A Theory of the Invariants Obtained from the Moduli Stacks of Stable Objects on a Smooth Polarized Surface

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Abstract

Let X be a smooth polarized algebraic surface over the compex number field. We discuss the invariants obtained from the moduli stacks of semistable sheaves of arbitrary ranks on X. For that purpose, we construct the virtual fundamental classes of some moduli stacks, and we show the transition formula of the integrals over the moduli stacks of the δ-stable Bradlow pairs for the variation of the parameter δ.

Then, we study the relation among the invariants. In the case $p_g = \dim H^2(X, \mathcal{O}_X) > 0$, we show that the invariants are independent of the choice of a polarization of X. We also show that the invariants can be reduced to the invariants obtained from the moduli of abelian pairs and the Hilbert schemes. In the case $p_g = 0$, we obtain the weak wall crossing formula and the weak intersection rounding formula, which describes the dependence of the invariants on the polarization.

Keywords: Moduli, Semistable sheaves, Donaldson invariant, Bradlow pair, Transition formula, Obstruction theory, Weak wall crossing formula

MSC: 14D20, 14J60, 14J80

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1 Introduction

1.1 Problems

1.1.1 Construction of the invariants

Let X be a smooth projective surface with an ample line bundle $\mathcal{O}_X(1)$ over the complex number field $\mathbb{C}$. We assume that X is simply connected in *Introduction*, for simplicity. Let y be an element of the cohomology group $H^*(X,\mathbb{Q})$, which is the Chern character of a torsion-free coherent sheaf E on X. Let a be the first Chern class determined by y. We take a line bundle $\mathcal{L}_a$ on X such that $c_1(\mathcal{L}_a) = a$. An oriented torsion-free sheaf on X of type y is defined to be a coherent torsion-free sheaf E such that $\mathrm{ch}(E) = y$ with an isomorphism $\rho : \det(E) \simeq \mathcal{L}_a$. Let $\mathcal{M}^{ss}(\widehat{y})$ (resp. $\mathcal{M}^{s}(\widehat{y})$) denote the moduli stack of the semistable (resp. stable) oriented torsion-free sheaves of type y. One of our main problems is the following.

Problem 1.1 *Construct an invariant from the moduli stack* $\mathcal{M}^{ss}(\widehat{y})$. ▮

Let $\widehat{E}^u$ denote the universal sheaf over $\mathcal{M}^{ss}(\widehat{y}) \times X$. Let $\Phi = P(\widehat{E}^u)$ be a polynomial of the slant products $\mathrm{ch}_i(\widehat{E}^u)/b$ for elements $b \in H^*(X)$ and $i \in \mathbb{Z}_{\geq 0}$. Naively speaking, we would like to obtain the number:

$$\Phi(\widehat{y}) := \int_{\mathcal{M}^{ss}(\widehat{y})} \Phi = \deg\bigl(\Phi \cap [\mathcal{M}^{ss}(\widehat{y})]\bigr) \tag{1}$$

Namely, we would like to obtain the 0-cycle by taking the cap product of Φ and the fundamental class $[\mathcal{M}(\widehat{y})]$, and we would like to obtain the number by taking the degree of the 0-cycle.

There are two main problems to make (1) well-defined.

(A) The moduli stack $\mathcal{M}^{ss}(\widehat{y})$ is not smooth, and hence it does not have the natural fundamental class, in general.

(B) Even if $\mathcal{M}^{ss}(\widehat{y})$ is smooth, the moduli stack $\mathcal{M}^{ss}(\widehat{y})$ is not Deligne-Mumford, in general. In such a situation, there are no known satisfactory definition of the degree of 0-cycles, or in other words, the push forward of cycles.

Remark 1.2 *We do not give detail in Introduction about how we consider the cohomology and the evaluation on the Deligne-Mumford stacks. See the subsection* 7.1. ▮

Remark 1.3 *The construction of such invariants was also discussed in* [36] *for real* 4*-dimensional manifolds from the differential geometric view point.* ▮

1.1.2 Virtual fundamental classes

The problem like (A) was well discussed and established in the course of their study of Gromov-Witten invariants ([5], [39], [16]). In this paper, we follow the method of Behrend and Fantechi ([5]). Namely, we will show that some interested moduli stacks are naturally provided with the perfect obstruction theories in the sense of [5]. (We will review the obstruction theory in the subsubsection 2.4.1.) And, we will obtain the virtual fundamental classes. Such an obstruction theory may be well known, perhaps. For example, it is a standard fact that the first cohomology group $H^1(X, End(E))$ gives the space of the infinitesimal deformations, and that the second cohomology group $H^2(X, End(E))$ gives the space of the obstruction, for any vector bundle E on X. However, the author does not know an appropriate reference to deal with the obstruction theory in the sense of [5] for the moduli stacks of reduced oriented L-Bradlow pairs and the master spaces, which is available for our arguments using the localization. Thus, we give a detailed argument in the section 5.

Remark 1.4 *Recently, the theory of "derived stacks" has been developed, which seems to provide us a general and powerful tool to construct obstruction theories for some stacks. (See* [56] *for an overview of the theory.) For example, the results in* [57] *implies the construction of the obstruction theory of the moduli stack of semistable sheaves. It is not clear to the author, at the present moment, whether we can directly apply their results to the moduli stacks of semistable oriented reduced L-Bradlow pairs and the master spaces. But, it would be quite hopeful to construct the obstruction theory of such stacks and to redo the argument in this paper, from that point of view. The author would like to come back to this problem in future.*

However, the author also expects that we will obtain the same "invariants", even if we adopt the other way of the construction of the obstruction theories. (See the subsubsection 1.7.1*.)* ▮

1.1.3 A naive idea for the construction of the invariants

To discuss the problem (B), we recall L-Bradlow pair and reduced L-Bradlow pair. Let L be a line bundle on X. Let U be a scheme, and let E be a torsion-free sheaf on X, which is flat over U. A morphism $\phi : p_X^* L \longrightarrow E$ is called an L-section. Such a pair (E, ϕ) is called an L-Bradlow pair.

Let M be a line bundle on U, and let $[\phi]$ be a morphism $p_X^* L \otimes p_U^* M \longrightarrow E$ such that $[\phi]_{|\{u\}\times X} \neq 0$ for any $u \in U$. Such a pair $(M, [\phi])$ is called a reduced L-section, and a pair $\bigl(E, (M, [\phi])\bigr)$ is called a reduced L-Bradlow pair. We will often omit to denote M, for simplicity.

Let $\mathcal{P}^{\mathrm{br}}$ denote the set of polynomials δ such that $\deg(\delta) \leq 1$ and $\delta(t) > 0$ for any sufficiently large t. Let δ be any element of $\mathcal{P}$. Recall that the δ-semistability and δ-stability conditions are defined for L-Bradlow pairs and reduced L-Bradlow pairs. (See the subsubsection 3.3.1.) We use the notation $\mathcal{M}^{ss}(y,L,\delta)$ (resp. $\mathcal{M}^{s}(y,L,\delta)$) to denote the moduli stack of the semistable (resp. stable) L-Bradlow pairs of type y. We also use the notation $\mathcal{M}^{ss}(\widehat{y},[L],\delta)$ (resp. $\mathcal{M}^{s}(\widehat{y},[L],\delta)$) to denote the moduli stack of the δ-semistable (resp. δ-stable) oriented reduced L-Bradlow pairs of type y.

We say that δ is critical, if $\mathcal{M}^{ss}(\widehat{y},[L],\delta)=\mathcal{M}^{s}(\widehat{y},[L],\delta)$ does not hold. It can be shown that there are only finitely many critical parameters. For a non-critical parameter δ, the moduli stack $\mathcal{M}^{ss}(\widehat{y},[L],\delta)$ is Deligne-Mumford. Moreover, it is naturally provided with the perfect obstruction theory. Therefore, we have the integrals of cohomology classes on $\mathcal{M}^{ss}(\widehat{y},[L],\delta)$.

Let us consider the case $L=\mathcal{O}(-m)$. If δ is sufficiently small, it is non-critical, and we have the naturally defined morphism $\pi:\mathcal{M}^{ss}(\widehat{y},[\mathcal{O}(-m)],\delta)\longrightarrow\mathcal{M}^{ss}(\widehat{y})$. It is easy to observe that the morphism π is smooth, if m is sufficiently large. The relative tangent bundle is denoted by T_{rel}. We put $H_y(m):=\int_X \mathrm{Td}(X)\cdot y\cdot \mathrm{ch}\big(\mathcal{O}(m)\big)$, which is same as $\dim H^0\big(X,E(m)\big)$ for any $(E,\rho)\in\mathcal{M}^{ss}(\widehat{y})$. We regard $\mathcal{M}^{ss}\big(\widehat{y},[\mathcal{O}(-m)],\delta\big)$ as a good approximation of $\mathcal{M}^{ss}(\widehat{y})$, and we would like to put as follows:

$$\Phi(\widehat{y}):=\int_{\mathcal{M}^{ss}(\widehat{y},[\mathcal{O}(-m)],\delta)}\Phi\cdot\frac{\mathrm{Eu}(T_{\mathrm{rel}})}{H_y(m)} \tag{2}$$

Needless to say, we should ask the following:

(C) Is (2) independent of the choice of m?

In the case $\mathcal{M}^{ss}(\widehat{y})=\mathcal{M}^{s}(\widehat{y})$, the morphism π is $\mathbb{P}^{H_y(m)-1}$-bundle. Hence (2) is independent of the choice of m, and it is compatible with the ordinary definition. We will obtain the affirmative answer of (C), in general.

1.1.4 Motivation of the study

The Donaldson invariants for smooth projective surfaces can be obtained from the moduli spaces of semistable sheaves of rank 2. It is natural to ask what invariants are obtained from the moduli stacks of semistable sheaves of higher ranks. That is one of our motivations for the study.

Donaldson invariant has been studied intensively since the 1980s, motivated by the application to the topology of real four dimensional manifolds. Nowadays, it is believed that the topological information contained in Donaldson invariant can be obtained from only Seiberg-Witten invariant, essentially. Similarly, even if we obtain the invariants from the moduli stacks of the objects with higher ranks, it is not so reasonable to expect a new exciting application to topology.

However, it seems interesting to investigate the relation among the invariants. For example, we can ask the following two problems, which are related with the Kotschick-Morgan conjecture and the Witten conjecture of Donaldson invariant.

Problem 1.5 *Clarify the dependence of $\Phi(\widehat{y})$ on the polarization $\mathcal{O}_X(1)$.* ∎

Problem 1.6 *Reduce $\Phi(\widehat{y})$ to the sum of the integrals over the products of the moduli spaces of the objects with rank one.* ∎

As for Problem 1.5, we will show that $\Phi(\widehat{y})$ is independent of the choice of $\mathcal{O}_X(1)$ in the case $p_g>0$, and we will obtain the weak wall crossing formula in the case $p_g=0$. As for Problem 1.6, we will give such a formula in the case $p_g>0$.

In principle, we can show the existence of the relations of the invariants as above, which are universal in some sense. Moreover, we *expect* that they can be described in terms of good functions such as modular forms. The author hopes that this work would, at least tentatively, provide a part of the foundation for such an interesting study.

We will explain our main results in the next subsections.

1.2 Transition Formulas in the Simple Cases

1.2.1 The case where the 2-stability condition is satisfied

Let $\mathcal{M}(\widehat{y},[L])$ denote the moduli stack of the oriented reduced L-Bradlow pairs of type y. We have the relative tautological line bundle $\mathcal{O}_{\mathrm{rel}}(1)$ on $\mathcal{M}(\widehat{y},[L])$. (See the subsubsection 3.1.5 for the definition.) The restriction to $\mathcal{M}^{ss}(\widehat{y},[L],\delta)$ is also denoted by $\mathcal{O}_{\mathrm{rel}}(1)$. Let ω denote the first Chern class of $\mathcal{O}_{\mathrm{rel}}(1)$. We consider the cohomology classes which is described as a sum of cohomology classes of the following form:

$$\Phi = P(\widehat{E}^u)\cdot\omega^k.$$

If δ is a non-critical parameter, i.e. $\mathcal{M}^{ss}(\widehat{y},[L],\delta)=\mathcal{M}^{s}(\widehat{y},[L],\delta)$, we put as follows:

$$\Phi(\widehat{y},[L],\delta) := \int_{\mathcal{M}^s(\widehat{y},[L],\delta)}\Phi$$

Let δ be critical. We take parameters $\delta_- < \delta < \delta_+$ such that $|\delta_\kappa-\delta|$ are sufficiently small. We would like to describe the transition $\Phi(\widehat{y},[L],\delta_+)-\Phi(\widehat{y},[L],\delta_-)$ as the sum of the integrals over the products of the moduli stacks of the objects with lower ranks. Such a description is called the transition formula.

If the following condition is satisfied, the problem is rather simple.

(2-stability condition) We say that the 2-stability condition holds for (y,L,δ), if the automorphism group of $(E,\phi)\in\mathcal{M}^{ss}(y,L,\delta)$ is $\{1\}$ or G_m.

To state the theorem, we need some preparation. Let $\mathcal{T}ype$ denote the set of cohomology classes of X obtained as the Chern character of some torsion-free sheaves on X. For any $y\in\mathcal{T}ype$, the $H^0(X)$-part is denoted by $\mathrm{rank}(y)$. We have the Hilbert polynomial $H_y(t)$ of y which satisfies the following for any integer m:

$$H_y(m) := \int_X \mathrm{Td}(X)\cdot y\cdot \mathrm{ch}\big(\mathcal{O}(m)\big).$$

The reduced Hilbert polynomial $H_y/\mathrm{rank}(y)$ is denoted by P_y. When a parameter δ is given, we put $P_y^\delta := (H_y+\delta)/\mathrm{rank}(y)$. We put as follows:

$$S(y,\delta) := \big\{(y_1,y_2)\in\mathcal{T}ype^2\,\big|\,y_1+y_2=y,\;\; P^\delta_y=P^\delta_{y_1}=P_{y_2}\big\}$$

For a given $(y_1,y_2)\in\mathcal{T}ype^2$, we put $r_i=\mathrm{rank}\,y_i$. We put as follows:

$$\mathcal{M}(y_1,\widehat{y}_2,L,\delta) := \mathcal{M}^{ss}(y_1,L,\delta)\times\mathcal{M}^{ss}(\widehat{y}_2)$$

On $\mathcal{M}(y_1,\widehat{y}_2,L,\delta)\times X$, we have the sheaf E_1^u which is obtained from the universal sheaf on $\mathcal{M}^{ss}(y_1,L,\delta)\times X$ via the natural projection. We also have the sheaf $\widehat{E}_2^u$ which is obtained from the universal sheaf on $\mathcal{M}^{ss}(\widehat{y}_2)\times X$ via the natural projection.

Let G_m denote the one dimensional torus. Let $e^{w\cdot t}$ denote the trivial line bundle with the G_m-action of weight w. We have the following element of the K-group of the G_m-equivariant coherent sheaves on $\mathcal{M}(y_1,\widehat{y}_2,L,\delta)$:

$$\begin{aligned}\mathfrak{N}_0(y_1,y_2) = -Rp_{X\,*}\Big(\mathcal{RH}om\big(E_1^u\cdot e^{-t},\,\widehat{E}_2^u\cdot e^{r_1(t-\omega_1)/r_2}\big)\Big) - Rp_{X\,*}\Big(\mathcal{RH}om\big(\widehat{E}_2^u\cdot e^{r_1(t-\omega_1)/r_2},\,E_1^u\cdot e^{-t}\big)\Big)\\ + Rp_{X\,*}\Big(\mathcal{H}om\big(L\cdot e^{-t},\,\widehat{E}_2^u\cdot e^{r_1(t-\omega_1)/r_2}\big)\Big)\end{aligned} \tag{3}$$

Here, we put $\omega_1 := c_1(\mathcal{O}r(E_1^u))/r_1$, and $e^{w\cdot\omega_1}$ denotes $\mathcal{O}r(E_1^u)^{w/r_1}$ formally.

As a special case of Theorem 7.12, we obtain the following theorem.

Theorem 1.7 *Assume that the 2-stability condition holds for (y,L,δ). Then, we have the following equality:*

$$\Phi(\widehat{y},[L],\delta_+)-\Phi(\widehat{y},[L],\delta_-) = \sum_{(y_1,y_2)\in S(y,\delta)}\int_{\mathcal{M}(y_1,\widehat{y}_2,L,\delta)}\operatorname*{Res}_{t=0}\left(\frac{P\big(E_1^u\cdot e^{-t}\oplus\widehat{E}_2^u\cdot e^{r_1(t-\omega_1)/r_2}\big)\cdot t^k}{\mathrm{Eu}\big(\mathfrak{N}_0(y_1,y_2)\big)}\right) \tag{4}$$

In the case $p_g>0$ and $\mathrm{rank}(y_1)>1$, the contributions from (y_1,y_2) are 0. ∎

The following condition is called the i-vanishing condition for (y, L, δ):

(i-vanishing condition) We have $H^j(X, L^{-1} \otimes E) = 0$ for any $j \geq i$ and for any $(E, \phi) \in \mathcal{M}^{ss}(y, L, \delta)$.

The problem of the transition is comparatively simple as in the following proposition.

Proposition 1.8 *Assume that the* 2*-stability condition and the* 2*-vanishing condition are satisfied for* (y, L, δ)*. Then, the transition at* δ *is trivial in the case* $p_g > 0$*, i.e., the equality* $\Phi(\widehat{y}, \delta_+) = \Phi(\widehat{y}, \delta_-)$ *holds.* ∎

If the 1-vanishing condition holds for (y, L, δ), we have the smooth morphism of $\mathcal{M}^{ss}(\widehat{y}, [L], \delta)$ to the moduli stack $\mathcal{M}(\widehat{y})$ of the oriented torsion-free sheaves of type y. The relative tangent bundle is denoted by T_{rel}. We put $N_L(y) := \int_X \mathrm{Td}(X) \cdot y \cdot \mathrm{ch}(L^{-1})$, which is same as $\mathrm{rank}\, T_{\mathrm{rel}} + 1$. We will be interested in the following integral:

$$\Phi_1(\widehat{y}, \delta) := \int_{\mathcal{M}^{ss}(\widehat{y},[L],\delta)} \Phi_1, \quad \Phi_1 := P(\widehat{E}^u) \cdot \frac{\mathrm{Eu}(T_{\mathrm{rel}})}{N_L(y)}$$

For $(y_1, y_2) \in S(y, \delta)$, we put as follows:

$$\mathcal{M}(\widehat{y}_1, \widehat{y}_2, [L], \delta) := \mathcal{M}^{ss}(\widehat{y}_1, [L], \delta) \times \mathcal{M}^{ss}(\widehat{y}_2).$$

On $\mathcal{M}(\widehat{y}_1, \widehat{y}_2, [L], \delta)$, we have the sheaf $\widehat{E}_1^u$ which is the pull back of the universal sheaf on $\mathcal{M}^{ss}(\widehat{y}_1, [L], \delta) \times X$ via the natural projection. Similarly, we have the sheaf $\widehat{E}_2^u$ which is the pull back of the universal sheaf on $\mathcal{M}^{ss}(\widehat{y}_2) \times X$ via the natural projection. Let $e^{w \cdot s}$ denote the trivial line bundle with the G_m-action of weight w. Let $e^{w \cdot s}$ denote the trivial line bundle on $\mathcal{M}(\widehat{y}_1, \widehat{y}_2, [L], \delta)$ with the G_m-action of weight w. We have the following element of the K-group of the G_m-equivariant coherent sheaves on $\mathcal{M}(\widehat{y}_1, \widehat{y}_2, [L], \delta)$:

$$-Rp_{X\,*}\Big(\mathcal{RH}om\big(\widehat{E}_1^u \cdot e^{-s/r_1},\, \widehat{E}_2^u \cdot e^{s/r_2}\big)\Big) - Rp_{X\,*}\Big(\mathcal{RH}om\big(\widehat{E}_2^u \cdot e^{s/r_2},\, \widehat{E}_1^u \cdot e^{-s/r_1}\big)\Big)$$

Let $Q\big(\widehat{E}_1 \cdot e^{-s/r_1}, \widehat{E}_2 \cdot e^{s/r_2}\big)$ denote the equivariant Euler class. As a special case of Theorem 7.15, we obtain the following.

Theorem 1.9 *Assume that the* 2*-stability condition and the* 1*-vanishing condition hold for* (y, L, δ)*. In the case* $p_g > 0$*, we have* $\Phi(\widehat{y}, [L], \delta_+) = \Phi(\widehat{y}, [L], \delta_-)$*. In the case* $p_g = 0$*, we have the following equality:*

$$\Phi(\widehat{y}, [L], \delta_+) - \Phi(\widehat{y}, [L], \delta_-) = \sum_{(y_1, y_2) \in S(y, \delta)} \int_{\mathcal{M}(\widehat{y}_1, \widehat{y}_2, [L], \delta)} \Psi(y_1, y_2)$$

The cohomology classes $\Psi(y_1, y_2)$ *are given as follows:*

$$\Psi(y_1, y_2) = \frac{N_L(y_1)}{N_L(y)} \cdot \operatorname*{Res}_{s=0} \left(\frac{P\big(\widehat{E}_1^u \cdot e^{-s/r_1} \oplus \widehat{E}_2^u \cdot e^{s/r_2}\big)}{Q\big(\widehat{E}_1^u \cdot e^{-s/r_1},\, \widehat{E}_2^u \cdot e^{s/r_2}\big)} \right) \cdot \frac{\mathrm{Eu}(T_{1,\mathrm{rel}})}{N_L(y_1)} \tag{5}$$

Here, $T_{1,\mathrm{rel}}$ *denote the relative tangent bundle of the smooth morphism* $\mathcal{M}^{ss}(\widehat{y}_1, [L], \delta) \longrightarrow \mathcal{M}(\widehat{y}_1)$*.* ∎

1.2.2 Reduced $\boldsymbol{L}$-Bradlow pair

Let $\boldsymbol{L} = (L_1, L_2)$ be a pair of line bundles on X. Let U be a scheme, and let E be a U-torsion-free sheaf on $U \times X$ which is flat over U. Let $[\boldsymbol{\phi}]$ be a pair $([\phi_1], [\phi_2])$ of reduced L_i-sections $[\phi_i]$ of E such that $[\phi_i]_{|\{u\} \times X} \neq 0$ for any $u \in U$. Such a pair $(E, [\boldsymbol{\phi}])$ is called a reduced $[\boldsymbol{L}]$-Bradlow pair. Let $\boldsymbol{\delta} = (\delta_1, \delta_2)$ be an element of $\mathcal{P}^{\mathrm{br}\ 2}$. We naturally have the notion of $\boldsymbol{\delta}$-semistability and $\boldsymbol{\delta}$-stability for reduced $\boldsymbol{L}$-Bradlow pairs. Let $\mathcal{M}^{ss}(\widehat{y}, [\boldsymbol{L}], \boldsymbol{\delta})$ (resp. $\mathcal{M}^{s}(\widehat{y}, [\boldsymbol{L}], \boldsymbol{\delta})$) denote the moduli stack of $\boldsymbol{\delta}$-semistable (resp. $\boldsymbol{\delta}$-stable) $\boldsymbol{L}$-Bradlow pairs of type y. When δ_i are sufficiently small, we have the morphism $\mathcal{M}^{ss}(\widehat{y}, [\boldsymbol{L}], \boldsymbol{\delta}) \longrightarrow \mathcal{M}^{ss}(\widehat{y})$.

Let us move δ_1 by fixing δ_2. We say that δ_1 is critical, if $\mathcal{M}^{ss}(\widehat{y}, [\boldsymbol{L}], \boldsymbol{\delta}) \neq \mathcal{M}^{s}(\widehat{y}, [\boldsymbol{L}], \boldsymbol{\delta})$ holds. As in the subsubsection 1.2.1, we have the notion of 2-stability condition for $(y, \boldsymbol{L}, \boldsymbol{\delta})$ (Definition 3.58). When δ_i are sufficiently small, it can be shown that the 2-stability condition is always satisfied even if δ_1 is critical, and we have a good transition formula.

Assume that δ_1 is critical. We take elements $\delta_-, \delta_+ \in \mathcal{P}^{\mathrm{br}}$ such that $\delta_- < \delta_1 < \delta_+$ and that $|\delta_\kappa - \delta_1|$ $(\kappa = \pm)$ are sufficiently small. We put $\boldsymbol{\delta}_\kappa = (\delta_\kappa, \delta_2)$ for $\kappa = \pm$. Let $T^{(1)}_{\mathrm{rel}}$ denote the relative tangent bundle of the smooth morphism of $\mathcal{M}^{ss}(\widehat{y}, [\boldsymbol{L}], \boldsymbol{\delta}_\kappa)$ to the moduli stack $\mathcal{M}(\widehat{y}, [L_2])$ of reduced L_2-Bradlow pairs. We consider the following cohomology class:

$$\Phi = \frac{\mathrm{Eu}(T^{(1)}_{\mathrm{rel}})}{N_{L_1}(y)} \cdot P(\widehat{E}^u)(\omega^{(2)})^k$$

Here, $\omega^{(2)}$ denote the first Chern class of the line bundle which is the pull back of the relative tautological line bundle of $\mathcal{M}(\widehat{y}, [L_2])$. We put as follows, for $\kappa = \pm$:

$$\Phi(\widehat{y}, [\boldsymbol{L}], \boldsymbol{\delta}_\kappa) := \int_{\mathcal{M}^{ss}(\widehat{y}, [\boldsymbol{L}], \boldsymbol{\delta}_\kappa)} \Phi$$

We would like to discuss the transition formula for $\Phi(\widehat{y}, [\boldsymbol{L}], \boldsymbol{\delta}_+) - \Phi(\widehat{y}, [\boldsymbol{L}], \boldsymbol{\delta}_-)$. We put as follows:

$$S(y, \boldsymbol{\delta}) := \big\{(y_1, y_2) \in \mathcal{T}ype^2 \,\big|\, P_{y_1} = P_{y_2},\ \ \delta_1/\operatorname{rank}(y_1) = \delta_2/\operatorname{rank}(y_2)\big\}$$

For any $(y_1, y_2) \in S(y, \boldsymbol{\delta})$, we put as follows:

$$\mathcal{M}(\widehat{y}_1, \widehat{y}_2, [\boldsymbol{L}], \boldsymbol{\delta}) := \mathcal{M}^{ss}(\widehat{y}_1, [L_1], \delta_1) \times \mathcal{M}^{ss}(\widehat{y}_2, [L_2], \delta_2)$$

Let $\widehat{E}_i$ denote the sheaf on $\mathcal{M}(\widehat{y}_1, \widehat{y}_2, [\boldsymbol{L}], \boldsymbol{\delta})$ which is the pull back of the universal sheaf over $\mathcal{M}^{ss}(\widehat{y}_i, [L_i], \delta_i) \times X$. Let $\mathcal{O}_{i,\mathrm{rel}}(1)$ denote the pull back of the relative tautological line bundles on $\mathcal{M}^{ss}(\widehat{y}_i, [L_i], \delta_i)$. We put $\omega_i := c_1\big(\mathcal{O}_{i,\mathrm{rel}}(1)\big)$. Let $T_{1,\mathrm{rel}}$ denote the bundle on $\mathcal{M}(\widehat{y}_1, \widehat{y}_2, [\boldsymbol{L}], \boldsymbol{\delta})$ induced by the relative tangent bundle of the smooth morphism $\mathcal{M}^{ss}(\widehat{y}_i, [L_i], \delta_i) \longrightarrow \mathcal{M}(\widehat{y}_i)$.

Let $e^{w \cdot s}$ denote the trivial line bundle with the G_m-action of weight w as above. We have the following element of the K-group $K^{G_m}\big(\mathcal{M}(\widehat{y}_1, \widehat{y}_2, [\boldsymbol{L}], \boldsymbol{\delta})\big)$ of G_m-equivariant coherent sheaves on $\mathcal{M}(\widehat{y}_1, \widehat{y}_2, [\boldsymbol{L}], \boldsymbol{\delta})$:

$$-Rp_{X\,*}\Big(\mathcal{RH}om\big(\widehat{E}^u_1 \cdot e^{-s/r_1},\ \widehat{E}^u_2 \cdot e^{s/r_2}\big)\Big) - Rp_{X\,*}\Big(\mathcal{RH}om\big(\widehat{E}^u_2 \cdot e^{s/r_2},\ \widehat{E}^u_1 \cdot e^{-s/r_1}\big)\Big)$$

The equivariant Euler class is denoted by $Q(\widehat{E}^u_1 \cdot e^{s/r_1}, \widehat{E}^u_2 \cdot e^{s/r_2})$. We also have the following element of $K^{G_m}(\mathcal{M}(\widehat{y}_1, \widehat{y}_2, [\boldsymbol{L}], \delta))$:

$$Rp_{X\,*}\mathcal{H}om(L_2, \widehat{E}^u_1) \cdot e^{-s/r_1 - s/r_2 + \omega_2}$$

The equivariant Euler class is denoted by $R(L_2 \cdot e^{-\omega_2 + s/r_2},\ \widehat{E}^u_1 \cdot e^{-s/r_1})$. We obtain the following proposition as the special case of Proposition 7.16 and Proposition 7.17.

Proposition 1.10 *In the case $p_g > 0$, we have the equality $\Phi(\widehat{y}, [\boldsymbol{L}], \boldsymbol{\delta}_+) = \Phi(\widehat{y}, [\boldsymbol{L}], \boldsymbol{\delta}_-)$. In the case $p_g = 0$, the following equality holds:*

$$\Phi(\widehat{y}, [\boldsymbol{L}], \boldsymbol{\delta}_+) - \Phi(\widehat{y}, [\boldsymbol{L}], \boldsymbol{\delta}_-) = \sum_{(y_1, y_2) \in S(y, \boldsymbol{\delta})} \int_{\mathcal{M}(\widehat{y}_1, \widehat{y}_2, [\boldsymbol{L}], \delta)} \Psi(y_1, y_2) \tag{6}$$

Here, $\Psi(y_1, y_2)$ are given as follows:

$$\Psi(y_1, y_2) := \frac{N_{L_1}(y_1)}{N_{L_1}(y)} \cdot \operatorname*{Res}_{s=0}\left(\frac{P(\widehat{E}^u_1 \cdot e^{-s/r_1} \oplus \widehat{E}^u_2 \cdot e^{s/r_2}) \cdot (\omega_2 - s/r_2)^k}{Q(\widehat{E}^u_1 \cdot e^{-s/r_1}, \widehat{E}^u_2 \cdot e^{s/r_2}) \cdot R(L_2 \cdot e^{-\omega_2 + s/r_2}, \widehat{E}^u_1 \cdot e^{-s/r_1})}\right) \cdot \frac{\mathrm{Eu}(T_{1,\mathrm{rel}})}{N_{L_1}(y_1)} \tag{7}$$

∎

As a special case, let us consider the integral of the following cohomology class, assuming that the 1-vanishing condition holds for (y, L_2):

$$\Phi = \frac{\mathrm{Eu}(T^{(1)}_{\mathrm{rel}})}{N_{L_1}(y)} \cdot \frac{\mathrm{Eu}(T^{(2)}_{\mathrm{rel}})}{N_{L_2}(y)} \cdot P(\widehat{E}^u) \tag{8}$$

Lemma 1.11 *Assume that the* 1*-vanishing condition holds for* (y, L_2)*. Let* Φ *be as in* (8)*. Then, the cohomology class* $\Psi(y_1, y_2)$ *in* (6) *is given as follows:*

$$\operatorname*{Res}_{s=0}\left(\frac{P(\widehat{E}_1^u \cdot e^{-s/r_1} \oplus \widehat{E}_2^u \cdot e^{s/r_2})}{Q(\widehat{E}_1^u \cdot e^{-s/r_1}, \widehat{E}_2^u \cdot e^{s/r_2})}\right) \cdot \frac{\mathrm{Eu}(T_{1,\mathrm{rel}})}{N_{L_1}(y_1)} \cdot \frac{\mathrm{Eu}(T_{2,\mathrm{rel}})}{N_{L_2}(y_2)}$$

▮

1.2.3 Well-definedness of (2)

In the case $p_g > 0$, we can show the well-definedness of (2) by using Proposition 1.10. Assume that L_1^{-1} is ample, and that the 1-vanishing condition holds for (y, L_2). If we take a sufficiently large integer m, the 1-vanishing condition holds for (y, L_1^m). We put $\boldsymbol{L}^{(m)} := (L_1^m, L_2)$. Let $T_{\mathrm{rel}}^{(1)}$ denote the relative tangent bundle of the smooth map $\mathcal{M}^{ss}(\widehat{y}, [\boldsymbol{L}^{(m)}], \boldsymbol{\delta}) \longrightarrow \mathcal{M}(\widehat{y}, [L_2])$. We use the notation $T_{\mathrm{rel}}^{(2)}$ in a similar meaning. We consider the following number:

$$g(L_1^m, L_2, \delta_1, \delta_2) := \int_{\mathcal{M}^{ss}(\widehat{y}, [\boldsymbol{L}^{(m)}], \boldsymbol{\delta})} P(\widehat{E}^u) \cdot \frac{\mathrm{Eu}(T_{\mathrm{rel}}^{(1)})}{N_{L_1^m}(y)} \cdot \frac{\mathrm{Eu}(T_{\mathrm{rel}}^{(2)})}{N_{L_2}(y)}$$

We assume that both of δ_i are sufficiently small. When δ_1 is sufficiently smaller than δ_2, we have the following:

$$g(L_1^m, L_2, \delta_1, \delta_2) = \int_{\mathcal{M}^{ss}(\widehat{y}, [L_2], \delta_2)} \Phi \cdot \frac{\mathrm{Eu}(T_{\mathrm{rel}}^{(2)})}{N_{L_2}(y)}$$

When δ_2 is sufficiently smaller than δ_1, we have the following:

$$g(L_1^m, L_2, \delta_1, \delta_2) = \int_{\mathcal{M}^{ss}(\widehat{y}, [L_1^m], \delta_1)} \Phi \cdot \frac{\mathrm{Eu}(T_{\mathrm{rel}}^{(1)})}{N_{L_1^m}(y)}$$

When we move δ_1, the transitions are trivial, due to Proposition 1.10. Therefore, we obtain the following:

$$\int_{\mathcal{M}^{ss}(\widehat{y}, [L_1^m], \delta_1)} \Phi \cdot \frac{\mathrm{Eu}(T_{\mathrm{rel}}^{(1)})}{N_{L_1^m}(y)} = \int_{\mathcal{M}^{ss}(\widehat{y}, [L_2], \delta_2)} \Phi \cdot \frac{\mathrm{Eu}(T_{\mathrm{rel}}^{(2)})}{N_{L_2}(y)}$$

In particular, we obtain that (2) is independent of the choice of m. Moreover, we can show the following equality, assuming the 2-vanishing condition for (y, L) $d := \int_X \mathrm{Td}(X) \cdot y \cdot \mathrm{ch}(L^{-1}) - 1 \geq 0$:

$$\Phi(\widehat{y}) = \int_{\mathcal{M}^{ss}(\widehat{y}, [L], \delta)} \Phi \cdot \omega^d \tag{9}$$

In the case $p_g = 0$, the problem is more subtle. We can derive it from Proposition 1.10 and Lemma 1.11 that (2) is independent of the choice of m in this case, too. However, we need some more additional argument.

Remark 1.12 *Although we do not discuss the parabolic structure in this section, we will consider the invariants obtained from the moduli stacks of the oriented parabolic torsion-free sheaves and the oriented parabolic reduced L-Bradlow pairs. And, it is not clear whether* (2) *is independent of the choice of* m*, in general. Instead, we can show the existence of the following limit, for a line bundle* L *such that* L^{-1} *is ample:*

$$\lim_{m \to \infty} \int_{\mathcal{M}^{ss}(\widehat{\boldsymbol{y}}, [L^m], \alpha_*, \delta)} \Phi \cdot \frac{\mathrm{Eu}(T_{\mathrm{rel}})}{N_{L^m}(y)} \tag{10}$$

Here, $\boldsymbol{y}$ *denotes a type of parabolic sheaves, and* α_* *denotes a system of weight. Moreover, the limit is independent of the choice of* L*. Hence, we may adopt* (10) *as the definition of* $\Phi(\widehat{\boldsymbol{y}}, \alpha_*)$*.* ▮

1.3 Rank 2 Case

1.3.1 Dependence on the polarizations

In the case $\mathrm{rank}(y) = 2$, the 2-stability condition is always satisfied. Hence, Theorem 1.7 provides us a tool to discuss the problems 1.5 and 1.6. We explain our result for Problem 1.5 in this subsubsection.

To distinguish the dependence on the polarization H, we use the notation $\mathcal{M}_H(\widehat{y})$ to denote the moduli stack of torsion-free sheaves of type y which are semistable with respect to H. We also assume $a^2 - 4n \leq \xi^2 < 0$. We put $W^\xi := \big\{c \in NS(X) \otimes \mathbb{R} \,\big|\, (c, \xi) = 0\big\}$, which is called the wall determined by ξ. It is well known that the ample cone is divided into the chambers by such walls, and the moduli $\mathcal{M}_H(\widehat{y})$ depends only on the chambers to which H belongs. We put as follows:

$$\Phi_H(\widehat{y}) := \int_{\mathcal{M}_H(\widehat{y})} \Phi$$

We would like to discuss how $\Phi_H(\widehat{y})$ changes when the polarizations vary across the wall W^ξ.

Let a and n be the first and second Chern classes of y. We put as follows:

$$S_0(y, \xi) := \big\{(a_0, a_1) \in NS(X)^2 \,\big|\, a_0 - a_1 = m \cdot \xi \ (m > 0)\big\}$$

For any $(a_0, a_1) \in S_0(y, \xi)$, we put as follows:

$$X(a_0, a_1) := \coprod_{n_0 + n_1 = n - a_0 \cdot a_1} X^{[n_0]} \times X^{[n_1]}$$

On $X^{[n_0]} \times X^{[n_1]} \times X$, we have the sheaf $\mathcal{I}_i^u$, which is the pull back of the universal ideal sheaf over $X^{[n_i]} \times X$ via the natural projection. Let e^{a_i} denote the holomorphic line bundle on X whose first Chern class is a_i. It is uniquely determined up to isomorphism, since we have assumed that X is simply connected in *Introduction.* Let $Q\big(\mathcal{I}_0^u e^{a_0 - s}, \mathcal{I}_1^u e^{a_1 + s}\big)$ be the equivariant Euler class of the following element of the K-group of the G_m-equivariant coherent sheaves on $X^{[n_0]} \times X^{[n_1]}$:

$$-Rp_{X\,*}\big(\mathcal{RH}om\big(\mathcal{I}_0^u \cdot e^{a_0 - s},\ \mathcal{I}_1^u \cdot e^{a_1 + s}\big)\big) - Rp_{X\,*}\big(\mathcal{RH}om\big(\mathcal{I}_1^u \cdot e^{a_1 + s},\ \mathcal{I}_0^u \cdot e^{a_0 - s}\big)\big)$$

Theorem 1.13 *Let C_+ and C_- be chambers which are divided by the wall W^ξ. Let H_+ and H_- be ample line bundles contained in C_+ and C_-, respectively. We assume $(H_-, \xi) < 0 < (H_+, \xi)$.*

- *In the case $p_g > 0$, we have $\Phi_{H_+}(\widehat{y}) = \Phi_{H_-}(\widehat{y})$. Namely, the invariant does not depend on the choice of generic polarizations.*
- *In the case $p_g = 0$, we have the following equality:*

$$\Phi_{H_+}(\widehat{y}) - \Phi_{H_-}(\widehat{y}) = \sum_{(a_0, a_1) \in S_0(y, \xi)} \int_{X(a_0, a_1)} \operatorname*{Res}_{s=0} \left(\frac{P\big(\mathcal{I}_0^u \cdot e^{a_0 - s} \oplus \mathcal{I}_1^u \cdot e^{a_1 + s}\big)}{Q\big(\mathcal{I}_0^u \cdot e^{a_0 - s},\ \mathcal{I}_1^u \cdot e^{a_1 + s}\big)} \right) \tag{11}$$

We call (11) *the weak wall crossing formula.* ▮

Remark 1.14 *Under the assumption that the wall W^ξ is good, the weak wall crossing formula was proved for the Donaldson invariant in* [10] *and* [15]*, which was refined in* [21]. ▮

Remark 1.15 *Remarkably, L. Göttsche-H. Nakajima-K. Yoshioka established the way to derive the wall crossing formula from the weak wall crossing formula. (See* [21]*.)* ▮

Remark 1.16 *K. Yamada proved the independence of the invariants from the polarizations in some cases.* ▮

1.3.2 Reduction to the integrals over Hilbert schemes

Let us discuss the problem 1.6 in the case $\mathrm{rank}(y)=2$. Let a and n denote the first and second Chern classes of y. We also assume $p_g>0$. For any element a_1 of the Neron-Severi group $NS^1(X)$, we put $a_2:=a-a_1$. Let e^{a_i} denote the holomorphic line bundle whose first Chern class is a_i. Let $\mathcal{I}_i^u$ denote the universal ideal sheaves over $X^{[n_i]}\times X$. Let $\mathcal{Z}_i$ denote the universal 0-scheme over $X^{[n_i]}\times X$. Let Ξ_i denote $p_{X\,*}\bigl(\mathcal{O}_{\mathcal{Z}_i}\otimes e^{a_i}\bigr)$. We use the same notation to denote the pull back of them via appropriate morphisms. In the case $\bigl(c_1(\mathcal{O}(1)),a_1\bigr)<\bigl(c_1(\mathcal{O}(1)),a_2\bigr)$, we put as follows:

$$\mathcal{A}(a_1,y):=\sum_{n_1+n_2=n-a_1\cdot a_2}\int_{X^{[n_1]}\times X^{[n_2]}}\operatorname*{Res}_{s=0}\left(\frac{P\bigl(\mathcal{I}_1^u\cdot e^{a_1-s}\oplus\mathcal{I}_2^u\cdot e^{a_2+s}\bigr)}{Q\bigl(\mathcal{I}_1^u\cdot e^{a_1-s},\mathcal{I}_2^u\cdot e^{a_2+s}\bigr)}\cdot\frac{\mathrm{Eu}(\Xi_1)\cdot\mathrm{Eu}(\Xi_2\cdot e^{2s})}{(2s)^{n_1+n_2-p_g}}\right)$$

In the case $\bigl(c_1(\mathcal{O}(1)),a_1\bigr)=\bigl(c_1(\mathcal{O}(1)),a_2\bigr)$, we put as follows:

$$\mathcal{A}(a_1,y):=\sum_{\substack{n_1+n_2=n-a_1\cdot a_2\\ n_1>n_2}}\int_{X^{[n_1]}\times X^{[n_2]}}\operatorname*{Res}_{s=0}\left(\frac{P\bigl(\mathcal{I}_1^u\cdot e^{a_1-s}\oplus\mathcal{I}_2^u\cdot e^{a_2+s}\bigr)}{Q\bigl(\mathcal{I}_1^u\cdot e^{a_1-s},\mathcal{I}_2^u\cdot e^{a_2+s}\bigr)}\cdot\frac{\mathrm{Eu}(\Xi_1)\cdot\mathrm{Eu}(\Xi_2\cdot e^{2s})}{(2s)^{n_1+n_2-p_g}}\right)$$

Recall that an abelian pair is defined to be a pair of a holomorphic line bundle $\mathcal{L}$ and a section $\phi:\mathcal{O}\longrightarrow\mathcal{L}$. Let $M(c)$ denote the moduli of abelian pairs $(\mathcal{L},\phi)$ such that $c_1(\mathcal{L})=c$. We can show the following proposition.

Proposition 1.17 (Proposition 6.29) *Assume $H^1(X,\mathcal{O}_X)=0$ and $p_g=\dim H^2(X,\mathcal{O}_X)>0$. Moreover, we assume that the virtual fundamental class of $M(c)$ is not 0. Then, the expected dimension of $M(c)$ is 0.*

We can regard the virtual fundamental class $[M(c)]$ as the number, and then it is same as the following:

$$\mathrm{SW}(c):=\frac{\prod_{i=1}^{d(c)}(i-p_g)}{d(c)!}$$

Here we put $d(c):=\dim M(c)=\dim H^0(X,e^c)-1$. ∎

Then, we put as follows:

$$\mathcal{SW}(X,y):=\bigl\{a_1\in NS(X)\bigm|\mathrm{SW}(a_1)\neq 0,\ \bigl(a_1,c_1(\mathcal{O}_X(1))\bigr)\leq\bigl(a,c_1(\mathcal{O}_X(1))\bigr)/2\bigr\}$$

Theorem 1.18 (Theorem 7.28) *Assume $p_g>0$ and $H^1(X,\mathcal{O})=0$. Assume $P_y>P_K$ and $\chi(y)-1\geq 0$, where K denotes the canonical line bundle of X, and we put $\chi(y)=\int\mathrm{Td}(X)\cdot y$ for the Todd class $\mathrm{Td}(X)$. Then, we have the following equality:*

$$\int_{\mathcal{M}(\widehat{y})}P(\widehat{E}^u)+\sum_{a_1\in\mathcal{SW}(X,y)}\mathrm{SW}(a_1)\cdot 2^{1-\chi(y)}\cdot\mathcal{A}(a_1,y)=0.$$

∎

1.4 Higher Rank Case

1.4.1 The case $p_g>0$

If $p_g>0$ is satisfied, the results in the rank 2 case can be rather easily generalized in the higher rank case. Actually, we have the formally same formula as (4).

Theorem 1.19 (Theorem 7.37) *Assume $p_g>0$. Then the following equality holds:*

$$\Phi(\widehat{y},[L],\delta_+)-\Phi(\widehat{y},[L],\delta_-)=\sum_{(y_1,y_2)\in S_1(y,\delta)}\int_{\mathcal{M}(y_1,\widehat{y}_2,L)}\operatorname*{Res}_{t=0}\left(\frac{P\bigl(E_1^u\cdot e^{-t}\oplus\widehat{E}_2^u\cdot e^{r_1(t-\omega_1)/r_2}\bigr)\cdot t^k}{\mathrm{Eu}\bigl(\mathfrak{N}_0(y_1,y_2)\bigr)}\right)\tag{12}$$

Here, we put $S_1(y,\delta):=\bigl\{(y_1,y_2)\in S(y,\delta)\bigm|\mathrm{rank}(y_1)=1\bigr\}$. We use the notation $\mathcal{M}(y_1,\widehat{y}_2,L)$ instead of $\mathcal{M}(y_1,\widehat{y}_2,L,\delta)$, because δ-semistability condition is trivial in the case $\mathrm{rank}(y_1)=1$. ∎

As for Problem 1.5, we can show the independence of the invariant from the polarization, by using the formula (12). Let $\Phi_H(\widehat{y})$ be as in the subsubsection 1.3.1.

Theorem 1.20 (Theorem 7.39) *The invariant $\Phi_H(\widehat{y})$ is independent of the choice of a generic polarization in the case $p_g > 0$.* ∎

As for Problem 1.6, we obtain an immediate generalization of Theorem 1.18. We do not reproduce it here. See the subsubsection 7.5.2 for detail.

1.4.2 Transition formula in the case $p_g = 0$

The transition formula is comparatively complicated in the case $p_g = 0$. We restrict ourselves to the case where the 1-vanishing condition holds for (y, L, δ), and we discuss the integral of the cohomology class of the following form:

$$\Phi = P(\widehat{E}^u) \cdot \frac{\mathrm{Eu}(T_{\mathrm{rel}})}{N_L(y)}$$

We put as follows:

$$\Phi(\delta) := \int_{\mathcal{M}^{ss}(\widehat{y},[L],\delta)} \Phi.$$

For each positive integer k we put as follows:

$$S_k(y,\delta) := \bigl\{\boldsymbol{Y} = (y_1, \dots, y_k) \in \mathcal{T}ype^k \bigm| P_{y_i} = P_y^\delta\bigr\}$$

For each element $\boldsymbol{Y} = (y_1, \dots, y_k) \in S_k(y,\delta)$, we put $|\boldsymbol{Y}| = \sum_{i=1}^k y_i$. We also put as follows:

$$W(\boldsymbol{Y}) := \prod_{i=1}^{k} \frac{\mathrm{rank}(y_i)}{\sum_{1\le j\le i} \mathrm{rank}(y_j)}$$

We put as follows:

$$\overline{S}(y,\delta) := \coprod_k \overline{S}_k(y,\delta), \qquad \overline{S}_k(y,\delta) := \bigl\{(y_0,\boldsymbol{Y}) \in \mathcal{T}ype \times S_k(y,\delta) \bigm| y_0 + |\boldsymbol{Y}| = y\bigr\}$$

For any $(y_0, \boldsymbol{Y}) \in \overline{S}(y,\delta)$, we put as follows:

$$\mathcal{M}(\widehat{y}_0, \widehat{\boldsymbol{Y}}, [L]) := \mathcal{M}^{ss}\bigl(\widehat{y}_0, [L], \delta_-\bigr) \times \prod_{i=1}^{k} \mathcal{M}^{ss}(\widehat{y}_i)$$

Let $\widehat{E}_0^u$ denote the sheaf over $\mathcal{M}(\widehat{y}_0, \widehat{\boldsymbol{Y}}, [L]) \times X$ which is obtained as the pull back of the universal sheaf over $\mathcal{M}(\widehat{y}_0, [L], \delta_-) \times X$ via the natural projection. We use the notation $\widehat{E}_i^u$ in a similar meaning.

When $(y_0, \boldsymbol{Y}) \in \overline{S}(y,\delta)$ is given, we put as follows:

$$T_0 = -\sum_{j>0} \frac{t_j}{\sum_{0\le h<j} \mathrm{rank}(y_h)}, \quad T_i = -\sum_{j>i} \frac{t_j}{\sum_{0\le h<j} \mathrm{rank}(y_h)} + \frac{t_i}{\mathrm{rank}(y_i)}$$

Here, $t_1, \dots, t_k$ are the variables. Let G denote the k-dimensional torus $\mathrm{Spec}\, k[\tau_1, \tau_1^{-1}, \dots, \tau_k, \tau_k^{-1}]$. Let $e^{w\cdot t_i}$ denote the trivial line bundle with G-action which is induced by the action of $\mathrm{Spec}\, k[\tau_i, \tau_i^{-1}]$ of weight w. We have the following element of the K-group of the G-equivariant coherent sheaves over $\mathcal{M}(\widehat{y}_0, \widehat{\boldsymbol{Y}}, [L])$:

$$-Rp_{X\,*}\mathcal{RH}om\bigl(\widehat{E}_i \cdot e^{T_i},\, \widehat{E}_j \cdot e^{T_j}\bigr) - Rp_{X\,*}\mathcal{RH}om\bigl(\widehat{E}_j \cdot e^{T_j},\, \widehat{E}_i \cdot e^{T_i}\bigr)$$

The equivariant Euler class is denoted by $Q\bigl(\widehat{E}_i \cdot e^{T_i}, \widehat{E}_j \cdot e^{T_j}\bigr)$. We put as follows:

$$Q\bigl(\widehat{E}_0 \cdot e^{T_0}, \widehat{E}_1 \cdot e^{T_1}, \dots, \widehat{E}_k \cdot e^{T_k}\bigr) := \prod_{i<j} Q\bigl(\widehat{E}_i \cdot e^{T_i}, \widehat{E}_j \cdot e^{T_j}\bigr)$$

Let $T_{0\ \mathrm{rel}}$ denote the vector bundle over $\mathcal{M}(\widehat{y}_0, \widehat{\boldsymbol{Y}}, [L])$ obtained from the relative tangent bundle of the smooth map $\mathcal{M}(\widehat{y}_0, [L], \delta) \longrightarrow \mathcal{M}(\widehat{y}_0, [L])$. Then, we put as follows:

$$\Psi(y_0, \boldsymbol{Y}) := \operatorname*{Res}_{t_1=0} \cdots \operatorname*{Res}_{t_k=0} \left(\frac{P\big(\bigoplus_{i=0}^{k} \widehat{E}_i^u \cdot e^{T_i}\big)}{Q\big(\widehat{E}_0 \cdot e^{T_0}, \ldots, \widehat{E}_k \cdot e^{T_k}\big)} \right) \cdot \frac{\mathrm{Eu}(T_{0,\mathrm{rel}})}{N_L(y_0)}$$

In a sense, much part of this paper is devoted to the proof of the following theorem.

Theorem 1.21 *We have the following formula:*

$$\Phi(\delta_+) - \Phi(\delta_-) = \sum_{(y_0, \boldsymbol{Y}) \in \overline{S}(y,\delta)} \frac{N_L(y_0)}{N_L(y)} \cdot W(\boldsymbol{Y}) \cdot \int_{\mathcal{M}(\widehat{y}_0, \widehat{\boldsymbol{Y}}, [L])} \Psi(y_0, \boldsymbol{Y}) \tag{13}$$

▮

1.4.3 Weak Intersection rounding formula

As for the problem 1.5, we easily obtain a generalization of the weak wall crossing formula by using (13). (See Theorem 7.47.) We do not reproduce it here. The formula itself is not so easy to deal with, partially because it contains the integrals over the moduli stacks of semistable sheaves with higher ranks. To derive a more accessible quantity from our invariants, we consider the "intersection rounding formula". The general case will be discussed in the subsection 7.8. In this subsubsection, we reproduce the result in the rank 3 case. See the subsubsection 7.7.4 for more detail.

We take an element $\boldsymbol{\xi} = (\xi_1, \xi_2) \in NS^1(X)^2$ such that ξ_1 and ξ_2 are linearly independent, and we put $W^{\boldsymbol{\xi}} := W^{\xi_1} \cap W^{\xi_2}$. A connected component T of $W^{\boldsymbol{\xi}} \setminus \bigcup_{W \neq W^{\xi_i}} W$ is called a tile. For each tile T, there exist four chambers C_{++}, C_{+-}, C_{--} and C_{-+} with the following properties:

- The closure of C_{κ_1,κ_2} contains T.
- Take an ample line bundle $H_{\kappa_1,\kappa_2} \in C_{\kappa_1,\kappa_2}$. Then the signature of the pairing $(H_{\kappa_1,\kappa_2}, \xi_i)$ is κ_i

Now, we put as follows:

$$\mathcal{D}_{\boldsymbol{\xi}}^{T} \Phi(\widehat{y}) := \Phi_{H_{++}}(\widehat{y}) - \Phi_{H_{+-}}(\widehat{y}) - \Phi_{H_{-+}}(\widehat{y}) + \Phi_{H_{--}}(\widehat{y}).$$

We would like to express $\mathcal{D}_{\boldsymbol{\xi}}^{T} \Phi(\widehat{y})$ as the sum of the integrals over the products of Hilbert schemes.

Let $S(2,1)$ be the set of $\boldsymbol{a} = (a_0, a_1, a_2) \in NS^1(X)^3$ with the following property:

- $a_0 + a_1 - 2a_2 = A_1 \cdot \xi_1$ and $a_0 - a_1 = A_2 \cdot \xi_2$ for some $A_i > 0$.

Let $S(1,2)$ be the set of $\boldsymbol{a} = (a_0, a_1, a_2) \in NS^1(X)^3$ with the following property:

- $2a_0 - (a_1 + a_2) = A_1 \cdot \xi_1$ and $a_1 - a_2 = A_2 \cdot \xi_2$ for some $A_i > 0$.

For each $\boldsymbol{a}$, we put as follows:

$$X(y, \boldsymbol{a}) := \coprod_{n_0+n_1+n_2=N(y,\boldsymbol{a})} \prod_{i=0}^{2} X^{[n_i]}, \quad N(y, \boldsymbol{a}) = n + \frac{a_0^2 + a_1^2 + a_2^2 - a^2}{2}$$

Here, a and n denote the first Chern class and the second Chern class of y, respectively.

Proposition 1.22

- $\mathcal{D}_{\boldsymbol{\xi}}^{T} \Phi(\widehat{y})$ *is independent of the choice of a tile T. Therefore, we can omit to denote T.*

- *The following equality holds:*

$$\mathcal{D}_{\boldsymbol{\xi}}\Phi(\widehat{y}) = \sum_{\boldsymbol{a}\in S(1,2)} \int_{X(y,\boldsymbol{a})} \operatorname*{Res}_{t_2} \operatorname*{Res}_{t_1} \left(\frac{P\big(\mathcal{I}_0^u e^{a_0-t_1} \oplus \mathcal{I}_1^u e^{a_1+t_1/2-t_2} \oplus \mathcal{I}_2^u e^{a_2+t_1/2+t_2}\big)}{Q\big(\mathcal{I}_0^u e^{a_0-t_1},\ \mathcal{I}_1^u e^{a_1+t_1/2-t_2},\ \mathcal{I}_2^u e^{a_2+t_1/2+t_2}\big)} \right)$$
$$+ \sum_{\boldsymbol{a}\in S(2,1)} \int_{X(y,\boldsymbol{a})} \operatorname*{Res}_{t_2} \operatorname*{Res}_{t_1} \left(\frac{P\big(\mathcal{I}_0^u e^{a_0-t_1/2-t_2} \oplus \mathcal{I}_1^u e^{a_1-t_1/2+t_2} \oplus \mathcal{I}_2^u e^{a_2+t_1}\big)}{Q\big(\mathcal{I}_0^u e^{a_0-t_1/2-t_2},\ \mathcal{I}_1^u e^{a_1-t_1/2+t_2},\ \mathcal{I}_2^u e^{a_2+t_1}\big)} \right) \tag{14}$$

The formula (14) *is called the weak intersection rounding formula in the rank* 3 *case. See Theorem* 7.52 *for the general case.* ▮

Remark 1.23 *The author expects that $\mathcal{D}_{\boldsymbol{\xi}}\Phi(\widehat{y})$ can be described more beautifully like the wall crossing formula in the rank* 2 *case (*[20], [21]*). It may be interesting to see the equality obtained by exchanging the roles of ξ_1 and ξ_2,* ▮

1.5 Master Space

1.5.1 The master space of Thaddeus

In this subsubsection, we recall the picture of the master space given by M. Thaddeus, which is one of the most important tools in this study. Let G be a linear reductive group. Let U be a projective variety with a G-action. Let $\mathcal{L}_i$ $(i = 1, 2)$ be G-polarizations of U. Then, we have the open subset $U^{ss}(\mathcal{L}_i)$ of semistable points of U with respect to $\mathcal{L}_i$. It is interesting to compare the stacks $\mathcal{M}_i := U^{ss}(\mathcal{L}_i)/G$ $(i = 1, 2)$.

For that purpose, Thaddeus introduced the idea of the master space. Let us consider the G-variety $\mathrm{TH} = \mathbb{P}(\mathcal{L}_1^{-1} \oplus \mathcal{L}_2^{-1})$ on U. We have the canonical polarization $\mathcal{O}_{\mathbb{P}}(1)$ on TH. We have the canonically defined G-action on TH, and $\mathcal{O}_{\mathbb{P}}(1)$ gives the G-polarization. The set of the semistable points is denoted by TH^{ss}. Then we obtain the stack $M = \mathrm{TH}^{ss}/G$.

We have the G_m-action on TH given by $\rho(t)([x : y]) = [t \cdot x : y]$, where $[x : y]$ denotes the homogeneous coordinate of TH along the fiber direction. The action ρ commutes with the action of G. Thus we have the G_m-action on M, which we denote by $\overline{\rho}$.

We have the natural inclusion $\mathrm{TH}_i = \mathbb{P}(\mathcal{L}_i^{-1}) \longrightarrow \mathrm{TH}$. Due to $\mathcal{O}_{\mathbb{P}}(1)_{|\mathrm{TH}_i} = \mathcal{L}_i$, it is easy to observe $\mathrm{TH}_i^{ss} = U(\mathcal{L}_i)^{ss}$. Thus we have the inclusion $\mathcal{M}_i \longrightarrow M$. Since TH_i is a component of the fixed point set with respect to the action ρ, the stacks $\mathcal{M}_i$ are the components of the fixed point set of the action $\overline{\rho}$.

We may have the fixed points of the action $\overline{\rho}$, which are not contained in $\mathcal{M}_1 \cup \mathcal{M}_2$. Let x be a point of TH^{ss}, which is not a fixed point of the action ρ. Assume $G_m \cdot x \subset G \cdot x$. Then the point $\pi(x)$ of M is a fixed point of the action $\overline{\rho}$, where π denotes the natural projection $\mathrm{TH}^{ss} \longrightarrow M$. The component of such fixed points is called the exceptional fixed point set. In a sense, the information on the difference of the quotient stacks $\mathcal{M}_i$ $(i = 1, 2)$ are concentrated at the exceptional fixed point set. We will later explain how to derive the information by using G_m-localization in our case.

Remark 1.24 *When we consider the categorical quotients $M_i := U^{ss}(\mathcal{L}_i)//G$ and $Y := \mathrm{TH}^{ss}(\mathcal{L}_i)//(G \times G_m)$, we obtain the morphisms $M_1 \longrightarrow Y \longleftarrow M_2$. The diagram is called Thaddeus Flip. It also provides us a significant tool to compare the spaces M_i. In particular, if M_i are smooth and each morphism $M_i \longrightarrow Y$ is the blow up along the smooth center, the flip is quite useful.* ▮

1.5.2 A construction of the master space when the 2-stability condition is satisfied

We explain the way of the construction of the master space for our moduli stacks, when the 2-stability condition is satisfied (the subsubsection 1.2.1). To begin with, we give a remark. It is known that the coarse moduli scheme of semistable torsion-free sheaves is obtained as the categorical quotient of the set of the semistable points of some projective variety provided with a reductive group action. But, we will discuss the moduli stacks of semistable *parabolic* sheaves or semistable *parabolic* L-Bradlow pairs, although we omit to mention parabolic structure in this *Introduction*. For such parabolic objects, the author does not know the reference to deal with the problem whether the moduli spaces have such descriptions. Hence, we need a modification of the construction of the master space.

Let y be an element of $\mathcal{T}ype$. We put $y(m) := \mathrm{ch}\big(\mathcal{O}(m)\big)$. Let H_y be the Hilbert polynomial associated to y. Let V_m be an $H_y(m)$-dimensional vector space. We put $V_{m,X} := V_m \otimes \mathcal{O}_X$. The projectivization of V_m is denoted by $\mathbb{P}_m$. Let $Q(m,y)$ denote the quot scheme. It is the moduli scheme of the quotient sheaves of $V_{m,X}$ whose Chern character is $y(m)$. It is easy to construct the moduli scheme $Q(m,\widehat{y})$ of the quotient sheaves of $V_{m,X}$ with *orientations* whose Chern character is $y(m)$. We fix an inclusion $\iota : \mathcal{O}(-m) \longrightarrow L$. Let δ be an element of $\mathcal{P}^{\mathrm{br}}$. Then, we obtain the subscheme $Q^{ss}(m,y,[L],\delta)$ of $Q^{\circ}(m,y)\times\mathbb{P}_m$, which is the moduli scheme of the quotient sheaves $(q : V_{m,X} \to \mathcal{E})$ with non-trivial reduced $[L]$-sections $\phi : L \longrightarrow \mathcal{E}(-m)$ with the following property:

- The Chern character of $\mathcal{E}$ is $y(m)$, and the pair $(\mathcal{E}(-m),\phi)$ is δ-semistable.

We put $Q^{ss}(m,\widehat{y},[L],\delta) := Q^{ss}(m,y,[L],\delta)\times_{Q(m,y)} Q(m,\widehat{y})$.

On the other hand, let $\det(y(m))$ denote the $H^2(X)$-part of $y(m)$. We take a line bundle $\mathfrak{L}_a$ such that $c_1(\mathfrak{L}_a) = y(m)$. We may assume $H^i(X,\mathfrak{L}_a) = 0$ $(i > 0)$. We denote by Z_m the projectivization of $H^0(X,\mathfrak{L}_a)$, which is called the Gieseker space. We put $\mathcal{A}_m := Z_m \times \mathbb{P}_m$

When a parameter $\delta\in\mathcal{P}^{\mathrm{br}}$ is given, the ample $\mathbb{Q}$-line bundle $\mathcal{L}$ is given on $\mathcal{A}_m$ as follows:

$$\mathcal{L} := \mathcal{O}_{Z_m}\big(P^{\delta}_y(m)\big)\otimes\mathcal{O}_{\mathbb{P}_m}\big(\delta(m)\big)$$

We have the naturally defined $\mathrm{SL}(V_m)$-action on $\mathcal{A}_m$, and $\mathcal{L}$ gives the $\mathrm{SL}(V_m)$-polarization. Let $\mathcal{A}_m(\mathcal{L})$ denote the open subset of the semistable points with respect to $\mathcal{L}$. It is rather standard to show the following proposition.

Proposition 1.25 (Proposition 4.2) *We have the* $\mathrm{SL}(V_m)$*-equivariant closed immersion* $Q^{ss}(m,y,[L],\delta)\longrightarrow \mathcal{A}^{ss}_m(\mathcal{L})$. ∎

We take a large integer k such that $\mathcal{L}^{\otimes k}$ is actually a line bundle. We take a rational number γ whose absolute value is sufficiently small. We put $\mathcal{L}_\gamma := \mathcal{L}^{\otimes k}\otimes\mathcal{O}_{\mathbb{P}_m}(\gamma)$. Let $\mathcal{A}^{ss}_m(\mathcal{L}_\gamma)$ denote the open subset of the semistable points with respect to $\mathcal{L}_\gamma$. On the other hand, we take $\delta_- < \delta < \delta_+$ sufficiently closely. It is not difficult to observe the following by using Proposition 1.25:

Lemma 1.26 *In the case* $\gamma < 0$*, we have the closed immersion* $Q^{ss}(m,y,[L],\delta_-)\longrightarrow\mathcal{A}^{ss}(\mathcal{L}_\gamma)$*. In the case* $\gamma > 0$*, we have the closed immersion* $Q^{ss}(m,y,[L],\delta_+)\longrightarrow\mathcal{A}^{ss}(\mathcal{L}_\gamma)$. ∎

We take rational numbers $\gamma_2 < 0 < \gamma_1$ such that $|\gamma_i|$ are sufficiently small. We take a large number k' such that $k'\cdot(\gamma_1-\gamma_2) = 1$. Let $\pi : \mathcal{A}_m\longrightarrow\mathbb{P}_m$ denote the projection. We put $\mathcal{B} := \mathbb{P}\big(\pi^*\mathcal{O}_{\mathbb{P}_m}(0)\oplus\mathcal{O}_{\mathbb{P}_m}(1)\big)$, which is $\mathbb{P}^1$-bundle over $\mathcal{A}$. The tautological line bundle is denoted by $\mathcal{O}_{\mathbb{P}}(1)$. We put $\mathcal{O}_{\mathcal{B}}(1) := \mathcal{O}_{\mathbb{P}}(1)\otimes\mathcal{L}_{\gamma_1}^{\otimes k'}$. It gives the $\mathrm{SL}(V_m)$-polarization of $\mathcal{B}$. Let $\mathcal{B}^{ss}$ denote the open subset of the semistable points of $\mathcal{B}$ with respect to $\mathcal{O}_{\mathcal{B}}(1)$.

We have the natural inclusion $\mathcal{B}_1 := \mathbb{P}\big(\pi^*\mathcal{O}_{\mathbb{P}_m}(0)\big)\subset\mathcal{B}$ and $\mathcal{B}_2 := \mathbb{P}\big(\pi^*\mathcal{O}_{\mathbb{P}_m}(1)\big)\subset\mathcal{B}$. We remark $\mathcal{O}_{\mathcal{B}}(1)_{|\mathcal{B}_i} = \mathcal{L}_{\gamma_i}^{\otimes k'}$. Let $\mathcal{B}_i^{ss}$ denote the semistable points of $\mathcal{B}_i$ with respect to $\mathcal{O}_{\mathcal{B}}(1)$. Then, we put as follows:

$$\widehat{\mathrm{TH}}^{ss} := Q^{ss}(m,\widehat{y},[L],\delta)\times_{\mathcal{A}}\mathcal{B}^{ss},\quad \widehat{\mathrm{TH}}^{ss}_i := Q^{ss}(m,\widehat{y},[L],\delta)\times_{\mathcal{A}}\mathcal{B}^{ss}_i.$$

We put $\widehat{M} := \widehat{\mathrm{TH}}^{ss}\big/\,\mathrm{GL}(V_m)$, which is the master space in this case. We also put $\widehat{M}_i := \widehat{\mathrm{TH}}^{ss}\big/\,\mathrm{GL}(V_m)$. Due to Lemma 1.26, we have $\widehat{M}_1\simeq\mathcal{M}^s(\widehat{y},[L],\delta_+)$ and $\widehat{M}_2\simeq\mathcal{M}^s(\widehat{y},[L],\delta_-)$.

