SIMPLICIAL CONSTRUCTIONS ASSOCIATED WITH S^2

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Abstract. Certain well-known facts from the theory of homotopy groups of spheres are considered from the point of view of commutator calculus in free groups and Lie algebras. The Leibniz analogs of the homotopy groups of the 2-sphere are considered.

1. Homotopy groups of the 2-sphere and group theory

1.1. **Milnor's $F[S^1]$-construction.** For the n-sphere S^n there are at least two classical ways how to associate a simplicial group whose homotopy groups will give the homotopy groups $\pi_*(S^n)$. The first one is Kan's construction GS^n, the second one is Milnor's construction $F[S^n]$ with geometric realization $|F[S^n]|$ weakly homotopically equivalent to the loop space $\Omega\Sigma S^n = \Omega S^{n+1}$. Different cell decompositions of S^n define different simplicial groups GS^n, which are, of course, weakly equivalent. The simplest construction of this kind from the point of view of simplicial structure is clearly GS^2 associated with cell decomposition $S^n = * \cup e_n$. Recall that, for a given pointed simplicial set K, the $F[K]$-construction is the simplicial group with $F[K]_n = F(K_n \setminus *)$, where $F(-)$ is the free group functor.

Consider the simplicial circle $S^1 = \Delta[1]/\partial\Delta[1]$:

$$S^1_0 = \{*\},\ S^1_1 = \{*, \sigma\},\ S^1_2 = \{*, s_0\sigma, s_1\sigma\}, \dots, S^1_n = \{*, x_0, \dots, x_n\},$$

where $x_i = s_n \dots \hat{s}_i \dots s_0\sigma$. The $F[S^1]$-construction then clearly has the following terms:

$$\begin{aligned}
&F[S^1] = 0,\\
&F[S^1]_1 = F(\sigma),\ \text{free abelian group generated by } \sigma,\\
&F[S^1]_2 = F(s_0\sigma, s_1\sigma),\\
&F[S^1]_3 = F(s_is_j\sigma \mid 0 \le j \le i \le 2),\\
&\dots
\end{aligned}$$

The face and degeneracy maps are determined naturally (with respect to the standard simplicial identities) for these simplicial groups. For example, the first nontrivial maps are defined as follows:

$$\begin{aligned}
&\partial_i : F[S^1]_2 \to F[S^1]_1,\ i = 0, 1, 2,\\
&\partial_0 : s_0\sigma \mapsto \sigma,\ s_1\sigma \mapsto 1,\\
&\partial_1 : s_0\sigma \mapsto \sigma,\ s_1\sigma \mapsto \sigma,\\
&\partial_2 : s_0\sigma \mapsto 1,\ s_1\sigma \mapsto \sigma.
\end{aligned}$$

The above construction gives a possibility to define the homotopy groups $\pi_n(S^2)$ combinatorially, in terms of free groups. Since the geometrical realization of $F[S^1]$ is weakly homotopically equivalent to the loop space ΩS^2, the homotopy groups $\pi_n(S^2)$ are naturally isomorphic to the homotopy groups of the Moore complex of $F[S^1]$: $\pi_{n+1}(S^2) \simeq \mathcal{Z}_n(F[S^1])/\mathcal{B}_n(F[S^1])$. Here

$\mathcal{Z}_n$ and $\mathcal{B}_n$ denote the cycles and the boundaries of the Moore complex of the correspondent simplicial group.

The explicit structure of the cycles and boundaries for $F[S^1]$ can be given in terms of certain normal subgroups in $F[S^1]$. This was realized by Jie Wu. Recall some notation from the work [4]. For a free group F and its normal subgroups $H_1, \dots, H_n$, denote

$$[[H_1, \dots, H_n]] := \prod_{(i_1,\dots,i_n)\in S_n} [H_{i_1}, \dots, H_{i_n}],$$

where the notation used is left-normalized. Let F_n be a free group with basis $x_0, x_1, \dots, x_{n-1}$ and $x_{-1} := x_0 x_1 \dots x_{n-1}$. Then there are the following natural equalities:

$$\begin{aligned} \mathcal{Z}_n &= \langle x_{-1}\rangle^{F_n} \cap \langle x_0\rangle^{F_0} \cap \dots \cap \langle x_{n-1}\rangle^{F_n}, \\ \mathcal{B}_n &= [[\langle x_{-1}\rangle^{F_n}, \langle x_0\rangle^{F_n}, \dots, \langle x_{n-1}\rangle^{F_n}]]. \end{aligned}$$

Hence, we have

$$\pi_{n+1}(S^2) \simeq \mathcal{Z}_n/\mathcal{B}_n, \ n \geq 2.$$

The main goal of these notes is to make explicit commutator computations, searching generators of the homotopy groups of S^2 inside the Moore complex of $F[S^1]$ and related constructions. The first step $n = 2$ is almost trivial. One can check that

$$\mathcal{Z}_2 = \gamma_2(F_2), \ \mathcal{B}_2 = \gamma_3(F_2),$$

where F_2 is a free group of rank 2. Hence

$$\pi_3(S^2) \simeq \gamma_2(F_2)/\gamma_3(F_2) \simeq \mathbb{Z}$$

and the generator of $\pi_3(S^2)$ is the commutator of two generators of F_2.

