kappa(a) = \frac{1}{p+q} y^i \partial_{dx^i} a$$

and else $\kappa(a) = 0$. One easily check that $\kappa^2 = 0$ and $\delta\kappa + \kappa\delta = \text{id} - \sigma$ where

$$\sigma : W \rightarrow C^\infty(M)[[\hbar]], a \mapsto a|_{dx^i=y^i=0}$$

is the projection onto 0-forms constant in the fibers.

Theorem 2.1 (Fedosov). *For any closed 2-form $\Omega = \omega + O(\hbar) \in Z^2(M)[[\hbar]]$ there exists a unique $r \in \Omega^1(M, W_3)$ such that $\kappa(r) = 0$ and*

$$D = \partial - \delta + \frac{1}{\hbar}[r, -]_\circ$$

has Weyl curvature Ω and is therefore flat.

Proof. First observe that $\Omega = \omega + R + \delta r - \partial r - \frac{1}{\hbar}r^2$ with $\kappa(r) = 0$ if and only if

$$(2.1) \quad r = \kappa(\Omega - \omega - R + \partial r + \frac{1}{\hbar}r^2).$$

Since ∂ preserves the filtration and κ raises its degree by 1 then $\kappa(\Omega - \omega - R) \in \Omega^1(M, W_3)$ and the sequence $(r_n)_{n \geq 3}$ defined by the iteration formula

$$(2.2) \quad r_{n+1} = r_n + \kappa(\partial r_n + \frac{1}{\hbar}r_n^2)$$

with $r_3 = \kappa(\Omega - \omega - R)$ converges to a unique element $r \in \Omega^1(M, W_3)$ which is a solution of equation (2.1). We have proved the existence.

Conversely, for any solution $r \in \Omega^1(M, W_3)$ of (2.1) define $r_k = r \bmod W_{k+1}$. Then $r_3 = \kappa(\Omega - \omega - R)$ and the sequence $(r_n)_{n \geq 3}$ satisfies (2.2). Unicity is proved. \square

Such a flat derivation D is called a *Fedosov connection (of ∇ -type)*. The previous theorem claims that they are in bijection with series Ω of closed two forms starting with ω .

Theorem 2.2 (Fedosov). *If D is a Fedosov connection then for any $f_0 \in C^\infty(M)[[\hbar]]$ there exists a unique D -closed section $f \in \Gamma(M, W)$ such that $\sigma(f) = f_0$. Hence σ establishes an isomorphism between $Z_D^0(W)$ and $C^\infty(M)[[\hbar]]$.*

Proof. Let $f_0 \in C^\infty(M)[[\hbar]]$. One has $D(f) = 0$ with $\sigma(f) = f_0$ if and only if

$$(2.3) \quad f = f_0 + \kappa(\partial f + \frac{1}{\hbar}[r, f]_\circ).$$

Like in the proof of Theorem 2.1 we can solve (uniquely) this equation with the help of an iteration formula: $f_{n+1} = f_0 + \kappa(\partial f_n + \frac{1}{\hbar}[r, f_n]_\circ)$. \square

Then $f * g = \sigma(\sigma^{-1}(f) \circ \sigma^{-1}(g))$ defines a star-product on $C^\infty(M)[[\hbar]]$ that quantizes (M, ω) . A star-product constructed this way is called a *Fedosov star-product (of ∇ -type)* and is uniquely determined, once ∇ is fixed, by its *characteristic 2-form*

$$\omega_\hbar := \frac{1}{\hbar}(\Omega - \omega) \in Z^2(M)[[\hbar]].$$

Moreover one can easily prove the following

Lemma 2.3 ([4]). *Let $\omega_\hbar^{(i)} = \sum_{k>0} \hbar^{k-1} \omega_k^{(i)} \in Z^2(M)[[\hbar]]$ ($i = 1, 2$) and denote by $*_i$ the Fedosov star-product with characteristic two-form $\omega_\hbar^{(i)}$. If $\omega_\hbar^{(1)} = \omega_\hbar^{(2)} + O(\hbar^k)$ then*

$$*_i^{(1)} - *_i^{(2)} = \hbar^{k+1} \pi^\#(\omega_k^{(1)} - \omega_k^{(2)}) + O(\hbar^{k+2}).$$

2.2. Fedosov's construction in the presence of symmetries. Let (M, ω) a symplectic manifold. Let us prove two results on the compatibility of Fedosov's construction with group actions and hamiltonians vector fields.

Proposition 2.4 (Fedosov). *Assume that a group G acts on (M, ω) by symplectomorphisms and is equipped with a G -invariant torsion free connection. Then for any $\omega_{\hbar} \in Z^2(M)^G[[\hbar]]$ the corresponding Fedosov star-product is G -invariant.*

Proof. First observe that starting from a G -invariant torsion free connection ∇ one can assume without loss of generality that it is symplectic (see the proof of Proposition 5.2.2 in [9], where all expressions become obviously G -invariant).