When the 2-stability condition is satisfied, it can be easily shown that $\widehat{M}$ is Deligne-Mumford and proper. We have the naturally defined G_m-action on $\widehat{M}$, as explained in the subsubsection 1.5.1. The fixed point set is as follows:

$$\widehat{M}_1\sqcup\widehat{M}_2\sqcup\coprod_{(y_1,y_2)\in S(y,\delta)}\widehat{M}^{G_m}(y_1,y_2)$$

Here, $\widehat{M}^{G_m}$ is isomorphic to the moduli stack of the objects $(E_1,\phi,E_2;\rho)$ with the following properties:

- (E_1,ϕ) is δ-stable L-Bradlow pair.

- E_2 is semistable torsion-free sheaf.
- ρ is an orientation of $E_1 \oplus E_2$.

It is easy to observe that $\widehat{M}^{G_m}$ is isomorphic to $\mathcal{M}^{ss}(y_1, L, \delta) \times \mathcal{M}^{ss}(\widehat{y}_2)$ up to etale finite morphisms.

1.5.3 The G_m-localization method in the case where the 2-stability condition is satisfied

We explain how to use the master space $\widehat{M}$ to obtain the transition formula (Theorem 1.7), when the 2-stability condition is satisfied for (y, L, δ). We use the notation in the subsubsection 1.2.1. Let $\varphi : \widehat{M} \longrightarrow \mathcal{M}(\widehat{y}, [L])$ be the naturally defined morphism. Let $\mathcal{T}(1)$ denote the trivial line bundle on $\mathcal{M}(\widehat{y}, [L])$ with the G_m-action of weight 1. We have the naturally defined G_m-action on $\varphi^*\widehat{E}^u$, $\varphi^*\mathcal{O}_{\mathrm{rel}}(1)$ and $\varphi^*\mathcal{T}(1)$. Therefore, we obtain the following G_m-equivariant cohomology classes on $\widehat{M}$:

$$\Phi_t := P\bigl(\varphi^*\widehat{E}^u\bigr) \cdot c_1\bigl(\varphi^*\mathcal{O}_{\mathrm{rel}}(1)\bigr)^k, \quad \overline{\Phi}_t := \Phi_t \cdot c_1\bigl(\varphi^*\mathcal{T}(1)\bigr)$$

The master space is naturally provided with the perfect obstruction theory, and hence we have the virtual fundamental class. Therefore, we can consider the polynomial $\int_{\widehat{M}} \overline{\Phi}_t \in \mathbb{Q}[t]$. Since $\varphi^*\mathcal{T}(1)$ is the trivial line bundle, the specialization at $t = 0$ is trivial. We use the localization theory of the virtual fundamental classes due to Graber-Pandharipande ([24]). We have the following equality in $\mathbb{Q}[t, t^{-1}]$:

$$\int_{\widehat{M}} \overline{\Phi}_t = \sum_{i=1,2} \int_{\widehat{M}_i} \frac{t \cdot \Phi_t}{\mathrm{Eu}(\mathfrak{N}(\widehat{M}_i))} + \sum_{(y_1,y_2) \in S(y,\delta)} \int_{\widehat{M}^{G_m}(y_1,y_2)} \frac{t \cdot \Phi_t}{\mathrm{Eu}\bigl(\mathfrak{N}(\widehat{M}^{G_m}(y_1, y_2))\bigr)}$$

Here, $\mathfrak{N}(\widehat{M}_i)$ and $\mathfrak{N}(\widehat{M}^{G_m}(y_1, y_2))$ denote the virtual normal bundles. Therefore, we obtain the following equality:

$$\sum_{i=1,2} \int_{\widehat{M}_i} \operatorname*{Res}_{t=0} \left(\frac{\Phi_t}{\mathrm{Eu}\bigl(\mathfrak{N}(\widehat{M}_i)\bigr)} \right) + \sum_{(y_1,y_2) \in S(y,\delta)} \int_{\widehat{M}^{G_m}(y_1,y_2)} \operatorname*{Res}_{t=0} \left(\frac{\Phi_t}{\mathrm{Eu}\bigl(\mathfrak{N}(\widehat{M}^{G_m}(y_1, y_2))\bigr)} \right) = 0. \tag{15}$$

It is easy to observe that the first term of the left hand side of (15) can be rewritten as follows:

$$-\int_{\widehat{M}_1} \Phi + \int_{\widehat{M}_2} \Phi = -\Phi(\widehat{y}, [L], \delta_+) + \Phi(\widehat{y}, [L], \delta_-)$$

Note that $\widehat{M}^{G_m}(y_1, y_2)$ is isomorphic to $\mathcal{M}^{ss}(y_1, \widehat{y}_2, L, \delta)$ up to etale finite morphisms. By a formal calculation, it can be shown that the second term is same as the right hand side of (4). Thus we obtain the transition formula in the case where the 2-stability condition is satisfied for (y, L, δ).

In the case $p_g > 0$, the transition problem is rather easy, because of the following proposition.

Proposition 1.27 (Proposition 6.24 and Proposition 6.25)

- *In the case* $\mathrm{rank}(y) > 1$ *and* $p_g > 0$*, the virtual fundamental class of* $\mathcal{M}^{ss}(y, L, \delta)$ *is trivial. (We remark that the virtual fundamental class of* $\mathcal{M}^{ss}(\widehat{y}, [L], \delta)$ *is not necessarily trivial.)*
- *Even in the case* $\mathrm{rank}(y) = 1$ *and* $p_g > 0$*, we have the vanishing* $[\mathcal{M}^{ss}(y, L, \delta)] = 0$*, if the* 2*-vanishing condition is satisfied for* (y, L, δ)*.*
- *On the other hand, we have* $\kappa^*\bigl([\mathcal{M}^{ss}(y, L, \delta)]\bigr) = [\mathcal{M}^{ss}(\widehat{y}, [L], \delta)]$*, where* κ *denotes the naturally defined etale finite morphism* $\mathcal{M}^{ss}(\widehat{y}, [L], \delta) \longrightarrow \mathcal{M}^{ss}(y, L, \delta)$ *in the case* $p_g = 0$*.* ∎

Due to Proposition 1.27, we obtain the vanishing of the contributions in (15) from (y_1, y_2) such that $\mathrm{rank}(y_1) > 1$, when $p_g > 0$. Even in the case $\mathrm{rank}(y_1) = 1$, the contributions vanish if L is sufficiently "negative".

1.5.4 The enhanced master space

When the 2-stability condition is not satisfied, $\widehat{M}$ in the subsubsection 1.5.2 is not Deligne-Mumford. So we need some modification, which we explain in this subsubsection. We use the notation in the subsubsection 1.5.2. We put as follows:

$$\widetilde{\mathcal{B}} := \mathcal{B} \times \mathrm{Flag}(V_m, \underline{N}), \quad \widetilde{\mathcal{B}}_i := \mathcal{B}_i \times \mathrm{Flag}(V_m, \underline{N}).$$

Here, $\mathrm{Flag}(V_m, \underline{N})$ denotes the full flag variety of V_m:

$$\mathrm{Flag}(V_m, \underline{N}) = \Big\{ \mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_N \,\Big|\, \dim \mathcal{F}_i / \mathcal{F}_{i-1} = 1 \Big\}$$

Let $G_l(V_m)$ denote the Grassmannian variety of the l-dimensional subspaces of V_m. We have the canonical polarization $\mathcal{O}_{G_l}(1)$. We have the morphism $\rho_l : \mathrm{Flag}(V_m, \underline{N}) \longrightarrow G_l(V_m)$ given by $\rho_l(\mathcal{F}_*) = \mathcal{F}_l$. Take small positive numbers n_i $(i = 1, 2, \ldots, N)$, then we obtain the $\mathrm{SL}(V_m)$-polarization of $\widetilde{\mathcal{B}}$:

$$\mathcal{O}_{\widetilde{\mathcal{B}}}(1) := \mathcal{O}_{\mathcal{B}}(1) \otimes \bigotimes_{l=1}^{N} \rho_l^* \mathcal{O}_{G_l(V_m)}(n_i)$$

Let $\widetilde{\mathcal{B}}^{ss}$ and $\widetilde{\mathcal{B}}_i^{ss}$ denote the set of semistable points with respect to $\mathcal{O}_{\widetilde{\mathcal{B}}}(1)$. In this case, we put as follows:

$$\widehat{\mathrm{TH}}^{ss} := Q^{ss}(m, \widehat{y}, [L], \delta) \times_{\mathcal{A}} \widetilde{\mathcal{B}}^{ss}, \quad \widehat{\mathrm{TH}}_i^{ss} := Q^{ss}(m, \widehat{y}, [L], \delta) \times_{\mathcal{A}} \widetilde{\mathcal{B}}_i^{ss}.$$

We use the notation $\widehat{M}$ to denote $\widehat{\mathrm{TH}}^{ss} / \mathrm{GL}(V_m)$, and we call it the enhanced master space. We also put $\widehat{M}_i := \widehat{\mathrm{TH}}_i^{ss} / \mathrm{GL}(V_m)$. Due to Lemma 1.26, it can be shown that $\widehat{M}_1$ is the full flag bundle over $\mathcal{M}^s(\widehat{y}, [L], \delta_+)$ associated to the vector bundle $p_{X\,*}\widehat{E}^u(m)$. Similarly, $\widehat{M}_2$ is the full flag bundle over $\mathcal{M}^s(\widehat{y}, [L], \delta_-)$.

We can show the following.

Proposition 1.28 (Proposition 4.21 and Proposition 4.40) *$\widehat{M}$ is Deligne-Mumford and proper.* ∎

We have the naturally defined G_m-action on $\widehat{M}$, as explained in the subsubsection 1.5.1. To describe the fixed point set, we need some preparation.

Definition 1.29 *A decomposition type is defined to be a datum $\mathfrak{I} := (y_1, y_2, I_1, I_2)$ as follows:*

- *$y = y_1 + y_2$ in $\mathcal{T}ype$ such that $P^{\delta}_{y_1} = P^{\delta}_{y}$.*
- *$\underline{N} = I_1 \sqcup I_2$ such that $|I_i| = H_{y_i}(m)$, here H_{y_i} denote the Hilbert polynomials associated to y_i.*

The set of the decomposition types is denoted by $\mathrm{Dec}(m, y, \delta)$. For $\mathfrak{I} = (y_1, y_2, I_1, I_2) \in \mathrm{Dec}(m, y, \delta)$, we put $\mathfrak{k}(\mathfrak{I}) := \min(I_2) - 1$. ∎

We introduce the notion of (δ, ℓ)-semistability.

Definition 1.30 *Let $(E, [\phi])$ be a reduced L-Bradlow pair on X, and let $\mathcal{F}$ be a full flag of $H^0\big(X, E(m)\big)$. Let ℓ be any positive integer. We say that $(E, [\phi], \mathcal{F})$ is (δ, ℓ)-semistable, if the following conditions are satisfied:*

- *$(E, [\phi])$ is δ-semistable.*
- *Take any Jordan-Hölder filtration of $(E_*, [\phi])$ with respect to δ-semistability:*

$$E^{(1)} \subset E^{(2)} \subset \cdots \subset E^{(i-1)} \subset (E^{(i)}, \phi) \subset \cdots \subset (E^{(k)}, \phi)$$

Then we have $\mathcal{F}_\ell \cap H^0\big(X, E^{(i-1)}(m)\big) = \{0\}$ and $\mathcal{F}_\ell \not\subset H^0\big(X, E^{(j)}(m)\big)$ for $j < k$.

We denote by $\widetilde{\mathcal{M}}^{ss}\big(y, [L], (\delta, \ell)\big)$, the moduli stack of such tuples $(E, [\phi], \mathcal{F})$. In the oriented case, we use the notation $\widetilde{\mathcal{M}}^{ss}\big(\widehat{y}, [L], \alpha_*, (\delta, \ell)\big)$ as usual.

Similarly, we have the notion of (δ, ℓ)-semistability for a L-Bradlow pairs (E, ϕ) with a full flag $\mathcal{F}$ of $H^0(X, E(m))$ such that $\phi \neq 0$. The moduli stack is denoted by $\widetilde{\mathcal{M}}^{ss}\big(y, L, (\delta, \ell)\big)$. ∎

Then, the fixed point set is as follows:

$$\widehat{M}_1 \sqcup \widehat{M}_2 \sqcup \coprod_{\mathfrak{I} \in \mathrm{Dec}(m,y,\delta)} \widehat{M}^{G_m}(\mathfrak{I})$$

Here, $\widehat{M}^{G_m}(\mathfrak{I})$ is isomorphic to the moduli stack of the objects $(E_1, \phi, E_2, \rho, \mathcal{F}^{(1)}, \mathcal{F}^{(2)})$ with the following properties:

- (E_1, ϕ) is a δ-semistable L-Bradlow pair.
- $\mathcal{F}^{(1)}$ is a full flag of $H^0(X, E_1(m))$ such that $(E_1, \phi, \mathcal{F}^{(1)})$ is $\bigl(\delta, \mathfrak{k}(\mathfrak{I})\bigr)$-semistable.
- E_2 is a semistable torsion-free sheaf.
- $\mathcal{F}^{(2)}$ is a full flag of $H^0(X, E_2(m))$ such that $\bigl(E_2, \mathcal{F}^{(2)}_{\min(I_2)}\bigr)$ is ϵ-semistable reduced $\mathcal{O}(-m)$-Bradlow pair, where ϵ denotes any sufficiently small positive number.
- ρ is an orientation of $E_1 \oplus E_2$.

Therefore, $\widehat{M}^{G_m}(\mathfrak{I})$ is isomorphic to $\widetilde{\mathcal{M}}^{ss}\bigl(y_1, L, (\delta, \mathfrak{k}(\mathfrak{I}))\bigr) \times \widetilde{\mathcal{M}}^{ss}(\widehat{y}_2, +)$ up to etale finite morphisms, where $\widetilde{\mathcal{M}}^{ss}(\widehat{y}_2, +)$ denotes oriented semistable sheaf (E_2, ρ_2) with a full flag $\mathcal{F}^{(2)}$ as in the condition above.

1.5.5 The G_m-localization method

We explain how to use the enhanced master space $\widehat{M}$ to obtain the transition formulas. The argument is essentially same as that in the subsubsection 1.5.3. Let $\mathcal{M}(m, \widehat{y}, [L])$ denote the open subset of $\mathcal{M}(\widehat{y}, [L])$ determined by the condition O_m. (See the subsubsection 3.1.2.) We have the vector bundle $p_{X\,*}\widehat{E}^u(m)$ on $\mathcal{M}(m, \widehat{y}, [L])$. The associated full flag bundle is denoted by $\widetilde{\mathcal{M}}(m, \widehat{y}, [L])$. Let $\widetilde{T}_{\mathrm{rel}}$ denote the relative tangent bundle of the smooth morphism $\widetilde{\mathcal{M}}(m, \widehat{y}, [L]) \longrightarrow \mathcal{M}(m, \widehat{y}, [L])$. We have the naturally defined morphism $\varphi : \widehat{M} \longrightarrow \widetilde{\mathcal{M}}(m, \widehat{y}, [L])$. Let $\mathcal{T}(1)$ denote the trivial line bundle on $\widetilde{\mathcal{M}}(m, \widehat{y}, [L])$ with the G_m-action of weight 1. We have the naturally defined G_m-action on $\varphi^*\widehat{E}^u$, $\varphi^*\mathcal{O}_{\mathrm{rel}}(1)$, $\varphi^*\widetilde{T}_{\mathrm{rel}}$ and $\varphi^*\mathcal{T}(1)$. We consider the following cohomology classes on $\widehat{M}$:

$$\widetilde{\Phi}_t := P(\varphi^*\widehat{E}^u) \cdot c_1\bigl(\varphi^*\mathcal{O}_{\mathrm{rel}}(1)\bigr)^k \cdot \frac{\mathrm{Eu}(\widetilde{T}_{\mathrm{rel}})}{H_y(m)!}, \quad \overline{\Phi}_t := \widetilde{\Phi}_t \cdot c_1\bigl(\varphi^*\mathcal{T}(1)\bigr)$$

By the argument in the subsubsection 1.5.3, we obtain the following equality:

$$\sum_{i=1,2} \int_{\widehat{M}_i} \operatorname*{Res}_{t=0} \left(\frac{\widetilde{\Phi}_t}{\mathrm{Eu}\bigl(\mathfrak{N}(\widehat{M}_i)\bigr)} \right) + \sum_{\mathfrak{I} \in \mathrm{Dec}(m,y,\delta)} \int_{\widehat{M}^{G_m}(\mathfrak{I})} \operatorname*{Res}_{t=0} \left(\frac{\widetilde{\Phi}_t}{\mathrm{Eu}(\mathfrak{N}(\widehat{M}^{G_m}(\mathfrak{I})))} \right) = 0. \tag{16}$$

Here $\mathfrak{N}(\widehat{M}_i)$ and $\mathfrak{N}(\widehat{M}^{G_m}(\mathfrak{I}))$ denote the virtual normal bundles. Since $\widehat{M}_i$ are the full flag bundles over $\mathcal{M}^{ss}(\widehat{y}, [L], \delta_\kappa)$ ($\kappa = \pm$), it is easy to observe that the first term of the left hand side of (15) can be rewritten as follows:

$$-\int_{\widehat{M}_1} \widetilde{\Phi} + \int_{\widehat{M}_2} \widetilde{\Phi} = -\Phi(\widehat{y}, [L], \delta_+) + \Phi(\widehat{y}, [L], \delta_-)$$

Let us see the second term in the case $p_g > 0$. Recall that $\widehat{M}^{G_m}(\mathfrak{I})$ is isomorphic to $\widetilde{\mathcal{M}}^{ss}\bigl(\widehat{y}_1, L, (\delta, \mathfrak{k}(\mathfrak{I}))\bigr) \times \widetilde{\mathcal{M}}^{ss}\bigl(\widehat{y}_2, +\bigr)$ up to etale finite morphisms. Due to the vanishing similar to Proposition 1.27, we obtain the vanishing of the contributions in (16) from $\mathfrak{I} = (y_1, y_2, I_1, I_2)$ such that $\mathrm{rank}(y_1) > 1$. In the case $\mathrm{rank}(y_1) = 1$, the $\bigl(\delta, \mathfrak{k}(\mathfrak{I})\bigr)$-semistability condition is trivial. Therefore, $\widetilde{\mathcal{M}}^{ss}\bigl(y_1, L, (\delta, \mathfrak{k}(\mathfrak{I}))\bigr)$ is the full flag bundle over $\mathcal{M}(y_1, L)$ associated to the vector bundle $p_{X\,*}E_1^u(m)$. On the other hand, $\widetilde{\mathcal{M}}^{ss}(\widehat{y}_2, +)$ is the flag variety bundle over $\mathcal{M}^{ss}\bigl(\widehat{y}_2, [\mathcal{O}(-m)], \epsilon\bigr)$ for any sufficiently small positive number ϵ. (See the subsubsection 4.6.1 for more detail.) Therefore, we can easily observe that the second term of the left hand side of (16) is same as the right hand side of (12). Thus, we obtain the transition formula in the case $p_g > 0$.

On the other hand, we need some more additional argument in the case $p_g = 0$. Due to the equality (16), we obtain the expression $\Phi(\widehat{y},[L],\delta_+) - \Phi(\widehat{y},[L],\delta_-)$ as the summation of the terms of the following form:

$$\int_{\widetilde{\mathcal{M}}^{ss}(\widehat{y}_1,[L],\delta,\ell)\times\mathcal{M}^{ss}(\widehat{y}_2)} \Psi(y_1,y_2)$$

Then, we take the enhanced master space connecting $\widetilde{\mathcal{M}}^{ss}\bigl(\widehat{y}_1,[L],(\delta,\ell)\bigr)$ and $\widetilde{\mathcal{M}}^{ss}\bigl(\widehat{y}_1,[L],\delta_-\bigr)$. Such a space can be constructed by the method in the subsubsection 1.5.4. We have only to choose appropriate numbers n_i $(i = 1,\ldots,N)$. Let $S(\ell)$ denote the set of decomposition types $\mathfrak{I}^{(2)} = (y_1^{(2)}, y_2^{(2)}, I_1^{(2)}, I_2^{(2)})$ with the following property:

- $y_1^{(2)} + y_2^{(2)} = y_1$.
- $I_1^{(2)} \sqcup I_2^{(2)} = \bigl\{1,\ldots,H_{y_1}(m)\bigr\}$ and $|I_i^{(2)}| = H_{y_i^{(2)}}(m)$.
- $\{1,\ldots,\ell\} \subset I_1^{(2)}$.

Then, applying the localization method again, we obtain the following:

$$\int_{\widetilde{\mathcal{M}}^{ss}(\widehat{y}_1,[L],\delta,\ell)\times\mathcal{M}^{ss}(\widehat{y}_2)} \Psi(y_1,y_2) = \int_{\widetilde{\mathcal{M}}^{ss}(\widehat{y}_1,[L],\delta_-)\times\mathcal{M}^{ss}(\widehat{y}_2)} \Psi(y_1,y_2) + \sum_{\mathfrak{I}^{(2)}\in S(\ell)} \int_{\widehat{M}^{G_m}(\mathfrak{I}^{(2)})} \Psi^{(2)}(y_1^{(2)},y_2^{(2)},y_2)$$

We can apply this procedure inductively. Note $\mathrm{rank}(y_1^{(2)}) < \mathrm{rank}(y_1) < \mathrm{rank}(y)$. Hence, the inductive process will stop. Thus, we can obtain the general transition formula, in principle, by using a rather easy combinatorial argument. However, the general formula would be comparatively complicated, and it is less interesting for the author at this moment. Hence, we restrict ourselves to the transition formula in the special case, as in Theorem 1.21.

1.6 Outline of the Paper

1.6.1 Section 2

In the subsection 2.1, we prepare some notation. In the subsection 2.2, we review basic results from the geometric invariant theory. In particular, we recall a sufficient condition for a quotient stack to be Deligne-Mumford and proper. We also recall the numerical criterion and calculate some easy examples. The results will be used in the section 4.

In the subsection 2.3, we review the basis of cotangent complex. Then, we recall how to calculate some cotangent complexes of quotient stacks in the subsubsection 2.3.2, which will be used in the section 5 in many times. We also recall some more examples in the subsubsection 2.3.3, which will be used in the subsections 6.3 and 6.6.

In the subsection 2.4, we review the obstruction theory of Behrend-Fantechi [5]. Then, we explain a naive strategy to construct the obstruction theory in the subsubsection 2.4.2. We recall the obstruction theory of the locally free subsheaves in the subsubsection 2.4.3. It gives the obstruction theory of the moduli of torsion-free quotient sheaves over a smooth projective surface. The result will be used in the subsection 5.6. We also obtain the smoothness of the moduli of quotient torsion-sheaves over a smooth projective curve, although we do not use it later. In the subsubsection 2.4.4, we recall the obstruction theory of the filtration of a vector bundle on a smooth projective curve. It will be used to see the obstruction theory of the parabolic structure.

In the subsection 2.5, we recall some easy results for equivariant complexes on Deligne-Mumford stacks with GIT construction, which will be used in the subsection 5.9. In the subsection 2.6, we give some elementary remarks on extreme sets, which are used in the subsections 4.3–4.4. In the subsection 2.7, we give easy remarks on the twist of line bundles.

1.6.2 Section 3

In the subsection 3.1, we review the basic notion. In the subsubsections 3.1.1–3.1.3, we recall the definition of some structure on torsion-free sheaves such as orientation, parabolic structure, L-section, and reduced L-section. In the subsubsection 3.1.4, we prepare the notion of type and the notation of some moduli stacks. In the subsubsection 3.1.5, we recall the notion of the relative tautological line bundle of the moduli stacks of oriented reduced L-Bradlow pair. We also see the relation of the moduli stack of oriented reduced L-Bradlow pair and the moduli stack of unoriented unreduced L-Bradlow pair.

In the subsection 3.2, we recall the Hilbert polynomials. Then, we have the naturally defined semistability conditions, which is discussed in the subsection 3.3. We recall the notion of Harder-Narasimhan filtration and partial Jordan-Hölder filtration in the subsubsection 3.3.2. Then, we introduce the notion of (δ, ℓ)-semistability in the subsubsection 3.3.3, which is useful to control the transition.

In the subsection 3.4, we review boundedness of some families. We recall foundational theorems in the subsubsection 3.4.1, Then, we recall the boundedness of semistable L-Bradlow pairs when the parameter is varied in the subsubsection 3.4.2. The important observation is due to M. Thaddeus. In the subsubsection 3.4.3, we see boundedness of Yokogawa family, which will be used to show properness of some morphisms in the section 4.

In the subsection 3.5, we recall the 1-stability and 2-stability conditions. In the subsection 3.6, we recall some moduli schemes of quotient sheaves with some structure.

1.6.3 Section 4

In the subsection 4.1, we review a basic result of the geometric invariant theory for the construction of the moduli stack of δ-semistable parabolic L-Bradlow pairs. In the subsection 4.2, we consider the perturbation of δ-semistability condition. Namely, we multiply the full flag bundle to the quot schemes, and we discuss what is obtained for small perturbation of the semistability conditions.

The results in the subsections 4.3–4.4 are the core of this paper, which are significant to discuss the transition formula. In the subsection 4.3, we construct the enhanced master space, and we show that it is Deligne-Mumford and proper. In the subsection 4.4, we see the fixed point set with respect to the torus action.

In the subsection 4.5, we construct the enhanced master space in the oriented case, and we give a description of the stack theoretic fixed point set with respect to the natural torus action. They are essentially reformulation of the results in the previous subsections. We give a more convenient description of the fixed point set in the subsection 4.6, i.e., we observe that they are isomorphic to the moduli stacks of objects with the lower ranks, up to etale finite morphisms.

In some cases, the problem is much simpler. The statements for such cases are given in the subsection 4.7.

1.6.4 Section 5

In the subsection 5.1, we discuss the deformation theory associated to torsion-free sheaf E on $U \times X$, where X is a smooth projective surface. We put $\mathfrak{g}(V.) := \mathcal{H}om(V., V.)^{\vee}[-1]$ and $\mathrm{Ob}(V.) := Rp_{X\,*}\big(\mathfrak{g}(V.) \otimes \omega_X\big)$ for a resolution $V.$ of E. In the subsubsection 5.1.1, we explain how we obtain the morphism $\mathfrak{g}(V.) \longrightarrow L_{U\times X/X}$, and hence $\mathrm{Ob}(V.) \longrightarrow L_U$. In the subsubsection 5.1.2, we see that $\mathfrak{g}(V.)$ is decomposed into the trace-free part and the diagonal part, and that the diagonal part is related to the determinant bundle. In the subsubsection 5.1.3, we give some factorization which will be useful in the construction of the obstruction theory of the master space. In the subsubsection 5.1.4, we give the obstruction theory of some open subset of the moduli stack of torsion-free sheaves, by directly applying the construction in the subsubsection 5.1.1. In the subsubsection 5.1.5, we discuss the case of the moduli of line bundles, which will be used in the construction of the relative obstruction theory for orientation in the subsection 5.2.

In the subsection 5.3, we discuss the relative obstruction theory for L-section. In the subsubsections 5.3.1–5.3.2, we give the construction and show the relative obstruction property. In the subsubsection 5.3.3, we give some factorization which will be useful to discuss the obstruction theory of the master space. In the subsection 5.4, we discuss the relative obstruction theory for reduced L-section. We need to make some modification to the construction in the subsection 5.3. In the subsection 5.5, we discuss the relative obstruction theory for parabolic structure. By pulling them together, we can easily construct the obstruction theory of the moduli stacks of parabolic L-Bradlow pairs and some related objects, which is explained in the subsection 5.6.

Then, we discuss the obstruction theory of the master space in the subsection 5.7. Once we have the factorizations as in the subsubsections 5.1.3, 5.3.3 and 5.4.3, the construction is easy. We also obtain the obstruction theories for some related stacks. In the subsubsections 5.7.6–5.7.7, we give only the statements in some easy cases for explanation.

In the subsection 5.8, we discuss the obstruction theory of the fixed point set. In the subsection 5.9, we discuss the equivariant obstruction theory of the master space and the induced obstruction theory of the fixed point set. We give the statements for the simple cases in the subsubsections 5.8.7–5.8.8 and 5.9.7–5.9.8.

1.6.5 Section 6

By showing the perfectness of the obstruction theories, we obtain the virtual fundamental classes for some stacks, which is discussed in the subsection 6.1. We compare the virtual fundamental classes of the moduli stack of the δ-stable oriented reduced L-Bradlow pairs and the moduli stack of δ-stable L-Bradlow pairs in the subsection 6.2. Although the moduli stacks are isomorphic up to etale finite morphisms, the obstruction theories are not same in general, and we obtain the vanishing of the virtual fundamental class of the moduli of δ-stable L-Bradlow pairs in the case $p_g > 0$.

In the subsection 6.3, we discuss the virtual fundamental classes of the moduli stack of the objects with rank one. In the subsubsection 6.3.1, we see the moduli of L-abelian pairs. In particular, we give a detailed description of the virtual fundamental class in the case where $H^2(X, \mathcal{O}) \neq 0$ and $H^1(X, \mathcal{O}) = 0$ are satisfied. In the subsubsection 6.3.2, we discuss the obstruction theory of the parabolic Hilbert schemes. In the rest of the subsection, we show the splitting given in Proposition 6.34.

In the subsections 6.4–6.6, we discuss the relations of the obstruction theories of some moduli stacks.

1.6.6 Section 7

In the subsection 7.1, we explain how we think the cohomology and the evaluation for our theory. In the subsection 7.2, we show the transition formulas in the simple cases. They are sufficiently useful for the construction of the invariants, which is discussed in the subsection 7.3. They also provide the sufficient tool to discuss the transition problem in the rank 2 case, which is done in the subsection 7.4.

In the subsection 7.5, we discuss the transition formula for the case $p_g > 0$. It is rather easy to show, and the formula is formally same as that in the simple case.

Then, we discuss the transition formula for the case $p_g = 0$, in the subsection 7.6. By using it, we obtain the weak wall crossing formula in the subsection 7.7. We write down the weak wall crossing formula and the weak intersection rounding formula for the rank 3 case in the subsubsection 7.7.3–7.7.4. We also give a transition formula for a critical parabolic weight in the subsubsection 7.7.5.

In the subsection 7.8, we derive the weak intersection rounding formula from the weak wall crossing formula.

1.7 Some Remarks

1.7.1 Further study

This note is tentative, partially because of the recent intensive development of the theory of stacks. We use only the rather old results in this paper. The author believes that it should be one of the main themes of the study of the stacks to make it easy to deal with the subject and the formalism in this work. Hence, he hopes that our arguments would be replaced with the new ways.

For example, as already mentioned in Remark 1.4, the powerful theory of “derived stack” has been developed (see [56] for overview). It seems to be applicable to a wide range of similar problems, contrast to our method in this paper. The author thinks that our method is more elementary. But the theory of “derived stack” will surely be standard in algebraic geometry in very near future (or, perhaps already?), and hence he hopes that our construction and argument would be refreshed from that point of view.

Then, it would be desired to compare the obstruction theories. It is not clear for the author whether such comparison is easy or not. However, he expects that the comparison of the invariants would be easily done, at least in the case $H^1(X,\mathcal{O})=0$, if the new construction would be done appropriately. We should have the transition formulas as given in the section 7, due to which it can be reduced to the comparison of the invariants obtained from the moduli stacks of the objects of rank one. We should have the splitting as in Proposition 6.34, and hence it can be reduced to the comparison of the invariants obtained from the abelian pairs. (See the subsubsection 7.5.2 for such a reduction in the case $H^2(X,\mathcal{O})\neq 0$. We may also obtain such a reduction in the case $H^2(X,\mathcal{O})=0$, although the formula would be more complicated.) In the case $H^1(X,\mathcal{O})=0$, the moduli of abelian pairs is smooth, and the obstruction theory is given by the obstruction bundle which should be as in the subsubsection 6.3.1. Thus, the comparison of the invariants could be done. The author hopes to clarify the points somewhere.

One of the important missing for our theory is the blow up formula, i.e., comparison of the invariants for X and a blow up of X. Originally, the author intended to develop the theory "without blow up". But it seems to contain an interesting problem for such a comparison, and the author would like to come back to this problem, if possible.

1.7.2 Difference from the previous version

In this second version, we obtained the comparatively satisfactory transition formula in the higher rank case, which made us to obtain the weak wall crossing formula. For that purpose, we introduced the notion of (δ,ℓ)-semistability. The other essential ideas are not changed. The author also hopes that the readability is improved.

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2 Preliminary

2.1 Notation

2.1.1 Vector bundles

Let Y be a variety. Let $g: T\longrightarrow U$ be a morphism of stacks. Then the naturally induced morphism $T\times Y\longrightarrow U\times Y$ is denoted by g_Y or simply by g.

Let V be a vector bundle on Y. The sheaf of local sections of V is also denoted by the same notation V, if there are no risk of confusion. But, we use some particular notation in the following case: Let V_1,V_2 be vector bundles. We have the sheaf $\mathcal{H}om(V_1,V_2)$ of the morphisms from V_1 to V_2. The corresponding vector bundle is denoted by $N(V_1,V_2)$.

Let F be a vector bundle on Y. The complement of the image of the 0-section in F is denoted by F^*, i.e., $F^*:=F-Y$, and the dual bundle of F is denoted by $F^\vee$. The projectivization of F is denoted by $\mathbb{P}(F^\vee)$ or $\mathbb{P}_F$.

2.1.2 Coherent sheaves on a product

Let X be a scheme over k, and let U be a stack over k. We denote by p_X the projection forgetting the X-component:

$$p_X: U\times X\longrightarrow U.$$

Similarly, p_U denotes the projection $U \times X \longrightarrow X$. We often denote $\{u\} \times X$ by X_u for any point $u \in U$.

A coherent sheaf E over $U \times X$ is called a U-coherent sheaf, if it is flat over U. A U-coherent sheaf E is called U-torsion free sheaf, if $E_{|\{u\}\times X}$ is torsion-free for each $u \in U$. We will often omit to denote "U-", if there are no risk of confusion.

For any coherent sheaf E on $U \times X$, we use the notation $E(m)$ to denote $E \otimes p_U^* \mathcal{O}_X(m)$ for a given polarization $\mathcal{O}_X(1)$ of X.

2.1.3 Signature

We follow the signature convention in [31]. We recall some of them for later use in our situation.

Let X be an algebraic stack. For two $\mathcal{O}_X$-complexes $C^{\cdot}$ and $D^{\cdot}$, let $\mathcal{H}om(C^{\cdot}, D^{\cdot})$ denote the complex whose i-th term is $\bigoplus_{k-j=i} \mathcal{H}om\big(C^j, D^k\big)$ and whose differentials are given as follows:

$$\mathcal{H}om\big(C^j, D^k\big) \longrightarrow \mathcal{H}om\big(C^j, D^{k+1}\big) \oplus \mathcal{H}om\big(C^{j-1}, D^k\big), \quad a \longmapsto \big(d_D \circ a,\ (-1)^{k-j+1} a \circ d_C\big).$$

Let us see some examples. For a complex $C^{\cdot}$, we denote the dual complex $\mathcal{H}om(C^{\cdot}, \mathcal{O}_X)$ by $C^{\cdot\vee}$. The differential is as follows:

$$\mathcal{H}om(C^n, \mathcal{O}_X) \longrightarrow \mathcal{H}om(C^{n-1}, \mathcal{O}_X), \quad a \longmapsto (-1)^{n+1} \cdot a \circ d_X$$

For two term complexes $C^{\cdot} = (C^{-1} \to C^0)$ and $D^{\cdot} = (D^{-1} \to D^0)$, the complex $\mathcal{H}om\big(C^{\cdot}, D^{\cdot}\big)$ is given as follows:

$$\mathcal{H}om\big(C^0, D^{-1}\big) \longrightarrow \mathcal{H}om\big(C^0, D^0\big) \oplus \mathcal{H}om\big(C^{-1}, D^{-1}\big), \quad a \longmapsto \big(d_D \circ a,\ a \circ d_C\big)$$

$$\mathcal{H}om\big(C^0, D^0\big) \oplus \mathcal{H}om\big(C^{-1}, D^{-1}\big) \longrightarrow \mathcal{H}om\big(C^{-1}, D^0\big), \quad (b_1, b_2) \longmapsto -b_1 \circ d_C + d_D \circ b_2$$

We will often use the dual $\mathcal{H}om(C^{\cdot}, D^{\cdot})^{\vee}$, which is given as follows:

$$\mathcal{H}om\big(D^0, C^{-1}\big) \longrightarrow \mathcal{H}om(D^0, C^0) \oplus \mathcal{H}om(D^{-1}, C^{-1}), \quad a \longmapsto (-d_C \circ a, a \circ d_D)$$

$$\mathcal{H}om(D^0, C^0) \oplus \mathcal{H}om(D^{-1}, C^{-1}) \longrightarrow \mathcal{H}om\big(D^{-1}, C^0\big), \quad (b_1, b_2) \longmapsto -b_1 \circ d_D - d_C \circ b_2$$

2.1.4 Compatible diagrams

Let $A_{i,j}$ $(i = 1, 2)$ $(j = 1, 2, 3, 4)$ be objects in some category. Assume that we are given morphisms $\varphi_j : A_{1,j} \longrightarrow A_{2,j}$. We also assume that we are given commutative diagrams $(CD)_i$:

$$\begin{tikzcd} A_{i,1} \arrow[r, "a_i"] \arrow[d, "b_i"'] & A_{i,2} \arrow[d, "c_i"] \\ A_{i,3} \arrow[r, "d_i"] & A_{i,4} \end{tikzcd}$$

We say that $(CD)_1$ and $(CD)_2$ are compatible with respect to the morphisms φ_j $(j = 1, 2, 3, 4)$, if every face of the naturally obtained cube is commutative. It is equivalent to the commutativity of the following diagrams:

$$\begin{tikzcd} A_{1,1} \arrow[r] \arrow[d] & A_{1,2} \arrow[d] \\ A_{2,1} \arrow[r] & A_{2,2} \end{tikzcd} \quad \begin{tikzcd} A_{1,1} \arrow[r] \arrow[d] & A_{1,3} \arrow[d] \\ A_{2,1} \arrow[r] & A_{2,3} \end{tikzcd} \quad \begin{tikzcd} A_{1,2} \arrow[r] \arrow[d] & A_{1,4} \arrow[d] \\ A_{2,2} \arrow[r] & A_{2,4} \end{tikzcd} \quad \begin{tikzcd} A_{1,3} \arrow[r] \arrow[d] & A_{1,4} \arrow[d] \\ A_{2,3} \arrow[r] & A_{2,4} \end{tikzcd}$$

2.1.5 Filtrations and complexes on a curve

Let $\mathcal{D}$ be a smooth projective curve over a stack S. Let E_a $(a = 1, 2)$ be coherent $\mathcal{O}_{\mathcal{D}}$-modules which are flat over S. Assume that we are given a decreasing filtration $F(E_a) = (F_i(E_a) \,|\, i = 1, \ldots, l)$ of E_a such that $\mathrm{Cok}_i(E_a) = E_a / F_{i+1}(E_a)$ are flat over S.

Let $V_{a,\cdot} = (V_{a,-1} \to V_{a,0})$ be a locally free resolution of E_a. We put $V_a^{(1)} := V_{a,0}$, $V_a^{(l+1)} = V_{a,-1}$ and $V_a^{(i)} := \mathrm{Ker}\big(V_{a,0} \longrightarrow \mathrm{Cok}_i(E_a)\big)$ $(i = 2, \ldots, l)$. Let $f_i : V_D^{(i+1)} \longrightarrow V_D^{(i)}$, $t_i : V_D^{(i)} \longrightarrow V_D^{(1)}$ and $s_i : V_D^{(l+1)} \longrightarrow V_D^{(i)}$ denote the inclusions.

Let us consider the complex $C_1(V_1^*, V_2^*)$ given as follows:

$$\mathcal{H}om\big(V_1^{(1)}, V_2^{(l+1)}\big) \xrightarrow{d^{-1}} \bigoplus_{i=1}^{l+1} \mathcal{H}om\big(V_1^{(i)}, V_2^{(i)}\big) \xrightarrow{d^0} \bigoplus_{i=1}^{l} \mathcal{H}om\big(V_1^{(i+1)}, V_2^{(i)}\big)$$

Here the first term stands in the degree -1, The differentials d_i are given as follows:

$$d^{-1}(a) = \big(s_i \circ a \circ t_i \,\big|\, i = 1, \ldots, l+1\big), \tag{17}$$

$$d^0(b_1, \ldots, b_l) = \big(-f_1 \circ b_1 + b_2 \circ f_1, -f_2 \circ b_2 + b_3 \circ f_2, \ldots, -f_l \circ b_l + b_{l+1} \circ f_l\big) \tag{18}$$

We have the naturally defined morphism:

$$\varphi = (\varphi_i) : C_1(V_1^*, V_2^*) \longrightarrow \mathcal{H}om\big(V_{1,\cdot}, V_{2,\cdot}\big) \tag{19}$$

The morphism φ_1 is given by $\varphi_1(a_i) = \sum s_{i+1} \circ a_i \cdot t_i$. The morphism φ_0 is the projection by the identification $V_0 = V^{(1)}$ and $V_{-1} = V^{(l+1)}$. The morphism φ_2 is the identity. It can be directly checked that φ is the morphism of complexes. We put $C_2(V_1^*, V_2^*) := \mathrm{Cone}(\varphi)[-1]$. The following lemma is easy to check.

Lemma 2.1 *The complexes $C_i(V_1^*, V_2^*)$ and the morphism $\varphi : C_1(V_1^*, V_2^*) \longrightarrow \mathcal{H}om(V_\cdot, V_\cdot)$ depend only on (E_1, F) and (E_2, F) in the derived category $D(\mathcal{D})$.* ▮

Notation 2.2 *We denote $C_i(V_1^*, V_2^*)$ by $\mathcal{RH}om'_i(E_{1*}, E_{2*})$.* ▮

When E_a and $E_a/F_j(E_a)$ are locally free sheaves, then we have $\mathcal{H}^i\big(\mathcal{H}om'_1(E_1, E_2)\big) = 0$ $(i \neq 0)$, and $\mathcal{H}^0\big(\mathcal{H}om'_1(E_1, E_2)\big)$ is isomorphic to the sheaf of homomorphisms of E_1 to E_2 which preserve the filtrations.

2.2 Geometric Invariant Theory

2.2.1 GIT quotient and algebraic stacks

Let k be an algebraically closed field with characteristic 0. Let G be a linear reductive group over k. Let Y be a projective variety over k, provided with a G-action ρ. Let L be an ample line bundle on Y, such that ρ can be lifted to the action on L.

We recall some basic definitions. A point $y \in Y$ is a semistable point with respect to L, if there exists a G-invariant section s of $L^{\otimes n}$ for some $n > 0$ such that $s(y) \neq 0$. A point $y \in Y$ is defined to be a stable point with respect to L, if there exists a G-invariant section of $L^{\otimes n}$ for some $n > 0$ such that $s(y) \neq 0$ and that any orbits of G contained in $Y - s^{-1}(0)$ are closed. Let $Y^s(L)$ (resp. $Y^{ss}(L)$) denote the set of the stable (resp. semistable) points with respect to L. The foundational theorem of D. Mumford is the following.

Proposition 2.3 ([48]) *There exists the uniform categorical quotient $\pi : Y \longrightarrow Y^{ss}//G$. Moreover the following holds:*

- *The map π is affine and universally submersive.*
- *$Y^{ss}//G$ is a projective variety.*
- *There exists the open subset $Y^s//G$ of $Y^{ss}//G$, such that $\pi^{-1}(Y^s//G) = Y^s$ and that $\pi : Y^s \longrightarrow Y^s//G$ is the universally geometric quotient of Y^s.*

Proof See Proposition 1.9, Theorem 1.10 and the page 40 in [48]. ▮

We combine it with some results of A. Vistoli in [58]. Let Y^{sf} denote the set of stable points of Y whose stabilizers are finite. In this situation, we obtain the quotient stack Y^{sf}/G, which is Deligne-Mumford. See the sections 2 and 7 of the paper of Vistoli [58] for more detail about such a quotient stack. We recall one of his results here.

Proposition 2.4 ([58]) *The naturally induced morphism $Y^{sf}/G \longrightarrow Y^{sf}//G$ is proper.*

Proof The map $Y^{sf} \longrightarrow Y^{sf}//G$ is universally geometric quotient. In particular, it is universally submersive, and the geometric fibers are precisely the orbits of the geometric points of X. Therefore, $Y^{sf}//G$ is a quotient of Y^{sf} by G in the sense of Vistoli (see the page 630 of [58]). Applying Proposition 2.11 of [58], we can conclude that the map $Y^{sf}/G \longrightarrow Y^{sf}//G$ is proper. ∎

Corollary 2.5 *Let Z be a variety over k with a G-action. Let $\Phi : Z \longrightarrow Y$ be a G-equivariant immersion with the following property:*

- *The stabilizer group of any point of Z is finite.*
- *The image $\Phi(Z)$ is contained in $Y^s(L)$.*
- *$\Phi : Z \longrightarrow Y^{ss}(L)$ is proper.*

Then the quotient stack Z/G is proper and Deligne-Mumford.

Proof We have the closed substack Z/G of Y^{sf}/G. We also have the closed subscheme $Z//G \longrightarrow Y^{sf}//G \subset Y^{ss}//G$. Since $Y^{ss}//G$ is projective, $Z//G$ is also projective. From the previous lemma, we obtain the properness of the morphism $Z/G \longrightarrow Z//G$. Therefore, we obtain the properness of Z/G. ∎

2.2.2 Mumford-Hilbert criterion and some elementary examples

Let Y and G be as above. Let $\lambda : G_m \longrightarrow G$ be a one-parameter subgroup. We put $P(\lambda) := \lim_{t\to 0} \lambda(t)\cdot P$. Then λ-acts on the fiber $L_{|P(\lambda)}$. The weight is denoted by $\mu_\lambda(P, L)$. The criterion says that the point P is semistable (resp. stable) with respect to L, if and only if $\mu_\lambda(P, L) \geq 0$ ($\mu_\lambda(P, L) > 0$) for any one-parameter subgroup λ.

Remark 2.6 *We use the convention to identify a vector bundle and the sheaf of its sections. Hence the above definition of μ is same as that given in* [48]. ∎

For later use, we recall some elementary examples. Let V be a vector space over an algebraically closed field k, with a base $u_1, \ldots, u_N$. Let λ be the one-parameter subgroup of $\mathrm{SL}(V)$ given by $\lambda(t)\cdot u_i = t^{w_i}\cdot u_i$, where $\sum w_i = 0$ and $w_i \leq w_{i+1}$. Let $V^{(i)}$ denote the subspace generated by $u_1, \ldots, u_i$. Let $V = \bigoplus V_w$ denote the weight decomposition of λ, i.e., λ preserves the decomposition, and the action on V_w is the multiplication of t^w. We put $\mathcal{G}_j := \bigoplus_{w\leq j} V_w$.

We denote a point of $\mathbb{P}(V^\vee)$ by $[v]$ by using a representative $v \in V - \{0\}$. Let us consider the right $\mathrm{SL}(V)$-action on $\mathbb{P}(V^\vee)$, given by $g\cdot[v] := [g^{-1}(v)]$, which can be lifted to the action on $\mathcal{O}_{\mathbb{P}(V^\vee)}(1)$.

Lemma 2.7 ([48]) *$\mu_\lambda\big([v], \mathcal{O}_{\mathbb{P}(V^\vee)}(1)\big)$ is same as $\min\big\{i \,|\, v_i \in \mathcal{G}_i\big\}$. It can be reworded as follows:*

$$\mu_\lambda\big([v], \mathcal{O}(1)\big) = \sum_i w_i \cdot \big(\dim V^{(i)} \cap \langle v\rangle - \dim V^{(i-1)} \cap \langle v\rangle\big) = \sum_j j\cdot \big(\dim \mathcal{G}_j \cap \langle v\rangle - \dim \mathcal{G}_{j-1} \cap \langle v\rangle\big).$$

Here $\langle v\rangle$ denotes the subspace generated by v.

Proof According to the weight decomposition $V = \bigoplus V_i$, we have the decomposition $v = \sum v_i$. In $\mathbb{P}(V^\vee)$, we have the following:

$$\lambda(t)[v] = \big[\lambda(t)^{-1} v\big] = \Big[\sum t^{-i}\cdot v_i\Big].$$

We put $i_0 := \max\big\{i \,|\, v_i \neq 0\big\} = \min\big\{i \,\big|\, v \in \mathcal{G}_i\big\}$. Then it is easy to see $\lim_{t\to 0}\lambda(t)[v] = [v_{i_0}]$. The weight of λ on $\mathcal{O}_{\mathbb{P}(V^\vee)}(1)_{|[v_{i_0}]}$ is i_0. Thus the first claim is obtained. The second claim follows from the first one. ∎

Let $G_l(V)$ denote the Grassmann variety of the l-dimensional subspaces of V:

$$G_l(V) := \big\{\iota : W \subset V \,\big|\, \dim W = l\big\}.$$

We have the Plücker embedding $G_l(V) \longrightarrow \mathbb{P}\big(\bigwedge^l V^\vee\big)$ given by $W \longmapsto \bigwedge^l W \subset \bigwedge^l V$. It gives the polarization $\mathcal{O}_{G_l(V)}(1)$ of $G_l(V)$. The $SL(V)$ has the right action on $G_l(V)$ given by $\iota \longmapsto g^{-1}\circ\iota$, which can be lifted to the action on $\mathcal{O}_{G_l(V)}(1)$.

Lemma 2.8 *For any point W of $G_l(V)$, we have the following:*

$$\mu_\lambda\big(W,\mathcal{O}_{G_l(V)}(1)\big)=\sum_{i=1}^{N}\big(\operatorname{rank}W\cap V^{(i)}-\operatorname{rank}W\cap V^{(i-1)}\big)\cdot w_i=\sum_{j\in\mathbb{Z}}j\cdot\dim\frac{W\cap\mathcal{G}_j}{W\cap\mathcal{G}_{j-1}}.$$

Proof For any $J=(j_1<j_2<\cdots<j_l)$, we put $u_J:=u_{j_1}\wedge\cdots\wedge u_{j_l}$, and $w_J:=\sum_{i=1}^{l}w_{j_i}$. Collection of such u_J gives the base of $\bigwedge^l V$. We have the naturally induced $\mathrm{SL}(V)$-action on $\bigwedge^l V$. Let $\widetilde{\lambda}$ denote the one-parameter subgroup of $\mathrm{SL}\big(\Lambda^l V\big)$ induced by λ. We have $\widetilde{\lambda}(t)(u_J)=t^{w_J}\cdot u_J$.

Let us take a base $v_1,\ldots,v_l$ of W such that $v_h=u_{i_h}+\sum_{j<i_h}a_{h,j}\cdot u_j$. Then $z:=v_1\wedge\cdots\wedge v_l$ is expressed as the sum $\sum a_J\cdot u_J$, where $a_J=1$ if $J=I=(i_1<\cdots<i_l)$ and $a_J=0$ if $w_J>w_I$. We have $\mu_\lambda\big(W,\mathcal{O}_{G_l}(1)\big)=\mu_{\widetilde{\lambda}}\big(z,\mathcal{O}_{\mathbb{P}(\bigwedge^l V^\vee)}(1)\big)=w_I$ due to Lemma 2.7. Then, it is easy to derive the claim of the lemma. ∎

We also have the Grassmann variety G'_l of quotients of l-dimensions:

$$G'_l:=\big\{q:V\longrightarrow Q\,\big|\,\dim Q=l\big\}.$$

We have the Plucker embedding $G'_l\longrightarrow\mathbb{P}\big(\bigwedge^{N-l}V\big)$ given by $q\longmapsto\bigwedge^{N-l}q:\bigwedge^{N-l}V\longrightarrow\bigwedge^{N-l}Q$. It gives the polarization $\mathcal{O}_{G'_l}(1)$.

Lemma 2.9 ([48], [41]) *Let $q:V\longrightarrow Q$ be a point of G'_l. We put $W:=\mathrm{Ker}(q)$. Then we have the following:*

$$\mu_\lambda\big(q,\mathcal{O}_{G'_l}(1)\big)=\sum_{i=1}^{N}w_i\cdot\big(\dim V^{(i)}\cap W-\dim V^{(i-1)}\cap W-1\big)=\sum_{j=1}^{N}j\cdot\left(\dim\frac{W\cap\mathcal{G}_j}{W\cap\mathcal{G}_{j-1}}-\dim\frac{\mathcal{G}_j}{\mathcal{G}_{j-1}}\right).$$

Proof We put $W^{(i)}:=\mathcal{G}_i\cap W\big/\mathcal{G}_{i-1}\cap W$. Since we have the natural isomorphism $\mathcal{G}_i/\mathcal{G}_{i-1}\simeq V_i$, we regard $W^{(i)}$ as the subspace of V_i. It is easy to see that the limit $\lim_{t\to0}\lambda(t)\cdot q$ is given by the quotient $\widehat{q}:V\longrightarrow\bigoplus V_i/W^{(i)}$. The weight of λ on $\mathcal{O}_{G'_l}(1)_{|\widehat{q}}$ is $-i\cdot\dim\big(V_i/W^{(i)}\big)$. Then it is easy to derive the claim. ∎

Remark 2.10 *We have the obvious isomorphism $G_l(V)\simeq G'_{N-l}(V)$. It does not preserve the semistability conditions on the varieties induced by the Plücker embeddings.* ∎

2.3 Cotangent Complex

2.3.1 Basis

Recall some fundamental properties of the cotangent complexes from [29], [37] and [50]. Let $\mathcal{X}$ and $\mathcal{Y}$ be Deligne-Mumford stacks with the etale site. For any morphism $f:\mathcal{X}\longrightarrow\mathcal{Y}$ of Delinge-Mumford stacks, the cotangent complex was constructed by L. Illusie [29] as a complex of $\mathcal{O}_{\mathcal{X}}$-modules, which is denoted by $L_{\mathcal{X}/\mathcal{Y}}$ or L_f. Recall that the cotangent complex controls the deformation theory (Section 3 [29]) in the following sense. Let T be a scheme over $\mathcal{Y}$, and let $h:T\longrightarrow\mathcal{X}$ be a $\mathcal{Y}$-morphism. Let $\overline{T}$ be a $\mathcal{Y}$-scheme such that T is a closed $\mathcal{Y}$-subscheme of $\overline{T}$ and the corresponding ideal J is square zero, i.e., $J^2=\{f\cdot g\,|\,f,g\in J\}=0$.

Proposition 2.11 (Illusie, [29]) *We have the obstruction class $o(h)$ in $Ext^1(h^*L_{\mathcal{X}/\mathcal{Y}},J)$ with the following property:*

- *The morphism h can be extended over $\overline{T}$, if and only if $o(h)$ vanishes.*

When $o(h)=0$, the set of the extension classes is the torsor over the group $Ext^0\big(h^*L_{\mathcal{X}/\mathcal{Y}},J\big)$. ∎

The cotangent complex has a nice functorial property. For example, if $g:\mathcal{Y}\longrightarrow\mathcal{Z}$ be a morphism, then we have the distinguished triangle, $f^*L_{\mathcal{Y}/\mathcal{Z}}\longrightarrow L_{\mathcal{X}/\mathcal{Z}}\longrightarrow L_{\mathcal{X}/\mathcal{Y}}\longrightarrow f^*L_{\mathcal{Y}/\mathcal{Z}}[1]$ in the derived category $D(\mathcal{X})$.

As for general Artin stacks with the lisse-étale site, the cotangent complex with some good functorial property was obtained by G. Laumon, L. Moret-Bailly and M. Olsson (Section 17 of [37] and Section 8 of [50]). For any

Artin stack $\mathcal{X}$, Olsson introduced the category $D'_{\mathrm{qcoh}}(\mathcal{X})$ of the projective systems $K = (\cdots \to K_{\geq -n-1} \to K_{\geq -n} \to \cdots \to K_{\geq 0})$ in $D^+(\mathcal{X})$ such that the morphisms $K_{\geq -n} \longrightarrow \tau_{\geq -n}K_{\geq -n}$ and $\tau_{\geq -n}K_{\geq -n-1} \longrightarrow \tau_{\geq -n}K_{\geq -n}$ are isomorphisms. Here $\tau_{\geq -n}$ denotes the canonical n-th truncation functor. See [50] for the functorial property of $D'_{\mathrm{qcoh}}(\mathcal{X})$. Let $f : \mathcal{X} \longrightarrow \mathcal{Y}$ be a quasi-compact and quasi-separated morphism of Artin stacks. Then, we can associate $L_{\mathcal{X}/\mathcal{Y}} = L_f = \left(\cdots \to L^{\geq -n-1}_{\mathcal{X}/\mathcal{Y}} \to L^{\geq -n}_{\mathcal{X}/\mathcal{Y}} \to \cdots L^{\geq 0}_{\mathcal{X}/\mathcal{Y}}\right) \in D'_{\mathrm{qcoh}}(\mathcal{X})$ to f with the following property (Theorem 8.1 [50]):

- If $\mathcal{X}$ and $\mathcal{Y}$ are algebraic spaces, $L^{\geq -n}_{\mathcal{X}/\mathcal{Y}}$ is isomorphic to $\tau_{\geq -n}L_{\mathcal{X}/\mathcal{Y}}$ in $D^+_{\mathrm{qcoh}}(\mathcal{X})$. Here the latter $L_{\mathcal{X}/\mathcal{Y}}$ denotes the usual cotangent complex defined by Illusie.
- When we are given a 2-commutative diagram of Artin stacks,

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{f} & \mathcal{X} \\ {\scriptstyle g}\downarrow & & \downarrow \\ \mathcal{Y}' & \xrightarrow{h} & \mathcal{Y} \end{array}$$

  we have the functorial morphism $Lf^*L_{\mathcal{X}/\mathcal{Y}} \longrightarrow L_{\mathcal{X}'/\mathcal{Y}'}$. If the diagram is 2-Cartesian, and if one of g or h is flat, then the morphism $Lf^*L_{\mathcal{X}/\mathcal{Y}} \longrightarrow L_{\mathcal{X}'/\mathcal{Y}'}$ is isomorphic.
- Let $f : \mathcal{X} \longrightarrow \mathcal{Y}$ be a morphism of Artin stacks. Let $g : \mathcal{Y} \longrightarrow \mathcal{Z}$ be another morphism. Then we have the distinguished triangle $Lf^*L_{\mathcal{Y}/\mathcal{Z}} \longrightarrow L_{\mathcal{X}/\mathcal{Z}} \longrightarrow L_{\mathcal{X}/\mathcal{Y}} \longrightarrow Lf^*L_{\mathcal{Y}/\mathcal{Z}}[1]$ in $D'_{\mathrm{qcoh}}(\mathcal{X})$.

The following properties can be derived directly from the construction. (See Section 8 of [50] for the construction of $L_{\mathcal{X}/\mathcal{Y}}$.)

- Each $L^{\geq -n}_{\mathcal{X}/\mathcal{Y}}$ is an object in $D^{[-n,1]}_{\mathrm{qcoh}}(\mathcal{X})$.
- If f is smooth and representable, then $L_{\mathcal{X}/\mathcal{Y}}$ is quasi isomorphic to the 0-th cohomology sheaf, which is isomorphic to the locally free sheaves of Kahler differentials $\Omega_{\mathcal{X}/\mathcal{Y}}$. In general, if f is smooth, any $L^{\geq -n}_{\mathcal{X}/\mathcal{Y}}$ is of perfect amplitude contained in $[0, 1]$. In particular, they are isomorphic to $L^{\geq 0}_{\mathcal{X}/\mathcal{Y}}$.

Remark 2.12 *M. Aoki* ([2]) *generalized the deformation theory of Illusie. He showed that a generalization of Proposition* 2.11 *holds for the Artin stacks.* ▮

2.3.2 Quotient stacks

Let G be a smooth group S-scheme. Let Y be a smooth S-scheme with a G-action. The quotient stack is denoted by Y_G. Let $f : Y \longrightarrow Y_G$ be a morphism. We have the corresponding G-torsor $P(f)$ over Y. Since f is smooth and representable, the cotangent complex L_f is isomorphic to the sheaf Ω_f of the relative Kahler differentials.

Lemma 2.13 *Ω_f is isomorphic to the sheaf of the G-invariant sections of $\Omega_{P(f)/Y}$.*

Proof Let $\pi : Y \longrightarrow Y_G$ denote the morphism corresponding to the trivial torsor. We have the following diagram:

$$\begin{array}{ccccc} P(f) \times_Y P(f) & \xrightarrow{\widetilde{p}_1, \widetilde{p}_2} & P(f) & \xrightarrow{\widetilde{\pi}} & Y \\ \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f} \\ Y \times_{Y_G} Y & \xrightarrow{p_1, p_2} & Y & \xrightarrow{\pi} & Y_G \end{array} \tag{20}$$

Then the sheaf Ω_f is the descent of Ω_{f_1} by the cocycle condition $\widetilde{p}_1^*\Omega_{f_1} \simeq \Omega_{f_2} \simeq \widetilde{p}_2^*\Omega_{f_2}$, where the isomorphisms are given by the naturally defined differentials. We have $P(f) \times_Y P(f) \simeq P(f) \times G$ for which p_1 and p_2 correspond to the natural projection and the G-action respectively. Then the claim is obvious. ▮

Let Z be an Artin stack over S with a morphism $F : Z \longrightarrow Y_G$. We have the corresponding G-torsor $P(F)$ over Z and the G-equivariant map $\widetilde{F} : P(F) \longrightarrow Y$.

$$\begin{array}{ccc} P(F) & \xrightarrow{\widetilde{F}} & Y \\ \Big\downarrow & & \pi\Big\downarrow \\ Z & \xrightarrow{F} & Y_G \end{array}$$

Let us describe the pull back of the cotangent complex $F^*L_{Y_G/S}$ on Z. We have the map $\alpha : \widetilde{F}^*\Omega_{Y/S} \longrightarrow \Omega_{P(F)/Z}$ on $P(F)$, which is the composite of the differential $\widetilde{F}^*\Omega_{Y/S} \longrightarrow \Omega_{P(F)/S}$ and the natural projection $\Omega_{P(F)/S} \longrightarrow \Omega_{P(F)/Z}$.

Proposition 2.14 *$F^*L_{Y_G/S}$ is represented by the decent of* $\mathrm{Cone}(-\alpha)[-1]$ *with respect to the natural G-action.*

Proof We recall the construction of $L_{Y_G/S}$ in this case. We put $Y^{(m)} := \overbrace{Y \times_{Y_G} \cdots \times_{Y_G} Y}^{m+1}$. We have the natural morphisms $Y^{(m)} \longrightarrow Y_G \longrightarrow S$. We have the complexes $C^{(m)} := \bigl(\Omega_{Y^{(m)}/S} \longrightarrow \Omega_{Y^{(m)}/Y_G}\bigr)$ on $Y^{(m)}$, where $\Omega_{Y^{(m)}/S}$ stands in the degree 0. We have the strictly simplicial structure given by the naturally defined quasi isomorphisms $\pi_i : \pi^*C^{(m-1)} \longrightarrow C^{(m)}$ $(i = 0, 1, \dots, m)$. Then $L_{Y_G/S} \in D'_{\mathrm{qcoh}}(Y_G)$ is obtained as the gluing of $\bigl(C^{(m)} \bigm| m = 0, 1, \dots\bigr)$ simplicially.

We put $P(F)^{(m)} := \overbrace{P(F) \times_Z \cdots \times_Z P(F)}^{m+1}$. We have the naturally defined morphisms $F^{(m)} : P(F)^{(m)} \longrightarrow Y^{(m)}$. Then $F^*L_{Y_G/S}$ is obtained as the gluing of $\bigl(F^{(m)*}C^{(m)} \bigm| m = 0, 1, \dots\bigr)$. We have the following commutative diagram:

$$\begin{array}{ccc} F^{(m)\,*}\Omega_{Y^{(m)}/S} & \longrightarrow & F^{(m)\,*}\Omega_{Y^{(m)}/Y_G} \\ =\Big\downarrow & & \simeq\Big\downarrow \\ F^{(m)\,*}\Omega_{Y^{(m)}/S} & \longrightarrow & \Omega_{P(F)^{(m)}/Z} \end{array}$$

Here the bottom morphism is same as the composite $F^{(m)\,*}\Omega_{Y^{(m)}/S} \longrightarrow \Omega_{P(F)^{(m)}/S} \longrightarrow \Omega_{P(F)^{(m)}/Z}$, where the first one is the differential and the second one is the natural projection.

Let $q_i : Y \times G^m \longrightarrow Y$ $(i = 0, 1, \dots, m)$ be a morphism given by $q_i(y, g_1, \dots, g_m) = y \cdot g_1 \cdot \cdots \cdot g_i$. They induce the isomorphism $Y \times G^m \longrightarrow Y^{(m)}$. Under the identification, q_i is the projection onto the i-th component. Similarly, we have the identification $P(F) \times G^m \simeq P(F)^{(m)}$, under which $F^{(m)}$ is given by $F^{(m)}(y, g_1, \dots, g_m) = \bigl(\widetilde{F}(y), g_1, \dots, g_m\bigr)$. Let ρ_m denote the projection of $P(F) \times G^m$ onto G^m. We have the subcomplex $\bigl(\rho_m^*\Omega_{G^m} \xrightarrow{\mathrm{id}} \rho_m^*\Omega_{G^m}\bigr)$ of $F^*C^{(m)}$. It is compatible with the simplicial structure. The quotients are denoted by $\widehat{C}^{(m)}$, and then the gluing of $\bigl(\widehat{C}^{(m)} \bigm| m = 0, 1, \dots\bigr)$ also gives $F^*L_{Y_G/S}$ in $D(Z)$. Then, it follows that $F^*L_{Y_G/S}$ is given as the descent of $\widehat{C}^{(0)} = \bigl(\widetilde{F}^*\Omega_{Y/S} \xrightarrow{\alpha} \Omega_{P(F)/Z}\bigr)$ with respect to the natural G-action. ∎

Let H denote the composite of F and the canonical map $Y_G \longrightarrow S_G$. Let $P(H)$ denote the G-torsor over Z corresponding to h. Since we have the natural isomorphism $P(H) \simeq P(F)$, we do not distinguish them. Let $\widetilde{H} : P(F) \longrightarrow S$ be the lift of H. Let π denote the projection $P(F) \longrightarrow Z$. We have the canonical isomorphism $\pi^*H^*L_{S_G/S}[1] \simeq \widetilde{H}^*\Omega_{S/S_G} \simeq \Omega_{P(F)/Z}$. We also have the canonical isomorphism $\pi^*F^*L_{Y_G/S_G} \simeq \widetilde{F}^*\Omega_{Y/S}$. We obtain the following corollary.

