Cubic forms and complex 3-folds

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Introduction

Nowadays, complex or algebraic manifolds are classified by Kodaira dimension. This classification is natural and fruitful, but in the complex case another point of view is possible. In this approach one starts with a topological or differentiable manifold X and asks for all complex or algebraic structures on X. Though this more traditional way of thinking can't replace the classification by Kodaira dimension, it remains useful and attractive and it has led to a number of wellknown if not famous problems. It suffices to recall Severi's problem: find all complex structures on $\mathbb{P}^2$, considered as a topological 4-manifold, or the same question asked for $S^2 \times S^2$ seen either as a topological or a differentiable manifold. For complex dimension 2 the work of Freedman on the topology of 4-folds as well as the work of Donaldson and many of his followers of course put this point of view very much at the centre of attention [O/V], [F/M].

In the past decades progress on the Kodaira classification for dimension 3 has been enormous ([Mo], [K/M/M], [Kol]), but the same can't be said about the relations between the topological and differentiable structures of 6-manifolds and the complex or algebraic structures they admit.

Let us restrict ourselves to the simplest case, the case of compact, oriented, simply-connected 6-manifolds without torsion. Their topological classification was carried out by Wall and Jupp ([W], [J]), who also determined which of them admit a differentiable structure, and for these showed that the differentiable classification coincides with the topological classification. This does not hold for the homotopy classification; in many cases there are even infinitely many homeomorphism classes of one and the same homotopy type. Apart of course from Stiefel-Whitney classes, Pontrjagin class and triangulation class the essential invariant is the cup form $H^2(X,\mathbb{Z}) \times H^2(X,\mathbb{Z}) \times H^2(X,\mathbb{Z}) \to H^6(X,\mathbb{Z})$ ($\cong \mathbb{Z}$). It is not difficult to characterize those forms which arise as cup forms of a 6-fold in question (below), but it remains very difficult to classify cubic forms up to $GL(\mathbb{Z})$-equivalence. Relatively few results are known in this direction, even for the lowest ranks.

The corresponding 4-folds are the simply connected ones, i.e. the 4-folds occuring in the work of Freedman and most of the papers of the Donaldson school. Here the crucial invariant is a unimodular form on $H^2(X,\mathbb{Z})$, namely the cup form $H^2(X,\mathbb{Z}) \times H^2(X,\mathbb{Z}) \to H^4(X,\mathbb{Z})$. For differentiable manifolds this form completely determines the homeomorphism type (this also holds in the topological case if the cupform is even, whereas for odd forms there are two homeomorphism types), but by no means for the diffeomorphism type. So considering the relation between the homotopy, the topological and the differential classification there is a big difference between dimensions 4 and 6. The next question: which topological 4-folds carry a complex structure, is equivalent to asking which unimodular, $\mathbb{Z}$-valued symmetric bilinear forms are realisable by complex or algebraic surfaces. It is related to the well-known inequality $c_1^2 \leq 3c_2$ and has been solved to a considerable extent.

Though in the case of 6-folds the corresponding question about the realisability of cubic forms is definitely weaker than the question which 6-folds carry a complex or algebraic structure, it still remains of much interest. In the second half of this paper we say something about algebra and arithmetic of cubic forms and consider the apparently largely untouched question of the realisability of **complex** forms by complex manifolds. A part from a considerable number of examples some conditions for Kähler manifolds are given. And to show how few 6-folds of the type in question actually carry Kähler structures, we add a theorem about Kähler structures on the set of 6-folds with $b_2 = 1, b_3 \leq$ constant and $w_2 \neq 0$.

The first part of this paper surveys the results of Wall and Jupp referred to before, and deals with the homotopy classification. By putting together (for the first time?) all this in a rather systematic way we hope to contribute to the knowledge of complex 3-folds from a topological point of view.

Acknowledgements: We would like to thank the following mathematicians for very helpful remarks and suggestions: F. Grunewald, G. Harder, F. Hirzebruch, and R. Schulze-Pillot. We also want to acknowledge support by the Science project "Geometry of Algebraic Varieties" SCI-0398-C(A), by the Max-Planck-Institut für Mathematik in Bonn, and by the Schweizer Nationalfond (Nr. 21-36111.92).

1. Topological classification of certain 6-manifolds

The topological classification of 1-connected, closed, oriented, 6-dimensional manifolds has been developed in a sequence of papers by C.T.C. Wall [W], P. Jupp [J], and A. Žubr [Z1], [Z2], [Z3]. Roughly speaking, their main result is that the topological classification of these 6-manifolds is equivalent to the arithmetic classification of certain systems of invariants naturally associated with them.
The aim of this section is to review these results and to reformulate the arithmetic classification problem in a way which makes it accessible to further investigation.

1.1 Homeomorphism types and C^∞-structures

Let X be a closed, oriented, 6-dimensional topological manifold; we assume that X is 1-connected with torsion-free homology. The **basic invariants** of X are [J]:

i) $H^2(X,\mathbb{Z})$, a finitely generated free abelian group;

ii) $b_3(X) = rk_{\mathbb{Z}} H^3(X,\mathbb{Z})$, a natural number which is even since $H^3(X, \mathbb{Z})$ admits a non-degenerate symplectic form;

iii) $F_X : H^2(X,\mathbb{Z}) \otimes H^2(X,\mathbb{Z}) \otimes H^2(X,\mathbb{Z}) \to \mathbb{Z}$, a symmetric trilinear form given by the cup-product evaluated on the orientation class;

iv) $p_1(X) \in H^4(X,\mathbb{Z})$, the first Pontrjagin class which is always integral because the inclusion of BO in $BTOP$ induces an isomorphism $H^4(BTOP,\mathbb{Z}) \to H^4(BO,\mathbb{Z})[J]$;

v) $w_2(X) \in H^2(X,\mathbb{Z}_{/2})$, the second Stiefel-Whitney class; $w_2(X)$ is determined by the Steenrod square $Sq^2 : H^4(X,\mathbb{Z}_{/2}) \to H^6(X,\mathbb{Z}_{/2})$, $Sq^2(\xi) = w_2(X)\cdot\xi \quad \forall\, \xi \in H^4(X,\mathbb{Z}_{/2})[W]$;

vi) $\tau(X) \in H^4(X,\mathbb{Z}_{/2})$, the triangulation class which is the obstruction to lifting the stable tangent bundle of Y to a PL bundle [J].

These invariants satisfy one **fundamental relation**

$$(*) \qquad W^3 \equiv (p_1(X) + T)\cdot W (\text{mod } 48)$$

for all integral classes $W \in H^2(X,\mathbb{Z}), T \in H^4(X,\mathbb{Z})$ with $\overline{W} \equiv w_2(X)(\text{mod } 2), \overline{T} \equiv \tau(x)(\text{mod } 2)$.

For smooth manifolds $(*)$ is simply the $\hat{A}$-integrality theorem of A. Borel and F. Hirzebruch [B/H], whereas for topological manifolds additional surgery arguments are necessary [J].
In the sequel we shall use Poincaré duality to identify $H^4(X,\mathbb{Z})$ with $\mathrm{Hom}_{\mathbb{Z}}(H^2(X,\mathbb{Z}),\mathbb{Z})$, so that $p_1(X)$ can be considered as a linear form on $H^2(X,\mathbb{Z})$, and we will write $x\cdot y\cdot z$ instead of $F_X(x\otimes y\otimes z)$ for elements $x,y,z\in H^2(X,\mathbb{Z})$.

Definition 1: *A* **system** *of* **invariants** *is a 6-tuple (r,H,w,τ,F,p) consisting of a non-negative integer r, a finitely generated free abelian group H, elements $w\in H/_{2H}$ and $\tau\in H^\vee/_{2H^\vee}$, a symmetric trilinear form $F\in S^3H^\vee$, and a linear form $p\in H^\vee$. The system (H,r,w,τ,F,p) is* **admissible** *iff for every $W\in H$ and $T\in H^\vee$ with $\overline{W}\equiv w(\mathrm{mod}\ 2)$ and $\overline{T}\equiv\tau(\mathrm{mod}\ 2)$ the follwing congruence holds:*

$$(*)\qquad W^3\equiv(p+24T)(W)(\mathrm{mod}\ 48).$$

Two systems of invariants (H,r,w,τ,F,p) and (H',r',w',τ',F',p') are **equivalent** *iff $r=r'$, and there exists an isomorphism $\alpha:H\to H'$ such that:*

$$\alpha(w)=w'\ ,\ \alpha^*(\tau')=\tau\ ,\ \alpha^*(F')=F\ ,\ \alpha^*(p')=p.$$

The main classification result can now be formulated in the following way:

Theorem 1 (Jupp): *The assignment $X\mapsto(\frac{b_3(X)}{2},H^2(X,\mathbb{Z}),w_2(X),\tau(X),F_X,p_1(X))$ induces a 1-1 correspondence between oriented homeomorphism classes of 1-connected, closed, oriented, 6-dimensional topological manifolds with torsion-free homology, and equivalence classes of admissible systems of invariants.*
Furthermore, a topological manifold X as above admits a C^∞-structure if and only if the triangulation class $\tau(X)$ vanishes; the C^∞-structure is then unique.

Remark 1: The classification theorem is due to C.T.C. Wall in the special case of differentiable spin-manifolds [W]; the final form above was obtained by P. Jupp [J].
A. Žubr generalized Wall's result in another direction; he proved a classification theorem for 1-connected, smooth spin-manifolds with not necessarily torsion-free homology [Z1]; in two further papers [Z2], [Z3] he also obtains P. Jupp's classification, and he asserts in addition, that algebraic isomorphisms of systems of invariants can always be realized by orientation preserving homeomorphisms (diffeomorphisms in the smooth case).

Note that the first invariant $\frac{b_3(X)}{2}$ of the system is completely independent of the remaining invariants, so that the following splitting theorem holds:

Corollary 1: *Every 1-connected, closed, oriented, 6-dimensional, topological (differentiable) manifold X with torsion-free homology admits a topological (differentiable) splitting $X = X_0 \sharp \frac{b_3(X)}{2}(S^3 \times S^3)$ as a connected sum of a core X_0 with $b_3(X_0) = 0$, and $\frac{b_3(X)}{2}$ copies of $S^3 \times S^3$. The oriented homeomorphism (diffeomorphism) type of X_0 is unique.*

Example 1: The 1-connected, closed, oriented 6-manifolds X with $H_2(X, \mathbb{Z}) = 0$ are S^6 and the connected sums $\sharp_r S^3 \times S^3$ of $r \geq 1$ copies of $S^3 \times S^3$[Sm].

1.2 Homotopy types

In order to describe the homotopy classification of the 6-manifolds above, we need some more preparations.
Let (H, F) be a pair consisting of a finitely generated free abelian group H, and a symmetric trilinear form F; consider the following subgroup of $H^\vee/_{48H^\vee}$:

$$U_F := \{l \in H^\vee/_{48H^\vee} | \exists u \in H \text{ with } l(x) \equiv 24u^2 \cdot x (\text{mod } 48) \ \ \forall x \in H\}.$$

If (H', F') is another such pair, and $\alpha : H \to H'$ an isomorphism with $\alpha^*(F') = F$, then there is an induced isomorphism

$$\alpha^* : H'^\vee/_{48H'^\vee}/_{U'_F} \to H^\vee/_{48H^\vee}/_{U_F}$$

of the quotients. Denote the class of a linear form $l \in H^\vee$ in the quotient $H^\vee/_{48H^\vee}/_{U_F}$ by $[l]$.

Definition 2: *Two systems of invariants (r, H, w, τ, F, p) and $(r', H', w', \tau', F', p')$ are* **weakly equivalent** *iff $r = r'$, and there exists an isomorphism $\alpha : H \to H'$ sucht that:*
$\alpha(w) = w', \alpha^*(F') = F$, and $\alpha^*[p' + 24T'] = [p + 24T]$ for all $T \in H^\vee, T' \in H'^\vee$ with $\overline{T} \equiv \tau (\text{mod } 2), \overline{T'} \equiv \tau' (\text{mod } 2)$.

With this definition we can phrase the homotopy classification in the following way:

Theorem 2 *(Žubr): The assignment $X \to \left(\frac{b_3(X)}{2}, H^2(X, \mathbb{Z}), w_2(X), \tau(X),\right.$ $\left. F_X, p_1(X)\right)$ induces a 1-1 correspondence between oriented homotopy classes of 1-connected, closed, oriented, 6-dimensional topological manifolds* *with torsion-free homology and weak equivalence classes of admissible systems of invariants.*

Remark 2: Žubr's theorem corrects and generalizes the homotopy classification in the papers by Wall [W] and Jupp [J]; he also treats manifolds with not necessarily torsion-free homology, and states without proof that algebraic isomorphisms of weak equivalence classes of systems of invariants are always realizable by orientation preserving homotopy equivalences [Z3].

Example 2: Manifolds with $b_2(X) = 1$.