Then, being a symplectomorphism of (M, ω) , any elements $g \in G$ acts via its differential dg on $(TM, \tilde{\pi})$ as a Poisson automorphism linear in the fibers. Then its dual map $g^* : T^*M \rightarrow T^*M$ defined by $\langle g^*\xi, x \rangle = \langle \xi, dg(x) \rangle$ extends to W as an automorphism.

Finally, we need to prove that g^* preserves the Fedosov connection with Weyl curvature $\Omega = \omega + \omega_{\hbar}$. On one hand the automorphism g^* commutes with ∂ (since ∇ is assumed to be G -invariant) and so $g^*R = R$. On the other hand g^* also commutes with δ and κ , thus if r is a solution of equation (2.1) with $\kappa(r) = 0$ then so is g^*r . By uniqueness $g^*r = r$. We are done. \square

Proposition 2.5 (Fedosov). *Let $H \in \mathcal{O}_M$ such that $\chi = \{H, \cdot\}$ preserves a torsion free connection on M . Then for any $\omega_{\hbar} \in Z^2(M)[[\hbar]]$ such that $\iota_{\chi}\omega_{\hbar} = 0$ the corresponding Fedosov star-product $*$ satisfies $H * f - f * H = \hbar(\chi \cdot f)$ for any $f \in \mathcal{O}_M$.*

Proof. First observes that $L_{\chi}\omega_{\hbar} = (d\iota_{\chi} + \iota_{\chi}d)\omega = 0$. Therefore, the infinitesimal version of the previous proof ensure us that the Fedosov connection D with Weyl curvature $\Omega = \omega + \omega_{\hbar}$ is L_{χ} -equivariant. Hence in local Darboux coordinates it writes $D = d + \frac{1}{\hbar}[\gamma, -]_{\circ}$ with $L_{\chi}\gamma = 0$, and $\Omega = -d\gamma - \frac{1}{\hbar}\gamma^2$. Let us compute

$$\begin{aligned} D(H - \iota_{\chi}\gamma) &= dH - d\iota_{\chi}\gamma - \frac{1}{\hbar}[\gamma, \iota_{\chi}\gamma]_{\circ} = \iota_{\chi}\Omega + \iota_{\chi}d\gamma + \frac{1}{\hbar}[\iota_{\chi}\gamma, \gamma]_{\circ} \\ &= \iota_{\chi}(\Omega + d\gamma + \frac{1}{\hbar}\gamma^2) = 0 \end{aligned}$$

Since $\sigma(H - \iota_{\chi}\gamma) = \sigma(H) = H$, it means that $\sigma^{-1}(H) = H - \iota_{\chi}\gamma$ in local Darboux coordinates. Consequently, for any $f \in \mathcal{O}_M[[\hbar]]$

$$\begin{aligned} H * f - f * H &= \sigma([H - \iota_{\chi}\gamma, \sigma^{-1}(f)]_{\circ}) = \sigma(-[\iota_{\chi}\gamma, \sigma^{-1}(f)]_{\circ}) \\ &= \sigma(-\iota_{\chi}\hbar(D - d)\sigma^{-1}(f)) = \hbar\sigma(\iota_{\chi}d\sigma^{-1}(f)) \\ &= \hbar\sigma(L_{\chi}\sigma^{-1}(f) - d\iota_{\chi}\sigma^{-1}(f)) = \hbar L_{\chi}(f) = \hbar(\chi \cdot f) \end{aligned}$$

The proposition is proved. \square

2.3. Proof of Theorem 0.1. Let $r : U \rightarrow \wedge^2 \mathfrak{g}$ a symplectic dynamical r -matrix. A basis \mathcal{B} of vector fields on $M = U \times G$ is given by $\mathcal{B} = (\dots, \partial_{\lambda^i}, \dots, \dots, \vec{e}_i, \dots)$ where $(\lambda^i)_i$ is a base of \mathfrak{h}^* and $(e_i)_i$ is a base of \mathfrak{g} . Then one defines a torsion free connection ∇ on M as

$$\nabla_b X = \frac{1}{2}[b, X] \quad (b \in \mathcal{B}, X \in \mathfrak{X}(M)).$$

Remark that $[\chi_h, b] \in \text{span}_{\mathbb{R}} \mathcal{B}$ for any $b \in \mathcal{B}$. Therefore it follows immediatly from the Jacobi identity that ∇ is \mathfrak{h} -invariant: for all $X, Y \in \mathfrak{X}(M)$ and $h \in \mathfrak{h}$,

$$[\chi_h, \nabla_X Y] = \nabla_{[\chi_h, X]} Y + \nabla_X [\chi_h, Y].$$

Thus from Proposition 2.5 the Fedosov star-product $*$ with the trivial characteristic 2-form is strongly \mathfrak{h} -invariant. Moreover ∇ is obviously G -invariant, hence Proposition 2.4 implies that $*$ is also G -invariant.