1.2. **Hopf fibration.** The Hopf fibration $S^3 \to S^2$ has a fibre S^1, hence, it induces isomorphisms of homotopy groups in dimension greater than 2. By Hopf fibration from the point of view of Milnor's F-construction we mean a homomorphism of simplicial groups

$$\eta : F(S^3) \to F(S^2),$$

which induces an isomorphism

$$\eta^* : \pi_2(F(S^3)) \to \pi_2(F(S^2)).$$

Since $\pi_3(F(S^3))$ can be viewed as a coset of σ, but $\pi_3(F(S^2))$ as a coset of $[s_0\sigma, s_1\sigma]$, we can define η at the first nontrivial level as

$$\eta : F(S^3)_3 \to F(S^2)_3, \ \sigma \mapsto [s_0\sigma, s_1\sigma]$$

and extend it naturally

$$\eta : F(S^3)_4 \to F(S^2)_4, \ s_i\sigma \mapsto [s_i s_0\sigma, s_i s_1\sigma]$$

etc. Clearly, the constructed η has the needed property.

Define the words $w_n, n \geq 2$ in abstract variables $y_0, y_1, \dots$ by setting $w_2(y_0, y_1) = [y_0, y_1]$ and inductively:

$$w_{n+1}(y_0, \dots, y_n) = [w_n(y_0, y_1, y_2, \dots, y_{n-1}y_n), w_n(y_0, y_1, \dots, y_{n-1})].$$

For example, we have

$$\begin{aligned}
&w_2(y_0, y_1) = [y_0, y_1],\\
&w_3(y_0, y_1, y_2) = [[y_0, y_1y_2], [y_0, y_1]],\\
&w_4(y_0, y_1, y_2, y_3) = [[[y_0, y_1y_2y_3], [y_0, y_1]], [[y_0, y_1y_2], [y_0, y_1]]].
\end{aligned}$$

It is easy to show that

$$w_n(x_0, \dots, x_{n-1}) \in \mathcal{Z}_n, \ n \geq 2.$$

These elements represent η-elements in $\pi_n(S^2)$. Consider the element $w_3(x_0, x_1, x_2) \in F_3$ modulo $\mathcal{B}_3$:

$$\begin{aligned}
&w_3(x_0, x_1, x_2) = [[x_0, x_1x_2], [x_0, x_1]] = [[x_0x_1x_2, x_1x_2]^{(x_1x_2)^{-1}}, [x_0, x_1]] =\\
&[[x_0x_1x_2, x_2]^{(x_1x_2)^{-1}}[x_0x_1x_2, x_1]^{x_1^{-1}}, [x_0, x_1]] =\\
&[[x_0x_1x_2, x_2]^{(x_1x_2)^{-1}}, [x_0, x_1]]^{[x_0x_1x_2, x_1]^{x_1^{-1}}}[[x_0x_1x_2, x_1]^{x_1^{-1}}, [x_0, x_1]] \equiv\\
&[[x_0x_1x_2, x_1]^{x_1^{-1}}, [x_0, x_1]] = [[x_0x_1x_2, x_1]^{x_1^{-1}}, [x_0, x_1x_2]^{x_2^{-1}}[x_0, x_2]^{-x_2^{-1}}] \equiv\\
&[[x_0x_1x_2, x_1]^{x_1^{-1}}, [x_0, x_1x_2]^{x_2^{-1}}] = [[x_0, x_1]^{x_1x_2x_1^{-1}}[x_1x_2, x_1]^{x_1^{-1}}, [x_0, x_1x_2]^{x_2^{-1}}] =\\
&[[x_0, x_1]^{x_1x_2x_1^{-1}}[x_2, x_1]^{x_1^{-1}}, [x_0, x_0x_1x_2]^{x_2^{-1}}] \equiv [[x_0, x_1]^{x_1x_2x_1^{-1}}, [x_0, x_0x_1x_2]^{x_2^{-1}}] \equiv\\
&[[x_0, x_1], [x_0, x_0x_1x_2]^{x_2^{-1}}] \equiv [[x_0, x_1], [x_0, x_0x_1x_2]] =\\
&[[x_0, x_1], [x_0, x_1x_2]] = w_3(x_0, x_1, x_2)^{-1}.
\end{aligned}$$

Hence, the element $w_3(x_0, x_1, x_2)$ is of order 2 modulo $\mathcal{B}_3$ and we have the well-known result first due to Whitehead:

$$\pi_4(S^2) \simeq \mathbb{Z}_2.$$

Analogically, one can prove that $w_n(x_0, x_1, \dots, x_{n-1})$ has the order 2 modulo $\mathcal{B}_n$.

We know from homotopy theory the following fact.

For $n = 3, 4$, the group $\mathcal{Z}_n/\mathcal{B}_n$ is cyclic and generated by $w_n(x_0, \dots, x_{n-1})\mathcal{B}_n$.

However, the purely algebraic proof of this statement is nontrivial.