Corollary 2.15 *The morphism $F^*L_{Y_G/S_G} \longrightarrow H^*L_{S_G/S}[1]$ on Z is obtained as the descent of $\alpha : \widetilde{F}^*\Omega_{Y/S} \longrightarrow \Omega_{P(F)/Z}$.*

Proof We have the distinguished triangle $H^*L_{S_G/S} \longrightarrow F^*L_{Y_G/S} \longrightarrow F^*L_{Y_G/S_G} \longrightarrow H^*L_{S_G/S}[1]$. Due to Proposition 2.14, we know the morphism $H^*L_{S_G/S} \longrightarrow F^*L_{Y_G/S}$. Then, we know the morphism $F^*L_{Y_G/S} \longrightarrow H^*L_{S_G/S}[1]$. ∎

Example 2.16 *Let E be a vector bundle on a variety X of rank R, and let $f : X \longrightarrow k_{\mathrm{GL}(R)}$ be the corresponding map. Then we have $f^* L_{k_{GL(R)}/k} \simeq \mathrm{End}(E)[-1]$.* ▮

Remark 2.17 *The expression in Proposition 2.14 is natural, in the following sense. Let Y_i $(i = 1, 2)$ be S-schemes with G-actions, and let $g : Y_1 \longrightarrow Y_2$ be a G-equivariant morphism. Let $g_G : Y_{1\,G} \longrightarrow Y_{2\,G}$ denote the induced morphism. Let $h_1 : Z \longrightarrow Y_{1\,G}$ be a morphism. The composite $g_G \circ h_1$ is denoted by h_2. We have the corresponding torsor P over Z and the G-equivariant morphisms $\widetilde{h}_i : P \longrightarrow Y_i$. We have the natural commutative diagram of G-equivariant sheaves on P:*

$$\begin{array}{ccc}
\widetilde{h}_2^* \Omega_{Y_2/S} & \xrightarrow{\alpha_2} & \Omega_{P/Z} \\
\Big\downarrow & & \Big\downarrow \\
\widetilde{h}_1^* \Omega_{Y_1/S} & \xrightarrow{\alpha_1} & \Omega_{P/Z}
\end{array}$$

Then the morphism $h_2^* L_{Y_{2\,G}/S} \longrightarrow h_1^* L_{Y_{1\,G}/S}$ is the descent of the induced morphism $\mathrm{Cone}(-\alpha_2)[-1] \longrightarrow \mathrm{Cone}(-\alpha_2)[-1]$. ▮

Remark 2.18 *Let G_1 be a smooth group scheme over S. Assume that Y is provided with the G_1-action, which commutes with the G-action. It induces the G_1-action on Y_G. Moreover, assume that Z is also provided with the G_1-action such that F is G_1-equivariant. Then, we have the naturally induced G_1-actions on the complex $\mathrm{Cone}(-\alpha)[-1]$, which commutes with the G-action. It induces the G_1-action on the descent of $\mathrm{Cone}(-\alpha)[-1]$ on Z. In particular, we obtain the G_1-equivariant representative of $F^* L_{Y_G/S}$.* ▮

Let $\pi : Y \longrightarrow Y_G$ denote the canonical projection. Due to Proposition 2.14, $L_{Y_G/S}$ on Y_G is the descent of $\big(\Omega_{Y/S} \xrightarrow{\alpha} \Omega_{Y/Y_G}\big)$ given on Y with the natural G-action, where $\Omega_{Y/S}$ stands in the degree 0. For simplicity, we consider the case $S = \mathrm{Spec}(k)$. Due to Lemma 2.13, we have $\Omega_{Y/Y_G} \simeq \mathfrak{g}^\vee \otimes \mathcal{O}_Y$, where $\mathfrak{g}$ denotes the tangent space of G at the unit element, or equivalently the vector space of the right invariant vector fields, and $\mathfrak{g}^\vee$ denotes the dual. Let $\Theta_{Y/S}$ denote the relative tangent sheaf of Y/S. The G-action on Y induces the map $A : \mathfrak{g} \otimes \mathcal{O}_Y \longrightarrow \Theta_{Y/S}$.

Lemma 2.19 *The map $\alpha : \Omega_{Y/S} \longrightarrow \mathfrak{g}^\vee \otimes \mathcal{O}_Y$ is given by the dual of $-A$. Namely, we have $\pi^* L_{Y_G/S} \simeq \mathrm{Cone}(A)[-1]$.*

Proof Let $p_i : Y \times_{Y_G} Y \longrightarrow Y$ denote the projection on the i-th component. We have the following factorization of $p_1^* \alpha$:

$$p_1^* \Omega_{Y/S} \longrightarrow \Omega_{Y \times_{Y_G} Y/S} \longrightarrow \Omega_{p_2} \simeq p_1^* \Omega_{Y/Y_G}$$

Each morphisms are induced by the natural differential. Let us take the identification $Y \times_{Y_G} Y \simeq Y \times G$, for which p_1 and p_2 correspond to the natural projection onto Y and the G-action, respectively.

Let y be any closed point of Y, and let e be the identity element of G. We have $p_1(y, e) = p_2(y, e) = y$. We denote the differential of p_i at (y, e) by $T_{(y,e)} p_i$. Let us consider the specialization of the dual of $p_1^* \alpha$ at (y, e). Then it is the composite of the inclusion of $\mathrm{Ker}(T_{(y,e)} p_2) \subset T_{(y,e)}(Y \times G)$ and the natural projection $T_{(y,e)} Y \times G \longrightarrow T_y Y$. Since we have $\mathrm{Ker}\big(T_{(y,e)} p_2\big) \simeq \big\{(-Av, v) \,\big|\, v \in \mathfrak{g}\big\} \simeq \mathfrak{g}$, the map is same as $-A$. Since α can be recovered from $p_1^* \alpha$, we are done. ▮

Remark 2.20 *Since $f^* L_{Y_G/S}$ is obtained as the decent of $\widetilde{f}^* \mathrm{Cone}(A)[-1]$ for a morphism $f : Z \longrightarrow Y_G$, Lemma 2.19 can be used in the calculation.* ▮

Example 2.21 *Let W_i $(i = -1, 0)$ be R_i-dimensional vector spaces over k. Let $N(W_{-1}, W_0)$ denote the vector space of linear maps from W_{-1} to W_0. We have the right $GL(W_{-1}) \times \mathrm{GL}(W_0)$-action on $N(W_{-1}, W_0)$ given by $(g_{-1}, g_0) \cdot f = g_0^{-1} \circ f \circ g_{-1}$. Hence we obtain the quotient space $Y(W.) := N(W_{-1}, W_0)_{\mathrm{GL}(W_{-1}) \times \mathrm{GL}(W_0)}$.*

Let X and U be stacks over k. Let V_i $(i = -1, 0)$ be vector bundles on $U \times X$ whose ranks are R_i. Let $f : V_{-1} \longrightarrow V_0$ be a morphism of $\mathcal{O}_{U \times X}$-modules. Then we obtain the morphism $\Phi_f : U \times X \longrightarrow Y(W.)$. The pull back of the cotangent complex $\Phi_f^* L_{Y(W.)/k}$ is quasi isomorphic to the following complex:

$$\mathcal{H}om\big(V_0, V_{-1}\big) \xrightarrow{\alpha} \mathcal{H}om\big(V_0, V_0\big) \oplus \mathcal{H}om(V_{-1}, V_{-1}).$$

Here $\mathcal{H}om\big(V_0, V_{-1}\big)$ stands in degree 0, and the map α is given by $\alpha(a) = \big(f \circ a, -a \circ f\big)$. We remark that it is isomorphic to $\mathcal{H}om\big(V., V.\big)^{\vee}_{\leq 0}[-1]$. (See the subsubsection 2.1.3.)

Actually, we have only to care the signature. We can see it formally. Let f be an element of $N(W_{-1}, W_0)$. The differential of the action of $\mathrm{GL}(W_{-1}) \times \mathrm{GL}(W_0)$ gives the map:

$$\mathrm{End}(W_{-1}) \oplus \mathrm{End}(W_0) \longrightarrow T_f N(W_{-1}, W_0) = N(W_{-1}, W_0), \quad (a_{-1}, a_0) \longmapsto -a_0 \circ f + f \circ a_{-1} \tag{21}$$

If we regard $W_{-1} \xrightarrow{f} W_0$ as a complex, (21) can be regarded as $\mathcal{H}om(W., W.)_{\geq 0}$. Then, Lemma 2.19 says that the cotangent complex corresponds to $\big(\mathcal{H}om(W., W.)_{\geq 0}\big)^{\vee}[-1]$. ∎

Let us consider the following diagram:

$$\begin{array}{ccc} Y & \xrightarrow{\widetilde{\psi}} & S \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi_1} \\ Y_G & \xrightarrow{\psi} & S_G \end{array}$$

We have the natural isomorphisms $\pi^* L_{Y_G/S_G} \simeq L_{Y/S}$ and $\pi^*\psi^* L_{S_G/S}[1] \simeq \widetilde{\psi}^* L_{S/S_G} \simeq L_{Y/Y_G}$. We identify them by the isomorphisms.

Lemma 2.22 *Under the identification above, the morphism $\pi^* L_{Y_G/S_G} \longrightarrow \pi^*\psi^* L_{S_G/S}[1]$ is same as the natural morphism $L_{Y/S} \longrightarrow L_{Y/Y_G}$.*

Proof We have the natural isomorphisms:

$$\pi^* L_{Y_G/S} \simeq \mathrm{Cone}\big(L_{Y/S} \longrightarrow L_{Y/Y_G}\big)[-1], \quad \pi^*\psi^* L_{S_G/S} \simeq \widetilde{\psi}^* L_{S/S_G}[-1]$$

The morphism $\pi^*\psi^* L_{S_G/S} \longrightarrow \pi^* L_{Y_G/S}$ is induced by $\widetilde{\psi}^* L_{S/S_G} \longrightarrow L_{Y/Y_G}$. Hence, the distinguished triangle $\pi^*\psi^* L_{S_G/S} \longrightarrow \pi^* L_{Y_G/S} \longrightarrow \pi^* L_{Y_G/S_G} \longrightarrow \pi^*\psi^* L_{S_G/S}[1]$ is the following:

$$\psi^* L_{S/S_G}[-1] \longrightarrow \mathrm{Cone}\big(L_{Y/S} \to L_{Y/Y_G}\big)[-1] \longrightarrow L_{Y/S} \longrightarrow \psi^* L_{S/S_G}$$

Then the claim of the lemma follows. ∎

2.3.3 Some more examples

The result in this subsubsection will be used in the subsections 6.3 and 6.6. The author recommends the reader to skip here. Let X be a smooth projective surface, and let U_2 be a quasi compact stack. We consider a pair of U_2-coherent sheaf $\mathcal{F}$ and a section $\varphi : \mathcal{O}_{U_2 \times X} \longrightarrow \mathcal{F}$. We assume that $p_{X\,*}\mathcal{F}$ is locally free. We have the induced section $\mathcal{H}(\varphi) : \mathcal{O}_{U_2 \times X} \longrightarrow p_{X\,*}\mathcal{F}$. Let U_1 be a subscheme of U_2 contained in the 0-set of $\mathcal{H}(\varphi)$.

Assume we are given a datum $(V., P., \phi, \widetilde{\phi})$ as follows:

- A locally free resolution of $\mathcal{F}$:

$$V_{-1} \xrightarrow{d_{-1}} V_0 \xrightarrow{d_0} V_1 \xrightarrow{\epsilon} \mathcal{F}$$

  Namely, the sequence $0 \longrightarrow V_{-1} \longrightarrow V_0 \longrightarrow V_1$ is exact, and $V_0/V_1 \simeq \mathcal{F}$.

- A morphism $\phi : \mathcal{O}_{U_2 \times X} \longrightarrow V_1$ such that $\epsilon \circ \phi = \varphi$. Such a ϕ is called a lift of φ.

- A resolution $P. = (P_{-1} \xrightarrow{\partial} P_0)$ of $\mathcal{O}_X$, i.e., $P_0/P_{-1} \simeq \mathcal{O}_X$.

- A morphism $\widetilde{\phi}_i : P_i \longrightarrow V_i$ $(i = 0, -1)$ such that the following diagram is commutative:

$$\begin{array}{ccccc} V_{-1} & \xrightarrow{d_{-1}} & V_0 & \xrightarrow{d_0} & V_1 \\ {\scriptstyle \widetilde{\phi}_{-1}}\uparrow & & {\scriptstyle \widetilde{\phi}_0}\uparrow & & {\scriptstyle \phi}\uparrow \\ P_{-1} & \xrightarrow{\partial} & P_0 & \longrightarrow & \mathcal{O} \end{array}$$

We put $\mathfrak{k}(V_{\cdot},P_{\cdot},\phi,\widetilde{\phi}) := \mathcal{H}om(p_{U_1}^*P_{\cdot},V_{\cdot|U_1\times X})^{\vee}$ on $U_1\times X$, and we will construct a morphism $\mathfrak{r}(V_{\cdot},P_{\cdot},\phi,\widetilde{\phi}):\mathfrak{k}(V_{\cdot},P_{\cdot},\phi,\widetilde{\phi})\longrightarrow L_{U_1\times X/U_2\times X}$ on U_1, which depends only on $(\mathcal{F},\varphi)$ in the derived category $D(U_1\times X)$.

We have the vector bundles $N(P_i,V_j)$ on $U_2\times X$. We have the following smooth morphism:

$$h_1 : N(P_0,V_0)\times_X N(P_{-1},V_{-1})\longrightarrow N(P_{-1},V_0),\quad h(a_0,a_{-1}) = f\circ a_{-1}-a_0\circ\partial$$

We put $Z_1(V_{\cdot},P_{\cdot}) := h^{-1}(0)$. We also have the following morphism:

$$h_2 : N(P_0,V_1)\longrightarrow N(P_{-1},V_1),\quad h_2(a) = -a\circ\partial.$$

We put $Z_2(V_{\cdot},P_{\cdot}) := h_2^{-1}(0)$. We remark $Z_2(V_{\cdot},P_{\cdot})$ is naturally isomorphic to $N(\mathcal{O},V_1)$.

We have the naturally defined morphism $\Gamma : Z_1(V_{\cdot},P_{\cdot})\longrightarrow Z_2(V_{\cdot},P_{\cdot})$. The morphisms ϕ and $\widetilde{\phi}$ give the sections $\Phi_i : U_i\times X\longrightarrow Z_i$, and we have the following commutative diagram:

$$\begin{array}{ccccc}
U_1\times X & \xrightarrow{\Phi_1} & Z_1(V_{\cdot},P_{\cdot}) & \longrightarrow & X\\
{\scriptstyle j_X}\downarrow & & \downarrow & & \downarrow\\
U_2\times X & \xrightarrow{\Phi_2} & Z_2(V_{\cdot},P_{\cdot}) & \longrightarrow & X
\end{array}\tag{22}$$

It is easy to see that $\Phi_1^*L_{Z_1(V_{\cdot},P_{\cdot})/Z_2(V_{\cdot},P_{\cdot})}$ is expressed by $\mathfrak{k}(V_{\cdot},P_{\cdot},\widetilde{\phi},\phi)_{\leq 0}$. The composite $\mathfrak{k}(V_{\cdot},P_{\cdot},\widetilde{\phi},\phi)\longrightarrow\Phi_1^*L_{Z_1(V_{\cdot},P_{\cdot})/Z_2(V_{\cdot},P_{\cdot})}\longrightarrow L_{U_1\times X/U_2\times X}$ is denoted by $\mathfrak{r}(V_{\cdot},P_{\cdot},\widetilde{\phi},\phi)$.

Lemma 2.23 *$\mathfrak{r}(V_{\cdot},P_{\cdot},\widetilde{\phi},\phi)$ and $\mathfrak{k}(V_{\cdot},P_{\cdot},\widetilde{\phi},\phi)$ depend only on $(\mathcal{F},\varphi)$ in the derived category $D(U_1\times X)$.*

Proof Let $(V^{(i)},P^{(i)},\phi^{(i)},\widetilde{\phi}^{(i)})$ $(i=1,2)$ be data as above. Let $V_1^{(3)}$ be the cokernel of $(\phi^{(1)},-\phi^{(2)}):\mathcal{O}_{U_2\times X}\longrightarrow V_1^{(1)}\oplus V_1^{(2)}$. Then $\epsilon^{(i)}$ $(i=1,2)$ naturally induce the morphism $\epsilon^{(3)}:V_1^{(3)}\longrightarrow\mathcal{F}$. We also have the morphism $\phi^{(3)}:\mathcal{O}_{U_2\times X}\longrightarrow V_1^{(3)}$, naturally. We take a surjection $A\longrightarrow\mathrm{Ker}(\epsilon^{(3)})$, appropriately. We have the naturally defined morphism $V_0^{(1)}\oplus V_0^{(2)}\longrightarrow\mathrm{Ker}(\epsilon^{(3)})$. We put $V_0^{(3)} := A\oplus V_0^{(1)}\oplus V_0^{(2)}$, and then we have the natural morphism $d_0^{(3)}:V_0^{(3)}\longrightarrow V_1^{(3)}$. The kernel of $d_0^{(3)}$ is locally free, and we put $V_{-1}^{(3)} := \mathrm{Ker}(d_0^{(3)})$. We put $P_0^{(3)} := P_0^{(1)}\oplus P_0^{(2)}$, and $P_{-1}^{(3)}$ denotes the kernel of the natural morphism $P_0^{(1)}\oplus P_0^{(2)}\longrightarrow\mathcal{O}_{U_2\times X}$. Then, we have the following compatible diagrams on $U_2\times X$:

$$\begin{array}{ccccccc}
V_{-1}^{(3)} & \longrightarrow & V_0^{(3)} & \longrightarrow & V_1^{(3)} & \longrightarrow & \mathcal{F}\\
{\scriptstyle a_3^{(i)}}\uparrow & & {\scriptstyle a_2^{(i)}}\uparrow & & {\scriptstyle a_1^{(i)}}\uparrow & & {\scriptstyle =}\uparrow\\
V_{-1}^{(i)} & \longrightarrow & V_0^{(i)} & \longrightarrow & V_1^{(i)} & \longrightarrow & \mathcal{F}
\end{array}
\qquad
\begin{array}{ccccc}
P_{-1}^{(3)} & \longrightarrow & P_0^{(3)} & \longrightarrow & \mathcal{O}\\
\uparrow & & \uparrow & & \uparrow\\
P_{-1}^{(i)} & \longrightarrow & P_0^{(i)} & \longrightarrow & \mathcal{O}
\end{array}$$

The cokernels of $a_j^{(i)}$ are locally free. The composite of $\phi^{(i)}$ and $a_1^{(i)}$ is same as $\phi^{(3)}$. On $U_1\times X$, $a_j^{(i)}$ and $\widetilde{\phi}_j$ are compatible. Then, we have only to compare $\mathfrak{r}(V^{(1)},P^{(1)},\phi^{(1)},\widetilde{\phi}^{(1)})$ and $\mathfrak{r}(V^{(3)},P^{(3)},\phi^{(3)},\widetilde{\phi}^{(3)})$.

We give only an indication. We regard $V_j^{(1)}\subset V_j^{(3)}$ and $P_j^{(1)}\subset P_j^{(3)}$ as the filtrations. Let $N'(P_j^{(3)},V_k^{(3)})$ denote the vector bundle corresponding to the locally free sheaves of filtration-preserving homomorphisms of $P_j^{(3)}$ to $V_k^{(3)}$. We construct the vector bundles $Z_i'(V_{\cdot},P_{\cdot})$ by using $N'(P_i^{(3)},V_j^{(3)})$ instead of $N(P_i^{(3)},V_j^{(3)})$. Then, we have the morphisms: $Z_i(V_{\cdot}^{(3)},P_{\cdot}^{(3)})\longrightarrow Z_i'(V_{\cdot}^{(3)},P_{\cdot}^{(3)})$ and $Z_i(V_{\cdot}^{(3)},P_{\cdot}^{(3)})\longrightarrow Z_i(V_{\cdot}^{(1)},P_{\cdot}^{(1)})$.

Let $\mathcal{H}om'(P_{\cdot},V_{\cdot})$ denote the subcomplex of $\mathcal{H}om(P_{\cdot},V_{\cdot})$ which consists of filtration-preserving homomorphisms. Let $\mathfrak{k}'(V_{\cdot}^{(3)},P_{\cdot}^{(3)},\phi^{(3)},\widetilde{\phi}^{(3)})$ denote the dual $\mathcal{H}om'(P_{\cdot}^{(3)},V_{\cdot})^{\vee}$. We obtain the following diagram on U:

$$\begin{array}{ccccc}
\mathfrak{k}(V_{\cdot}^{(3)},P_{\cdot}^{(3)},\phi^{(3)},\widetilde{\phi}^{(3)}) & \xrightarrow{b_1} & \mathfrak{k}'(V_{\cdot}^{(3)},P_{\cdot}^{(3)},\phi^{(3)},\widetilde{\phi}^{(3)}) & \xleftarrow{b_2} & \mathfrak{k}(V_{\cdot}^{(1)},P_{\cdot}^{(1)},\phi^{(1)},\widetilde{\phi}^{(1)})\\
\downarrow & & \downarrow & & \downarrow\\
L_{Z_1(V_{\cdot}^{(3)},P_{\cdot}^{(3)})/Z_2(V_{\cdot}^{(3)},P_{\cdot}^{(3)})} & \longrightarrow & L_{Z_1'(V_{\cdot}^{(3)},P_{\cdot}^{(3)})/Z_2(V_{\cdot}^{(3)},P_{\cdot}^{(3)})} & \longleftarrow & L_{Z_1(V_{\cdot}^{(1)},P_{\cdot}^{(1)})/Z_2(V_{\cdot}^{(1)},P_{\cdot}^{(1)})}\\
\downarrow & & \downarrow & & \downarrow\\
L_{U_1\times X/U_2\times X} & \xrightarrow{=} & L_{U_1\times X/U_2\times X} & \xleftarrow{=} & L_{U_1\times X/U_2\times X}
\end{array}$$

Then, it is clear that the morphisms b_i are quasi-isomorphic. ∎

Now, we use the notation $\mathfrak{k}(\mathcal{F},\varphi)$ to denote $\mathfrak{k}(V_\cdot,P_\cdot,\phi,\widetilde{\phi})$, in the following argument. We also put as follows:

$$\mathrm{Ob}^H_{\mathrm{rel}}(\mathcal{F},\varphi) := Rp_{X\,*}\big(\mathfrak{k}(\mathcal{F},\varphi)\otimes\omega_X\big)$$

Here, ω_X denotes the dualizing complex of X. Let $\mathrm{ob}^H_{\mathrm{rel}}(\mathcal{F},\varphi)$ denote the composite of the morphisms

$$\mathrm{Ob}^H_{\mathrm{rel}}(\mathcal{F},\varphi) \longrightarrow Rp_{X\,*}\big(L_{U_1\times X/U_2\times X}\otimes\omega_X\big) \longrightarrow L_{U_1/U_2}$$

Let $(\mathcal{F}_i,\varphi_i)$ $(i=1,2)$ be as above. Assume we are given a morphism with a morphism $f:\mathcal{F}_1\longrightarrow\mathcal{F}_2$ such that $\varphi_2=f\circ\varphi_1$.

Lemma 2.24 *We have the induced morphism $\mathfrak{s}(f):\mathfrak{k}(\mathcal{F}_2,\varphi_2)\longrightarrow\mathfrak{k}(\mathcal{F}_1,\varphi_1)$ such that $\mathfrak{r}(\mathcal{F}_1,\varphi_1)\circ\mathfrak{s}(f)=\mathfrak{r}(\mathcal{F}_2,\varphi_2)$.*

Proof Before going into the proof, we give a rather canonical construction of a datum for a given $(\mathcal{F},\varphi)$ as in the beginning of this subsubsection. We take a sufficiently large integer m_1, and we put $V_1:=p_X^*\big(p_{X\,*}\mathcal{F}(m)\big)\otimes\mathcal{O}(-m_1)\oplus\mathcal{O}_{U_2\times X}$. The canonically defined morphism $p_X^*\big(p_{X\,*}\mathcal{F}(m_1)\big)\otimes\mathcal{O}(-m_1)\longrightarrow\mathcal{F}$ and $\varphi:\mathcal{O}_{U_2\times X}\longrightarrow\mathcal{F}$ gives a surjection $\epsilon:V_1\longrightarrow\mathcal{F}$. We also have the lift $\phi:\mathcal{O}_{U_2\times X}\longrightarrow V_1$. Then, we take a sufficiently large integer m_0, and we put $V_0:=p_X^*\big(p_{X\,*}\mathrm{Ker}(\epsilon)(m_0)\big)\otimes\mathcal{O}_X(-m_0)$. Then, we have the surjection $V_0\longrightarrow\mathrm{Ker}(\epsilon)$, which induces $d_0:V_0\longrightarrow V_1$. Since the kernel is locally free, we put $V_{-1}:=\ker(d_0)$. We take a resolution $P_\cdot$ such that P_0 is a direct sum of $\mathcal{O}(-m_0)$. Then we canonically obtain the morphism $\widetilde{\phi}_i:P_i\longrightarrow V_i$ $(i=0,-1)$ on $U_1\times X$. Thus, we obtain a datum as above.

We take $(V_\cdot^{(i)},P_\cdot,\phi^{(i)},\widetilde{\phi}^{(i)})$ for $(\mathcal{F}_i,\varphi_i)$ by applying the construction explained above. Then the morphism f is canonically lifted to $\widetilde{f}:V_\cdot^{(1)}\longrightarrow V_\cdot^{(2)}$ such that $\phi^{(2)}=\widetilde{f}\circ\phi^{(1)}$ and $\widetilde{\phi}_i^{(2)}=\widetilde{f}\circ\widetilde{\phi}_i^{(1)}$. Then, we obtain the following diagram:

$$\begin{array}{ccccc}
U_2\times X & \longrightarrow & Z_2(V_\cdot^{(1)},P_\cdot) & \longrightarrow & Z_2(V_\cdot^{(2)},P_\cdot)\\
\uparrow & & \uparrow & & \uparrow\\
U_1\times X & \longrightarrow & Z_1(V_\cdot^{(1)},P_\cdot) & \longrightarrow & Z_1(V_\cdot^{(2)},P_\cdot)
\end{array}$$

Then, the claim of the lemma is clear. ∎

Now, we assume $R^ip_{X\,*}\mathcal{F}=0$ for $i>0$. We put $\mathfrak{V}:=p_{X\,*}\mathcal{F}$. The morphism $\varphi:\mathcal{O}_{U_2\times X}\longrightarrow\mathcal{F}$ induces the section $\overline{\phi}$ of $\mathfrak{V}$. We have the following diagram:

$$\begin{array}{ccc}
U_1 & \xrightarrow{j_2} & U_2\\
{\scriptstyle j_1}\big\downarrow & & {\scriptstyle i}\big\downarrow\\
U_2 & \xrightarrow{\overline{\phi}} & \mathfrak{V}
\end{array}\qquad (23)$$

Here i denotes the 0-section.

Proposition 2.25 $\mathrm{Ob}^H_{\mathrm{rel}}(\mathcal{F},\varphi)$ *is isomorphic to $j_2^*L_{U_2/\mathfrak{V}}\simeq\mathfrak{V}^\vee[1]$, and the morphism $\mathfrak{r}(\mathcal{F},\varphi)$ is same as the morphism $\kappa:j_2^*L_{U_2/\mathfrak{V}}\longrightarrow L_{U_1/U_2}$ induced from the diagram* (23).

Proof We have the naturally defined morphism $a_1:p_X^*\mathfrak{V}\longrightarrow\mathcal{F}$, for which we have $\varphi=\overline{\phi}\circ a_1$. Due to Lemma 2.24, we have the following diagram:

$$\mathfrak{k}(\mathcal{F},\varphi)\xrightarrow{\mathfrak{s}(a_1)}\mathfrak{k}(p_X^*\mathfrak{V},\overline{\phi})\longrightarrow L_{U_1\times X/U_2\times X}$$

It induces the following morphisms:

$$\mathrm{Ob}^H_{\mathrm{rel}}(\mathcal{F},\varphi)\xrightarrow{b_0}\mathrm{Ob}^H_{\mathrm{rel}}(p_X^*\mathfrak{V},p_X^*\overline{\phi})\xrightarrow{b_1}L_{U_1/U_2}$$

Let us see b_1 more closely. In the construction for $(p_X^*\mathfrak{V}, p_X^*\overline{\phi})$, we can put $V_1 = p_X^*\mathcal{F}$, $V_i = U_2 \times X$ $(i = 0, -1)$, $P_0 = \mathcal{O}_{U_1 \times X}$ and $P_{-1} = U_2 \times X$. Then, $Z_1 = U_2 \times X$ and $Z_2 = p_X^*\mathfrak{V}$. The diagram (22) is given as follows:

$$\begin{array}{ccccc} U_1 \times X & \xrightarrow{j_1} & Z_1 & = & U_2 \times X \\ {\scriptstyle j_2}\downarrow & & {\scriptstyle i}\downarrow & & \downarrow \\ U_2 \times X & \xrightarrow{\overline{\phi}} & Z_2 & = & \mathfrak{V} \end{array}$$

Here i denotes the 0-section. We have $\mathfrak{k} = j_1^* L_{Z_1/Z_2} \simeq p_X^*\mathfrak{V}_{|U_1 \times X}[1]$, and the morphism $\mathfrak{r} : \mathfrak{k} \longrightarrow L_{U_1 \times X/U_2 \times X}$ is same as the pull back of κ. In particular, we have the following factorization of b_1:

$$\mathrm{Ob}^H(p_X^*\mathfrak{V}, p_X^*\overline{\phi}) = \mathfrak{V}[1] \otimes Rp_{X\,*}\bigl(p_{U_2}^*\omega_X\bigr) \xrightarrow{b_2} \mathfrak{V}[1] \xrightarrow{\kappa} L_{U_1/U_2}$$

It is easy to see that the composite $b_2 \circ a_0$ is isomorphic, under the assumption $R^i p_{X\,*}\mathcal{F} = 0$ $(i > 0)$. Thus the proof of Proposition 2.25 is finished. ∎

We can obtain a similar result for a smooth projective curve. The argument is similar and simpler, and hence we omit to give a proof.

Let D be a smooth projective curve. Let $\mathcal{F}$ be a U_2-coherent sheaf on $U_2 \times D$ such that $p_{D\,*}\mathcal{F}$ is locally free. Let φ be a morphism $\mathcal{O}_{U_2 \times D} \longrightarrow \mathcal{F}$. It induces the section $\mathcal{H}(\varphi)$ of $p_{D\,*}\mathcal{F}$. Let U_1 be a subscheme of U_2 contained in $\mathcal{H}(\varphi)^{-1}(0)$.

Assume that we have a locally free resolution $(V_0 \to V_1)$ of $\mathcal{F}$ such that there exists a lift $\phi : \mathcal{O}_{U_2 \times D} \longrightarrow V_1$ of φ. The morphism $\widetilde{\phi} : \mathcal{O}_{U_1 \times D} \longrightarrow V_{0\,|\,U_1 \times D}$ is induced. We put $\mathfrak{k}(V_\cdot, \phi) := \mathcal{H}om(\mathcal{O}_{U_1 \times D}, V_{\cdot|U_1 \times D})^\vee$.

Let us construct a morphism $\mathfrak{r}(V_\cdot, \phi) : \mathfrak{k}(V_\cdot, \phi) \longrightarrow L_{U_1 \times D/U_2 \times D}$. We put $Z_1 := N(\mathcal{O}, V_0)$ and $Z_2 := N(\mathcal{O}, V_1)$. Then we have the naturally defined morphism $Z_1 \longrightarrow Z_2$. The sections ϕ and $\widetilde{\phi}$ induces the following commutative diagram:

$$\begin{array}{ccc} U_1 \times D & \xrightarrow{j} & Z_1 \\ \downarrow & & \downarrow \\ U_2 \times D & \longrightarrow & Z_2 \end{array}$$

It induces the morphism $\mathfrak{r}(V_\cdot, \phi) : \mathfrak{k}(V_\cdot, \phi) \simeq j^* L_{Z_1/Z_2} \longrightarrow L_{U_1 \times D/U_2 \times D}$. It can be shown that $\mathfrak{r}(V_\cdot, \phi)$ and $\mathfrak{k}(V_\cdot, \phi)$ depends only on $(\mathcal{F}, \varphi)$. Therefore, we use the notation $\mathfrak{r}(\mathcal{F}, \varphi)$ and $\mathfrak{k}(\mathcal{F}, \varphi)$. We put as follows:

$$\mathrm{Ob}^H_{\mathrm{rel}}(\mathcal{F}, \varphi) := Rp_{D\,*}\bigl(\mathfrak{k}(\mathcal{F}, \varphi) \otimes \omega_D\bigr)$$

Then, we have the induced morphism $\mathrm{ob}^H_{\mathrm{rel}}(\mathcal{F}, \varphi) : \mathrm{Ob}^H_{\mathrm{rel}}(\mathcal{F}, \varphi) \longrightarrow L_{U_1/U_2}$.

We put $\mathfrak{V} := p_{X\,*}\mathcal{F}$. We have the induced section $\overline{\phi}$. Then, we obtain the diagram (23). It induces the morphism $\kappa : \mathfrak{V}^\vee[1] \longrightarrow L_{U_1/U_2}$.

Proposition 2.26 *Assume $R^i p_{D\,*}\mathcal{F} = 0$ for $i > 0$. Then, we have the following commutative diagram:*

$$\begin{array}{ccc} \mathrm{Ob}^H_{\mathrm{rel}}(\mathcal{F}, \varphi) & \xrightarrow{\mathrm{ob}_{\mathrm{rel}}(\mathcal{F},\varphi)} & L_{U_1/U_2} \\ {\scriptstyle \simeq}\downarrow & & {\scriptstyle =}\downarrow \\ \mathfrak{V}^\vee[1] & \xrightarrow{\kappa} & L_{U_1/U_2} \end{array}$$

Proof It can be shown by an argument similar to the proof of Proposition 2.25. ∎

2.4 Obstruction Theory

2.4.1 Definition and the foundational theorem of Behrend-Fantechi

In the study of Gromov-Witten theory, Li-Tian and Behrend-Fantechi introduced the notion of virtual fundamental class of moduli stacks with some good structure. (See [39] and [5].) In this paper, we follow the work of Behrend-Fantechi. Let us recall their notion of obstruction theory of an algebraic stack with a minor generalization.

Definition 2.27 *Let $\mathcal{X}$ be an algebraic stack over an algebraic stack S. Let $E^{\cdot}$ be an object in $D(\mathcal{X})$ such that $\mathcal{H}^i(E^{\cdot})$ are coherent $(i=-1,0,1)$. A homomorphism $\phi : E^{\cdot} \longrightarrow L_{\mathcal{X}/S}$ is called an obstruction theory for $\mathcal{X}/S$, if $\mathcal{H}^i(\phi)$ $(i \geq 0)$ are isomorphic and $\mathcal{H}^{-1}(\phi)$ is surjective. In that case, $E^{\cdot}$ is also called an obstruction theory for $\mathcal{X}/S$.* ▮

Since we have $\mathcal{H}^i(L_{\mathcal{X}}) = 0$ for $i > 1$, the condition implies $\mathcal{H}^i(E^{\cdot}) = 0$ for $i > 1$. If $\mathcal{X}$ is Deligne-Mumford, we also have $\mathcal{H}^1(E^{\cdot}) = \mathcal{H}^1(L_{\mathcal{X}}) = 0$.

We will often use the following theorem.

Proposition 2.28 (Behrend-Fantechi, Theorem 4.5, [5]) *Let $\mathcal{X}$ be a Deligne-Mumford stack over S. Let $\phi : E^{\cdot} \longrightarrow L_{\mathcal{X}/S}$ be a morphism in $D(\mathcal{X})$. The following conditions are equivalent.*

- *ϕ is an obstruction theory.*
- *Let T and $\overline{T}$ be S-schemes such that T is a closed subscheme of $\overline{T}$ whose ideal sheaf J is square 0. Let $g : T \longrightarrow \mathcal{X}$ be a morphism over S.*

  (A1) *g can be extended to a morphism $\overline{g} : \overline{T} \longrightarrow \mathcal{X}$ over S, if and only if $\phi^*\bigl(o(g)\bigr) = 0$ in $\mathrm{Ext}^1\bigl(g^*E^{\cdot}, J\bigr)$, where $o(g)$ is the obstruction class of g. (See Proposition 2.11.)*

  (A2) *If $\phi^*\bigl(o(g)\bigr) = 0$, the set of the extension classes of g is the torsor over the group $\mathrm{Ext}^0\bigl(g^*E^{\cdot}, J\bigr)$.* ▮

We recall the perfectness of the obstruction theory in the sense of Behrend-Fantechi with a minor generalization.

Definition 2.29 *Let $\phi : E^{\cdot} \longrightarrow L_{\mathcal{X}/S}$ be an obstruction theory of an algebraic stack $\mathcal{X}$ over S. It is called perfect, if it is quasi isomorphic to a complex of locally free sheaves $F^{-1} \to F^0 \to F^1$ in the derived category $D(\mathcal{X})$.* ▮

In that case, the number $-\operatorname{rank} F^1 + \operatorname{rank} F^0 - \operatorname{rank} F^{-1}$ is well defined on each connected component of $\mathcal{X}$. The number is called the expected dimension of $\mathcal{X}$ over S with respect to ϕ.

If $\mathcal{X}$ is Deligne-Mumford, we have $\mathcal{H}^1(E^{\cdot}) = \mathcal{H}^1(L_{\mathcal{X}}) = 0$ for the obstruction theory $E^{\cdot}$. Hence, a perfect obstruction theory is quasi isomorphic to $F^{-1} \to F^0$. The important and foundational theorem of Behrend and Fantechi is the following. (See also [39].) Let $A_*(\mathcal{X})$ denote the Chow group of $\mathcal{X}$ with rational coefficient.

Proposition 2.30 (Behrend-Fantechi, Section 5, [5]) *Let $\mathcal{X}$ be a Deligne-Mumford stack over a smooth scheme S. A perfect obstruction theory $\phi : E^{\cdot} \longrightarrow L_{\mathcal{X}/S}$ induces the virtual fundamental class $[\mathcal{X}, \phi] \in A_d(\mathcal{X})$, where d is the expected dimension with respect to ϕ.*

When $\mathcal{X}$ is smooth, $[\mathcal{X}, \phi]$ is given by the Euler class of the vector bundle $\mathcal{H}^1\bigl(E^{\cdot\vee}\bigr)$. ▮

Let $\mathcal{X}_i$ $(i = 1, 2)$ be algebraic stacks over S with obstruction theories $\phi_i : E_i^{\cdot} \longrightarrow L_{\mathcal{X}_i/S}$. Assume we have the following commutative diagram:

$$\begin{tikzcd} \mathcal{X}_1 \arrow[r, "f"] \arrow[d, "g"'] & \mathcal{X}_2 \arrow[d] & \\ \mathcal{Y}_1 \arrow[r, "h"] & \mathcal{Y}_2 \arrow[r] & S \end{tikzcd}$$

Recall the following definition in [5].

Definition 2.31 *We say that ϕ_i are compatible over h, if we have the following morphism of distinguished triangles on $\mathcal{X}_1$:*

$$\begin{tikzcd} f^*E_2^{\cdot} \arrow[r] \arrow[d] & E_1^{\cdot} \arrow[r] \arrow[d] & g^*L_{\mathcal{Y}_1/\mathcal{Y}_2} \arrow[r] \arrow[d] & f^*E_2^{\cdot}[1] \arrow[d] \\ g^*L_{\mathcal{X}_2/S} \arrow[r] & L_{\mathcal{X}_1/S} \arrow[r] & L_{\mathcal{X}_1/\mathcal{X}_2} \arrow[r] & g^*L_{\mathcal{X}_2/S}[1] \end{tikzcd}$$

▮

We recall the following theorem for later use.

Proposition 2.32 (Behrend-Fantechi, Proposition 7.5, [5]) *Assume $\mathcal{X}_i$ $(i=1,2)$ are Deligne-Mumford, and the obstruction theories ϕ_i are perfect. If ϕ_i are compatible over h, then $h^![\mathcal{X}_2,\phi_2]=[\mathcal{X}_1,\phi_1]$, at least if h is smooth or $\mathcal{Y}_i$ are smooth over S* ∎

See [5] for more detail about virtual fundamental classes.

2.4.2 Easy example

Let X be a smooth variety over k. We would like to construct an obstruction theory of the moduli spaces $\mathcal{M}$ of some objects on X. Our naive strategy is summarized as follows (See [5], for example):

1. Take the classifying stack Y of such objects over X. It means that such objects over $U\times X$ bijectively correspond to morphisms $\Phi : U\times X \longrightarrow Y$ over X. For example, recall that a vector bundle of rank R over $U\times X$ corresponds to a map $U\times X\longrightarrow X_{\mathrm{GL}(R)}$ over X.
2. For any classifying map $\Phi : U\times X\longrightarrow Y$, we obtain the morphism $\Phi^*L_{Y/X}\longrightarrow L_{U\times X/X}$ on $U\times X$. Let ω_X denote the dualizing complex on X, i.e., it is the canonical sheaf shifted by the dimension of X. Then, we obtain the morphisms on U:
$$\mathrm{Ob}_U := Rp_{X*}\big(\Phi^*L_{Y/X}\otimes\omega_X\big)\longrightarrow Rp_{X*}\big(p_X^*L_{U/k}\otimes\omega_X\big)\longrightarrow L_{U/k}.$$
In particular, we obtain the morphism $\mathrm{Ob}_{\mathcal{M}}\longrightarrow L_{\mathcal{M}}$ on $\mathcal{M}$.
3. Then we hope that the morphism $\mathrm{Ob}_{\mathcal{M}}\longrightarrow L_{\mathcal{M}}$ gives the obstruction theory, in some cases. Note that the property is local, once the morphism is given globally. Thus we have only to check the claim for the sufficiently small etale open sets of $\mathcal{M}$. The tool for checking is Proposition 2.28.

Remark 2.33 *In general, we need some modification for the construction of* $\mathrm{Ob}_{\mathcal{M}}$ *to obtain the good obstruction theory.* ∎

Let us see the easiest example. Let F and V be vector bundles defined on X. Let U be any scheme over k, and let $f : p_U^*(F)\longrightarrow p_U^*(V)$ be a morphism of $\mathcal{O}_{U\times X}$-modules over $U\times X$. It is easy to see that such a morphism f corresponds to a morphism $\Phi_f : U\times X\longrightarrow N(F,V)$ over X. Thus we obtain the complex $\mathfrak{g}(f):=\Phi_f^*L_{N(F,V)/X}$ and the morphism $\mathfrak{g}(f)\longrightarrow L_{U\times X/X}$ in the derived category $D(U\times X)$.

Lemma 2.34 *The complex $\mathfrak{g}(f)$ is quasi isomorphic to $p_U^*\mathcal{H}om(V,F)$.*

Proof Let $\pi : N(F,V)\longrightarrow X$ denote the natural projection. Since the morphism $N(F,V)\longrightarrow X$ is smooth, the cotangent complex $L_{N(F,V)/X}$ is quasi isomorphic to $\Omega_{N(F,V)/X}\simeq\pi^*\mathcal{H}om(V,F)$. Thus we obtain the quasi isomorphism $\Phi_f^*L_{N(F,V)/X}\simeq p_U^*\mathcal{H}om(V,F)$. ∎

We put $\mathrm{Ob}(f):=Rp_{X*}\big(\mathfrak{g}(f)\otimes\omega_X\big)$. Then, we obtain the morphisms $\mathrm{Ob}(f)\longrightarrow Rp_{X*}\big(L_{U\times X/X}\otimes\omega_X\big)\longrightarrow L_U$ in the derived category $D(U)$. The composite is denoted by $\mathrm{ob}(f)$.

Now, let $M(F,V)$ denote the moduli scheme of the morphisms $F\longrightarrow V$, i.e., maps $U\longrightarrow M(F,V)$ correspond to $f : p_U^*(F)\longrightarrow p_U^*(V)$ on $U\times X$. It is easy to see that $M(F,V)$ is isomorphic to the vector space $H^0\big(X,\mathcal{H}om(F,V)\big)$. We have the universal morphism $f^u : p_{M(F,V)}^*(F)\longrightarrow p_{M(F,V)}^*(V)$ over $M(F,V)\times X$. It induces the morphism $\mathrm{ob}(f^u):\mathrm{Ob}(f^u)\longrightarrow L_{M(F,V)}$.

Lemma 2.35 *The morphism $ob(f^u)$ gives the obstruction theory of $M(F,V)$.*

Proof It is almost obvious from the universal properties of $N(F,V)$ and $M(F,V)$. But, we give an argument for the explanation of our later discussion. We have only to check the conditions (A1) and (A2) in Proposition 2.28.

Since the claim is local, we can check the claim for any sufficiently small open subset U of $M(F,V)$. Let T be an affine scheme over k. A morphism $g : T\longrightarrow U$ induces the morphism $g_X : T\times X\longrightarrow U\times X$ and

$\widetilde{g}_X = \Phi \circ g_X : T \times X \longrightarrow N(F,V)$ over X. Let $\overline{T}$ denote a scheme such that T is embedded in $\overline{T}$ whose ideal J is square zero. The deformation theory of the morphisms g and $\widetilde{g}_X$ is controlled by the groups $\mathrm{Ext}^i\big(g^* L_{U/k}, J\big)$ and $\mathrm{Ext}^i\big(\widetilde{g}_X^* L_{N(F,V)/X}, J_X\big)$ respectively. We have the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Ext}^i\big(g^* L_{U/k}, J\big) \arrow[r, "h"] \arrow[d] & \mathrm{Ext}^i\big(g^* \mathrm{Ob}(f^u), J\big) & \\
\mathrm{Ext}^i\big(g_X^* L_{U\times X/X}, J_X\big) \arrow[r] & \mathrm{Ext}^i\big(g_X^*(\mathfrak{g}), J\big) \arrow[u, "\simeq"] \arrow[r, equal] & \mathrm{Ext}^i\big(\widetilde{g}_X^* L_{N(F,V)/X}, J\big)
\end{tikzcd}$$

We have the obstruction classes $o(g) \in \mathrm{Ext}^1\big(g^* L_{U/k}, J\big)$ and $o(\widetilde{g}_X) \in \mathrm{Ext}^1\big(g_X^* \mathfrak{g}, J\big)$ of the morphisms g and $\widetilde{g}_X$ respectively. By the functoriality of the cotangent complex, the obstruction class $o(g)$ is mapped to the obstruction class $o(\widetilde{g}_X)$ in the diagram above.

If the image $h\big(o(g)\big)$ is 0, the class $o(\widetilde{g}_X)$ is 0. Thus $\widetilde{g}_X$ can be extended to a morphism $\overline{T} \times X \longrightarrow N(F,V)$, which induces a morphism of $p_{\overline{T}}^*(F) \longrightarrow p_{\overline{T}}^*(V)$ on $\overline{T} \times X$. By the universal property of $M(F,V)$, we obtain a morphism $\overline{T} \longrightarrow M(F,V)$, which is the extension of g. Therefore, the condition (A1) is satisfied.

Similarly, we know that the morphism $\mathrm{Ext}^0\big(g^* L_{U/k}, J\big) \longrightarrow \mathrm{Ext}^0\big(\widetilde{g}_X^* L_{N(F,V)/X}, J\big)$ is isomorphic from the universality of $M(F,V)$ and $N(F,V)$. Hence the condition (A2) is also satisfied. Thus we are done. ∎

2.4.3 Obstruction theory for locally free subsheaves

Let X be a smooth projective variety over k with an ample line bundle $\mathcal{O}_X(1)$. Let V be a locally free sheaf on X. Let W denote an R-dimensional k-vector space. We denote $W \otimes \mathcal{O}_X$ by W_X. We have the natural right $\mathrm{GL}(W)$-action on $N(W_X, V)$. The quotient stack is denoted by $Y_{\mathrm{quo}}(W.)$.

We consider the deformation theory of locally free subsheaves $F \subset V$ of rank R. Let U be any k-scheme. Any locally free subsheaf $f : F \longrightarrow p_U^* V$ on $U \times X$ induces the morphism $\Phi(F,f) : U \times X \longrightarrow Y_{\mathrm{quo}}(W.)$ over X. We put $\mathfrak{g}(F,f) := \Phi(F,f)^* L_{Y_{\mathrm{quo}}(W.)/X}$, and $\mathrm{Ob}(F,f) := Rp_{X*}\big(\mathfrak{g}(F,f) \otimes \omega_X\big)$. Then, we have the morphism $\mathfrak{g}(F,f) \longrightarrow L_{U\times X/X}$ on $U \times X$, which induces $\mathrm{ob}(F,f) : \mathrm{Ob}(F,f) \longrightarrow L_U$ on U. The following lemma can be shown by the argument explained in the subsection 2.3.2.

Lemma 2.36 *$\mathfrak{g}(F,f)$ is represented by $\mathrm{Cone}(\alpha)[-1]$ of the morphism $\alpha : \mathcal{H}om(p_U^* V, F) \longrightarrow \mathcal{H}om(F,F)$, where α is given by $\alpha(a) = a \circ f$.* ∎

Remark 2.37 *We put $V_{-1} := F$ and $V_0 := p_U^* V$, and we regard $V. = (V_{-1} \to V_0)$ as a complex, where V_0 stands in the degree 0. Then, $\mathrm{Cone}(\alpha)$ is naturally isomorphic to $\mathcal{H}om(V_{-1}[1], V.)^\vee[-1]$.* ∎

Let H be a polynomial. We have the moduli of quotients $(q : V \longrightarrow Q)$ of V such that the Hilbert polynomials of Q are H. Let $M(V,H)$ denote the open subscheme which consists of the points $(q : V \longrightarrow Q)$ such that $\mathrm{Ker}(q)$ are locally free. Then, we have the universal family $f^u : F^u \longrightarrow p_{M(V,H)}^*(V)$ defined over $M(V,H) \times X$. We obtain the morphism $\mathrm{ob}(F^u, f^u) : \mathrm{Ob}(F^u, f^u) \longrightarrow L_{M(V,H)}$.

Proposition 2.38 *The morphism $\mathrm{ob}(F^u, f^u)$ gives the obstruction theory of $M(V,H)$.*

Proof Let N be a sufficiently large number satisfying the condition O_N for the family F^u, i.e., we have $H^i\big(X, F^u(N)_{|\{q\}\times X}\big) = 0$ for any $q \in M(V,H)$ and $i > 0$, and $F^u_{|\{q\}\times X}(N)$ are globally generating for any $q \in M(V,H)$. We put $\widetilde{F}^u := p_X^* p_{X*}\big(F^u(N)\big) \otimes \mathcal{O}(-N)$. We have the natural surjection $g : \widetilde{F}^u \longrightarrow F^u$.

We put $\overline{F} = \mathcal{O}(-N)^{\oplus d}$, where $d = \mathrm{rank}\, \widetilde{F}^u$. We have the Grassmaniann bundle $\pi : Gr(\overline{F}, R) \longrightarrow X$ associated to the vector bundle $\overline{F}$, i.e., the fiber of π over a point $x \in X$ is the Grassmann variety of the R-dimensional quotient spaces of the vector space $\overline{F}_{|x}$. We denote the universal quotient bundle over $Gr(\overline{F}, R)$ by Q. Then, we have the vector bundle $\widetilde{Y}_{\mathrm{quo}} := N(Q, \pi^* V)$ over $Gr(\overline{F}, R)$, which is a variety smooth over X. We have the natural morphism $\pi_1 : \widetilde{Y}_{\mathrm{quo}} \longrightarrow Y_{\mathrm{quo}}(W.)$.

We would like to check the conditions (A1) and (A2) in Proposition 2.28. Let U be any sufficiently small open set of $M(V,H)$, on which we can assume that there exists an isomorphism $\widetilde{F}^u \simeq p_U^* \overline{F}$. Thus, the morphism $\alpha : p_U^* \overline{F} \longrightarrow F^u$ is given on $U \times X$. From α and f^u, we obtain the morphism $\Phi(\alpha, F^u, f^u) : U \times X \longrightarrow \widetilde{Y}_{\mathrm{quo}}$ over

X. By the argument in the subsubsection 2.3.2, we can show that the complex $\Phi(\alpha, F^u, f^u)^* L_{\widetilde{Y}_{\mathrm{quo}}/X}$ is quasi isomorphic to $\mathrm{Cone}(\beta)[-1]$ for the morphism $\beta : \mathcal{H}om(p_U^* V, F^u) \oplus \mathcal{H}om(F^u, p_U^* \overline{F}) \longrightarrow \mathcal{H}om(F^u, F^u)$, where $\beta(b_1, b_2) = b_1 \circ f^u - f^u \circ b_2$. We can also show that the natural morphism $\mathrm{Cone}(\alpha)[-1] \longrightarrow \mathrm{Cone}(\beta)[-1] \longrightarrow L_{U\times X/X}$ corresponds to the factorization $\Phi(F^u, f^u)^* L_{Y_{\mathrm{quo}}/X} \longrightarrow \Phi(\alpha, F^u, f^u)^* L_{\widetilde{Y}_{\mathrm{quo}}/X} \longrightarrow L_{U\times X/X}$ associated to $U \times X \longrightarrow \widetilde{Y}_{\mathrm{quo}} \longrightarrow Y_{\mathrm{quo}}(W.)$. We put as follows:

$$\mathfrak{g}(\alpha, F^u, f^u) := \Phi(\alpha, F^u, f^u)^* L_{\widetilde{Y}_{\mathrm{quo}}/X}, \quad \mathrm{Ob}(\alpha, F^u, f^u) := Rp_{X*}\big(\mathfrak{g}(\alpha, F^u, f^u) \otimes \omega_X\big).$$

Let T be an affine scheme, and let $g : T \longrightarrow U$ be a morphism. We put $\widetilde{g}_X := \Phi(F^u, f^u) \circ g$ and $\widehat{g}_X := \Phi(\alpha, F^u, f^u) \circ g$. For any coherent sheaf J on T, we have the following diagram:

$$\begin{CD}
\mathrm{Ext}^i(g^* L_{U/k}, J) @>{h_1^i}>> \mathrm{Ext}^i\big(g^* \mathrm{Ob}(\alpha, F^u, f^u), J\big) @>{h_2^i}>> \mathrm{Ext}^i\big(g^* \mathrm{Ob}(F^u, f^u), J\big) \\
@VVV @A{\simeq}AA @A{\simeq}AA \\
\mathrm{Ext}^i(g_X^* L_{U\times X/X}, J_X) @>>> \mathrm{Ext}^i\big(g_X^* \mathfrak{g}(\alpha, F^u, f^u), J\big) @>>> \mathrm{Ext}^i\big(g_X^* \mathfrak{g}(F^u, f^u), J\big)
\end{CD} \tag{24}$$

Let $\overline{T}$ be an affine scheme into which T is embedded closedly such that the corresponding ideal J is square zero. Due to the deformation theory of Illusie, we have the obstruction classes of the morphisms g and $\widehat{g}_X$ in the groups $\mathrm{Ext}^1(g^* L_{U/k}, J)$ and $\mathrm{Ext}^1\big(g_X^* \mathfrak{g}(\alpha, F^u, f^u), J\big)$ respectively. The classes are denoted by $o(g)$ and $o(\widehat{g}_X)$. Due to the functoriality, the class $o(g)$ is mapped to the class $o(\widehat{g}_X)$ in the diagram (24). If $h_1^1(o(g))$ is 0, then the morphism $\widehat{g}_X$ can be extended.

Note that the cohomology sheaves $R^i p_{X*}\big(\mathcal{H}om(F^u, p_U^* \overline{F}) \otimes \omega_X\big)$ vanish unless $i = 0$, because of our choice of N. Thus, we have the isomorphism $\mathrm{Ext}^i\big(g^* \mathrm{Ob}(\alpha, F^u, f^u), J\big) \simeq \mathrm{Ext}^i\big(g^* \mathrm{Ob}(F^u, f^u), J\big)$ for any $i > 0$ and for any coherent sheaf J on T. Hence $h_2^1 \circ h_1^1(o(g)) = 0$ implies $h_1^1\big(o(g)\big) = 0$. Then the morphism $\widehat{g}_X$ can be extended over $\overline{T} \times X$, and hence $\widetilde{g}_X$ can also be extended over $\overline{T} \times X$. Therefore, we obtain a locally free subsheaf $\widehat{F}$ of $p_{\overline{T}}^*(V)$ on $\overline{T} \times X$, which is the extension of $g_X^* F^u$. Due to the universal property of $M(V, H)$, the morphism g can be extended over $\overline{T}$. Therefore, the condition (A1) is satisfied.

Let us check the condition (A2). We put as follows:

$$H_0 := \mathrm{Ext}^0\Big(p_{X*}\big(g_X^* \mathcal{H}om(F^u, p_U^* \overline{F}) \otimes \omega_X\big), J\Big) = H^0\Big(T, g^* \mathcal{E}nd\big(p_{X*}\big(F^u(m)\big)\big) \otimes J\Big)$$

$$H_1 := \mathrm{Ext}^0\big(g^* \mathrm{Ob}(g, F^u, f^u), J\big) = \mathrm{Ext}^0\big(\widehat{g}_X^* L_{\widetilde{Y}_{\mathrm{quo}}/X}, J_X\big), \quad H_2 := \mathrm{Ext}^0\big(g^* \mathrm{Ob}(F^u, f^u), J\big)$$

We obtain the exact sequence $0 \longrightarrow H_0 \longrightarrow H_1 \longrightarrow H_2 \longrightarrow 0$. Due to the theory of Illusie, H_1 parameterizes the set of extensions $\widehat{g}'_X : \overline{T} \times X \longrightarrow \widetilde{Y}_{\mathrm{quo}}$ of $\widehat{g}_X$. The natural action of H_0 on H_1 determines the equivalence relation on H_1, and it is easy to see that $\widehat{g}'_X \sim \widehat{g}''_X$ if and only if $\pi_1 \circ \widehat{g}'_X = \pi_1 \circ \widehat{g}''_X$, because H_0 parameterizes the deformation of the morphisms $\overline{F} \longrightarrow F^u$. Thus the set of the extensions of the morphisms $T \times X \longrightarrow Y_{\mathrm{quo}}(W.)$ over $\overline{T} \times X$ is the torsor over the group H_2.

By the universal property of $M(V, H)$ and $Y_{\mathrm{quo}}(W.)$, the set of the extensions of g over $\overline{T}$ is also the torsor over H_2. Namely the condition (A2) is satisfied. Therefore we are done. ∎

Usually, we consider the deformation theory of quotients of V. Let H be a polynomial, and let $\mathrm{Quot}(V, H)$ denote the quot scheme which parameterizes the quotient sheaves of V whose Hilbert polynomials are H. We have the universal quotient $q^u : p_{\mathrm{Quot}(V,H)}^*(V) \longrightarrow Q^u$ on $\mathrm{Quot}(V, H) \times X$. We denote the kernel of q^u by F^u, and the inclusion $F^u \longrightarrow p_{\mathrm{Quot}(V,H)}^*(V)$ is denoted by f^u.

Let us consider the case $\dim X = 1$. Let H_V denote the Hilbert polynomial of V. Then $\mathrm{Quot}(V, H)$ parameterizes the locally free subsheaves of V whose Hilbert polynomials is $H_V - H$. Therefore, we have obtained the obstruction theory $\mathrm{ob}(f^u) : \mathrm{Ob}(f^u) \longrightarrow L_{\mathrm{Quot}(V,H)}$.

Proposition 2.39 *In the case* $\dim(X) = 1$*, the obstruction theory* $\mathrm{ob}(f^u)$ *is perfect. The scheme* $\mathrm{Quot}(V, H)$ *is smooth, if H is a constant, i.e., H is a Hilbert polynomial of sheaves of finite length.*

Proof To show the perfectness of $\mathrm{Ob}(f^u)$, we have only to show that $Rp_{X\,*}\big(\mathfrak{g}(F^u, f^u)^\vee\big)$ is perfect of amplitude contained in $[0,1]$. Let q be any point of $Q(V,H)$. We put $F := F^u_{|\{q\}\times X}$ and $Q := V/F$. The complex $\mathfrak{g}(F^u, f^u)^\vee_{|\{q\}\times X}$ is $\mathrm{Cone}(\gamma)[-1]$ for the natural morphism $\gamma : \mathcal{H}om(F,F) \longrightarrow \mathcal{H}om(F,V)$, which is quasi-isomorphic to $\mathcal{H}om(F,Q)$. Hence we have $H^i\big(X, \mathfrak{g}(F^u,f^u)^\vee_{|\{q\}\times X}\big) = 0$ unless $i = 0,1$ for any point $q \in \mathrm{Quot}(V,H)$. Then the desired perfectness easily follows.

Let us show the second claim. When H is a constant, i.e., Q is a torsion sheaf, we always have the vanishing $H^1\big(X, \mathfrak{g}(F^u,f^u)^\vee_{|\{q\}\times X}\big) = 0$. Let T be any affine scheme over k, and let $g : T \longrightarrow \mathrm{Quot}(V,H)$ be a morphism. Then we obtain the vanishing $\mathrm{Ext}^1\big(g^*\,\mathrm{Ob}(f^u), J\big) = 0$ for any coherent $\mathcal{O}_T$-module J, and hence the vanishing of any obstruction class. Thus we obtain the smoothness. ∎

Remark 2.40 *Let us consider the case* $\dim X = 2$. *Let* $Q^{tf}(V,H)$ *denote the open subset of* $Q(F,H)$ *corresponding to the torsion-free quotient. It gives an open subset of the moduli stack of locally free subsheaves of* V. *Then,* $F^u_{|\{q\}\times X}$ *are locally free for any* $q \in Q^{tf}(V,H)$. *Therefore, we obtain the obstruction theory* $\mathrm{ob}(F^u,f^u) : \mathrm{Ob}(F^u,f^u) \longrightarrow L_{Q^{tf}(V,H)/k}$ *from Proposition* 2.38. ∎

2.4.4 Obstruction theory for filtrations of a vector bundle on a curve

Let S be a scheme over k, and let $\mathcal{D}$ be a smooth projective curve over S provided with an ample line bundle $\mathcal{O}(1)$. The projection $\mathcal{D} \longrightarrow S$ is denoted by p. For any point $s \in S$, the fiber over s is denoted by $\mathcal{D}_s$. Let V and F be a locally free sheaf on $\mathcal{D}$ provided with an injective morphism $f : F \longrightarrow V$. Assume that the quotient is S-flat.

Let H_i be polynomials. For an S-scheme T, let $F(T)$ denote the set of the data (g, V^*) as follows:

- g is a morphism $T \longrightarrow \mathcal{D}$ over S.
- V^* denotes a filtration $g^*V = V^{(1)} \supset V^{(2)} \supset \cdots \supset V^{(l)} \supset V^{(l+1)} = g^*F$. We assume that the quotients $\mathrm{Cok}_i := V^{(1)}/V^{(i+1)}$ are T-flat.
- The Hilbert polynomials of $\mathrm{Cok}_{i|\,\mathcal{D}_s}$ are H_i for any $i = 1, \ldots, l$ and $s \in S$.

The functor is representable by a scheme, which can be shown by the standard technique using the quot schemes. Let $M(H_*)$ denote the moduli scheme. Let $p_{M(H_*)}$ denote the projection $M(H_*) \times_S \mathcal{D} \longrightarrow \mathcal{D}$. We have the universal filtration on $M(H_*) \times_S \mathcal{D}$:

$$p^*_{M(H_*)}V = \mathcal{V}^{(1)} \supset \mathcal{V}^{(2)} \supset \cdots \supset \mathcal{V}^{(l)} \supset \mathcal{V}^{(l+1)} = p^*_{M(H_*)}F.$$

To discuss an obstruction theory of $M(H_*)$, we introduce some stacks. Take vector spaces W_i $(i = 2, \ldots, l)$ over k such that $\mathrm{rank}\, W_i = \mathrm{rank}\, \mathcal{V}^{(i)} =: r_i$. We put $W^{(i)} := W_i \otimes \mathcal{O}_{\mathcal{D}}$ $(i = 2, \ldots, l)$. We put $W^{(1)} = V$ and $W^{(l+1)} = F$. We put $Y_0 := N(W^{(l+1)}, W^{(1)})$ and $R_1 := \prod_{i=1}^{l} N(W^{(i+1)}, W^{(i)})$. We put $G(W_*) := \prod_{i=2}^{l} \mathrm{GL}(W_i)$. We have the natural right $G(W_*)$-action on R_1. Let Y_1 denote the quotient stack of R_1 by the $G(W_*)$-action. By the composition of the maps, we obtain the morphism $\phi : R_1 \longrightarrow Y_0$, which induces $Y_1 \longrightarrow Y_0$. We also put $Y_2 := \mathcal{D}$. Then the morphism $F \longrightarrow V$ induces the morphism $Y_2 \longrightarrow Y_0$. We put $Y := Y_1 \times_{Y_0} Y_2$.

Let V^* denote the above filtered vector bundle on $T \times_S \mathcal{D}$. We have the naturally defined morphism:

$$G(V^*) : \Phi_0(V^*)^* L_{Y_0/\mathcal{D}} \longrightarrow \bigoplus_{i=1,2} \Phi_i(V^*)^* L_{Y_i/\mathcal{D}}$$

We use the notation in the subsubsection 2.1.5. We put $\mathfrak{g}(V^*) := C_2(V^*, V^*)^\vee[-1]$. We obtain the morphism $\Phi_i(V^*) : T \times_S \mathcal{D} \longrightarrow Y_i$. By the argument in the subsubsection 2.3.2, the cone of $G(V^*)$ is expressed by the complex $\mathfrak{g}(V^*)$. Thus, we have the naturally defined morphism $\mathfrak{g}(V^*) \longrightarrow L_{T\times_S\mathcal{D}/\mathcal{D}}$. We put $\mathrm{Ob}(V^*) := Rp_*\big(\mathfrak{g}(V^*) \otimes \omega_{\mathcal{D}/S}\big)$, and then we obtain the morphism $\mathrm{ob}(V^*) : \mathrm{Ob}(V^*) \longrightarrow L_T$.

Lemma 2.41 *The morphism* $\mathrm{ob}(\mathcal{V}^*)$ *gives an obstruction theory for* $M(H_*)$.

Proof Let us take locally free sheaves $J^{(i)}$ $(i=2,\ldots,l)$ on $\mathcal{D}$ such that $H^1\big(\mathcal{D}_s, \mathcal{H}om(J^{(i)}, \mathcal{V}^{(i)}_{|\mathcal{D}_s})\big)=0$ for any $s\in\mathcal{D}$. For any S-scheme T, let $\widetilde{F}(T)$ denote the set of the data (g,V^*,φ_*) as follows:

- g denotes a morphism $T\longrightarrow\mathcal{D}$, and V^* denotes a filtration as above.
- φ_* denotes a tuple of surjections of $g^*J^{(i)}$ onto $V^{(i)}$.

The functor $\widetilde{F}$ is representable by the scheme which is denoted by $\widetilde{M}(H_*)$. It is easily described. We have the locally free sheaf $N_i=\mathcal{H}om\big(p^*_{M(H_*)}J^{(i)},\mathcal{V}^{(i)}\big)$ on $M(H_*)\times_S\mathcal{D}$. Then $\widetilde{M}(H_*)$ is isomorphic to an open subset of $\bigoplus p_*N_i$. On $\widetilde{M}(H_*)\times_S\mathcal{D}$, we have the universal filtration $\mathcal{V}^*$ with the tuple of surjective morphisms φ^u_*.

Let $Gr(J^{(i)},r_i)$ be the Grassmannian bundles of r_i-dimensional quotient spaces associated to the vector bundles $J^{(i)}$. We have the universal quotient bundle Q_i. We put $Z:=\prod_{i=2}^l Gr(J^{(i)},r_i)$, where the fiber product is taken over $\mathcal{D}$. The pull back of Q_i via the projection $Z\longrightarrow Gr(J^{(i)},r_i)$ is denoted by $\widetilde{W}^{(i)}$ $(i=2,\ldots,l)$. The pull back of V and F via the projection $Z\longrightarrow\mathcal{D}$ are denoted by $\widetilde{W}^{(1)}$ and $\widetilde{W}^{(l+1)}$ respectively. Then we put $\widetilde{Y}_0:=N\big(\widetilde{W}^{(l+1)},\widetilde{W}^{(1)}\big)$, $\widetilde{Y}_1:=\prod_{i=1}^l N(\widetilde{W}^{(i+1)},\widetilde{W}^{(i)})$ and $\widetilde{Y}_2:=Z$. We have the natural morphisms $\widetilde{Y}_i\longrightarrow\widetilde{Y}_0$ $(i=1,2)$ as above, the fiber product $\widetilde{Y}_1\times_{\widetilde{Y}_0}\widetilde{Y}_2$ is denoted by $\widetilde{Y}$. The inclusions $\widetilde{Y}\longrightarrow\widetilde{Y}_i$ are denoted by j_i. On $\widetilde{Y}$, we have the natural morphism $j_0^*L_{\widetilde{Y}_0/\mathcal{D}}\longrightarrow\bigoplus_{i=1,2}j_i^*L_{\widetilde{Y}_i/\mathcal{D}}$. The cone of the morphism is denoted by $\mathrm{Ob}(\widetilde{Y})$. Then we have the naturally defined morphism $\mathrm{ob}(\widetilde{Y}):\mathrm{Ob}(\widetilde{Y})\longrightarrow L_{\widetilde{Y}/\mathcal{D}}$, and it gives an obstruction theory for $\widetilde{Y}$ over $\mathcal{D}$. (Basic example in [5]).