Finally, we apply Proposition 1.7 to construct a compatible quantization of π_r . We are done. \square

3. CLASSIFICATION

Let $r : \mathfrak{h}^* \supset U \rightarrow \wedge^2 \mathfrak{g}$ a dynamical r -matrix. Denote by π_r the corresponding H -invariant \mathfrak{g} -quasi-Poisson structure (0.1) on $M = U \times G$ (together with $Z \in (\wedge^3 \mathfrak{g})^{\mathfrak{g}}$).

3.1. Strongly invariant equivalences and obstructions. By a *strongly invariant equivalence* between two strongly \mathfrak{h} -invariant quantizations of π_r , we mean a H -invariant equivalence Q (nameky, $Q = \text{id} + O(\hbar) \in \text{Diff}(M)^{G \times H}[[\hbar]]$) satisfying $Q(h) = h$ for any $h \in \mathfrak{h} \subset \mathcal{O}_M$. We will now develop an analogue of the usual obstruction theory in this context.

Let us denote by b the Hochschild coboundary operator for cochains on the (commutative) algebra \mathcal{O}_M . We start with the following result which is a variant of a standard one.

Proposition 3.1. *Suppose that $*_1$ and $*_2$ are two strongly invariant quantizations of π_r :*

$$f *_i g = \sum_{k \geq 0} \hbar^k C_k^i(f, g) \quad (i = 1, 2).$$

Assume that $*_1$ and $*_2$ coincide up to order n , i.e. $C_k^1 = C_k^2$ if $k \leq n$. Then

- (1) *there exists $B \in \Gamma(M, \wedge^2 TM)^{G \times H}$ and $E \in \text{Diff}(M)^{G \times H}$ such that $B(h, -) = 0$ and $E(h) = 0$ if $h \in \mathfrak{h} \subset \mathcal{O}_{\mathfrak{h}^*}$, $[\pi_r, B] = 0$, and satisfying*

$$(C_{n+1}^1 - C_{n+1}^2)(f, g) = B(f, g) + (bE)(f, g) \quad (f, g \in \mathcal{O}_M);$$

- (2) *there exists $P \in \text{Diff}(M)^{G \times H}$ such that $C_1 = \pi_r + bP$ and $P(h) = 0$ for $h \in \mathfrak{h}$;*
(3) *if $B = [\pi_r, X]$, $X \in \mathfrak{X}(M)^{G \times H}$ such that $X(h) = 0$, then the strongly invariant equivalence $Q = 1 + \hbar^n X + \hbar^{n+1}(E + [X, P])$ transforms $*_1$ into another strongly invariant star-product which coincides with $*_2$ up to order $n + 1$.*

Proof. We use a similar argument as in [13, 3, 4].

- (1) It is well-known that $b(C_{n+1}^1 - C_{n+1}^2) = 0$. Hence we may write

$$C_{n+1}^1 - C_{n+1}^2 = B + b(E_0)$$

where $B \in \Gamma(M, \wedge^2 TM)^{G \times H}$ is the skew-symmetric part of $C_{n+1}^1 - C_{n+1}^2$ and $E_0 \in \text{Diff}(M)$. Moreover, one knows (see e.g. [4]) that $[\pi_r, B] = 0$, and it follows directly from the strong \mathfrak{h} -invariance property for $*_1$ and $*_2$ that $B(h, -) = 0$ if $h \in \mathfrak{h}$.

Since $U \times G$ admits a $G \times H$ -invariant connection and $b(E_0)$ is obviously $G \times H$ -invariant, then according to [3, Proposition 2.1] we can assume that E_0 is $G \times H$ -invariant. In particular E_0 is G -invariant and hence $E_0(f)$, $f \in \mathcal{O}_U$, is a function on U only. Thus we can define a H -invariant vector field \vec{v} on U as follows: $\langle dh, \vec{v} \rangle = E_0(h)$ for any $h \in \mathfrak{h} \subset \mathcal{O}_U$. Now $E := E_0 - \vec{v}$ satisfies all the required properties and $b(E) = b(E_0) - b(\vec{v}) = b(E_0)$.

- (2) It is standard that $C_1 = \pi_r + b(P_0)$. By repeating a similar argument as in (1) we can prove that P_0 can be chosen so that $P_0 = P \in \text{Diff}(M)^{G \times H}$ and satisfies $P(h) = 0$ for any $h \in \mathfrak{h}$.

The third statement (3) follows from an easy (and standard) calculation. \square

This proposition means that obstructions to strongly invariant equivalences are in the second cohomology group of the subcomplex $C^\infty(U, \wedge^* \mathfrak{g})^{\mathfrak{h}}$ in the H -invariant quasi-Poisson cochain complex of (M, π_r, Z) . On such cochains the (quasi-)Poisson coboundary operator $[\pi_r, -]$ reduces to $[r, -]$.