Let X be a 1-connected, closed oriented, 6-dimensional manifold with $H_2(X,\mathbb{Z}) \cong \mathbb{Z}$. Splitting off possible copies of $S^3 \times S^3$ we may assume $b_3(X) = 0$. Choosing a $\mathbb{Z}$-basis of $H^2(X,\mathbb{Z})$ we see that systems of invariants can be identified with 4-tuples $(\overline{W}, \overline{T}, d, p) \in \mathbb{Z}_{/2} \times \mathbb{Z}_{/2} \times \mathbb{Z} \times \mathbb{Z}$ where the 'degree' d corresponds to the cubic form. Such a 4-tuple is admissible iff $d(2x+W)^3 \equiv (p+24T)\cdot(2x+W)(\text{mod } 48)$ holds for every integer x. This is equivalent to $p \equiv 4d(\text{mod } 24)$ if $\overline{W} = 0$, and to $p \equiv d + 24T(\text{mod } 48)$ with $d \equiv 0(\text{mod } 2)$ if $\overline{W} \neq 0$.

Two admissible 4-tuples $(\overline{W}, \overline{T}, d, p)$ and $(\overline{W}', \overline{T}', d', p')$ are equivalent iff $\overline{W}' = \overline{W}, \overline{T}' = \overline{T}$ and $(d', p') = \pm(d, p)$. Taking the degree d non-negative, we find:

Proposition 1: *There is a 1-1 correspondence between oriented homeomorphism types of cores X_0 with $b_2(X_0) = 1$, and 4-tuples $(\overline{W}, \overline{T}, d, p)$, normalized so that $d \geq 0$, and $p \geq 0$ if $d = 0$, which satisfy $p \equiv 4d(\text{mod } 24)$ if $\overline{W} = 0$, and $d \equiv 0(\text{mod } 2), p \equiv d + 24T(\text{mod } 48)$ if $\overline{W} \neq 0$.*

In order to classify the associated homotopy types we first have to determine the subgroup U_F associated to a given cubic form F. By definition we find $U_F = 0$ if $d \equiv 0(\text{mod } 2), U_F = \mathbb{Z}_{/2}$ if $d \equiv 1(\text{mod } 2)$. Two normalized 4-tuples $(\overline{W}, \overline{T}, d, p)$ and $(\overline{W}', \overline{T}', d', p')$ are weakly equivalent iff $d' = d, \overline{W'} = \overline{W}$, and $p + 24T \equiv p' + 24T'(\text{mod } 48)$ if $d \equiv 0(\text{mod } 2), p \equiv p'(\text{mod } 24)$ if $d \equiv 1(\text{mod } 2)$.

Putting everything together, we find a single oriented homotopy type for every odd degree $d \geq 0$, which is necessarily spin, and 3 oriented homotopy types for every even degree $d \geq 0$; one of these 3 types has $\overline{W} \neq 0$, the other two are spin, and they are distinguished by $p + 24T(\text{mod } 48)$ i.e. $p \equiv 4d(\text{mod } 48)$, or $p \equiv 4d + 24(\text{mod } 48)$.

2. Realization of cubic forms

In the previous section the (homotopy) topological classification of 1-connected, closed, oriented, 6-dimensional manifolds with torsion-free homology has been transformed into an arithmetical moduli problem: to describe the sets of (weak) equivalence classes of admissible systems of invariants. In this section we begin to investigate the latter problem; we give a simple criterion for the realizability of cubic forms by smooth manifolds, and we describe, at least in principle, the classification of homotopy types of manifolds with a given cohomology ring.

2.1 Cohomology rings of 6-manifolds

Let (r, H, w, τ, F, p) be a system of invariants as in section 1; recall that it is admissible iff for every $W \in H, T \in H^\vee$ with $\overline{W} = w (\mathrm{mod}\ 2), \overline{T} \equiv \tau (\mathrm{mod}\ 2)$ the following congruence holds:
$(*)\qquad W^3 \equiv (p + 24T)(W)\ (\mathrm{mod}\ 48).$

Lemma 1: *(r, H, w, τ, F, p) is admissible if and only if there exist $W_\circ \in H, T_\circ \in H^\vee$ with $\overline{W}_\circ \equiv w(\mathrm{mod}\ 2), \overline{T}_\circ \equiv \tau(\mathrm{mod}\ 2)$, such that*

i) $W_\circ^3 \equiv (p + 24T_\circ)(W_\circ)(\mathrm{mod}\ 48)$

ii) $p(x) \equiv 4x^3 + 6x^2W_\circ + 3xW_\circ^2(\mathrm{mod}\ 24)\ \forall\ x \in H.$

Proof: Obvious since the set of integral lifts of w is a coset $W_\circ + 2H$.

Definition 3: *Let $F \in S^3H^\vee$ be a symmetric trilinear form on a finitely generated free abelian group H. An element $W \in H$ is* **characteristic** *for F iff*
$(**)\qquad x \cdot y \cdot (x + y + W) \equiv 0(\mathrm{mod}\ 2)\ \forall\ x, y \in H.$

Lemma 2: *$W \in H$ is a characteristic element for $F \in S^3H^\vee$ if and only if the function $l_W : H \to \mathbb{Z}, l_W(x) := 4x^3 + 6x^2W + 3xW^2$ is linear in x modulo 24.*

Proof: $l_W(x + y) = l_W(x) + l_W(y) + 12(x^2y + xy^2 + xyW)$, whence the assertion.

The existence of characteristic elements is a necessary and sufficient condition for a cubic form $F \in S^3H^\vee$ to be realizable by a manifold. In fact, we have:

Proposition 2: *A given cubic form $F \in S^3H^\vee$ on a finitely generated free abelian group H is realizable as cup-form of a 1-connected, closed, oriented, 6-dimensional manifold with torsion-free homology if and only if it possesses a characteristic element.*

Proof: If (r, H, w, τ, F, p) is an admissible system of invariants, and $W_\circ \in H$ any integral lift of w, then we have
$p(x) \equiv 4x^3 + 6x^2W_\circ + 3xW_\circ^2 (\text{mod } 24)\ \forall x \in H$. i.e. the function $l_{W_\circ} : H \rightarrow \mathbb{Z}$ is linear modulo 24, and $W_\circ$ is therefore characteristic for F. Conversely, suppose $W_\circ \in H$ is a characteristic element for a cubic form $F \in S^3H^\vee$; let $w := \overline{W}_\circ (\text{mod } 2), r := 0$.

By the main lemma we have to construct linear forms $p, T \in H^\vee$, such that

i) $W_\circ^3 \equiv (p + 24T)(W_\circ)(\text{mod } 48)$

ii) $p(x) \equiv 4x^3 + 6x^2W_\circ + 3xW_\circ^2 (\text{mod } 24)\ \forall\ x \in H$.

The function $l_{W_\circ} : H \rightarrow \mathbb{Z}, l_{W_\circ}(x) = 4x^3 + 6x^2W_\circ + 3xW_\circ^2$ is linear modulo 24 since $W_\circ$ is a characteristic element for F; we therefore choose a linear form $p_\circ \in H^\vee$ with $p_\circ(x) \equiv l_{W_\circ}(x)(\text{mod } 24)\ \forall x \in H$. Substituting $x = W_\circ$ we find $p_\circ(W_\circ) \equiv 13W_\circ^3(\text{mod } 24)$; but since $W_\circ$ is characteristic we have $W_\circ^3 \equiv 0(\text{mod } 2)$, thus $p_\circ(W_\circ) \equiv W_\circ^3(\text{mod } 24)$. Write $p_\circ(W_\circ) = W_\circ^3 + 24k$ for some $k \in \mathbb{Z}$.

case 1) $k \equiv 0(\text{mod } 2)$: define $p := p_0, T := 0$.

case 2) $k \equiv 1(\text{mod } 2)$: we must find a linear form $T_\circ \in H^\vee$ with $T_\circ(W_\circ) \equiv 1(\text{mod } 2)$; clearly this can be done if and only if $W_\circ$ is not divisible by 2. If $W_\circ$ were divisible by 2, $W_\circ = 2V_\circ$ for some $V_\circ \in H$, then $2p_\circ(V_\circ) = p_\circ(W_\circ) = W_\circ^3 + 24k = 8V_\circ^3 + 24k$ would give $p_\circ(V_\circ) = 4V_\circ^3 + 12k$; then, using $p_\circ(V_\circ) \equiv 4V_\circ^3 + 6V_\circ^2W_\circ + 3V_\circ W_\circ^2 \equiv 4V_\circ^3(\text{mod } 24)$ we would find $k \equiv 0(\text{mod } 2)$, which is not the case by assumption.

This shows that $F \in S^3H^\vee$ is realizable by a topological manifold with Pontrjagin class $p_\circ$ and non-vanishing triangulation obstruction $\tau_\circ := \overline{T}_\circ$ (mod 2). In order to realize F by a smooth manifold, one can take $p := p_\circ + 24T_\circ$, and $\tau := \circ$.

Remark 3: The topological counterpart of the existence of a characteristic element for a given cubic form $F \in S^3H^\vee$ is the existence of a mod-2 Steenrod-algebra structure, which is a necessary condition for a ring to be a cohomology ring.

The existence and the classification of characteristic elements for a given cubic form is essentially a linear algebra problem over $\mathbb{Z}/_2$. To see this, let

$F \in S^3H^\vee$ be a fixed cubic form on a finitely generated free abelian group H. Associated with F we have a linear map
$F^t : H \to S^2H^\vee$ sending an element $h \in H$ to the bilinear form $F^t(h) : H \otimes H \to \mathbb{Z}, (x,y) \to x \cdot y \cdot h$. Let $\overline{H} := H/_{2H}, \overline{F} \in S^3\overline{H}^\vee$ be the reductions of H and F modulo 2, and let $- : H \to \overline{H}$ be the natural epimorphism. The symmetric trilinear form $\overline{F}$ on the $\mathbb{Z}/_2$-module $\overline{H}$ defines a natural symmetric bilinear form $q_{\overline{F}} \in S^2\overline{H}^\vee$ given by $q_{\overline{F}}(\overline{x},\overline{y}) := \overline{x} \cdot \overline{y} \cdot (\overline{x} + \overline{y})$.

Lemma 3: *$F \in S^3H^\vee$ admits characteristic elements if and only if $q_{\overline{F}}$ lies in the image of $\overline{F}^t \in Hom_{\mathbb{Z}}(H, S^2\overline{H}^\vee)$. The set of all characteristic elements for F is a coset of the form $W_\circ +$ Ker$(\overline{F}^t)$.*
Proof: $W_\circ$ is characteristic for F if and only if $q_{\overline{F}} = \overline{F}^t(W_\circ)$.

In terms of a $\mathbb{Z}$-basis $\{e_1, \ldots, e_b\}$ for H the condition $q_{\overline{F}} \in \mathrm{Im}(\overline{F}^t)$ translates into a simple rank condition over $\mathbb{Z}/_2$: the $\mathbb{Z}/_2$-rank of the $b \times \binom{b+1}{2}$-matrix A representing $\overline{F}^t$ must be equal to the $\mathbb{Z}/_2$-rank of the matrix A extended by the column $(\overline{e}_i \cdot \overline{e}_j \cdot (\overline{e}_i + \overline{e}_j))_{1 \le i \le j \le b}$

Example 3: Let $H = \mathbb{Z}e_1 \oplus \mathbb{Z}_{e_2}$ be free of rank 2, $F \in S^3H^\vee$ given by $e_1^3 = a, e_1^2e_2 = b, e_1e_2^2 = c, e_2^3 = d$ with $a, b, c, d \in \mathbb{Z}$. The rank condition becomes

$$rk_2 \begin{bmatrix} \overline{a} & \overline{b} \\ \overline{c} & \overline{d} \\ \overline{b} & \overline{c} \end{bmatrix} = rk_2 \begin{bmatrix} \overline{a} & \overline{b} & \overline{0} \\ \overline{c} & \overline{d} & \overline{0} \\ \overline{b} & \overline{c} & \overline{b+c} \end{bmatrix}$$

2.2 Homotopy types with a given cohomology ring

Our next task is to describe the set of oriented homotopy types of 1-connected, closed, oriented, 6-dimensional manifolds with a fixed torsion-free cohomology ring.
From Žubr's classification theorem we know that in algebraic terms this means the following: fix a non-negative integer $r_\circ$, a finitely generated free abelian group $H_\circ$, and a symmetric trilinear form $F_\circ \in S^3H_\circ^\vee$ which admits characteristic elements.
Let $\mathcal{M}(r_\circ, H_\circ, F_\circ)$ be the set of 1-connected, closed, oriented, 6-dimensional manifolds X with $b_3(X) = 2r_\circ$, such that there exists an isomorphism $\alpha : H_\circ \to H^2(X, \mathbb{Z})$ with $\alpha^* F_X = F_\circ$. Denote by Aut$(F_\circ)$ the subgroup of $\mathbb{Z}$-isomorphisms of $H_\circ$ which leave $F_\circ \in S^3H_\circ^\vee$ invariant; Aut$(F_\circ)$ acts on pairs $(w, [l]) \in \overline{H}_\circ \times H_\circ^\vee/_{48H_\circ^\vee}/_{U_{F_\circ}}$ in a natural way:

$$\gamma \cdot (w, [l]) := (\gamma(w), (\gamma^{-1})^*[l]).$$

Let ${}_{\mathrm{Aut}(F_\circ)}\backslash \overline{H}_\circ \times H_\circ^\vee/_{48H_\circ^\vee}/_{U_{F_\circ}}$ be the set of $\mathrm{Aut}(F_\circ)$-orbits.
A manifold X in $\mathcal{M}(r_\circ, H_\circ, F_\circ)$ and an isomorphism $\alpha : H_\circ \to H^2(X, \mathbb{Z})$ with $\alpha^* F_X = F_\circ$ yields a well-defined $\mathrm{Aut}(F_\circ)$-orbit: $(\alpha^{-1}(w_2(X)), \alpha^*[p_1(X)+24T])$ (modulo $\mathrm{Aut}(F_\circ)$), where $T \in H^4(X, \mathbb{Z})$ is an arbitrary integral lifting of $\tau(X) \in H^4(X, \mathbb{Z}_{/2})$.
The set of oreinted homotopy types $\mathcal{M}(r_\circ, H_\circ. F_\circ)/_\simeq$ of manifolds in $\mathcal{M}(r_\circ, H_\circ, F_\circ)$ can now be described in the following way:

Proposition 3: *The assignment* $X \mapsto (\alpha^{-1}(w_2(X)), \alpha^*[p_1(X)+24T])$ *(modulo* $\mathrm{Aut}(F_\circ)$*) defines an injection*
$I : \mathcal{M}(r_\circ, H_\circ, F_\circ)_\simeq \to {}_{\mathrm{Aut}(F_\circ)} \backslash \overline{H}_\circ \times H_\circ^\vee/_{48H_\circ^\vee}/_{U_{F_\circ}}$.