Let T be an S-scheme. From (V^*,φ_*), we obtain the morphism $\Phi_i(V^*,\varphi_*):T\times_S\mathcal{D}\longrightarrow\widetilde{Y}_i$. Therefore, we obtain $\Phi(V^*,\varphi_*)^*\,\mathrm{Ob}(\widetilde{Y})\longrightarrow L_{T\times_S\mathcal{D}/S}$. We put $\widetilde{\mathrm{Ob}}(V^*,\varphi_*):=Rp_*\big(\mathrm{Ob}(\widetilde{Y})\otimes\omega_{\mathcal{D}/S}\big)$, and then we obtain the morphism $\widetilde{\mathrm{ob}}(V^*,\varphi_*):\widetilde{\mathrm{Ob}}(V^*,\varphi_*)\longrightarrow L_{T/S}$.

Let us describe the complex $\widetilde{\mathrm{Ob}}(V^*,\varphi_*)$. We have the morphism $\mathcal{H}om\big(V^{(i)},J^{(i)}\big)\longrightarrow\mathcal{H}om\big(V^{(i)},V^{(i)}\big)$ given by $a_i\longmapsto\varphi_i\circ a_i$. It induces the morphism of the complexes $\alpha:\bigoplus_{i=2}^l\mathcal{H}om\big(V^{(i)},J^{(i)}\big)[-1]\longrightarrow\mathfrak{g}(V^*)$. We put $\widetilde{\mathfrak{g}}(V^*):=\mathrm{Cone}(\alpha)$. By using the argument in the subsubsection 2.3.2, we can show that $\widetilde{\mathfrak{g}}(V^*)$ expresses $\Phi(V^*,\varphi_*)^*\,\mathrm{Ob}(\widetilde{Y})$.

Applying the above construction to $(\mathcal{V}^*,\varphi^u_*)$, we obtain the morphism $\widetilde{\mathrm{ob}}(\mathcal{V}^*,\varphi^u_*):\widetilde{\mathrm{Ob}}(\mathcal{V}^*,\varphi^u_*)\longrightarrow L_{\widetilde{M}/S}$.

Lemma 2.42 *The morphism $\widetilde{\mathrm{ob}}(\mathcal{V}^*,\varphi^u_*)$ gives an obstruction theory of $\widetilde{M}(H_*)$ over S.*

Proof Let $h:T\longrightarrow\widetilde{M}(H_*)$ be a morphism, and let J be a coherent sheaf on T. The pull back of J via $T\times_S\mathcal{D}\longrightarrow T$ is denoted by $J_{\mathcal{D}}$. We put $\widehat{h}_{\mathcal{D}}:=\Phi(\mathcal{V}^*,\varphi^u_*)\circ h_{\mathcal{D}}$. We have the following commutative diagram:

$$\begin{array}{ccc}
\mathrm{Ext}^1\big(h^*L_{\widetilde{M}(H_*)/S},\,J\big) & \xrightarrow{\psi} & \mathrm{Ext}^1\big(h^*\widetilde{\mathrm{Ob}},\,J\big)\\
\big\downarrow & & \simeq\big\uparrow\\
\mathrm{Ext}^1\big(h^*_{\mathcal{D}}L_{\widetilde{M}(H_*)\times_S\mathcal{D}/\mathcal{D}},\,J_{\mathcal{D}}\big) & \longrightarrow & \mathrm{Ext}^1\big(h^*_{\mathcal{D}}\widetilde{\mathfrak{g}}(\mathcal{V}^*),\,J_{\mathcal{D}}\big)
\end{array}$$

Let $\overline{T}$ be an S-scheme such that T is embedded as a closed subscheme and that the corresponding ideal J is square 0. We have the obstruction classes $o(h)$ and $o(\widehat{h}_{\mathcal{D}})$ in $\mathrm{Ext}^1\big(h^*L_{\widetilde{M}(H_*)/S},\,J\big)$ and $\mathrm{Ext}^1\big(\widehat{h}^*_{\mathcal{D}}L_{\widetilde{Y}/S},J_X\big)$. It is easy to see that $\psi\big(\mathrm{ob}(h)\big)\in\mathrm{Ext}^1\big(h^*\widetilde{\mathrm{Ob}},J\big)$ is same as the image of $o(\widehat{h}_{\mathcal{D}})$ via the composite of the following morphisms:

$$\mathrm{Ext}^1\big(\widehat{h}^*_{\mathcal{D}}L_{\widetilde{Y}/S},J_{\mathcal{D}}\big)\xrightarrow{b_1}\mathrm{Ext}^1\big(h^*_{\mathcal{D}}\widetilde{\mathfrak{g}}(\mathcal{V}^*),J_{\mathcal{D}}\big)\xrightarrow{b_2}\mathrm{Ext}^1\big(h^*\widetilde{\mathrm{Ob}},J\big)$$

Hence the vanishing of $\psi\big(o(h)\big)$ implies $b_1\big(o(\widehat{h}_{\mathcal{D}})\big)=0$. Since $\widetilde{\mathrm{Ob}}$ gives the obstruction theory for $\widetilde{Y}$, it implies that $\widehat{h}$ can be extended to a morphism $\overline{T}\times_S\mathcal{D}\longrightarrow\widetilde{Y}$. Then we obtain the extension of h to the morphism $\overline{T}\longrightarrow\widetilde{M}(H_*)$ due to the universal property of $\widetilde{M}(H_*)$. Therefore, the condition (A1) of Proposition 2.28 is checked. The condition (A2) can also be checked easily, and the proof of Lemma 2.42 is finished. ∎

Let π denote the projection $\widetilde{M}(H_*) \longrightarrow M(H_*)$, which is smooth. We have the following commutative diagram:

$$\begin{tikzcd}
\widetilde{M}(H_*) \times_S \mathcal{D} \arrow[r, "\Phi_i(\mathcal{V}^*,\varphi_*^u)"] \arrow[d, "\pi"'] & \widetilde{Y}_i \arrow[d] \\
M(H_*) \times_S \mathcal{D} \arrow[r, "\Phi_i(\mathcal{V}^*)"] & Y_i
\end{tikzcd}$$

Then, we obtain the following morphism of the distinguished triangles on $\widetilde{M}(H_*) \times_S \mathcal{D}$:

$$\begin{tikzcd}
\pi^* G(\mathcal{V}^*) \arrow[r] \arrow[d] & \Phi(\mathcal{V}^*,\varphi_*^u)^* \operatorname{Ob}(\widetilde{Y}) \arrow[r] \arrow[d] & \bigoplus_{i=2}^{l} \mathcal{H}om(\mathcal{V}^{(i)}, J^{(i)}) \arrow[r] \arrow[d] & \pi^* G(\mathcal{V}^*)[1] \arrow[d] \\
\pi^* L_{M(H_*)\times_S \mathcal{D}/\mathcal{D}} \arrow[r] & L_{\widetilde{M}(H_*)\times_S \mathcal{D}/\mathcal{D}} \arrow[r] & L_{\widetilde{M}(H_*)\times_S \mathcal{D}/M(H_*)\times_S \mathcal{D}} \arrow[r] & \pi^* L_{M(H_*)\times_S \mathcal{D}/\mathcal{D}}[1]
\end{tikzcd}$$

Hence, we obtain the following morphism of the distinguished triangles:

$$\begin{tikzcd}
\pi^* \operatorname{Ob}(\mathcal{V}^*) \arrow[r] \arrow[d] & \widetilde{\operatorname{Ob}}(\mathcal{V}^*, \varphi^*) \arrow[r] \arrow[d] & \bigoplus_i \bigl(p_* \mathcal{H}om(J^{(i)}, \mathcal{V}^{(i)})\bigr)^\vee \arrow[r] \arrow[d, "\varphi"'] & \pi^* \operatorname{Ob}(\mathcal{V}^*)[1] \arrow[d] \\
\pi^* L_{M(H_*)} \arrow[r] & L_{\widetilde{M}(H_*)/S} \arrow[r] & L_{\widetilde{M}(H_*)/M(H_*)} \arrow[r] & \pi^* L_{M(H_*)/S}[1]
\end{tikzcd}$$

It is easy to see that both of $L_{\widetilde{M}(H_*/S)}$ and $\bigoplus_i \bigl(p_* \mathcal{H}om(J^{(i)}, \mathcal{V}^{(i)})\bigr)^\vee$ are isomorphic to the 0-th cohomology sheaves, and that the morphism φ is isomorphic. Then, the claim of Lemma 2.41 immediately follows from Lemma 2.42. ∎

2.5 Equivariant Complexes on Deligne-Mumford Stacks with GIT Construction

The results in this subsection will be used when we discuss the equivariant obstruction theory of the master space in the subsection 5.8.

2.5.1 Locally free resolution

Let G_i $(i = 1, 2)$ be a linear reductive group over k. Let U be a quasi projective variety over k provided with $G_1 \times G_2$. We assume that there exists a $G_1 \times G_2$-embedding into some projective space $\mathbb{P}^N$. The closure of U in $\mathbb{P}^N$ is denoted by $\overline{U}$. The $G_1 \times G_2$-equivariant polarization is denoted by $\mathcal{O}(1)$. We assume that U is contained in the open subset of the stable points of $\overline{U}$ with respect to the polarization $\mathcal{O}(1)$ and the G_2-action. We assume that $\mathcal{M} = U/G_2$ is a separated Deligne-Mumford stack. The projection $U \longrightarrow \mathcal{M}$ is denoted by π.

Lemma 2.43 *Let $\mathcal{F}$ be a G_1-equivariant quasi coherent sheaf on $\mathcal{M}$. Then there exists a G_1-equivariant locally free sheaf $\mathcal{V}$ on $\mathcal{M}$ and a G_1-equivariant surjection $\phi : \mathcal{V} \longrightarrow \mathcal{F}$.*

Proof There exists a coherent sheaf $\mathcal{G}$ on $\overline{U}$ such that $\mathcal{G}_{|U} = \pi^*\mathcal{F}$. There exists a large number N such that $\mathcal{G}(N)$ is globally generating. Then $\pi^*\mathcal{F}(N)$ is also globally generating. We may take a $G_1 \times G_2$-equivariant subspace W of $H^0\bigl(U, \pi^*\mathcal{F}(N)\bigr)$ such that $W \otimes \mathcal{O}(-N) \longrightarrow \pi^*\mathcal{F}$ is surjective. Then we have only to take the descent of $W \otimes \mathcal{O}(-N)$ and the morphism. ∎

Corollary 2.44 *Let $\mathcal{F}^{\cdot}$ be a bounded G_1-equivariant complex of coherent sheaves on $\mathcal{M}$. Assume that there exist integers M_1 and M_2 such that the following holds:*

- *For any point of $\mathcal{M}$, there exists a neighbourhood $\mathcal{U}$ such that $\mathcal{F}^{\cdot}_{|\mathcal{U}}$ is isomorphic to a G_1-equivariant coherent locally free complex $\mathcal{G}^{\mathcal{U}}_{\cdot}$ in $D(\mathcal{U})$ where $\mathcal{G}^{\mathcal{U}}_i = 0$ unless $M_1 \leq i \leq M_2$.*

Then there exists a global G_1-equivariant coherent locally free complex $\mathcal{G}^{\cdot} \simeq \mathcal{F}$ in $D(\mathcal{M})$, where $\mathcal{G}_i = 0$ unless $M_1 \leq i \leq M_2$. ∎

2.5.2 Equivariant representative

We recall that the morphism of $\mathcal{M}$ to the coarse scheme is finite (Proposition 2.4).

Lemma 2.45 *Let $C_i\cdot$ $(i=1,2)$ be G_1-equivariant bounded complexes of coherent sheaves on $\mathcal{M}$. We assume that $C_1\cdot$ is perfect. Let φ be an element of the G_1-invariant part of $\mathrm{Ext}^0(C_1\cdot, C_2\cdot)$. Then, we can take a G_1-equivariant perfect complex $\widetilde{C}_1\cdot$ with a G_1-equivariant morphism $\psi : \widetilde{C}_1\cdot \longrightarrow C_2$, such that $\widetilde{C}_1\cdot$ is G_1-equivariantly quasi isomorphic to $C_1\cdot$, and that ψ represents φ.*

Proof We give only an indication. We may assume that $C_{2,i}=0$ unless $|i|<N$. We take a sufficiently large number N_1, and we replace $C_1\cdot$ with a G_1-equivariant quasi isomorphic complex $\widetilde{C}_1\cdot$ with the property $\mathrm{Ext}^k\bigl(C_{1,i}, C_{2,j}\bigr)=0$ for any $k>0$ and $i>-N_1$, and for any j. Then, $\mathrm{Ext}^0(C_1,C_2)\simeq\mathrm{Ext}^0(\widetilde{C}_1,C_2)$ is isomorphic to the first cohomology of the following:

$$\bigoplus_{-i+j=-1}\mathrm{Ext}^0(\widetilde{C}_{1,i},C_{2,j})\longrightarrow\bigoplus_{-i+j=0}\mathrm{Ext}^0(\widetilde{C}_{1,i},C_{2,j})\longrightarrow\bigoplus_{-i+j=1}\mathrm{Ext}^0(\widetilde{C}_{1,i},C_{2,j})$$

Since G_1 is assumed to be reductive, the claim is clear. ∎

Let $B^{(i)}$ $(i=1,2)$ be G_1-equivariant bounded complexes on $\mathcal{M}$. We assume that $B^{(1)}$ is perfect. Let ϕ be an element of the G_1-invariant part of $\mathrm{Ext}^0\bigl(B^{(1)},B^{(2)}\bigr)$. We take a G_1-equivariant perfect complex $\widetilde{B}^{(1)}$ with G_1-equivariant morphisms $a_i:\widetilde{B}^{(1)}\longrightarrow B^{(i)}$ such that a_1 is quasi isomorphic, and that the diagram $B^{(1)}\xleftarrow{a_1}\widetilde{B}^{(1)}\xrightarrow{a_2}B^{(2)}$ represents ϕ. We have the natural G_1-equivariant structure on the cone $\mathrm{Cone}(a_2)$. Assume we have other G_1-equivariant complex $\widehat{B}^{(1)}$ with G_1-equivariant morphisms $\widehat{a}_i:\widehat{B}^{(1)}\longrightarrow B^{(i)}$ such that the diagram $B^{(1)}\xleftarrow{\widehat{a}_1}\widehat{B}^{(1)}\xrightarrow{\widehat{a}_2}B^{(2)}$ represents ϕ. Then, there exists G_1-equivariant complex $\overline{B}^{(1)}$ with G_1-equivariant morphisms with morphisms $f:\overline{B}^{(1)}\longrightarrow\widetilde{B}^{(1)}$ and $g:\overline{B}^{(1)}\longrightarrow\widehat{B}^{(1)}$ such that the following diagrams are commutative up to homotopy for $i=1,2$:

$$\begin{array}{ccc}\overline{B}^{(1)} & \xrightarrow{\ f\ } & \widetilde{B}^{(1)}\\ {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle a_i}\\ \widehat{B}^{(1)} & \xrightarrow{\ \widehat{a}_i\ } & B^{(i)}\end{array}$$

Due to an argument similar to the proof of Lemma 2.45, we may assume that the homotopy is also G_1-equivariant. Then, we have the G_1-equivariant quasi isomorphisms:

$$\mathrm{Cone}(\widehat{a}_2)\longleftarrow\mathrm{Cone}(\widehat{a}_2\circ g)\simeq\mathrm{Cone}(a_2\circ f)\longrightarrow\mathrm{Cone}(a_2)$$

In this sense, the G_1-equivariant complex $\mathrm{Cone}(a_2)$ is uniquely determined up to G_1-equivariant quasi isomorphisms. We denote it by $\mathrm{Cone}(\varphi)$.

2.6 Elementary Remarks on some Extreme Sets

The results in this subsection will be used when we discuss the geometric invariant theory for the enhanced master space in the subsections 4.3–4.4.

2.6.1 Preparation for the proof of Proposition 4.21

Let us consider a vector space $\mathcal{U}=\bigoplus_{i=1}^N\mathbb{Q}\cdot e_i$. We put $f_j:=(j-N)\sum_{i\leq j}e_i+j\cdot\sum_{i>j}e_i$. The following lemma is well known and easy to prove.

Lemma 2.46 *Take any element $\rho=\sum_{i=1}^N a_i\cdot e_i\in\mathcal{U}$ satisfying $\sum_{j=1}^N a_j=0$ and $a_1\leq a_2\leq\cdots\leq a_N$. Then there exist non-negative rational numbers b_j such that $\rho=\sum b_j\cdot f_j$.* ∎

Let $r_1, \dots, r_s$ be positive integers such that $\sum_{j=1}^{s} r_j = N$. We put $R_j = \sum_{i \le j} r_i$. We put as follows:

$$v_j := \sum_{R_{j-1} < i \le R_j} e_i, \quad (j = 1, \dots, s).$$

For an integer j such that $1 \le j \le s$, we put as follows:

$$y(j) := -(N - R_j) \sum_{h \le j} v_h + R_j \cdot \sum_{h > j} v_h.$$

For a pair of integers (i_1, i_2) such that $1 \le i_1 < i_2 \le s$, we put as follows:

$$x(i_1, i_2) := -(N - R_{i_2}) \sum_{h \le i_1} v_h + R_{i_1} \sum_{i_2 < h} v_h.$$

For an integer i_0 such that $1 \le i_0 \le s$, we put as follows:

$$\mathcal{S}(i_0) := \Big\{ (i_1, i_2) \in \mathbb{Z}^2 \,\Big|\, 1 \le i_1 < i_0 < i_2 \le s \Big\}.$$

Lemma 2.47 *Let $v = \sum_{j=1}^{s} a_j \cdot v_j$ be an element of $\mathcal{U}$ satisfying the following:*

$$a_1 \le a_2 \le \dots \le a_s, \quad \sum_{j=1}^{s} r_j \cdot a_j = 0. \tag{25}$$

Take an integer i_0 such that $1 \le i_0 \le s$.

- *Assume $a_{i_0} > 0$. Then there exist the non-negative rational numbers $b(i_1, i_2) \in \mathbb{Q}_{\ge 0}$ for $(i_1, i_2) \in \mathcal{S}(i_0)$ and the non-negative rational numbers c_j $(1 \le j < i_0)$ such that the following equality holds:*

$$v = \sum_{(i_1, i_2) \in \mathcal{S}(i_0)} b(i_1, i_2) \cdot x(i_1, i_2) + \sum_{j=1}^{i_0 - 1} c_j \cdot y(j). \tag{26}$$

- *Assume $a_{i_0} = 0$. Then there exist the non-negative rational numbers $b(i_1, i_2) \in \mathbb{Q}_{\ge 0}$ for $(i_1, i_2) \in \mathcal{S}(i_0)$ such that the following holds:*

$$v = \sum_{(i_1, i_2) \in \mathcal{S}(i_0)} b(i_1, i_2) \cdot x(i_1, i_2).$$

  One of $b(i_1, i_2)$ is not 0.

- *Assume $a_{i_0} < 0$. Then there exist the non-negative rational numbers $b(i_1, i_2) \in \mathbb{Q}_{\ge 0}$ for $(i_1, i_2) \in \mathcal{S}(i_0)$ and the non-negative rational numbers c_j $(i_0 < j \le N)$ such that the following holds:*

$$v = \sum_{(i_1, i_2) \in \mathcal{S}(i_0)} b(i_1, i_2) \cdot x(i_1, i_2) + \sum_{j = i_0 + 1}^{N} c_j \cdot y(j).$$

Proof We use an induction on the number $d(v) := \#\big\{ i \,\big|\, a_i \ne a_{i+1} \big\}$. In the case $d(v) = 0$, the claim is obvious. Let v be as in the lemma such that $d(v) = m + 1$. Take the integers h_1 and h_2 satisfying the following:

$$a_1 = a_2 = \dots = a_{h_1} < a_{h_1 + 1}, \quad a_s = a_{s-1} = \dots = a_{h_2 + 1} > a_{h_2}.$$

We remark the following:

- In the case $a_{i_0} > 0$, we have $h_1 < i_0$.
- In the case $a_{i_0} = 0$, we have $h_1 < i_0 < h_2$.

- In the case $a_{i_0} < 0$, we have $i_0 < h_2$.

Let us discuss the case $a_{i_0} > 0$. If we have $i_0 \leq h_2$, we put as follows:

$$v' := v - f \cdot x(h_1, h_2) = \sum a'_i \cdot v_i, \quad f := \min \left\{ \frac{a_{h_1+1} - a_{h_1}}{N - R_{h_2}}, \frac{a_{h_2+1} - a_{h_2}}{R_{h_1}} \right\}$$

If we have $i_0 \geq h_2 + 1$, we put as follows:

$$v' := v - g \cdot y(h_2), \quad g := \frac{a_{i_0} - a_{h_2}}{2R_{h_2}}$$

It is easy to see that the numbers a'_i satisfy the condition (25), and that we have $d(v') \leq d(v) - 1$. Due to the hypothesis of the induction, we have the expression for v' as in (26) with the non-negative coefficients. Then we obtain the desired expression for v.

The cases $a_{i_0} = 0$ or $a_{i_0} < 0$ can be discussed similarly. ∎

2.6.2 Preparation for the proof of Proposition 4.35

Let $N^{(\alpha)}$ $(\alpha = 1, 2)$ be positive integers. Let us consider a vector space as follows:

$$\mathcal{U} = \mathcal{U}^{(1)} \oplus \mathcal{U}^{(2)}, \quad \mathcal{U}^{(\alpha)} = \bigoplus_{i=1}^{N^{(\alpha)}} \mathbb{Q} \cdot e_i^{(\alpha)}.$$

Let $r_1^{(\alpha)}, \ldots, r_{s(\alpha)}^{(\alpha)}$ $(\alpha = 1, 2)$ be positive integers such that $\sum_{j=1}^{s(\alpha)} r_j^{(\alpha)} = N^{(\alpha)}$. We put $R_j^{(\alpha)} = \sum_{h \leq j} r_h^{(\alpha)}$. We put $\Omega^{(\alpha)} = \sum_i e_i^{(\alpha)}$. We also put as follows:

$$v_j^{(\alpha)} := \sum_{R_{j-1}^{(\alpha)} < i \leq R_j^{(\alpha)}} e_i^{(\alpha)}, \quad \bigl(j = 1, \ldots, s(\alpha)\bigr).$$

For each integer j such that $1 \leq j \leq s(2)$, we put as follows:

$$y^{(2)}(j) = -(N - R_j^{(2)}) \cdot \sum_{h \leq j} v_h^{(2)} + R_j^{(2)} \cdot \sum_{h > j} v_h^{(2)}.$$

For each integer j such that $1 \leq j \leq s(1)$, we put as follows:

$$x_1(j) := -N^{(2)} \cdot \sum_{h \leq j} v_h^{(1)} + R_j^{(1)} \cdot \Omega^{(2)}, \quad x_2(j) := N^{(2)} \cdot \sum_{h \geq j} v_h^{(1)} + (R_{j-1}^{(1)} - N^{(1)}) \cdot \Omega^{(2)}.$$

Lemma 2.48 *Let $v = \sum_{\alpha=1,2} \sum_j a_j^{(\alpha)} \cdot v_j^{(\alpha)}$ be any element of $\mathcal{U}$ satisfying the following conditions:*

$$a_1^{(\alpha)} \leq a_2^{(\alpha)} \leq \cdots \leq a_{s(\alpha)}^{(\alpha)}, \quad \sum_{\alpha=1,2} \sum_j r_j^{(\alpha)} \cdot a_j^{(\alpha)} = 0. \tag{27}$$

Take an integer i_0 such that $1 \leq i_0 \leq s(1)$. Then, there exist non-negative rational numbers $c(j) \geq 0$ $(j = 1, \ldots, s(2))$, $d_1(i) \geq 0$ $(i = 1, \ldots, i_0)$, $d_2(i) \geq 0$ $(i = i_0 + 1, \ldots, s(1))$ and a rational number A such that the following holds:

$$v = \sum_{j=1}^{s(2)} c(j) \cdot y^{(2)}(j) + \sum_{i < i_0} d_1(i) \cdot x_1(i) + \sum_{i > i_0} d_2(i) \cdot x_2(i) + A \cdot \bigl(N^{(2)}\Omega^{(1)} - N^{(1)} \cdot \Omega^{(2)}\bigr). \tag{28}$$

Proof Due to Lemma 2.46, we may assume $a_1^{(2)} = \cdots = a_{s(2)}^{(2)}$ from the beginning. We use an induction on the number $d(v) = \#\{i \mid a_i^{(1)} \neq a_{i+1}^{(1)}\}$. In the case $d(v) = 0$, we have $v = A \cdot \bigl(N^{(2)} \cdot \Omega^{(1)} - N^{(1)} \cdot \Omega^{(1)}\bigr)$ for some A, and hence the claim is clear. Let v be an element as in the lemma such that $d(v) = m + 1 > 0$. Let us take the integer h_1 satisfying $a_1^{(1)} = a_{h_1}^{(1)} < a_{h_1+1}^{(1)}$. In the case $i_0 > h_1$, we put as follows:

$$v' := v - \frac{a_{h_1+1}^{(1)} - a_{h_1}^{(1)}}{N^{(2)}} x_1(h_1)$$

In the case $i_0 \leq h_1$, we put as follows:

$$v' = v - \frac{a_{h_1+1}^{(1)} - a_{h_1}^{(1)}}{N^{(2)}} \cdot x_2(h_1).$$

Then v' satisfies the condition (27) and $d(v') < d(v)$. Due to the hypothesis of the induction, we have the expression for v' as in (28). Hence we obtain the desired expression for v. ∎

2.7 Twist of Line Bundles

This subsubsection is a preparation for the subsection 4.6.

2.7.1 Construction

Let Y be an algebraic stack over a field k. Let G_m denote the one dimensional algebraic torus $\operatorname{Spec} k[t, t^{-1}]$. Let I denote the trivial line bundle on Y. A point of I is denoted by (y, u) where $y \in Y$ and $u \in I_{|y}$. For each integer n, $\mathcal{T}(n)$ denote the line bundle I with the G_m-action by $t \cdot (y, u) := (y, t^n \cdot u)$.

Let L be any line bundle on Y. Let L^* denote the complement of the image of the 0-section, i.e., $L^* := L - Y$. Let $\pi : L^* \longrightarrow Y$ denote the naturally defined projection. A point of L^* is also denoted by (y, v), where $y \in Y$ and $v \in \pi^{-1}(y)$.

Let us fix an integer r. We consider the G_m-action on L^* given by $t \cdot (y, v) := (y, t^r \cdot v)$. We have the naturally defined G_m-action on $\pi^*\mathcal{T}(n)$. It induces the line bundle $\mathcal{I}_n$ on the algebraic stack L^*/G_m. Let $\varphi : L^*/G_m \longrightarrow Y$ denote the naturally induced morphism.

Lemma 2.49 *We have the canonical isomorphism $\mathcal{I}_n \otimes \mathcal{I}_m \simeq \mathcal{I}_{n+m}$ and $\mathcal{I}_{-n} \simeq \mathcal{I}_n^{-1}$ and $\mathcal{I}_0 \simeq \mathcal{O}_{Y/G_m}$. We also have the canonical isomorphism $\mathcal{I}_{-r} \simeq \varphi^* L$.*

Proof The first claim is obvious. Let us show the second claim. Let us denote a point of π^*L by (y, v, u'), where $y \in Y$, $v \in \pi^{-1}(y)$ and $u' \in L_{|y}$. The trivial G_m-action on L induces the G_m-action on π^*L over L^*, which is given by $t \cdot (y, u, v') = \bigl(y, t^r \cdot u, v'\bigr)$.

On the other hand, let us denote a point of $\pi^*\mathcal{T}(-r)$ by (y, v, u) where $y \in Y$, $v \in \pi^{-1}(y)$ and $u \in \mathcal{T}(-r)_{|y}$. The action is denoted by $t \cdot (y, v, u) = \bigl(y, t^r v, t^{-r} u\bigr)$.

We have the naturally defined isomorphism $\pi^*\mathcal{T}(-r) \longrightarrow \pi^*L$ given by $(y, u, v) \longmapsto (y, u, u \cdot v)$, which is G_m-equivariant. Therefore, we obtain the isomorphism $\mathcal{I}_{-r} \simeq \varphi^* L$. ∎

2.7.2 The weight of the induced action

We use the notation in the previous subsubsection. Let $G_m^{(i)}$ denote the torus $\operatorname{Spec} k[t_i, t_i^{-1}]$. Let us consider the action of $G_m^{(1)} \times G_m^{(2)}$ on L given by $(t_1, t_2) \cdot (y, v) := \bigl(y, t_1 \cdot t_2 \cdot v\bigr)$.

Let $\mathcal{T}(n_1, n_2)$ denote the trivial line bundle I with the $G_m^{(1)} \times G_m^{(2)}$ given by $(t_1, t_2) \cdot (y, u) = \bigl(y, t_1^{n_1} \cdot t_2^{n_2} \cdot u\bigr)$. Then we have the $G_m^{(1)} \times G_m^{(2)}$-line bundle $\pi^*\mathcal{T}(n_1, n_2)$ on L^*. We obtain the line bundle $\mathcal{I}_{n_2}$ on $L^*/G_m^{(2)}$, and we have the induced $G_m^{(1)}$-action on $\mathcal{I}_{n_2}$.

Lemma 2.50 *The weight of the $G_m^{(1)}$-action on $\mathcal{I}_{n_2}$ is $n_1 - n_2$.*

Proof We put $\widetilde{G}_m^{(i)} := \operatorname{Spec} k[s_i]$. Let us consider the morphism $\widetilde{G}_m^{(1)} \times \widetilde{G}_m^{(2)} \longrightarrow G_m^{(1)} \times G_m^{(2)}$ given by $(s_1, s_2) \longmapsto \bigl(s_1, s_1^{-1} \cdot s_2\bigr)$. The induced $\widetilde{G}_m^{(1)} \times \widetilde{G}_m^{(2)}$-action on L^* and $\mathcal{T}(n_1, n_2)$ is given by $(s_1, s_2) \cdot (y, v) = \bigl(y, s_2^r \cdot v\bigr)$ and $(s_1, s_2) \cdot (y, u) = \bigl(y, s_1^{n_1 - n_2} \cdot s_2^{n_2} \cdot u\bigr)$. Therefore, the weight of the $G_m^{(1)}$-action on $\mathcal{I}_{n_2}$ is given by $n_1 - n_2$. ∎

3 Parabolic L-Bradlow Pairs

In this section, we recall some definitions. All of them are more or less familiar. The purpose is to fix the meaning in this paper. In the following of this section, X will denote a smooth irreducible projective variety over an algebraically closed field k of characteristic 0. Let Pic_X denote the Picard variety of X. We fix a base point $x_0 \in X$, due to which we have the Poincaré bundle $\mathcal{P}oin_X$ on $\mathrm{Pic}_X \times X$.

3.1 Sheaves with some Structure and their Moduli Stacks

3.1.1 Orientation

Let E be a U-coherent sheaf on $U \times X$. Then we have the morphism $\det_E : U \longrightarrow \mathrm{Pic}_X$ induced by the determinant line bundle $\det(E)$ of E, which satisfies the condition $\det(E)_{|\{u\}\times X} \simeq \mathcal{P}oin_{X\,|\,\{\det_E(u)\}\times X}$. The morphism will be denoted by simply det, if there are no risk of confusion. In general, the line bundles $\det_E^* \mathcal{P}oin_X$ and $\det(E)$ are not isomorphic.

Example 3.1 Let c be an element of the second cohomology group $H^2(X)$, and let $\mathrm{Pic}_X(c)$ denote the Picard variety of the line bundles whose first Chern classes are c. Assume $H^i(X, L) = 0$ $(i > 0)$ for any line bundle $L \in \mathrm{Pic}_X(c)$. Then, $p_{X*}(\mathcal{P}oin_X)$ gives the vector bundle on $\mathrm{Pic}_X(c)$. We obtain the associated projective space bundle $\mathbb{P}_c = \mathbb{P}\big(p_{X*}(\mathcal{P}oin_X)^\vee\big)$ on $\mathrm{Pic}_X(c)$.

Let π denote the natural projection $\mathbb{P}_c \times X \longrightarrow \mathrm{Pic}_X \times X$. We have the line bundle $L(a) = \pi^*\mathcal{P}oin_X \otimes p_X^*\mathcal{O}_{\mathbb{P}_c}(a)$ for each $a \in \mathbb{Z}$. Here $\mathcal{O}_{\mathbb{P}_c}(1)$ denotes the tautological bundle of the projective space bundle $\mathbb{P}_c \longrightarrow \mathrm{Pic}_X(c)$, and $\mathcal{O}_{\mathbb{P}_c}(a) = \mathcal{O}_{\mathbb{P}_c}(1)^{\otimes a}$. The determinant bundle $\det\big(L(a)\big)$ is obviously $L(a)$ itself. On the other hand, $\det_{L(a)}$ is given by the projection π. Thus, $L(a)$ and $\det_{L(a)}^* \mathcal{P}oin_X$ are not isomorphic, if $a \neq 0$. ∎

Definition 3.2 (Orientation) *An orientation of a U-coherent sheaf E on $U \times X$ is defined to be an isomorphism $\rho : \det(E) \longrightarrow \det_E^* \mathcal{P}oin_X$ on U. A tuple (E, ρ) is called an oriented U-coherent sheaf.*

An isomorphism of two oriented sheaves (E, ρ) and (E', ρ') is defined to be an isomorphism $\chi : E \longrightarrow E'$ satisfying $\rho' = \chi^*(\rho) := \rho \circ \chi$. ∎

The restrictions $\big(\det_E^* \mathcal{P}oin_X\big)_{|\{u\}\times X}$ and $\det(E)_{|\{u\}\times X}$ are isomorphic for any point $u \in U$ by definition of $\det_E$, so that the push-forward $p_{X*}\mathcal{H}om\big(\det(E), \det_E^* \mathcal{P}oin_X\big)$ is the line bundle on U.

Definition 3.3 (Orientation bundle) *The line bundle $p_{X*}\mathcal{H}om\big(\det(E), \det_E^* \mathcal{P}oin_X\big)$ is called the orientation bundle of E. We denote it by $\mathcal{O}r(E)$.* ∎

If E is oriented, then the orientation bundle $\mathcal{O}r(E)$ is naturally isomorphic to the trivial line bundle $\mathcal{O}_U$, i.e., an orientation is equivalent to a trivialization of $\mathcal{O}r(E)$.

Example 3.4 Let $L(a)$ be given in Example 3.1. The orientation bundle $\mathcal{O}r(L(a), \mathbb{P}_c)$ induced by $L(a)$ is isomorphic to $p_{X*}\Big(\mathcal{H}om\big(L(a), \det{}^*_{L(a)}\mathcal{P}oin\big)\Big) \simeq p_{X*} \circ p_X^*\big(\mathcal{O}_{\mathbb{P}_c}(-a)\big) \simeq \mathcal{O}_{\mathbb{P}_c}(-a)$. ∎

We have the following additive property of the orientation bundles.

Lemma 3.5 *Let E_i $(i = 1, 2)$ be U-coherent sheaves on $U \times X$. Then we have the natural isomorphism $Or(E_1 \oplus E_2) \simeq Or(E_1) \otimes Or(E_2)$.*

Proof The natural isomorphism $\det(E_1 \oplus E_2) \simeq \det(E_1) \otimes \det(E_2)$ is given, and hence $\det_{E_1}^* \mathcal{P}oin_X \otimes \det_{E_2}^* \mathcal{P}oin_X \simeq \det_{E_1\oplus E_2}^* \mathcal{P}oin_X$. It induces the following isomorphism:

$$\mathcal{H}om\Big(\det(E_1 \oplus E_2), \det{}^*_{E_1\oplus E_2}\mathcal{P}oin_X\Big) \simeq \mathcal{H}om\big(\det(E_1), \det{}^*_{E_1}\mathcal{P}oin_X\big) \otimes \mathcal{H}om\big(\det(E_2), \det{}^*_{E_2}\mathcal{P}oin_X\big).$$

Therefore, we obtain the natural morphism $\mathcal{O}r(E_1) \otimes \mathcal{O}r(E_2) \longrightarrow \mathcal{O}r(E_1 \oplus E_2)$, which is isomorphic. ∎

3.1.2 Parabolic structure

See [41] and [61] for detail on the notion of parabolic sheaf. Our terminology slightly differs from theirs. We remark that it is different from that in the author's other papers ([46], for example.) Let D be a Cartier divisor of X. A U-parabolic sheaf, or simply parabolic sheaf, on $U \times (X, D)$ is defined to be a tuple $\big(E, F_*(E), \alpha_*\big)$:

- E is a U-coherent sheaf on $U \times X$.
- $F_*(E)$ denotes a filtration of E:

$$E = F_1(E) \supset F_2(E) \supset \cdots \supset F_l(E) \supset F_{l+1}(E) = E(-D).$$

  Here $E(-D)$ denotes $E \otimes p_U^* \mathcal{O}(-D)$. We assume that $\mathrm{Cok}_i(E) := E/F_{i+1}(E)$ are flat on U.
- $\alpha_* = (\alpha_1, \ldots, \alpha_l)$ is a tuple of numbers $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_l \leq 1$. It is called a system of weights.

A tuple(E, F_*, α_*) will be often denoted simply by E_*. The filtration F_* is called a quasi-parabolic structure. Isomorphisms of parabolic sheaves are defined naturally.

The number l is called the depth of the parabolic structure. The tuple α_* will be called a weight of the parabolic structure. For any parabolic sheaf E_*, we put $Gr_i(E) := F_i(E)/F_{i+1}(E)$, which are the U-coherent sheaves on $U \times D$.

Remark 3.6 *We will often use the word "parabolic" even when a system of weights is not given.* ∎

(Subobject and quotient object)
Let E_* be a parabolic torsion-free sheaf defined over $U \times (X, D)$. For any subsheaf $E' \subset E$ and any quotient sheaf $E \longrightarrow E''$, we have the induced parabolic structures on E' and E''. Namely we put $F_i(E') = F_i(E) \cap E'$, and $F_i(E'') = \mathrm{Im}(F_i(E) \longrightarrow E'')$. The parabolic structures are called the induced parabolic structures. We always consider the induced parabolic structures on the subsheaves and the quotient sheaves.

(The Condition O_m)
Let (E, F_*) be a U-quasi-parabolic sheaf on $U \times (X, D)$. Let m be a positive integer. We say that (E, F_*) satisfies the condition O_m, if the following holds:

- $F_i(E)(m)_{|\{u\} \times X}$ and $\mathrm{Cok}_i(E)(m)_{|\{u\} \times X}$ are generated by its global sections, for all $i = 1, \ldots, l+1$ and for all $u \in U$.
- The higher cohomology groups of $F_i(E)(m)_{|\{u\} \times X}$ and $\mathrm{Cok}_i(E)(m)_{|\{u\} \times X}$ vanish, for all $i = 1, \ldots, l+1$ and for all $u \in U$.

When we are given a U-quasi-parabolic sheaf (E, F_*) on $U \times (X, D)$, the open subset U' is determined by the condition O_m.

(Twist)
Let m be an integer, and let E be a U-coherent sheaf defined over $U \times X$. Recall that $E(m)$ denotes the coherent sheaf $E \otimes p_U^* \mathcal{O}_X(m)$. If E has a quasi parabolic structure $F_*(E)$, we have the naturally induced parabolic structure $F_*\big(E(m)\big)$ of $E(m)$. The tuple $\big(E(m), F_*(E(m))\big)$ is denoted by $E_*(m)$.

3.1.3 L-Bradlow pairs and reduced L-Bradlow pairs

Let L be a line bundle over X.

Definition 3.7 (L-section) *Let E be a U-coherent sheaf on $U \times X$. A morphism $\phi : p_U^* L \longrightarrow E$ is called an L-section. An $\mathcal{O}_X$-section is simply called a section.* ∎

Definition 3.8 *Let (E, ϕ) be a pair of a U-coherent sheaf on X and an L-section. We say that ϕ is non-trivial everywhere, if $\phi_{|\{u\} \times X} \neq 0$ for every point $u \in U$.* ∎

Definition 3.9

(L-Bradlow pair) *A parabolic L-Bradlow pair (E_*, ϕ) on $U \times (X, D)$ is a pair of a U-torsion-free parabolic sheaf E_* and an L-section $\phi : \pi_U^* L \longrightarrow E$.*

An isomorphism between two such pairs (E_*, ϕ) and (E'_*, ϕ') is defined to be an isomorphism $\chi : E_* \longrightarrow E'_*$ satisfying $\phi' = \chi_*(\phi) := \chi \circ \phi$.

(Oriented L-Bradlow pair) *An oriented parabolic L-Bradlow pair (E_*, ϕ, ρ) on $U \times (X, D)$ is a pair of a parabolic L-Bradlow pair (E_*, ϕ) and an orientation of E. An isomorphism of two such pairs is defined naturally.* ▮

Remark 3.10 *We are mainly interested in parabolic L-Bradlow pairs (E_*, ϕ) such that ϕ is non-trivial everywhere. Sometimes, we will assume it without mention.* ▮

Definition 3.11 *Let (E_*, ϕ) and (E'_*, ϕ') be parabolic L-Bradlow pairs on X. We say that (E'_*, ϕ') is a subobject of (E_*, ϕ) if the following conditions hold:*

- *E' is a subsheaf of E, and the parabolic structure is same as the induced one.*
- *If the image of ϕ is contained in E, we have $\phi' = \phi$. Otherwise, $\phi' = 0$.* ▮

We also introduce the notion of reduced L-section.

Definition 3.12 (Reduced L-section) *Let L be a line bundle over X. Let E be a U-coherent sheaf on $U \times X$. A reduced L-section of E is defined to be a pair $(M, [\phi])$ of a line bundle M on U and a morphism $[\phi] : p_X^*(M) \otimes p_U^*(L) \longrightarrow E$.*

A reduced L-section is often denoted simply by $[\phi]$ instead of $(M, [\phi])$, if there are no risk of confusion. ▮

Definition 3.13 *Let $(E, [\phi])$ be a pair of a U-coherent sheaf on $U \times X$ and a reduced L-section. We say that $[\phi]$ is non-trivial everywhere, if $[\phi]_{|\{u\} \times X} \neq 0$ for each $u \in U$.* ▮

Let $(M, [\phi])$ be a reduced L-section of E which is non-trivial everywhere. Then, $(M, [\phi])$ induces the morphism $\overline{[\phi]} : M \longrightarrow p_{X*}\mathcal{H}om(L, E)$. Under the vanishing $H^i\big(X, \mathcal{H}om(L, E)_{|\{u\} \times X}\big) = 0$ $(i > 0)$ for each $u \in U$, we obtain the naturally induced section $U \longrightarrow \mathbb{P}\big(p_{X\,*}(\mathcal{H}om(L, E))^\vee\big)$. On the other hand, such a section $U \longrightarrow \mathbb{P}\big(p_{X\,*}(\mathcal{H}om(L, E))^\vee\big)$ induces a reduced L-section which is non-trivial everywhere.

Definition 3.14 (Parabolic reduced L-Bradlow pair)

- *A parabolic reduced L-Bradlow pair $(E_*, M, [\phi])$ on $U \times (X, D)$ is defined to be a pair of a torsion-free parabolic sheaf E_* on $U \times (X, D)$ and a reduced L-section $(M, [\phi])$ which is non-trivial everywhere. It is often denoted by $(E_*, [\phi])$ instead of $(E_*, M, [\phi])$.*

  An isomorphism of two reduced L-Bradlow pairs $(E_{i\,*}, M_i, [\phi_i])$ $(i = 1, 2)$ is defined to be a pair (χ, η) of an isomorphism $\chi : E_{1\,*} \longrightarrow E_{2\,*}$ and $\eta : M_1 \simeq M_2$ such that the following diagram is commutative:

$$\begin{array}{ccc} p_X^* M_1 \otimes p_U^* L & \xrightarrow{\phi_1} & E_1 \\ {\scriptstyle \eta \otimes \mathrm{id}_L}\big\downarrow & & {\scriptstyle \chi}\big\downarrow \\ p_X^* M_2 \otimes p_U^* L & \xrightarrow{\phi_2} & E_2 \end{array}$$

- *An oriented parabolic reduced L-Bradlow pair is defined to be a tuple of a parabolic reduced L-Bradlow pair $(E_*, [\phi])$ with an orientation ρ of E.*

  An isomorphism between two oriented reduced L-Bradlow pair is naturally defined. ▮

Remark 3.15 *In the definition, we assume that $[\phi]$ is non-trivial everywhere, contrast to the definition of non-reduced L-Bradlow pair.*

Remark 3.16 *In the case $U = \mathrm{Spec}(k)$, a parabolic reduced L-Bradlow pair is just a parabolic L-Bradlow pair whose L-section is non-trivial, up to isomorphisms.* ▮

Remark 3.17 *We will often use the word "L-Bradlow pair" and "reduced L-Bradlow pair" instead of "parabolic L-Bradlow pair" and "parabolic reduced L-Bradlow pair", if there are no risk of confusion.*

We will also use the word "quasi-parabolic L-Bradlow pair" and "quasi-parabolic reduced L-Bradlow pair", if a system of weights is not given. ▮

We will use the following notion subordinately.

Definition 3.18 *Let $\boldsymbol{L}=(L_1,L_2)$ be a tuple of line bundles on X.*

- *A parabolic $\boldsymbol{L}$-Bradlow pair on $U\times(X,D)$ is defined to be a tuple $(E_*,\boldsymbol{\phi})$ of a U-parabolic torsion-free sheaf E_* on $U\times(X,D)$ and a pair $\boldsymbol{\phi}$ of L_i-sections ϕ_i $(i=1,2)$.*
- *An oriented parabolic reduced $\boldsymbol{L}$-Bradlow pair on $U\times(X,D)$ is defined to be a tuple $(E_*,[\boldsymbol{\phi}],\rho)$ of a U-parabolic torsion-free sheaf E_* on $U\times(X,D)$, a pair $[\boldsymbol{\phi}]$ of reduced L_i-sections $[\phi_i]$ which are non-trivial everywhere, and an orientation ρ. Isomorphisms are defined naturally.* ▮

3.1.4 Type and the moduli stacks

Let H^* denote an appropriate cohomology theory with the appropriate Chern class for vector bundles. We put $H^{\mathrm{ev}}(X):=\bigoplus_{i=0}^{\dim X}H^{2i}(X)$. If D is a smooth divisor, we put as follows:

$$\widetilde{\mathcal{T}ype}:=\Bigl\{(y,y_1,y_2,\ldots\ldots)\in H^{\mathrm{ev}}(X)\oplus\bigoplus_{i=1}^{\infty}H^{\mathrm{ev}}(Y)\,\Bigm|\,\sum_{i\geq 1}y_i=y_{|D}\Bigr\}.$$

In general, we put as follows:

$$\widetilde{\mathcal{T}ype}:=\Bigl\{(y,y_1,y_2,\ldots\ldots)\in H^{\mathrm{ev}}(X)\oplus\bigoplus_{i=1}^{\infty}H^{\mathrm{ev}}(X)\,\Bigm|\,\sum_{i\geq 1}y_i=y\cdot\Bigl(1-\mathrm{ch}\bigl(\mathcal{O}(-D)\bigr)\Bigr)\Bigr\}.$$

In the following, $y\cdot\Bigl(1-\mathrm{ch}\bigl(\mathcal{O}(-D)\bigr)\Bigr)$ is denoted by $y_{|D}$ for simplicity of the notation, even when D is not necessarily smooth.

For any quasi-parabolic sheaf (E,F_*) on X of depth l, we obtain the element of $\widetilde{\mathcal{T}ype}$:

$$\mathrm{type}\bigl(E,F_*\bigr):=\Bigl(\mathrm{ch}(E),\mathrm{ch}\bigl(\mathrm{Gr}_1(E)\bigr),\ldots,\mathrm{ch}\bigl(\mathrm{Gr}_{l-1}(E)\bigr),\mathrm{ch}\bigl(\mathrm{Gr}_l(E)\bigr),0,\ldots\Bigr)\in\widetilde{\mathcal{T}ype}$$

Let $\mathcal{T}ype$ denote the subset of $\widetilde{\mathcal{T}ype}$ which consists of $\mathrm{type}\bigl(E,F_*\bigr)$ for some quasi parabolic sheaves (E,F_*).

Let $\boldsymbol{y}=(y,y_1,y_2,\ldots\ldots)$ be any element of $\mathcal{T}ype$. The number $\mathrm{depth}(\boldsymbol{y}):=\max\{i\,|\,y_i\neq 0\}$ is called the depth of $\boldsymbol{y}$. The element y is called the $H^{\mathrm{ev}}(X)$-component of $\boldsymbol{y}$, and $(y_1,y_2,\ldots,)$ is called the parabolic part of $\boldsymbol{y}$. The $H^0(X)$-component of y is called the rank of $\boldsymbol{y}$, and it is denoted by $\mathrm{rank}(\boldsymbol{y})$ or $\mathrm{rank}\,y$. We put $\mathcal{T}ype_r:=\bigl\{\boldsymbol{y}\in\mathcal{T}ype\bigm|\mathrm{rank}(\boldsymbol{y})=r\bigr\}$. We denote by $\mathcal{T}ype^{\circ}$ the set of types whose parabolic components are trivial, i.e., $y_1=y_{|D}$ and $y_i=0$ for $i>1$. We often identify y and $\boldsymbol{y}$ in that case, and we regard $\mathcal{T}ype^{\circ}$ as the subset of $H^*(X)$. We put $\mathcal{T}ype^{\circ}_r:=\mathcal{T}ype^{\circ}\cap\mathcal{T}ype_r$.

We have the sum $\boldsymbol{y}^{(1)}+\boldsymbol{y}^{(2)}$ of two elements $\boldsymbol{y}^{(i)}\in\mathcal{T}ype$ $(i=1,2)$ by taking the component summation.

For any quasi-parabolic sheaf (E,F_*) on X, we obtain the element $\mathrm{type}\bigl(E,F_*\bigr)$ of $\mathcal{T}ype$. In general, let (E,F_*) be a U-quasi-parabolic sheaf on $U\times X$. When we have the element $\boldsymbol{y}\in\mathcal{T}ype$ such that $\mathrm{type}(E_{*|u})=\boldsymbol{y}$ for any closed point $u\in U$, then E_* is called of type $\boldsymbol{y}$. When U is connected, such an element always exists.

Definition 3.19 *The type of an (oriented) parabolic L-Bradlow pairs is defined to be the type of the underlying quasi-parabolic sheaf.* ▮

We introduce the following notation.

Notation 3.20 *In each line, the left hand side denotes the moduli stack of the object in the right hand side:*

$\mathcal{M}(\boldsymbol{y})$: *Parabolic sheaves of type $\boldsymbol{y}$.*

$\mathcal{M}(\widehat{\boldsymbol{y}})$**:** *Oriented parabolic sheaves of type* $\boldsymbol{y}$*.*

$\mathcal{M}(\boldsymbol{y},L)$**:** *Parabolic L-Bradlow pairs of type* $\boldsymbol{y}$ *whose L-sections are non-trivial everywhere.*

$\mathcal{M}(\widehat{\boldsymbol{y}},L)$**:** *Oriented parabolic L-Bradlow pairs of type* $\boldsymbol{y}$ *whose L-sections are non-trivial everywhere.*

$\mathcal{M}(\boldsymbol{y},[L])$**:** *Parabolic reduced L-Bradlow pairs of type* $\boldsymbol{y}$*.*

$\mathcal{M}(\widehat{\boldsymbol{y}},[L])$**:** *Oriented parabolic reduced L-Bradlow pairs of type* $\boldsymbol{y}$*.*

$\mathcal{M}(\widehat{\boldsymbol{y}},[\boldsymbol{L}])$**:** *Oriented parabolic reduced* $\boldsymbol{L}$*-Bradlow pairs of type* $\boldsymbol{y}$*.* ∎

The condition O_m determines the open substack of each moduli stack. They are denoted by $\mathcal{M}(m,\boldsymbol{y})$, $\mathcal{M}(m,\widehat{\boldsymbol{y}})$, $\mathcal{M}(m,\boldsymbol{y},L)$, $\mathcal{M}(m,\widehat{\boldsymbol{y}},L)$ $\mathcal{M}(m,\boldsymbol{y},[L])$, $\mathcal{M}(m,\widehat{\boldsymbol{y}},[L])$ and $\mathcal{M}(m,\widehat{\boldsymbol{y}},[\boldsymbol{L}])$ respectively. When the parabolic part of $\boldsymbol{y}$ is trivial, we often use the notation $\mathcal{M}(y)$, $\mathcal{M}(\widehat{y})$, $\mathcal{M}(m,y)$, $\mathcal{M}(m,\widehat{y})$, etc.

3.1.5 The tautological line bundle and the relations among some moduli stacks

Let $\boldsymbol{y}$ be an element of $\mathcal{T}ype$, and let L be a line bundle on X. Let $\widehat{\mathcal{E}}^u(L)$, $\mathcal{E}^u(L)$ and $\widehat{\mathcal{E}}^u[L]$ denote the universal sheaves over $\mathcal{M}(\widehat{\boldsymbol{y}},L)\times X$, $\mathcal{M}(\boldsymbol{y},L)\times X$ and $\mathcal{M}(\widehat{\boldsymbol{y}},[L])\times X$ respectively. The universal L-sections of $\widehat{\mathcal{E}}^u(L)$ and $\mathcal{E}^u(L)$ are denoted by $\widehat{\phi}^u$ and ϕ^u, respectively. The universal reduced L-section of $\widehat{\mathcal{E}}^u[L]$ is denoted by $[\widehat{\phi}^u]$.

We have the G_m-action ρ_1 on $\mathcal{M}(\widehat{\boldsymbol{y}},L)$ given by $\rho_1(t)\cdot(E,F_*,\phi,\rho):=\bigl(E,F_*,t\cdot\phi,\rho\bigr)$. It is easy to observe that the quotient stack is isomorphic to $\mathcal{M}(\widehat{\boldsymbol{y}},[L])$. Therefore, we can regard $\mathcal{M}(\widehat{\boldsymbol{y}},L)$ as the G_m-torsor over $\mathcal{M}(\widehat{\boldsymbol{y}},[L])$. The associated line bundle is denoted by $\mathcal{O}_{\mathrm{rel}}(-1)$. We put $\mathcal{O}_{\mathrm{rel}}(1):=\mathcal{O}_{\mathrm{rel}}(-1)^{\vee}$, and we obtain $\mathcal{O}_{\mathrm{rel}}(n)$ in the obvious manner.

Definition 3.21 *The line bundle $\mathcal{O}_{\mathrm{rel}}(1)$ is called the relative tautological line bundle of $\mathcal{M}(\widehat{\boldsymbol{y}},[L])$. It is also called the tautological line bundle.* ∎

We can obtain the same line bundles by the way of the subsubsection 2.7.1. Namely, let $\mathcal{T}(n)$ denote the trivial line bundle on $\mathcal{M}(\widehat{\boldsymbol{y}},[L])$ provided with the G_m-action of the weight n. Let $\pi_1:\mathcal{M}(\widehat{\boldsymbol{y}},L)\longrightarrow\mathcal{M}(\widehat{\boldsymbol{y}},[L])$ denote the natural projection. Then we obtain the G_m-line bundle $\pi_1^*\mathcal{T}(n)$. By taking the descent, we obtain the line bundle $\mathcal{I}_n$. Then we have $\mathcal{I}_n\simeq\mathcal{O}_{\mathrm{rel}}(n)$.

We remark that $\mathcal{O}_{\mathrm{rel}}(-1)$ appears in the domain of the universal reduced L-section $[\widehat{\phi}^u]$. Namely, $[\widehat{\phi}^u]$ is the morphism of $p^*_{\mathcal{M}(\widehat{\boldsymbol{y}},[L])}L\otimes p_X^*\mathcal{O}_{\mathrm{rel}}(-1)\longrightarrow\widehat{\mathcal{E}}^u[L]$. To see that, we have only to observe that $\pi^*[\widehat{\phi}^u]$: $p^*_{\mathcal{M}(\widehat{\boldsymbol{y}},L)}L\otimes\pi_{1,X}^*\mathcal{T}(-1)\longrightarrow\pi_{1,X}^*\widehat{\mathcal{E}}^u[L]$ is equivariant with respect to ρ_1, and they give the universal objects over $\mathcal{M}(\widehat{\boldsymbol{y}},L)\times X$.

Remark 3.22 *For example, we have the projective space bundle such as $\mathcal{M}(m,\widehat{\boldsymbol{y}},[\mathcal{O}(-m)])\longrightarrow\mathcal{M}(m,\widehat{\boldsymbol{y}})$, and the restriction of $\mathcal{O}_{\mathrm{rel}}$ to $\mathcal{M}(m,\widehat{\boldsymbol{y}},[\mathcal{O}(-m)])$ is the relative tautological line bundle of this bundle.* ∎

On the other hand, we have the G_m-action ρ_2 on $\mathcal{M}(\widehat{\boldsymbol{y}},L)$ given by $\rho_2(t)\cdot(E,F_*,\phi,\rho):=(E,F_*,\phi,t\cdot\rho)$. It is easy to observe that the quotient stack is isomorphic to $\mathcal{M}(\boldsymbol{y},L)$. Thus, we can regard $\mathcal{M}(\widehat{\boldsymbol{y}},L)$ as the G_m-torsor on $\mathcal{M}(\boldsymbol{y},L)$. The associated line bundle is clearly isomorphic to the orientation bundle $\mathcal{O}r\bigl(\mathcal{E}^u(L)\bigr)$.

Let r be the rank of $\boldsymbol{y}$. The obvious multiplication of E gives the isomorphism $(E,t^{-1}\cdot\phi,t^r\rho)\simeq(E,\phi,\rho)$. To see it, note the following: Let $f:E_1\longrightarrow E_2$ be an isomorphism. Then the orientation ρ of E_2 induces the orientation of E_1 as follows:

$$\det(E_1)\xrightarrow{\det(f)}\det(E_2)\xrightarrow{\ \rho\ }\det{}^*\mathcal{P}oin.$$

On the other hand, the L-morphism ϕ of E_2 induces the L-morphism of E_1 as follows:

$$L\xrightarrow{\ \phi\ }E_2\xrightarrow{f^{-1}}E_1.$$

Therefore, we have $\rho_1(t)=\rho_2(t^r)$. Hence we have the naturally defined morphism $\kappa:\mathcal{M}(\widehat{\boldsymbol{y}},[L])\longrightarrow\mathcal{M}(\boldsymbol{y},L)$, which is etale and finite of degree r^{-1}. However, the morphism κ does not preserve the universal object. We have the relation $\kappa_X^*\mathcal{E}^u(L)\simeq\widehat{\mathcal{E}}^u[L]\otimes\mathcal{O}_{\mathrm{rel}}(1)$, and hence $\kappa^*\mathcal{O}r\bigl(\mathcal{E}^u(L)\bigr)\simeq\mathcal{O}_{\mathrm{rel}}(r)$.

3.2 Hilbert Polynomials

3.2.1 Hilbert polynomials of coherent sheaves

Let $\mathcal{O}_X(1)$ be an ample line bundle on a projective variety X. Let E be a coherent sheaf on X. In this paper, the non-reduced Hilbert polynomial of E is denoted by H_E, i.e., H_E is the unique polynomial of $\mathbb{Q}$-coefficients such that $H_E(m)$ is same as the Euler number $\sum(-1)^i \dim H^i\big(X, E(m)\big)$. In the case $\operatorname{rank}(E) > 0$, the reduced Hilbert polynomial of E is denoted by P_E, i.e., $P_E := H_E/\operatorname{rank}(E)$.

We also use the notation $h^0(E)$ to denote $\dim H^0(X,E)$.

3.2.2 Hilbert polynomials of parabolic sheaves

We recall the parabolic Hilbert polynomials and the parabolic degree, which were introduced by Maruyama and Yokogawa in [41]. See [41] for more detail.

Let $E_* := (E, F_*, \alpha_*)$ be a parabolic sheaves of depth l. We put $\epsilon_i := \alpha_{i+1} - \alpha_i$ $(i = 1, \ldots, l)$. Recall that the non-reduced parabolic Hilbert polynomial H_{E_*} is defined to be as follows:

$$H_{E_*}(t) := H_{E(-D)}(t) + \sum_{i=1}^{l} \alpha_i \cdot H_{\mathrm{Gr}_i(E)} = H_E(t) - \sum_{i=1}^{l} \epsilon_i \cdot H_{\mathrm{Cok}_i(E)}.$$

The reduced parabolic Hilbert polynomial $P_{E_*}(t)$ is defined to be $P_{E_*}(t) := H_{E_*}(t)\big/\operatorname{rank}(E)$.

Since we have the equality $H_E = H_{E(-D)} + \sum_{i=1}^{l} H_{\mathrm{Gr}_i(E)}$, we obtain the following lemma.

Lemma 3.23 *We have the inequality $H_{E_*}(t) \leq H_E(t)$ and $P_{E_*}(t) \leq P_E(t)$ for any sufficiently large t.* ▮

The parabolic degree is defined to be as follows:

$$\operatorname{par-deg}(E_*) = \deg(E) + (\dim X - 1)! \times \Big(\text{the coefficient of } t^{\dim X - 1} \text{ of the polynomial } \textstyle\sum_{i=1}^{l} \alpha_i \cdot H_{\mathrm{Gr}_i}(t)\Big)$$

Note that we have the inequality $\deg(E) \leq \operatorname{par-deg}(E_*)$.

The parabolic slope $\mu(E_*)$ is defined to be $\operatorname{par-deg}(E_*)/\operatorname{rank}(E)$. Then we have the inequality $\mu(E) \leq \mu(E_*) \leq \mu(E) + \deg(D)$ for the usual slope $\mu(E)$ of a torsion-free sheaf E.

We also put $h^0(E_*) := \alpha_1 h^0\big(E(-D)\big) + \sum \epsilon_i h^0\big(F_{i+1}(E)\big)$.

3.2.3 Hilbert polynomial for parabolic L-Bradlow pairs

We recall the Hilbert polynomials for Bradlow pairs, following [54].

Notation 3.24 *Let $\mathcal{P}^{\mathrm{br}}$ denote the set of polynomials δ of $\mathbb{R}$-coefficients such that $\deg(\delta) \leq \dim X - 1$ and that $\delta(t) > 0$ for any sufficiently large t*

For any element $\delta \in \mathcal{P}^{\mathrm{br}}$, the coefficient of t^{d-1} in δ is denoted by δ_{top}, which may be 0.

The total order $\leq$ on the set $\mathcal{P}^{\mathrm{br}}$ is defined as follows: Let δ and δ' be elements of $\mathcal{P}^{\mathrm{br}}$. Then $\delta \leq \delta'$ if and only if $\delta(t) \leq \delta'(t)$ for any sufficiently large t. ▮

Let (E_*, ϕ) be a parabolic L-Bradlow pair on (X, D). For any $\delta \in \mathcal{P}^{\mathrm{br}}$, the non-reduced δ-Hilbert polynomial $H^{\delta}_{(E_*,\phi)}$ of (E_*, ϕ) is defined as follows:

$$H^{\delta}_{(E_*,\phi)} := H_{E_*} + \epsilon(E_*, \phi) \cdot \delta, \quad \epsilon(E_*, \phi) := \begin{cases} 1 & (\phi \neq 0) \\ 0 & (\phi = 0) \end{cases} \tag{29}$$

The reduced δ-Hilbert polynomial is defined to be $P^{\delta}_{(E_*,\phi)} := H^{\delta}_{(E_*,\phi)}/\operatorname{rank} E$. Similarly, the slope $\mu^{\delta}_{(E_*,\phi)}$ is defined by $\mu^{\delta}(E_*, \phi) := \mu(E_*) + \epsilon(\phi) \cdot \delta_{\mathrm{top}}/\operatorname{rank}(E)$.

Let $\boldsymbol{L} = (L_1, L_2)$ be a pair of line bundles on X. We take $\boldsymbol{\delta} := (\delta_1, \delta_2) \in \big(\mathcal{P}^{\mathrm{br}}\big)^2$. Let $(E_*, \boldsymbol{\phi})$ on (X, D) be a parabolic $\boldsymbol{L}$-Bradlow pair. The $\boldsymbol{\delta}$-Hilbert polynomial is defined by $H^{\boldsymbol{\delta}}_{(E_*,\boldsymbol{\phi})} := H_{E_*} + \sum_{i=1,2} \epsilon(E_*, \phi_i) \cdot \delta_i$, where $\epsilon(E_*, \phi_i)$ are given as in (29). We also put $P^{\boldsymbol{\delta}}_{(E_*,\boldsymbol{\phi})} = H^{\boldsymbol{\delta}}_{(E_*,\boldsymbol{\phi})}/\operatorname{rank} E$.

3.2.4 The Hilbert polynomials associated to a type

Let z be any element of $\bigoplus_{i\geq 1} H^i(X)$. Since we have $z^k = 0$ for a large integer k, we have the polynomial $\exp(t\cdot z) = \sum_{i=0}^{\infty}(k!)^{-1}(t\cdot z)^k$. In the case $c = c_1\bigl(\mathcal{O}_X(1)\bigr)$, it will be denoted by $\mathrm{ch}\bigl(\mathcal{O}_X(t)\bigr)$ in the following. When we substitute $t = m$ for some integer m, it is same as the ordinary meaning.

Let $\boldsymbol{y} = (y, y_1, y_2, \dots,)$ be an element of $\mathcal{T}ype$. We put as follows:

$$H_{\boldsymbol{y}}(t) := \int_X \mathrm{Td}(X)\cdot \mathrm{ch}\bigl(\mathcal{O}_X(t)\bigr)\cdot y, \quad H_{\boldsymbol{y},i}(t) := \int_X \mathrm{Td}(X)\cdot \mathrm{ch}\bigl(\mathcal{O}_X(t)\bigr)\cdot \sum_{j\leq i} y_j.$$

When D is smooth and we regard y_i as an element of $H^*(D)$, we put $H_{\boldsymbol{y},i}(t) := \int_D \mathrm{Td}(D)\cdot\mathrm{ch}\bigl(\mathcal{O}_X(t)\bigr)\cdot\sum_{j\leq i} y_j$. When the parabolic part of $\boldsymbol{y}$ is trivial, we use the notation $H_y(t)$ instead of $H_{\boldsymbol{y}}(t)$.

If we are given a system of weights α_*, we put $\epsilon_i := \alpha_{i+1} - \alpha_i$. And we put as follows:

$$H_{\boldsymbol{y}}^{\alpha_*} := H_{\boldsymbol{y}} - \sum \epsilon_i \cdot H_{\boldsymbol{y},i}, \quad P_{\boldsymbol{y}}^{\alpha_*} := \frac{H_{\boldsymbol{y}}^{\alpha_*}}{\mathrm{rank}\,\boldsymbol{y}}.$$

We also put as follows:

$$\deg(\boldsymbol{y},\alpha_*) := \int_X y\cdot c_1\bigl(\mathcal{O}_X(1)\bigr) - \sum\epsilon_i\cdot\int_X\sum_{j\leq i} y_j\cdot c_1\bigl(\mathcal{O}_X(1)\bigr), \quad \mu(\boldsymbol{y},\alpha_*) := \frac{\deg(\boldsymbol{y},\alpha_*)}{\mathrm{rank}\,\boldsymbol{y}}$$

If we are given an element $\delta\in\mathcal{P}^{\mathrm{br}}$, we put as follows:

$$H_{\boldsymbol{y}}^{\alpha_*\delta} := H^{\alpha_*} + \delta, \quad P_{\boldsymbol{y}}^{\alpha_*\delta} := \frac{H_{\boldsymbol{y}}^{\alpha_*\delta}}{\mathrm{rank}\,\boldsymbol{y}}, \quad \mu(\boldsymbol{y},\alpha_*,\delta) := \mu(\boldsymbol{y},\alpha_*) + \frac{\delta_{top}}{\mathrm{rank}(\boldsymbol{y})}$$

3.3 Semistability

3.3.1 Semistability and the moduli stacks

Let (E_*,ϕ) be a parabolic L-Bradlow pair on (X,D), and let δ be an element of $\mathcal{P}^{\mathrm{br}}$. Recall that (E_*,ϕ) is called δ-semistable, if the following inequality holds for each sub-objects (E'_*,ϕ') of (E,ϕ):

$$P^{\delta}_{(E'_*,\phi')}(t) \leq P^{\delta}_{(E_*,\phi)}(t) \quad (t \text{ is sufficiently large}) \tag{30}$$

If the strict inequality holds in (30) for each subobject, (E_*,ϕ) is called δ-stable.