Definition 3.2. The cohomology $H_r^*(U, \mathfrak{g})$ of this cochain complex is called the *dynamical r -matrix cohomology* associated to $r : U \rightarrow \wedge^2 \mathfrak{g}$.

3.2. Classification of strongly invariant star-products. Now assume that the quasi-Poisson manifold (M, π_r) is actually symplectic and denote by ω_r the symplectic form; it is $G \times H$ -invariant and satisfies $\iota_{\chi_h} \omega = 0$ for any $h \in \mathfrak{h} \subset \mathcal{O}_{\mathfrak{h}^*}$. The $G \times H$ -invariant isomorphism

$$\pi_r^\# : T^*M \xrightarrow{\sim} TM$$

extends to a $G \times H$ -invariant isomorphism of cochain complexes

$$(\Omega^*(M), d) \xrightarrow{\sim} (\Gamma(M, \wedge^* TM), [\pi_r, -])$$

which restricts to an isomorphism

$$(\Omega_{\mathfrak{h}}^*(M)^G, d) \xrightarrow{\sim} (C^\infty(U, \wedge^* \mathfrak{g})^{\mathfrak{h}}, [r, -]),$$

where $\Omega_{\mathfrak{h}}^*(M) := \{\alpha \in \Omega^*(M)^H \mid \iota_{\chi_h} \alpha = 0, \forall h \in \mathfrak{h}\}$.

Let us fix once and for all a symplectic $G \times H$ -invariant connection ∇ on M (we know it exists) and remember from the previous section that for any $\omega_{\hbar} \in \hbar \Omega^2(M)^{G \times H}[[\hbar]]$ such that $d\omega_{\hbar} = 0$ there exists a (unique) $G \times H$ -invariant Fedosov star-product $*$ with characteristic 2-form ω_{\hbar} . Moreover, if $\omega_{\hbar} \in \Omega_{\mathfrak{h}}^2(M)[[\hbar]]$ then Proposition 2.5 implies that $*$ is strongly \mathfrak{h} -invariant. Therefore we can associate a strongly invariant quantization of π_r (which is actually a Fedosov star-product) to any closed two form $\omega_{\hbar} \in \Omega_{\mathfrak{h}}^2(M)^G[[\hbar]]$.

In the rest of the section, all Fedosov star-products are assumed to be of ∇ -type and G -invariant (since they quantize the \mathfrak{g} -quasi-Poisson structure π_r).

Theorem 3.3. *Two strongly invariant Fedosov star-products are equivalent by a strongly invariant equivalence if and only if their characteristic 2-forms lies in the same cohomology class in $H_{\mathfrak{h}}^{G,2}(M)[[\hbar]]$.*

Proof. Let $*_0$ and $*_1$ two strongly invariant Fedosov star-products with respective characteristic 2-form $\omega_{\hbar}^{(0)}$ and $\omega_{\hbar}^{(1)}$.

First assume that $\omega_{\hbar}^{(0)} = \omega_{\hbar}^{(1)} + d\alpha$ for some $\alpha = \sum_k \hbar^k \alpha^{(k)} \in \Omega_{\mathfrak{h}}^1(M)^G[[\hbar]]$, and define $\omega_{\hbar}(t) = \omega_{\hbar}^{(0)} + t d\alpha$. Let $D_t = \partial - \delta + \frac{1}{\hbar}[r(t), -]$ be the Fedosov differential with Weyl curvature $\Omega(t) = \omega + \hbar \omega_{\hbar}(t)$. Let $H(t) \in \Gamma(M, W)$ be the solution of the equation

$$D_t H(t) = -\alpha + \dot{r}(t)$$

with $\sigma(H(t)) = 0$. Then $H(t)$ is $G \times H$ -invariant since D_t , α and $r(t)$ are. According to [9, Theorem 5.5.3] the solution of the Heisenberg equation

$$\frac{dF(t)}{dt} + [H(t), F(t)]_{\circ} = 0, \quad F(0) = f$$

establishes an isomorphism $Z_{D_0}^0(W) \rightarrow Z_{D_1}^0(W)$, $f \mapsto F(1)$ and then the corresponding series of differential operators $Q : (\mathcal{O}_M[[\hbar]], *_0) \rightarrow (\mathcal{O}_M[[\hbar]], *_1)$ is obviously $G \times H$ -invariant.

Remember from the proof of Proposition 2.5 that in local Darboux coordinates the Fedosov differential writes $D_t = d + \frac{1}{\hbar}[\gamma(t), -]_{\circ}$ and $\sigma_t^{-1}(h) = h - \iota_{\chi_h} \gamma(t)$ if $h \in \mathfrak{h}$. Now remark that $\dot{\gamma}(t) = \dot{r}(t)$ and that $\iota_{\chi_h} r(t)$ is independant of t . Hence $\sigma_t^{-1}(h)$ does not depend on t and thus $Q(h) = h$.