1.3. **From free groups to free Lie algebras.** Clearly, Lie algebras are much more convenient from the point of view of commutator calculus than groups. It follows from the simplification of main commutator identities in groups:

$$\begin{aligned}
&[ab, c] = [a, c]^b[b, c] \ \rightsquigarrow \ [a + b, c] = [a, c] + [b, c];\\
&[a, bc] = [a, c][b, c]^a \ \rightsquigarrow \ [a, b + c] = [a, b] + [a, c];\\
&[a, b^{-1}, c]^b[b, c^{-1}, a]^c[c, a^{-1}, b]^a = 1 \ \rightsquigarrow \ [a, b, c] + [b, c, a] + [c, a, b] = 0.
\end{aligned}$$

For a given group G, denote by $\{\gamma_n(G)\}_{n\geq 1}$ the lower central series of G. Recall that a simplicial group G (or simplicial Lie algebra) is called *connected* if $G_0 = 1$ (resp. $G_0 = 0$). The following result is due to Curtis [2].

Theorem 1. *Let F be a connected simplicial group or Lie algebra (over $\mathbb{Z}$) then $\gamma_r(F)$ is $log_2 r$-connected.*

Let $L : \mathcal{G}r \to \mathcal{L}ie$ be the Lie-functor:

$$L : G \mapsto \bigoplus_{i\geq 1} \gamma_i(G)/\gamma_{i+1}(G).$$

Clearly, this functor can be extended to the functor between categories of simplicial groups and simplicial Lie algebras.

Theorem 2. *Let F be a connected free simplicial group. Then there are natural isomorphisms of abelian groups:*

$$\pi_i(F) \simeq \pi_i(L(F)), \ i \geq 1,$$

where $L(F)$ is a free simplicial Lie algebra, constructed functorially from F.

Proof. Let us prove by induction on n, that there are the natural isomorphism of abelian groups

$$\pi_i(F/\gamma_n(F)) \simeq \pi_i(L(F)/L^n(F)), \ i \geq 1, \ n \geq 2. \tag{1.1}$$

First, due to Magnus-Witt and Hall's theorems, we have the natural isomorphsim of abelian simplicial groups

$$\gamma_n(F)/\gamma_{n+1}(F) \simeq L^n(F)/L^{n+1}(F)$$

and (1.1) follows for $n = 2$. Suppose we have the natural isomorphisms (1.1) for a given n. Then the isomorphism for (1.1) follows from the natural commutative diagram of abelian groups:

$$\begin{array}{ccccc}
\pi_i(\gamma_n(F)/\gamma_{n+1}(G)) & \longrightarrow & \pi_i(G/\gamma_{n+1}(F)) & \longrightarrow & \pi_i(G/\gamma_n(G)) \\
\| & & \downarrow & & \| \\
\pi_i(L^n(F)/L^{n+1}(F)) & \longrightarrow & \pi_i(L(F)/L^{n+1}(F)) & \longrightarrow & \pi_i(L(F)/L^n(F)).
\end{array}$$

Theorem 1 implies that there are the natural isomorphisms:

$$\pi_i(F) \simeq \pi_i(F/\gamma_r(F)), \ \pi_i(L(F)) \simeq \pi_i(L(F)/L^r(F)), \ i < log_2 r,$$

□

The same way as for a group, for a given Lie algebra and its ideals $I_1, \dots, I_n$, define

$$[[I_1, \dots, I_n]] = \sum_{(i_1,\dots,i_n)\in S_n} [I_{i_1}, \dots, I_{i_n}].$$

Let L_n be a free Lie algebra over $\mathbb{Z}$ with basis $x_0, \dots, x_{n+1}$. Denote

$$x_{-1} = x_0 + x_1 + \dots + x_{n-1}$$

and I_i the ideal in L_n, generated by $x_i, \ i = -1, 0, \dots, n-1$. Define the ideals:

$$\mathcal{Z}_n = I_{-1} \cap I_0 \cap \dots \cap I_{n-1},$$
$$\mathcal{B}_n = [[I_{-1}, I_0, \dots, I_{n-1}]].$$

(We use the same notation as in the case of free groups.)

2. Computation of $\pi_4(S^2)$

Consider a free Lie algebra L_3 with basis x_0, x_1, x_2. Curtis Theorem 1 together with Milnor's F-construction give the following description of the fourth homotopy group of S^2:

$$\pi_4(S^2) \simeq \mathcal{Z}_3/(\mathcal{B}_3 + L_3^5).$$

Let us compute this quotient.

Theorem 3. $\mathcal{Z}_3/(\mathcal{B}_3 + L_3^5) \simeq \mathbb{Z}_2$.

Proof. Let $w \in Z_3$. Clearly, $w \in [[I_0, I_1, I_2]] \in L_5^3$. Suppose $w \notin L_5^4$. Then w can be written as

$$w \equiv a_1[x_0, x_1, x_2] + a_2[x_1, x_2, x_0] \mod L_3^4.$$

Taking the quotient of L_3 by the ideal $I_{-1} = \langle x_0 + x_1 + x_2 \rangle L_3$. We get

$$-a_1[x_0, x_2, x_2] - a_2[x_0, x_2, x_0] \in L_2^4,\ a_1, a_2 \in \mathbb{Z},$$

where L_2 is a free Lie algebra with generators x_0, x_2. Hence, $a_1, a_2 = 0$, since $[x_0, x_2, x_2], [x_0, x_2, x_0]$ are Hall's basis commutators. We conclude that $w \in L_3^4$.