Since parabolic reduced L-Bradlow pairs on X are just parabolic L-Bradlow pairs whose L-section is non-trivial, the notion of δ-(semi)stability of parabolic reduced L-Bradlow pairs is also given. The δ-(semi)stability of oriented parabolic (reduced) L-Bradlow pairs on X is defined by the δ-(semi)stability of the underlying parabolic (reduced) L-Bradlow pairs.

Remark 3.25 *Let E be a torsion-free sheaf on X. We can regard it as the parabolic sheaf E_* canonically. The quasi-parabolic structure is given by $F_2 = E(-D) \subset F_1 = E$. The weight is given by $\alpha_1 = 1$. Then we have $H_{E_*} = H_E$, and hence the semistability of (E_*,ϕ) is equivalent to the semistability of (E,ϕ).*

If the weight is given by $\alpha_1 = 0$, then we have $H_{E_*} = H_{E(-D)}$, and hence the semistability of (E_*,ϕ) and (E,ϕ) are not equivalent, as remarked in Remark 1.1.1 *in* [41]. ∎

We also have the δ-μ-semistability in the standard way. Namely, a parabolic L-Bradlow pair (E_*,ϕ) on (X,D) is called δ-μ-semistable, if the inequality $\mu^{\delta}(E'_*,\phi') \leq \mu^{\delta}(E_*,\phi)$ holds for any sub-objects $(E'_*,\phi')\subset(E_*,\phi)$. If the strict inequality holds for any subjects, (E_*,ϕ) is called δ-μ-stable. It is easy to check the implication:

$$\delta\text{-}\mu\text{-stable} \Longrightarrow \delta\text{-stable} \Longrightarrow \delta\text{-semistable} \Longrightarrow \delta\text{-}\mu\text{-semistable}$$

Similarly, we have the notion of δ-μ-semistability and δ-μ-stability of parabolic reduced L-Bradlow pairs.

In the family case, a U-parabolic L-Bradlow pair (E_*,ϕ) on $U\times X$ is called δ-(semi)stable, if $(E_*,\phi)_{|\{u\}\times X}$ is δ-(semi)stable for each $u\in U$. We obtain the δ-(semi)stability in the oriented case and in the reduced case similarly.

Remark 3.26 *As usual, we have only to consider sub-objects* $(E'_*, \phi') \subset (E_*, \phi)$ *such that* E' *is saturated, when we check the* δ*-(semi)stability of* (E_*, ϕ). ∎

Let L_i $(i = 1, 2)$ be line bundles on X, and let $\iota : L_1 \longrightarrow L_2$ be a non-trivial morphism. Let (E_*, ϕ) be a parabolic L_2-Bradlow pair on (X, D). Then we obtain the parabolic L_1-Bradlow pair $(E, \phi \circ \iota)$. When we consider the semistable condition for a parabolic L-Bradlow pair, the choice of L is not essential in the following sense.

Lemma 3.27 *A pair* (E_*, ϕ) *is* δ*-semistable, if and only if* $(E_*, \phi \circ \iota)$ *is* δ*-semistable.*

Proof Let F be a saturated subsheaf of E. The image $\phi \circ \iota(L_1)$ is contained in F if and only if the image $\phi(L_2)$ is contained in F. Thus the claim is clear. ∎

Lemma 3.28 *Let* $(E_{i\,*}, \phi_i)$ $(i = 1, 2)$ *be* δ*-semistable parabolic* L*-Bradlow pairs with* $P^{\delta}_{(E_{1\,*}, \phi_1)}(t) = P^{\delta}_{(E_{2\,*}, \phi_2)}(t)$. *Let* $f : (E_{1\,*}, \phi_1) \longrightarrow (E_{2\,*}, \phi_2)$ *be a non-trivial morphism. We have the induced* L*-Bradlow pairs* $(\mathrm{Ker}(f)_*, \phi')$, $(\mathrm{Im}(f)_*, \phi'')$ *and* $(\mathrm{Cok}(f)_*, \phi''')$. *Then they are also* δ*-semistable.*

A similar claim holds for δ*-*μ*-semistability.*

Proof We put $(E_{3\,*}, \phi_3) := \big(\mathrm{Im}(f)_*, \phi''\big)$. From the δ-semistability of $(E_{i\,*}, \phi_i)$ $(i = 1, 2)$, we obtain the inequality for sufficiently large t:

$$P^{\delta}_{(E_{1\,*}, \phi_1)}(t) \leq P^{\delta}_{(E_{3\,*}, \phi_3)}(t) \leq P^{\delta}_{(E_{2\,*}, \phi_2)}(t). \tag{31}$$

Due to the assumption $P^{\delta}_{(E_{1\,*}, \phi_1)}(t) = P^{\delta}_{(E_{2\,*}, \phi_2)}(t)$, the equalities hold in (31). Then, it is easy to derive the claims of the lemma by definition of semistability. ∎

Corollary 3.29 *Any automorphisms of stable objects are constant multiplication.* ∎

We introduce the following notation. In each line, the left hand side denotes the moduli stack of the object in the right hand side:

$\mathcal{M}^s(\boldsymbol{y}, \alpha_*)$: stable parabolic sheaves of type $\boldsymbol{y}$ with weight α_*.

$\mathcal{M}^s(\widehat{\boldsymbol{y}}, \alpha_*)$: stable oriented parabolic sheaves of type $\boldsymbol{y}$ with weight α_*.

$\mathcal{M}^s(\boldsymbol{y}, L, \alpha_*, \delta)$: δ-stable L-Bradlow pairs of type $\boldsymbol{y}$ with weight α_*, whose L-sections are non-trivial everywhere.

$\mathcal{M}^s(\widehat{\boldsymbol{y}}, L, \alpha_*, \delta)$: δ-stable oriented L-Bradlow pairs of type $\boldsymbol{y}$ with weight α_*, whose L-sections are non-trivial everywhere.

$\mathcal{M}^s(\boldsymbol{y}, [L], \alpha_*, \delta)$: δ-stable reduced L-Bradlow pairs of type $\boldsymbol{y}$ with weight α_*.

$\mathcal{M}^s(\widehat{\boldsymbol{y}}, [L], \alpha_*, \delta)$: δ-stable oriented reduced L-Bradlow pairs of type $\boldsymbol{y}$ with weight α_*.

For the moduli stack of semistable objects, we use the notation $\mathcal{M}^{ss}(\boldsymbol{y}, \alpha_*)$, $\mathcal{M}^{ss}(\widehat{\boldsymbol{y}}, \alpha_*)$, $\mathcal{M}^{ss}(\boldsymbol{y}, L, \alpha_*, \delta)$, $\mathcal{M}^{ss}(\widehat{\boldsymbol{y}}, [L], \alpha_*, \delta)$, etc.. When the parabolic structure of $\boldsymbol{y}$ is trivial, we often use the notation $\mathcal{M}^s(\widehat{y})$, $\mathcal{M}^s(\widehat{y}, [L], \delta)$, $\mathcal{M}^s(y, L, \delta)$, etc..

Similarly, we also have the notion of $\boldsymbol{\delta}$-(semi)stability for (oriented, reduced) $\boldsymbol{L}$-Bradlow pairs. We also have $\boldsymbol{\delta}$-μ-(semi)stability. We denote by $\mathcal{M}^s(\widehat{\boldsymbol{y}}, [\boldsymbol{L}], \alpha_*, \boldsymbol{\delta})$ the moduli stack of (δ_1, δ_2)-stable oriented reduced $\boldsymbol{L}$-Bradlow pairs of type $\boldsymbol{y}$ with weight α_*, for example.

When any $E_* \in \mathcal{M}^s(\widehat{\boldsymbol{y}}, \alpha_*)$ satisfies the condition O_m, we denote by $\widetilde{\mathcal{M}}^s_m(\widehat{\boldsymbol{y}}, \alpha_*)$ the full flag bundle associated to the vector bundle $p_{X\,*}\widehat{E}^u(m)$, where $\widehat{E}^u$ denotes the universal bundle over $\mathcal{M}^s_m(\widehat{\boldsymbol{y}}, \alpha_*) \times X$. We use the notation $\widetilde{\mathcal{M}}^s_m(\widehat{\boldsymbol{y}}, [L], \alpha_*, \delta)$ and $\widetilde{\mathcal{M}}^s_m(\widehat{\boldsymbol{y}}, [\boldsymbol{L}], \alpha_*, \boldsymbol{\delta})$, etc. in similar ways. If there are no risk of confusion, we will often omit to denote m.

3.3.2 Harder-Narasimhan filtration and partial Jordan-Hölder filtration

Let (E_*, ϕ) be a δ-semistable parabolic L-Bradlow pair. In this paper, a filtration

$$(E_{1\,*}, \phi_1) \subset (E_{2\,*}, \phi_2) \subset \cdots \subset (E_{k\,*}, \phi_k) = (E_*, \phi)$$

is called a partial Jordan-Hölder filtration with respect to δ-semistability, if each $(E_{i\,*}, \phi_i)$ is δ-semistable such that $P^{\delta}_{(E_{i\,*},\phi_i)} = P^{\delta}_{(E_*,\phi)}$. Each $\mathrm{Gr}_i(E) := E_i/E_{i-1}$ has the induced parabolic structure and the L-section $\mathrm{Gr}_i(\phi)$, and the parabolic L-Bradlow pair $\big(\mathrm{Gr}_{i\,*}(E), \mathrm{Gr}_i(\phi)\big)$ is δ-semistable with $P^{\delta}_{(\mathrm{Gr}_{i\,*}(E),\mathrm{Gr}_i(\phi))} = P^{\delta}_{(E_*,\phi)}$.

If each $\big(\mathrm{Gr}_{i\,*}(E), \mathrm{Gr}_i(\phi)\big)$ is δ-stable, the filtration is called a Jordan-Hölder filtration with respect to δ-stability. It can be shown that the length of Jordan-Hölder filtration and the collection of graded objects $\big\{\big(\mathrm{Gr}_{i\,*}(E), \mathrm{Gr}_i(\phi)\big)\big\}$ are independent of the choice of Jordan-Hölder filtration.

Similarly, we have the notion of partial Jordan-Hölder filtration with respect to δ-μ-semistability and Jordan-Hölder filtration.

Lemma 3.30 *Let (E_*, ϕ) be a parabolic L-Bradlow pair. There exists the unique increasing filtration $F = \big(F_i(E)\,|\, i = 1, 2, \dots\big)$ of E with the following properties:*

- *The induced objects $\big(\mathrm{Gr}^F_i(E), \phi'_i\big)$ are δ-semistable.*
- *The inequalities $P^{\delta}_{(\mathrm{Gr}^F_i(E),\phi'_i)}(t) > P^{\delta}_{(\mathrm{Gr}^F_{i+1}(E),\phi'_{i+1})}(t)$ hold, for any sufficiently large t.*

The filtration is called the Harder-Narasimhan filtration with respect to the δ-semistability.

We also have the unique Harder-Narasimhan filtration with respect to the δ-μ-semistability.

Proof We give only an outline. We use an induction on $\mathrm{rank}(E)$. In the case $\mathrm{rank}(E) = 1$, the claim is trivial. Take a sufficiently negative number C. We know that the family of saturated subsheaves E' of E such that $\deg(E') \geq C$ is bounded (Proposition 3.34). Therefore, the family of saturated subsheaves E' of E such that $P^{\delta}_{(E'_*,\phi')} \geq P^{\delta}_{(E_*,\phi)}$ is bounded, where (E'_*, ϕ') denotes the subobject of (E_*, ϕ). Hence we obtain the finiteness of the set S of the polynomials P with the following properties:

- $P(t) \geq P_{(E_*,\phi)}(t)$ for any sufficiently large t.
- There exist a saturated subsheaf E' of E such that $P^{\delta}_{(E'_*,\phi')} = P$.

We say $P \leq' Q$ if $P(t) \leq Q(t)$ for any sufficiently large t. It gives the total order of S. Let P_0 be the maximum. Let $T(P_0)$ denote the family of saturated subsheaves E' of E such that $P^{\delta}_{(E'_*,\phi)} = P_0$. Then it is easy to see that (E'_*, ϕ) is δ-semistable for any $E' \in T(P_0)$. It is also easy to see that $E'_1 + E'_2 \in T(P_0)$ for $E'_i \in T(P_0)$. Therefore, we have the maximum E_1 in $T(P_0)$ with respect to the inclusion.

We put $\widehat{E} := E/E_1$, and let $\pi : E \longrightarrow \widehat{E}$ denote the natural projection. Let $\widehat{E}' \subset \widehat{E}$ be any saturated subsheaf. If $P^{\delta}_{(\widehat{E}',\widehat{\phi}')} \geq P^{\delta}_{(E_1,\phi_1)}$, then we have $P^{\delta}_{(\pi^{-1}(\widehat{E}'),\phi'')} \geq P^{\delta}_{(E_1,\phi_1)}$, which contradicts our choice of E_1. Therefore, we have $P^{\delta}_{(\widehat{E}',\widehat{\phi}')} < P^{\delta}_{(E_1,\phi_1)}$. By applying the hypothesis of the induction, we have the Harder-Narasimhan filtration of $(\widehat{E}_*, \widehat{\phi})$ with respect to δ-semistability. Together with the above remark, we obtain the Harder-Narasimhan filtration of (E_*, ϕ) with respect to δ-semistability. ∎

3.3.3 (δ, ℓ)-Semistability

Let P be a polynomial, and let r be a positive integer. Let m be a sufficiently large integer satisfying the following condition:

- Let (E_*, ϕ) be a δ-parabolic L-Bradlow pair with $P^{\delta}_{(E_*,\phi)} = P$ and $\mathrm{rank}\, E \leq r$. Then E_* satisfies the condition O_m.

Definition 3.31 *Let $(E_*, [\phi])$ be a parabolic reduced L-Bradlow pair on X with $P^{\delta}_{(E_*,\phi)} = P$, and let $\mathcal{F}$ be a full flag of $H^0\big(X, E(m)\big)$. Let ℓ be any positive integer. We say that $(E_*, [\phi], \mathcal{F})$ is (δ, ℓ)-semistable, if the following condition holds:*

- $(E_*, [\phi])$ *is* δ*-semistable.*
- *Take any partial Jordan-Hölder filtration of* $(E_*, [\phi])$ *with respect to* δ*-semistability:*
$$E_*^{(1)} \subset E_*^{(2)} \subset \cdots \subset E_*^{(i-1)} \subset (E_*^{(i)}, \phi) \subset \cdots \subset (E_*^{(k)}, \phi)$$
Then we have $\mathcal{F}_\ell \cap H^0\big(X, E^{(i-1)}(m)\big) = \{0\}$ *and* $\mathcal{F}_\ell \not\subset H^0\big(X, E^{(j)}(m)\big)$ *for* $j < k$.

We denote by $\widetilde{\mathcal{M}}_m^{ss}\big(\boldsymbol{y}, [L], \alpha_*, (\delta, \ell)\big)$*, the moduli stack of such tuples* $(E_*, [\phi], \mathcal{F})$*. In the oriented case, we use the notation* $\widetilde{\mathcal{M}}_m^{ss}\big(\widehat{\boldsymbol{y}}, [L], \alpha_*, (\delta, \ell)\big)$ *as usual.*

Similarly, we have the notion of (δ, ℓ)*-semistability for a* L*-Bradlow pair* (E_*, ϕ) *such that* $\phi \neq 0$ *and a full flag* $\mathcal{F}$ *of* $H^0(X, E(m))$*. The moduli stack is denoted by* $\widetilde{\mathcal{M}}_m^{ss}\big(\boldsymbol{y}, L, \alpha_*, (\delta, \ell)\big)$*. When there are no risk of confusion, we omit to denote* m*.* ∎

Remark 3.32 *When* ℓ *is sufficiently large, the second condition is trivial. The first condition is equivalent to* $i = 1$ *for any J-H filtration. Therefore,* $(q, E_*, \phi, \mathcal{F})$ *is* (δ, ℓ)*-semistable if and only if* (E_*, ϕ) *is* δ_-*-semistable for a parameter* $\delta_- < \delta$ *such that* $|\delta - \delta_-|$ *is sufficiently small.* ∎

Lemma 3.33 *Let* $(E_*, [\phi], \mathcal{F})$ *be a reduced* L*-Bradlow pair with a full flag* $\mathcal{F}$ *of* $H^0\big(X, E(m)\big)$*. We assume that it is* (δ, ℓ)*-semistable. Then the automorphism group of* $(E_*, [\phi], \mathcal{F})$ *is* G_m*.*

Proof Let f be an endomorphism of $(E_*, [\phi], \mathcal{F})$. We have the generalized eigen decomposition $(E_*, [\phi], \mathcal{F}) = (E_{1\,*}, [\phi_1], \mathcal{F}^{(1)}) \oplus \bigoplus_{i=2}^k (E_{i\,*}, \mathcal{F}^{(i)})$. But, the condition of (δ, ℓ) is not satisfied if $k \geq 2$. Therefore, f has the unique eigenvalue. Let N be the nilpotent part of f. Let h be the integer such that $N^h \neq 0$ and $N^{h+1} = 0$. Assume $h \geq 0$, and we will derive the contradiction. We put $E_1 := \operatorname{Im} N^h$ and $E_2 := \operatorname{Ker} N^h$. We have the naturally induced parabolic structures and the L-sections ϕ_i of E_i. Since we have $N(\phi) = 0$, $\phi_1 = 0$. Then, we obtain the partial Jordan-Hölder filtration $E_{1\,*} \subset (E_{2\,*}, \phi_2) \subset (E_*, \phi)$ with respect to δ-semistability, due to Lemma 3.28.

We have the induced filtrations $\mathcal{F}^{(1)}$ on $H^0\big(X, E_1(m)\big)$ and $\mathcal{F}^{(3)}$ on $H^0\big(X, E/E_1(m)\big)$. We have the induced isomorphism $H^0\big(X, E/E_1(m)\big) \longrightarrow H^0\big(X, E_1(m)\big)$, which has to preserve the filtration $\mathcal{F}^{(i)}$ above. However, we have $\mathcal{F}_\ell^{(1)} = 0$ and $\mathcal{F}_\ell^{(3)} \neq 0$ due to the (δ, ℓ)-semistability. Thus we arrive at the contradiction, and hence we have $h = 0$. ∎

3.4 Some Boundedness

3.4.1 Foundational theorems

Let $\mathcal{O}_X(1)$ be a very ample line bundle. Let D denote a Cartier divisor of X. We recall several foundational theorems, by following D. Huybrecht-M. Lehn ([28]).

Proposition 3.34 (A. Grothendieck, Lemma 2.5, [25]) *Let* $\mathcal{D}$ *be a bounded family of coherent sheaf* F *on* X*. Let* C *be any positive number. Then we have the boundedness of the family of torsion-free coherent sheaves* F'' *with the following property:*

- $\deg(F'') \leq C$.
- *There exists a member* $F \in \mathcal{D}$ *such that* F'' *is a quotient sheaf of* F*.* ∎

For any torsion-free coherent sheaf E on X, we have the Harder-Narasimhan filtration with respect to the standard semistability. We denote the slope of the first (resp. last) term of the Harder-Narasimhan filtration by $\mu_{\max}(E)$ (resp. $\mu_{\min}(E)$).

Proposition 3.35 (M. Maruyama [40]) *Let* H *be a polynomial, and let* C *be a constant. We have the boundedness of the family of torsion-free coherent sheaves* F *on* X *satisfying* $\mu_{\max}(F) \leq C$ *and* $H_F = H$*.* ∎

We use the notation $[x]_+ = \max\{x, 0\}$ for any real number x.

Proposition 3.36 (C. Simpson [53]) *Let r be a positive integer. Then there is a positive constant c such that the following inequality holds for every μ-semistable sheaf F satisfying* $\mathrm{rank}(F) < r$ *and* $\mu(F) < \mu$*:*

$$\frac{h^0(F)}{\mathrm{rank}(F)} \leq \frac{1}{g^{d-1}d!}([\mu + c]_+^d).$$

Here d denotes the dimension of X, and g denotes the number $c_1\big(\mathcal{O}_X(1)\big)^d \cap [X]$. ∎

3.4.2 Boundedness of semistable L-Bradlow pairs

Let $\boldsymbol{y}$ be an element of $\mathcal{T}ype$, and let α_* be a system of weights. Let L be a line bundle over X, and let $\delta^{(0)}$ be an element of $\mathcal{P}^{\mathrm{br}}$. Let $\mathcal{SS}(\boldsymbol{y}, L, \alpha_*, \delta^{(0)})$ denote the family of parabolic L-Bradlow pairs (E_*, ϕ) such that $\phi \neq 0$ with the following properties:

- The type of E_* is $\boldsymbol{y}$, and the weight of the parabolic structure is given by α_*.
- (E_*, ϕ) is δ-μ-semistable for some $\delta \leq \delta^{(0)}$ in $\mathcal{P}^{\mathrm{br}}$.

Lemma 3.37 *The family $\mathcal{SS}(\boldsymbol{y}, L, \alpha_*, \delta^{(0)})$ is bounded.*

Proof Let (E_*, ϕ) be a member of $\mathcal{SS}(\boldsymbol{y}, L, \alpha_*, \delta^{(0)})$. Assume that it is δ-μ-semistable for $\delta \in \mathcal{P}^{\mathrm{br}}$. Let E' be the first member of the Harder-Narasimhan filtration of E with respect to the standard semistability. Then we have the following inequalities:

$$\mu_{\max}(E) = \mu(E') \leq \mu(E'_*) + \frac{\epsilon(E', \phi) \cdot \delta_{top}}{\mathrm{rank}\, E'} \leq \mu(E_*) + \frac{\epsilon(E, \phi) \cdot \delta_{top}}{\mathrm{rank}\, E} \leq \mu(E_*) + \frac{\delta_{top}^{(0)}}{\mathrm{rank}\, E}.$$

The last term depends only on $(\boldsymbol{y}, \delta^{(0)})$. Thus we obtain the boundedness from Proposition 3.35. ∎

Recall the following important observation due to Thaddeus [54].

Proposition 3.38 *Take an element $\boldsymbol{y} \in \mathcal{T}ype$ and a line bundle L on X. Assume $r = \mathrm{rank}(\boldsymbol{y}) > 1$. Let δ be an element of $\mathcal{P}^{\mathrm{br}}$ satisfying the following condition:*

$$\delta_{top} > \frac{r}{r-1}\big(\mu(\boldsymbol{y}, \alpha_*) - \deg(L)\big). \tag{32}$$

Then, there does not exist δ-semistable parabolic L-Bradlow pair (E_*, ϕ) of type $\boldsymbol{y}$ such that $\phi \neq 0$.

Proof Let (E_*, ϕ) be a δ-semistable L-Bradlow pair such that $\phi \neq 0$. The L-section ϕ generates the subsheaf E' of E with $\mathrm{rank}\, E' = 1$. Due to the stability, we have the following inequality:

$$\mu(E'_*) + \delta_{top} \leq \mu(E_*) + \frac{\delta_{top}}{r}.$$

We also have $\mu(E'_*) \geq \mu(E') = \deg(E') \geq \deg(L)$. Therefore, we obtain the following:

$$\left(1 - \frac{1}{r}\right) \cdot \delta_{top} \leq \mu(E_*) - \deg(L).$$

Thus we are done. ∎

Corollary 3.39 *The family $\mathcal{SS}\big(\boldsymbol{y}, L, \alpha_*\big) := \bigcup_{\delta \in \mathcal{P}^{\mathrm{br}}} \mathcal{SS}\big(\boldsymbol{y}, L, \alpha_*, \delta\big)$ is bounded.*

Proof It follows from Proposition 3.38 and Lemma 3.37. ∎

Let $\boldsymbol{L} = (L_1, L_2)$ be a pair of line bundles on X. Let $\boldsymbol{\delta}^{(0)} = (\delta_1^{(0)}, \delta_2^{(0)})$ be an element of $\mathcal{P}^{\mathrm{br}}$. Let $\mathcal{SS}(\boldsymbol{y}, \boldsymbol{L}, \alpha_*, \boldsymbol{\delta}^{(0)})$ denote the family of parabolic $\boldsymbol{L}$-Bradlow pairs $(E_*, \boldsymbol{\phi})$ of type $\boldsymbol{y}$ with weight α_* such that $\phi_i \neq 0$, which is (δ_1, δ_2)-μ-semistable for some $\delta_i \leq \delta_i^{(0)}$. By an argument similar to the proof of Lemma 3.37, we can show the following:

Lemma 3.40 *The family $\mathcal{SS}(\boldsymbol{L},\alpha_*,\boldsymbol{\delta})$ is bounded.* ∎

By an argument similar to the proof of Proposition 3.38, we can show the following lemma.

Lemma 3.41 *There exists $\delta_2^{(0)}$ such that the following holds for any $\delta_2 \geq \delta_2^{(0)}$ and for any δ_1:*

- *There does not exist any (δ_1,δ_2)-semistable $\boldsymbol{L}$-Bradlow pair (E_*,ϕ_1,ϕ_2) of type $\boldsymbol{y}$ such that $\phi_2 \neq 0$.*

In particular, we obtain the boundedness of the family $\mathcal{SS}(\boldsymbol{L},\alpha_*,\delta_1) := \bigcup_{\delta_2} \mathcal{SS}\bigl(\boldsymbol{L},\alpha_*,(\delta_1,\delta_2)\bigr)$. ∎

3.4.3 Boundedness of Yokogawa family

Let $\boldsymbol{y}$ be an element of $\mathcal{T}ype$, and let L be a line bundle on X. Let us fix a system of weights α_*, and we put $\epsilon_i := \alpha_{i+1} - \alpha_i$. For each positive integer m, let us take an $H_{\boldsymbol{y}}(m)$-dimensional vector space V_m. Let $\mathbb{P}_m$ denote the projectivization of V_m, i.e., $\mathbb{P}_m := \mathbb{P}(V_m^\vee)$. We also take an inclusion $\iota : \mathcal{O}(-m) \longrightarrow L$.

Definition 3.42 *A Yokogawa datum of type $(\boldsymbol{y},m)$ is defined to be a tuple $\bigl(q,E_*,\phi,W_*,[\widetilde{\phi}]\bigr)$ as follows:*

- *(E_*,ϕ) is a parabolic L-Bradlow pair over X, such that* $\operatorname{type}(E_*) = \boldsymbol{y}$.
- *q is a generically surjective morphism $V_{m,X} \longrightarrow E(m)$, where we put $V_{m,X} := V_m \otimes \mathcal{O}_X$.*
- *$W_* = (W_1,\ldots,W_l)$ is a tuple of subspaces of V_m such that* $\dim W_i = H_{\boldsymbol{y}}(m) - H_{\boldsymbol{y},i}(m)$. *We assume that $H^0(q)(W_i)$ is contained in $H^0\bigl(X,F_{i+1}(E)(m)\bigr)$.*
- *$[\widetilde{\phi}]$ is a point of $\mathbb{P}_m$, and there exists a non-zero scalar c such that $H^0(q)(\widetilde{\phi}) = c\cdot\iota(\phi)$, where $\iota(\phi)$ denotes the section of $E(m)$ induced by ι and ϕ.* ∎

Definition 3.43 *Let δ be an element of $\mathcal{P}^{\mathrm{br}}$. Let K be any non-negative number. Let $\widetilde{\mathrm{YOK}}(m,K,\boldsymbol{y},L,\delta)$ denotes the set of Yokogawa data $\bigl(q,E_*,\phi,W_*,[\widetilde{\phi}]\bigr)$ such that the following inequality holds for any subspace $W \subset V_m$:*

$$P_{\boldsymbol{y}}^{\delta,\alpha_*}(m)\cdot \operatorname{rank}\mathcal{E}_W - \epsilon(W,[\widetilde{\phi}])\cdot\delta(m) - \sum_{i=1}^{l}\epsilon_i\cdot\dim(W_i\cap W) - \alpha_1\cdot\dim(W) + K \geq 0. \tag{33}$$

Here, $\mathcal{E}_W$ denotes the subsheaf of $E(m)$ generated by W and q, and $\epsilon(W,[\widetilde{\phi}])$ is defined to be 1 ($[\widetilde{\phi}] \in \mathbb{P}_W$) or 0 ($[\widetilde{\phi}] \notin \mathbb{P}_W$). We remark $\epsilon(W,[\widetilde{\phi}]) = \epsilon(\mathcal{E}_W(-m),\phi)$, where $\epsilon(\mathcal{E}_W(-m),\phi)$ is given as in (29). ∎

For each positive integer N, we put $\widetilde{\mathcal{YOK}}(N,K,\boldsymbol{y},L,\delta) := \bigcup_{m\geq N}\widetilde{\mathrm{YOK}}(m,K,\boldsymbol{y},L,\delta)$. Following Yokogawa [61], we consider the family $\mathcal{YOK}(N,K,\boldsymbol{y},L,\delta)$ of quasi-parabolic L-Bradlow pairs (E,F_*,ϕ) of type $\boldsymbol{y}$ such that there exists a lift $\bigl(q,E,F_*,\phi,W_*,[\widetilde{\phi}]\bigr) \in \widetilde{\mathcal{YOK}}(N,K,\boldsymbol{y},L,\delta)$. The family will be called the Yokogawa family.

Proposition 3.44 *There exists a small positive number $K_0 = K_0(\boldsymbol{y},L,\delta)$ and a large integer $N_0 = N_0(\boldsymbol{y},L,\delta)$ such that the following holds:*

- *The family $\mathcal{YOK}\bigl(N_0,K_0,\boldsymbol{y},L,\delta\bigr)$ is bounded, and it satisfies the condition O_m for any $m \geq N_0$.*
- *For any $(q,E_*,\phi,W_*,[\widetilde{\phi}]) \in \mathcal{YOK}\bigl(N_0,K_0,\boldsymbol{y},L,\delta\bigr)$, the morphism q is surjective. In particular, $H^0(q) : V_m \longrightarrow H^0(X,E(m))$ is isomorphic.*
- *All members of $\mathcal{YOK}(N_0,K_0,\boldsymbol{y},L,\delta)$ are δ-semistable.*

Proof We follow the argument of Yokogawa [61]. We begin with the following lemma.

Lemma 3.45 *Let $\bigl(E_*,\phi\bigr)$ be a parabolic L-Bradlow pair contained in the family $\mathcal{YOK}(N,K,\boldsymbol{y},L,\delta)$ with a lift $\bigl(q,E_*,\phi,W_*,[\widetilde{\phi}]\bigr)$. Let E' denote a quotient sheaf of E. Then the following inequality holds:*

$$P_{\boldsymbol{y}}^{\delta,\alpha_*}(m) \leq \frac{h^0\bigl(E'_*(m)\bigr) + \epsilon(E',\phi)\cdot\delta(m) + K}{\operatorname{rank}(E')}. \tag{34}$$

Proof Let W denote the kernel of the composite of the morphisms:

$$V_m \xrightarrow{H^0(q)} H^0\bigl(X, E(m)\bigr) \longrightarrow H^0\bigl(X, E'(m)\bigr)$$

Let $\mathcal{E}_W$ denote the subsheaf of $E(m)$ generated by the image of W via q. Due to (33), we obtain the following inequality:

$$P_{\boldsymbol{y}}^{\delta,\alpha_*}(m)\cdot \operatorname{rank}(\mathcal{E}_W) + \epsilon(W,\widetilde{\phi})\cdot\delta(m) - \sum_i \epsilon_i \dim(W_i\cap W) - \alpha_1\dim(W) + K \geq 0.$$

We have the inequalities $\dim(W_i\cap W)\geq \dim(W_i) - h^0\bigl(F_{i+1}(E')(m)\bigr)$ for $i=1,\ldots,l$. We also have the equalities $\epsilon(W,[\widetilde{\phi}]) + \epsilon(E',[\phi']) = 1$, where ϕ' is the induced L-section of E' by ϕ. Since q is generically surjective, we have $\operatorname{rank}(E') + \operatorname{rank}(\mathcal{E}_W) = \operatorname{rank}(E)$. Thus, we obtain the following inequality:

$$\begin{aligned} 0 \leq P_{\boldsymbol{y}}^{\delta,\alpha_*}(m)\cdot\operatorname{rank}E - \delta(m) - \sum_i \epsilon_i\cdot\dim(W_i) + \sum_i \epsilon_i\cdot h^0\bigl(F_{i+1}(E')(m)\bigr) - \alpha_1\dim(W)\\ - P_{\boldsymbol{y}}^{\delta,\alpha_*}(m)\cdot\operatorname{rank}E' + \epsilon(E',\phi')\cdot\delta(m) + K. \end{aligned} \tag{35}$$

We have the following equality:

$$\begin{aligned} &P_{\boldsymbol{y}}^{\delta,\alpha_*}(m)\cdot\operatorname{rank}E - \delta(m) - \sum_i \epsilon_i\cdot\dim(W_i) - \alpha_1\dim(W)\\ &\qquad = H(m) - \sum \epsilon_i\cdot H_i(m) - \sum_i \epsilon_i\cdot\dim(W_i) - \alpha_1\dim(V) + \alpha_1\dim(V/W). \end{aligned} \tag{36}$$

Since we have the equalities $\dim(W_i) = H(m) - H_i(m)$ and $\dim(V) = H(m)$, the right hand side equals to $\alpha_1\dim(V/W)$. We have the inequality $\dim(V/W)\leq \dim H^0\bigl(X,E'(m)\bigr)$. Hence we obtain the following inequality:

$$0\leq -\operatorname{rank}(E')\cdot P^{\delta,\alpha_*}(m) + h^0\bigl(E'_*(m)\bigr) + \epsilon(E',\phi)\cdot\delta(m) + K.$$

Then (34) immediately follows. ∎

Lemma 3.46 *There exists an integer N_1 such that the Yokogawa family $\mathcal{YOK}(N_1,K,\boldsymbol{y},L,\delta)$ is bounded.*

Proof We put $d := \dim X$ and $g := c_1\bigl(\mathcal{O}_X(1)\bigr)^d\cap[X]$. Take a sufficiently negative number C satisfying the following inequality for any sufficiently large t:

$$\frac{1}{g^{d-1}d!}\bigl(C+tg+c\bigr)^d + \delta(t) + K < P^{\delta,\alpha_*}(t).$$

Take a large integer N_1 such that the following inequalities hold for any $m > N_1$:

$$C+mg+c>0,\quad \delta(m)>0,\quad \frac{\bigl(C+mg+c\bigr)^d}{g^{d-1}\cdot d!} + \delta(m) + K < P^{\alpha_*,\delta}(m).$$

Then we will show that $\mathcal{YOK}(N_1,K,\boldsymbol{y},L,\delta)$ is bounded.

Let $\bigl(E_*,\phi\bigr)$ be a parabolic L-Bradlow pair contained in the family $\mathcal{YOK}(N_1,K,\boldsymbol{y},L,\delta)$. Let E' denote the last member of the Harder-Narasimhan filtration of E with respect to the standard semistability. Assume $\mu(E')<C$, and we will derive the contradiction. Due to Proposition 3.36, we have the following inequality:

$$\frac{h^0\bigl(E'_*(m)\bigr)}{\operatorname{rank}(E')} \leq \frac{h^0(E'(m))}{\operatorname{rank}(E')} \leq \frac{\bigl[\mu(E')+mg+c\bigr]_+^d}{g^{d-1}d!}.$$

By the assumption $\mu(E')<C$, we obtain the following inequality:

$$\frac{h^0\bigl(E'_*(m)\bigr) + \epsilon(E',\phi)\cdot\delta(m) + K}{\operatorname{rank}(E')} \leq \frac{(C+mg+c)^d}{g^{d-1}d!} + \delta(m) + K < P^{\delta,\alpha_*}(m). \tag{37}$$

However, it contradicts with (34). Thus we obtain $\mu(E') > C$. It implies $\mu_{\max}(E) < C'$ for some constant C', and thus we obtain that the family $\mathcal{YOK}(N_1, K, \boldsymbol{y}, \delta)$ is bounded due to Proposition 3.35. ∎

Then, there exists an integer N_2 such that the family $\mathcal{YOK}(N_2, K, \boldsymbol{y}, L, \delta)$ satisfies the condition O_m for any $m \geq N_2$.

Lemma 3.47 *Assume K_1 is strictly smaller than α_1. Then, the map $H^0(q) : V_m \longrightarrow H^0\bigl(X, E(m)\bigr)$ is isomorphic for any $\bigl(q, E, F_*, \phi, W_*, [\widetilde{\phi}]\bigr) \in \widetilde{\mathcal{YOK}}(N_2, K_1, \boldsymbol{y}, L, \delta)$. In particular, q is surjective.*

Proof We have only to check that $H^0(q)$ is injective. Let W denote the kernel of $H^0(q)$. Then we obtain the following inequality from (33):

$$-\sum \epsilon_i \cdot \dim(W_i \cap W) - \alpha_1 \cdot \dim(W) + K_1 \geq 0.$$

Since K_1 is strictly smaller than α_1, we obtain $\dim(W) = 0$, i.e., $H^0(q)$ is injective. ∎

Let us finish the proof of Proposition 3.44. Let us consider the family $\mathcal{S}_1$ of parabolic L-Bradlow pairs (E'_*, ϕ') with the following property:

- There exists some $(E_*, \phi) \in \mathcal{YOK}(N_2, K_1, \boldsymbol{y}, L, \delta)$ such that (E'_*, ϕ') is isomorphic to the last member of Harder Narasimhan filtration of (E_*, ϕ) with respect to δ-semistability.

Since $\mathcal{S}_1$ is bounded, the number of the Hilbert polynomials of members in $\mathcal{S}_1$ is finite. Therefore, there exists a small positive number $K_0 < K_1$ and a large integer $N_0 \geq N_2$ such that the following holds for any $m \geq N_0$.

- Let (E_*, ϕ) be a member of $\mathcal{YOK}(N_0, K_0, \boldsymbol{y}, L, \delta)$, which is not δ-semistable. Let (E'_*, ϕ') denote last member of the Harder-Narasimhan filtration of (E_*, ϕ) with respect to δ-semistability. Then, the inequality $P^{\delta_0}_{(E'_*, \phi')}(t) + K_0 < P^{\delta_0, \alpha_*}_{\boldsymbol{y}}(t)$ holds for any $t \geq m$.
- The family $\mathcal{S}_1$ satisfies the condition O_m.

Let (E_*, ϕ) be a member of $\mathcal{YOK}(N_0, K_0, \boldsymbol{y}, L, \delta)$, and let $(q, E_*, \phi, W_*, [\widetilde{\phi}])$ be its lift in $\widetilde{\mathcal{YOK}}(N_0, K_0, \boldsymbol{y}, L, \delta)$. Assume (E_*, ϕ) is not δ-semistable, and let (E'_*, ϕ') be the last member of the Harder-Narasimhan filtration of (E_*, ϕ) with respect to the δ-semistability. Then, we obtain the following inequality from (34) and the second condition:

$$P^{\delta, \alpha_*}_{\boldsymbol{y}}(m) \leq \frac{h^0\bigl(E'_*(m)\bigr) + \epsilon(E', \phi) \cdot \delta(m) + K_0}{\operatorname{rank}(E')} \leq P^{\delta}_{(E'_*, \phi')}(m) + K_0$$

It contradicts with the first condition. Thus we are done. ∎

3.5 1-Stability Condition and 2-Stability Condition

3.5.1 Parabolic sheaf

Let $\boldsymbol{y} = (y, y_1, \ldots, y_l)$ be an element of $\mathcal{T}ype$, and let $\alpha_* = (\alpha_1, \ldots, \alpha_l)$ be a system of weights.

Definition 3.48

- *We say that the 1-stability condition holds for $(\boldsymbol{y}, \alpha_*)$, if $\mathcal{M}^s(\boldsymbol{y}, \alpha_*) = \mathcal{M}^{ss}(\boldsymbol{y}, \alpha_*)$ holds.*
- *We say that the 2-stability condition holds for $(\boldsymbol{y}, \alpha_*)$, if the automorphism group of $E_* \in \mathcal{M}^{ss}(\boldsymbol{y}, \alpha_*)$ is G_m or G_m^2.* ∎

Lemma 3.49 *The 2-stability condition for $(\boldsymbol{y}, \alpha_*)$ is equivalent to the following condition:*

- *Let E_* be a parabolic sheaf of type $\boldsymbol{y}$ with weight α_*, which is polystable but not stable. Then we have the unique decomposition $E_* = E_{1\,*} \oplus E_{2\,*}$, where $E_{i\,*}$ are stable parabolic sheaves with weight α_* such that $E_{1\,*} \not\simeq E_{2\,*}$.*

Proof Assume that the 2-stability condition holds. Let E_* be a polystable parabolic sheaf of type $\boldsymbol{y}$ with weight α_*. We have a decomposition $E_* = E_{1\,*} \oplus E_{2\,*}$. If one of $E_{i\,*}$ is not stable, then by taking the graduation of a Jordan-Hölder filtration, we obtain a polystable parabolic sheaf $E'_* = E'_{1\,*} \oplus E'_{2\,*} \oplus E'_{3\,*}$ of type $\boldsymbol{y}$ with weight α_*. However, the automorphism of E'_* is G_m^3, which contradicts with the 2-stability condition.

Assume that the condition above holds. Let E_* be a semistable parabolic sheaf of type $\boldsymbol{y}$ with weight α_*. Let $f : E_* \longrightarrow E_*$ be an endomorphism. The eigenvalues of f are constant. Let $E_* = \bigoplus_{i=1}^N E_{i,*}$ be the generalized eigen decomposition of f. We have the decomposition $f = \bigoplus_{i=1}^N f_i$. If $N \geq 3$, the length of a Jordan-Hölder filtration is longer than 3, and hence we have a polystable object which has more than three stable components. Hence $N \leq 2$.

In the case $N = 2$, it can be shown that $E_{i\,*}$ are stable by the same argument. Hence the automorphism group is G_m^2.

Let us consider the case $N = 1$. If E_* is stable, the automorphism group is G_m. In the case E_* is not stable, the length of the Harder-Narasimhan filtration is 2. Moreover, the graded components are not mutually isomorphic. Hence the automorphism group is G_m. ∎

In the above argument, the following corollary is proved.

Corollary 3.50 *Assume that the* 2*-stability condition holds for* $(\boldsymbol{y}, \alpha_*)$*. Let* E_* *be a semistable parabolic sheaf of type* $\boldsymbol{y}$ *with weight* α_**. Then one of the following holds:*

- E_* *is stable.*
- E_* *is uniquely decomposed into* $E_{1\,*} \oplus E_{2\,*}$*, where* $E_{i\,*}$ *are stable such that* $E_{1\,*} \not\simeq E_{2\,*}$*.*
- *We have the non-split exact sequence* $0 \longrightarrow E_{1\,*} \longrightarrow E_* \longrightarrow E_{2\,*} \longrightarrow 0$*, where* $E_{i\,*}$ *are stable such that* $E_{1\,*} \not\simeq E_{2\,*}$*.* ∎

3.5.2 Parabolic L-Bradlow pair

Let L be a line bundle on X, and let δ be an element of $\mathcal{P}^{\mathrm{br}}$.

Definition 3.51

- *We say that the* 1*-stability condition holds for* $(\boldsymbol{y}, \alpha_*, L, \delta)$*, if* $\mathcal{M}^s(\boldsymbol{y}, [L], \alpha_*, \delta) = \mathcal{M}^{ss}(\boldsymbol{y}, [L], \alpha_*, \delta)$ *holds.*
- *We say that the* 2*-stability condition holds for* $(\boldsymbol{y}, \alpha_*, [L], \delta)$*, if the automorphism group of* $(E_*, \phi) \in \mathcal{M}^{ss}(\boldsymbol{y}, [L], \alpha_*, \delta)$ *is* G_m *or* G_m^2*.* ∎

Definition 3.52 *Fix a type* $\boldsymbol{y} \in \mathcal{T}ype$*, a system of weights* α_* *and a line bundle* L*. A parameter* $\delta \in \mathcal{P}^{\mathrm{br}}$ *is called critical for* $(\boldsymbol{y}, \alpha_*, L)$*, if the* 1*-stability condition does not hold for* $(\boldsymbol{y}, \alpha_*, L, \delta)$*. The set of such critical parameters is denoted by* $\mathrm{Cr}(\boldsymbol{y}, \alpha_*, L)$*.* ∎

Lemma 3.53 *The set* $\mathrm{Cr}(\boldsymbol{y}, L, \alpha_*)$ *of critical values is finite.*

Proof We may assume $\mathrm{rank}(\boldsymbol{y}) > 1$ and $\mu(\boldsymbol{y}, \alpha_*) \geq 0$. Recall Proposition 3.38. We can take a sufficiently negative number C such that there does not exist δ-semistable L-Bradlow pairs for any $\delta_{top} \geq -C$.

Let $\mathcal{S}_1$ denote the family of L-Bradlow pairs (E'_*, ϕ') with the following property:

- There exists a member (E_*, ϕ) of $\mathcal{SS}(\boldsymbol{y}, L, \alpha_*)$ such that (E'_*, ϕ') is a saturated subobject of (E_*, ϕ).
- $\deg(E'_*) \geq C$.

Since $\mathcal{S}_1$ is bounded (Proposition 3.34), we obtain the finiteness of the set $\mathcal{T}_1$ of the polynomials H such that There exists a member $(E'_*, \phi') \in \mathcal{S}_1$ satisfying $H_{E'_*} = H$.

Let δ be an element of $\mathrm{Cr}(\boldsymbol{y}, L, \alpha_*)$. Then there exists a δ-semistable (E_*, ϕ) such that $\phi \neq 0$, which has a non-trivial partial Jordan-Hölder filtration $(E'_*, \phi') \subset (E_*, \phi)$. We have the following equality:

$$\frac{\deg(E'_*) + \epsilon(E', \phi') \cdot \delta_{top}}{\mathrm{rank}\, E'} = \frac{\deg(E_*) + \delta_{top}}{\mathrm{rank}\, E}$$

Since we have $\delta_{top} \leq -C$, we obtain the following inequality:

$$\deg(E'_*) \geq \operatorname{rank}(E') \cdot \mu(E_*) - \delta_{top} \geq C$$

Hence, (E'_*, ϕ') is a member of $\mathcal{S}_1$. We have the equality $P^{\delta}_{(E'_*,\phi')} = P^{\delta}_{(E_*,\phi)}$. Hence there exist $H \in \mathcal{T}_1$, an integer r_1 and an integer ϵ satisfying the following:

$$\left(\frac{1}{r} - \frac{\epsilon}{r_1}\right) \cdot \delta = \frac{H_{E_*}}{r} - \frac{H}{r_1}, \quad 0 < r_1 < r, \quad \epsilon = 0, \text{ or } 1$$

Thus we obtain the finiteness of $\mathrm{Cr}(\boldsymbol{y}, L, \alpha_*)$ from the finiteness of $\mathcal{T}_1$ ∎

Corollary 3.54 *If $\delta' \neq \delta$ is sufficiently close to some element $\delta \in \mathcal{P}^{\mathrm{br}}$, the 1-stability condition holds for $(\boldsymbol{y}, L, \alpha_*, \delta')$. If δ is sufficiently close to 0, the 1-stability condition for $(\boldsymbol{y}, L, \alpha_*, \delta')$.* ∎

Lemma 3.55 *Let δ_0 be an element of $\mathcal{P}^{\mathrm{br}}$. If $\delta_1 \in \mathcal{P}^{\mathrm{br}}$ is sufficiently close to δ_0, any δ_1-semistable L-Bradlow pair is also δ_0-semistable.*

Proof Let $\mathcal{S}$, $\mathcal{S}_1$ and $\mathcal{T}_1$ be as in the proof of Lemma 3.53. Let r_1 and ϵ be integers such that $0 < r_1 < r$ and $\epsilon = 0, 1$. We put $P(H, r_1, \epsilon, \delta) := r_1^{-1}\big(H + \epsilon \cdot \delta\big)$ for any $\delta \in \mathcal{P}^{\mathrm{br}}$ and $H \in \mathcal{T}_1$. If δ_1 is sufficiently close to δ_0, the following holds:

(A) $P(H, r_1, \epsilon, \delta_1)(t) > 0$ implies $P(H, r_1, \epsilon, \delta_0)(t) \geq 0$ for any sufficiently large t.

Let δ_1 be as above. We may assume that the 1-stability condition holds for $(\boldsymbol{y}, L, \alpha_*, \delta_1)$. Let (E_*, ϕ) be δ_1-stable L-Bradlow pair of type $\boldsymbol{y}$, and we assume that it is not δ-semistable. Then there exists a saturated subobject (E'_*, ϕ') of (E_*, ϕ) with the following property:

- $P^{\delta_0}_{(E'_*,\phi')}(t) \geq P^{\delta_0}_{(E_*,\phi)}(t)$ for any sufficiently large t.
- $P^{\delta_1}_{(E'_*,\phi')}(t) \leq P^{\delta_1}_{(E_*,\phi)}(t)$ for any sufficiently large t.

Then (E'_*, ϕ') is a member of $\mathcal{S}_1$ due to the first inequality. Therefore, the two inequalities contradict with the condition (A) above. Thus we are done. ∎

By a similar argument, we can show the following.

Lemma 3.56 *Let $\boldsymbol{y}$ be an element of $\mathcal{T}ype$, and let α_* be a system of weights. Assume δ is sufficiently small. Then, E_* is semistable if (E_*, ϕ) be a δ-stable L-Bradlow pair.*

Proof Let $\mathcal{S}$ be a family of μ-semistable parabolic torsion-free sheaves of type $\boldsymbol{y}$. Let $\overline{\mathcal{S}}$ denote the family of parabolic torsion-free sheaves E'_* with the following property:

- There exists $E_* \in \mathcal{S}$ such that E'_* is a saturated subobject of E_*. Moreover, we have $\mu(E'_*) = \mu(\boldsymbol{y}, \alpha_*)$.

Let $\mathcal{T}$ denote the set of the polynomials P such that $P \neq P^{\alpha_*}_{\boldsymbol{y}}$ and $P = P_{E'_*}$ for some $E'_* \in \overline{\mathcal{S}}$. Since the families $\mathcal{S}$ and $\overline{\mathcal{S}}$ are bounded, the set $\mathcal{T}$ is finite. We take a positive number δ_1 satisfying the following:

$$0 < \delta_1 < \frac{1}{10 \cdot \operatorname{rank}(\boldsymbol{y})} \min\big\{|P - P^{\alpha_*}_{\boldsymbol{y}}| \,\big|\, P \in \mathcal{T}\big\}$$

We regard δ_1 as the polynomial of degree 0.

Let (E_*, ϕ) be a δ_1-semistable L-Bradlow pair of type $\boldsymbol{y}$ with weight α_*. It is easy to observe that E_* is μ-semistable. Let E'_* be a subobject of E_*. In the case $\mu(E'_*) < \mu(E_*)$, we obviously have $P_{E'_*} < P_{E_*}$. Assume $\mu(E'_*) = \mu(E_*)$ and $P_{E'_*} \neq P_{E_*}$. Then E'_* is a member of $\overline{\mathcal{S}}$, and $P_{E'_*}$ is a member of $\mathcal{T}$. Due to δ_1-semistability of (E_*, ϕ), we have the following:

$$P_{E'_*} + \frac{\epsilon(E'_*, \phi) \cdot \delta_1}{\operatorname{rank} E'} \leq P_{E_*} + \frac{\delta_1}{\operatorname{rank} E}$$

It implies $P_{E'_*} < P_{E_*}$ due to our choice of δ_1. ∎

By an argument similar to the proof of Lemma 3.49, we obtain the following lemma.

Lemma 3.57 *The* 2*-stability condition for* $(\boldsymbol{y}, \alpha_*, L, \delta)$ *is equivalent to the following:*

- *Let* (E_*, ϕ) *be a parabolic* L*-Bradlow pair of type* $\boldsymbol{y}$ *with weight* α_* *such that* $\phi \neq 0$*, which is* δ*-polystable but not* δ*-stable. Then we have the unique decomposition* $(E_*, \phi) = (E_{1\,*}, \phi_1) \oplus E_{2\,*}$*, where* $(E_{1\,*}, \phi_1)$ *is* δ*-stable and* $E_{2\,*}$ *is stable.*

Moreover, when the 2*-stability condition holds for* $(\boldsymbol{y}, \alpha_*, L, \delta)$*, one of the following holds for any* δ*-semistable parabolic* L*-Bradlow pair* (E_*, ϕ) *of type* $\boldsymbol{y}$ *with weight* α_* *such that* $\phi \neq 0$*.*

- (E_*, ϕ) *is* δ*-stable.*
- (E_*, ϕ) *is uniquely decomposed into* $(E_*, \phi) = (E_{1\,*}, \phi_1) \oplus E_{2\,*}$*, where* $(E_{1\,*}, \phi)$ *is* δ*-stable and* $E_{2\,*}$ *is stable.*
- *We have the non-split exact sequence* $0 \longrightarrow (E_{1\,*}, \phi_1) \longrightarrow (E_*, \phi) \longrightarrow E_{2\,*} \longrightarrow 0$ *or* $0 \longrightarrow E_{2\,*} \longrightarrow (E_*, \phi) \longrightarrow (E_{1\,*}, \phi_1) \longrightarrow 0$*, where* $(E_{1\,*}, \phi)$ *is* δ*-stable and* $E_{2\,*}$ *is stable.* ∎

3.5.3 Parabolic $\boldsymbol{L}$-Bradlow pair

Let $\boldsymbol{L} = (L_1, L_2)$ be a pair of line bundles on X, and let $\boldsymbol{\delta} = (\delta_1, \delta_2)$ be elements of $\left(\mathcal{P}^{\mathrm{br}}\right)^2$. Similarly, we have the notion of 1-stability and 2-stability conditions.

Definition 3.58

- *We say that the* 1*-stability condition holds for* $(\boldsymbol{y}, \alpha_*, \boldsymbol{L}, \boldsymbol{\delta})$*, if* $\mathcal{M}^s(\boldsymbol{y}, [\boldsymbol{L}], \alpha_*, \boldsymbol{\delta}) = \mathcal{M}^{ss}(\boldsymbol{y}, [\boldsymbol{L}], \alpha_*, \boldsymbol{\delta})$*.*
- *We say that the* 2*-stability condition holds for* $(\boldsymbol{y}, \alpha_*, \boldsymbol{L}, \boldsymbol{\delta})$*, if the automorphism group of any* $(E_*, \boldsymbol{\phi}) \in \mathcal{M}^{ss}(\boldsymbol{y}, [\boldsymbol{L}], \alpha_*, \boldsymbol{\delta})$ *is* G_m *or* G_m^2*.* ∎

Definition 3.59

- *Fix a type* $\boldsymbol{y} \in \mathcal{T}ype$*, a system of weights* α_* *and a pair of line bundles* $\boldsymbol{L}$*. A parameter* $\boldsymbol{\delta} \in \mathcal{P}^{\mathrm{br}\ 2}$ *is called critical for* $(\boldsymbol{y}, \alpha_*, \boldsymbol{L})$*, if the* 1*-stability condition does not hold for* $(\boldsymbol{y}, \alpha_*, \boldsymbol{L}, \boldsymbol{\delta})$*. The set of such critical parameters is denoted by* $\mathrm{Cr}(\boldsymbol{y}, \alpha_*, \boldsymbol{L})$*.*
- *We also put as follows:*

$$\mathrm{Cr}(\boldsymbol{y}, \alpha_*, \boldsymbol{L}, \delta_1) := \left\{\delta_2 \in \mathcal{P}^{\mathrm{br}} \,\middle|\, (\delta_1, \delta_2) \in \mathrm{Cr}(\boldsymbol{y}, \alpha_*, \boldsymbol{L})\right\}$$

  Any element $\delta_2 \in \mathrm{Cr}(\boldsymbol{y}, \alpha_*, \boldsymbol{L}, \delta_1)$ *is called critical for* $(\boldsymbol{y}, \alpha_*, \boldsymbol{L}, \delta_1)$*.* ∎

We can show the following lemma by using Lemma 3.41 and an argument similar to the proof of Lemma 3.53.

Lemma 3.60 *The set* $\mathrm{Cr}(\boldsymbol{y}, \boldsymbol{L}, \alpha_*, \delta_1)$ *is finite.* ∎

By an argument similar to the proof of Lemma 3.55, we can show the following lemma.

Lemma 3.61 *If* δ_2' *are sufficiently close to* δ_2*, any* (δ_1, δ_2')*-semistable* $\boldsymbol{L}$*-Bradlow pair is also* (δ_1, δ_2)*-semistable.*
If δ_2' *is sufficiently close to* 0*, the* L_1*-Bradlow pair* (E_*, ϕ_1) *is* δ_1*-semistable for any* (δ_1, δ_2)*-semistable* $\boldsymbol{L}$*-Bradlow pair* (E_*, ϕ_1, ϕ_2)*.* ∎

Lemma 3.62 *Assume that* $\delta_1 + \delta_2$ *is sufficiently small as in Lemma* 3.56*. Then, if* (E_*, ϕ_1, ϕ_2)*is a* (δ_1, δ_2)*-semistable* $\boldsymbol{L}$*-Bradlow pair,* E_* *is semistable.* ∎

Lemma 3.63 *If both of* δ_i *are sufficiently small, the following claims hold:*

- *If the* 1*-stability condition holds for* $(\boldsymbol{y}, \alpha_*)$*, then the* 1*-stability condition holds also for* $(\boldsymbol{y}, \alpha_*, \boldsymbol{L}, \boldsymbol{\delta})$*.*
- *Even if the* 1*-stability condition does not hold for* $(\boldsymbol{y}, \alpha_*)$*, the* 2*-stability condition holds for* $(\boldsymbol{y}, \alpha_*, \boldsymbol{L}, \boldsymbol{\delta})$*.*

- *If the 1-stability condition does not hold for $\boldsymbol{\delta} = (\delta_1, \delta_2)$, the equality $\delta_1/r_1 = \delta_2/r_2$ holds for some decomposition $r_1 + r_2 = \operatorname{rank}(\boldsymbol{y})$.*

Proof We take δ_i as in the proof of Lemma 3.56. The first claim is clear. Let us see the second claim. Let (E_*, ϕ_1, ϕ_2) be a (δ_1, δ_2)-polystable (L_1, L_2)-Bradlow pair of type $\boldsymbol{y}$ with weight α_* such that $\phi_i \neq 0$. We remark that E_* is semistable. Since δ_i are sufficiently small, the number of stable components are at most 2. If it is decomposed, it is of the form $(E_{1\,*}, \phi_1, 0) \oplus (E_{2\,*}, 0, \phi_2)$, where $(E_{i\,*}, \phi_i)$ are δ_i-stable with $\phi_i \neq 0$. The components are not isomorphic. Hence the 2-stability condition holds. The third claim also immediately follows. ∎

3.6 Quot Schemes

3.6.1 Preliminary

Let y be an element of $H^{\mathrm{ev}}(X)$. We will denote $y \cdot \operatorname{ch}\big(\mathcal{O}(m)\big)$ by $y(m)$ for any integer m. We also have the element $\det(y) = y_1$, where y_1 denote the $H^2(X)$-component of y.

Let H_y be the associated Hilbert polynomial to y (the subsubsection 3.2.4). Take a large integer m such that $H_y(m) > 0$. We also assume that any line bundles M with $c_1(M) = \det\big(y(m)\big)$ satisfy $H^i(X, M) = 0$ $(i > 0)$.

We take an $H_y(m)$-dimensional vector space V_m over k. We denote $V_m \otimes \mathcal{O}_X$ by $V_{m,X}$.

3.6.2 Quotient sheaves

A pair of U-coherent sheaf $\mathcal{E}$ on $U \times X$ and a surjection $q : p_U^* V_{m,X} \longrightarrow \mathcal{E}$ is called a U-quotient sheaf of $V_{m,X}$, which is denoted by $(q, \mathcal{E})$ or simply by q. A U-quotient sheaf $(q, \mathcal{E})$ with an orientation ρ is called an oriented U-quotient sheaf of $V_{m,X}$. The type of $(q, \mathcal{E})$ or $(q, \mathcal{E}, \rho)$ is defined to be the type of $\mathcal{E}(-m)$. (See the subsubsection 3.1.4 for the type.)

Recall that the moduli functor of quotient sheaves of $V_{m,X}$ with type y is representable, and the moduli scheme is projective. ([25]). We denote it by $Q(m, y)$. Let $(q^u, \mathcal{E}^u)$ denote the universal quotient sheaf of $V_{m,X}$ on $Q(m, y) \times X$. A point of $Q(m, y)$ is denoted by the corresponding quotient $\big(q, \mathcal{E}\big)$. We have the right action of $GL(V_m)$ on $Q(m, y)$ given by $g \cdot (q, \mathcal{E}) := (q \circ g, \mathcal{E})$.

Let $(q, \mathcal{E})$ be a U-quotient sheaf of $V_{m,X}$ with type y defined over $U \times X$. We say that $\big(q, \mathcal{E}\big)$ satisfies the (TFV)-condition, if the following holds for any $u \in U$:

(TFV): The sheaf $\mathcal{E}_{|\{u\} \times X}$ is torsion-free, the induced map $V_m \longrightarrow H^0\big(X, \mathcal{E}_{|\{u\} \times X}\big)$ is isomorphic, and $H^i\big(X, \mathcal{E}_{|\{u\} \times X}\big)$ vanish for any $i > 0$.

In general, the condition determines the maximal open subset of U on which the (TFV)-condition holds. In particular, it determines the open subset of $Q(m, y)$, which is denoted by $Q^{\circ}(m, y)$.

Let $\mathcal{O}r(\mathcal{E}^u)$ denote the orientation bundle, which is the line bundle on $Q(m, y)$. The moduli functor of oriented quotient sheaves of $V_{m,X}$ with type y is representable by $Q(m, \widehat{y}) := \mathcal{O}r(\mathcal{E}^u)^*$. We have the induced right action of $GL(m)$ on $Q(m, \widehat{y})$. We put $Q^{\circ}(m, \widehat{y}) := Q^{\circ}(m, y) \times_{Q(m,y)} Q(m, \widehat{y})$.

3.6.3 Quotient quasi-parabolic sheaves and the Maruyama-Yokogawa construction

Let D be a Cartier divisor of X. Let y and $V_{m,X}$ be given as above. A U-quasi-parabolic quotient sheaf $(q, \mathcal{E}, F_*)$ of $V_{m,X}$ on $U \times (X, D)$ is defined to be a U-quotient sheaf $(q, \mathcal{E})$ of $V_{m,X}$ with a U-quasi-parabolic structure F_* of $\mathcal{E}$ at D. The type of U-quasi parabolic quotient sheaf $(q, \mathcal{E}, F_*)$ of $V_{m,X}$ is defined to be the type of the underlying U-quasi-parabolic sheaf $\big(\mathcal{E}(-m), F_*\big)$. (See the subsubsection 3.1.4 for the type.)

Let $\boldsymbol{y}$ be an element of $\mathcal{T}ype$, whose $H^*(X)$-component is y. Let $\big(q, \mathcal{E}, F_*\big)$ be a U-quotient quasi-parabolic sheaf of $V_{m,X}$ with type $\boldsymbol{y}$ on $U \times (X, D)$. We say that $\big(q, \mathcal{E}, F_*\big)$ satisfies the (TFV)-condition for quasi-parabolic shaves, if the following holds for any $u \in U$ and for any i:

(TFV): $(q, \mathcal{E})$ satisfies the (TFV)-condition. Moreover, the sheaves $F_i(\mathcal{E})_{|\{u\} \times X}$ and $\operatorname{Cok}_i(\mathcal{E})_{|\{u\} \times X}$ are generated by global sections, and the higher cohomology groups of $F_i(\mathcal{E})_{|\{u\} \times X}$ and $\operatorname{Cok}_i(\mathcal{E})_{|\{u\} \times X}$ vanish.

In general, the condition determines the maximal open subset of U on which the (TFV)-condition holds.

We put $H_i := H_{\boldsymbol{y},i}$. Let $Q_{m,i}$ denote the scheme representing the moduli functor of the quotient sheaves of $V_{m,X}$ whose Hilbert polynomials are H_i. We have the natural right $GL(V_m)$-action on $Q_{m,i}$. We have the open subset $U_{m,i}$ of $Q_{m,i}$ given by the conditions that $H^j\big(X,\mathcal{E}_i\big)=0$ for any $j>0$ and that $V_m \longrightarrow H^0\big(X,\mathcal{E}_i\big)$ is surjective. Let $G_{m,i}$ denote the Grassmannian variety, which is the moduli of $H_i(m)$-dimensional quotient space of the vector space V_m. We have the $GL(V_m)$-equivariant morphism of $U_{m,i}$ to $G_{m,i}$ by the correspondence:

$$\big(q_i,\mathcal{E}_i\big) \longmapsto \Big(H^0(q_i): V_m \to H^0\big(X,E_i(m)\big)\Big).$$

Let $Q^{tf}(m,y)$ denote the open subset of $Q(m,y)$ consisting of the points corresponding to the torsion-free quotients. For the construction of the moduli of semistable parabolic sheaves, Maruyama and Yokogawa constructed the scheme Γ which is obtained as the subscheme of $Q^{tf}(m,y)\times\prod_i^l U_{m,i}$. (See section 3 of [41].) The scheme Γ is the moduli of quotient quasi-parabolic sheaves $(q,\mathcal{E},F_*)$ of $V_{m,X}$ with type $\boldsymbol{y}$ satisfying the following conditions: (i) $\mathcal{E}$ is torsion-free. (ii) The higher cohomology groups of $\mathrm{Cok}_i(\mathcal{E})$ vanish, and $V_m \longrightarrow H^0\big(X,\mathrm{Cok}_i(\mathcal{E})\big)$ are surjective for any i.

Moreover, the (TFV)-condition determines the open subset of Γ, which is denoted by $Q^\circ(m,\boldsymbol{y})$. The scheme $Q^\circ(m,\boldsymbol{y})$ represents the moduli functor of quotient quasi-parabolic sheaves of $V_{m,X}$ with type $\boldsymbol{y}$ satisfying (TFV)-conditions. We have the universal objects on $Q^\circ(m,\boldsymbol{y})\times X$, which is denoted by $\big(q^u,\mathcal{E}^u,F^u_*\big)$. We have the right $GL(V_m)$-action on $Q^\circ(m,\boldsymbol{y})$ given by $g\cdot(q,\mathcal{E},F_*) := (q\circ g,\mathcal{E},F_*)$.

Let $\mathrm{Pic}_X\big(c\big)$ denote the component of the Picard variety of X such that any line bundle $M\in\mathrm{Pic}_X\big(c\big)$ satisfy $c_1(M)=c$. Let $\mathcal{P}oin_X\big(c\big)$ denote the Poincaré bundle on $\mathrm{Pic}_X\big(c\big)\times X$. Then we obtain the locally free sheaf:

$$\widehat{Z}_m := p_{X\,*}\mathcal{H}om\Big(\bigwedge^{\mathrm{rank}\,y} V_{m,X}, \mathcal{P}oin_X\big(\det y(m)\big)\Big). \tag{38}$$

The projectivization Z_m is called the Gieseker space. We have the natural right action of $GL(V_m)$ on Z_m.

It is known that the correspondence $(q,\mathcal{E}) \longmapsto H^0\big(\bigwedge^r q\big)$ gives the $GL(V_m)$-equivariant morphism of $Q^\circ(m,y)$ to Z_m, which is known to be an immersion. Therefore, we obtain the morphism $Q^\circ(m,y)\times\prod_i U_{m,i} \longrightarrow Z_m\times\prod_i G_{m,i}$. By the composition of the inclusion $Q^\circ(m,\boldsymbol{y}) \longrightarrow Q^\circ(m,y)\times\prod_i U_{m,i}$, we obtain the $GL(V_m)$-equivariant morphism $Q^\circ(m,\boldsymbol{y}) \longrightarrow Z_m\times\prod_i G_{m,i}$. The following lemma was shown in [41].