Conversely, assume that $*_0$ and $*_1$ are related by a strongly invariant equivalence with $[\omega_{\hbar}^{(0)}] \neq [\omega_{\hbar}^{(1)}]$ in $H_{\mathfrak{h}}^{G,2}(M)[[\hbar]]$. Write $\omega_{\hbar}^{(i)} = \sum_{k>0} \hbar^{k-1} \omega_k^{(i)}$ ($i = 0, 1$) and denote by l the lowest integer for which $[\omega_l^{(0)}] \neq [\omega_l^{(1)}]$ in $H_{\mathfrak{h}}^{G,2}(M)$. Let us then define

$$\omega_{\hbar}^{(2)} = \sum_{0 < k < l} \hbar^{k-1} \omega_k^{(0)} + \sum_{k \geq l} \hbar^{k-1} \omega_k^{(1)}$$

and $*_2$ the strongly invariant Fedosov star-product with characteristic 2-form $\omega_{\hbar}^{(2)}$. Since $[\omega_{\hbar}^{(2)}] = [\omega_{\hbar}^{(1)}] \in H_{\mathfrak{h}}^{G,2}(M)[[\hbar]]$ then $*_2$ is equivalent to $*_1$, and hence to $*_0$, by a strongly

invariant equivalence. Then we deduce from Lemma 2.3 that $C_{l+1}^2 - C_{l+1}^0 = \pi_r^\#(\omega_l^{(1)} - \omega_l^{(0)})$, where C_k^i 's ($i = 0, 2$) are the cochains defining $*_i$. Thus it follows from Proposition 3.1 that $\omega_l^{(1)} - \omega_l^{(0)}$ is exact and we obtain a contradiction. \square

Theorem 3.4. *Any strongly invariant quantization of π_r is equivalent to a strongly invariant Fedosov star-product by a strongly invariant equivalence. Therefore the set of strongly invariant star-products quantizing π_r up to strongly invariant equivalences is an affine space modeled on $H_{\mathfrak{h}}^{G,2}(M)[[\hbar]] = H_r^2(U, \mathfrak{g})$.*

Proof. We follow [4, Proposition 4.1].

Let $*$ be an arbitrary strongly invariant quantization of π_r . Denote by $*_0$ the strongly invariant Fedosov star-product with the trivial characteristic 2-form, which coincides with $*$ up to order 0. Moreover, the skew-symmetric part of the first order term in $* - *_0$ vanishes, hence it follows from Proposition 3.1 that there exists a strongly invariant equivalence $Q^{(0)} = 1 + \hbar Q_0$ that transforms $*$ into a new strongly invariant quantization $*^{(0)}$ which coincides with $*_0$ up to order 1. Now the skew-symmetric part of the second order term in $*^{(0)} - *_0$ yields a closed form $\omega_1 \in \Omega_{\mathfrak{h}}^2(M)^G$.

Denote by $*_1$ the strongly invariant Fedosov star-product with characteristic 2-form ω_1 . Lemma 2.3 tells us that the skew-symmetric part of the second order term in $*^{(0)} - *_1$ vanishes, hence it follows from Proposition 3.1 that there exists a strongly invariant equivalence $Q^{(1)} = 1 + \hbar^2 Q_1$ that transforms $*^{(0)}$ into a new strongly invariant quantization $*^{(1)}$ which coincides with $*_1$ up to order 2.

Repeating this procedure we get a sequence of strongly invariant equivalences $Q^{(k)} = 1 + \hbar^{k+1} Q_k$ ($k \geq 0$) and a sequence of closed forms ω_k ($k > 0$) such that the strongly invariant quantization $*^{(k)}$ obtained from $*$ by applying successively $Q^{(0)}, \dots, Q^{(k)}$ coincides up to order $k+1$ with the strongly invariant Fedosov star-product $*_k$ with characteristic 2-form $\omega_1 + \dots + \hbar^{k-1} \omega_k$.

Finally, the strongly invariant equivalence $Q := \dots Q^{(2)} Q^{(1)} Q^{(0)}$ transform $*$ into the strongly invariant Fedosov star-product with characteristic 2-form $\omega_{\mathfrak{h}} = \sum_{k>0} \hbar^{k-1} \omega_k$. \square

3.3. Classification of dynamical twist quantizations. Let T be a gauge equivalence of dynamical twist quantizations J_1 and J_2 of r . One can view T as an element in $\text{Diff}(U \times G)^{G \times H}[[\hbar]]$ such that $T(u) = u$ for any $u \in \mathcal{O}_{\mathfrak{h}^*}$. Moreover if we denote by $*'_i$ the compatible quantization of π_r corresponding to J_i ($i = 1, 2$) then it follows from an easy calculation that

$$T(f *_1' g) = T(f) *_2' T(g).$$

Conversely, any $G \times H$ -invariant gauge equivalence T from $*'_1$ to $*'_2$ which is such that $T(u) = u$ for any $u \in \mathcal{O}_{\mathfrak{h}^*}$, that we will call from now a *compatible equivalence*, obviously gives rise to a gauge equivalence of the dynamical twist quantizations J_1 and J_2 .