Hall basis theorem says that L_3^4/L_3^5 is a free abelian group freely generated by the following set of basic commutators:

$$\begin{aligned}
&e_1 = [x_2, x_0, x_0, x_1],\ e_2 = [x_2, x_0, x_1, x_1],\ e_3 = [x_1, x_0, x_2, x_2],\\
&e_4 = [x_1, x_0, x_0, x_2],\ e_5 = [x_1, x_0, x_1, x_2],\ e_6 = [x_2, x_0, x_1, x_2],\\
&e_7 = [[x_2, x_0], [x_1, x_0]],\ e_8 = [[x_2, x_1], [x_1, x_0]],\ e_9 = [[x_2, x_1]], [x_2, x_0]],\\
&e_{10} = [x_2, x_1, x_1, x_1],\ e_{11} = [x_1, x_0, x_0, x_0],\ e_{12} = [x_2, x_0, x_0, x_0],\\
&e_{13} = [x_1, x_0, x_1, x_1],\ e_{14} = [x_1, x_0, x_0, x_1], e_{15} = [x_2, x_0, x_2, x_2],\\
&e_{16} = [x_2, x_0, x_0, x_2],\ e_{17} = [x_2, x_1, x_2, x_2],\ e_{18} = [x_2, x_1, x_1, x_2].
\end{aligned}$$

It is easy to see that

$$e_7 \equiv e_8 \equiv e_9 \equiv -e_7 \equiv -e_8 \equiv -e_9 \mod \mathcal{B}_3 + L_3^5$$

and $e_7 \in \mathcal{Z}_3$. Suppose w is not equivalent to e_7 modulo $\mathcal{B}_3 + L_3^5$. Then w modulo $\langle e_7 \rangle \mathcal{B}_3 + L_3^5$ can be written as

$$w \equiv \sum_{i=1}^{6} a_i e_i \mod \langle e_7 \rangle + \mathcal{B}_3 + L_3^5,\ a_i \in \mathbb{Z}.$$

Observe that

$$\begin{aligned}
e_5 &\equiv -e_6 \mod \mathcal{B}_3,\\
e_4 &\equiv -e_5 - e_3 \mod \mathcal{B}_3,\\
e_1 &\equiv -e_2 - e_6 \mod \langle e_7 \rangle + \mathcal{B}_3.
\end{aligned}$$

Therefore,

$$w \equiv b_1 e_2 + b_2 e_3 + b_3 e_6 \mod \langle e_7 \rangle + B_3 + L_3^5.$$

Taking the quotient of L_3 by I_{-1}, we get

$$-b_1[x_2, x_1, x_1, x_1] - b_2[x_1, x_2, x_2, x_2] - b_3[x_2, x_1, x_1, x_2] \in L_2^5, \tag{2.1}$$

where L_2 is a free Lie algebra with generators x_1, x_2. The commutators in (2.1) present different basic commutators, hence $b_1, b_2, b_3 = 0$ and therefore,

$$w \in \langle e_7 \rangle + B_3 + L_3^5.$$

Now let us prove that

$$[[x_0, x_2], [x_0, x_1]] \notin \mathcal{B}_3 + L_3^5.$$

It is easy to check that B_3 modulo L_3^5 is generated by the following elements:

$$\alpha_1 = [x_0 + x_1 + x_2, x_0, x_1, x_2], \ \alpha_2 = [x_0 + x_1 + x_2, x_0, x_2, x_1],$$
$$\alpha_3 = [x_0 + x_1 + x_2, x_1, x_0, x_2], \ \alpha_4 = [x_0 + x_1 + x_2, x_1, x_2, x_0],$$
$$\alpha_5 = [x_0 + x_1 + x_2, x_2, x_0, x_1], \ \alpha_6 = [x_0 + x_1 + x_2, x_2, x_1, x_0],$$
$$\alpha_7 = [x_0, x_1, x_0 + x_1 + x_2, x_2], \ \alpha_8 = [x_0, x_2, x_0 + x_1 + x_2, x_1],$$
$$\alpha_9 = [x_1, x_2, x_0 + x_1 + x_2, x_0].$$

These elements can be written modulo L_3^5 in terms of Hall's basis in the following way:

$$\alpha_1 = e_5 + e_6,$$
$$\alpha_2 = e_5 + e_6 - e_8 - e_9,$$
$$\alpha_3 = -e_3 + e_4 + e_6 + e_7 + e_9,$$
$$\alpha_4 = -e_3 - e_4 + e_6 + e_7 + e_9,$$
$$\alpha_5 = -e_1 - e_2 + e_5 - e_8,$$
$$\alpha_6 = -e_1 + e_5 - e_7 - 2e_8,$$
$$\alpha_7 = -e_3 - e_4 - e_5,$$
$$\alpha_8 = -e_1 - e_2 - e_6 + e_9,$$
$$\alpha_9 = -e_1 + e_2 + e_3 + e_4 - e_5 - e_6 - 2e_7 - e_9.$$