Proof: Suppose X and X' are manifolds in $\mathcal{M}(r_\circ, H_\circ, F_\circ), \alpha : H_\circ \to H^2(X, \mathbb{Z})$ and $\alpha' : H_\circ \to H^2(X', \mathbb{Z})$ isomorphisms with $\alpha^* F_X = F_\circ$ and $(\alpha')^* F_{X'} = F_\circ$. X and X' have the same image under I iff there exists an automorphism $\gamma \in \mathrm{Aut}(F_\circ)$ with $\gamma\alpha^{-1}(w_2(X)) = (\alpha')^{-1} w_2(X')$ and $(\gamma^{-1})^* \alpha^*[p_1(X) + 24T] = (\alpha')^*[p_1(X') + 24T']$. Consider $\beta := \alpha \circ \gamma \circ \alpha^{-1} : H^2(X, \mathbb{Z}) \to H^2(X', \mathbb{Z}); \beta$ is obviously an isomorphism with $\beta^* F_{X'} = F_X, \beta w_2(X) = w_2(X')$, and $\beta^*[p_1(X') + 24T'] = [p_1(X) + 24T]$; but this means that the systems of invariants associated with X and X' are weakly equivalent, and therefore X and X' oriented homotopy equivalent.

A complete description of the set $\mathcal{M}(r_\circ, H_\circ, F_\circ)/_\simeq$ i.e. of the image of I is only possible if the automorphism group $\mathrm{Aut}(F_\circ)$ is known; this can be a serious problem, but we will see that the 'general' automorphism group is finite (and usually small), so that the next proposition gives a reasonable estimate for the number of elements in $\mathcal{M}(r_\circ, H_\circ, F_\circ)/_\simeq$:

Proposition 4: *Fix* $r_\circ \in \mathbb{N}$, *a finitely generated free abelian group* $H_\circ$, *and a symmetric trilinear form* $F_\circ \in S^3 H_\circ^\vee$ *which admits characteristic elements. Set* $b := rk_{\mathbb{Z}} H_\circ, s := rk_{\mathbb{Z}_{/2}}(\overline{F}_\circ^t)$, *and let* $t := rk_{\mathbb{Z}_{/2}}(\cdot_{\overline{F}_\circ})$ *be the* $\mathbb{Z}_{/2}$*-rank of the* $\mathbb{Z}_{/2}$*-linear square map* $\cdot_{\overline{F}_\circ} : \overline{H}_\circ \to \overline{H}_\circ^\vee$ *sending* $\overline{u} \in \overline{H}_\circ$ *to* $\overline{u}^2 \in \overline{H}_\circ^\vee$. *Then* $\mathcal{M}(r_\circ, H_\circ, F_\circ)/_\simeq$ *contains at most* 2^{2b-s-t} *elements.*

Proof: Fix any admissible system of invariants $(r_\circ, H_\circ, w_\circ, \tau_\circ, F_\circ, p_\circ)$ for a manifold in $\mathcal{M}(r_\circ, H_\circ, F_\circ)$. Given $(r_\circ, H_\circ, F_\circ)$, we know from the last lemma that the possible elements $w_\circ$ form a coset of $\mathrm{Ker}(\overline{F}_\circ^t)$ in $\overline{H}_\circ$, so that there exist precisely 2^{b-s} such elements. It remains to count the classes

$[l] \in H_\circ^\vee/_{48H_\circ^\vee}/_{U_{F_\circ}}$, such that the $\mathrm{Aut}(F_\circ)$-orbit of $(w_\circ, [p_\circ + 24T_\circ + l])$ lies in the image of I.
To understand the latter condition we fix integral liftings $W_\circ, \in H_\circ, T_\circ \in H_\circ^\vee$ of $w_\circ$ and $\tau_\circ$ satisfying the admissibility conditions

i) $W_\circ^3 \equiv (p_\circ + 24T_\circ)(W_\circ)(\mathrm{mod}\ 48)$

ii) $p_\circ(x) \equiv 4x^3 + 6x^2W_\circ + 3xW_\circ^2(\mathrm{mod}\ 24)\ \forall x \in H_\circ$.

Clearly the $\mathrm{Aut}(F_\circ)$-orbit of $(w_\circ, [p_\circ + 24T_\circ + l])$ lies in the image of I if and only if

i') $W_\circ^3 \equiv (p_\circ + 24T_\circ + l)(W_\circ)(\mathrm{mod}\ 48)$,

ii') $(p_\circ + l)(x) \equiv 4x^3 + 6x^2W_\circ + 3xW_\circ^2(\mathrm{mod}\ 24)\ \forall x \in H_\circ$,

which is equivalent to $l(W_\circ) \equiv 0(\mathrm{mod}\ 48)$, and $l \equiv 0(\mathrm{mod}\ 24H_\circ^\vee)$ because of i) and ii).
Now, by definition of the subgroup $U_{F_\circ} \subset H_\circ^\vee/_{48H_\circ^\vee}$ we have the following commutative diagram with exact rows and columns:

$$
\begin{array}{ccccccccc}
 & & \mathrm{Ker}(\cdot_{\overline{F}_\circ}) & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 \to \mathrm{Ker}(24\cdot_{\overline{F}_\circ}) & \hookrightarrow & H_\circ/_{2H_\circ} & \xrightarrow{24\cdot_{\overline{F}_\circ}} & U_{F_\circ} & \to 0 & & & \\
 & & {}^{\cdot_{\overline{F}_\circ}}\downarrow & & \downarrow & & & & \\
 & 0 \to & H_\circ^\vee/_{2H_\circ^\vee} & \xrightarrow{24} & H_\circ^\vee/_{48H_\circ^\vee} & \to & H_\circ^\vee/_{24H_\circ^\vee} & \to 0 & \\
 & & \downarrow & & \downarrow & & \| & & \\
 & 0 \to & \mathrm{Coker}(\cdot_{\overline{F}_\circ}) & \to & H_\circ^\vee/_{48H_\circ^\vee}/_{U_{F_\circ}} & \to & H_\circ^\vee/_{24H_\circ^\vee} & \to 0 & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

The number of elements $[l] \in H_\circ^\vee/_{48H_\circ^\vee}/_{U_{F_\circ}}$ to be counted coincides therefore with the cardinality of the kernel of the map $ev(w_\circ)$: $\mathrm{Coker}(\cdot_{\overline{F}_\circ}) \to \mathbb{Z}/_2$ induced by evaluation in $w_\circ$. This is number is at most $2^{b-t}(2^{b-t-1}$ if $w_\circ \neq 0$ and $t \neq b)$.

Corollary 2: *If the $\mathbb{Z}/_2$-rank $s = rk_{\mathbb{Z}/_2}(\cdot_{\overline{F}_\circ})$ is maximal, then $\mathcal{M}(r_\circ, H_\circ, F_\circ)/_\simeq$ contains at most one class.*
Proof: Suppose $\cdot_{\overline{F}_\circ} : \overline{H}_\circ \to \overline{H}_\circ^\vee$ is surjective; then $\overline{F}_\circ^t : \overline{H}_\circ \to S^2\overline{H}_\circ^\vee$ must have a trivial kernel, since $\overline{h}\overline{x}^2 = 0$ for all $\overline{x} \in \overline{H}_\circ$ implies $\overline{h} = 0$ if every linear form is a square. But this means $s = t = b$, so that $\mathcal{M}(r_\circ, H_\circ, F_\circ)/_\simeq$ has at most one element.

Example 4: Let $H_\circ := \mathbb{Z}e_1 \oplus \mathbb{Z}e_2, e_1^3 = a, e_1^2e_2 = b, e_1e_2^2 = c, e_2^3 = d$. If $\bar{b} \equiv \bar{c}(mod2)$, and $\bar{a}\bar{d} - \bar{b}\bar{c} \equiv 1(\text{mod } 2)$, then $\mathcal{M}(r_\circ, H_\circ, F_\circ)/_\simeq$ contains precisely one class for every $r_\circ \geq 0$.

3. Algebra and arithmetic of cubic forms

Let H be a finitely generated free $\mathbb{Z}$-module of rank b. In this section we want to study algebraic and arithmetic properties of symmetric trilinear forms $F \in S^3H^\vee$ on H which admit characteristic elements; ultimately we would like to describe the classification of those forms under the action of the general linear group $GL(H)$, i.e. we like to investigate (part of) the quotient $S^3H^\vee/_{GL(H)}$.

From what we have said in sections 1. and 2., this is clearly equivalent to classifying the cohomology rings of 1-connected, closed, oriented, 6-dimensional manifolds without torsion, and with $b_2 = b, b_3 = 0$. Furthermore, up to finite indeterminancy, this is also equivalent to classifying the homotopy types of these manifolds.

The proper setting for this arithmetic moduli problem can be found in C. Seshadri's paper [S]; here we investigate only its set-theoretic aspects. Let $H_\mathbb{C} := H \otimes_\mathbb{Z} \mathbb{C}$ be the complexification of H, and let $S^3H_\mathbb{C}^\vee/_{SL(H_\mathbb{C})}$ be the GIT quotient of the reductive group $SL(H_\mathbb{C})$. We obtain a natural map $c : S^3H^\vee/_{SL(H)} \to S^3H_\mathbb{C}^\vee/_{SL(H_\mathbb{C})}$, which allows us to break up the problem into three parts: the description of the quotient $S^3H_\mathbb{C}^\vee/_{SL(H_\mathbb{C})}$, the investigation of the fibers of c, and the study of the remaining $\mathbb{Z}/_2$-action on $S^3H^\vee/_{SL(H)}$ which is induced by the choice of an arbitrary automorphism $A_\circ \in GL(H)$ of determinant $\det A_\circ = -1$.

3.1 Algebraic properties of cubic forms

Let $H_\mathbb{C} = H \otimes_\mathbb{Z} \mathbb{C}$ be as above, and denote by $\mathbb{C}[H_\mathbb{C}]_3$ the space of homogeneous polynomials of degree 3 on $H_\mathbb{C}$. There exists a linear polarization operator Pol $: \mathbb{C}[H_\mathbb{C}]_3 \to S^3H_\mathbb{C}^\vee$, sending a homogeneous cubic polynomial $f \in \mathbb{C}[H_\mathbb{C}]_3$ to the symmetric trilinear form $F = \text{Pol}(f) \in S^3H_\mathbb{C}^\vee$ which is related to f by the identity $F(h,h,h) = 6f(h)$. We will usually not distinguish between a cubic polynomial f and its associated form $F = Pol(f)$. On $S^3H_\mathbb{C}^\vee$ there exists a polynomial function $\Delta : S^3H_\mathbb{C}^\vee \to \mathbb{C}$, the discriminant, which is homogeneous of degree $b \cdot 2^{b-1}$, and vanishes in a form F if and only if the associated cubic hypersuface $(f)_o \subset \mathbb{P}(H_{\mathbb{C}})$ has a singular point; Δ can be defined over $\mathbb{Z}$ and is clearly invariant under the natural action of $GL(H_{\mathbb{C}})$.

Remark 4: Of course, a discriminant function Δ exists for forms of arbitrary degree d; in the general case Δ is homogeneous of degree $b \cdot (d-1)^{b-1}$ on $S^d H_{\mathbb{C}}^{\vee}$.

Proposition 5: *Fix a symmetric trilinear form $F \in S^3 H_{\mathbb{C}}^{\vee}$ and an element $h \in H_{\mathbb{C}} \setminus \{0\}$ with $f(h) = 0$. The associated point $< h > \in \mathbb{P}(H_{\mathbb{C}})$ is a singular point of the cubic hypersurface $(f)_o \subset \mathbb{P}(H_{\mathbb{C}})$ if and only if the linear form $h^2 \in H_{\mathbb{C}}^{\vee}$ is zero. The existence of at least one such point is equivalent to the vanishing of the discriminant.*

Proof: From $f(h+tv) = f(h)+3th^2 \cdot v+3t^2 h \cdot v^2+t^3 v^3$ for every $v \in H_{\mathbb{C}}, t \in \mathbb{C}$ we find $\frac{d}{dt}|_o f(h+tv) = 3h^2 \cdot v$, i.e. $h^2 \in H_{\mathbb{C}}^{\vee}$ defines the differential of f in h.