Lemma 3.64 ([41], Proposition 3.2) *The morphism $Q^\circ(m,\boldsymbol{y}) \longrightarrow Z_m\times\prod_i G_{m,i}$ is an immersion.*

Proof We give only some remarks. Since the morphism $Q^\circ(m,x) \longrightarrow Z_m$ is an immersion, we have only to care the morphism of $Q^\circ(m,\boldsymbol{y})$ to $Q^\circ(m,y)\times\prod G_{m,i}$. Recall the precise result of Maruyama and Yokogawa: Let α_* be a system of weights. Let Γ^{ss} denote the open subscheme of Γ such that the corresponding parabolic sheaves $(\mathcal{E},F_*,\alpha_*)$ are semistable. We may assume that the (TFV)-condition holds for each member Γ^{ss}.

They showed that the morphism of Γ^{ss} to $Q^\circ(m,y)\times\prod_i G_{m,i}$ is immersion. Their argument can be summarized as follows:

(i) Construct a subscheme $\mathcal{Z}$ of $Q(m,y)\times\prod_i Q_{m,i}\times\prod_i G_{m,i}$ such that the projection of $\mathcal{Z}$ to $Q(m,y)\times\prod_i G_{m,i}$ is immersive. ($\mathcal{Z}$ is denoted as $\Delta_1^0\times_Q\times_Q\Delta_2^0\times_Q\cdots\times_Q\Delta_l^0$ in [41].)

(ii) The morphism $\Gamma^{ss}\longrightarrow\prod_i G_{m,i}$ gives the graph $\mathcal{G}$, which is a subset of $Q(m,y)\times\prod_i Q_{m,i}\times\prod_i G_{m,i}$. It can be shown that $\mathcal{G}$ is contained in $\mathcal{Z}$. Then the projection of $\mathcal{G}$ to $Q(m,y)\times\prod_i G_{m,i}$ is immersive. Hence the morphism $\Gamma^{ss}\longrightarrow Q^{tf}(m,y)\times\prod_i G_{m,i}$ is immersion.

For the argument, we need the only fact that each member of Γ^{ss} satisfies the (TFV)-condition. (It is given as the conditions (3.0.1) and (3.0.2) in [41].) Thus the morphism $Q^\circ(m,\boldsymbol{y})\longrightarrow Z_m\times\prod_i G_{m,i}$ is immersive. See [41] for more detail. ∎

3.6.4 Quotient parabolic L-Bradlow pair

Let L be a line bundle on X. Fix a non-trivial morphism $\iota : \mathcal{O}(-m) \longrightarrow L$. Let $q : p_U^* V_{m,X} \longrightarrow \mathcal{E}$ be a U-quotient sheaf with type y defined over $U \times X$ satisfying (TFV)-condition. We have the morphism $\mathcal{E}(-m) \otimes L^{-1} \longrightarrow \mathcal{E}$ induced by ι. Let ϕ be an L-section of $\mathcal{E}(-m)$. Then ι and ϕ induce the $\mathcal{O}$-section $\iota(\phi)$ of $\mathcal{E}$.

Definition 3.65 (Quotient L-section) *Let $(q, \mathcal{E})$ be a quotient sheaf of $V_{m,X}$ defined over $U \times X$. An $\mathcal{O}_X$-section $\overline{\phi}$ of $p_U^* V_{m,X}$ is called a quotient L-section of $\big(q, \mathcal{E}\big)$ with respect to ι, if there exists an L-section of $\mathcal{E}(-m)$ such that $\iota(\phi) = q \circ \overline{\phi}$.*

The condition means that the element $q \circ \overline{\phi} \in \Gamma\big(U \times X, \mathcal{E}\big)$ is contained in $\Gamma\big(U \times X, \mathcal{E}(-m) \otimes L^{-1}\big)$, where the inclusion $\mathcal{E}(-m) \otimes L^{-1} \longrightarrow \mathcal{E}$ is given by ι. ▮

Definition 3.66 *A quotient U-quasi-parabolic L-Bradlow pair of $V_{m,X}$ with type $\boldsymbol{y}$ on $U \times (X, D)$ is defined to be a pair $\big(q, \mathcal{E}, F_*, \overline{\phi}\big)$ of quotient U-quasi parabolic sheaf $(q, \mathcal{E}, F_*)$ of $V_{m,X}$ with type $\boldsymbol{y}$ over $U \times (X, D)$ and a quotient L-section $\overline{\phi}$ with respect to ι.* ▮

Let us construct the moduli scheme $Q^\circ(m, \boldsymbol{y}, L)$ of quotient quasi-parabolic L-Bradlow pairs of $V_{m,X}$ with type $\boldsymbol{y}$ satisfying (TFV)-condition, whose L-section is non-trivial everywhere. We put $Q^\circ\big(m, \boldsymbol{y}, \mathcal{O}(-m)\big) := Q^\circ\big(m, \boldsymbol{y}\big) \times V_m^*$. Let π denote the projection of $Q^\circ\big(m, \boldsymbol{y}, \mathcal{O}(-m)\big) \times X$ onto $Q^\circ\big(m, \boldsymbol{y}\big) \times X$. On $Q^\circ\big(m, \boldsymbol{y}, \mathcal{O}(-m)\big) \times X$, we have the quotient quasi-parabolic sheaf $\pi^*(q^u, \mathcal{E}^u, F_*^u)$ of $V_{m,X}$ with type $\boldsymbol{y}$. We also have the canonical $\mathcal{O}_X$-section $\overline{\phi}^u$ of $p^*_{Q^\circ(m,\boldsymbol{y},\mathcal{O}(-m))} V_{m,X}$, which is induced by the identity of V_m.

We have the composite Λ of the following morphisms on $Q^\circ\big(m, \boldsymbol{y}, \mathcal{O}(-m)\big) \times X$, where the last quotient sheaf is induced by ι:

$$\mathcal{O}_{Q^\circ(m,\boldsymbol{y},\mathcal{O}(-m))} \xrightarrow{\overline{\phi}^u} p^*_{Q^\circ(m,\boldsymbol{y},\mathcal{O}(-m))} V_{m,X} \xrightarrow{\pi^* q^u} \pi^* \mathcal{E}^u \longrightarrow \frac{\pi^* \mathcal{E}^u}{\pi^* \mathcal{E}^u(-m) \otimes L^{-1}}.$$

Recall the following result (Lemma 4.3 of [60]) due to Yokogawa.

Lemma 3.67 *Let $f : \mathcal{X} \longrightarrow S$ be a proper morphism of Noetherian schemes and $\phi : I \longrightarrow F$ be an $\mathcal{O}_{\mathcal{X}}$-morphism of coherent sheaves. Assume that F is flat over S. Then there exists a unique closed subscheme Z of S such that for all morphism $g : T \longrightarrow S$, $g^*\phi = 0$ if and only if g factors through Z.* ▮

Remark the following easy lemma.

Lemma 3.68 *Let E be a U-coherent sheaf on $X \times U$ such that $E_{|X}$ is torsion-free for any $u \in U$. Let D be a Cartier divisor of X. Then $E \otimes \mathcal{O}_D$ is flat over U.*

Proof Let f be any local section of $\mathcal{O}_X$, and let $\mathcal{F}$ be any $\mathcal{O}_U$-coherent sheaf. We have only to show the injectivity of the endomorphism of $E \otimes_{\mathcal{O}_U} \mathcal{F}$ induced by f. By considering the support of $\pi_X\big(\mathrm{Ker}(f)\big)$, we can reduce the case $U = \mathrm{Spec}(K)$ and $\mathcal{F} = \mathcal{O}_U$ for some field K. In the case, the claim is trivial. ▮

Due to the above two lemmas, the vanishing condition of Λ gives the closed subscheme of $Q^\circ\big(m, \boldsymbol{y}, \mathcal{O}(-m)\big)$, which is $Q^\circ(m, \boldsymbol{y}, L)$. By the construction, it is easy to see that $Q^\circ(m, \boldsymbol{y}, L)$ has the desired property. We denote the universal pair on $Q^\circ(m, \boldsymbol{y}, L) \times X$ by $\big(q^u, \mathcal{E}^u, F_*^u, \overline{\phi}^u\big)$. We remark that we have the unique L-section ϕ^u of $\mathcal{E}^u(-m)$ such that $\iota(\phi^u) = q_*(\overline{\phi}^u)$.

The right $GL(V_m)$-action on $Q^\circ\big(m, \boldsymbol{y}, \mathcal{O}(-m)\big)$ is given by $g \cdot (q, \mathcal{E}, F_*, \phi) = \big(q \circ g, \mathcal{E}, F_*, g^{-1} \circ \phi\big)$. The action can be naturally lifted to the action on the universal object. Note that $Q^\circ\big(m, \boldsymbol{y}, L\big)$ is a closed $GL(V_m)$-stable subscheme of $Q^\circ\big(m, \boldsymbol{y}, \mathcal{O}(-m)\big)$.

3.6.5 Quotient reduced L-Bradlow pair

Let $\iota : \mathcal{O} \longrightarrow L(m)$ be the fixed non-trivial morphism. Let $q : p_U^* V_{m,X} \longrightarrow \mathcal{E}$ be a U-quotient sheaf of $V_{m,X}$ defined over $U \times X$. A reduced $\mathcal{O}_X$-section $[\overline{\phi}]$ of $p_U^* V_{m,X}$ and q induce the reduced $\mathcal{O}_X$-section $q_*([\phi])$ of $\mathcal{E}$. On the other hand, a reduced L-section $[\phi]$ of $\mathcal{E}(-m)$ and ι induce the reduced $\mathcal{O}_X$-section $\iota([\phi])$ of $\mathcal{E}$.

Definition 3.69

- *A reduced $\mathcal{O}_X$-section $[\bar{\phi}]$ of $p_U^* V_{m,X}$ is called a quotient reduced L-section if there exists a reduced L-section of $\mathcal{E}(-m)$ such that $\iota([\phi]) = q_*([\overline{\phi}])$.*

- *A quotient U-quasi-parabolic reduced L-Bradlow pair of $V_{m,X}$ with type $\boldsymbol{y}$ on $U \times X$ is defined to be a tuple $\bigl(q, \mathcal{E}, F_*, [\overline{\phi}]\bigr)$ of quotient U-quasi-parabolic sheaf $(q, \mathcal{E}, F_*)$ of $V_{m,X}$ with type $\boldsymbol{y}$ and a reduced L-section $[\overline{\phi}]$ of $\mathcal{E}(-m)$. We also assume that $[\overline{\phi}]$ is non-trivial everywhere.* ∎

Let us construct the scheme $Q^\circ(m, \boldsymbol{y}, [L])$ representing the moduli functor of quotient quasi parabolic reduced L-Bradlow pair of $V_{m,X}$ with type $\boldsymbol{y}$ satisfying (TFV)-condition. We have the free G_m-action on the scheme $Q^\circ(m, \boldsymbol{y}, L)$ defined by $t \cdot (q, \mathcal{E}, F_*, \overline{\phi}) = \bigl(q, \mathcal{E}, F_*, t \cdot \overline{\phi}\bigr)$. Then we put $Q^\circ(m, \boldsymbol{y}, [L]) := Q^\circ(m, \boldsymbol{y}, L)/G_m$. It is the closed subscheme of $Q^\circ(m, \boldsymbol{y}) \times \mathbb{P}_m$. Due to the naturally defined morphism $\pi : Q^\circ(m, \boldsymbol{y}, [L]) \times X \longrightarrow Q^\circ(m, \boldsymbol{y}) \times X$, we have the quotient quasi parabolic sheaf $\bigl(\widehat{q}^u, \widehat{\mathcal{E}}^u, \widehat{F}^u_*\bigr) := (\pi^* q^u, \pi^* \mathcal{E}^u, \pi^* F^u_*)$. The morphism $\overline{\phi}^u$ naturally induces the reduced $\mathcal{O}_X$-section $[\overline{\phi}^u] : p^*_{Q^\circ(m,\boldsymbol{y},[L])} \mathcal{O}_X \otimes \mathcal{O}_{\mathbb{P}_m}(-1) \longrightarrow p^*_{Q^\circ(m,\boldsymbol{y},[L])} V_{m,X}$. We also have the reduced L-section $[\phi^u] : p^*_{Q^\circ(m,\boldsymbol{y},[L])} L \otimes \mathcal{O}_{\mathbb{P}_m}(-1) \longrightarrow p^*_{Q^\circ(m,\boldsymbol{y},[L])} \widehat{\mathcal{E}}(-m)$.

Lemma 3.70 *$Q^\circ(m, \boldsymbol{y}, [L])$ has the desired universal property, and $\bigl(\widehat{q}^u, \widehat{\mathcal{E}}^u, \widehat{F}^u_*, [\phi^u]\bigr)$ is the universal object.*

Proof We give only an outline. Due to Lemma 3.67, we can reduce the case $L = \mathcal{O}(-m)$ and ι is the identity. Let $(q, \mathcal{E}, F_*, [\overline{\phi}])$ denote a U-quotient quasi-parabolic reduced L-Bradlow pair, satisfying (TFV)-condition. We have the map $F : U \longrightarrow Q^\circ(m, \boldsymbol{y})$ corresponding to $(q, \mathcal{E}, F_*)$.

We have the locally free sheaf $p_{X\,*}\mathcal{E}$ and $p_{X\,*}\widehat{\mathcal{E}}^u$ on U and $Q^\circ(m, \boldsymbol{y})$ respectively. Let $\mathbb{P}_1$ and $\mathbb{P}_2$ denote the projectivization of them. We have $\Psi^* \mathbb{P}_2 \simeq \mathbb{P}_1$, and we have the natural morphism $\widetilde{\Psi} : \mathbb{P}_1 \longrightarrow \mathbb{P}_2$. We remark that $\mathbb{P}_2$ is naturally isomorphic to $Q^\circ\bigl(m, \boldsymbol{y}, [\mathcal{O}(-m)]\bigr) = Q^\circ\bigl(m, \boldsymbol{y}\bigr) \times \mathbb{P}_m$. The pull back of the naturally defined reduced L-section $[\overline{\phi}^u]$ on $\mathbb{P}_2 \times X$ is same as the naturally defined reduced L-section on $\mathbb{P}_1 \times X$.

On the other hand, the reduced L-morphism $[\overline{\phi}]$ induces the section $f : U \longrightarrow \mathbb{P}_1$. It is easy to see that $[\overline{\phi}]$ is the pull back of the naturally defined reduced L section on $\mathbb{P}_1 \times X$. Thus we are done. ∎

Since the above G_m-action and the $GL(V_m)$-action on $Q^\circ(m, \boldsymbol{y}, L)$ are commutative, we have the induced right $GL(V_m)$-action on $Q^\circ(m, \boldsymbol{y}, [L])$ and the universal object. We also have the $GL(V_m)$-equivariant immersion of $Q^\circ(m, \boldsymbol{y}, [L])$ to $Z_m \times \prod_i G_{m,i} \times \mathbb{P}(V_m^\vee)$.

3.6.6 Quotient reduced $\boldsymbol{L}$-Bradlow pair

Let $\boldsymbol{L} = (L_1, L_2)$ be a pair of line bundles on X. Quotient quasi-parabolic reduced $\boldsymbol{L}$-Bradlow pairs of $V_{m,X}$ with type $\boldsymbol{y}$ can be given as in Definition 3.69. It is easy to construct the scheme $Q^\circ(m, \boldsymbol{y}, [\boldsymbol{L}])$ representing the moduli functor of quotient quasi-parabolic reduced $\boldsymbol{L}$-Bradlow pairs of $V_{m,X}$ with type $\boldsymbol{y}$ satisfying (TFV)-conditions. In fact, we have only to take the fiber product of $Q^\circ(m, \boldsymbol{y}, [L_i])$ $(i = 1, 2)$ over $Q^\circ(m, \boldsymbol{y})$. We have the natural right $GL(V_m)$-action on $Q^\circ(m, \boldsymbol{y}, [\boldsymbol{L}])$ and the $GL(V_m)$-equivariant immersion $Q^\circ(m, \boldsymbol{y}, [\boldsymbol{L}]) \longrightarrow Z_m \times \prod_i G_{m,i} \times \mathbb{P}_m \times \mathbb{P}_m$.

3.6.7 Oriented objects

We put $Q^\circ(m, \widehat{\boldsymbol{y}}, [L]) := Q^\circ(m, \boldsymbol{y}, [L]) \times_{Q^\circ(m,y)} Q^\circ(m, \widehat{y})$, which represents the moduli functor of quotient oriented quasi-parabolic reduced L-Bradlow pairs of $V_{m,X}$ with type $\boldsymbol{y}$ satisfying (TFV)-condition. We naturally have the universal object. Similarly, the schemes $Q^\circ(m, \widehat{\boldsymbol{y}})$, $Q^\circ(m, \widehat{\boldsymbol{y}}, L)$, $Q^\circ(m, \widehat{\boldsymbol{y}}, [L_1], [L_2])$, etc. are given, which represent appropriate functors respectively.

Let $\widehat{Z}_m$ be as in (38). We have the following naturally induced Cartesian diagram:

$$\begin{array}{ccc} Q^\circ(m, \widehat{y}) & \longrightarrow & \widehat{Z}_m \\ \downarrow & & \downarrow \\ Q^\circ(m, y) & \longrightarrow & Z_m \end{array}$$

The morphisms are $\mathrm{GL}(V_m)$-equivariant. Hence, we obtain the $\mathrm{GL}(V_m)$-equivariant morphism:

$$Q^\circ(m,\widehat{\boldsymbol{y}}) \longrightarrow \widehat{Z}_m \times \prod_i G_{m,i}$$

3.6.8 Quotient stacks

We have the universal quotient quasi-parabolic sheaf $\bigl(q^u,\mathcal{E}^u,F_*\bigr)$ of $p_{Q^\circ(m,\boldsymbol{y})}V_{m,X}$ over $Q^\circ(m,\boldsymbol{y})\times X$. The $\mathrm{GL}(V)$-action on $Q^\circ(m,\boldsymbol{y})$ is naturally lifted to the action on $\bigl(q^u,\mathcal{E}^u,F_*\bigr)$. Then, we obtain the quasi-parabolic sheaf (E^u,F_*) on $\bigl(Q^\circ(m,\boldsymbol{y})/\mathrm{GL}(V_m)\bigr)\times(X,D)$ by taking the descent of $\bigl(\mathcal{E}^u(-m),F_*\bigr)$. The following lemma is well known.

Lemma 3.71 *Let $\boldsymbol{y}$ be an element of $\mathcal{T}ype$. The quotient stack $Q^\circ(m,\boldsymbol{y})/\mathrm{GL}(V_m)$ is isomorphic to $\mathcal{M}(m,\boldsymbol{y})$, and the quasi parabolic sheaf (E^u,F_*) is the universal one.*

Proof Let $g:T\longrightarrow\mathcal{M}(m,\boldsymbol{y})$ be a morphism. Then we have the corresponding T-quasi-parabolic sheaf (E,F_*) on $T\times X$ of type $\boldsymbol{y}$, satisfying the condition O_m. Then we obtain the vector bundle $\mathcal{V}:=p_{X\,*}E(m)$ on T. Let P denote the associated principal $\mathrm{GL}(V_m)$-bundle, and let $\pi:P\longrightarrow T$ denote the projection. On P, we have the equivariant trivialization $\pi_X^*\mathcal{V}\simeq V_m\otimes\mathcal{O}_T$. Thus we obtain the equivariant morphism $q:p_P^*V_{m,X}\longrightarrow\pi_X^*E(m)$. Therefore, we obtain the quotient quasi-parabolic sheaf $\bigl(q,\pi_X^*E(m),F_*\bigr)$ on $T\times X$ which is naturally $\mathrm{GL}(V_m)$-equivariant. It also satisfies the (TFV)-condition. Therefore, we obtain the $\mathrm{GL}(V_m)$-equivariant morphism $P\longrightarrow Q^\circ(m,\boldsymbol{y})$, in other words, the morphism $T\longrightarrow Q^\circ(m,\boldsymbol{y})/\mathrm{GL}(V_m)$. In particular, we obtain $\mathcal{M}(m,\boldsymbol{y})\longrightarrow Q^\circ(m,\boldsymbol{y})/\mathrm{GL}(V_m)$.

On the other hand, let $g:T\longrightarrow Q^\circ(m,\boldsymbol{y})/\mathrm{GL}(V_m)$. We have the corresponding $\mathrm{GL}(V_m)$-torsor $P(g)$ over T. On $P(g)\times X$, we have the quotient $g_X^*(q^u):p_T^*V_{m,X}\longrightarrow g_X^*\mathcal{E}^u$ with a quasi parabolic structure t_X^*F, which is $\mathrm{GL}(V_m)$-equivariant. By taking the descent with respect to the action, we obtain the T-quasi parabolic sheaf $\bigl(\mathcal{E}^u(-m),F_*\bigr)$ on $T\times X$. It satisfies the condition O_m. Therefore, we obtain the morphism of T to $\mathcal{M}(m,\boldsymbol{y})$. In particular, we obtain $Q^\circ(m,\boldsymbol{y})/\mathrm{GL}(V_m)\longrightarrow\mathcal{M}(m,\boldsymbol{y})$.

It is easy to check that they are mutually inverse. ∎

By the same argument, we obtain the description of the moduli stacks $\mathcal{M}(m,\widehat{\boldsymbol{y}})$, $\mathcal{M}(m,\boldsymbol{y},L)$, $\mathcal{M}(m,\boldsymbol{y},[L])$, $\mathcal{M}(m,\widehat{\boldsymbol{y}},[L])$, $\mathcal{M}(m,\widehat{\boldsymbol{y}},[\boldsymbol{L}])$, etc. as the quotient stacks of $Q^\circ(m,\widehat{\boldsymbol{y}})$, $Q^\circ(m,\boldsymbol{y},L)$, $Q^\circ(m,\boldsymbol{y},[L])$, $Q^\circ(m,\widehat{\boldsymbol{y}},[L])$, $Q^\circ(m,\widehat{\boldsymbol{y}},[\boldsymbol{L}])$, etc.. The universal objects are constructed by the same procedure.

4 Geometric Invariant Theory and Enhanced Master Space

4.1 δ-Semistability and Numerical Criterion

4.1.1 Statement

Let X be a smooth d-dimensional projective variety over an algebraically closed field k of characteristic 0, and let $\mathcal{O}_X(1)$ be a very ample line bundle. We denote the number $c_1(\mathcal{O}_X(1))^d\cap[X]$ by g. Let D denote a Cartier divisor of X. We do not have to assume the smoothness of D in this section. We use the notation in the subsection 3.6.

Let $\boldsymbol{y}=(y,y_1,y_2,\ldots,y_l)$ be an element of $\mathcal{T}ype$. (See the subsubsection 3.1.4.) We have the associated Hilbert polynomials $H_{\boldsymbol{y},*}=(H_{\boldsymbol{y}},H_{\boldsymbol{y},1},H_{\boldsymbol{y},2},\ldots,H_{\boldsymbol{y},l})$. (See the subsubsection 3.2.4.) Let V_m be an $H_{\boldsymbol{y}}(m)$-dimensional vector space over k, and let $V_{m,X}$ denote $V_m\otimes\mathcal{O}_X$. Let Z_m denote the Gieseker space over $\mathrm{Pic}\bigl(\det(y(m))\bigr)$, and let $G_{m,i}$ denote the Grassmann variety of $H_{\boldsymbol{y}}(m)-H_{\boldsymbol{y},i}(m)$-dimensional quotients of V_m. We put as follows:

$$\mathcal{A}_m(\boldsymbol{y}):=Z_m\times\prod_i G_{m,i},\quad \mathcal{A}_m(\boldsymbol{y},[L]):=\mathcal{A}_m(\boldsymbol{y})\times\mathbb{P}_m,\quad \mathcal{A}_m(\boldsymbol{y},[\boldsymbol{L}]):=\mathcal{A}_m(\boldsymbol{y})\times\mathbb{P}_m^{(1)}\times\mathbb{P}_m^{(2)}.$$

Here L denotes a line bundle on X, and $\boldsymbol{L}=(L_1,L_2)$ denotes a pair of line bundles on X. We also put $\mathbb{P}_m^{(i)}=\mathbb{P}_m$. Recall that we have obtained the $SL(V_m)$-equivariant immersions Ψ_m of $Q^\circ(m,\boldsymbol{y})$ to $\mathcal{A}_m(\boldsymbol{y})$.

(See the subsubsection 3.6.3). We take an inclusion $\iota : \mathcal{O}(-m) \longrightarrow L$. Then we have the closed immersion $Q^\circ(m,\boldsymbol{y},[L]) \longrightarrow Q^\circ\big(m,\boldsymbol{y},[\mathcal{O}(-m)]\big) = Q^\circ(m,\boldsymbol{y}) \times \mathbb{P}_m$. Therefore, we obtain the equivariant immersion of $Q^\circ(m,\boldsymbol{y},[L])$ to $\mathcal{A}_m(\boldsymbol{y},[L])$. Similarly, we can obtain the equivariant immersion of $Q^\circ(m,\boldsymbol{y},[\boldsymbol{L}])$ to $\mathcal{A}_m(\boldsymbol{y},[\boldsymbol{L}])$ by taking an inclusion $\iota_i : \mathcal{O}(-m) \longrightarrow L_i$. The immersions are denoted by Ψ_m.

Since Z_m is the projective space bundle on $\mathrm{Pic}_X\big(\det(y(m))\big)$, we have the relative tautological bundle $\mathcal{O}_{Z_m}(1)$. We also have the canonical polarizations $\mathcal{O}_{G_{m,i}}(1)$ and $\mathcal{O}_{\mathbb{P}_m}(1)$ of $G_{m,i}$ and $\mathbb{P}_m$ respectively. They are $SL(V_m)$-equivariant line bundles.

For a positive number A and a tuple of positive numbers $B_* = (B_i \,|\, i = 1,\dots,l)$, we formally consider the line bundles on $\mathcal{A}_m(\boldsymbol{y})$ given as follows, although it is not precisely a line bundle when the numbers are not integers:

$$\mathcal{L}_{\boldsymbol{y}}(A,B_*) := \mathcal{O}_{Z_m}(A) \otimes \bigotimes_{i=1}^{l} \mathcal{O}_{G_{m,i}}(B_i).$$

Similarly, for positive numbers C and C_j $(j = 1,2)$, we formally consider $\mathcal{L}_{\boldsymbol{y},L}(A,B_*,C) := \mathcal{L}_{\boldsymbol{y}}(A,B_*) \otimes \mathcal{O}_{\mathbb{P}_m}(C)$ on $\mathcal{A}_m(\boldsymbol{y},L)$ and $\mathcal{L}_{\boldsymbol{y},\boldsymbol{L}}(A,B_*,C_1,C_2) := \mathcal{L}_{\boldsymbol{y}}(A,B_*) \otimes \mathcal{O}_{\mathbb{P}^{(1)}_m}(C_1) \otimes \mathcal{O}_{\mathbb{P}^{(2)}_m}(C_2)$ on $\mathcal{A}_m(\boldsymbol{y},\boldsymbol{L})$.

When the numbers are positive rational numbers, $\mathcal{L}_{\boldsymbol{y}}(A,B_*)$ gives $SL(V_m)$-equivariant $\mathbb{Q}$-polarizations of $\mathcal{A}_m(\boldsymbol{y})$. Even if the numbers are not rational, the Hilbert-Mumford criterion formally provides us the semistability condition with respect to $\mathcal{L}_{\boldsymbol{y}}(A,B_*)$. So, let $\mathcal{A}^{ss}_m(\boldsymbol{y},A,B_*)$ denote the set of semistable points of the $SL(V_m)$-action on $\mathcal{A}_m(\boldsymbol{y})$ with respect to $\mathcal{L}_{\boldsymbol{y}}(A,B_*)$. Similarly, $\mathcal{A}^{ss}_m(\boldsymbol{y},L,A,B_*,C)$ and $\mathcal{A}^{ss}_m(\boldsymbol{y},\boldsymbol{L},A,B_*,C_1,C_2)$ are given.

The semistability (resp. stability) condition with respect to the system of weights α_* determines the open subset $Q^{ss}(m,\boldsymbol{y},\alpha_*)$ (resp. $Q^{s}(m,\boldsymbol{y},\alpha_*)$) of $Q^\circ(m,\boldsymbol{y})$. Namely, it is the maximal open subset of $Q^\circ(m,\boldsymbol{y})$ which consists of the points $(q,\mathcal{E},F_*)$ such that the parabolic sheaf $(\mathcal{E}(-m),F_*,\alpha_*)$ is semistable. Similarly, we have the open subset $Q^{ss}(m,\boldsymbol{y},[L],\alpha_*,\delta)$ (resp. $Q^{s}(m,\boldsymbol{y},[L],\alpha_*,\delta)$) of $Q^\circ(m,\boldsymbol{y},[L])$ determined by the semistability (stability) condition with respect to the system of weights α_* and a parameter δ. We also have the open subsets $Q^{ss}(m,\boldsymbol{y},[\boldsymbol{L}],\alpha_*,\boldsymbol{\delta})$ and $Q^{s}(m,\boldsymbol{y},[\boldsymbol{L}],\alpha_*,\boldsymbol{\delta})$ of $Q^\circ(m,\boldsymbol{y},[\boldsymbol{L}])$ in a similar way.

The first claim of the following proposition was proved by Maruyama-Yokogawa [41] and the second claim was proved by Yokogawa in [61].

Proposition 4.1 *There exists an integer $N(\boldsymbol{y},\alpha_*)$ such that the following holds for any $m \geq N(\boldsymbol{y},\alpha_*)$:*

1. *The image of $Q^{ss}(m,\boldsymbol{y},\alpha_*)$ via the morphism Ψ_m is contained in $\mathcal{A}^{ss}\big(m,\boldsymbol{y},P^{\alpha_*}_{\boldsymbol{y}}(m),\epsilon_*\big)$. Thus we obtain the morphism $\widehat{\Psi}_m : Q^{ss}(m,\boldsymbol{y},\alpha_*) \longrightarrow \mathcal{A}^{ss}\big(m,\boldsymbol{y},P^{\alpha_*}_{\boldsymbol{y}}(m),\epsilon_*\big)$.*
2. *The morphism $\widehat{\Psi}_m$ above is proper. In particular, it is a closed immersion.* ▮

By the same argument, we can show the following proposition.

Proposition 4.2 *There exists an integer $N_1(\boldsymbol{y},L,\alpha_*,\delta)$ such that the following claims hold for any $m \geq N_1(\boldsymbol{y},L,\alpha_*,\delta)$:*

1. *The image of $Q^{ss}(m,\boldsymbol{y},[L],\alpha_*,\delta)$ via the morphism Ψ_m is contained in $\mathcal{A}^{ss}_m\big(P^{\alpha_*,\delta}_{\boldsymbol{y}}(m),\epsilon_*,\delta(m)\big)$. Thus we obtain the morphism $\widehat{\Psi}_m : Q^{ss}(m,\boldsymbol{y},[L],\alpha_*,\delta) \longrightarrow \mathcal{A}^{ss}_m\big(P^{\alpha_*,\delta}_{\boldsymbol{y}}(m),\epsilon_*,\delta(m)\big)$.*
2. *The morphism $\widehat{\Psi}_m$ above is proper. In particular, it is a closed immersion.*

Similarly, there exists a large integer $N_1(\boldsymbol{y},\boldsymbol{L},\alpha_*,\boldsymbol{\delta})$ such that we have the $SL(V_m)$-equivariant closed immersion for any $m \geq N_1(\boldsymbol{y},\boldsymbol{L},\alpha_*,\boldsymbol{\delta})$:

$$\widehat{\Psi}_m : Q^{ss}(m,\boldsymbol{y},[\boldsymbol{L}],\alpha_*,\boldsymbol{\delta}) \longrightarrow \mathcal{A}^{ss}_m\big(\boldsymbol{y},[\boldsymbol{L}],P^{\alpha_*,\boldsymbol{\delta}}_{\boldsymbol{y}}(m),\epsilon_*,\boldsymbol{\delta}(m)\big).$$

Here, we put $\boldsymbol{\delta}(m) := (\delta_1(m),\delta_2(m))$.

Although we need only a minor modification, we will later give a rather detailed proof of the claims for $(\boldsymbol{y}, L, \alpha_*, \delta)$ in Proposition 4.2, for the convenience of the reader. We closely follow the arguments of [41] and [61]. We also use the argument in [28]. Since the claim for $(\boldsymbol{y}, \boldsymbol{L}, \alpha_*, \boldsymbol{\delta})$ can be shown similarly, we omit to give the proof of it.

We also obtain the following.

Proposition 4.3 *The image of $Q^s(m, \boldsymbol{y}, [L], \alpha_*, \delta)$ via Ψ_m is contained in $\mathcal{A}^s_m(\boldsymbol{y}, [L], \alpha_*, \delta)$. Similar claims hold for $Q^s(m, \boldsymbol{y}, \alpha_*)$ and $Q^s(m, \boldsymbol{y}, [\boldsymbol{L}], \alpha_*, \boldsymbol{\delta})$.*

The claim for $Q^s(m, \boldsymbol{y})$ was proved by Maruyama-Yokogawa in [41]. Since the argument is similar to the proof of the first claim of Proposition 4.2, we give just some remarks in the proof of Proposition 4.2.

Remark 4.4 *When α_i and the coefficients of δ and δ_i $(i = 1, 2)$ are rational, it is standard to obtain the projective coarse moduli scheme of δ-semistable parabolic L-Bradlow pairs or $\boldsymbol{\delta}$-semistable parabolic $\boldsymbol{L}$-Bradlow pairs from Proposition 4.2.*

Even if the numbers are not rational, we can obtain the coarse moduli schemes of δ-stable parabolic reduced L-Bradlow pairs, if the 1-stability condition holds for $(\boldsymbol{y}, L, \alpha_*, \delta)$. Take a sufficiently large $N > 0$, and take rational numbers A, $B_* = (B_i \,|\, i = 1, \ldots, l)$ and C which are close to $N \cdot P^{\delta,\alpha_*}_{\boldsymbol{y}}(m)$, $N \cdot \epsilon_*$ and $N \cdot \delta(m)$ respectively. If a point of $\mathcal{A}(m, \boldsymbol{y}, [L])$ is stable with respect to $\mathcal{L}_{\boldsymbol{y},L}\big(P^{\delta,\alpha_*}_{\boldsymbol{y}}(m), \epsilon_*, \delta(m)\big)$, then it is stable with respect to $\mathcal{L}_{\boldsymbol{y},L}(A, B_*, C)$. Thus we can obtain the coarse scheme by the geometric invariant theory (Proposition 2.3). Therefore, when the 1-stability condition holds, we obtain the projective coarse moduli scheme of semistable ones. ∎

Before going into the proof of Proposition 4.2, we give some consequence about the property of the moduli stacks.

Corollary 4.5 *When the 1-stability condition holds for $(\boldsymbol{y}, \alpha_*)$, the moduli stack $\mathcal{M}^s(\widehat{\boldsymbol{y}}, \alpha_*)$ is Deligne-Mumford and proper.*

Proof Under the assumption, we obtain $Q^{ss}(\boldsymbol{y}, \alpha_*) = Q^s(\boldsymbol{y}, \alpha_*)$. It is easy to see the finiteness of the stabilizer of any point $z \in Q^{ss}(\boldsymbol{y}, \alpha_*)$ with respect to the $\mathrm{SL}(V_m)$-action from Corollary 3.29. Then, the quotient stack $Q^s(\boldsymbol{y}, \alpha_*)/\,\mathrm{SL}(V_m)$ is Deligne-Mumford and proper, due to Proposition 4.1, Proposition 4.3 and Proposition 2.4.

Since we have the etale finite morphisms $Q^s(\boldsymbol{y}, \alpha_*)/\,\mathrm{SL}(V_m) \longrightarrow Q^s(\boldsymbol{y}, \alpha_*)/\,\mathrm{PGL}(V_m)$ and $\mathcal{M}(\widehat{\boldsymbol{y}}, \alpha_*) \longrightarrow Q^s(\boldsymbol{y}, \alpha_*)/\,\mathrm{PGL}(V_m)$, it is easy to derive the claimed property of $\mathcal{M}(\widehat{\boldsymbol{y}}, \alpha_*)$. ∎

Similarly, we obtain the following.

Proposition 4.6

- *If the 1-stability condition holds for $(\boldsymbol{y}, L, \alpha_*, \delta)$, the moduli stacks $\mathcal{M}^s(\widehat{\boldsymbol{y}}, [L], \alpha_*, \delta)$ and $\mathcal{M}^s(\boldsymbol{y}, L, \alpha_*, \delta)$ are Deligne-Mumford and proper.*
- *If the 1-stability condition holds for $(\boldsymbol{y}, \boldsymbol{L}, \alpha_*, \boldsymbol{\delta})$, the moduli stack $\mathcal{M}^s(\widehat{\boldsymbol{y}}, [\boldsymbol{L}], \alpha_*, \boldsymbol{\delta})$ is Deligne-Mumford and proper.* ∎

4.1.2 The numerical criterion

Let us start the proof of Proposition 4.2. Let $(q, E_*, \phi, W_*, [\widetilde{\phi}])$ be a Yokogawa datum of type $(\boldsymbol{y}, m)$ (Definition 3.42). From W_*, we obtain the tuple of the quotients $\big(V_m/W_i \,|\, i = 1, \ldots, l\big)$, which gives the point of $\prod_i G_{m,i}$. We denote the point by V/W_*. Then we obtain the following element of $\mathcal{A}_m(\boldsymbol{y}, [L])$:

$$\Psi\big(q, E_*, \phi, W_*, [\widetilde{\phi}]\big) := \left(H^0\Big(\bigwedge^r q\Big),\, V_m/W_*, [\widetilde{\phi}]\right) \in \mathcal{A}_m(\boldsymbol{y}, [L]). \tag{39}$$

Let W be a subspace of V_m. The number $\epsilon(W, [\widetilde{\phi}])$ is given as in Definition 3.43. Let $\mathcal{E}_W$ denote the subsheaf of $E(m)$ generated by the image of W via q.

Lemma 4.7 *The point $\Psi\big(q, E_*, \phi, W_*, [\widetilde{\phi}]\big)$ is contained in $\mathcal{A}^{ss}_m\big(P^{\alpha_*,\delta}_{\boldsymbol{y}}(m), \epsilon_*, \delta(m)\big)$, if and only if the following inequalities hold for any non-trivial subspace $W \subset V_m$:*

$$P^{\alpha_*,\delta}_{\boldsymbol{y}}(m) \cdot \operatorname{rank}(\mathcal{E}_W) - \epsilon(W, [\phi]) \cdot \delta(m) - \sum \epsilon_i \cdot \dim(W_i \cap W) - \alpha_1 \cdot \dim(W) \geq 0. \tag{40}$$

In other words, $\Psi(q, E_*, \phi, W_*, [\widetilde{\phi}])$ is semistable point, if and only if $(q, E_*, \phi, W_*, [\widetilde{\phi}]) \in \widetilde{\mathrm{YOK}}(m, 0, \boldsymbol{y}, L, \delta)$.

The point $\Psi\big(q, E_*, \phi, W_*, [\widetilde{\phi}]\big)$ is contained in $\mathcal{A}^{s}_m\big(P^{\alpha_*,\delta}_{\boldsymbol{y}}(m), \epsilon_*, \delta(m)\big)$, if and only if the strict inequalities hold in (40) for any subspace $W \subset V_m$.

Proof We give only an indication. We put $\mathcal{L}_1 := \mathcal{O}_Z\big(P^{\alpha_*}_{\boldsymbol{y}}(m)\big) \otimes \bigotimes_{i=1}^{l} \mathcal{O}_{G_m\, i}(\epsilon_i)$ and $\mathcal{L}_2 := \mathcal{O}_Z\big(r^{-1} \cdot \delta(m)\big) \otimes \mathcal{O}_{\mathbb{P}_m}(\delta(m))$. Then we have $\mathcal{L}\big(P^{\alpha_*,\delta}_{\boldsymbol{y}}(m), \epsilon_*, \delta(m)\big) = \mathcal{L}_1 \otimes \mathcal{L}_2$.

We put $N = \dim V_m$. Let $u_1, \ldots, u_N$ be the vectors satisfying the following:

$$\lambda(t) \cdot u_i = t^{w_i} \cdot u_i, \quad w_1 \leq w_2 \leq \cdots \leq w_N, \quad \sum w_i = 0. \tag{41}$$

We remark that we can regard (41) as the condition for λ, when we fix $u_1, \ldots, u_N$.

The number $\mu_\lambda(P, \mathcal{L}_1)$ is calculated in [41] (see also [6]):

$$\mu_\lambda(P, \mathcal{L}_1) = -P^{\alpha_*}_{\boldsymbol{y}}(m) \sum_{i=1}^{N} \big(\operatorname{rank} \mathcal{E}^{(i)} - \operatorname{rank} \mathcal{E}^{(i-1)}\big) w_i - \sum_{j=1}^{l} \epsilon_j \sum_{i=1}^{N} \Big(\operatorname{rank} W_j \cap \mathcal{V}^{(i-1)} - \operatorname{rank} W_j \cap \mathcal{V}^{(i)} + 1\Big) \cdot w_i. \tag{42}$$

Here, $\mathcal{E}^{(i)}$ denote the subsheaves of $E(m)$ generated by $u_1, \ldots, u_i$ via q, and $\mathcal{V}^{(i)}$ denote the subspaces of V_m generated by $u_1, \ldots, u_i$. The number $\mu_\lambda(P, \mathcal{L}_2)$ is as follows:

$$\delta(m) \cdot \left(-\frac{1}{r} \sum_{i=1}^{N} \big(\operatorname{rank} \mathcal{E}^{(i)} - \operatorname{rank} \mathcal{E}^{(i-1)}\big) \cdot w_i + \sum_{i=1}^{N} \Big(\dim\langle\widetilde{\phi}\rangle \cap \mathcal{V}^{(i)} - \dim\langle\widetilde{\phi}\rangle \cap \mathcal{V}^{(i-1)}\Big) \cdot w_i \right) \tag{43}$$

Here, $\langle\widetilde{\phi}\rangle$ denotes the subspace of V_m generated by $\widetilde{\phi}$. The number $\mu_\lambda(P, \mathcal{L})$ can be obtained as the sum of (42) and (43). We write it as the reference for the later argument:

$$\begin{aligned} \mu_\lambda(P, \mathcal{L}) = -P^{\alpha_*}_{\boldsymbol{y}}(m) \cdot \sum_{i=1}^{N} \big(\operatorname{rank} \mathcal{E}^{(i)} - \operatorname{rank} \mathcal{E}^{(i-1)}\big) \cdot w_i - \sum_{j=1}^{l} \epsilon_j \sum_{i=1}^{N} \Big(\operatorname{rank} W_j \cap \mathcal{V}^{(i-1)} - \operatorname{rank} W_j \cap \mathcal{V}^{(i)} + 1\Big) \cdot w_i \\ + \delta(m) \left(-\frac{1}{r} \sum_{i=1}^{N} \big(\operatorname{rank} \mathcal{E}^{(i)} - \operatorname{rank} \mathcal{E}^{(i-1)}\big) \cdot w_i + \sum_{i=1}^{N} \big(\dim\langle\widetilde{\phi}\rangle \cap \mathcal{V}^{(i)} - \dim\langle\widetilde{\phi}\rangle \cap \mathcal{V}^{(i-1)}\big) \cdot w_i \right). \end{aligned} \tag{44}$$

The right hand side of (44) is linear with respect to $(w_1, \ldots, w_N)$. Therefore, $\mu_\lambda(P, \mathcal{L}) \geq 0$ holds for any λ satisfying (41), if and only if $\mu_{f_k}(P, \mathcal{L}) \geq 0$ for any $k = 1, \ldots, N-1$, where $f_k = (\overbrace{k-N, \ldots, k-N}^{k}, k \ldots, k)$. Due to the calculation of Maruyama and Yokogawa, we have the following:

$$\mu_{f_k}(P, \mathcal{L}_1) = H(m) \cdot \left(P^{\alpha_*}_{\boldsymbol{y}}(m) \cdot \operatorname{rank} \mathcal{E}^{(k)} - \sum_{i=1}^{l} \epsilon_i \cdot \dim W_i \cap \mathcal{V}^{(k)} - \alpha_1 \cdot \dim \mathcal{V}^{(k)}. \right) \tag{45}$$

By a direct calculation, we have the following:

$$\mu_{f_k}(P, \mathcal{L}_2) = H(m) \cdot \delta(m) \cdot \left(\frac{\operatorname{rank} \mathcal{E}^{(k)}}{r} - \epsilon\big(\mathcal{V}^{(k)}, [\widetilde{\phi}]\big) \right). \tag{46}$$

Hence $\mu_{f_k}(P, \mathcal{L})$ depends only on $\mathcal{V}^{(k)}$. Therefore, we obtain the function $F_{(P,\mathcal{L})}$ on the sets of the non-trivial subspaces $\{0 \neq W \subsetneq V\}$, and P is semistable with respect to $\mathcal{L}$ if and only if $F_{(P,\mathcal{L})}(W) \geq 0$ for any subspace W. Since $F_{(P,\mathcal{L})}(W)$ is the left hand side of (40) multiplied with the positive number $H(m)$, we are done. ∎

4.1.3 A lemma

Following Huybrecht-Lehn [28], we show the following lemma.

Lemma 4.8 *Let $\boldsymbol{y}$ be an element of $\mathcal{T}ype$. There exists an integer N_4 with the following property:*

- *Let (E_*, ϕ) be a δ-semistable parabolic L-Bradlow pair of type $\boldsymbol{y}$ such that $\phi \neq 0$. Then the following inequality holds for any integer $m \geq N_4$ and for any subsheaf $F \subset E$:*
$$\frac{h^0\big(F_*(m)\big) + \epsilon(\phi, F) \cdot \delta(m)}{\operatorname{rank}(F)} \leq P^{\delta}_{(E_*,\phi)}(m). \tag{47}$$
- *If the equality holds in (47), (F_*, ϕ') is δ-semistable with $P^{\delta}_{(F_*,\phi')} = P^{\delta}_{(E_*,\phi)}$.*
- *If (E_*, ϕ) is δ-stable, the strict inequality holds in (47).*

Proof We have the inequality $\mu(F) \leq \mu^{\delta}(F_*) \leq \mu^{\delta}(E_*)$ for any subsheaf $F \subset E$, and hence $\mu_{\max}(F) \leq \mu(E_*) + \delta_{top} = C_0$. On the other hand, we have the inequality $\mu_{\min}(F) \leq \mu(F)$ by definition. Hence we obtain the following inequality by using Proposition 3.36 ([28]):
$$\frac{h^0(F(m))}{\operatorname{rank}(F)} \leq \frac{1}{g^{d-1}d!}\left(\Big(1 - \frac{1}{\operatorname{rank}(F)}\Big)\big[C_0 + mg + c\big]^d_+ + \frac{1}{\operatorname{rank}(F)}\big[\mu(F) + mg + c\big]^d_+\right).$$

We can take a sufficiently negative number C with the following property:

- For any positive integer r' such that $r' < \operatorname{rank}(E)$ and for any sufficiently large $t > 0$, the following inequalities hold:
$$\frac{1}{g^{d-1}d!}\left(\Big(1 - \frac{1}{r'}\Big)(C_0 + tg + c)^d + \frac{1}{r'}(C + tg + c)^d\right) + \frac{\delta(t)}{r'} \leq P^{\alpha_*}_{\boldsymbol{y}}(t). \tag{48}$$

Note that the coefficient of t^d of the both sides are same by the construction, and thus we can take such C. Let N_5 be a large number such that $-C + mg + c > 0$ for any $m > N_5$.

Let $\mathcal{S}$ be the family of the sheaves F with the following property:

- There exists a δ-semistable parabolic reduced L-Bradlow pairs (E_*, ϕ) of type $\boldsymbol{y}$ with weight α_* such that F is a saturated subsheaf of E.

We divide $\mathcal{S}$ into two families by the following conditions: (i) $\mu(F) < C$ or (ii) $\mu(F) \geq C$. If F is contained in the family (i), then the desired inequality holds for any $m \geq N_5$ because of our choices of C and N_5. On the other hand, the family (ii) is bounded (Proposition 3.34). Thus we can take a large number N_6 such that the family (ii) satisfies the condition (O_m) for any $m \geq N_6$. Then the desired inequalities hold for any $m \geq N_6$, because of the δ-semistability of (E_*, ϕ). Thus we have only to put $N_4 := \max\{N_5, N_6\}$. ∎

4.1.4 Proof of the claim 1 in Proposition 4.2

Let $N_1(\boldsymbol{y}, L, \alpha_*, \delta)$ be an integer larger than N_4 in Lemma 4.8 and $N_0(\boldsymbol{y}, L, \delta)$ in Proposition 3.44. Let $\big(q, E(m), F_*, [\widetilde{\phi}]\big)$ be a point of $Q^{ss}(m, \boldsymbol{y}, [L], \alpha_*, \delta)$. The reduced L-section $[\phi]$ of E is induced by $[\widetilde{\phi}]$ and ι. By definition, $H^0(q)$ gives the isomorphism $V_m \simeq H^0\big(X, E(m)\big)$. Let W be a subspace of V_m, which generates the subsheaf $\mathcal{E}_W$ of $E(m)$ via q. Due to Lemma 4.8, we have the following inequality:
$$\frac{h^0(\mathcal{E}_{W,*}) + \epsilon\big(\mathcal{E}_W(-m), \phi\big) \cdot \delta(m)}{\operatorname{rank}(\mathcal{E}_W)} \leq P^{\delta,\alpha_*}_{\boldsymbol{y}}(m). \tag{49}$$

We have the equalities $\epsilon([\phi], W) = \epsilon([\phi], \mathcal{E}_W)$ and the inequalities $\dim(W) \leq h^0(\mathcal{E}_W)$ and $\dim(W \cap W_i) \leq h^0(F_i(\mathcal{E}_W))$. Thus we obtain the following inequality:
$$\alpha_1 \dim(W) + \sum \epsilon_i \dim(W_i \cap W) \leq \alpha_1 \cdot h^0(\mathcal{E}_W) + \sum \epsilon_i \cdot h^0(F_i(\mathcal{E}_W)) = h^0\big(\mathcal{E}_{W*}\big). \tag{50}$$

By substituting (50) to (49), we obtain the inequality (40). Hence $\Psi_m(q, E(m), F_*, [\widetilde{\phi}])$ gives the point contained in $\mathcal{A}^{ss}_m\big(P^{\alpha_*,\delta}_{\boldsymbol{y}}(m), \epsilon_*, \delta(m)\big)$. ∎

Remark 4.9 *We obtain Proposition* 4.3 *by using the same argument and the second claims in Lemma* 4.7 *and Lemma* 4.8. ∎

4.1.5 Proof of the claim 2 of Proposition 4.2

Take a discrete valuation ring R over k. We denote the quotient field by K. Assume that we have the following diagram:

$$\begin{array}{ccc} \mathrm{Spec}(K) & \xrightarrow{f} & Q^{ss}(m,\boldsymbol{y},[L],\alpha_*,\delta) \\ \Big\downarrow & & \Big\downarrow{\scriptstyle\widehat{\Phi}} \\ \mathrm{Spec}(R) & \xrightarrow{g} & \mathcal{A}_m^{ss}\big(P_{\boldsymbol{y}}^{\alpha_*,\delta}(m),\epsilon_*,\delta(m)\big) \end{array} \tag{51}$$

We have only to show the existence of a lift $\mathrm{Spec}(R) \longrightarrow Q^{ss}(m,\boldsymbol{y},[L],\alpha_*,\delta)$. Let X_K and X_R denote $X\times\mathrm{Spec}\,K$ and $X\times\mathrm{Spec}\,R$ respectively.

The morphism f corresponds to the tuple $(q_K, E_{K*}(m), [\widetilde{\phi}_K])$ of the quotient parabolic sheaf $\big(q_K, E_{K*}(m)\big)$ and the quotient reduced L-section $[\widetilde{\phi}_K]$ defined over X_K. The tuple $(q_K, E_{K*}(m))$ satisfies the (TFV)-condition. The L-section $[\phi_K]$ of E_K is induced by $[\widetilde{\phi}_K]$ and ι, and the parabolic L-Bradlow pair $(E_{K\,*},\phi_K)$ with weight α_* is δ-semistable.

As in [61] pp. 502–503, $E_{K*}(m)$ can be extended to the parabolic torsion-free sheaf $E_{R*}(m)$ over X_R. The morphism q_K can be extended to the morphism $q_R : V_m\otimes\mathcal{O}_{X_R}\longrightarrow E_R(m)$ such that the restriction of q_R to the closed fiber is generically surjective. Since $\mathbb{P}_m$ is proper, we can extend $[\widetilde{\phi}_K]$ to $[\widetilde{\phi}_R]$ over R. We put $W_{K,i} := H^0\big(X\otimes K, F_{i+1}(E(m))\big)$ $(i=1,\ldots,l)$ which give the subspaces of $V_m\otimes K$. Since $\prod_i G_{m,i}$ are proper, we obtain the subbundle $W_{R,i}$ of $V_m\otimes R$ over $\mathrm{Spec}\,R$. We put $W_{R,*} = (W_{R,i}\,|\,i=1,\ldots,l)$. The family $\big(q_R, E_{R\,*}, W_{R\,*}, [\widetilde{\phi}_R]\big)$ induces the morphism $\mathrm{Spec}(R)\longrightarrow\mathcal{A}_m(\boldsymbol{y},[L])$ as in (39). By separatedness of $\mathcal{A}_m(\boldsymbol{y},[L])$, it coincides with g in the diagram (51).

Let $\big(q_0, E_{0*}, W_{0*}, [\widetilde{\phi}_0]\big)$ denote the specialization of $\big(q_R, E_{R*}, W_{R*}, [\widetilde{\phi}_R]\big)$ to the closed point of $\mathrm{Spec}\,R$. We also have the induced L-section ϕ_0 of E_0. The tuple $\big(q_0, E_{0*}, \phi_0, W_{0*}, [\widetilde{\phi}_0]\big)$ is the Yokogawa datum such that $\Psi\big(q_0, E_{0\,*}, \phi_0, W_{0\,*}, [\widetilde{\phi}_0]\big)$ is contained in $\mathcal{A}_m^{ss}\big(P_{\boldsymbol{y}}^{\alpha_*,\delta}(m),\epsilon_*,\delta(m)\big)$. Due to Lemma 4.7, $\big(q_0, E_{0*}, \phi_0, W_{0*}, [\widetilde{\phi}_0]\big)$ is contained in $\widehat{\mathrm{YOK}}(m,0,\boldsymbol{y},L,\delta)$. Recall $m\geq N_1(\boldsymbol{y},L,\alpha_*,\delta)\geq N_0(\boldsymbol{y},L,\delta)$, where $N_0(\boldsymbol{y},L,\delta)$ is as in Proposition 3.44. Hence, we know that q_0 is surjective, that $E_{0,*}(m)$ satisfies the TFV-condition, and that the parabolic L-Bradlow pair $(E_{0,*},\phi_0)$ is δ-semistable. Hence $\big(q_R, E_{R\,*}, W_{R*}, [\phi_R]\big)$ gives a map $f_R:\mathrm{Spec}(R)\longrightarrow Q^{ss}\big(m,\boldsymbol{y},[L],\alpha_*,\delta\big)$, whose restriction to $\mathrm{Spec}(K)$ is f. It is clear that f_R gives the lift of g. Hence the claim 2 of Proposition 4.2 is proved. ∎

4.1.6 Complement

We give a consequence of the proof. We put $V := V_m$, $Q := Q^{ss}(m,\boldsymbol{y},[L],\alpha_*,\delta)$. We also put as follows:

$$\mathcal{L} := \mathcal{L}_{\boldsymbol{y},L}(P_{\boldsymbol{y}}^{\delta,\alpha_*}(m),\epsilon_*,\delta(m))$$

Lemma 4.10 *Let $z=(q,E,F_*,[\phi])$ be a point of Q. Let $V=V'\oplus V''$ be a decomposition, and let λ be the one-parameter subgroup of* $\mathrm{SL}(V)$ *given by $t^{-\mathrm{rank}\,V''}\cdot\mathrm{id}_{V'}\oplus t^{\mathrm{rank}\,V'}\cdot\mathrm{id}_{V''}$. Let $E'(m)$ denote the subsheaf generated by V'. We have the induced parabolic structure and the L-Bradlow pair ϕ' of E'. Then (E'_*,ϕ') is δ-semistable with $P^{\delta}_{(E'_*,\phi')}=P_{\boldsymbol{y}}^{\alpha_*,\delta}$, if and only if $\mu_\lambda(z,\mathcal{L})=0$ hold.*

Proof Assume $\mu_\lambda(z,\mathcal{L})=0$. We put $W:=H^0\big(X,E'(m)\big)$. Let W_i denote the kernel of $H^0\big(X,E(m)\big)\longrightarrow H^0\big(X,\mathrm{Cok}_{i+1}(m)\big)$. From the calculation in the proof of Lemma 4.7, we have the following:

$$0=\mu_\lambda(z,\mathcal{L})=H(m)\cdot\Big(P_{\boldsymbol{y}}^{\alpha_*,\delta}(m)\cdot\mathrm{rank}\,E'-\epsilon(E',\phi')\cdot\delta(m)-\sum\epsilon_i\cdot\dim(W_i\cap W)-\alpha_1\cdot\dim(W)\Big). \tag{52}$$

Let $\mathcal{E}_W$ denote the subsheaf of $E(m)$ generated by W via q. Therefore, we obtain the following inequality from (52):

$$P_{\boldsymbol{y}}^{\alpha_*,\delta}(m)=\frac{\alpha_1\cdot\dim(W)+\sum\epsilon_i\cdot\dim(W_i\cap W)+\epsilon(E',\phi')\cdot\delta(m)}{\operatorname{rank}E'}\leq\frac{h^0(\mathcal{E}_{W\,*})+\epsilon(\mathcal{E}_W,\phi)\cdot\delta(m)}{\operatorname{rank}\mathcal{E}_W}$$
$$\leq\frac{h^0(E'_*(m))+\epsilon(E',\phi)\cdot\delta(m)}{\operatorname{rank}E'}\leq P_{\boldsymbol{y}}^{\alpha_*,\delta}(m)\quad(53)$$

Here, the first inequality is obtained in (50), the second inequality follows from $\mathcal{E}_W\subset E'(m)$, and the third inequality follows from Lemma 4.8. We can conclude that the equality holds in (53). Then (E'_*,ϕ') is δ-semistable due to the second claim of Lemma 4.8.

Assume (E'_*,ϕ') is δ-semistable. Note that the condition O_m holds for (E'_*,ϕ'), because it holds for $(E'_*,\phi')\oplus\big((E/E')_*,\phi''\big)$, where ϕ'' denotes the induced L-section on the quotient E/E'. Hence we have $h^0(E'_*)=\alpha_1\cdot\dim V'+\sum\epsilon_i\cdot\dim W_i\cap V'$. Then $\mu_\lambda(z,\mathcal{L})=0$ follows from the calculation of $\mu_\lambda(z,\mathcal{L})$ in the proof of Lemma 4.7. See (45) and (46). We remark $\lambda=f_k$ and $V'=V^{(k)}$ in this case. ∎

Corollary 4.11 *Let $z=(q,E,F_*,[\phi])$ be a point of Q. Let λ be a one parameter subgroup of* $\mathrm{SL}(V)$*. Let $V=\bigoplus\mathcal{V}_i$ be the weight decomposition of λ, i.e., λ preserves the decomposition, and the weight on $\mathcal{V}_i$ is i. Let $\mathcal{E}^{(i)}$ be the subsheaf of $E(m)$ generated by $\mathcal{V}_j$ $(j\leq i)$ via q. We have the induced L-section ϕ_i and the parabolic structure of $\mathcal{E}^{(i)}(-m)$. Then all $\big(\mathcal{E}^{(i)}(-m)_*,\phi_i\big)$ are δ-semistable with $P^{\delta}_{(\mathcal{E}^{(i)}(-m)_*,\phi_i)}=P^{\delta}_{(E_*,\phi)}$, if and only if $\mu_\lambda(z,\mathcal{L})=0$ holds.*

Proof We put $U_i=\bigoplus_{j\leq i}\mathcal{V}_j$ and $U'_i:=\bigoplus_{j>i}\mathcal{V}_i$. Let λ_i be the one-parameter subgroup of $\mathrm{SL}(V)$ given by $t^{-\operatorname{rank}U''_i}\cdot id_{U'_i}\oplus t^{\operatorname{rank}U'_i}\cdot\mathrm{id}_{U''_i}$. It is easy to see that λ can be expressed as $\prod\lambda_i^{a_i}$ with $a_i\in\mathbb{Q}_{>0}$. The condition $\mu_\lambda(z,\mathcal{L})=0$ implies $\mu_{\lambda_i}(z,\mathcal{L})=0$. Therefore, the claim immediately follows from Lemma 4.10. ∎

4.2 Perturbation of δ-Semistability

4.2.1 Preliminary

We continue to use the notation in the subsubsection 4.1.1. We put $V:=V_m$, $Q:=Q^{ss}(m,\boldsymbol{y},[L],\alpha_*,\delta)$ and $\mathcal{L}:=\mathcal{L}_{\boldsymbol{y},L}(P_{\boldsymbol{y}}^{\delta,\alpha_*}(m),\epsilon_*,\delta(m))$. For simplicity, we assume that α_i and the coefficients of δ are rational.

Take a sufficiently large number k such that $\mathcal{L}^{\otimes k}$ is a line bundle on $\mathcal{A}$. For a rational number γ, we put $\mathcal{L}_\gamma:=\mathcal{L}^{\otimes k}\otimes\mathcal{O}_{\mathbb{P}_m}(\gamma)$. Let $\mathcal{A}^{ss}(\mathcal{L}_\gamma)$ (resp. $\mathcal{A}^s(\mathcal{L}_\gamma)$) denote the set of the semistable (resp. stable) points of $\mathcal{A}$ with respect to $\mathcal{L}_\gamma$.

Let $\mathrm{Flag}(V,\underline{N})$ denote the full flag variety:

$$\Big\{\mathcal{F}_*=\big(0\subset\mathcal{F}_1\subset\mathcal{F}_2\subset\cdots\subset\mathcal{F}_N=V\big)\,\Big|\,\dim\mathcal{F}_i/\mathcal{F}_{i-1}=1\Big\}.\quad(54)$$

Let $G_l(V)$ denote the Grassmann variety of l-dimensional subspaces of V. We have the natural morphism $\rho_l:\mathrm{Flag}(V,\underline{N})\longrightarrow G_l(V)$. Let $\mathcal{O}_{G_l(V)}(1)$ denote the canonical polarization of $G_l(V)$. For a tuple of positive rational numbers $n_*=(n_1,n_2,\ldots,n_N)$, we put as follows:

$$\mathcal{O}_{\mathrm{Flag}}(n_*):=\bigotimes_{i=1}^{N}\rho_i^*\mathcal{O}_{G_i(V)}(n_i).$$

We put $\widetilde{Q}:=Q\times\mathrm{Flag}(V,\underline{N})$ and $\widetilde{\mathcal{A}}:=\mathcal{A}\times\mathrm{Flag}(V,\underline{N})$. We have the induced map $\widetilde{\Psi}_m:\widetilde{Q}\longrightarrow\widetilde{\mathcal{A}}$. For a tuple n_* and a rational number γ, let us consider the following $\mathbb{Q}$-line bundle:

$$\widetilde{\mathcal{L}}(\gamma,n_*):=\mathcal{L}_\gamma\otimes\mathcal{O}_{\mathrm{Flag}}(n_*)=\mathcal{L}^{\otimes k}\otimes\mathcal{O}_{\mathbb{P}_m}(\gamma)\otimes\mathcal{O}_{\mathrm{Flag}}(n_*).$$

Let $\widetilde{\mathcal{A}}^{ss}(\gamma,n_*)$ denote the set of the semistable points with respect to $\widetilde{\mathcal{L}}(\gamma,n_*)$. We will be interested in the open subset $\widetilde{\Psi}_m^{-1}\big(\widetilde{\mathcal{A}}^{ss}(\gamma,n_*)\big)$.

4.2.2 δ_+-semistability and δ_--semistability

Let δ_+ and δ_- denote elements of $\mathcal{P}^{\mathrm{br}}$ with $\delta_- < \delta < \delta_+$ such that δ_+ and δ_- are sufficiently close to δ. The following lemma is clear from Lemma 3.55.

Lemma 4.12 *When δ_+ (resp. δ_-) is sufficiently close to δ, a parabolic L-Bradlow pair (E_*, ϕ) is δ_+-semistable (δ_--semistable) if and only if the following condition holds:*

- *Take any partial Jordan-Hölder filtration of (E_*, ϕ) with respect to δ-semistability:*
$$\big(E_*^{(1)}, \phi^{(1)}\big) \subset \big(E_*^{(2)}, \phi^{(2)}\big) \subset \cdots \subset \big(E_*^{(k)}, \phi^{(k)}\big) = \big(E_*, \phi\big).$$
Then we have $\phi^{(i)}(0) = 0$ for $i < k$ (resp. $\phi^{(1)} \neq 0$).

Moreover, any δ_+-semistable (resp. δ_--semistable) L-Bradlow pair is also δ_+-stable (resp. δ_--stable). ∎

We put $Q_+ := Q^{ss}\big(m, \boldsymbol{y}, [L], \alpha_*, \delta_+\big)$ and $Q_- := Q^{ss}\big(m, \boldsymbol{y}, [L], \alpha_*, \delta_-\big)$. They are independent of choices of δ_+ and δ_- when δ_+ and δ_- are sufficiently close to δ due to the previous lemma. We denote the signature of γ by $\mathrm{sign}(\gamma)$. The absolute value of γ is denoted by $|\gamma|$.

Proposition 4.13 *Assume that $|\gamma|$ is sufficiently small, and that n_i are sufficiently smaller than $|\gamma|$. Then, we have $\widetilde{\Psi}_m^{-1}\big(\widetilde{\mathcal{A}}^{ss}(\gamma, n_*)\big) = Q_{\mathrm{sign}(\gamma)} \times \mathrm{Flag}(V, \underline{N})$. In particular, we have the closed immersion $Q_{\mathrm{sign}(\gamma)} \times \mathrm{Flag}(V, \underline{N}) \longrightarrow \widetilde{\mathcal{A}}^{ss}(\gamma, n_*)$. Moreover, the image is contained in $\widetilde{\mathcal{A}}^{s}(\gamma, n_*)$.*

Proof Let us begin with the following lemma.

Lemma 4.14 *Assume that the absolute value of $\gamma \neq 0$ is sufficiently small.*

- *Then, we have $\Psi_m^{-1}\big(\mathcal{A}^{ss}(\mathcal{L}_\gamma)\big) = Q^{ss}_{\mathrm{sign}(\gamma)}$.*
- *The induced morphism $\Psi_m : Q^{ss}_{\mathrm{sign}(\gamma)} \longrightarrow \mathcal{A}^{ss}(\mathcal{L}_\gamma)$ is a closed immersion. Moreover, the image is contained in $\mathcal{A}^{s}(\mathcal{L}_\gamma)$.*

Proof Let us show the first claim. Let z denote a point $(q, E_*, \phi) \in Q$. As a preparation, we consider the following two cases:

(A) There exists a partial Jordan-Hölder filtration $E'_* \subset (E_*, \phi)$ with respect to δ-semistability.

(B) There exists a partial Jordan-Hölder filtration $(E''_*, \phi) \subset (E_*, \phi)$ with respect to δ-semistability.

In the case (A), we put $V' := H^0\big(X, E'(m)\big)$, and take a complement V'' of V' in V. We consider the one parameter subgroup λ given by $t^{-\mathrm{rank}\, V''} \cdot \mathrm{id}_{V'} \oplus t^{\mathrm{rank}\, V'} \cdot \mathrm{id}_{V''}$. Since we have $\mu_\lambda(z, \mathcal{L}) = 0$, the equality $\mu_\lambda\big(z, \mathcal{L}_\gamma\big) = \gamma \cdot \mu_\lambda\big(z, \mathcal{O}_{\mathbb{P}_m}(1)\big) = \gamma \cdot \mathrm{rank}\, V'$ holds.

In the case (B), we put $V'' := H^0\big(X, E''(m)\big)$, and take a complement V' of V'' in V. Let us consider the one parameter subgroup λ given by $t^{-\mathrm{rank}\, V'} \cdot \mathrm{id}_{V''} \oplus t^{\mathrm{rank}\, V''} \cdot \mathrm{id}_{V'}$. As before, we have $\mu(z, \mathcal{L}_\gamma) = -\gamma \cdot \mathrm{rank}\, V'$.