Define the ideal

$$D := \langle e_2, e_3, e_4, e_5, e_6, e_7 + e_8, e_8 + e_9, 2e_7 \rangle L_3.$$

Then $\mathcal{B}_3/(D+L_3^5) = \langle \alpha_1, \dots, \alpha_9 \rangle L_3/(D+L_3^5)$ is a free abelian group generated by element e_1+e_7. Hence e_7 is nontrivial in $\mathcal{B}_3/(D+L_3^5)$ and the needed statement follows, since $[[x_0, x_2], [x_0, x_1]] = e_7$. $\square$

3. Elements v_n

The same way as for the words $w_n(y_0, \dots, y_{n-1})$ in free groups, we define the words $v_n(z_0, \dots, z_{n-1})$ in free Lie algebras, by setting

$$v_2(z_0, z_1) = [z_0, z_1],$$
$$v_3(z_0, z_1, z_2) = [[z_0, z_2], [z_0, z_1]],$$
$$v_4(z_0, z_1, z_2, z_3) = [[[z_0, z_3], [z_0, z_1]], [[z_0, z_2], [z_0, z_1]]],$$
$$v_{n+1}(z_0, \dots, z_n) = [v_n(z_0, \dots, z_{n-2}, z_n), v_n(z_0, \dots, z_{n-1})].$$

Obviously, in a free Lie algebra L_n:

$$v_n(x_0, \dots, x_{n-1}) \in \mathcal{Z}_n, \ v_n(x_0, \dots, x_{n-1})^2 \in \mathcal{B}_n.$$

The situation with these elements is very interesting from the algebraic point of view:

$$v_2(x_0, x_1) \notin \mathcal{B}_2,$$
$$v_3(x_0, x_1, x_2) \notin \mathcal{B}_3,$$
$$v_4(x_0, x_1, x_2, x_3) \notin \mathcal{B}_4,$$
$$v_5(x_0, x_1, x_2, x_3, x_4) \notin \mathcal{B}_5,$$
$$v_6(x_0, x_1, x_2, x_3, x_4, x_5) \in \mathcal{B}_6!$$

It seems to be very strange from the algebraic point of view! The reason for that is that the composition of η-maps:

$$S^7 \to S^6 \to S^5 \to S^4 \to S^3 \to S^2$$

is null-homotopical. This is an algebraic interpretation of the simplest case of Nishida's Nilpotency Theorem.

4. 3-torsion of $\pi_6(S^2)$

Let L_5 be a free Lie algebra over $\mathbb{Z}$ with generators x_0, x_1, x_2, x_3, x_4. Consider the following ideals: I_i is the ideal in L_5, generated by x_i, $i = 0, \dots, x_4$, I_5 is the ideal in L_5, generated by the element $x_0 + x_1 + x_2 + x_3 + x_4 + x_5$. We know from the homotopy theory that

$$\pi_6(S^2) \simeq \frac{I_{-1} \cap \dots \cap I_4}{[[I_{-1}, \dots, I_4]]} \simeq \mathbb{Z}_{12}. \tag{4.1}$$

The construction of the 3-torsion element in the above quotient seems to be non-trivial. Consider the following element in L_5:

$$\begin{aligned}\Psi :=&[[x_0, x_1], [x_0, x_2], [x_3, x_4]] + [[x_0, x_3], [x_0, x_1], [x_2, x_4]]+\\ &[[x_0, x_1], [x_0, x_4], [x_2, x_3]] + [[x_0, x_2], [x_0, x_3], [x_1, x_4]]+\\ &[[x_0, x_4], [x_0, x_2], [x_1, x_3]] + [[x_0, x_3], [x_0, x_4], [x_1, x_2]].\end{aligned}$$

The fact that $\Psi \in I_0 \cap I_1 \cap I_2 \cap I_3 \cap I_4$ is obvious. Let us check that $\Psi \in I_{-1}$. Taking the quotient by I_{-1}, we have the following image of Ψ:

$$\begin{aligned}\bar{\Psi} =& - [[x_0, x_1], [x_0, x_2], [x_3, x_0 + x_1 + x_2]] - [[x_0, x_3], [x_0, x_1], [x_2, x_0 + x_1 + x_3]]-\\ &- [[x_0, x_1], [x_0, x_2 + x_3], [x_2, x_3]] - [[x_0, x_2], [x_0, x_3], [x_1, x_0 + x_2 + x_3]]-\\ &- [[x_0, x_1 + x_3], [x_0, x_2], [x_1, x_3]] - [[x_0, x_3], [x_0, x_1 + x_2], [x_1, x_2]] =\\ &[[x_0, x_1], [x_0, x_2], [x_0, x_3]] + [[x_0, x_3], [x_0, x_1], [x_0, x_2]] + [[x_0, x_2], [x_0, x_3], [x_0, x_1]] = 0.\end{aligned}$$

The element Ψ is the natural candidate for the role of the 3-torsion in (4.1).