Remark 5: $\mathbb{Q}$-rational points in $(f)_o \subset \mathbb{P}(H_{\mathbb{C}})$, and $\mathbb{Q}$-rational singularities of $(f)_o$ have geometric significance if the cubic f is defined by the cup-form of a 6-manifold X. In fact, integral classes $h \in H^2(X, \mathbb{Z})$ correspond to homotopy classes of maps to $\mathbb{P}^3_{\mathbb{C}}$; such a map factors over $\mathbb{P}^2_{\mathbb{C}} \subset \mathbb{P}^3_{\mathbb{C}}$ if and only if $h^3 = 0$; if it factors over $\mathbb{P}^1_{\mathbb{C}} \subset \mathbb{P}^3_{\mathbb{C}}$, then clearly $h^2 = 0$. The converse will probably not always be true since, in general, the cohomolgy ring does not determine the homotopy type.

In addition to the invariant discriminant $\Delta(f)$ of a polynomial f, we will also need a fundamental covariant H_f, the Hessian of f. Let $F \doteq \mathrm{Pol}(f) \in S^3 H_{\mathbb{C}}^{\vee}$ be the polarization of $f \in \mathbb{C}[H_{\mathbb{C}}]_3$; the Hessian of f can then be defined as the composition $H_f : H_{\mathbb{C}} \xrightarrow{F'} S^2 H_{\mathbb{C}}^{\vee} \xrightarrow{\mathrm{disc}} \mathbb{C}$, i.e. H_f is the homogeneous polynomial function of degree b on $H_{\mathbb{C}}$ given by $H_f(h) = \mathrm{disc}(F^t(h))$. In terms of linear coordinates $\xi_1, \cdots, \xi_b$ on H one finds the more familiar expression $H_f = \det(\frac{\partial^2}{\partial \xi_i \partial \xi_j} f)$.

Proposition 6: *Let $F \in S^3 H_{\mathbb{C}}^{\vee}$ be a symmetric trilinear form. The Hessian of F is identically zero if and only if there exist no element $h \in H_{\mathbb{C}}$ for which the map $\cdot h : H_{\mathbb{C}} \longrightarrow H_{\mathbb{C}}^{\vee}$ is an isomorphism.*

Proof: H_f is identically zero if and only if the symmetric bilinear forms $F^t(h) \in S^2 H_{\mathbb{C}}^{\vee}$ are degenerate for every $h \in H_{\mathbb{C}}$. But this means that none of the maps $\cdot h : H_{\mathbb{C}} \longrightarrow H_{\mathbb{C}}^{\vee}$ is an isomorphism.

Corollary 3: *Let $F \in S^3H_{\mathbb{C}}^{\vee}$ be a form whose associated map $F^t : H_{\mathbb{C}} \to S^2H_{\mathbb{C}}^{\vee}$ is not injective. Then we have $H_f = 0$.*
Proof: Let $k \in \mathrm{Ker}(F^t)$ be a non-zero element, and consider an arbitrary element $h \in H_{\mathbb{C}}$. By definition of k we have $F(k, h, v) = 0$ for all $v \in H_{\mathbb{C}}$, i.e. $k \cdot h \in H_{\mathbb{C}}^{\vee}$ is zero.

Remark 6: It is not difficult to show that F^t is not injective if and only if there exists a proper quotient $\overline{H}_{\mathbb{C}}$ of $H_{\mathbb{C}}$, and a form $\overline{F} \in S^3\overline{H}_{\mathbb{C}}^{\vee}$ whose pull-back to $H_{\mathbb{C}}$ is the given form F. This means that the Hessians of cubic polynomials $f \in \mathbb{C}[H_{\mathbb{C}}]_3$ which 'do not depend on all variables' are automatically zero.
The converse holds for forms in $b \leq 4$ variables, but not in general [G/N].

3.2 The GIT quotient $S^3H_{\mathbb{C}}^{\vee}/_{SL(H_{\mathbb{C}})}$

Let $V := S^3H_{\mathbb{C}}^{\vee}$ be the vector space of complex cubic forms. The reductive group $G := SL(H_{\mathbb{C}})$ acts rationally on V, and therefore has a finitely generated ring $\mathbb{C}[V]^G$ of invariants [H]. The inclusion $\mathbb{C}[V]^G \subset \mathbb{C}[V]$ induces a regular map $\pi : V \to V/_G$ onto the affine variety $V/_G$ with coordinate ring $\mathbb{C}[V]^G$. It is well known that π is a categorical quotient, which is G-closed and G-separating, so that $V/_G$ parametrizes precisely the closed G-orbits in V. Recall that a point $v \in V$ is semi-stable if $o \notin \overline{G \cdot v}$, and that v is stable if $G \cdot v$ is closed in V and the isotropy group G_v is finite [M/F]. Denote the G-invariant, open subsets of semistable (stable) points in V by $V^{ss}(V^s)$.
The complement $V \setminus V^{ss} = \pi^{-1}(\pi(0))$ consists of 'Nullformen' , i.e. forms for which all polynomial invariants vanish. The open subset of stable points, which includes in particular all non-singular forms, has a geometric quotient, given by the restricted map $\pi|V^s : V^s \to \pi(V^s)$.

Remark 7: Let $A_o \in GL(H)$ be a fixed automorphism of determinant $\det A_o = -1$, e.g. $A_o = -id_H$ if b is odd. A_o induces a $\mathbb{Z}/_2$-action on $S^3H^{\vee}/_{SL(H)}$ and on $S^3H_{\mathbb{C}}^{\vee}/_{SL(H_{\mathbb{C}})}$, for which the map c is equivariant. Let $\hat{G} \subset GL(H_{\mathbb{C}})$ be the semi-direct product of $SL(H_{\mathbb{C}})$ and $\mathbb{Z}/_2$ generated by A_o and $SL(H_{\mathbb{C}})$. The invariant ring $\mathbb{C}[V]^{\hat{G}}$ has an important topological interpretation: it consists of all polynomial invariants of complex cohomology rings of 1-connected, closed, oriented 6-dimensional manifolds with torsion-free homology.

Example 5: Binary cubics ($b = 2$)
Choose linear coordinates X, Y on $H_{\mathbf{C}}$, and write a cubic polynomial $f \in \mathbb{C}[X, Y]_3$ in the form $f = a_o X^3 + 3a_1 X^2 Y + 3a_2 XY^2 + a_3 Y^3$.
We use a_0, a_1, a_2, a_3 as coordinates on $S^3 H_{\mathbf{C}}^{\vee}$, so that $\mathbb{C}[S^3 H_{\mathbf{C}}^{\vee}] = \mathbb{C}[a_0, a_1, a_2, a_3]$. The discriminant $\Delta(f)$ of f is a homogeneous polynomial of degree 4 in the coefficients a_0, a_1, a_2, a_4, explicitly given by $\Delta(f) = a_0^2 a_3^2 - 3a_1^2 a_2^2 - 6a_0 a_1 a_2 a_3 + 4a_0 a_2^3 + 4a_1^3 a_3$.
The discriminant generates the ring of $SL(H_{\mathbf{C}})$-invariants, $\mathbb{C}[S^3 H_{\mathbf{C}}^{\vee}]^{SL(H_{\mathbf{C}})} = \mathbb{C}[\Delta]$, and it is easy to see that Δ is also $\mathbb{Z}_{/2}$-invariant. A cubic form f is stable if and only if it is semistable, if and only if it is non-singular [N]. The cone of nullforms $\pi^{-1}(\pi(0))$ is the affine hypersurface $(\Delta)_o \subset S^3 H_{\mathbf{C}}^{\vee}$; it has a nice geometric interpretation in terms of the Hessian. The Hessian of the cubic f is the quadratic form $H_f = 6^2[(a_0 a_2 - a_1^2) X^2 + (a_0 a_3 - a_1 a_2) XY + (a_1 a_3 - a_2^2) Y^2]$. The set of forms f with vanishing Hessians H_f form the affine cone over the rational normal curve in $\mathbb{P}(S^3 H_{\mathbf{C}}^{\vee})$; the hypersurface of nullforms is the cone over the tangential scroll of this curve. There are 4 different types of $SL(H_{\mathbf{C}})$-orbits in $S^3 H_{\mathbf{C}}^{\vee}$, represented by the normal forms $XY(X + \lambda Y), X^2 Y, X^3, 0$. The first type is stable, the others are nullforms, the orbits of X^3 and 0 have vanishing Hessians.

Example 6: Ternary cubics (b=3)
The ring of $SL(H_{\mathbf{C}})$-invariants of ternary cubics is a weighted polynomial ring in 2 variables, $\mathbb{C}[S^3 H_{\mathbf{C}}^{\vee}]^{SL(H_{\mathbf{C}})} = \mathbb{C}[S, T]$ whose generators S, T have been found by S. Aronhold [A]. S is a homogeneous polynomial of degree 4 in the coefficients of a cubic f, T is homogeneous of degree 6, both polynomials are $\mathbb{Z}_{/2}$-invariant. For a cubic of the form $f = aX^3 + bY^3 + cZ^3 + 6dXYZ$, S and T are given by $S = 4d(d^3 - abc)$ and $T = 8d^6 + 20abc(d^3 - abc)$ respectively [P]. The general formulae, which take two pages to write down, can be found in the book of Sturmfels [St]. The discriminant of a form f is homogeneous of degree 12 in the coefficients of f; in terms of Aronhold's invariants S, T it is simply given by $\Delta = S^3 - T^2$.
We obtain the following overall picture: The GIT quotient for ternary cubics is an affine plane $\mathbf{A}^2$ with coordinates S, T. The complement $\mathbf{A}^2 \setminus (\Delta)_o$ of the discriminant curve is the geometric quotient of stable cubics. The π-fibers over a point $(S, T) \neq (0, 0)$ on the discriminant curve $(\Delta)_o$ consist of 3 types of $SL(H_{\mathbf{C}})$-orbits: nodal cubics with normal form $X^3 + Y^3 + 6\alpha XYZ$, reducible cubics formed by a smooth conic and a transversal line (normal form:$X^3 + 6\alpha XYZ$), and cubics consisting of three lines in general position (normal form: $6\alpha XYZ$); these cubics are proberly semistable for $\alpha \neq 0$ with Aronhold invariants $S = 4\alpha^4, T = 8\alpha^6$. The fiber of π over 0 contains 6 orbits with normal forms $Y^2 Z - X^3, Y(X^2 - YZ), XY(X+$

$Y), X^2Y, X^3$, and 0, of which the last 4 types have vanishing Hessians. For more details we refer to H. Kraft's book [Kr].

Remark 8: The natural $\mathbb{C}^*$-action $f \to \lambda \cdot f$ on cubic forms induces a weighted action on the GIT quotient $S^3H_{\mathbb{C}}^{\vee}/_{SL(H_{\mathbb{C}})}$, $\lambda\cdot(S,T) = (\lambda^4 S, \lambda^6 T)$. The associated weighted projective space $\mathbb{P}^1(4,6)$ with homogeneous coordinates $< S, T >$ is the good quotient for semi-stable plane cubic curves. Its affine part $\mathbb{P}^1 \setminus (\Delta)_o$ is the moduli space of genus-1 curves. The $PGL(H_{\mathbb{C}})$-invariant $J := \frac{S^3}{\Delta}$ gives the J-invariant of the corresponding curve.