Therefore, the set of dynamical twist quantization of r up to gauge equivalences is in bijection with the set of compatible quantizations of π_r up to compatible equivalences.

Remember from Proposition 1.7 and Remark 1.8 that any strongly invariant quantization is equivalent to a compatible one by a strongly invariant equivalence. Moreover the PBW star-product has the following nice property: for any $h \in \mathfrak{h}$, $h^{*PBW^n} = h^n$. Hence any strongly invariant equivalence between two compatible quantizations is actually a compatible equivalence. Consequently:

Proposition 3.5. *There is a bijection*

$$\frac{\{\text{strongly invariant quantizations of } \pi_r\}}{\text{strongly invariant equivalences}} \longleftrightarrow \frac{\{\text{compatible quantizations of } \pi_r\}}{\text{compatible equivalences}}$$

End of the proof of Theorem 0.2. Assume that the dynamical r -matrix is symplectic. Then Theorem 0.2 follows from Proposition 3.5 and Theorem 3.4. \square

4. THE QUANTUM COMPOSITION FORMULA

In this section we assume that $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{m}$ is a nondegenerate reductive splitting and we denote by $r_{\mathfrak{t}}^{\mathfrak{m}} : \mathfrak{t}^* \supset V \rightarrow \wedge^2 \mathfrak{h}$ the corresponding symplectic dynamical r -matrix. Let $p : \mathfrak{h} \rightarrow \mathfrak{t}$ be the \mathfrak{t} -invariant projection along \mathfrak{m} . For any function f on \mathfrak{h}^* with values in a \mathfrak{h} -module L we write $f_{|\mathfrak{t}^*}$ for the function $f \circ p^*$ on \mathfrak{t}^* with values in L viewed as a \mathfrak{t} -module; in particular if f is \mathfrak{h} -invariant then $f_{|\mathfrak{t}^*}$ is \mathfrak{t} -invariant.

4.1. The classical composition formula (proof of Proposition 0.3). Let $\rho : U \rightarrow \wedge^2 \mathfrak{g}$ be a dynamical r -matrix with $Z \in (\wedge^3 \mathfrak{g})^{\mathfrak{g}}$. Then $\pi := \pi_{r_{\mathfrak{t}}^{\mathfrak{m}}} + \pi_{\rho}$ defines a \mathfrak{g} -quasi-Poisson structure (with the same Z) on the manifold $X = V \times H \times U \times G$ which is

- H -invariant with respect to left multiplication on H ,
- H -invariant with respect to the right action on $U \times G$,
- T -invariant with respect to the right action on $V \times H$.

The right diagonal H -action, given by $(\tau, x, \lambda, y) \cdot q = (\tau, q^{-1}x, \text{Ad}_q^* \lambda, yq)$, actually comes from a momentum map:

$$\begin{aligned} \mu : X &\longrightarrow \mathfrak{h}^* \\ (\tau, x, \lambda, y) &\longmapsto \lambda - \text{Ad}_{x^{-1}}^*(p^* \tau). \end{aligned}$$

Consequently we can apply the reduction with respect to μ . The right H -invariant smooth map

$$\begin{aligned} \psi : X = V \times H \times U \times G &\longrightarrow M := U \cap V \times G \\ (\tau, x, \lambda, y) &\longmapsto (\tau, yx) \end{aligned}$$

restricts to a diffeomorphism $\mu_2^{-1}(0)/H \rightarrow M$ with inverse given by

$$(\tau, y) \longmapsto \overline{(\tau, 1, p^* \tau, y)} \quad (\tau \in U \cap V, y \in G).$$

Remark 4.1. From an algebraic viewpoint, we have an injective map of commutative algebras $\psi^* : \mathcal{O}_M \rightarrow \mathcal{O}_X$ with values in $\mathcal{O}_X^{\mathfrak{h}} = \mathcal{O}_{X/H}$ and such that, composed with the projection $\mathcal{O}_X^{\mathfrak{h}} \rightarrow \mathcal{O}_X^{\mathfrak{h}} / (\mathcal{O}_X^{\mathfrak{h}} \cap \langle \text{im}(\mu^*) \rangle) = \mathcal{O}_{\mu^{-1}(0)/H}$, it becomes an isomorphism.