One can generalize the construction of the element Ψ and define for a given $n \geq 2$ the element Ψ_n on $2n + 1$ generators $x_0, \dots, x_{2n}$, which is the sum of the elements of the form

$$[[*, *], [*, *], \dots, [*, *]],$$

and which lies in the intersection of the ideals

$$I_{-1} \cap \dots \cap I_{2n+1}$$

in the free Lie algebra L_{2n+1}. For the prime $n+1$, such elements are natural candidates for the generators of the p-torsion of the homotopy group $\pi_{2p}(S^2)$, which is non-trivial due to Serre.

5. Mod-2 case: unstable Adams spectral sequence

The case of mod-p homotopy groups of connected simplicial groups is much simpler. For that reason one can use so-called unstable Adams spectral sequence, introduced in [1]. The general description of E^1-term as Λ-algebra in [1] gives a possibility to describe the lower generators in terms of commutators in free groups. In this section for a given group G, $\{\gamma_{n,2}(G)\}$ denote the mod-2 lower central series of G, where $\gamma_{n_2}(G)$ is defined as the normal closure in G of the left-normalized brackets $[a_1,\dots,a_s]^{2^t}$ with $s2^t \geq n$. The mod-2 Lie functor now is the mod-2 lower central quotient.

Consider the unstable Adams spectral sequence for S^2 from [1]. The first term $E^1(S^2)$ can be described as the (mod-2) Lie-functor applied to arbitrary free abelian simplicial $K(\mathbb{Z},1)$:

$$E^1_{p,q}(S^2) = \pi_q(\mathcal{L}^p(AK(\mathbb{Z},1))) \Longrightarrow \pi_{q+1}(S^2,2).$$

One can take $\mathcal{L}^p(AK(\mathbb{Z},1))) = \gamma_{2^p,2}(GS^2)/\gamma_{2^{p+1},2}(GS^2)$. By [1] we have the description of the structure of $E^1(S^2)$ in terms of generators of the Λ-algebra. In terms of these generators (denoted by λ_i) we have:

$\pi_3(S^2,2)$ is generated by $\lambda_1 \in \pi_2(\gamma_{2,2}(GS^2)/\gamma_{4,2}(GS^2))$,

$\pi_4(S^2,2)$ is generated by $\lambda_1\circ\lambda_1 = \lambda_1\circ\Sigma\lambda_1 \in \pi_3(\gamma_{4,2}(GS^2)/\gamma_{8,2}(GS^2))$ (Σ is suspension),

$\pi_5(S^2,2) = E^\infty_{3,4}(S^2)\oplus E^\infty_{4,4}(S^2)$, where $E^\infty_{3,4}(S^2)(=0)$ is generated by

$\lambda_1\circ\lambda_2 \in \pi_4(\gamma_{4,2}(GS^2)/\gamma_{8,2}(GS^2))$,

$E^\infty_{4,4}(S^2)$ is generated by $\lambda_1\circ\lambda_1\circ\lambda_1 = \lambda_1\circ\Sigma\lambda_1\circ\Sigma^2\lambda_1 \in \pi_4(\gamma_{8,2}(GS^2)/\gamma_{16,2}(GS^2))$,

$\pi_6(S^2,2)$ is generated by $\lambda_1\circ\lambda_1\circ\ \lambda_1\circ\lambda_1 = \lambda_1\circ\Sigma\lambda_1\circ\Sigma^2\lambda_1\circ\Sigma^3\lambda_1 \in \pi_5(\gamma_{16,2}(GS^2)/\gamma_{32,2}(GS^2))$,

together with $\lambda_1\circ\lambda_2\circ\lambda_1 = \lambda_1\circ\lambda_2\circ\Sigma^3\lambda_1 \in \pi_5(\gamma_{8,2}(GS^2)/\gamma_{16,2}(GS^2))$,

$\lambda_1\circ\lambda_1\circ\lambda_2 = \lambda_1\circ\Sigma\lambda_1\circ\Sigma\lambda_2 \in \pi_5(\gamma_{8,2}(GS^2)/\gamma_{16,2}(GS^2))$

On the commutator language the elements λ_1, $\lambda_1\circ\lambda_1$, $\lambda_1\circ\lambda_1\circ\lambda_1$, etc can be presented by commutators

$$[x_0,x_1],\ [[x_0,x_1],[x_0,x_2]],\ [[[x_0,x_1],[x_0,x_3]],[[x_0,x_1],[x_1,x_2]]],\ \dots$$

These are suspensions over Hopf, therefore, they are nontrivial in π_3,π_4,π_5,π_6. Direct computations give the following structure of the representatives of the above elements:

$$\lambda_1\circ\lambda_2 = [[x_0,x_1],[x_2,x_3]] - [[x_0,x_2],[x_1,x_3]] + [[x_0,x_3],[x_1,x_2]],$$

$$\begin{aligned}\lambda_1\circ\lambda_1\circ\lambda_2 = [[[x_0,x_1],[x_0,x_2]],[[x_0,x_3],[x_0,x_4]]] - [[[x_0,x_1],[x_0,x_3]],[[x_0,x_2],[x_0,x_4]]]\\ + [[[x_0,x_1],[x_0,x_4]],[[x_0,x_2],[x_0,x_3]]].\end{aligned}$$

$$\begin{aligned}\lambda_1\circ\lambda_2\circ\lambda_1 = [[[x_0,x_1],[x_2,x_3]],[[x_0,x_1],[x_2,x_4]]] - [[[x_0,x_2],[x_1,x_3]],[[x_0,x_2],[x_1,x_4]]]\\ + [[[x_0,x_3],[x_1,x_2]],[[x_0,x_4],[x_1,x_2]]]\end{aligned}$$