3.3 Arithmetical aspects

Let $c : S^3H^{\vee}/_{SL(H)} \to S^3H_{\mathbb{C}}^{\vee}/_{SL(H)}$ be the map which associated to the $SL(H)$-orbit $< F >$ of a symmetric trilinear form $F \in S^3H^{\vee}$ the $SL(H_{\mathbb{C}})$-orbit $< F >_{\mathbb{C}}$ of its complexification. The c-fiber over $< F >_{\mathbb{C}}$ can be identified with the subset $(SL(H_{\mathbb{C}}) \cdot F \cap S^3H^{\vee})/_{SL(H)}$ of $S^3H^{\vee}/_{SL(H)}$. C. Jordan has shown that these subsets are finite provided the cubic form $f \in \mathbb{C}[H_{\mathbb{C}}]_3$ associated to F has a non-vanishing discriminant [J1]. Jordan's original proof, which is only two pages long, is somewhat hard to follow. The following theorem of A. Borel and Harish-Chandra provides, however, a vast generalization of Jordan's finiteness result:

Theorem 3 *(Borel/Harish-Chandra): Let G be a reductive $\mathbb{Q}$-group, $\Gamma \subset G$ an arithmetic subgroup, $\xi : G \to GL(V)$ a $\mathbb{Q}$-morphism, and $L \subset V$ a Γ-invariant sublattice of $V_{\mathbb{Q}}$. If $v \in V$ has a closed $G-$orbit in V, then $G_v \cap L/_{\Gamma}$ is a finite set.*
Proof: $[B]$

Corollary 4: *Let $F \in S^3H^{\vee}$ be a symmetric trilinear form on H. If the $SL(H_{\mathbb{C}})$-orbit of F in $S^3H_{\mathbb{C}}^{\vee}$ is closed, then the fiber $c^{-1}(< F >_{\mathbb{C}})$ over $< F >_{\mathbb{C}}$ is finite.*

To check whether a $SL(H_{\mathbb{C}})$-orbit $SL(H_{\mathbb{C}}) \cdot F$ is closed in $S^3H_{\mathbb{C}}^{\vee}$, one has a generalization of the Hilbert-criterion [Kr]: $SL(H_{\mathbb{C}}) \cdot F$ is closed in $S^3H_{\mathbb{C}}^{\vee}$ if and only if for every 1-parameter subgroup $\lambda : \mathbb{C}^* \to SL(H_{\mathbb{C}})$, for which $lim_{t\to 0}\lambda(t) \cdot F$ exist in $S^3H_{\mathbb{C}}^{\vee}$, this limit is already contained in $SL(H_{\mathbb{C}}) \cdot F$. A sufficient condition for $SL(H_{\mathbb{C}}) \cdot F$ to be closed follows from another result of C. Jordan [J2]:

Theorem 4 *(Jordan): Let $f \in \mathbb{C}[H_{\mathbb{C}}]_d$ be a homogeneous polynomial of degree $d \geq 3$. If its discriminant $\Delta(f)$ is non-zero, then f has a finite isotropy group $SL(H_{\mathbb{C}})_f$.*

Corollary 5: *Let $F \in S^3H^\vee$ be a form whose associated cubic polynomial $f \in \mathbb{C}[H_{\mathbb{C}}]_3$ has $\Delta(f) \neq 0$. Then $SL(H_{\mathbb{C}}) \cdot F$ is closed in $S^3H^\vee_{\mathbb{C}}$.*
Proof: Standard arguments, cf. [Bo].

Remark 9: Closedness of the $SL(H_{\mathbb{C}})$-orbit of F is only a sufficient condition for the finiteness of the fiber $c^{-1}(< F >_{\mathbb{C}})$. There exist other finiteness theorems for special types of forms, like e.g. forms which decompose into linear factors.
Some of these results are surveyed in volume III of L. Dickson's book [D].

We say that two forms $F, F' \in S^3H^\vee$ belong to the same (proper) equivalence class if they lie in the same $(SL(H)\text{-})GL(H)$-orbit. The group $\mathbb{Z}_{/2} = GL(H)/_{SL(H)}$ acts on the set $S^3H^\vee/_{SL(H)}$ of proper classes, and the quotient becomes the orbit space $S^3H^\vee/_{GL(H)}$.
The $\mathbb{Z}_{/2}$-action is not free in general, but for finiteness properties this plays no rôle.

Example 7: Binary cubics
Let H be a free $\mathbb{Z}$-module of rank $b = 2$. There exist only finitely many classes of symmetric trilinear forms $F \in S^3H^\vee$ with a given non-zero discriminant Δ. Of course, Δ must be integral, and a square modulo 4, in order to be realizable by an integral form. For some small values of $\Delta \neq 0$ the number of classes is known. Results in this direction go back to a paper by F. Arndt [A]; his tables have been rearranged by A. Cayley [Cay]. It should certainly be possible to go much further using modern computers.

Example 8: Ternary cubics
Let H be a free $\mathbb{Z}$-module of rank 3 with coordinates X, Y, Z. The cubic polynomials with closed $SL(H_{\mathbb{C}})$-orbits are the non-singular cubics, and the polynomials in the orbits of $6\alpha XYZ$ for all $\alpha \in \mathbb{C}$.
The number of integral classes in these orbits is therefore finite. We have, however, an even stronger finiteness theorem for stable ternary cubics:

Proposition 7: *Let H be a free $\mathbb{Z}$-module of rank 3. There exist only finitely many classes of symmetric trilinear forms $F \in S^3H^\vee$ with a fixed discriminant $\Delta \neq 0$.*

Proof: In terms of Arnhold's invariants S and T, Δ is given by $\Delta = S^3 - T^2$. By a theorem of C. Siegel [Si], the diophantine equation $S^3 - T^2 = \Delta$ has only finitely many integral solution (S, T) for any integer $\Delta \neq 0$. For each of these solutions the corresponding point in $S^3 H_{\mathbb{C}}^{\vee} / _{SL(H_{\mathbb{C}})}$ lies outside of the discriminant curve, so that the π-fiber over it is a closed $SL(H_{\mathbb{C}})$-orbit. The finiteness of the class number then follows from the Borel/Harisch-Chandra theorem.

A famous special case of Siegel's theorem is Bachet's equation $S^3 - T^2 = 2$; it has only the two obvious solutions $(3, \pm 5)$.

Remark 10: To get finiteness results for ternary cubic forms it is not sufficient to fix the J-invariant (instead of the discriminant): The forms $f_m = X^3 + XZ^2 + Z^3 + mY^2Z, m \in \mathbb{Z} \setminus \{0\}$, all have the same J-invariant, but they are not equivalent, even over $\mathbb{Q}$, since they have bad reduction at different primes $p|m$.

4. Invariants of complex 3-folds

In this section we begin to investigate the topology of 1-connected, compact, complex 3-folds. After a brief discussion of the possible systems of Chern numbers of almost complex 6-manifolds, we study the behaviour of the topological invariants of complex 3-folds under certain standard constructions, like e.g. branched coverings, or blow-ups of points and curves. Then we describe some interesting examples of 1-connected, non-Kählerian 3-folds, including a new construction method which generalizes the Calabi-Eckmann manifolds. These examples will be needed in the next section in order to realizes complex types of cubic forms as cup-forms of complex 3-folds.

4.1 Chern numbers of almost complex structures

Let X be a closed, oriented, 6-dimensional differentiable manifold. The tangent bundle of X is induced by a classifying map $t_X : X \to BSO(6)$ which is unique up to homotopy. By an almost complex structure on X we mean the homotopy class $[\tilde{t}_X]$ of a lifting $\tilde{t}_X : X \to BU(3)$ of t_X to $BU(3)$.

Proposition 8: *Every closed, oriented, 6-dimensional C^∞-manifold X without 2-torsion in $H^3(X, \mathbb{Z})$ admits an almost complex structure. There* *is a 1-1 correspondence between almost complex structures on X and integral lifts $W \in H^2(X,\mathbb{Z})$ of $w_2(X)$. The Chern classes c_i of the almost complex manifold (X,W) are given by $c_1 = W, c_2 = \frac{1}{2}(W^2 - p_1(X))$.*

Proof: (cf.[W]). The obstructions against lifting t_X to $BU(3)$ lie in the cohomology groups $H^{i+1}(X,\pi_i(SO(6)/_{U(3)}), i = 0,1,\ldots,5$. Since $SO(6)/_{U(3)} = \mathbb{P}^3$ has only one nontrivial homotopy group $\pi_2(SO(6)/_{U(3)}) \cong \mathbb{Z}$ in dimensions $i \leq 5$, there is in fact only one obstruction $o(t_X) \in H^3(X,\mathbb{Z})$, and this obstruction can be identified with the image of $w_2(X)$ under the Bockstein homomorphism $\beta : H^2(X,\mathbb{Z}_{/2}) \to H^3(X, \mathbb{Z})$. Since $H^3(X,\mathbb{Z})$ has no 2-torsion by assumption, $\beta w_2(X)$ must be equal to zero, so that X has at least one almost complex structure $[\tilde{t}_X] \in [X, BU(3)]$. Standard homotopy arguments show now that the map, which asigns to an almost complex structure $[\tilde{t}_X]$ its first Chern class $\tilde{t}_X^* c_1$, induces a 1-1 correspondence between integral lifts $W \in H^2(X,\mathbb{Z})$ of $w_2(X)$ and homotopy classes of liftings of $[t_X]$ to $BU(3)$.
The second Chern class c_2 of the almost complex manifold (X,W) is determined by $W^2 - 2c_2 = p_1(X)$.

The Chern numbers c_1^3, c_1c_2, c_3 of an almost complex manifold X of real dimension 6 satisfy the following congruences: $c_1^3 \equiv 0(\text{mod } 2), c_1c_2 \equiv 0(\text{mod } 24), c_3 \equiv 0(\text{mod } 2)$. Conversely, given a triple (a,b,c) of integers $a \equiv 0(\text{mod } 2), b \equiv 0(\text{mod } 24)$, and $c \equiv 0(\text{mod } 2)$, there always exist an almost complex manifold X of dimension 6 with Chern mumbers $c_1^3 = a, c_1c_2 = b, c_3 = c$.
It is not totally clear, however, that one can find a **connected** manifold X with prescribed Chern numbers [H1].

Proposition 9: *Every tripel $(a,b,c) \in \mathbb{Z}^{\oplus 3}$ satisfying $a \equiv 0(\text{mod } 2), b \equiv 0(\text{mod } 24), c \equiv 0(\text{mod } 2)$ is realizable as the Chern numbers of an almost complex 6-manifold.*

Proof: Consider the complete intersection $V(f,g) \subset \mathbb{P}^5$ defined by the polynomials $f(z) = z_0^2 + z_1^2 + 2z_2^2 - z_3^2 - z_4^2 - 2z_5^2$, and $g(z) = z_0^4 + z_1^4 + 2z_2^4 - z_3^4 - z_4^4 - 2z_5^4$ [Wc]. $V(f,g)$ is a singular 3-fold with 90 ordinary double points, and every small resolution V of these nodes is a (not neccessarily projective) Calabi-Yau 3-fold with Euler number 4. Suppose now that a prescribed triple $(a,b,c) \in \mathbb{Z}^{\oplus 3}$ is realized by a possibly disconnected almost complex manifold $X = \coprod_{i\in I} X_i$. If we form the connected sum $X' = \sharp_{i\in I} X_i$, we obtain a connected almost complex manifold X' with Chern numbers $c_1^3 = a, c_1c_2 = b$, but with $c_3 = c - 2(|I| - 1)$.
If $|I| > 1$ take the connected sum of X' with $|I| - 1$ copies of the complex

manifold V. Since V is Calabi-Yau, the Chern numbers c_1^3 and c_1c_2 remain unchanged, whereas the Euler number of $X'\sharp_{|I|-1}V$ becomes $c_3 = c$.

Remark 11: The above argument has been suggested by F. Hirzebruch after talk at the MPI, in which one of us had sketched a less geometric proof of the proposition.

There is another question which is related to the result above: Fix a closed, oriented, 6-dimensional differentiable manifold X. Which pairs (a, b) of integers with $a \equiv 0(\text{mod } 2)$ and $b \equiv 0(\text{mod } 24)$ occur as Chern numbers c_1^3 and c_1c_2 of almost complex structures on X, and in how many ways?
For manifolds with $b_2(X) = 1$ the Chern numbers determine the almost complex structure. For manifolds with $b_2 > 1$ this is no longer true. It is possible to construct infinitely many distinct almost complex structures with the same Chern numbers on a hypersurface of bidegree (3,3) in $\mathbb{P}^2 \times \mathbb{P}^2$.

An almost complex structure $[\tilde{t}_X]$ on a differentiable 6-manifold X is said to be integrable if $\tilde{t}_X$ is homotopic to the classifying map of a complex 3-fold. We are not aware of any example of an almost complex 6-manifold which is known not be integrable. On the other hand, it is also unknown whether or not the Chern numbers c_1^3, c_1c_2 of integrable almost complex manifold are topological invariants. The following remark might therefore be of some interest:

Proposition 10: *If the Chern numbers of complex 3-folds are topological invariants, then there exist almost complex structures which are not integrable.*

Proof: Consider a closed, oriented differentiable 6-manifold X without 2-torsion in $H^3(X, \mathbb{Z})$. Fix any almost complex structure on X with first Chern class $W \in H^2(X, \mathbb{Z})$.
Every element $x \in H^2(X, \mathbb{Z})$ defines a new almost complex structure on X with first Chern class $W + 2x$, and it is easy to see that these two almost complex structures have the same Chern numbers if and only if x satisfies the equations $p_1(X) \cdot x = 0$, and $3W^2 \cdot x + 6W \cdot x^2 + 4x^3 = 0$.
Suppose now (X, W) is integrable, $p_1(X) \neq 0$, and choose $x \in H^2(X, \mathbb{Z})$ such that $p_1(X) \cdot x \neq 0$. Then clearly, either none of the almost complex manifolds $(X, W + 2x)$ is integrable, or the Chern numbers of complex 3-folds are not topologically invariant.

Remark 12: It is very likely that there exist non-integrable almost complex structures on manifolds X as above, but probably this is hard to prove. It is also not unlikely that the Chern numbers of complex 3-folds are not topological invariants. A possible way to check this would be, to run a computer search for 3-folds given by certain standard constructions.

4.2 Standard constructions

For later use we investigate the topological invariants of complex 3-fold which can be obtained by certain simple standard constructions like complete intersections, simple cyclic coverings, blow-ups of points and curves, and projective bundles.