From the above considerations, we easily obtain $\Psi_m^{-1}(\mathcal{A}^{ss}(\mathcal{L}_\gamma)) \subset Q^{ss}_{\mathrm{sign}(\gamma)}$.

Let us show the reverse implication. We use the argument in the proof of Lemma 4.7. Let $u_1, \ldots, u_N$ be any base of V. Let $(w_1, \ldots, w_N)$ be an element of $\mathbb{Z}^N$ such that $w_i \leq w_{i+1}$ and $\sum_{i=1}^N w_i = 0$. Let λ be the one parameter subgroup of $\mathrm{SL}(V)$ given by $\lambda(t) \cdot u_i = t^{w_i} \cdot u_i$. We have $\mu_\lambda(z, \mathcal{L}_\gamma) = k \cdot \mu_\lambda(z, \mathcal{L}) + \gamma \cdot \mu_\lambda\big(z, \mathcal{O}_{\mathbb{P}_m}(1)\big)$. As seen in the proof of Lemma 4.7, $\mu_\lambda(z, \mathcal{L})$ is linear with respect to $(w_1, \ldots, w_N)$. It is also well known that $\mu_\lambda\big(z, \mathcal{O}_{\mathbb{P}_m}(1)\big)$ is linear with respect to $(w_1, \ldots, w_N)$. (See Lemma 2.7, for example.) Therefore, we have only to show $\mu_{f_h}\big(z, \mathcal{L}_\gamma\big) > 0$ for any $h = 1, \ldots, N-1$.

In the case $\mu_{f_h}\big(z, \mathcal{L}\big) > 0$, we have $k \cdot \mu_{f_h}\big(z, \mathcal{L}\big) \geq 1$. On the other hand, the absolute value of $\mu_{f_h}\big(z, \mathcal{O}_{\mathbb{P}_m}(1)\big)$ is dominated by $\dim V$. Therefore, if γ is sufficiently small, we have $\mu_{f_h}\big(z, \mathcal{L}_\gamma\big) > 0$.

Let us consider the case $\mu_{f_h}(z, \mathcal{L}) = 0$. Let W denote the subspace of V generated by $u_1, \ldots, u_h$. Let $\mathcal{E}_W$ denote the subsheaf of $E(m)$ generated by W and q. We put $\phi' := \phi$ if the image of ϕ is contained in $\mathcal{E}_W$, and

$\phi' := 0$ otherwise. Then $(\mathcal{E}'(-m), \phi')$ is δ-semistable (Lemma 4.10). Due to the considerations (A) and (B), we obtain $\mu_{f_h}(z, \mathcal{L}_\gamma) > 0$ in the case (E_*, ϕ) is $\delta_{\mathrm{sign}(\gamma)}$-semistable. Thus the first claim is proved.

In the above argument, we showed $\mu_\lambda(\Psi_m(z), \mathcal{L}_\gamma) > 0$ for any point z of $Q_{\mathrm{sign}(\gamma)}$. Thus we have already obtained $\Psi_m(Q_{\mathrm{sign}(\gamma)}) \subset \mathcal{A}^s(\mathcal{L}_\gamma)$. Since the morphism $\Psi_m : Q \longrightarrow \mathcal{A}^{ss}(\mathcal{L}_0)$ is proper (Proposition 4.2), the properness of $Q_{\mathrm{sign}(\gamma)} \longrightarrow \mathcal{A}^{ss}(\mathcal{L}_\gamma)$ follows from the first claim. Thus the proof of Lemma 4.14 is finished. ∎

Let us return to the proof of Proposition 4.13. Let z be any point of $Q \times \mathrm{Flag}(V, \underline{N})$. Let λ be the one parameter subgroup as in the proof of Lemma 4.14. Due to Lemma 2.8, $\mu_\lambda\big(z, \mathcal{O}_{\mathrm{Flag}}(n_*)\big)$ is linear with respect to $w_1, \dots w_N$. Hence $\mu_\lambda\big(z, \mathcal{L}(\gamma, n_*)\big)$ is also linear. Therefore, we have only to show $\mu_{f_h}\big(z, \mathcal{L}(\gamma, n_*)\big) > 0$ for any $h = 1, \dots, N-1$. We have the following:

$$\mu_{f_h}\big(z, \mathcal{L}_\gamma\big) + \sum_j n_h \cdot \mu_{f_h}\big(z, \rho_j^* \mathcal{O}_{G_j(V)}(1)\big).$$

If we take a large integer k' such that $\mathcal{L}_\gamma^{\otimes k'}$ is a line bundle, then we have $\mu_{f_h}\big(z, \mathcal{L}_\gamma^{\otimes k'}\big) \geq 1$, because $\mu_{f_h}\big(z, \mathcal{L}_\gamma^{\otimes k'}\big)$ is a positive integer. On the other hand, $\mu_{f_h}\big(z, \rho_i^* \mathcal{O}_{G_i(V)}(1)\big)$ is dominated by $2 \dim V^2$. Therefore, if n_i are sufficiently small, the contribution of $\mathcal{O}_{\mathrm{Flag}}(n_*)$ to $\mu_{f_h}\big(z, \mathcal{L}(\gamma, n_*)\big)$ is sufficiently small. Thus we are done. ∎

4.2.3 (δ, ℓ)-semistability

Let ℓ be a positive integer. Let $\widetilde{Q}^{ss}(\delta, \ell)$ denote the maximal subset of $\widetilde{Q}$, which consists of the points $\big(q, E_*, [\phi], \mathcal{F}\big)$ such that $(E_*, [\phi], \mathcal{F})$ is (δ, ℓ)-semistable. (See the subsubsection 3.3.3 for (δ, ℓ)-semistability.)

Proposition 4.15 *There exist negative rational number γ and a tuple of positive rational numbers n_*, for which the following holds:*

- *We have $\widetilde{Q}^{ss}(\delta, \ell) = \widetilde{\Psi}_m^{-1}\big(\widetilde{\mathcal{A}}^{ss}(\gamma, n_*)\big)$.*
- *The induced morphism $\widetilde{Q}^{ss}(\delta, \ell) \longrightarrow \widetilde{\mathcal{A}}^{ss}(\gamma, n_*)$ is a closed immersion. The image is contained in $\widetilde{\mathcal{A}}^s(\gamma, n_*)$.*

Before going into the proof of Proposition 4.15, we give a consequence.

Corollary 4.16 *The moduli stack $\widetilde{\mathcal{M}}^{ss}\big(\boldsymbol{y}, [L], \alpha_*, (\delta, \ell)\big)$ of the (δ, ℓ)-semistable objects is Deligne-Mumford and proper.*

Proof From Lemma 3.33, Proposition 4.15 and Proposition 2.4, the quotient stack $\widetilde{Q}^{ss}(\delta, \ell)/\,\mathrm{SL}(V_m)$ is Deligne-Mumford and proper. Since we have the etale and finite morphisms $\widetilde{Q}^{ss}(\delta, \ell)/\,\mathrm{SL}(V_m) \longrightarrow \widetilde{Q}^{ss}(\delta, \ell)/\,\mathrm{PGL}(V_m)$ and $\widetilde{\mathcal{M}}\big(\widehat{\boldsymbol{y}}, [L], \alpha_*, (\delta, \ell)\big) \longrightarrow \widetilde{Q}^{ss}(\delta, \ell)/\,\mathrm{PGL}(V_m)$, we easily obtain the desired properties for $\widetilde{\mathcal{M}}\big(\widehat{\boldsymbol{y}}, [L], \alpha_*, (\delta, \ell)\big)$ by using the valuative criterion. ∎

Let us start the proof of Proposition 4.15. We put $N := \dim V_m = H_{\boldsymbol{y}}(m)$. When ℓ is larger than N, (δ, ℓ)-semistability is same as δ_--semistability (Remark 3.32). Hence the claim follows from Proposition 4.13. Therefore, we will assume $\ell < N$ in the following argument.

We take γ and n_* satisfying the following condition:

Condition 4.17 *Let $K_0(\boldsymbol{y}, L, \delta)$ be the number as in Proposition* 3.44. *Take a small rational number ϵ satisfying the following:*

$$0 < \epsilon < \frac{K_0(\boldsymbol{y}, L, \delta)}{100 \cdot N^{100}}$$

Take an irrational number $a > 0$ satisfying the following:

$$\left(\ell - \frac{1}{\ell}\right) \cdot a < \epsilon < \ell \cdot a.$$

Take mutually distinct prime numbers $p_1, \ldots, p_\ell$ such that $p_1 > 100 \cdot N^{100}$ and $p_i > 100 \cdot N^{100} \cdot p_{i-1}$. We also assume the following:

$$\sum \frac{i}{p_i} < \min \left\{ \left|\ell \cdot a - \epsilon\right|, \ \left|\epsilon - (\ell - \ell^{-1}) \cdot a\right| \right\}$$

Take rational numbers $q_1, \ldots, q_\ell$ such that the following holds:

$$\left| \frac{q_i}{p_i} - \frac{a}{i} \right| < \frac{1}{p_i}, \quad (i = 1, \ldots, \ell).$$

We put $\gamma := -\epsilon$ and $n_i := q_i / p_i$ $(i = 1, \ldots, \ell)$. We remark $n_1 > n_2 > \cdots > n_\ell$ and the following inequalities:

$$\gamma + \sum_{i=1}^{\ell} i \cdot n_i > 0, \quad \gamma + \sum_{\substack{1 \leq i \leq \ell \\ i \neq i_0}} i \cdot n_i + n_{i_0} \cdot (i_0 - 1) < 0$$

We also take prime numbers p_i $(i = \ell + 1, \ldots, N)$ satisfying $p_i > 100 \cdot N^{100} \cdot p_{i-1}$ $(i = \ell + 1, \ldots, N)$ and the following:

$$\sum_{i=\ell+1}^{N} \frac{i}{p_i} \cdot 100 \cdot N^{100} < \min \left\{ \left| \sum_{i=1}^{\ell} i \cdot n_i - \epsilon \right|, \ \left| \sum_{\substack{1 \leq i \leq \ell \\ i \neq i_0}} i \cdot n_i + n_{i_0} \cdot (i_0 - 1) - \epsilon \right| \right\}$$

We put $n_i = p_i^{-1}$ $(i = \ell + 1, \ldots, N)$. ∎

We remark the following elementary fact.

Lemma 4.18 *Let $p_1, \ldots, p_N$ be mutually distinct prime numbers. Let $q_i \neq 0$ be an integer which is coprime to p_i, for each i. Then the sum $\sum_{i=1}^{N} q_i / p_i$ cannot be an integer.*

Proof Assume that $\sum_{i=1}^{N} q_i / p_i = a$ is an integer. Then we obtain the relation:

$$q_1 \cdot \prod_{j > 1} p_j + p_1 \cdot \left(a \cdot \prod_{i=2}^{N} p_i + \sum_{j=2}^{N} q_j \cdot \prod_{\substack{i \geq j, \\ i \neq j}} p_i \right) = 0.$$

It contradicts with the assumption that q_1 and p_1 are coprime. ∎

Let us start the proof of Proposition 4.15. Let $z = (q, E_*, \phi, \mathcal{F})$ be a point of $\widetilde{Q}$. We give preliminary considerations.

(A) If there exists a partial Jordan-Hölder filtration $E'_* \subset (E_*, \phi)$ with respect to δ-semistability, we put $V' := H^0\big(X, E'(m)\big)$, and we take a complement V'' of V' in V. Let λ denote the one parameter subgroup of $\mathrm{SL}(V)$ given by $t^{-\operatorname{rank} V''} \cdot \mathrm{id}_{V'} \oplus t^{\operatorname{rank} V'} \cdot \mathrm{id}_{V''}$. Then we have the following:

$$\begin{aligned}
\mu_\lambda\big(z, \widetilde{\mathcal{L}}(\gamma, n_*)\big) &= k \cdot \mu_\lambda(z, \mathcal{L}) + \gamma \cdot \mu_\lambda\big(z, \mathcal{O}_{\mathbb{P}}(1)\big) + \mu_\lambda\big(z, \mathcal{O}_{\mathrm{Flag}}(n_*)\big) \\
&= \gamma \cdot \operatorname{rank} V' + \sum n_i \cdot \Big(-\dim\big(\mathcal{F}_i \cap V'\big) \cdot \operatorname{rank} V'' + \dim\Big(\mathcal{F}_i / \mathcal{F}_i \cap V'\Big) \cdot \operatorname{rank} V' \Big) \\
&= \left(\gamma + \sum_{i=1}^{\ell} n_i \cdot \dim\big(\mathcal{F}_i / \mathcal{F}_i \cap V'\big) \right) \cdot \operatorname{rank} V' - \sum_{i=1}^{\ell} n_i \cdot \dim\big(\mathcal{F}_i \cap V'\big) \cdot \operatorname{rank} V'' \\
&\qquad + \sum_{i=\ell+1}^{N} n_i \cdot \big(-\dim(\mathcal{F}_i \cap V') \cdot \operatorname{rank} V'' + \dim\big(\mathcal{F}_i / \mathcal{F}_i \cap V'\big) \cdot \operatorname{rank} V' \big). \quad (55)
\end{aligned}$$

Due to our choice of ϵ and n_*, the right hand side is larger than 0, if and only if $\mathcal{F}_\ell \cap V'_1 = \{0\}$. Moreover, if it is larger than 0, it is strictly larger than 0 due to Lemma 4.18.

(B) Let us consider the case where there exists a partial Jordan-Hölder filtration $(E''_*,\phi)\subset(E_*,\phi)$ with respect to δ-semistability. We put $V'':=H^0\bigl(X,E''(m)\bigr)$ and take a complement V' of V'' in V. Consider the one parameter subgroup λ given by $\bigl\{t^{-\operatorname{rank}V''}\cdot\mathrm{id}_{V'}\oplus t^{\operatorname{rank}V'}\cdot\mathrm{id}_{V''}\bigr\}$. Then we have the following:

$$\begin{aligned}\mu_\lambda\bigl(z,\widetilde{\mathcal{L}}(\gamma,n_*)\bigr)&=-\gamma\cdot\operatorname{rank}V'+\sum_{i=1}^{N}n_i\cdot\bigl(-\dim\bigl(\mathcal{F}_i\cap V''\bigr)\cdot\operatorname{rank}V'+\dim\bigl(\mathcal{F}_i/\mathcal{F}_i\cap V''\bigr)\cdot\operatorname{rank}V''\bigr)\\&=\left(-\gamma-\sum_{i=1}^{\ell}n_i\cdot\dim\bigl(\mathcal{F}_i\cap V''\bigr)\right)\cdot\operatorname{rank}V'+\sum_{i=1}^{\ell}n_i\cdot\dim\bigl(\mathcal{F}_i/\mathcal{F}_i\cap V''\bigr)\cdot\operatorname{rank}V''\\&\qquad+\sum_{i=\ell+1}^{N}n_i\cdot\Bigl(-\dim(\mathcal{F}_i\cap V'')\cdot\operatorname{rank}V'+\dim\bigl(\mathcal{F}_i/\mathcal{F}_i\cap V''\bigr)\cdot\operatorname{rank}V''\Bigr).\end{aligned}\tag{56}$$

Due to our choice of γ and n_*, it is strictly smaller than 0, if and only if $\mathcal{F}_\ell\subset V''$. Namely, it is larger than 0 if and only if $\mathcal{F}_\ell\not\subset V''$. Moreover, if it is larger than 0, it is strictly larger than 0 due to Lemma 4.18.

From the above preliminary consideration, we obtain $\widetilde{\Psi}^{-1}\bigl(\widetilde{\mathcal{A}}^{ss}(\delta,n_*)\bigr)\subset\widetilde{Q}^{ss}(\delta,\ell)$. Let us show the reverse implication. We use the standard argument as in the proof of Lemma 4.7. Let $z=(q,E_*,\phi,\mathcal{F})$ be a point of $\widetilde{Q}^{ss}(\delta,\ell)$. Let $u_1,\dots,u_N$ be a base of V, and let $(w_1,\dots,w_N)$ be an element of $\mathbb{Z}^N$ such that $w_i\leq w_{i+1}$ and $\sum w_i=0$. Let λ be the one-parameter subgroup of $\mathrm{SL}(V)$ given by $\lambda(t)\cdot u_i=t^{w_i}\cdot u_i$. Then $\mu_\lambda\bigl(z,\mathcal{L}(\gamma,n_*)\bigr)$ is linear with respect to $(w_1,\dots,w_N)$. Hence we have only to show $\mu_{f_h}\bigl(z,\mathcal{L}(\gamma,n_*)\bigr)>0$ for any h.

First, let us consider the case $\mu_{f_h}\bigl(z,\mathcal{L}\bigr)>0$. We have $\mu_{f_h}\bigl(z,\mathcal{L}^{\otimes k}\bigr)\geq 1$. Since $|\gamma|$ and n_i are smaller than $100^{-1}\cdot(\dim V)^{-100}$, we have $\mu_{f_h}\bigl(z,\mathcal{O}_{\mathbb{P}}(\gamma)\otimes\mathcal{O}_{\mathrm{Flag}}(n_*)\bigr)<10^{-1}$. Hence we obtain $\mu_{f_h}\bigl(z,\mathcal{L}(\gamma,n_*)\bigr)>0$. Next, let us consider the case $\mu_{f_h}\bigl(z,\mathcal{L}\bigr)=0$. Let $\mathcal{E}'$ denote the subsheaf of $E(m)$ generated by $u_1,\dots,u_h$. We put $\phi':=\phi$ if the image of ϕ is contained in $\mathcal{E}'(-m)$, and $\phi':=0$ otherwise. Then $\bigl(\mathcal{E}'(-m)_*,\phi'\bigr)\subset(E_*,\phi)$ is a partial Jordan-Hölder filtration as in (A) or (B). Therefore, we have $\mu_{f_h}\bigl(z,\mathcal{L}(\gamma,n_*)\bigr)>0$ in this case, too. Hence the first claim of Proposition 4.15 is obtained.

Since we have shown that $\mu_{f_h}\bigl(z,\mathcal{L}(\gamma,n_*)\bigr)>0$ for any h, we obtain that the image $\widetilde{\Psi}(\widetilde{Q}^{ss}(\delta,\ell))$ is contained in $\widetilde{\mathcal{A}}^s(\gamma,n_*)$. Let us show the properness of $\widetilde{\Psi}:\widetilde{Q}^{ss}(\delta,\ell)\longrightarrow\widetilde{\mathcal{A}}^{ss}(\gamma,n_*)$. We use the argument in the subsubsection 4.1.5. Assume that we have the following diagram:

$$\begin{array}{ccc}\operatorname{Spec}K&\xrightarrow{f}&\widetilde{Q}^{ss}(\delta,\ell)\\\downarrow&&\downarrow\\\operatorname{Spec}R&\xrightarrow{g}&\widetilde{\mathcal{A}}^{ss}(\gamma,n_*)\end{array}$$

Let X_K and X_R denote $X\times\operatorname{Spec}K$ and $X\times\operatorname{Spec}R$ respectively. We denote the projection $\widetilde{\mathcal{A}}\longrightarrow\mathcal{A}$ by π.

Let $\bigl(q_K,E_{K\,*},[\widetilde{\phi}_K],\mathcal{F}_{K,*}\bigr)$ be the objects on X_K corresponding to f. As in the subsubsection 4.1.5, we obtain the objects $(q_R,E_{R\,*},\phi_R,W_{R\,*},[\widetilde{\phi}_R])$ on X_R. It induces the morphism $f_1:\operatorname{Spec}R\longrightarrow\mathcal{A}$, which is same as $\pi\circ g$. We also obtain the Yokogawa datum $(q_0,E_{0\,*},\phi_0,W_{0,*},[\widetilde{\phi}_0])$.

Since $|\gamma|$ and n_i are sufficiently small, the tuple $\bigl(q_0,E_{0*},\phi_0,W_{0\,*},[\widetilde{\phi}_0]\bigr)$ is contained in the Yokogawa family $\mathcal{YOK}(N_0,K_0,\boldsymbol{y},L,\delta)$ for $N_0=N_0(\boldsymbol{y},L,\delta)$ and $K_0=K_0(\boldsymbol{y},L,\delta)$. Hence $\bigl(q_0,E_{0*},[\widehat{\phi}]\bigr)$ gives a point of $Q=Q^{ss}(m,\boldsymbol{y},[L],\alpha_*,\delta)$, due to Proposition 3.44. It implies the image of $f_1=\pi\circ g$ is contained in $\mathcal{A}^{ss}(\mathcal{L}_0)$. Then the desired properness immediately follows from the first claim. ∎

4.3 Enhanced Master Space

4.3.1 The construction

We use the notation in the subsection 4.2. Take a rational number $\gamma_2<0$, whose absolute value is sufficiently small. We will consider the following two situations:

(I) γ_1 is a sufficiently small positive rational number, and n_i are sufficiently smaller than γ_1.

(II) γ_1 and n_* are as in Condition 4.17.

In the both cases, we assume $|\gamma_1|$ is sufficiently smaller than $|\gamma_2|$. We also assume the following:

$$\sum_i i\cdot n_i+|\gamma_2|\leq K_0(\boldsymbol{y},L,\delta). \tag{57}$$

Here $K_0(\boldsymbol{y},L,\delta)$ denotes the constant in Proposition 3.44.

Let k' be a number such that $k'\cdot(\gamma_1-\gamma_2)=1$. We consider the following $\mathbb{Q}$-line bundles on $\widetilde{Q}$:

$$\widetilde{\mathcal{L}}_1:=\widetilde{\mathcal{L}}(\gamma_1,n_*)^{\otimes k'},\quad \widetilde{\mathcal{L}}_2:=\widetilde{\mathcal{L}}(\gamma_2,n_*)^{\otimes k'}.$$

Then we have $\widetilde{\mathcal{L}}_2=\widetilde{\mathcal{L}}_1\otimes\mathcal{O}_{\mathbb{P}_m}(-1)$.

Let $\pi_1:\widetilde{Q}\longrightarrow\mathbb{P}_m$ denote the projection. We put $\widetilde{\mathcal{B}}:=\mathbb{P}\bigl(\pi_1^*\mathcal{O}_{\mathbb{P}_m}(0)\oplus\pi_1^*\mathcal{O}_{\mathbb{P}_m}(1)\bigr)$ over $\widetilde{Q}$. We put $\mathcal{O}_{\widetilde{\mathcal{B}}}(1):=\mathcal{O}_{\mathbb{P}_m}(1)\otimes\widetilde{\mathcal{L}}_1$, where $\mathcal{O}_{\mathbb{P}_m}(1)$ denotes the tautological bundle of $\mathbb{P}\bigl(\pi_1^*\mathcal{O}_{\mathbb{P}_m}(0)\oplus\pi_1^*\mathcal{O}_{\mathbb{P}_m}(1)\bigr)$ over $\widetilde{Q}$. Let $\widetilde{\mathcal{B}}^{ss}$ (resp. $\widetilde{\mathcal{B}}^{s}$) denote the set of the semistable points (resp. the stable points) with respect to $\mathcal{O}_{\widetilde{\mathcal{B}}}(1)$.

We put $\widetilde{\mathcal{B}}_1:=\mathbb{P}\bigl(\pi_1^*\mathcal{O}_{\mathbb{P}_m}(0)\bigr)$ and $\widetilde{\mathcal{B}}_2:=\mathbb{P}\bigl(\pi_1^*\mathcal{O}_{\mathbb{P}_m}(1)\bigr)$. We naturally regard $\widetilde{\mathcal{B}}_i$ as the closed subscheme of $\widetilde{\mathcal{B}}$. The following lemma is clear from the construction.

Lemma 4.19 *The restriction of $\mathcal{O}_{\widetilde{\mathcal{B}}}(1)$ to $\widetilde{\mathcal{B}}_i$ is same as $\widetilde{\mathcal{L}}_i$. The $\mathbb{Q}$-line bundle $\mathcal{O}_{\widetilde{\mathcal{B}}}(1)$ gives a $\mathrm{GL}(V_m)$-equivariant polarization.* ∎

We put $\mathrm{TH}:=\widetilde{\mathcal{B}}\times_{\widetilde{\mathcal{A}}}\widetilde{Q}$, $\mathrm{TH}_i:=\widetilde{\mathcal{B}}_i\times_{\widetilde{\mathcal{A}}}\widetilde{Q}$ and $\mathrm{TH}^*:=\mathrm{TH}-\bigl(\mathrm{TH}_1\cup\mathrm{TH}_2\bigr)$. We remark that TH^* is isomorphic to $Q^{ss}(m,\boldsymbol{y},L,\alpha_*,\delta)\times\mathrm{Flag}(V,\underline{N})$. We also put $\mathrm{TH}^{ss}=\widetilde{\mathcal{B}}^{ss}\times_{\widetilde{\mathcal{A}}}\widetilde{Q}$. The following lemma is obvious from Proposition 4.13, Proposition 4.15 and our choice of the constants.

Lemma 4.20

- *In the case (I), we have* $\mathrm{TH}^{ss}\times_{\mathrm{TH}}\mathrm{TH}_1=Q_+^{ss}\times\mathrm{Flag}(V,\underline{N})$.
- *In the case (II), we have* $\mathrm{TH}^{ss}\times_{\mathrm{TH}}\mathrm{TH}_1=\widetilde{Q}(\delta,\ell)$.
- *In the both cases, we have* $\mathrm{TH}^{ss}\times_{\mathrm{TH}}\mathrm{TH}_2=Q_-^{ss}\times\mathrm{Flag}(V,\underline{N})$. ∎

The quotient stack $\mathrm{TH}^{ss}/\mathrm{SL}(V)$ is called the enhanced master space. In the rest of this subsection, we will show the following proposition.

Proposition 4.21 *The stack $\mathrm{TH}^{ss}/\mathrm{SL}(V)$ is Deligne-Mumford and proper.*

Due to Corollary 2.5, the proposition is obtained from the following three lemmas.

Lemma 4.22 *Let us consider the $\mathrm{SL}(V)$-action on TH^{ss}. The stabilizer of any point $z\in\mathrm{TH}^{ss}$ is finite and reduced. As a result, $\mathrm{TH}^{ss}/\mathrm{SL}(V)$ is Deligne-Mumford.*

Lemma 4.23 *If m is sufficiently large, then the morphism $\mathrm{TH}^{ss}\longrightarrow\widetilde{\mathcal{B}}^{ss}$ is proper.*

Lemma 4.24 *The image of $\mathrm{TH}^{ss}\longrightarrow\widetilde{\mathcal{B}}^{ss}$ is contained in $\widetilde{\mathcal{B}}^{s}$.*

4.3.2 Proof of Lemma 4.22

The claim is obvious for any point $z\in\mathrm{TH}^{ss}\cap\bigl(\mathrm{TH}_1\cup\mathrm{TH}_2\bigr)$. So we discuss the stabilizer of a point $z=\bigl(q,E_*,[\phi],\mathcal{F}_*,u\bigr)\in\mathrm{TH}^{ss}\cap\mathrm{TH}^*$. Let $g\in\mathrm{SL}(V)$ be any element such that $g\cdot z=z$. Let $V=\bigoplus V^{(i)}$ denote the generalized eigen decomposition of g. Correspondingly, we have the decomposition:

$$\bigl(q,E_*,[\phi],\mathcal{F}\bigr)=\bigl(q^{(1)},E_*^{(1)},[\phi],\mathcal{F}^{(1)}\bigr)\oplus\bigoplus_{i=2}^{l}\bigl(q^{(i)},E_*^{(i)},\mathcal{F}^{(i)}\bigr).$$

Lemma 4.25 $l \leq 2$.

Proof Assume $l > 2$, and we will derive a contradiction. Let us consider the one parameter subgroup λ given by $t^{-\operatorname{rank} V^{(3)}} \operatorname{id}_{V^{(2)}} \oplus t^{\operatorname{rank} V^{(2)}} \operatorname{id}_{V^{(3)}}$. It is easy to see that λ fixes the point z. Since we have $\mu_\lambda(z, \mathcal{L}) = \mu_\lambda\bigl(z, \mathcal{O}_{\mathbb{P}_m}(1)\bigr) = 0$, we have the following equality:

$$\mu_\lambda\bigl(z, \mathcal{O}_{\widetilde{\mathcal{B}}}(1)\bigr) = \sum_{i=1}^{N} n_i \cdot \Bigl(-\operatorname{rank} V^{(3)} \cdot \operatorname{rank} \mathcal{F}_i^{(2)} + \operatorname{rank} V^{(2)} \cdot \operatorname{rank} \mathcal{F}_i^{(3)}\Bigr). \tag{58}$$

Due to our choice of n_*, the right hand side of (58) can be 0, if and only if the following equality holds for any i:

$$-\operatorname{rank} V^{(3)} \cdot \operatorname{rank} \mathcal{F}_i^{(2)} + \operatorname{rank} V^{(2)} \cdot \operatorname{rank} \mathcal{F}_i^{(3)} = 0.$$

However, there exists a number i_0 such that $\operatorname{rank} \mathcal{F}^{(2)}_{i_0+1} = \operatorname{rank} \mathcal{F}^{(2)}_{i_0} + 1$ and $\operatorname{rank} \mathcal{F}^{(3)}_{i_0+1} = \operatorname{rank} \mathcal{F}^{(3)}_{i_0}$. Thus $\mu_\lambda\bigl(z, \mathcal{O}_{\widetilde{\mathcal{B}}}(1)\bigr) \neq 0$. Let λ^{-1} denote the one-parameter subgroup given by $\lambda^{-1}(t) = \lambda(t)^{-1}$. Then one of $\mu_\lambda\bigl(z, \mathcal{O}_{\widetilde{\mathcal{B}}}(1)\bigr)$ or $\mu_{\lambda^{-1}}\bigl(z, \mathcal{O}_{\widetilde{\mathcal{B}}}(1)\bigr)$ is negative. Hence z cannot be semistable. ∎

Lemma 4.26 *Let N be a nilpotent endomorphism of $(E_*, [\phi], \mathcal{F})$. Then $N = 0$.*

Proof Assume $N \neq 0$, and we will derive a contradiction. There exists the integer such that $N^j \neq 0$ and $N^{j+1} = 0$. It is easy to obtain $N(\phi) = 0$. We obtain the subsheaves $\operatorname{Im} N^j$ and $\operatorname{Ker}(N^j)$ of E. Then we obtain the naturally induced parabolic L-Bradlow pairs $(\operatorname{Im} N_*^j, \phi') \subset (\operatorname{Ker} N_*^j, \phi'') \subset (E_*, \phi)$, which gives the partial Jordan-Hölder filtration with respect to δ-semistability, due to Lemma 3.28. We take subspaces $\mathcal{K}_i$ $(i = 1, 2, 3)$ of V satisfying the following condition:

$$\mathcal{K}_1 = H^0\bigl(X, \operatorname{Im} N^j\bigr), \quad \mathcal{K}_1 \oplus \mathcal{K}_2 = H^0\bigl(X, \operatorname{Ker} N^j\bigr), \quad \mathcal{K}_1 \oplus \mathcal{K}_2 \oplus \mathcal{K}_3 = V.$$

We remark that N^j induces the isomorphism $\mathcal{K}_3 \longrightarrow \mathcal{K}_1$. From the inclusion $\mathcal{K}_1 \subset V$, we have the induced filtration $\mathcal{F}_{\mathcal{K}_1}$. From the isomorphism $\mathcal{K}_3 \simeq V\big/(\mathcal{K}_1 \oplus \mathcal{K}_2)$, the filtration $\mathcal{F}_{\mathcal{K}_3}$ is induced. We remark $N\bigl(\mathcal{F}_{\mathcal{K}_3\, h}\bigr) \subset \mathcal{F}_{\mathcal{K}_1\, h}$. Let us consider the one-parameter subspace λ given by $t^{-1} \operatorname{id}_{\mathcal{K}_1} \oplus t \operatorname{id}_{\mathcal{K}_3}$, for which we have $\mu_\lambda(z, \mathcal{L}) = \mu_\lambda\bigl(z, \mathcal{O}_{\mathbb{P}_m}(1)\bigr) = 0$. From $N\bigl(\mathcal{F}_{\mathcal{K}_3\, h}\bigr) \subset \mathcal{F}_{\mathcal{K}_1\, h}$ and $n_i > n_{i+1}$, we obtain $\mu_\lambda\bigl(z, \mathcal{O}_{\mathrm{Flag}}(n_*)\bigr) < 0$. Thus the point z cannot be semistable. ∎

From the lemmas 4.25 and 4.26, we obtain the following lemma.

Lemma 4.27 *Let $(q, E_*, [\phi], \mathcal{F})$ be a point of $\widetilde{Q}$ such that $z = (q, E_*, [\phi], \mathcal{F}, u) \in \mathrm{TH}^{ss} \cap \mathrm{TH}^*$. Then, either one of the following holds:*

1. *The automorphism group of $(E_*, [\phi], \mathcal{F})$ is G_m.*
2. *There exists the unique decomposition $(q, E_*, [\phi], \mathcal{F}) = (q^{(1)}, E_*^{(1)}, [\phi^{(1)}], \mathcal{F}^{(1)}) \oplus (q^{(2)}, E_*^{(2)}, \mathcal{F}^{(2)})$, and the automorphism group of $(q, E_*, [\phi], \mathcal{F})$ is G_m^2.* ∎

In the first case, the stabilizer of $(q, E_*, [\phi], \mathcal{F}, u)$ with respect to the $SL(V)$-action is trivial. In the second case, we put $V^{(i)} := H^0\bigl(X, E^{(i)}(m)\bigr) \subset V$. Then the intersection $G_m^2 \cap \mathrm{SL}(V)$ consists of the elements $\rho(t) = t^a \cdot \operatorname{id}_{V^{(1)}} \oplus t^b \cdot \operatorname{id}_{V^{(2)}}$ satisfying $a \cdot \operatorname{rank} V^{(1)} + b \cdot \operatorname{rank} V^{(2)} = 0$. By considering the action along the direction of the fiber $\mathrm{TH}/\widetilde{Q}$, which is given by $\rho(t)u = t^a \cdot u$, we obtain that the stabilizer is finite. ∎

4.3.3 Proof of Lemma 4.23

Let $(q, E_*, \phi, W_*, [\widetilde{\phi}])$ be a Yokogawa datum. Recall that we obtain the element $\Psi(q, E_*, \phi, W_*, [\widetilde{\phi}]) \in \mathcal{A}$.

Lemma 4.28 *Assume that m is larger than the constant $N(\boldsymbol{y}, L, \delta)$ in Proposition 3.44. Let z be a point of $\widetilde{\mathcal{B}}^{ss}$ such that $\pi(z) = \Psi\bigl(q, E_*, [\phi], W_*, [\widetilde{\phi}]\bigr)$, where π denotes the naturally defined projection $\widetilde{\mathcal{B}} \longrightarrow \mathcal{A}$. Then, $(E_*, [\phi])$ is δ-semistable, q is onto, and the condition O_m holds for $(E_*, [\phi])$.*

Proof Let W be any subspace of V. Take a complement W' of W in V. Consider the one-parameter subgroup of $\mathrm{SL}(V)$ given by $t^{-\operatorname{rank} W'}\,\mathrm{id}_W \oplus t^{\operatorname{rank} W}\,\mathrm{id}_{W'}$. We have the following:

$$\mu_\lambda\big(z,\mathcal{O}_{\widetilde{\mathcal{B}}}(1)\big)=k\cdot k'\mu_\lambda\big(z,\mathcal{L}\big)+k'\sum n_j\cdot\mu_\lambda\big(z,\mathcal{O}_{G_l(V)}(1)\big)+k'\max\Big\{\gamma_1\cdot\mu_\lambda\big(z,\mathcal{O}_{\mathbb{P}_m}(1)\big),\ \gamma_2\cdot\mu_\lambda\big(z,\mathcal{O}_{\mathbb{P}_m}(1)\big)\Big\}. \quad (59)$$

The first term in the right hand side is as follows:

$$k\cdot k'\cdot H(m)\cdot\Big(P^{\alpha_*,\delta}(m)\cdot\operatorname{rank}\mathcal{E}_W-\sum\epsilon_i\dim(W_i\cap W)-\alpha_1\dim(W)-\epsilon(W,[\phi])\cdot\delta(m)\Big).$$

The absolute value of the second term can be dominated by $k'\sum_{i=1}^n n_i\cdot i\cdot\dim V$. The absolute value of the third term can be dominated by $k'\cdot|\gamma_2|\cdot\dim V$. Recall $\dim V=H(m)$. Since we have assumed that $|\gamma_i|$ and n_j are sufficiently small as in (57), we obtain the following inequality:

$$P^{\alpha_*,\delta}_{\boldsymbol{y}}(m)\cdot\operatorname{rank}\mathcal{E}_W-\sum\epsilon_i\cdot\dim(W_i\cap W)-\alpha_1\dim(W)-\epsilon(W,[\phi])\cdot\delta(m)+K\geq 0. \quad (60)$$

Here $K=K_0(\boldsymbol{y},L,\delta)$ denotes the constant in Proposition 3.44. Namely, $(q,E_*,\phi,W_*,[\widetilde{\phi}])$ is contained in $\widetilde{\mathrm{YOK}}(m,K,\boldsymbol{y},L,\delta)$. Therefore, the claim of the lemma follows from Proposition 3.44. ∎

Now, we use the same argument as the last part of the proof of Proposition 4.15. Assume that we have a diagram:

$$\begin{array}{ccccc}
\operatorname{Spec}K & \xrightarrow{f} & \mathrm{TH}^{ss} & & \\
\big\downarrow & & \big\downarrow & & \\
\operatorname{Spec}R & \xrightarrow{g} & \widetilde{\mathcal{B}}^{ss} & \longrightarrow & \mathcal{A}
\end{array}$$

Then we can show the composite $\pi\circ g$ is contained in $\mathcal{A}^{ss}(\mathcal{L}_0)$, by using Lemma 4.28. Then the desired properness follows from the definition of TH^{ss}. ∎

4.3.4 Proof of Lemma 4.24, Step 1

Let us show that the image of TH^{ss} is contained in $\widetilde{\mathcal{B}}^s$. Let $z=(q,E_*,[\phi],\mathcal{F},u)$ be a point of TH^{ss}. We have only to consider the case $u\neq 0$.

Let $u_1,\ldots,u_N$ be a base of V, and let $(w_1,\ldots,w_N)$ be an element of $\mathbb{Z}^N$ such that $w_i\leq w_{i+1}$ and $\sum w_i=0$. Let λ be the one-parameter subgroup of $\mathrm{SL}(V)$ given by $\lambda(t)\cdot u_i=t^{w_i}\cdot u_i$. We will not distinguish the elements $w=(w_1,\ldots,w_N)$ and λ. We have the following:

$$\mu_\lambda\big(z,\mathcal{O}_{\widetilde{\mathcal{B}}}(1)\big)=k\cdot k'\cdot\mu_\lambda\big(z,\mathcal{L}\big)+k'\cdot\mu_\lambda\big(z,\mathcal{O}_{\mathrm{Flag}}(n_*)\big)+k'\max_{i=1,2}\Big\{\gamma_i\cdot\mu_\lambda\big(z,\mathcal{O}_{\mathbb{P}_m}(1)\big)\Big\}.$$

Recall that we have the expression $\lambda=\sum a_j\cdot f_j$ for $a_j\geq 0$, where $f_j=\big(\overbrace{j-N,\ldots,j-N}^{j},j\ldots,j\big)$.

Lemma 4.29 *If $\mu_\lambda(z,\mathcal{L})=0$, then $\mu_{f_h}(z,\mathcal{L})=0$ for any h such that $a_h\neq 0$.*

Proof Since $(E_*,[\phi])$ is δ-semistable, the claim immediately follows. ∎

We put $S_1:=\big\{j\,\big|\,\mu_{f_j}(z,\mathcal{L})=0\big\}$ and $S_2:=\big\{j\,\big|\,\mu_{f_j}(z,\mathcal{L})>0\big\}$.

Lemma 4.30 *For any element $0\neq\rho=\sum_{j\in S_2}a_j\cdot f_j$, we have the following:*

$$k\cdot\mu_\rho\big(z,\mathcal{L}\big)+\mu_\rho\big(z,\mathcal{O}_{\mathrm{Flag}}(n_*)\big)+\min_{i=1,2}\Big\{\gamma_i\cdot\mu_\rho\big(z,\mathcal{O}_{\mathbb{P}_m}(1)\big)\Big\}>0.$$

Proof We put $F_i(\rho):=k\cdot\mu_\rho(z,\mathcal{L})+\mu_\rho(z,\mathcal{O}_{\mathrm{Flag}}(n_*))+\gamma_i\cdot\mu_\rho(z,\mathcal{O}_{\mathbb{P}_m}(1))$ for $i=1,2$. We have only to show $F_i(\rho)>0$. Since F_i are linear with respect to ρ, we have only to show $F_i(f_j)>0$ for any $j\in S_2$. We remark $k\cdot\mu_{f_j}(z,\mathcal{L})\geq 1$. The number $\mu_{f_j}(z,\mathcal{O}_{\mathrm{Flag}}(n_*))$ is dominated by $\dim(V)$ and n_i $(i=1,\ldots,N)$. The number $\gamma_i\cdot\mu_{f_j}(z,\mathcal{O}_{\mathbb{P}_m}(1))$ is dominated by $\dim(V)$ and γ_i. Since n_j and γ_i are sufficiently small, the claim is clear. ∎

Lemma 4.31 *To show $\mu_\lambda\big(z,\mathcal{O}_{\widetilde{\mathcal{B}}}(1)\big)>0$ for any λ, we have only to show the following inequality for $0\neq\rho=\sum_{j\in S_1}a_j\cdot f_j$:*

$$k\cdot\mu_\rho(z,\mathcal{L})+\mu_\rho\big(z,\mathcal{O}_{\mathrm{Flag}}(n_*)\big)+\max_{i=1,2}\Big\{\gamma_i\cdot\mu_\rho(z,\mathcal{O}_{\mathbb{P}_m}(1))\Big\}>0. \tag{61}$$

Proof We have the decomposition $\lambda=\lambda^{(1)}+\lambda^{(2)}$, where $\lambda^{(i)}=\sum_{j\in S_i}a_j\cdot f_j$. Assume (61) holds for $\lambda^{(1)}$. Then we obtain the following:

$$\begin{aligned}\frac{1}{k'}\mu_\rho\big(z,\mathcal{O}_{\widetilde{\mathcal{B}}}(1)\big)&=\sum_{i=1,2}\Big(k\cdot\mu_{\rho^{(i)}}(z,\mathcal{L})+\mu_{\rho^{(i)}}(z,\mathcal{O}_{\mathrm{Flag}}(n_*))\Big)+\max_{j=1,2}\big\{\gamma_j\cdot\mu_{\rho^{(1)}}(z,\mathcal{O}_{\mathbb{P}_m}(1))+\gamma_j\cdot\mu_{\rho^{(2)}}(z,\mathcal{O}_{\mathbb{P}_m}(1))\big\}\\&\geq k\cdot\mu_{\rho^{(1)}}(z,\mathcal{L})+\mu_{\rho^{(1)}}(z,\mathcal{O}_{\mathrm{Flag}}(n_*))+\min_{j=1,2}\big\{\gamma_j\cdot\mu_{\rho^{(1)}}(z,\mathcal{O}_{\mathbb{P}_m}(1))\big\}\\&\qquad+k\cdot\mu_{\rho^{(2)}}\big(z,\mathcal{L}\big)+\mu_{\rho^{(2)}}\big(z,\mathcal{O}_{\mathrm{Flag}}(n_*)\big)+\max_{j=1,2}\big\{\gamma_j\cdot\mu_{\rho^{(2)}}\big(z,\mathcal{O}_{\mathbb{P}_m}(1)\big)\big\}>0.\end{aligned}\tag{62}$$

Thus we are done. ∎

4.3.5 Proof of Lemma 4.24, Step 2

To show (61), we give some preliminary consideration.

(A) If there exists a partial Jordan-Hölder filtration $E'_*\subset(E_*,\phi)$ with respect to δ-semistability, take a decomposition $V=V'\oplus V''$ such that $V'=H^0\big(X,E'(m)\big)$ and $V=V'\oplus V''$. Consider the one parameter subgroup λ given by $t^{-\operatorname{rank}V''}\mathrm{id}_{V'}\oplus t^{\operatorname{rank}V'}\cdot\mathrm{id}_{V''}$. Then we have $\mu_\lambda(z,\mathcal{L})=0$, $\mu_\lambda\big(z,\mathcal{O}_{\mathbb{P}_m}(1)\big)=\operatorname{rank}V'>0$, and the following equality:

$$\mu_T(z,\mathcal{O}_{\mathrm{Flag}}(n_*))=\sum n_j\cdot\Big(-\operatorname{rank}V''\cdot\dim\mathcal{F}_j\cap V'+\operatorname{rank}V'\cdot\dim\frac{\mathcal{F}_j}{\mathcal{F}_j\cap V'}\Big).$$

From the semistability of z, we have the following:

$$\gamma_1\cdot\operatorname{rank}V'+\sum n_j\cdot\Big(-\operatorname{rank}V''\cdot\dim(\mathcal{F}_j\cap V')+\operatorname{rank}V'\cdot\dim\frac{\mathcal{F}_j}{\mathcal{F}_j\cap V'}\Big)\geq 0. \tag{63}$$

We remark that the strict inequality holds in (63). In the case (I), it is obvious. In the case (II), it follows from our choice of n_* and Lemma 4.18. Hence $\mu_\lambda\big(z,\mathcal{O}_{\widetilde{\mathcal{B}}}(1)\big)>0$ in this case.

(B) If there exists a partial Jordan-Hölder filtration $(E''_*,\phi'')\subset(E_*,\phi)$ with respect to δ-semistability such that $\phi''\neq 0$, let us take a decomposition $V=V'\oplus V''$ such that $V''=H^0\big(X,E''(m)\big)$. Consider the one-parameter subgroup λ given by $t^{-\operatorname{rank}V'}\mathrm{id}_{V''}\oplus t^{\operatorname{rank}V''}\mathrm{id}_{V'}$. We have $\mu_\lambda(z,\mathcal{L})=0$, $\mu_\lambda(z,\mathcal{O}_{\mathbb{P}_m}(1))=-\operatorname{rank}V'$ and the following:

$$\mu_\lambda(z,\mathcal{O}_{\mathrm{Flag}}(n_*))=\sum n_j\cdot\left(-\operatorname{rank}V'\cdot\operatorname{rank}\big(\mathcal{F}_j\cap V''\big)+\operatorname{rank}V''\cdot\operatorname{rank}\frac{\mathcal{F}_j}{\mathcal{F}_j\cap V''}\right).$$

From the semistability of z, we obtain the following:

$$-\gamma_2\cdot\dim V'+\sum n_j\cdot\left(-\operatorname{rank}V'\cdot\operatorname{rank}\big(\mathcal{F}_j\cap V''\big)+\operatorname{rank}V''\cdot\operatorname{rank}\frac{\mathcal{F}_j}{\mathcal{F}_j\cap V''}\right)\geq 0 \tag{64}$$

In the both cases of (I) and (II), the strict inequality holds in (64). Hence we have $\mu_\lambda\big(z,\mathcal{O}_{\widetilde{\mathcal{B}}}(1)\big)>0$ in this case.

(C) Let us consider the case that there exists a partial Jordan-Hölder filtration $E'_*\subset(E''_*,\phi)\subset(E_*,\phi)$ with respect to δ-semistability. We take a decomposition $V=V'\oplus V''\oplus V'''$ such that $V'=H^0(X,E'(m))$ and $V'\oplus V''=H^0(X,E''(m))$. Consider the one-parameter subgroup λ given by $t^{-\operatorname{rank}V'''}\cdot\mathrm{id}_{V'}\oplus t^{\operatorname{rank}V'}\cdot\mathrm{id}_{V'''}$.

Then we have $\mu_\lambda(z,\mathcal{L})=\mu_\lambda(z,\mathcal{O}_{\mathbb{P}_m}(1))=0$. Hence, we obtain the following inequality from the semistability of the point z:

$$\sum n_j\cdot\left(-\operatorname{rank}V''\cdot\operatorname{rank}(\mathcal{F}_j\cap V')+\operatorname{rank}V'\cdot\operatorname{rank}\left(\frac{\mathcal{F}_j}{\mathcal{F}_j\cap V''}\right)\right)\geq 0. \tag{65}$$

Due to our choice of n_* and Lemma 4.18, the strict inequality holds in (65). Therefore, we have $\mu_\lambda(z,\mathcal{O}_{\widetilde{\mathcal{B}}}(1))>0$ in this case.

4.3.6 Proof of Lemma 4.24, Step 3

For $\rho=\sum_{f_j\in S_1}a_j\cdot f_j$, let $V=\bigoplus V_i$ be the weight decomposition. We have the number i_0 such that $\phi\in\mathcal{G}_{i_0}-\mathcal{G}_{i_0-1}$. We put $r_i=\dim V_i$.

We use the notation in the subsubsection 2.6.1. We regard ρ as an element of $\mathcal{U}=\bigoplus_{i=1}^n\mathbb{Q}\cdot e_i$. Then we have the expression $\rho=\sum a'_j\cdot v_j$ such that $a'_1\leq a'_2\leq\cdots\leq a'_s$ and $\sum r_i\cdot a'_i=0$.

Assume $a'_{i_0}>0$. Then we have the expression, due to Lemma 2.47:

$$\rho=\sum_{(i_1,i_2)\in\mathcal{S}(i_0)}b(i_1,i_2)\cdot x(i_1,i_2)+\sum_{j=1}^{i_0-1}c_j\cdot y(j).$$

Here the coefficients $b(i_1,i_2)$ and c_j are non-negative, and one of $b(i_1,i_2)$ or c_j is positive. We remark that we have the following linearity:

$$\max_{i=1,2}\bigl\{\gamma_i\cdot\mu_\rho(z,\mathcal{O}_{\mathbb{P}_m}(1))\bigr\}=\gamma_1\cdot\sum_{j=1}^{i_0-1}c_j\cdot\mu_{y(j)}(z,\mathcal{O}_{\mathbb{P}_m}(1)).$$

Hence the following holds:

$$\mu_\rho\bigl(z,\mathcal{O}_{\widetilde{\mathcal{B}}}(1)\bigr)=\sum b(i_1,i_2)\cdot\mu_{x(i_1,i_2)}\bigl(z,\mathcal{O}_{\widetilde{\mathcal{B}}}(1)\bigr)+\sum_{j=1}^{i_0-1}c_j\cdot\mu_{y(j)}\bigl(z,\mathcal{O}_{\widetilde{\mathcal{B}}}(1)\bigr).$$

We have the positivity $\mu_{x(i_1,i_2)}\bigl(z,\mathcal{O}_{\widetilde{\mathcal{B}}}(1)\bigr)>0$ and $\mu_{y(j)}\bigl(z,\mathcal{O}_{\widetilde{\mathcal{B}}}(1)\bigr)>0$ from (C) and (A), respectively. Therefore, we obtain the positivity $\mu_\rho\bigl(z,\mathcal{O}_{\widetilde{\mathcal{B}}}(1)\bigr)>0$.

By similar arguments, we can show the desired positivity in the cases $a'_{i_0}=0$ and $a'_{i_0}<0$. Thus the image of TH^{ss} is contained in $\widetilde{\mathcal{B}}^s$. Therefore, we obtain Lemma 4.24 and hence Proposition 4.21. ∎

4.4 Fixed Point Set of the Torus Action on Enhanced Master Space

4.4.1 Preliminary

We continue to use the notation in the subsection 4.3. We have the G_m-action ρ on $\widetilde{\mathcal{B}}=\mathbb{P}\bigl(\mathcal{O}_{\mathbb{P}_m}(0)\oplus\mathcal{O}_{\mathbb{P}_m}(1)\bigr)$ given by $\rho(t)\cdot[u_1:u_2]=[t\cdot u_1:u_2]$. It induces the G_m-action on TH, which is also denoted by ρ. Since it commutes with the $\mathrm{SL}(V)$-action, we obtain the induced action $\overline{\rho}$ on $\mathrm{TH}^{ss}/\mathrm{SL}(V)$.

We would like to discuss the fixed point set of the enhanced master space. The stack theoretic fixed point set (see [24]) is given in our case, as follows: We have the $\mathrm{SL}(V)\times G_m$-equivariant closed immersion $\mathrm{TH}^{ss}\longrightarrow\widetilde{\mathcal{B}}^s$. Therefore, we have an open neighbourhood $\mathcal{U}$ of $\mathrm{TH}^{ss}/\mathrm{SL}(V)$ in $\widetilde{\mathcal{B}}^s/\mathrm{SL}(V)$, which is G_m-invariant, Deligne-Mumford and smooth. The embedding $\mathrm{TH}^{ss}/\mathrm{SL}(V)\longrightarrow\mathcal{U}$ is G_m-equivariant. The fixed point set $\mathcal{U}^{G_m}$ of $\mathcal{U}$ is defined to be the 0-set of the vector field induced by the G_m-action. Then, the stack theoretic fixed point set M^{G_m} of M is defined to be the intersection $M\cap\mathcal{U}^{G_m}$.

However, we restrict ourselves to the set theoretic fixed point set in this subsection. In other words, we will consider only the closed points of the fixed point set, although it is not difficult to prove the result for the stack theoretic fixed point set. We will later discuss the stack theoretic fixed point set of the enhanced master space in the oriented case.

Let π be the projection $\mathrm{TH}^{ss} \longrightarrow \mathrm{TH}^{ss} / \mathrm{SL}(V)$. We will use the notation $\bigl(q, E_*, [\phi], \mathcal{F}, u\bigr)$ to denote a point of TH, where $(q, E_*, [\phi], \mathcal{F})$ denotes a point of $\widetilde{Q}$, and u denotes a point of the fiber of $\mathrm{TH} \longrightarrow \widetilde{Q}$ over $(q, E_*, [\phi], \mathcal{F})$.

Lemma 4.32 *Let $z = \bigl(q, E_*, [\phi], \mathcal{F}, u\bigr)$ be a point of TH^{ss}. The point $\pi(z)$ is contained in the fixed point set if and only if one of the following holds:*

1. *$z \in \mathrm{TH}_1 \cup \mathrm{TH}_2$.*
2. *We have the unique decomposition $(q, E_*, [\phi], \mathcal{F}) = (q^{(1)}, E_*^{(1)}, [\phi^{(1)}], \mathcal{F}^{(1)}) \oplus (q^{(2)}, E_*^{(2)}, \mathcal{F}^{(2)})$.*

Proof We have only to consider the case $z \in \mathrm{TH}^{ss} \cap \mathrm{TH}^*$. Assume that the condition 2 holds. We put $V^{(i)} := H^0\bigl(X, E^{(i)}(m)\bigr)$, and we consider the one parameter subgroup λ of $\mathrm{SL}(V)$ given by $t^{-\operatorname{rank} V^{(2)}}\, \mathrm{id}_{V^{(1)}} \oplus t^{\operatorname{rank} V^{(1)}} \cdot \mathrm{id}_{V^{(2)}}$. It fixes $\bigl(q, E_*, [\phi], \mathcal{F}\bigr)$, and it acts non-trivially along the direction of the fiber $\mathrm{TH} \longrightarrow \widetilde{Q}$, as $\lambda(t)u = t^{-\operatorname{rank} V^{(2)}} u$. Therefore, the action $\overline{\rho}$ fixes $[z]$.

On the other hand, if $\pi(z)$ is a fixed point with respect to $\overline{\rho}$, then we obtain the one-parameter subgroup of $\mathrm{SL}(V)$ which fixes $(q, E_*, [\phi], \mathcal{F})$ due to Lemma 4.27. Hence, it has the decomposition. The uniqueness follows from Lemma 4.22. ∎

Let $z = (q, E_*, [\phi], \mathcal{F})$ be a point of $\mathrm{TH}^{ss} \cap \mathrm{TH}^*$ such that $\pi(z) \in M^{G_m}$. We have the decomposition as in Lemma 4.32. Then, we obtain the types $\boldsymbol{y}_1 = \mathrm{type}(E_*^{(1)})$ and $\boldsymbol{y}_2 = \mathrm{type}(E_*^{(2)})$. We also obtain the decomposition $I_1 \sqcup I_2 = \underline{N} = \{1, \ldots, N\}$:

$$I_j := \bigl\{ i \in \underline{N} \bigm| \mathcal{F}_i^{(j)} / \mathcal{F}_{i-1}^{(j)} \neq 0 \bigr\}$$

The datum $(\boldsymbol{y}_1, \boldsymbol{y}_2, I_1, I_2)$ is called the decomposition type of z. Thus, we prepare the following definition.

Definition 4.33 *A decomposition type is defined to be a datum $\mathfrak{I} := (\boldsymbol{y}_1, \boldsymbol{y}_2, I_1, I_2)$ as follows:*

- *$\boldsymbol{y} = \boldsymbol{y}_1 + \boldsymbol{y}_2$ in $\mathcal{T}ype$ such that $P^{\alpha_*,\delta}_{\boldsymbol{y}_1} = P^{\alpha_*,\delta}_{\boldsymbol{y}}$.*
- *$\underline{N} = I_1 \sqcup I_2$ such that $|I_i| = H_{y_i}(m)$.*

The set of the decomposition types is denoted by $\mathrm{Dec}(m, \boldsymbol{y}, \alpha_*, \delta)$. ∎

We remark that the condition O_m-holds for $\mathcal{M}^{ss}(\boldsymbol{y}_1, L, \alpha_*, \delta)$ and $\mathcal{M}^{ss}(\boldsymbol{y}_2, \alpha_*)$ for a decomposition type $(\boldsymbol{y}_1, \boldsymbol{y}_2, I_1, I_2)$, if $\mathcal{M}^{ss}(\boldsymbol{y}_1, L, \alpha_*, \delta) \times \mathcal{M}^{ss}(\boldsymbol{y}_2, \alpha_*) \neq \emptyset$.

4.4.2 Statement

Let $\mathfrak{I} := (\boldsymbol{y}_1, \boldsymbol{y}_2, I_1, I_2)$ be a decomposition type. Take a decomposition $V = V^{(1)} \oplus V^{(2)}$ such that $\dim V^{(i)} = H_{\boldsymbol{y}_i}(m)$. We put $\mathbb{P}^{(i)} := \mathbb{P}(V^{(i)\vee})$. Then, we put $Q^{(1)} := Q^{ss}\bigl(m, \boldsymbol{y}_1, [L], \alpha_*, \delta\bigr)$ and $Q^{(2)} := Q^{ss}(m, \boldsymbol{y}_2, \alpha_*)$. We put as follows:

$$\mathrm{Flag}^{(i)} := \mathrm{Flag}(V^{(i)}, I_i) := \Bigl\{ \mathcal{F}_*^{(i)} \Bigm| \text{filtration indexed by } \underline{N},\ \dim \mathrm{Gr}_j^{\mathcal{F}^{(i)}} = 1\ (j \in I_i), \text{or } = 0\ (j \notin I_i) \Bigr\}.$$

Clearly, $\mathrm{Flag}^{(i)}$ are isomorphic to the full flag varieties of $V^{(i)}$. We put $\widetilde{Q}^{(1)} := Q^{(1)} \times \mathrm{Flag}^{(1)}$, $\widetilde{Q}^{(2)} := Q^{(2)} \times \mathrm{Flag}^{(2)}$, and $\widetilde{Q}_{\mathrm{split}}(\mathfrak{I}) := \widetilde{Q}^{(1)} \times \widetilde{Q}^{(2)}$. Then, we have the naturally defined morphism $\widetilde{Q}_{\mathrm{split}}(\mathfrak{I}) \longrightarrow \widetilde{Q}$. We put $\mathrm{TH}_{\mathrm{split}}(\mathfrak{I}) := \mathrm{TH} \times_{\widetilde{Q}} \widetilde{Q}_{\mathrm{split}}(\mathfrak{I})$, $\mathrm{TH}^*_{\mathrm{split}}(\mathfrak{I}) := \mathrm{TH}^* \times_{\widetilde{Q}} \widetilde{Q}_{\mathrm{split}}(\mathfrak{I})$ and $\mathrm{TH}_{i,\mathrm{split}}(\mathfrak{I}) := \mathrm{TH}_i \times_{\widetilde{Q}} \widetilde{Q}_{\mathrm{split}}(\mathfrak{I})$. We have the closed immersion $\iota : \mathrm{TH}_{\mathrm{split}}(\mathfrak{I}) \longrightarrow \mathrm{TH}$.

Let $z = \bigl((q_1, E_*^{(1)}, [\phi], \mathcal{F}^{(1)}), (q_2, E_*^{(2)}, \mathcal{F}^{(2)}), u\bigr)$ be a point of $\mathrm{TH}_{\mathrm{split}}(\mathfrak{I})$. Let $\min(I_2)$ denote the minimum of I_2. We will prove the following lemma, later (the subsubsection 4.4.5).

Lemma 4.34 *In the case (II) (the subsubsection* 4.3.1*), if $\iota(z)$ is contained in TH^{ss}, then we have $\min(I_2) > \ell$.*

We put $\mathcal{F}^{(2)}_{\min} := \mathcal{F}^{(2)}_{\min(I_2)}$. We remark $\mathcal{F}^{(2)}_{\min-1} = 0$, and $\dim \mathcal{F}^{(2)}_{\min} = 1$. We also remark that the pair $\bigl(E^{(2)}_*, \mathcal{F}^{(2)}_{\min}\bigr)$ can be regarded as a reduced $\mathcal{O}_X(-m)$-Bradlow pair on X. We will also prove the following proposition.

Proposition 4.35 *$\iota(z)$ is contained in TH^{ss}, if and only if the following conditions hold:*

- $z \in \mathrm{TH}^*_{\mathrm{split}}$
- $\bigl(E^{(2)}_*, \mathcal{F}^{(2)}_{\min}\bigr)$ *is an ϵ-semistable reduced $[\mathcal{O}(-m)]$-Bradlow pair for any sufficiently small $\epsilon > 0$.*
- $(E^{(1)}_*, [\phi], \mathcal{F}^{(1)})$ *is $\bigl(\delta, \min(I_2) - 1\bigr)$-semistable.*

4.4.3 Step 1

Let G_1 denote the subgroup of $\mathrm{GL}(V_1) \times \mathrm{GL}(V_2)$ determined by $\det(g) = 1$, i.e., $G = \{(g_1, g_2) \in \mathrm{GL}(V_1) \times \mathrm{GL}(V_2) \mid \det(g_1) \cdot \det(g_2) = 1\}$.

Lemma 4.36 *The following two conditions are equivalent:*

- $\mu_\lambda\bigl(\iota(z), \mathcal{O}_{\widetilde{\mathcal{B}}}(1)\bigr) \geq 0$ *for any one parameter subgroup λ of $\mathrm{SL}(V)$.*
- $\mu_\lambda\bigl(\iota(z), \mathcal{O}_{\widetilde{\mathcal{B}}}(1)\bigr) \geq 0$ *for any one parameter subgroup λ of G_1.*

Proof The first condition clearly implies the second condition. Let us show the reverse implication. Assume the second condition holds. Let $\lambda : G_m \longrightarrow \mathrm{SL}(V)$ be a one-parameter subgroup. We have the decomposition $\lambda = \lambda^{(1)} + \lambda^{(2)}$ such that $\mu_{\lambda^{(1)}}\bigl(z, \mathcal{L}\bigr) = 0$ and $\mu_{\lambda^{(2)}}\bigl(z, \mathcal{L}\bigr) > 0$. By the same argument as the proof of Lemma 4.31, we have only to show the following inequalities:

$$k \cdot \mu_{\lambda^{(1)}}(z, \mathcal{L}) + \max_{j=1,2}\bigl\{\gamma_j \cdot \mu_{\lambda^{(1)}}(z, \mathcal{O}_{\mathbb{P}}(1))\bigr\} + \mu_{\lambda^{(1)}}(z, \mathcal{O}_{\mathrm{Flag}}(n_*)) > 0. \tag{66}$$

$$k \cdot \mu_{\lambda^{(2)}}(z, \mathcal{L}) + \min_{j=1,2}\bigl\{\gamma_j \cdot \mu_{\lambda^{(2)}}(z, \mathcal{O}_{\mathbb{P}}(1))\bigr\} + \mu_{\lambda^{(2)}}(z, \mathcal{O}_{\mathrm{Flag}}(n_*)) > 0. \tag{67}$$

Since n_i and γ_j are sufficiently small, we can show that the inequality (67) always holds by the same argument as the proof of Lemma 4.30. So we may and will assume $\mu_\lambda(z, \mathcal{L}) = 0$.

Let $V = \bigoplus_i \mathcal{V}_i$ denote the weight decomposition of λ. We put $\mathcal{G}_j = \bigoplus_{i \leq j} \mathcal{V}_i$. We have the number i_0 determined by $\phi \in \mathcal{G}_{i_0} - \mathcal{G}_{i_0 - 1}$. We have only to show the following:

$$\max_{j=1,2}\bigl\{\gamma_j \cdot i_0\bigr\} + \sum_j n_j \sum_i \dim \frac{\mathcal{F}_j \cap \mathcal{G}_i}{\mathcal{F}_j \cap \mathcal{G}_{i-1}} \geq 0.$$

We put $\mathcal{H}_0 = V^{(1)}$ and $\mathcal{H}_1 = V$, which gives the filtration of V. We have the natural identification $\mathrm{Gr}^{\mathcal{H}}(V) \simeq V$. Since $\mathcal{F}$ is compatible with the decomposition $V = V^{(1)} \oplus V^{(2)}$, the induced filtration by $\mathcal{F}$ is same as $\mathcal{F}$. Let $\mathcal{G}'$ denote the induced filtration by $\mathcal{G}$ on $\mathrm{Gr}^{\mathcal{H}}(V) \simeq V$:

$$\mathcal{G}'_i = \mathcal{G}'_i \cap V^{(1)} \oplus \mathcal{G}'_i \cap V^{(2)}, \quad \mathcal{G}'_i \cap V^{(1)} = \mathcal{G}_i \cap V^{(1)}, \quad \mathcal{G}_i \cap V^{(2)} = \frac{\mathcal{G}_i}{\mathcal{G}_i \cap V^{(1)}}$$

Because of $\phi \in V^{(1)}$, we have $\phi \in \mathcal{G}'_{i_0} - \mathcal{G}'_{i_0 - 1}$.

Let us take any decomposition $V^{(a)} = \bigoplus \mathcal{K}^{(a)}_i$ $(a = 1, 2)$ such that $\mathcal{G}'_j \cap V^{(a)} = \bigoplus_{i \leq j} \mathcal{K}^{(a)}_i$. We put $\mathcal{K}_i = \mathcal{K}^{(1)}_i \oplus \mathcal{K}^{(2)}_i$. Then we obtain the one parameter subgroup λ' of G_1 whose weight decomposition is $\bigoplus \mathcal{K}_i$.

Lemma 4.37 *We have $\mu_{\lambda'}\bigl(z, \mathcal{L}\bigr) = 0$.*

Proof Let $\mathcal{E}_i(m)$ denote the subsheaf of $E(m)$ generated by $\mathcal{F}_i$. Each $\mathcal{E}_i$ has the induced parabolic structure and the reduced L-section $[\phi_i]$. Then the filtration $\cdots\subset(\mathcal{E}_{i\,*},[\phi_i])\subset(\mathcal{E}_{i+1\,*},[\phi_{i+1}])\subset\cdots$ is a partial Jordan-Hölder filtration due to Corollary 4.11.

Let us consider the filtration $\mathcal{H}$ of E given by $\mathcal{H}_0=E^{(1)}$ and $\mathcal{H}_1=E$. On $\mathrm{Gr}^{\mathcal{H}}(\mathcal{E}_i)$, we have the induced parabolic structure and the reduced L-section $[\mathrm{Gr}^{\mathcal{H}}(\phi_i)]$. Then, the tuple $\bigl(\mathrm{Gr}^{\mathcal{H}}(\mathcal{E}_i)_*,\mathrm{Gr}^{\mathcal{H}}(\phi_i)\bigr)$ is δ-semistable.