Since ψ is obviously left G -invariant then it remains to show that the induced \mathfrak{g} -quasi-Poisson structure on M is $\pi_{\rho_{|\mathfrak{t}^*} + r_{\mathfrak{t}}^{\mathfrak{m}}}$. Let $t, t' \in \mathfrak{t} \subset \mathcal{O}_{\mathfrak{t}^*}$ and $f, g \in \mathcal{O}_G$. First of all we have

$$\{\psi^* t, \psi^* t'\}_X = \{t, t'\}_X = [t, t'] = \psi^* [t, t'],$$

hence $\{t, t'\}_M = [t, t']$. Then

$$\{\psi^* t, \psi^* f\}_X = \{t, f(yx)\}_X = \overrightarrow{t}^H \cdot (f(yx)) = (\overrightarrow{t} \cdot f)(yx) = \psi^* (\overrightarrow{t} \cdot f).$$

The third equality follows from the left H -invariance of \overrightarrow{t}^H . Thus $\{t, f\}_M = \overrightarrow{t} \cdot f$. Finally

$$\begin{aligned} \{\psi^* f, \psi^* g\}_X(\tau, x, \lambda, y) &= \overrightarrow{r_{\mathfrak{t}}^{\mathfrak{m}}(\tau)}^H \cdot (f(yx), g(yx)) + \overrightarrow{\rho(\lambda)}^G \cdot (f(yx), g(yx)) \\ &= \overrightarrow{(r_{\mathfrak{t}}^{\mathfrak{m}}(\tau) \cdot (f, g))}(yx) + \overrightarrow{(\rho(\text{Ad}_x^* \lambda) \cdot (f, g))}(yx). \end{aligned}$$

Therefore, when restricting to $\mu_2^{-1}(0)$ one obtains

$$\begin{aligned} \{\psi^* f, \psi^* g\}_X(\tau, x, \text{Ad}_{x^{-1}}^*(p^* \tau), y) &= \overrightarrow{(r_{\mathfrak{t}}^{\mathfrak{m}}(\tau) \cdot (f, g))}(yx) + \overrightarrow{(\rho(p^* \tau) \cdot (f, g))}(yx) \\ &= \psi^* (\overrightarrow{(r_{\mathfrak{t}}^{\mathfrak{m}} + \rho_{|\mathfrak{t}^*}) \cdot (f, g)}). \end{aligned}$$

Therefore $\{f, g\}_M = \overrightarrow{(r_{\mathfrak{t}}^{\mathfrak{m}} + \rho_{|\mathfrak{t}^*}) \cdot (f, g)}$. This ends the proof of Proposition 0.3. \square

4.2. Quantization of the momentum map μ . Let us first consider $(V \times H, \pi_{r_t^m})$. There is a momentum map

$$\begin{aligned} \nu : V \times H &\longrightarrow \mathfrak{h}^* \\ (\tau, x) &\longmapsto -\text{Ad}_{x^{-1}}(p^*\tau) \end{aligned}$$

with corresponding right H -action on $V \times H$ given by $(\tau, x) \cdot q = (\tau, q^{-1}x)$.

Like in subsection 2.3 one has a T -invariant and H -invariant torsion free connexion on $V \times H$, therefore from Proposition 2.5 the corresponding Fedosov star-product $*$ is both strongly \mathfrak{h} -invariant and strongly \mathfrak{t} -invariant⁵.

Then Proposition 1.7 tells us that there exists a strongly \mathfrak{t} -invariant (and H -invariant) equivalence Q such that $*' := *^{(Q)}$ is a compatible quantization of $\pi_{r_t^m}$. Consequently we can define the following algebra morphism:

$$\mathbf{N} := Q^{-1} \circ U(\nu^*) \circ \text{sym} : (\mathcal{O}_{\mathfrak{h}^*}[[\hbar]], *_PBW) \rightarrow (\mathcal{O}_{V \times H}[[\hbar]], *').$$

It is obviously a quantization of the Poisson map ν and, moreover, for any $h \in \mathfrak{h}$ and any $f \in \mathcal{O}_{V \times H}$ one has

$$[\mathbf{N}(h), f]_{*'} = Q^{-1}([\nu^*h, Q(f)]_*) = Q^{-1}(\hbar\{\nu^*h, Q(f)\}) = \hbar\{\nu^*h, f\}.$$

In other words, \mathbf{N} is a quantum momentum map quantizing ν .

Let us now assume that we know a dynamical twist quantization $J(\lambda) : U \rightarrow \otimes^2 U\mathfrak{g}[[\hbar]]$ of $\rho(\lambda)$ (with some associator Φ) and denote by $*_J$ the corresponding compatible quantization of π_ρ on $U \times G$. Together with $*'$ it induces a quantization $*'_J$ of $\pi_{r_t^m} + \pi_\rho$ on X (with the same Φ).