Proposition 11: (Libgober/Wood): *Let $X \subset \mathbb{P}^{3+r}$ be a smooth complete intersection of multidegree $\underline{d} = (d_1, \ldots, d_r)$. Choose a normalized basis $e \in H^2(X, \mathbb{Z})$, and let $\varepsilon \in H^4(X, \mathbb{Z})$ be defined by $\varepsilon(e) = 1$. Then the invariants of X are:*
$F_X(xe) = dx^3$ *where* $d = \prod_{i=1}^r d_i, w_2(X) \equiv (4 + r - \sum_{i=1}^r d_i)e$,
$p_1(X) = d(4 + r - \sum_{i=1}^r d_i^2)\varepsilon$, *and*
$b_3(X) = 4 - \frac{d}{6}[(4 + r - \sum_{i=1}^r d_i)^3 - 3(4 + r - \sum_{i=1}^r d_i)(4 + r - \sum_{i=1}^r d_i^2) + 2(4 + r - \sum_{i=1}^r d_i^3)]$.
Proof: [L/W].

Proposition 12: *Let X be a smooth, 1-connected, complex projective 3-fold, and let $\pi : X' \to X$ be a simple cyclic covering of degree d branched along a non-singular ample divisor $B \in |L^{\otimes d}|$.X' is smooth, projective, 1-connected, and $\pi^* : H^2(X, \mathbb{Z}) \to H^2(X', \mathbb{Z})$ is an isomorphism. The invariants of X and X' are related by the formulae:*
$(\pi^*)^* F_{X'} = dF_X$, $w_2(X') - \pi^* w_2(X) \equiv (d-1)\pi^* c_1(L)$,
$p_1(X') - \pi^* p_1(X) = (1-d)(1+d)\pi^* c_1(L)^2$, *and*
$b_3(X') = db_3(X) + (d-1)(b_2(B) - 2b_2(X))$.

Proof: X' is clearly smooth and projective. By a theorem of M. Cornalba $\pi : X' \to X$ is a 3-equivalence, i.e. $\pi_* : \pi_i(X') \to \pi_i(X)$ is bijective for $i \leq 2$, and surjective for $i = 3$[Co]. X' is therefore 1-connected, and $\pi^* : H^2(X, \mathbb{Z}) \to H^2(X', \mathbb{Z})$ is an isomorphism. The relation between $F_{X'}$ and F_X is obvious, whereas the formula for $b_3(X')$ follows from $\pi_1(B) = \{1\}$ and standard properties of Euler numbers.
In order to calculate $w_2(X')$ and $p_1(X')$ we compute the Chern classes of X' : $c_1(X') - \pi^* c_1(X) = (1-d)\pi^* c_1(L), c_2(X') - \pi^* c_2(X) = (1-d)\pi^*[c_1(X)c_1(L) - dc_1(L)^2]$.
The latter formulae follow from the description of X' as a divisor in the total space of the line bundle L.

Example 9: Let X be a d-fold, simple cyclic covering of $\mathbb{P}^3$ branched along a smooth surface $B \subset \mathbb{P}^3$ of degree $dl, l \geq 1$. Let $e \in H^2(X, \mathbb{Z})$ correspond to the preimage of a plane in $\mathbb{P}^3$. The invariants of X are then given by: $F_X(xe) = dx^3, w_2(X) \equiv (4 + (1-d)l)e, p_1(X) = d[4 + (1-d)(1+d)l^2]\varepsilon$ $(\varepsilon(e) = 1), b_3(X) = (d-1)(d^2l^2 - 4dl + 6)dl$.

Proposition 13: *Let $\sigma : \hat{X} \to X$ be the blow-up of a complex 3-fold X in a point, and let $e \in H^2(\hat{X}, \mathbb{Z})$ be the class of the exceptional divisor. The invariants of $\hat{X}$ and X are related by the following formulae:*
$F_{\hat{X}}(\sigma^*h + xe) = F_X(h) + x^3$ *for every* $h \in H^2(X, \mathbb{Z}), x \in \mathbb{Z}, w_2(\hat{X}) = \sigma^* w_2(X), p_1(\hat{X}) = \sigma^* p_1(X) + 4(e^2 - \sigma^* c_1(X) \cdot e), b_3(\hat{X}) = b_3(X)$.

Proof: Standard arguments, see [G/H]. The Chern classes are related by $c_1(\hat{X}) = \sigma^* c_1(X) - 2e, c_2(\hat{X}) = \sigma^* c_2(X)$.

Proposition 14: *Let $\sigma : \hat{X} \to X$ be the blow-up of a complex 3-fold X along a smooth curve C of genus g, and let $e \in H^2(\hat{X}, \mathbb{Z})$ be the class of the exceptional divisor. The invariants of $\hat{X}$ and X are related by:*
$F_{\hat{X}}(\sigma^*h + xe) = F_X(h) - 3h \cdot Cx^2 - \deg N_{C/X} x^3$ *for every* $h \in H^2(X, \mathbb{Z}), x \in \mathbb{Z}, w_2(\hat{X}) \equiv \sigma^* w_2(X) + e, p_1(\hat{X}) = \sigma^* p_1(X) + (e^2 - 2\sigma^* C), b_3(\hat{X}) = b_3(X) + 2g$.

Proof: $[G/H]$. The Chern classes are given by $c_1(\hat{X}) = \sigma^* c_1(X) - e, c_2(\hat{X}) = \sigma^*(c_2(X) + C) - \sigma^* c_1(X) \cdot e$.

Proposition 15: *Let E be a holomorphic vector bundle of rank 2 with Chern classes $c_i(E), i = 1, 2$ over a 1-connected, compact complex surface Y, and let $\pi : \mathbb{P}(E) \to Y$ be the projective bundle of lines in the fibers of E. The cup-form of $\mathbb{P}(E)$ is given by $F_{\mathbb{P}(E)}(h + x\xi) = x[(3h^2) - (3c_1(E) \cdot h)x + (c_1(E)^2 - c_2(E))x^2]$, where $\xi = c_1(\mathcal{O}_{\mathbb{P}(E)}(1)), h \in H^2(Y, \mathbb{Z})$, and $x \in \mathbb{Z}$. The other topological invariants of $\mathbb{P}(E)$ are: $w_2(\mathbb{P}(E)) \equiv \pi^*(w_2(Y) + c_1(E)), p_1(\mathbb{P}(E)) = \pi^*[c_1(Y)^2 - 2c_2(Y) + c_1(E)^2 - 4c_2(E)], b_3(\mathbb{P}(E)) = 0$.*

Proof: The Leray-Hirsch theorem identifies the cohomology ring $H^*(\mathbb{P}(E), \mathbb{Z})$ with the ring $H^*(Y, \mathbb{Z})[\xi]/_{<\xi^2 + c_1(E)\cdot\xi + c_2(E)>}$; this determines the cup-form. In order to calculate the characteristic classes one uses the exact sequence $0 \to \mathcal{O}_{\mathbb{P}(E)} \to \pi^* E \otimes \mathcal{O}_{\mathbb{P}(E)}(1) \to T_{\mathbb{P}(E)} \to \pi^* T_Y \to 0$. $b_3(\mathbb{P}(E)) = 0$ follows from $b_1(Y) = 0$ and the Leray-Hirsch theorem.

4.3 Examples of 1-connected non-Kählerian 3-folds

Recall that the Hessian of a symmetric trilinear form $F \in S^3 H^\vee$ on a free $\mathbb{Z}$-module H of finite rank was defined as the composition

$\mathcal{H}_F : H \xrightarrow{F^t} S^2H^\vee \xrightarrow{\text{disc}} \mathbb{Z}$. In terms of coordinates $\xi_1, \ldots, \xi_b$ on H it is given by the determinant $\det(\frac{\partial^2 f}{\partial\xi_i\partial\xi_j})$, where $f \in \mathbb{C}[H_{\mathbb{C}}]_3$ is the homogeneous cubic polynomial associated with F.

Proposition 16: *Let F be a symmetric trilinear form whose Hessian vanishes identically. Then F is not realizable as cup-form of Kählerian 3-fold.*

Proof: Let X be a complex 3-fold with a Kähler metric g. The Kähler class $[\omega_g] \in H^2(X, \mathbb{R})$ defines a multiplication map $\cdot[\omega_g] : H^2(X, \mathbb{R}) \to H^4(X, \mathbb{R})$, which is an isomorphism by the Hard Lefschetz Theorem [G/H]. In section 3.1 we have seen that this is not possible if the Hessian of the cup-form vanishes.

Corollary 6: *Cubic forms $f \in \mathbb{C}[H_{\mathbb{C}}]_3$ which depend on strictly less than $b = rk_{\mathbb{Z}}H$ variables are not realizable as cup-forms of Kählerian 3-folds with $b_2 = b$.*

By considering the Hessian of a cup-form over the reals one obtains further conditions.

Definition 4: *Let $F \in S^3H^\vee$ be a symmetric trilinear form on a free $\mathbb{Z}$-module of rank b.*
The **Hesse cone** *of F is the subset $\mathcal{H}_F \subset H_{\mathbb{R}}$ defined by $\mathcal{H}_F := \{h \in H_{\mathbb{R}} | (-1)^b \det (F^t(h)) < 0\}$.*
The **index cone** *I_F of F is the subset $I_F := \{h \in H_{\mathbb{R}} | F^t(h) \in S^2H_{\mathbb{R}}^\vee$ has signature $(1, -1, \ldots, -1)\}$.*

Clearly I_F is an open subcone of $\mathcal{H}_F$ which coincides with $\mathcal{H}_F$ iff $b \leq 2$.

Theorem 5: *Let $F_X \in S^3H^2(X, \mathbb{Z})^\vee$ be the cup-form of a smooth projective 3-fold with $h^{0,2}(X) = 0$. Then F_X has a non-empty index cone.*

Proof: Let $h \in H^2(X, \mathbb{Z})$ be the dual class of a hyperplane section Y in some projective embedding. The inclusion $i : Y \hookrightarrow X$ induces a monomorphism $i^* : H^2(X, \mathbb{Z}) \to H^2(Y, \mathbb{Z})$ by the weak Lefschetz theorem. The symmetric bilinear form $F_X^t(h) \in S^2H^2(X, \mathbb{Z})^\vee$ is simply the pull-back of the cup-form of Y under the inclusion i^*; it is therefore non-degenerate by the Hard Lefschetz theorem [L]. Applying the Hodge index theorem to Y we see that the real bilinear form $F_X^t(h) \in S^2H^2(X, \mathbb{R})^\vee$ must have one positive and $b - 1$ negative eigenvalues. In other words: $h \in I_{F_X}$.

Remark 13: This result has two applications: if provides topological

'upper bounds' for the ample cone of a projective 3-fold with $h^{0,2} = 0$, and if gives further restrictions on symmetric trilinear forms to be realizable as cup-forms of projective 3-folds with $h^{0,2} = 0$ if $b \geq 4$.
These applications will be discussed in section 5.

We will now describe examples of 1-connected, non-Kählerian, complex 3-folds and fit them into the topological classification.

Example 10 (Calabi-Eckmann):
E. Calabi and B. Eckmann have defined complex structures X_τ, depending on a parameter τ, on the product $S^3 \times S^3$[C/E]. Their manifolds are principal fiber bundles over $\mathbb{P}^1 \times \mathbb{P}^1$ whose fiber and structure group is the elliptic curve $E_\tau = \mathbb{C}/_{\mathbb{Z} \oplus \mathbb{Z}\tau}$, $\mathrm{Im}(\tau) > 0$.
The Calabi-Eckmann manifolds are homogeneous non-Kählerian 3-folds of algebraic dimension 2.

Example 11 (Maeda):
H. Maeda has generalized the Calabi-Eckmann construction. He constructed fiber bundles X'_τ over Hirzebruch surfaces $\mathbb{F}_n, n \geq 0$, whose fibers and structure groups are an elliptic curve E_τ and $\mathrm{Aut}(E_\tau)$ respectively [M]. X'_τ is again diffeomorphic to $S^3 \times S^3$, and therefore non-Kählerian. Maeda's manifolds X'_τ are homogeneous if and only if $n = 0$ in which case they are Calabi-Eckmann 3-folds.

The Calabi-Eckmann construction can also be generalized in the following way:
Let $S^2 \tilde{\times} S^4$ be the non-trivial S^4-bundle over S^2, i.e. $S^2 \tilde{\times} S^4$ is the unique 1-connected, closed, oriented, differentiable 6-manifold with $H_2(S^2 \tilde{\times} S^4, \mathbb{Z}) \doteq \mathbb{Z}$ and $b_3 = 0$, whose cup-form and Pontrjagin class vanish, but whose Stiefel-Whitney class w_2 is non-zero.

Theorem 6: *For any integer $b \geq 0$ there exist compact complex 3-folds X_b, and $X_b^\sim$ if $b \geq 1$, which are homeomorphic to $\sharp_b S^2 \times S^4 \sharp_{b+1} S^3 \times S^3$, and $S^2 \tilde{\times} S^4 \sharp_{b-1} S^2 \times S^4 \sharp_{b+1} S^3 \times S^3$.*
Proof: Let Y be a 1-connected, compact complex surface with $p_g(Y) = 0$ and $b_2(Y) \geq 2$, and let $E = \mathbb{C}/_\Gamma$ be the elliptic curve associated to the lattice $\Gamma \subset \mathbb{C}$. We want to construct the required 3-folds as total spaces of principal E-bundles over Y : Let $\underline{c} : H_2(Y, \mathbb{Z}) \to \Gamma$ be an arbitrary epimorphism. The corresponding cohomology class $c \in H^2(Y, \Gamma)$ defines a topological principal bundle over Y with fiber and structure group $E = \mathbb{C}/_\Gamma$ as follows immediately from the identification of the classifying space $BE \simeq K(\Gamma, 2)$.