6. Leibniz homotopy groups

Let R be a commutative ring with identity. Recall that a *(left) Leibniz algebra* is an R-vector space $\mathfrak{g}$ equipped with a bilinear map $[-,-]:\mathfrak{g}\otimes\mathfrak{g}\to\mathfrak{g}$ satisfying the identity:

$$[[x,y],z] = [x,[y,z]] - [y,[x,z]].$$

Clearly, Lie algebras are Leibniz algebras satisfying $[x,x]=0$.

We shall use the left-normalized notation. First observe that for any $x, y \in \mathfrak{g}$,

$$[x, x, y] = [x, [x, y]] - [x, [x, y]] = 0.$$

However, the identity $[x, [x, x]] = 0$ does not follow.

Let $R = \mathbb{Z}$. Consider the cyclic Lebniz algebra, generated by single element x:

$$L_1 = \langle x \rangle.$$

Proposition 1. *The set of right-normalized commutators:*

$$x, [x, x], [x, [x, x]], [x, [x, [x, x]]], [x, [x, [x, [x, x]]]], \ldots$$

form a $\mathbb{Z}$-basis of L_1.

Consider the Leibniz analogy $Lei[S^1]$ of Milnor's $F[S^1]$-construction. This is a simplicial Leibniz algebra with $L_0 = 0$, L_1 - the cyclic Leibniz algebra and L_n - the n generated free Leibniz algebra (all Leibniz algebras we consider over $\mathbb{Z}$). The face and degeneracy maps can be written naturally, the same way as for the case of simplicial Lie algebra's analogy of $F[S^1]$. We define *Leibniz homotopy groups of* S^2 as

$$\pi_i^{Lei}(S^2) = \pi_{i+1}(Lei[S^1]), \ i \geq 1.$$

The homotopy groups of $Lei[S^1]$ are defined naturally through its Moore complex. It is not clear, however, that these are abelian groups.

For a given Leibniz algebra L and two its ideals H, K, define the ideal

$$[H, K] := \langle [x, y], [y, x], \ x \in H, \ y \in K \rangle.$$

Then for ideals $H_1, \ldots, H_n$ in L, define inductively

$$[H_1, \ldots, H_n] := [[H_1, \ldots, H_{n-1}], H_n] + [H_n, [H_1, \ldots, H_{n-1}]]$$

and

$$[[H_1, \ldots, H_n]] = \oplus_{(i_1, \ldots, i_n) \in S_n} [H_{i_1}, \ldots, H_{i_n}].$$

The second term of our simplicial Leibniz algebra is L_2, a free Leibniz algebra with two generators x_0 and x_1. We have three homomorphisms:

$$\begin{aligned}
&\partial_i : L_2 \to L_1, \ i = 0, 1, 2, \\
&\partial_0 : x_0 \mapsto 0, \ x_1 \mapsto x, \\
&\partial_1 : x_0 \mapsto x, \ x_1 \mapsto x, \\
&\partial_2 : x_0 \mapsto x, \ x_1 \mapsto 0.
\end{aligned}$$

By definition, the second Leibniz homotopy is

$$\pi_2^{Lei}(S^2) := L_1 / im(\partial_0 : ker(\partial_1) \cap ker(\partial_2) \to L_1).$$

Proposition 2.

$$ker(\partial_1) \cap ker(\partial_2) = \langle x_0 - x_1 \rangle \cap \langle x_1 \rangle$$

is the ideal in L_2, generated by brackets

$$[x_0 - x_1, x_1], [x_1, x_0 - x_1].$$

Clearly, both brackets $[x_0 - x_1, x_1], [x_1, x_0 - x_1]$ after applying ∂_0, go to the element $[x, x]$. Thus, we have

$$\pi_2^{Lei}(S^2) = \langle x\rangle / \langle [x, x]\rangle \simeq \mathbb{Z}$$

That is, the second Leibniz homotopy group coincides with classical one.

Now let us make the next step. Take L_3, a free Leibniz algebra with generators x_0, x_1, x_2 and consider four homomorphisms

$$\begin{aligned} &d_i : L_3 \to L_2,\ i = 0, 1, 2, 3,\\ &d_0 : x_0 \mapsto x_0,\ x_1 \mapsto 0,\ x_2 \mapsto x_1,\\ &d_1 : x_0 \mapsto x_0,\ x_1 \mapsto x_1,\ x_2 \mapsto x_1,\\ &d_2 : x_0 \mapsto x_0,\ x_1 \mapsto x_1,\ x_2 \mapsto x_0,\\ &d_3 : x_0 \mapsto 0,\ x_1 \mapsto x_1,\ x_2 \mapsto x_0. \end{aligned}$$