Remark 4.2. Actually $*'_J$ is the compatible quantization corresponding to the dynamical twist quantization $\mathbf{J}(\tau, \lambda) := J_t^m(\tau)J(\lambda) : (\mathfrak{t} \oplus \mathfrak{h})^* \supset V \times U \longrightarrow \otimes^2 U(\mathfrak{h} \oplus \mathfrak{g})[[\hbar]]$ of the dynamical r -matrix $\mathbf{r}(\tau, \lambda) := r_t^m(\tau) + \rho(\lambda) : V \times U \longrightarrow \wedge^2(\mathfrak{h} \oplus \mathfrak{g})$. Here J_t^m is the dynamical twist quantizing r_t^m .

For any $f \in \mathcal{O}_{\mathfrak{h}^*}$ we define $\mathbf{M}(f) := (Q \otimes \text{inc}) \circ \Delta(f) \in (\mathcal{O}_{V \times H} \otimes \mathcal{O}_{U \times G})[[\hbar]] = \mathcal{O}_X[[\hbar]]$. Here $\text{inc} : \mathcal{O}_{\mathfrak{h}^*} \hookrightarrow \mathcal{O}_{U \times G}$ is the natural inclusion and $\Delta : \mathcal{O}_{\mathfrak{h}^*} \rightarrow \mathcal{O}_{\mathfrak{h}^*} \otimes \mathcal{O}_{\mathfrak{h}^*} = \mathcal{O}_{\mathfrak{h}^* \times \mathfrak{h}^*}$ is defined by $\Delta(f)(\lambda_1, \lambda_2) = f(\lambda_1 + \lambda_2)$.

Proposition 4.3. *The algebra morphism*

$$\mathbf{M} : (\mathcal{O}_{\mathfrak{h}^*}[[\hbar]], *_PBW) \longrightarrow (\mathcal{O}_X[[\hbar]], *'_J)$$

is a quantum momentum map quantizing μ .

4.3. Quantization of the composition formula (proof of Theorem 0.4). Let us assume that J is a dynamical twist quantization of ρ and keep the notations of the previous subsection.

Denote by \mathcal{I} the right ideal generated by $\text{im}(\mathbf{M})$ in $(\mathcal{O}_X[[\hbar]], *'_J)$ and consider the reduced algebra $\mathcal{A} := \mathcal{O}_X^{\mathfrak{h}}[[\hbar]]/\mathcal{O}_X^{\mathfrak{h}}[[\hbar]] \cap \mathcal{I}$. Let $\Psi = \psi^* + O(\hbar)$ be the composition of $\psi^* : \mathcal{O}_M[[\hbar]] \rightarrow \mathcal{O}_X^{\mathfrak{h}}[[\hbar]]$ with the projection $\mathcal{O}_X^{\mathfrak{h}}[[\hbar]] \rightarrow \mathcal{A} \cong \mathcal{O}_{\mu^{-1}(0)/H}[[\hbar]]$. It is obviously bijective and G -invariant (since ψ^* is), therefore it defines a quantization $\tilde{*}$ of the quasi-Poisson structure $\pi_{r_t^m + \rho|_{\mathfrak{t}^*}}$. We end the proof of Theorem 0.4 using the following proposition:

Proposition 4.4. *$\tilde{*}$ is a compatible quantization.*

Proof. First of all for any $u, v \in \mathcal{O}_{\mathfrak{t}^*}$ one has

$$(\psi^*u) *'_J (\psi^*v) = u *'_J v = u *_PBW v = \psi^*(u *_PBW v).$$

Consequently $u\tilde{*}v = \Psi^{-1}(\Psi(u) \cdot_{\mathcal{A}} \Psi(v)) = u *_PBW v$.

⁵Remind that we also have a momentum map $V \times H \rightarrow \mathfrak{t}^*$; $(\tau, x) \mapsto \tau$ with corresponding right T -action given by $(\tau, x) \cdot b = (\text{Ad}_{\mathfrak{h}}^*(\tau), xb)$.

Then let $u \in \mathcal{O}_{t^*}$ and $f \in \mathcal{O}_G$. On one hand

$$(\psi^* f) *_J' \psi^* u = (f(yx)) *_J' u = f(yx)u = \psi^*(fu)$$

and thus $f\tilde{*}u = fu$. On the other hand for $u = t^n$ ($t \in \mathfrak{t}$) one has

$$\begin{aligned} \psi^*(t^n) *_J' (\psi^* f) &= (t^n) *_J' (f(yx)) = \sum_{k=0}^n \hbar^k C_n^k((\vec{t}^H)^k \cdot (f(yx))) t^{n-k} \\ &= \sum_{k=0}^n \hbar^k C_n^k(\vec{t}^k \cdot f)(yx) t^{n-k} = \psi^* \left(\sum_{k=0}^n \hbar^k C_n^k(\vec{t}^k \cdot f) t^{n-k} \right). \end{aligned}$$

Therefore $t^n \tilde{*} f = \sum_{k=0}^n \hbar^k C_n^k(\vec{t}^k \cdot f) t^{n-k}$. The proposition is proved. \square

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