Let $\mathcal{O}_Y(E)$ be the sheaf of germs of holomorphic maps from Y to E. We have a short exact sequence $0 \to \underline{\Gamma} \to \mathcal{O}_Y \to \mathcal{O}_Y(E) \to 0$ and a corresponding exact cohomology sequence

$\to H^1(Y, \mathcal{O}_Y) \to H^1(Y, \mathcal{O}_Y(E)) \xrightarrow{\delta} H^2(Y, \Gamma) \to H^2(Y, \mathcal{O}_Y) \to$

By our assumptions δ is an isomorphism, so that every topological principal E-bundle admits a holomorphic structure. Let X be the total space of such a bundle corresponding to a surjective map $\underline{c} : H_2(Y, \mathbb{Z}) \to \Gamma$. The homotopy sequence of the fibration $p : X \to Y$ yields the sequence

$0 \to \pi_2(X) \xrightarrow{p_*} \pi_2(Y) \to \pi_1(E) \to \pi_1(X) \xrightarrow{p_*} \pi_1(Y) \to 0.$

Since Y is 1-connected, $\pi_2(Y)$ can be identified with $H_2(Y, \mathbb{Z})$, and then the boundary map $\pi_2(Y) \to \pi_1(E)$ becomes the characteristic map $\underline{c}$: $H_2(Y, \mathbb{Z}) \to \Gamma$ of the bundle. This implies $\pi_1(X) = \{1\}$, whereas $H_2(X, \mathbb{Z})$ is given by: $0 \to H_2(X, \mathbb{Z}) \xrightarrow{p_*} H_2(Y, \mathbb{Z}) \xrightarrow{\underline{c}} \Gamma \to 0$.

In particular, $H_2(X, \mathbb{Z})$ is free as a submodule of $H_2(Y, \mathbb{Z})$, and by dualizing the last sequence we obtain an identification

$H^2(X, \mathbb{Z}) = H^2(Y, \mathbb{Z})/_{\Gamma^\vee} \text{via} p^*$.

The cup-form F_X of X is therefore trivial. In order to calculate $p_1(X)$ and $w_2(X)$, we use the exact sequence of tangent sheaves: $0 \to T_{X/Y} \to T_X \to p^*T_Y \to 0$. Since $T_{X/Y}$ is a trivial bundle, the characteristic classes of X are simply the pullbacks of the corresponding classes of Y. But the map $p^* : H^4(Y, \mathbb{Z}) \to H^4(X, \mathbb{Z})$ is zero, since $< p^*(\varepsilon) \cup p^*(\alpha), [X] > = < \varepsilon \cup \alpha, p_*[X] > = 0$ for all classes $\varepsilon \in H^4(Y, \mathbb{Z})$, and $\alpha \in H^2(Y, \mathbb{Z})$.

Thus $p_1(X) = 0$, and $w_2(X)$ is the residue of $w_2(Y) \in H^2(Y, \mathbb{Z}_{/2})$ modulo $\Gamma^\vee/_{2\Gamma^\vee}$.

The Euler characteristic of X is zero, so that from $b_2(X) = b_2(Y) - 2$ we find $b_3(X) = 2(b_2(Y) - 1)$. The system of invariants associated to the manifold X is therefore given by $(b_2(Y) - 1, H^2(Y, \mathbb{Z})/_{\Gamma^\vee}, w_2(Y) \, (\text{mod} \Gamma^\vee/_{2\Gamma^\vee}), 0, 0, 0)$, i.e. X is diffeomorphic to $\sharp_{b_2(Y)-2} S^2 \times S^4 \sharp_{b_2(Y)-1} S^3 \times S^3$ if $w_2(Y) \in \Gamma^\vee/_{2\Gamma^\vee}$, and to $S^2 \tilde{\times} S^4 \sharp_{b_2(Y)-3} S^2 \times S^4 \sharp_{b_2(Y)-1} S^3 \times S^3$ if $b_2(Y) \geq 3$, and $w_2(Y) \notin \Gamma^\vee/_{2\Gamma^\vee}$.

Example 12 (Kato):

In the two papers [K1], [K2] M. Kato studies the class of compact, complex 3-folds X containing smooth rational curves with neighborhoods biholomorphic to those of projective lines in $\mathbb{P}^3$. On this class of 3-folds, called class L, he defines a semi-group structure $+$ with neutral element $\mathbb{P}^3$.

Kato's connecting operation $+$ is defined by removing 'lines' $L_i \subset X_i$ from 3-folds $X_i, i = 1, 2$, and by identifying the complements $X_i \setminus L_i$ along open sets $U_i \setminus L_i$ obtained from suitable neighborhoods $U_i \subset X_i$.

Starting with a certain elliptic fiber space X_1 over the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$

in a point, he constructs a sequence of 3-folds $X_n := X_1 + X_{n-1}, n \geq 2$. The 3-folds X_n are 1-connected spin-manifolds with $H_2(X_n, \mathbb{Z}) = \mathbb{Z}$. Their cup-forms F_{X_n}, and their Pontrjagin classes $p_1(X_n)$ are in terms of a (normalized) generator $e_n \in H^2(X_n, \mathbb{Z})$ and its dual class $\varepsilon_n \in H^4(X_n, \mathbb{Z})$ given by $F_{X_n}(xe_n) = (n-1)x^3$, and $p_1(X_n) = 4(n-1)\varepsilon_n$ $(\varepsilon_n(e_n) = 1)$. The third Betti-number of X_n is $4n$.

In particular, X_1 is diffeomorphic to $S^2 \times S^4 \sharp_2 S^3 \times S^3$, and X_2 is diffeomorphic to $\mathbb{P}^3 \sharp_4 S^3 \times S^3$. It is interesting to note that the Chern-numbers $c_1^3, c_1 c_2$ of the X'_{ns} are $c_1^3 = 64(1-n), c_1 c_2 = 24(1-n)$, i.e. they satisfy $8c_1c_2 = 3c_1^3$. For projective manifolds of general type this equality is characteristic for ball quotients [Y].

Example 13 (Twistor spaces):

Let $p : Z \to M$ be the twistor fibration of a closed, oriented Riemannian 4-manifold (M, g). Z carries a natural almost complex structure which is integrable if and only if g is self-dual [A/H/S].

Examples of 1-connected 4-manifolds which admit self-dual structures are $S^4, \sharp_n \mathbb{P}^2$, and $K3$-surfaces.

The total spaces of their twistor fibrations are 1-connected complex 3-folds which may be Moishezon for S^4 and $\sharp_n \mathbb{P}^2$ [C], but which are usually non-Kähler [Hi]. We leave it to the reader to calculate the topological invariants of these 3-folds. There is an interesting relation between Twistor spaces of connected sums and Kato's connection operation + for class L manifolds [K2], [D/F].

Example 14 (Oguiso):

In a recent preprint [O1] K. Oguiso constructs examples of 1-connected, Moishezon Calabi-Yau 3-folds with very interesting cup-forms. He proves that for every integer $d \geq 1$ there exists a smooth complete intersection X'_d of type (2,4) in $\mathbb{P}^5$ which contains a non-singular rational curve C_d of degree d with normal bundle $N_{C_d/X_d} = \mathcal{O}_{\mathbf{C}_d}(-1)^{\oplus 2}$.

The 3-fold X'_d can now be flopped along C_d, i.e. C_d can be blown up to $\mathbb{P}(N_{C_d/X_d}) \cong \mathbb{P}^1 \times \mathbb{P}^1$, and then 'blown down in the other direction'. The resulting 3-fold X_d is a 1-connected Moishezon manifold with trivial canonical bundle and cup-form F_{X_d} given by $F_{X_d}(xe_d) = (d^3 - 8)x^3$. Here $e_d \in H^2(X_d, \mathbb{Z})$ is the normalized generator corresponding to the strict transform of the negative of a hyperplane section of X'_d. The Pontrjagin class of X_d is $p_1(X_d) = (112 + 4d)\varepsilon_d$ where $\varepsilon_d \in H^4(X_d, \mathbb{Z})$ denotes the generator with $\varepsilon_d(e_d) = 1$. Since the Euler-number does not change under a flop we have $b_3(X_d) = 180$ for every d.

5. Complex 3-folds with small b_2

In this section we investigate the following natural problem: Which cubic forms can be realized as cup-forms of compact complex 3-folds? For small b_2 something can be said: Any core of a 1-connected, closed, oriented differentiable 6-manifold with $H_2(X,\mathbb{Z}) \cong \mathbb{Z}$ is homotopy equivalent to the core of a 1-connected complex 3-fold. In the case $b_2 = 2$, at least every discriminant Δ is realizable by a complex manifold. If $b_2 = 3$ we can realize all types of complex cubics with one exception, the union of a smooth conic and a tangent line. In addition to these realization results we prove a finiteness theorem for 3-folds with $b_2 = 1, w_2 \neq 0$, and we give examples which show that the condition $I_{F_X} \neq \emptyset$ for the index cone of a projective 3-fold with $h^{0,2} = 0$ is non-trivial in general.

5.1 3-folds with $b_2 = 1$

Recall from section 1.1 that every closed, oriented, 1-connected differentiable 6-manifold X with torsion-free homology has a connected sum decomposition $X \cong X_\circ \sharp_r S^3 \times S^3$ where $r = \frac{b_3(X)}{2}$, which is unique up to orientation preserving diffeomorphisms; the manifold $X_\circ$ with $b_3(X_\circ) = 0$ is the core of X.

Theorem 6: *Let $X_\circ$ be a 1-connected, closed, oriented differentiable 6-manifold with $H_2(X_\circ,\mathbb{Z}) \cong \mathbb{Z}$ and $b_3(X_\circ) = 0$. There exists a compact complex 3-fold X whose core is orientation preserving homotopy equivalent to $X_\circ$.*

Proof: The oriented homotopy type of $X_\circ$ is determined by the invariants d, w_2, and $p_1(\text{mod } 48)$; more precisely: for $d \equiv 1(\text{mod } 2)$ there is a single homotopy type whereas for $d \equiv 0(\text{mod } 2)$ there are three; one of these 3 types has $w_2 \neq 0$, the other two are spin, they are distinguished by $p_1 \equiv 4d(\text{mod } 48), p_1 \equiv 4d + 24(\text{mod } 48)$ respectively. In order to realize these homotopy types as cores of complex 3-folds we first look at simple cyclic coverings of $\mathbb{P}^3$. Given a positive integer d, let $\pi : X \to \mathbb{P}^3$ be a simple cyclic covering of $\mathbb{P}^3$ branched along a smooth surface B of degree dl. Then X has the correct 'degree' d and the characteristic classes $w_2 \equiv (d-1)l(\text{mod } 2)$, and $p_1 = 4d + (1-d)(1+d)dl^2$, see 4.2. For odd d there is nothing to prove. For even d we can realize $w_2 = 0$ or $w_2 \neq 0$ by choosing $l \equiv 0(\text{mod } 2)$ or $l \equiv 1(\text{mod } 2)$. Taking $l \equiv 0(\text{mod } 4)$ gives $w_2 = 0, p_1 \equiv 4d(\text{mod } 48)$, taking $l \equiv 2(\text{mod } 4)$ yields $w_2 = 0$, and $p_1 \equiv 4d + 24(\text{mod } 48)$. It remains to treat the special case $d = 0$, where the 3 homotopy types are given by $w_2 \neq 0$, by $w_2 = 0, p_1 \equiv 0(\text{mod } 16)$, and by $w_2 = 0, p_1 \equiv 8(\text{mod } 16)$. The first two

homotopy types are realizable as cores of elliptic fiber bundles over the projective plane blown up in two points.
The third homotopy type is realized by the core of Oguiso's Calabi-Yau 3-fold X_2 with vanishing cup-form and $p_1(X_2) = 120\varepsilon_2$.