These homomorphisms naturally correspond to the face maps in $F[S^1]$. By definition,

$$\pi_3^{Lei}(S^2) := \frac{ker(\partial_1) \cap ker(\partial_2) \cap ker(\partial_3)}{im(d_0 : ker(d_1) \cap ker(d_2) \cap ker(d_3) \to L_2)}.$$

Proposition 3. *Leibniz cycle*

$$ker(\partial_0) \cap ker(\partial_1) \cap ker(\partial_2) = \langle x_0\rangle \cap \langle x_1\rangle \cap \langle x_0 - x_1\rangle$$

is the ideal in L_2, generated by elements

(6.1) $$[x_0, x_1, x_0],\ [x_0, x_1] - [x_1, x_0]$$

and the ideal $[[\langle x_0\rangle, \langle x_1\rangle, \langle x_0 - x_1\rangle]]$.

Now consider the "Leibniz boundaries". We have the following:

Proposition 4. *In L_3:*

$$ker(d_1) = \langle x_1 - x_2\rangle,\ ker(d_2) = \langle x_0 - x_2\rangle,\ ker(d_3) = \langle x_0\rangle.$$

$$ker(d_1) \cap ker(d_2) \cap ker(d_3)$$

is the ideal in L_3 equal to

$$[[\langle x_0\rangle, \langle x_1 - x_2\rangle, \langle x_0 - x_2\rangle]].$$

Clearly, the image of $[[\langle x_0\rangle, \langle x_1 - x_2\rangle, \langle x_0 - x_2\rangle]]$ under the map d_0 is the ideal

$$[[\langle x_0\rangle, \langle x_1\rangle, \langle x_0 - x_1\rangle]]$$

in L_2. Therefore, we have

Corollary 1.

$$\pi_3^{Lei}(S^2) \simeq \mathbb{Z} \oplus \mathbb{Z}.$$

The generators of this group in L_2 can be written, for example, as $[x_0, x_1, x_0]$ and $[x_0, x_1] - [x_1, x_0]$.

Probably, the general Leibniz homotopy group can be written in style of Wu formula:

(6.2) $$\pi_{n+1}^{Lei}(S^2) = \frac{\langle x_0\rangle \cap \ldots \langle x_n\rangle \cap \langle x_0 + \cdots + x_n\rangle}{[[\langle x_0\rangle, \ldots, \langle x_n\rangle, \langle x_0 + \cdots + x_n\rangle]]}$$

in the free Leinbiz algebra with generators $x_0, \ldots, x_n$. Furthermore, we can take (6.2) as a definition of Leibniz homotopy groups. Note that the concept of *Leibniz sphere* as a certain differential graded Leibniz algebra was considered in [3]. However, the approach described in [3] is completely different from one given here.

The following question rises naturally: what is $\pi_4^{Lei}(S^2)$? In any case, we need elements w from L_3, such that

$$w \in \langle x_0\rangle \cap \langle x_1\rangle \cap \langle x_2\rangle \cap \langle x_0+x_1+x_2\rangle \setminus [[\langle x_0\rangle, \langle x_1\rangle, \langle x_2\rangle, \langle x_0+x_1+x_2\rangle]].$$

One candidate is the element

$$w = [[x_0, x_1], [x_0, x_2], x_0],$$

which clearly lies in $\langle x_0\rangle \cap \langle x_1\rangle \cap \langle x_2\rangle \cap \langle x_0+x_1+x_2\rangle$.

7. Problems

1. It is known that 2-torsion of $\pi_6(S^2)$ is $\mathbb{Z}_4$ and the generator ν' is given by triadic Toda bracket. It is also known that $2\nu' = \eta\circ\eta\circ\eta\circ\eta$. That is, there exists an element v' of L_5, such that

$$2v' \equiv v_5(x_0, x_1, x_2, x_3, x_4) \mod \mathcal{B}_5.$$

Find v' (Toda's bracket $(\eta_3, 2i_4, \eta_4)$).

2. Prove purely algebraically that $v_6(x_0, x_1, x_2, x_3, x_4, x_5) \in \mathcal{B}_6$. It will give an algebraic prove of the fact that $\eta^5 : S^7 \to S^2$ is null-homotopical.

3. One can define the Leibniz homotopy groups of $S^n, n \geq 2$, changing free groups in $F[S^{n-1}]$ by free Leibniz algebras. Do we have an isomorphism $\pi_i^{Lei}(S^3) \simeq \pi_i^{Lei}(S^2),\ i \geq 3$. What does Leibniz analog of Hopf fibration mean?

4. Can one give an algebraic proof of Nishida's Nilpotency Theorem using only Milnor's $F[S^n]$-construction and Curtis' Theorem (Theorem 1)?

5. Compute $\pi_n^{Lei}(S^2)$ for $n = 3, 4, 5$.

6. Find an analog of Curtis Theorem for simplicial Leibniz algebras.

7. Does it make sense to define the analogs of homotopy groups of S^2 using Jordan algebras, Malcev algebras etc, "algebras closed to associative"? It seems, that it is possible to define an analog of $\pi_n(S^2)$ for k-Lie (or k-Leibniz) algebras for $k \geq n-1$.

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