The result just proven suggests a natural question: given a manifold $X_\circ$ as above, which (even) integers $b_3 \geq 0$ occur as the third Betti numbers of complex 3-folds X whose core is homotopy equivalent to $X_\circ$?
There will certainly be some gaps for algebraic 3-folds. In order to show this, we prove the following finiteness theorem for families of Kähler structures:

Theorem 7: *Fix a positive constant c. There exist only finitely many families of 1-connected, smooth projective 3-folds X with $H_2(X,\mathbb{Z}) \cong \mathbb{Z}, w_2(X) \neq 0$, and with $b_3(X) \leq c$.*

Proof: Let X be a smooth projective 3-fold with $H_1(X,\mathbb{Z}) = \{0\}$, $H_2(X, \mathbb{Z}) \cong \mathbb{Z}$, and with $w_2(X) \neq 0$. Clearly $\mathrm{Pic}(X) \cong H^2(X,\mathbb{Z})$, and we can choose a basis $e \in H^2(X,\mathbb{Z})$ corresponding to the ample generator of $\mathrm{Pic}(X)$.
Let $c_1(X) = c_1 e, c_2(X) = c_2\varepsilon$ where $e^2 = d\varepsilon, \varepsilon(e) \doteq 1$. If c_1 is positive, then X is Fano, and there are only finitely many possibilities [Mu]. The case $c_1 = 0$ is excluded, so that we are left with $c_1 < 0$, i.e. the canonical bundle of X is ample.
The Riemann-Roch formula $\chi(X,\mathcal{O}_X) = 1 - h^3(X,\mathcal{O}_X) = \frac{1}{24}c_1c_2$ shows that the set of possible Chern numbers c_1c_2 is bounded from below: $24(1-c) \leq c_1c_2$. Using Yau's inequality $8c_1(X)c_2(X) \leq 3c_1(X)^3$ we find that $d|c_1|^3 \leq 64(c-1)$, i.e. the degree d and the order of divisibility $|c_1|$ of $c_1(X)$ is bounded. Now Kollar's finiteness theorem [$Ko2$] yields the assertion.

Example 15: Let X be a 1-connected, smooth projective 3-fold with $H_2(X,\mathbb{Z}) \cong \mathbb{Z}$ and $w_2(X) \neq 0$. If $b_3(X) \leq 2$, then $h^3(X,\mathcal{O}_X) \leq 1$ and X must be Fano of index 1 or 3. For $b_3(X) = 4$ we have that X is either Fano, or $h^3(X,\mathcal{O}_X) = 2$ and X is of general type with $d|c_1|^3 \leq 64$.

Note that the assumption $w_2 \neq 0$ was only used to exclude Calabi-Yau 3-folds.

5.2 3-folds with $b_2 = 2$

Let X be a 1-connected, closed, oriented, 6-dimensional differentiable manifold with $H_2(X,\mathbb{Z}) \cong \mathbb{Z}^2$.

We choose a basis e_1, e_2 for $H^2(X, \mathbb{Z})$ and set $a_0 = e_1^3, a_1 = e_1^2 e_2, a_2 = e_1 e_2^2, a_3 = e_2^3$; the cubic polynomial f associated to the cup-form of X is then given by $f = a_0 X^3 + 3a_1 X^2 Y + 3a_2 XY^2 + a_3 Y^3$. The discriminant of f is by definition $\Delta(f) = a_0^2 a_3^2 - 3a_1^2 a_2^2 - 6a_0 a_1 a_2 a_3 + 4a_0 a_2^3 + 4a_1^3 a_3$; up to a factor it is simply the discriminant of the Hessian $H_f = 6^2[(a_0 a_2 - a_1^2)X^2 + (a_0 a_3 - a_1 a_2)XY + (a_1 a_3 - a_2^2)Y^2]$ of $f : \Delta(f) = (a_0 a_3 - a_1 a_2)^2 - 4(a_0 a_2 - a_1^2)(a_1 a_3 - a_2^2)$.
The last identity shows that $\Delta(f)$ is always a square modulo 4, i.e. $\Delta(f) \equiv 0, 1 (\bmod 4)$.

Proposition 17: *Every integer $\Delta \equiv 0, 1 (\bmod 4)$ is realizable as discriminant of a compact complex 3-fold.*
Proof: Consider the projectivization $X = \mathbb{P}_{\mathbb{P}^2}(E)$ of a holomorphic rank-2 vector bundle E over the plane. In terms of the standard basis of $H^2(X, \mathbb{Z})$ $(e_1 = \pi^* h, e_2 = c_1(\mathcal{O}_{\mathbb{P}(E)}(1)))$ the cubic polynomial associated to X is given by $f = (c_1^2 - c_2)X^3 + 3(-c_1)X^2 Y + 3XY^2$, where $c_i = c_i(E)$ are the Chern classes of E considered as integers. Inserting this into the discriminant formula yields $\Delta(f) = c_1^2 - 4c_2$. Since every pair c_1, c_2 occurs as pair of Chern classes of a holomorphic rank-2 bundle on $\mathbb{P}^2$, every integer $\Delta \equiv 0, 1 (\bmod 4)$ can be realized as discriminant of a holomorphic projective bundle $\mathbb{P}_{\mathbb{P}^2}(E)$.

Recall from section 3.2 that there are 4 different types of $SL(2)$-orbits of complex binary cubics: non-singular forms f (with $\Delta(f) \neq 0$), and three orbits of singular cubics, represented by the normal forms $X^2 Y, X^3$, and 0.

Proposition 18: *All four types of complex binary cubics are realizable by complex 3-folds.*
Proof: We have seen this already for non-singular cubics. Clearly the product $\mathbb{P}^1 \times \mathbb{P}^2$ realizes the normal form $X^2 Y$. The cubics of normal forms X^3 or 0 are degenerate, i.e. their Hessians vanish identically. Therefore they can only be realized by non-Kählerian 3-folds. To realize X^3 one can blow up a point in an elliptic fiber bundle over a surface Y with $b_2(Y) = 3$; the trivial form occurs for elliptic fiber bundles over a surface with $b_2 = 4$.

More detailed investigations of the possible homotopy types of real or complex manifolds with $b_2 = 2$ will appear elsewhere.
Here we only want to illustrate an interesting phenomenon which relates the ample cone of a projective 3-fold with $b_2 = 2$ to the Hessian of its cup-form.

Proposition 19: *Let X be a smooth projective 3-fold with $b_2(X) = 2$.*

The ample cone C_X is contained in the Hesse cone $\mathcal{H}_F := \{h \in H^2(X,\mathbb{R})| \det(F^t(h)) < 0\}$.

Proof: This is only a special case of our general result in section 4.3.

Remark 14: The Hessian of a binary form $F \in S^3H^\vee$ is identically zero iff F is degenerate; it is negative semi-definite if F is non-degenerate and $\Delta(F) \leq 0$; it is indefinite iff $\Delta(F) > 0[Ca]$. Only in the indefinite case $\Delta(F) > 0$ can the closure $\overline{\mathcal{H}}_F := \{h \in H_{\mathbb{R}}| \det F^t(h) \leq 0\}$ of the Hesse cone be a proper subset of $H_{\mathbb{R}}$.

Example 16: Let $P = \mathbb{P}_{\mathbb{P}^2}(E)$ be the projectivization of a rank-2 vector bundle E with Chern classes $c_i = c_i(E)$. The cup-form of P yields the cubic polynomial $f = (c_1^2 - c_2)X^2 + 3(-c_1)X^2Y + 3XY^2$ whose Hessian is $H_f = (-c_2)X^2 + c_1XY - Y^2$. Rewriting H_f as $H_f = -\frac{1}{4}[(2Y - c_1X)^2 + X^2(4c_2 - c_1^2)] = \frac{-1}{4}[(2Y - c_1X)^2 - \Delta(f)X^2]$ we find 3 possibilities for the Hesse cone:

i) $\Delta(f) < 0 : \mathcal{H}_f = H^2(P,\mathbb{R}) \setminus \{0\}$

ii) $\Delta(f) = 0 : \mathcal{H}_f = H^2(P,\mathbb{R}) \setminus L_{c_1}$ for a real line L_{c_1} depending on c_1 ($L_{c_1} = \mathbb{R}(2, c_1)$ in the coordinates X, Y)

iii) $\Delta(f) > 0 : \mathcal{H}_f$ is an open cone whose angle is determined by $\Delta(f)$ $((Z + \sqrt{\Delta(f)}X)(Z - \sqrt{\Delta(f)}X) > 0$ in coordinates $X, Z := 2Y - c_1X$).

5.3 3-folds with $b_2 \geq 3$

Let X be a 1-connected, compact complex 3-fold with $H_2(X,\mathbb{Z}) \cong \mathbb{Z}^{\oplus 3}$. The cup-form of X gives rise to a curve C_X of degree 3 in the projective plane $\mathbb{P}(H^2(X,\mathbb{C}))$:
$C_X := \{< h > \in \mathbb{P}(H^2(X,\mathbb{C}))| h^3 = 0\}$.
A first natural question is which types of plane cubic curves occur in this way?
Recall that there are 10 types of plane cubics, namely: 1) non-singular cubics, 2) irreducible cubics with a node, 3) irreducible cubics with a cusp, 4) reducible cubics consisting a smooth conic and a transversal line, 5) smooth conics with a tangent line, 6) three lines forming a triangle, 7) three distinct lines through a common point, 8) a double line with a third skew line, 9) a triple line, 10) the trivial 'cubic' with equation 0.

Lemma 4: *If the 3-fold X has a non-trivial Hodge number $h^{2,0}(X) \neq 0$, then C_X is of type 4), 6) 9) or 10).*

Proof: Choose basis vectors $e^{k,l} \in H^{k,l}(X)$, so that every $h \in H^2(X,\mathbb{C})$ can be uniquely written as $h = xe^{2,0} + ye^{1,1} + ze^{0,2}$.

Then clearly $h^3 = y[y^2(e^{1,1})^3 + 6xz(e^{2,0} \cdot e^{1,1} \cdot e^{0,2})]$.

We now realize the cubics of types 7) - 10). These cubics are degenerate, i.e. they are cones, and therefore their Hessians vanish identically. From section 4.3 we know that they can not be realized by Kählerian 3-folds.

Proposition 20: *The plane cubics of types 7) - 10) can all be realized by 1-connected, non-Kählerian 3-folds.*

Proof: 'Cubics' of type 10) can be realized by elliptic fibre bundles over surfaces Y with $b_2(Y) = 5$. In order to realize cubics of type 9) or 7) one blows up one or two points in an elliptic fibre bundle over a surface with $b_2 = 4$ or 3 respectively. The realization of a type 8) cubic is a little trickier: One starts with an elliptic fibre bundle over a surface Y with $b_2(Y) = 3$, and blows up one of its fibers. The resulting 3-fold X' has $b_2(X') = 2$ and $F_{X'} \equiv 0$. Now choose a line l in the exceptional divisor E of X', and let X be the blow-up of X' along l. The cup-form of X yields the cubic polynomial $x^2[y(-3l \cdot E) - x(\deg N_{C/X'})]$ with a non-zero coefficient $-3l \cdot E = 3$.

There are four types of complex cubics which we have been able to realize by projective 3-folds.

Proposition 21: *Cubics of type 1), 3), 4) and 6) are realizable by 1-connected projective 3-folds.*

Proof: Type 1) occurs for blow-ups of complete intersections in two distinct points. The product $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ realizes a triangle, whereas most projective bundles over a surface with $b_2 = 2$ lead to a smooth conic union a transversal line.

Irreducible cubics with a cusp can be obtained by blowing-up a line and a point in $\mathbb{P}^3$. The resulting 3-fold yields the cubic polynomial $X^3 - 3XY^2 - 2Y^3 + Z^3 = (X+Y)^2(X-2Y) + Z^3$.

The remaining two types of cubics are cubics with a node (type 2)), and smooth conics with tangent a line (type 5)). We do not know if these types are realizable by projective 3-folds. A non-Kählerian 3-fold whose cup-form yields a nodal cubic can be constructed: one just takes the blow-up of two suitable curves in Oguiso's Calabi-Yau 3-fold with $b_2 = 1$ and vanishing cup-form.

Finally we like to show that the non-emptiness condition on the index cone of a projective 3-fold with $h^{0,2} = 0$ gives non-trivial restrictions for the possible cup-forms if $b_2 \geq 4$. Further investigations of this condition will appear elsewhere.

Example 17: Let H be a free $\mathbb{Z}$-module of rank 4 with basis $(e_i)_{i=1,\dots,4}$. Consider a trilinear form $F \in S^3H^\vee$ and its adjoint map $F^t : H \to S^2H^\vee$. The image $F^t(h)$ of an element $h \in H$ is in terms of the chosen basis $(e_i)_{i=1,\dots,4}$ represented by the symmetric 4×4-matrix $[[he_ie_j]]_{i,j=1,\dots,4}$. Suppose this matrix is a diagonal sum $[[he_ie_j]]_{i,j=1,2} \oplus [[he_ke_l]]_{k,l=3,4}$ such that the determinants of both 2×2-matrices are negative for every $h \in H \setminus \{0\}$.
In this case $F^t(h)$ were of signature (1,-1,1,-1) for every $h \in H \setminus \{0\}$, and we would have $I_F = \mathcal{H}_F = \emptyset$.
All these conditions can be met, e.g. by setting $e_1^2e_2 = e_2^3 = e_3^2e_4 = e_4^3 = 1, e_1e_2^2 = e_3e_4^2 = 2$, and $e_ie_je_k = 0$ otherwise. In this particular case the image of $h = \sum_{i=1}^4 h_ie_i$ under F^t in represented by the matrix

$$\left[\begin{array}{cc|cc} h_2 & h_1 + 2h_2 & 0 & 0 \\ h_1 + 2h_2 & 2h_1 + h_2 & 0 & 0 \\ \hline 0 & 0 & h_4 & h_3 + 2h_4 \\ 0 & 0 & h_3 + 2h_4 & 2h_3 + h_4 \end{array}\right],$$

which has a positive determinant unless $h = 0$.

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