We have the canonical isomorphism $\mathrm{Gr}^{\mathcal{H}}(E)\simeq E$. We regard $\mathrm{Gr}^{\mathcal{H}}(\mathcal{E}_i)(m)$ as the subsheaf of $E(m)$. Then, $\mathcal{G}'_i$ generates $\mathrm{Gr}^{\mathcal{H}}(\mathcal{E}_i)(m)$. Then, we obtain $\mu_{\lambda'}(z,\mathcal{L})=0$ from Corollary 4.11 ∎

Let us return to the proof of Lemma 4.36. Since we have assumed the second condition in the lemma, we have $\mu_{\lambda'}\bigl(z,\mathcal{O}_{\widetilde{\mathcal{B}}}(1)\bigr)\leq 0$. Then, we obtain the following inequality from Lemma 4.37:

$$\max_{j=1,2}\bigl\{\gamma_j\cdot i_0\bigr\}+\sum_j n_j\sum_i\dim\frac{\mathcal{F}_j\cap\mathcal{G}'_i}{\mathcal{F}_j\cap\mathcal{G}'_{i-1}}\geq 0.$$

Hence we have only to show the following inequality:

$$\sum_i i\cdot\dim\frac{\mathcal{F}_j\cap\mathcal{G}'_i}{\mathcal{F}_j\cap\mathcal{G}'_{i-1}}\leq\sum_i i\cdot\dim\frac{\mathcal{F}_j\cap\mathcal{G}_i}{\mathcal{F}_j\cap\mathcal{G}_{i-1}}.$$

It can be shown by the geometric argument on the $\mathrm{SL}(V)$-action on the Grassmann variety of $N-H_j(m)$-dimensional quotients of V. But, we give a more elementary proof. We put $M:=\max\{i\,|\,\mathcal{V}^{(i)}\neq 0\}$, and we put $H:=\mathcal{F}_j$. Then we have the following:

$$\begin{aligned}\sum i\cdot\dim\frac{H\cap\mathcal{G}_i}{H\cap\mathcal{G}_{i-1}}=\sum_{i\leq M}i\cdot\dim\frac{H\cap\mathcal{G}_i}{H\cap\mathcal{G}_{i-1}}=\sum_{i\leq M}i\cdot\dim(H\cap\mathcal{G}_i)-\sum_{i\leq M-1}(i+1)\dim H\cap\mathcal{G}_i\\ =\dim H\cdot M-\sum_{i\leq M-1}\dim H\cap\mathcal{G}_i.\end{aligned}\tag{68}$$

Hence we have only to show $\dim H\cap\mathcal{G}_i\leq\dim H\cap\mathcal{G}'_i$ for each i. But we have the equality $H\cap\mathcal{G}_i\cap V^{(1)}=H\cap\mathcal{G}'_i\cap V^{(1)}$ and the following inclusion:

$$\frac{H\cap\mathcal{G}_i}{H\cap\mathcal{G}_i\cap V^{(1)}}\subset H\cap\mathcal{G}'_i\cap V^{(2)}.$$

Hence, we obtain the desired inequality, and we are done. ∎

4.4.4 Step 2

We give some preliminary consideration.

(O1) Assume that there exists a partial Jordan-Hölder filtration $E^{(2)}_{1\,*}\subset E^{(2)}_{2\,*}$. We take a decomposition $V^{(2)}=V^{(2)}_1\oplus V^{(2)}_2$ such that $V^{(2)}_1=H^0\bigl(X,E^{(2)}_1(m)\bigr)$, and we consider the one-parameter subgroup λ given by $t^{-\operatorname{rank}V^{(2)}_2}\,\mathrm{id}_{V^{(2)}_1}\oplus t^{\operatorname{rank}V^{(2)}_1}\,\mathrm{id}_{V^{(2)}}$. In the case, we have the following:

$$\mu_\lambda\bigl(z,\mathcal{O}_{\widetilde{\mathcal{B}}}(1)\bigr)=\sum_{j=\min(I_2)}^{N}n_j\cdot\Bigl(-\operatorname{rank}V^{(2)}_2\cdot\dim\mathcal{F}^{(2)}_j\cap V^{(2)}_1+\operatorname{rank}V^{(2)}_1\cdot\dim\frac{\mathcal{F}^{(2)}_j}{\mathcal{F}^{(2)}_j\cap V^{(2)}_1}\Bigr).\tag{69}$$

Since $n_{\min(I_2)}$ is sufficiently larger than n_j $(j>\min(I_2))$, (69) is larger than 0, if and only if $\mathcal{F}^{(2)}_{\min(I_2)}\cap V^{(2)}_1=\{0\}$. We also remark that (69) cannot be 0.

In particular, we obtain the following:

Lemma 4.38 *If $\iota(z)$ is contained in $\widetilde{\mathcal{B}}^{ss}$, then the reduced $\mathcal{O}(-m)$-Bradlow pair $(E^{(2)}_*,\mathcal{F}_{\min})$ is ϵ-stable for any sufficiently small number $\epsilon>0$.* ∎

4.4.5 Step 3

We give some preliminary consideration.

(O2) Let us consider the one-parameter subgroup λ of G_1 given by $t^{\operatorname{rank} V^{(2)}} \operatorname{id}_{V^{(1)}} \oplus t^{-\operatorname{rank} V^{(1)}} \operatorname{id}_{V^{(2)}}$. Let π_i denote the projection of $\mathrm{TH}_{\mathrm{split}}$ onto $\mathrm{TH}_{\mathrm{split}} \cap \mathrm{TH}_i$. Let us consider points $z_i := \pi_i(z)$ which are fixed with respect to λ. We have $\mu_\lambda(z_i, \mathcal{L}) = 0$ for $i = 1, 2$. We have $\mu_\lambda\big(z_2, \mathcal{O}_{\mathbb{P}_m}(\gamma)\big) = \gamma_2 \cdot \operatorname{rank} V^{(2)}$. Since n_j are sufficiently smaller than $|\gamma_2|$, we always have the inequality $\mu_\lambda\big(z_2, \mathcal{O}_{\widetilde{\mathcal{B}}_{\mathrm{split}}}(1)\big) < 0$.

Let us consider the condition $\mu_\lambda\big(z_1, \mathcal{O}_{\widetilde{\mathcal{B}}}(1)\big) \geq 0$. It is equivalent to the following inequality:

$$\mu_\lambda\big(z_1, \mathcal{O}_{\mathbb{P}_m}(\gamma_1)\big) + \mu_\lambda\big(z_1, \mathcal{O}_{\mathrm{Flag}}(n_*)\big) \geq 0. \tag{70}$$

In the case (I), we have $\gamma_1 > 0$, and n_i are sufficiently smaller than γ_1. Hence, the inequality (70) always holds. More strongly, the strict inequality holds.

In the case (II), the inequality (70) can be rewritten as follows:

$$\begin{aligned}
&\gamma_1 \cdot \operatorname{rank} V^{(2)} + \sum_i n_i \cdot \left(\dim(\mathcal{F}_i \cap V^{(1)}) \cdot \operatorname{rank} V^{(2)} - \dim\left(\frac{\mathcal{F}_i}{\mathcal{F}_i \cap V^{(1)}}\right) \cdot \operatorname{rank} V^{(1)} \right) \\
&\quad = \left(\gamma_1 + \sum_{i=1}^{\ell} (\dim \mathcal{F}_i \cap V^{(1)}) \cdot n_i \right) \cdot \operatorname{rank} V^{(2)} - \sum_{i=1}^{\ell} \dim\left(\frac{\mathcal{F}_i}{\mathcal{F}_i \cap V^{(1)}}\right) \cdot \operatorname{rank} V^{(1)} \cdot n_i \\
&\qquad + \sum_{i \geq \ell+1} n_i \cdot \left(\dim(\mathcal{F}_i \cap V^{(1)}) \cdot \operatorname{rank} V^{(2)} - \dim\left(\frac{\mathcal{F}_i}{\mathcal{F}_i \cap V^{(1)}}\right) \cdot \operatorname{rank} V^{(1)} \right) \geq 0
\end{aligned} \tag{71}$$

The inequality (71) is equivalent to the condition $\mathcal{F}_\ell \subset V^{(1)}$, due to our choice of γ_1 and n_*. Moreover, if the inequality holds, the strict inequality holds due to our choice of n_*.

Now, we give a proof of Lemma 4.34. If $\iota(z)$ is contained in $\widetilde{\mathcal{B}}^{ss}$, we have $\mu_\lambda\big(z, \mathcal{O}_{\widetilde{\mathcal{B}}}(1)\big) \geq 0$ for λ as above. Therefore, we obtain the inequality $\mu_\lambda\big(z_1, \mathcal{O}_{\widetilde{\mathcal{B}}}(1)\big) \geq 0$, and hence $\mathcal{F}_\ell \subset V^{(1)}$. It means $\min(I_2) > \ell$. Thus Lemma 4.34 is proved.

4.4.6 Step 4

We put $k := \min(I_2) - 1$ in the following argument. We give some more preliminary considerations.

(O3) Assume there exists a partial Jordan-Hölder filtration $E^{(1)}_{1\,*} \subset (E^{(1)}_*, \phi)$ with respect to δ-semistability. We take a decomposition $V^{(1)} = V^{(1)}_1 \oplus V^{(1)}_2$ such that $V^{(1)}_1 = H^0\big(X, E^{(1)}_1(m)\big)$, and we consider the one-parameter subgroup λ given by $t^{-\operatorname{rank} V^{(2)}} \operatorname{id}_{V^{(1)}_1} \oplus t^{\operatorname{rank} V^{(1)}_1} \operatorname{id}_{V^{(2)}}$. We have $\mu_\lambda(z, \mathcal{L}) = \mu_\lambda\big(z, \mathcal{O}_{\mathbb{P}_m}(1)\big) = 0$ for such one parameter subgroups. Therefore, the condition $\mu_\lambda\big(z, \mathcal{O}_{\widetilde{\mathcal{B}}}(1)\big) \geq 0$ is equivalent to the following inequality:

$$\begin{aligned}
0 \leq \mu_\lambda\big(z, \mathcal{O}_{\mathrm{Flag}}(n_*)\big) &= \sum_i n_i \cdot \Big(-\operatorname{rank} V^{(2)} \cdot \dim\big(\mathcal{F}^{(1)}_i \cap V^{(1)}_1\big) + \operatorname{rank} V^{(1)}_1 \cdot \dim \mathcal{F}^{(2)}_i \Big) \\
&= -\sum_{i=1}^{k-1} n_i \cdot \operatorname{rank} V^{(2)} \cdot \dim\big(\mathcal{F}^{(1)}_i \cap V^{(1)}_1\big) + n_k \cdot \Big(-\operatorname{rank} V^{(2)} \cdot \dim\big(\mathcal{F}^{(1)}_k \cap V^{(1)}_1\big) + \operatorname{rank} V^{(1)}_1 \cdot \dim \mathcal{F}^{(2)}_k \Big) \\
&\qquad + \sum_{i > k} n_i \cdot \Big(-\operatorname{rank} V^{(2)} \cdot \dim\big(\mathcal{F}^{(1)}_i \cap V^{(1)}_1\big) + \operatorname{rank} V^{(1)}_1 \cdot \dim \mathcal{F}^{(2)}_i \Big).
\end{aligned} \tag{72}$$

Recall that n_i are sufficiently smaller than n_{i-1}. Hence the inequality (72) holds, if and only if $\mathcal{F}^{(1)}_{k-1} \cap V^{(1)}_1 = \{0\}$.

(O4) Assume that there exists a partial Jordan-Hölder filtration $(E^{(1)}_{2\,*}, \phi) \subset (E^{(1)}_*, \phi)$ with respect to δ-semistability. We take a decomposition $V^{(1)} = V^{(1)}_2 \oplus V^{(1)}_1$ such that $V^{(1)}_2 = H^0\big(X, E^{(1)}_2(m)\big)$, and we take the one-parameter subgroup λ given by $t^{\operatorname{rank} V^{(2)}} \operatorname{id}_{V^{(1)}_1} \oplus t^{-\operatorname{rank} V^{(1)}_1} \cdot \operatorname{id}_{V^{(2)}}$. Then, we have $\mu_\lambda(z, \mathcal{L}) =$

$\mu_\lambda\big(z, \mathcal{O}_{\mathbb{P}_m}(1)\big) = 0$ for such one-parameter subgroups. Therefore, the condition $\mu_\lambda\big(z, \mathcal{O}_{\widetilde{\mathcal{B}}}(1)\big) \geq 0$ is equivalent to the following conditions:

$$\begin{aligned}
0 &\leq \sum_i n_i \left(\operatorname{rank} V^{(2)} \cdot \operatorname{rank}\left(\frac{\mathcal{F}_i^{(1)}}{\mathcal{F}_i^{(1)} \cap V_2^{(1)}} \right) - \operatorname{rank} V_1^{(1)} \cdot \operatorname{rank} \mathcal{F}_i^{(2)} \right) \\
&= \sum_{i=1}^{k-1} n_i \cdot \operatorname{rank} V^{(2)} \operatorname{rank}\left(\frac{\mathcal{F}_i^{(1)}}{\mathcal{F}_i^{(1)} \cap V_2^{(1)}} \right) + n_k \left(\operatorname{rank} V^{(2)} \cdot \operatorname{rank}\left(\frac{\mathcal{F}_k^{(1)}}{\mathcal{F}_k^{(1)} \cap V_2^{(1)}} \right) - \operatorname{rank} V_1^{(1)} \cdot \operatorname{rank} \mathcal{F}_k^{(2)} \right) \\
&\qquad + \sum_{i>k} n_i \left(\operatorname{rank} V^{(2)} \cdot \operatorname{rank}\left(\frac{\mathcal{F}_i^{(1)}}{\mathcal{F}_i^{(1)} \cap V_2^{(1)}} \right) - \operatorname{rank} V_1^{(1)} \cdot \operatorname{rank} \mathcal{F}_i^{(2)} \right)
\end{aligned} \tag{73}$$

Since n_i is sufficiently smaller than n_{i-1}, the inequality (73) holds if and only if $\mathcal{F}_{k-1}^{(1)} \not\subset V_2^{(1)}$.

We obtain the following claim from the above preliminary considerations.

Lemma 4.39 *If $\iota(z)$ is contained in $\widetilde{\mathcal{B}}^{ss}$, the tuple $(E_*, \phi, \mathcal{F}^{(1)})$ is $\big(\delta, \min(I_2) - 1\big)$-semistable.* ∎

4.4.7 End of the proof of Proposition 4.35

When $\iota(z)$ is contained in $\widetilde{\mathcal{B}}^{ss}$, it is easy to see $z \in \mathrm{TH}^*_{\mathrm{split}}$. We have already seen the other two conditions are satisfied (Lemma 4.38 and Lemma 4.39).

Let $z \in \mathrm{TH}_{\mathrm{split}}$ be a point which satisfies the conditions in Proposition 4.35. Let $u_1, \ldots, u_{N^{(1)}}$ be a base of $V^{(1)}$, and let $u_{N^{(1)}+1}, \ldots, u_N$ be a base of V. Let $(w_1, \ldots, w_N)$ be an element of $\mathbb{Z}^N$ such that $w_i \leq w_{i+1}$ and $\sum w_i = 0$. Let λ be the one-parameter subgroup of G, given by $\lambda(t) \cdot u_i = t^{w_i} \cdot u_i$. We do not distinguish λ and $(w_1, \ldots, w_N)$. We have the following:

$$\mu_\lambda\big(z, \mathcal{O}_{\widetilde{\mathcal{B}}}(1)\big) = k \cdot k' \cdot \mu_\lambda\big(z, \mathcal{L}\big) + k' \mu_\lambda\big(z, \mathcal{O}_{\mathrm{Flag}}(n_*)\big) + k' \max_{i=1,2} \Big\{ \gamma_i \cdot \mu_\lambda(z, \mathcal{O}_{\mathbb{P}_m}(1)) \Big\}.$$

We put $S_1 := \big\{ h \,\big|\, \mu_{f_h}(z, \mathcal{L}) = 0 \big\}$ and $S_2 := \big\{ h \,\big|\, \mu_{f_h}(z, \mathcal{L}) > 0 \big\}$, where $f_h = \big(\overbrace{h-N, \ldots, h-N}^{h}, h, \ldots, h\big)$. Since n_j and γ_i are sufficiently small, we can show that the following inequality holds for any $h \in S_2$ by the same argument as the proof of Lemma 4.30:

$$k \cdot \mu_{f_h}(z, \mathcal{L}) + \mu_{f_h}(z, \mathcal{O}_{\mathrm{Flag}}(n_*)) + \min_{i=1,2} \big\{ \gamma_i \cdot \mu_{f_h}(z, \mathcal{O}_{\mathbb{P}_m}(1)) \big\} > 0. \tag{74}$$

By the same argument as the proof of Lemma 4.31, it can be shown that we have only to show the following inequalities, for any $0 \neq \rho = \sum_{j \in S_1} a_j \cdot f_j$ with $a_j \geq 0$:

$$F(\rho) := k \cdot \mu_\rho(z, \mathcal{L}) + \mu_\rho(z, \mathcal{O}_{\mathrm{Flag}}(n_*)) + \max_{i=1,2} \big\{ \gamma_i \cdot \mu_\rho(z, \mathcal{O}_{\mathbb{P}_m}(1)) \big\} > 0. \tag{75}$$

Let us show (75). We have the weight decomposition $V^{(\alpha)} = \bigoplus_j^{s(\alpha)} \mathcal{V}_j^{(\alpha)}$ of ρ. We put $N^{(\alpha)} := \dim V^{(\alpha)}$. We put $r_j^{(\alpha)} := \dim \mathcal{V}_j^{(\alpha)}$. We use the notation in the subsubsection 2.6.2. Then ρ can be expressed as $\sum a_j^{(\alpha)} \cdot v_j^{(\alpha)}$ satisfying (27). Let i_0 be the number determined by $\phi \in \bigoplus_{i \leq i_0} \mathcal{V}_i - \bigoplus_{i \leq i_0 - 1} \mathcal{V}_i$. Due to Lemma 2.48, we have the expression:

$$\rho = \sum_{j=1}^{s(2)} c(j) \cdot y^{(2)}(j) + \sum_{i<i_0} d_1(i) \cdot x_1(i) + \sum_{i>i_0} d_2(i) \cdot x_2(i) + A \cdot \big(N^{(2)} \Omega^{(1)} - N^{(1)} \cdot \Omega^{(2)}\big). \tag{76}$$

Here $c(j)$, $d_1(i)$ and $d_2(i)$ are non-negative numbers, and A is a rational number. One of $c(j)$, $d_1(i)$, $d_2(i)$ or A is not zero. Due to $\mu_\kappa(z, \mathcal{O}_{\mathbb{P}_m}(1)) = 0$ for $\kappa = y^{(2)}(j), x_1(i), x_2(i)$, we have the following linearity:

$$F(\rho) = \sum_{j=1}^{s(2)} c(j) \cdot F\big(y^{(2)}(j)\big) + \sum_{i<i_0} d_1(i) \cdot F\big(x_1(i)\big) + \sum_{i>i_0} d_2(i) \cdot F\big(x_2(i)\big) + F\Big(A \cdot \big(N^{(2)} \Omega^{(1)} - N^{(1)} \cdot \Omega^{(2)}\big)\Big).$$

We obtain $F\big(y^{(2)}(j)\big) > 0$, $F\big(x_1(i)\big) > 0$ and $F\big(x_2(i)\big) > 0$ from the preliminary considerations (O1), (O3) and (O4) respectively. We have $F\Big(A \cdot \big(N^{(2)}\Omega^{(1)} - N^{(1)} \cdot \Omega^{(2)}\big)\Big) > 0$ in the case $A \neq 0$, from the preliminary consideration (O2). Therefore, we obtain the desired positivity, and the proof of Proposition 4.35 is finished. ▮

4.5 Enhanced Master Space in the Oriented Case

4.5.1 The construction

We put $\widehat{Q} := Q^{ss}(\widehat{\boldsymbol{y}}, [L], \alpha_*, \delta)$ and $\widehat{\widehat{Q}} := \widehat{Q} \times \mathrm{Flag}(V, \underline{N})$. Then, we put $\widehat{\mathrm{TH}} := \mathrm{TH} \times_{\widehat{Q}} \widehat{\widehat{Q}}$ and $\widehat{\mathrm{TH}}^* := \mathrm{TH}^* \times_{\mathrm{TH}} \widehat{\mathrm{TH}}$. We remark that $\widehat{\mathrm{TH}}^*$ is isomorphic to $Q^{ss}(m, \widehat{\boldsymbol{y}}, L, \alpha_*, \delta) \times \mathrm{Flag}(V, \underline{N})$. We also put $\widehat{\mathrm{TH}}^{ss} = \mathrm{TH}^{ss} \times_{\mathrm{TH}} \widehat{\mathrm{TH}}$. We have the natural $\mathrm{GL}(V)$-action on $\widehat{\mathrm{TH}}^{ss}$. The quotient stack is called the enhanced master space in the oriented case, and it is denoted by $\widehat{M}$. It is also called the master space for abbreviation.

Proposition 4.40 *$\widehat{M}$ is Deligne-Mumford and proper.*

Proof From Proposition 4.21, the stack $\mathrm{TH}^{ss} \,/\, \mathrm{PGL}(V)$ is Deligne-Mumford and proper. We have the naturally defined morphism $\widehat{M} \longrightarrow \mathrm{TH}^{ss} \,/\, \mathrm{PGL}(V)$, which is etale and finite. Then, we obtain that $\widehat{M}$ is also Deligne-Mumford and proper. ▮

We have the universal quotient objects $\big(q^u, \mathcal{E}^u, F_*, [\phi^u], \rho^u\big)$ on $\widehat{Q} \times X$. It induces the oriented reduced L-Bradlow pair $\pi_X^*\big(\mathcal{E}^u(-m), F_*, [\phi^u], \rho^u\big)$ on $\widehat{\mathrm{TH}} \times X$, where $\pi_X : \widehat{\mathrm{TH}} \times X \longrightarrow \widehat{Q} \times X$ denotes the natural projection. By taking the descent with respect to the $\mathrm{GL}(V)$-action, we obtain the oriented reduced L-Bradlow pair $(\widehat{E}^{\widehat{M}}, F_*^{\widehat{M}}, [\phi^{\widehat{M}}], \rho^{\widehat{M}})$. We also have the induced full flag $\mathcal{F}^{\widehat{M}}$ of $p_{X\,*}\widehat{E}^{\widehat{M}}(m)$.

The G_m action ρ on TH induces the G_m-action on $\widehat{\mathrm{TH}}$, which is also denoted by ρ. It induces the G_m-action on $\widehat{M}$, which is denoted by $\overline{\rho}$. Let us see the stack theoretic fixed point set of $\widehat{M}$ in the following subsubsections.

4.5.2 The obvious fixed point sets

We put $\widehat{\mathrm{TH}}_i^{ss} = \mathrm{TH}_i \times_{\mathrm{TH}} \widehat{\mathrm{TH}}^{ss}$. We have the substacks $\widehat{M}_i := \widehat{\mathrm{TH}}_i^{ss} \,/\, \mathrm{GL}(V)$ $(i = 1, 2)$. The stack $\widehat{M}_2$ can be easily described. By construction, $\widehat{M}_2$ gives the moduli stack $\widetilde{\mathcal{M}}^s(\widehat{\boldsymbol{y}}, [L], \alpha_*, \delta_-)$ of the tuples $(E_*, [\phi], \rho, \mathcal{F})$, where $(E_*, [\phi], \rho)$ is an oriented δ_--stable reduced L-Bradlow pairs of type $\boldsymbol{y}$, and $\mathcal{F}$ is a full flag of $H^0(X, E(m))$. It is easily related with the moduli stack $\mathcal{M}^s(\widehat{\boldsymbol{y}}, [L], \alpha_*, \delta_-)$ of δ_--semistable oriented reduced L-Bradlow pairs. We have the universal sheaf $\widehat{E}^u$ over $\mathcal{M}^s(\widehat{\boldsymbol{y}}, [L], \alpha_*, \delta_-) \times X$. We obtain the locally free sheaf $p_{X\,*}\widehat{E}^u(m)$ on $\mathcal{M}^s(\widehat{\boldsymbol{y}}, [L], \alpha_*, \delta_-)$. The associated full flag bundle is isomorphic to $\widehat{M}_2$.

Let us see $\widehat{M}_1$. In the case (I), $\widehat{M}_1$ is isomorphic to the moduli stack of the tuples $(E_*, [\phi], \rho, \mathcal{F})$, where $(E_*, [\phi], \rho)$ is a δ_+-semistable oriented reduced L-Bradlow pair of type $\boldsymbol{y}$, and $\mathcal{F}$ is a full flag of $H^0(X, E(m))$. It is related with the moduli stack $\mathcal{M}^s\big(\widehat{\boldsymbol{y}}, [L], \alpha_*, \delta_+\big)$, as in the case of $\widehat{M}_2$. In the case (II), $\widehat{M}_1$ is the moduli stack of (δ, ℓ)-semistable tuples $(E_*, [\phi], \rho, \mathcal{F})$.

We have the restriction of the oriented reduced L-Bradlow pair $(\widehat{E}^{\widehat{M}}, F_*^{\widehat{M}}, [\phi^{\widehat{M}}], \rho^{\widehat{M}})$ to $\widehat{M}_i \times X$. We also have the universal objects on $\widehat{M}_i \times X$ by the moduli theoretic meaning of $\widehat{M}_i$. It is clear that they are isomorphic, by the construction of $\widehat{E}^{\widehat{M}}$. It is also easy to observe that the weight of $\overline{\rho}$ on $\widehat{E}^{\widehat{M}}_{|\widehat{M}_i \times X}$ is 0.

4.5.3 Fixed point sets associated to decomposition types

Let us describe the components of the fixed point set contained in $\widehat{M}^* := \widehat{M} - (\widehat{M}_1 \cup \widehat{M}_2)$. We use the notation in the subsubsection 4.4.2. Let $\mathfrak{I} = (\boldsymbol{y}_1, \boldsymbol{y}_2, I_1, I_2)$ be a decomposition type. In the case (II) (the subsubsection 4.3.1), we assume $\underline{\ell} \subset I_1$. We put $k := \min(I_2) - 1$. Let $\widetilde{Q}^{(1)}(\delta, k)$ denote the maximal open subset of $\widetilde{Q}^{(1)}$ determined by the (δ, k)-semistability. (It is open due to Proposition 4.15.) Let $\widetilde{Q}_+^{(2)}$ denote the maximal open subset determined by the condition that the reduced $\mathcal{O}(-m)$-Bradlow pair $\big(E_*^{(2)}, \mathcal{F}_{k+1}\big)$ is ϵ-stable for any sufficiently small $\epsilon > 0$. We have the naturally defined morphism $\widetilde{Q}^{(1)}(\delta, k) \times \widetilde{Q}_+^{(2)} \longrightarrow \widetilde{Q}$. Let $\widehat{\widehat{Q}}_{\mathrm{split}}(\mathfrak{I})$

denote the fiber product of $\widetilde{Q}^{(1)}(\delta,k)\times\widetilde{Q}^{(2)}_{+}$ and $\widehat{\widetilde{Q}}$ over $\widetilde{Q}$. We also put $\widehat{\mathrm{TH}}^{ss}_{\mathrm{split}}(\mathfrak{I}):=\widehat{\mathrm{TH}}^{*}\times_{\widehat{\widetilde{Q}}}\widehat{\widetilde{Q}}_{\mathrm{split}}(\mathfrak{I})$. We have the natural $\mathrm{GL}(V^{(1)})\times\mathrm{GL}(V^{(2)})$-action on $\widehat{\mathrm{TH}}^{ss}_{\mathrm{split}}(\mathfrak{I})$. The quotient stack is denoted by $\widehat{M}^{G_m}(\mathfrak{I})$. Due to Proposition 4.35, we have the naturally defined morphism $\varphi_{\mathfrak{I}}:\widehat{M}^{G_m}(\mathfrak{I})\longrightarrow\widehat{M}$.

Lemma 4.41

1. *$\widehat{M}^{G_m}(\mathfrak{I})$ is isomorphic to the moduli stack of the objects $\big((E^{(1)}_*,\phi,\mathcal{F}^{(1)}),(E^{(2)}_*,\mathcal{F}^{(2)}),\rho\big)$ as follows:*
    - *$(E^{(1)}_*,\phi)$ is δ-semistable parabolic L-Bradlow pair of type $\boldsymbol{y}_1$, and $\mathcal{F}^{(1)}$ is a full flag of $p_{X\,*}E^{(1)}(m)$ such that $(E^{(1)}_*,\phi,\mathcal{F}^{(1)})$ is (δ,k)-semistable.*
    - *$E^{(2)}_*$ is semistable parabolic sheaf of type $\boldsymbol{y}_2$, and $\mathcal{F}^{(2)}$ is a full flag of $p_{X\,*}E^{(2)}(m)$ such that $(E^{(2)},\mathcal{F}^{(2)}_{\min})$ is a reduced $\mathcal{O}(-m)$-Bradlow pair.*
    - *ρ is an orientation of $E^{(1)}\oplus E^{(2)}$.*
2. *We have the decomposition $\varphi_{\mathfrak{I}}^*\widehat{E}^{\widehat{M}}_*=E^{\widehat{M}}_1\oplus E^{\widehat{M}}_2$ and $\varphi_{\mathfrak{I}}^*\mathcal{F}^{\widehat{M}}=\mathcal{F}^{\widehat{M}}_1\oplus\mathcal{F}^{\widehat{M}}_2$. The pull back of the reduced L-section $[\phi^{\widehat{M}}]$ of $\widehat{E}^{\widehat{M}}$ naturally gives the L-section $\phi^{\widehat{M}}_1$ of $E^{\widehat{M}}_1$.*

    Then, the object $\big((E^{\widehat{M}}_{1\,*},\phi^{\widehat{M}}_1,\mathcal{F}^{\widehat{M}}_1),(E^{\widehat{M}}_{2\,*},\mathcal{F}^{\widehat{M}}_2),\varphi_{\mathfrak{I}}^*\rho^{\widehat{M}}\big)$ gives the universal object over $\widehat{M}^{G_m}\times X$ in the moduli theoretic meaning above.

Proof We recall the fibration $\mathrm{TH}^*\longrightarrow\widetilde{Q}$ is $\mathcal{O}_{\mathbb{P}_m}(-1)^*$. Then, the claim is clear by construction. ∎

We will later give a more convenient description of $\widehat{M}^{G_m}(\mathfrak{I})$ and the restriction of $\big(\widehat{E}^{\widehat{M}},F_*,[\phi],\rho\big)$ to $\widehat{M}^{G_m}(\mathfrak{I})$.

4.5.4 Statement

We put as follows:

$$\mathcal{S}(m,\boldsymbol{y}):=\begin{cases}\mathrm{Dec}(m,\boldsymbol{y},\alpha_*,\delta) & \text{(the case (I))}\\[1ex] \big\{\mathfrak{I}=(\boldsymbol{y}_1,\boldsymbol{y}_2,I_1,I_2)\in\mathrm{Dec}(m,\boldsymbol{y},\alpha_*,\delta)\,\big|\,\underline{\ell}\subset I_1\big\} & \text{(the case (II))}\end{cases}$$

Proposition 4.42 *Let $\widehat{M}^{G_m}$ denote the stack theoretic fixed point set of $\widehat{M}$ with respect to the action $\overline{\rho}$. Then we have the following isomorphism:*

$$\widehat{M}^{G_m}\simeq\widehat{M}_1\sqcup\widehat{M}_2\sqcup\coprod_{\mathfrak{I}\in\mathcal{S}(m,\boldsymbol{y})}\widehat{M}^{G_m}(\mathfrak{I}).$$

We will prove the proposition in the following subsubsections.

4.5.5 Ambient stack

We have the Poincaré bundle $\mathcal{P}oin$ on $\mathrm{Pic}\big(\det(y(m))\big)\times X$. Then, we obtain the vector bundle $\widehat{Z}_m:=p_{X\,*}\big(\mathcal{H}om(\bigwedge^r V_{m,X},\mathcal{P}oin)\big)$ on $\mathrm{Pic}\big(\det(y(m))\big)$. Recall that the Gieseker space is the projectivization of $\widehat{Z}_m$. We have the natural projection $\widetilde{\mathcal{B}}\longrightarrow Z_m$. Thus, we put $\widehat{\widetilde{\mathcal{B}}}^{ss}:=\widehat{Z}_m\times_{Z_m}\widetilde{\mathcal{B}}^{ss}$. We have the naturally defined $\mathrm{GL}(V_m)$-action on $\widehat{\widetilde{\mathcal{B}}}^{ss}$. The quotient stack is denoted by $\mathfrak{B}'$. The G_m-action ρ on $\widetilde{\mathcal{B}}^{ss}$ induces the action on $\widehat{\widetilde{\mathcal{B}}}^{ss}$. Since it commutes with the action of $\mathrm{GL}(V)$, we obtain the G_m-action $\overline{\rho}$ on $\mathfrak{B}'$.

We have the naturally defined closed immersion $\widehat{\mathrm{TH}}^{ss}\longrightarrow\widehat{\widetilde{\mathcal{B}}}^{ss}$, which is $\mathrm{GL}(V_m)\times G_m$-equivariant. Therefore, we obtain the closed G_m-equivariant immersion $\widehat{M}\longrightarrow\mathfrak{B}'$. Since $\widehat{M}$ is Deligne-Mumford and separated, we can take an equivariant open neighbourhood $\mathfrak{B}$ of $\widehat{M}$ in $\mathfrak{B}'$, which is Deligne-Mumford and separated. We remark that $\mathfrak{B}$ is also smooth. Let $\mathfrak{B}^{G_m}$ denote the fixed point set of $\mathfrak{B}$ with respect to G_m, i.e., the 0-set of the vector field induced by the G_m-action. Recall that $\widehat{M}^{G_m}$ is defined to be $\mathfrak{B}^{G_m}\times_{\mathfrak{B}}\widehat{M}$.

4.5.6 Fixed point set of the ambient space

We also put $\widehat{\widetilde{\mathcal{B}}}_i^{ss} := \widetilde{\mathcal{B}}_i \times_{\widetilde{\mathcal{B}}} \widehat{\widetilde{\mathcal{B}}}^{ss}$. The quotient stack $\widehat{\widetilde{\mathcal{B}}}_i^{ss}$ is denoted by $\mathfrak{B}_i$. It gives an open subset of $\mathfrak{B}^{G_m}$. It is easy to see $\widehat{M}_i = \mathfrak{B}_i \times_{\mathfrak{B}} \widehat{M}$.

We put $N = H_y(m)$. Let $\boldsymbol{N} = (N_1, N_2)$, $\boldsymbol{r} = (r_1, r_2)$, $\boldsymbol{k}_* := \bigl(k_{1,j}, k_{2,j} \,\big|\, j = 1, \ldots, l\bigr)$, $\boldsymbol{I} := (I_1, I_2)$ be a datum as follows:

- N_i are positive integers such that $N_1 + N_2 = N$.
- r_i are positive integers such that $r_1 + r_2 = r$.
- $k_{i,j}$ are positive integers such that $k_{1,j} + k_{2,j} = H_y(m) - H_{\boldsymbol{y},j}(m)$.
- $I_1 \sqcup I_2 = \underline{N}$.

Such a tuple $\mathfrak{Q} := (\boldsymbol{N}, \boldsymbol{r}, \boldsymbol{k}_*, \boldsymbol{I})$ is called a decomposition type for $\widetilde{\mathcal{A}}$. We remark that a decomposition type $\mathfrak{I} = (\boldsymbol{y}_1, \boldsymbol{y}_2, I_1, I_2)$ induces the decomposition type $\mathfrak{Q}(\mathfrak{I})$ for $\widetilde{\mathcal{A}}$, as follows:

$$N_i = |I_i|, \quad r_i = \mathrm{rank}(\boldsymbol{y}_i), \quad k_{i,j} = H_{y_i}(m) - H_{\boldsymbol{y}_i,j}(m).$$

We put $\widehat{\widetilde{\mathcal{A}}} := \widetilde{\mathcal{A}} \times_{Z_m} \widehat{Z}_m$. For a decomposition type $\mathfrak{Q} = (\boldsymbol{N}, \boldsymbol{r}, \boldsymbol{k}_*, \boldsymbol{I})$ for $\widetilde{\mathcal{A}}$, the locally closed regular subvariety $\mathfrak{C}_1(\mathfrak{Q})$ of $\widehat{\widetilde{\mathcal{A}}}$ is the set of the points $(\mathfrak{f}, K_*, [\phi], \mathcal{F})$ satisfying the following conditions:

- There exists the unique decomposition $V = V^{(1)} \oplus V^{(2)}$.
- $\mathfrak{f} \in \widehat{Z}_m$ is contained in $H^0\Bigl(X, \mathcal{H}om\Bigl(\bigwedge^{r_1} V_X^{(1)} \otimes \bigwedge^{r_2} V_X^{(2)}, L\Bigr)\Bigr)$ for some line bundle $L \in \mathrm{Pic}\bigl(\det(y(m))\bigr)$, where $V_X^{(i)} := V^{(i)} \otimes \mathcal{O}_X$.
- $K_* = \bigl(K_i \,\big|\, i = 1, \ldots, l\bigr) \in \prod_i Q_{m,i}$ is compatible with the decomposition $V = V^{(1)} \oplus V^{(2)}$, i.e., we have the decomposition $K_j = K_j^{(1)} \oplus K_j^{(2)}$ such that $K_j^{(i)}$ are quotients of $V^{(i)}$. We also assume $\dim K_j^{(i)} = k_{i,j}$.
- $[\phi]$ is contained in the projectivization of $V^{(1)}$.
- We have the decomposition $\mathcal{F} = \mathcal{F}^{(1)} \oplus \mathcal{F}^{(2)}$ compatible with the decomposition $V = V^{(1)} \oplus V^{(2)}$. Moreover, $\mathcal{F}_j^{(i)} / \mathcal{F}_{j-1}^{(i)} \neq 0$ if and only if $j \in I^{(i)}$.

We put $\widetilde{\mathcal{B}}^* := \widetilde{\mathcal{B}} - \bigcup \widetilde{\mathcal{B}}_i$. We put $\mathfrak{C}_2(\mathfrak{Q}) := \widetilde{\mathcal{B}}^* \times_{\widehat{\widetilde{\mathcal{A}}}} \mathfrak{C}_1(\mathfrak{Q})$. We have the natural $\mathrm{GL}(V)$-action on $\mathfrak{C}_2(\mathfrak{Q})$, and the quotient stack is denoted by $\mathfrak{C}_3'(\mathfrak{Q})$, which is the closed substack of $\mathfrak{B}'$. We put $\mathfrak{C}_3(\mathfrak{Q}) = \mathfrak{C}_3'(\mathfrak{Q}) \cap \mathfrak{B}$. The following lemma can be checked easily.

Lemma 4.43 $\mathfrak{C}_3(\mathfrak{Q})$ *are open subsets of* $\mathfrak{B}^{G_m}$. ∎

4.5.7 Proof of Proposition 4.42

We put as follows:

$$\mathfrak{B}_0^{G_m} := \coprod_{\mathfrak{I} \in \mathcal{S}(m, \boldsymbol{y})} \mathfrak{C}_3\bigl(\mathfrak{Q}(\mathfrak{I})\bigr)$$

We have the natural morphism $\widehat{M}^{G_m}(\mathfrak{I}) \longrightarrow \mathfrak{C}_3\bigl(\mathfrak{Q}(\mathfrak{I})\bigr)$. Therefore, we have the morphism $\widehat{M}^{G_m}(\mathfrak{I}) \longrightarrow \mathfrak{B}_0^{G_m}$. We obtain the following morphism:

$$\psi_1 : \coprod \widehat{M}^{G_m}(\mathfrak{I}) \longrightarrow \widehat{M} \times_{\mathfrak{B}} \mathfrak{B}_0^{G_m}$$

Lemma 4.44 ψ_1 *is isomorphic.*

Proof Let $g : T \longrightarrow \widehat{M} \times_{\mathfrak{B}} \mathfrak{B}_0^{G_m}$ be a morphism. Then we have the $\mathrm{GL}(V)$-equivariant torsor $P(g)$. We also obtain the following data from $g : T \longrightarrow \mathfrak{B}_0^{G_m}$.

- We have the $\mathrm{GL}(V)$-equivariant decomposition $\mathcal{V}_1 \oplus \mathcal{V}_2$ of $p_{P(g)}^* V_X$ over $X \times P(f)$.

- $\boldsymbol{f} : \bigwedge^r \mathcal{V} \longrightarrow \det^* \mathcal{P}oin$, which is contained in $\mathrm{Hom}\big(\bigwedge^{r_1} \mathcal{V}_1 \otimes \bigwedge^{r_2} \mathcal{V}_2, \det^* \mathcal{P}oin\big)$.

The composite $T \longrightarrow \widehat{M} \times_{\mathfrak{B}} \mathfrak{B}_0^{G_m} \longrightarrow \widehat{M}$ is contained in $\widehat{M}^*$, and hence in $Q^{ss}(m, \widehat{\boldsymbol{y}}, L, \alpha_*, \delta)/\,\mathrm{GL}(V_m)$. Therefore, we obtain the following data:

- The oriented quotient parabolic L-Bradlow pair $\big(q, E_*(m), \rho, \phi\big)$

- $q : p_{P(g)}^* V_{m,X} \longrightarrow E(m)$ satisfies (TFV)-condition.

We have $\rho \circ \bigwedge^r q = \mathfrak{f}$. Then, we obtain the decomposition $E = E^{(1)} \oplus E^{(2)} = q\big(\mathcal{V}^{(1)}\big) \oplus q\big(\mathcal{V}^{(2)}\big)$. The claim is clear on the open subset where E is locally free. Since E is torsion-free and q is surjective, the decomposition is obtained on the whole space.

By taking the decent with respect to the $\mathrm{GL}(V)$-action, we obtain $\big((q_1, E_*^{(1)}, \phi_1, \mathcal{F}^{(1)}), (q_2, E_*^{(2)}, \mathcal{F}^{(2)}), \rho\big)$ on T. We remark that the decomposition data is determined on each connected component of T. The conditions in Proposition 4.35 is satisfied for the specialization of $(q_1, E_*^{(1)}, \phi_1, \mathcal{F}^{(1)})$ and $(q_2, E_*^{(2)}, \mathcal{F}^{(2)})$ to any closed fibers $\{u\} \times X$. Hence we obtain the morphism $T \longrightarrow \coprod \widehat{M}^{G_m}(\mathfrak{I})$. In particular, we obtain the morphism $\psi_2 : \widehat{M} \times_{\mathfrak{B}} \mathfrak{B}_0^{G_m} \longrightarrow \coprod \widehat{M}^{G_m}(\mathfrak{I})$. It is easy to see that ψ_1 and ψ_2 are mutually inverse. ∎

Then, we obtain the following:

$$\widehat{M}_1 \sqcup \widehat{M}_2 \sqcup \coprod \widehat{M}^{G_m}(\mathfrak{I}) = \widehat{M}^{G_m} \times_{\mathfrak{B}} \big(\mathfrak{B}_1 \cup \mathfrak{B}_2 \cup \mathfrak{B}_0^{G_m}\big) = \widehat{M}^{G_m} \times_{\mathfrak{B}} \mathfrak{B}^{G_m}$$

In particular, $\widehat{M}_1 \sqcup \widehat{M}_2 \sqcup \coprod \widehat{M}^{G_m}(\mathfrak{I})$ is the closed substack of $\widehat{M}$. Since $\mathfrak{B}(\mathfrak{Q})$ and $\mathfrak{B}_i$ are open in $\mathfrak{B}^{G_m}$, it is easy to see that $\widehat{M}_i$ and $\widehat{M}^{G_m}(\mathfrak{I})$ are unions of connected components of the fixed point set. Thus, the proof of Proposition 4.42 is finished. ∎

4.6 Decomposition into Product of Two Moduli Stacks

4.6.1 Statement

Let $\mathfrak{I} = (\boldsymbol{y}_1, \boldsymbol{y}_2, I_1, I_2)$ be a decomposition type. We would like to decompose $\widehat{M}^{G_m}(\mathfrak{I})$ into the product of two moduli stacks up to etale finite morphisms. We put $k := \min(I_2) - 1$. We use the notation in the subsubsection 4.5.3. We introduce some more moduli stacks.

Let $\widetilde{\mathcal{M}}^{ss}\big(\widehat{\boldsymbol{y}}_1, [L], \alpha_*, (\delta, k)\big)$ denote the moduli stack of the objects $(E_*^{(1)}, [\phi^{(1)}], \rho^{(1)}, \mathcal{F}^{(1)})$ as follows:

- $(E_*^{(1)}, [\phi^{(1)}], \rho^{(1)})$ is a δ-semistable oriented reduced L-Bradlow pair of type $\boldsymbol{y}_1$, and $\mathcal{F}^{(1)}$ is a full flag of $p_{X\,*}E^{(1)}(m)$ such that $(E_*^{(1)}, [\phi^{(1)}], \mathcal{F})$ is (δ, k)-semistable.

The universal object over $\widetilde{\mathcal{M}}^{ss}\big(\widehat{\boldsymbol{y}}_1, [L], \alpha_*, (\delta, k)\big) \times X$ is denoted by $\big(\widehat{E}_{1*}^u, \phi_1^u, \rho_1^u, \mathcal{F}_1^u\big)$.

Let $\widetilde{\mathcal{M}}^{ss}\big(\boldsymbol{y}_1, L, \alpha_*, (\delta, k)\big)$ denote the moduli stack of the objects $(E_*^{(1)}, \phi^{(1)}, \mathcal{F}^{(1)})$ as follows:

- $(E_*^{(1)}, \phi^{(1)})$ is a δ-semistable L-Bradlow pair of type $\boldsymbol{y}_1$ such that $\phi^{(1)}$ is non-trivial everywhere, and $\mathcal{F}^{(1)}$ is a full flag of $p_{X\,*}E^{(1)}(m)$ such that $(E_*^{(1)}, \phi^{(1)}, \mathcal{F})$ is (δ, k)-semistable.

The universal object over $\widetilde{\mathcal{M}}^{ss}\big(\boldsymbol{y}_1, L, \alpha_*, (\delta, k)\big) \times X$ is denoted by $(E_{1\,*}^u, \phi^u, \mathcal{F}_1^u)$.

Let $\widetilde{\mathcal{M}}^{ss}\big(\widehat{\boldsymbol{y}}_2, \alpha_*, +\big)$ denote the moduli stack of the objects $(E_*^{(2)}, \rho^{(2)}, \mathcal{F}^{(2)})$ as follows:

- $(E_*^{(2)}, \rho^{(2)})$ is semistable oriented parabolic sheaf of type $\boldsymbol{y}_2$, and $\mathcal{F}^{(2)}$ is a full flag of $p_{X\,*}E^{(2)}(m)$.

- $(E_*^{(2)}, \mathcal{F}_{\min}^{(2)})$ is an ϵ-semistable reduced $\mathcal{O}(-m)$-Bradlow pair for any sufficiently small $\epsilon > 0$.

The universal object over $\widetilde{\mathcal{M}}^{ss}(\widehat{\boldsymbol{y}}_2, \alpha_*, +) \times X$ is denoted by $(\widehat{E}_2^u, \rho_2^u, \mathcal{F}_2^u)$.

Proposition 4.45 *We put $r_i := \operatorname{rank} \boldsymbol{y}_i$. There exists the algebraic stack $\mathcal{S}$ with the following properties:*

- *There exist the following diagram:*

$$\widehat{M}^{G_m}(\mathfrak{I}) \xleftarrow{\ F\ } \mathcal{S} \xrightarrow{\ G\ } \widetilde{\mathcal{M}}^{ss}\big(\widehat{\boldsymbol{y}}_1, [L], \alpha_*, (\delta, k)\big) \times \widetilde{\mathcal{M}}^{ss}\big(\widehat{\boldsymbol{y}}_2, \alpha_*, +\big) \tag{77}$$

The morphisms F and G are etale and finite of degree $(r_1 \cdot r_2)^{-1}$ and r_2^{-1}, respectively.

- *Let $\mathcal{O}_{1,\mathrm{rel}}(1)$ denote the tautological line bundle of $\widetilde{\mathcal{M}}^{ss}(\widehat{\boldsymbol{y}}_1, [L], \alpha_*, \delta)$. There exists the line bundle $\mathcal{O}_{1,\mathrm{rel}}(1/r_2)$ on $\mathcal{S}$ such that $\mathcal{O}_{1,\mathrm{rel}}(1/r_2)^{r_2} = G^*\mathcal{O}_{1,\mathrm{rel}}(1)$, and we have the following relations:*

$$F^* E_1^{\widehat{M}} \simeq G^* \widehat{E}_1^u \otimes \mathcal{O}_{1,\mathrm{rel}}(1), \quad F^* E_2^{\widehat{M}} \simeq G^* \widehat{E}_2^u \otimes \mathcal{O}_{1,\mathrm{rel}}(-r_1/r_2) \tag{78}$$

The weight of the G_m-action $\overline{\rho}$ on $E_1^{\widehat{M}}$ and $E_2^{\widehat{M}}$ are -1 and r_1/r_2, respectively.

Corollary 4.46 *We have the following diagram:*

$$\widehat{M}^{G_m}(\mathfrak{I}) \xleftarrow{\ F\ } \mathcal{S} \xrightarrow{\ G'\ } \widetilde{\mathcal{M}}^{ss}\big(\boldsymbol{y}_1, L, \alpha_*, (\delta, k)\big) \times \widetilde{\mathcal{M}}^{ss}\big(\widehat{\boldsymbol{y}}_2, \alpha_*, +\big) \tag{79}$$

Here G' is etale and finite of degree $(r_1 \cdot r_2)^{-1}$. We have the following relations

$$F^* E_1^{\widehat{M}} \simeq G'^* E_1^u, \quad F^* E_2^{\widehat{M}} \simeq G'^* \widehat{E}_2^u \otimes \mathcal{O}r(E_1^u)^{-1/r_2}$$

Here, we put $\mathcal{O}r(E_1^u)^{-1/r_2} := \mathcal{O}_{1,\mathrm{rel}}(-r_1/r_2)$. ∎

Before going into the proof, we give some remark on the inductive process to investigate the transition of the moduli stacks, heuristically. Let δ be an element of $\mathrm{Cr}(\boldsymbol{y}, L, \alpha_*)$. Let δ_- and δ_+ be sufficiently close to δ such that $\delta_- < \delta < \delta_+$. We will be interested in the difference of $\mathcal{M}^{ss}(\widehat{\boldsymbol{y}}, [L], \alpha_*, \delta_-)$ and $\mathcal{M}^{ss}(\widehat{\boldsymbol{y}}, [L], \alpha_*, \delta_+)$. We make the enhanced master space. Then $\widehat{M}_i$ are isomorphic to the full flag variety bundles over $\widetilde{\mathcal{M}}^{ss}(\widehat{\boldsymbol{y}}, [L], \alpha_*, \delta_-)$ and $\mathcal{M}^{ss}(\widehat{\boldsymbol{y}}, [L], \alpha_*, \delta_+)$. So we can derive some information from the fixed point sets $\widehat{M}^{G_m}(\mathfrak{I})$ by the localization technique, and $\widehat{M}^{G_m}(\mathfrak{I})$ is isomorphic to $\widetilde{\mathcal{M}}^{ss}(\widehat{\boldsymbol{y}}_1, [L], \alpha_*, \delta, \ell) \times \widetilde{\mathcal{M}}^{ss}(\widehat{\boldsymbol{y}}_2, \alpha_*, +)$ up to finite and etale morphisms, where $\ell = \min(I_2) - 1$.

The structure of $\widetilde{\mathcal{M}}^{s}(\widehat{\boldsymbol{y}}_2, \alpha_*, +)$ can be easily related with the moduli stack $\mathcal{M} = \widetilde{\mathcal{M}}^{s}(\widehat{\boldsymbol{y}}_2, [\mathcal{O}(-m)], \alpha_*, \epsilon)$, where ϵ is a sufficiently small number. We have the universal oriented sheaf $\widehat{E}^u$ over $\mathcal{M} \times X$ with the universal reduced section $[\phi_1^u]$. Then we obtain the vector bundle $p_{X\,*}\big(\widehat{E}^u(m)\big)$ with the line subbundle $\mathcal{Q} \subset p_{X\,*}\big(\widehat{E}^u(m)\big)$ induced by $[\phi^u]$. We put $\mathcal{C} := p_{X\,*}\big(\widehat{E}^u(m)\big)/\mathcal{Q}$. Then the associated full flag bundle to $\mathcal{C}$ is isomorphic to $\widetilde{\mathcal{M}}^{ss}\big(\widehat{\boldsymbol{y}}_2, [\mathcal{O}(-m)], \alpha_*, \epsilon\big)$.

On the other hand, the structure of $\widetilde{\mathcal{M}}^{ss}\big(\widehat{\boldsymbol{y}}_1, [L], \alpha_*, \delta, \ell\big)$ is not so easy to describe. However, we can make the enhanced master space $\widehat{M}^{(1)}$ again, so that $\widehat{M}_1^{(1)}$ and $\widehat{M}_2^{(1)}$ are isomorphic to $\widetilde{\mathcal{M}}^{ss}\big(\widehat{\boldsymbol{y}}_1, [L], \alpha*, \delta, \ell\big)$ and the full flag variety bundle over $\widetilde{\mathcal{M}}^{ss}\big(\widehat{\boldsymbol{y}}_1, [L], \alpha_*, \delta_-\big)$. Thus, we can proceed inductively.

We remark $\mathrm{rank}(\boldsymbol{y}_1) < \mathrm{rank}(\boldsymbol{y})$. Therefore, the process will stop, and we will arrive the description of the difference of $\mathcal{M}^{ss}(\widehat{\boldsymbol{y}}, [L], \alpha_*, \delta_+)$ and $\mathcal{M}^{ss}(\widehat{\boldsymbol{y}}, [L], \alpha_*, \delta_-)$ in terms of the products of the moduli stacks of semistable objects with lower ranks. We use such an argument in the subsection 7.6.

4.6.2 Preliminary

We use the notation in the subsection 4.4. Let $\widehat{\widehat{Q}}^{(1)}$ denote the maximal open subset of $Q^\circ(m, \widehat{\boldsymbol{y}}_1, L) \times \mathrm{Flag}^{(1)}$ determined by the (δ, k)-semistability. Let $\widehat{\widehat{Q}}^{(2)}$ denote the maximal open subset of $Q^\circ(m, \widehat{\boldsymbol{y}}_2) \times \mathrm{Flag}^{(2)}$, which consists of the points $(q_2, E_{2\,*}, \rho_2, \mathcal{F}^{(2)})$ such that $(E_{2\,*}, \mathcal{F}^{(2)}_{\min})$ is ϵ-semistable for any small $\epsilon > 0$. We also put $T_1 := \operatorname{Spec} k[t_1, t_1^{-1}]$. We have the T_1-action on $\widehat{\widehat{Q}}^{(1)} \times \widehat{\widehat{Q}}^{(2)}$, given as follows:

$$t_1 \cdot \Big(\big(E_{1\,*}, \phi, \rho, \mathcal{F}^{(1)}\big), \big(E_{2\,*}, \phi, \rho, \mathcal{F}^{(2)}\big) \Big) = \Big(\big(E_{1\,*}, \phi,\, t_1 \cdot \rho,\, \mathcal{F}^{(1)}\big), \big(E_{2\,*}, \phi,\, t_1^{-1} \cdot \rho,\, \mathcal{F}^{(2)}\big) \Big)$$

By construction, we have the isomorphism $\widehat{\mathrm{TH}}^{ss}_{\mathrm{split}}(\mathfrak{I}) \simeq \big(\widehat{\widehat{Q}}^{(1)} \times \widehat{\widehat{Q}}^{(2)}\big)/T_1$. We have the naturally defined actions $\mathrm{GL}(V_i)$ on $\widehat{\widehat{Q}}^{(i)}$. We put $\mathcal{M}_i := \widehat{\widehat{Q}}^{(i)} / \mathrm{GL}(V_i)$. Then we obtain the following description:

$$\widehat{M}^{G_m}(\mathfrak{I}) \simeq \frac{\widehat{\widehat{Q}}^{(1)} \times \widehat{\widehat{Q}}^{(2)}}{\mathrm{GL}(V_1) \times \mathrm{GL}(V_2) \times T_1} \simeq \frac{\mathcal{M}_1 \times \mathcal{M}_2}{T_1}$$

Let us see the right hand side more closely. The stack $\mathcal{M}_1$ is isomorphic to the moduli stack of the objects $(E^{(1)}_*, \phi^{(1)}, \rho^{(1)}, \mathcal{F}^{(1)})$ as follows:

- $(E^{(1)}_*, \phi^{(1)}, \rho^{(1)})$ is a δ-semistable oriented L-Bradlow pair of type $\boldsymbol{y}_1$, and $\mathcal{F}^{(1)}$ is a full flag of $p_{X\,*}E^{(1)}(m)$ such that $(E^{(1)}_*, \phi^{(1)}, \mathcal{F}^{(1)})$ is (δ, k_0)-semistable.

The quotient stack $\mathcal{M}_2$ is isomorphic to the moduli stack $\widetilde{\mathcal{M}}^{ss}\big(\widehat{\boldsymbol{y}}_2, \alpha_*, +\big)$.

The T_1-action on $\mathcal{M}_1$ is given by $t_1 \cdot \big(E^{(1)}_*, \phi^{(1)}, \rho^{(1)}, \mathcal{F}^{(1)}\big) = \big(E^{(1)}_*, \phi^{(1)}, t_1 {\cdot} \rho^{(1)}, \mathcal{F}^{(1)}\big)$. The T_1-action on $\mathcal{M}_2$ is given by $t_1 \cdot \big(E^{(2)}_*, \rho^{(2)}, \mathcal{F}^{(2)}\big) = \big(E^{(2)}_*, t_1^{-1} {\cdot} \rho^{(2)}, \mathcal{F}^{(2)}\big)$.

4.6.3 Construction of $\mathcal{S}$

We put $\widetilde{T}_1 := \operatorname{Spec} k[s_1, s_1^{-1}]$. Let $\widetilde{T}_1 \longrightarrow T_1$ be the morphism given by $t_1 = s_1^{r_1 r_2}$, where $r_i = \operatorname{rank} \boldsymbol{y}_i$. We have the naturally induced $\widetilde{T}_1$-action on $\mathcal{M}_1 \times \mathcal{M}_2$. Let $\mathcal{S}$ denote the quotient stack $(\mathcal{M}_1 \times \mathcal{M}_2)/\widetilde{T}_1$. Then, we have the following morphism:

$$\mathcal{S} \xrightarrow{\;F\;} (\mathcal{M}_1 \times \mathcal{M}_2)/T_1 \simeq \widehat{M}^{G_m}(\mathfrak{I})$$

Here F is etale and finite of degree $(r_1 \cdot r_2)^{-1}$.

Let us see the $\widetilde{T}_1$-action on $\mathcal{M}_i$ $(i = 1, 2)$. The induced $\widetilde{T}_1$-action on $\mathcal{M}_2$ is trivial, i.e., $s_1 \cdot \big(E^{(2)}_*, \rho^{(2)}, \mathcal{F}^{(2)}\big) = \big(E^{(2)}_*, s_1^{-r_1 \cdot r_2} {\cdot} \rho^{(2)}, \mathcal{F}^{(2)}\big) \simeq \big(E^{(2)}_*, \rho^{(2)}, \mathcal{F}^{(2)}\big)$. The isomorphism is given by the following diagram:

$$\begin{array}{ccccc} E^{(2)} & & \det(E^{(2)}) & \xrightarrow{s_1^{-r_1 r_2} \rho^{(2)}} & \det^* \mathcal{P}oin \\ {\scriptstyle s_1^{-r_1}}\big\downarrow & & {\scriptstyle s_1^{-r_1 r_2}}\big\downarrow & & {\scriptstyle \mathrm{id}}\big\downarrow \\ E^{(2)} & & \det(E^{(2)}) & \xrightarrow{\rho^{(2)}} & \det^* \mathcal{P}oin \end{array} \tag{80}$$

The induced $\widetilde{T}_1$-action on $\mathcal{M}_1$ is given as follows:

$$s_1 \cdot \big(E^{(1)}_*, \phi^{(1)}, \rho^{(1)}, \mathcal{F}^{(1)}\big) = \big(E^{(1)}_*, \phi^{(1)}, s_1^{r_1 r_2} {\cdot} \rho^{(1)}, \mathcal{F}^{(1)}\big) \simeq \big(E^{(1)}_*, s_1^{r_2} {\cdot} \phi^{(1)}, \rho^{(1)}, \mathcal{F}^{(1)}\big).$$

The isomorphism is given by the following diagram:

$$\begin{array}{ccccccc} L & \longrightarrow & E^{(1)} & & \det(E^{(1)}) & \xrightarrow{s_1^{r_1 r_2} \rho^{(1)}} & \det^* \mathcal{P}oin \\ {\scriptstyle \mathrm{id}}\big\downarrow & & {\scriptstyle s_1^{r_2}}\big\downarrow & & {\scriptstyle s_1^{r_1 r_2}}\big\downarrow & & {\scriptstyle \mathrm{id}}\big\downarrow \\ L & \xrightarrow{s_1^{r_2} \phi} & E^{(1)} & & \det(E^{(1)}) & \xrightarrow{\rho^{(1)}} & \det^* \mathcal{P}oin \end{array} \tag{81}$$

In particular, since $\widetilde{T}_1$-action on $\mathcal{M}_2$ is trivial, we obtain the following morphism G:

$$\mathcal{S} = (\mathcal{M}_1/\widetilde{T}_1) \times \mathcal{M}_2 \xrightarrow{\;G\;} (\mathcal{M}_1/T_1) \times \mathcal{M}_2 = \widetilde{\mathcal{M}}^{ss}\big(\widehat{\boldsymbol{y}}_1, [L], \alpha_*, (\delta, k_0)\big) \times \widetilde{\mathcal{M}}^{ss}(\boldsymbol{y}_2, \alpha_*, I_2)$$

The morphism G is etale and finite of degree $1/r_2$.

4.6.4 The universal sheaf

We put $\overline{\mathcal{M}}_1 := \widetilde{\mathcal{M}}^{ss}\big(\widehat{\boldsymbol{y}}_1, [L], \alpha_*, (\delta, k_0)\big)$. We remark that $\pi : \mathcal{M}_1 \times \mathcal{M}_2 \longrightarrow \overline{\mathcal{M}}_1 \times \mathcal{M}_2$ is isomorphic to $\mathcal{O}_{1,\mathrm{rel}}(-1)^* \longrightarrow \overline{\mathcal{M}}_1 \times \mathcal{M}_2$, and the $\widetilde{T}_1$-action on $\mathcal{O}_{1,\mathrm{rel}}(-1)^*$ is given by $s \cdot v = s^{r_2} v$ on each fiber. We use the argument in the subsection 2.7. Let $\mathcal{T}(n)$ denote the trivial line bundle on $\overline{\mathcal{M}}_1 \times \mathcal{M}_2$ with the $\widetilde{T}_1$-action of weight n. It induces the $\widetilde{T}_1$-line bundle $\pi^*\mathcal{T}(n)$ over $\mathcal{M}_1 \times \mathcal{M}_2$. By the descent, we obtain the line bundle $\mathcal{I}_n$ over $\mathcal{M}_1/\widetilde{T}_1 \times \mathcal{M}_2$. We put $\mathcal{O}_{1,\mathrm{rel}}(1/r_2) := \mathcal{I}_1$. It satisfies $\mathcal{O}_{1,\mathrm{rel}}(1/r_2)^{r_2} \simeq G^*\mathcal{O}_{1,\mathrm{rel}}(1)$, due to Lemma 2.49.

Let $\widehat{E}_1'^{\,u}$ and $\widehat{E}_2'^{\,u}$ denote the pull back of $\widehat{E}_1^u$ via the morphism $\mathcal{M}_1 \times \mathcal{M}_2 \times X \longrightarrow \overline{\mathcal{M}}_1 \times \mathcal{M}_2 \times X$. By the construction, we have the $\widetilde{T}_1$-equivariant sheaves $\widehat{E}_1'^{\,\mathrm{TH}} \oplus \widehat{E}_2'^{\,\mathrm{TH}}$ on $\mathcal{M}_1 \times \mathcal{M}_2 \times X$, which is induced by the sheaf on $\widehat{\mathrm{TH}}^{ss}$ as in the subsubsection 4.5.1. When we take it into account of the G_m-action $\overline{\rho}$, we have $\widehat{E}_1'^{\,\mathrm{TH}} \simeq \widehat{E}_1'^{\,u} \otimes \pi^*\mathcal{T}(r_2)$ and $\widehat{E}_2'^{\,\mathrm{TH}} \simeq \widehat{E}_2'^{\,u} \otimes \pi^*\mathcal{T}(-r_1)$ due to the diagrams (80) and (81). Therefore, we obtain (78).

We put $\widetilde{T}_2 := \operatorname{Spec} k[s_2, s_2^{-1}]$, and let $\widetilde{T}_2 \longrightarrow G_m$ is a morphism given by $t = s_2^{r_2}$. We have the induced action $\widetilde{\rho}$ of $\widetilde{T}_2$ on $\widehat{\mathrm{TH}}^{ss}$. On $\widehat{\widehat{Q}}^{(1)} \times \widehat{\widehat{Q}}^{(2)}$, the induced action is given as follows:

$$s_2 \cdot \big(E_{1\,*}, \phi, \rho_1, \mathcal{F}_1, E_{2\,*}, \rho_2, \mathcal{F}_2\big) = \big(E_{1\,*}, s_2^{r_2}\phi, \rho_1, \mathcal{F}_1, E_{2\,*}, \rho_2, \mathcal{F}_2\big).$$

Therefore, the action of $\widetilde{T}_1$ and $\widetilde{T}_2$ on $\mathcal{M}_1 \times \mathcal{M}_2$ are same. On the other hand, the induced bundles over $\mathcal{M}_1 \times \mathcal{M}_2 \times X$ are $\widehat{E}_1'^{\,u}$ and $\widehat{E}_2'^{\,u}$ with respect to the $\widetilde{\rho}$-action. Therefore, the weight of $\widetilde{\rho}$ on $\varphi^*\widehat{E}_1^u \otimes \mathcal{I}_{r_2}$ is $-r_2$, and the weight of $\widetilde{\rho}$ on $\varphi^*\widehat{E}_2^u \otimes \mathcal{I}_{-r_1}$ is r_1, due to Lemma 2.50. Thus, the proof of Proposition 4.45 is finished. ∎

4.7 Simple Cases

4.7.1 The case where the 2-stability condition is satisfied

We give some indication about what happens when the 2-stability condition is satisfied for $(\boldsymbol{y}, L, \alpha_*, \delta)$, without proof. In this case, we do not have to consider the enhanced master space and (δ, ℓ)-semistability. Hence the problem is simpler.

We use the notation in the subsubsections 4.1.1 and 4.2.1. We take a positive rational number γ_1 and a negative rational number γ_2 such that $|\gamma_i|$ are sufficiently small. We take a large rational number k' such that $k' \cdot (\gamma_1 - \gamma_2) = 1$. We put $\mathcal{L}_i := \mathcal{L}_{\gamma_i}^{\otimes k'}$. We have $\mathcal{L}_2 = \mathcal{L}_1 \otimes \mathcal{O}_{\mathbb{P}_m}(-1)$.

Let us consider $\mathcal{B} := \mathbb{P}\big(\mathcal{O}_{\mathbb{P}_m}(0) \oplus \mathcal{O}_{\mathbb{P}_m}(1)\big)$ over $\mathcal{A}$. We put $\mathcal{B}_1 = \mathbb{P}\big(\mathcal{O}_{\mathbb{P}_m}(0)\big)$ and $\mathcal{B}_2 = \mathbb{P}\big(\mathcal{O}_{\mathbb{P}_m}(1)\big)$, which are naturally regarded as the closed subscheme of $\mathcal{B}$. We have the tautological line bundle $\mathcal{O}_{\mathbb{P}}(1)$, and we put $\mathcal{O}_{\mathcal{B}}(1) := \mathcal{O}_{\mathbb{P}}(1) \otimes \mathcal{L}_1$. We have the natural $\mathrm{SL}(V)$-action on $\mathcal{B}$, and $\mathcal{O}_{\mathcal{B}}(1)$ gives the equivariant polarization. Let $\mathcal{B}^{ss}$ denote the set of the semistable points of $\mathcal{B}$ with respect to $\mathcal{O}_{\mathcal{B}}(1)$. We put $\mathrm{TH}^{ss} := Q \times_{\mathcal{A}} \mathcal{B}^{ss}$. We have the natural $\mathrm{SL}(V)$-action on TH^{ss}. The following proposition can be shown by an argument similar to the proof of Proposition 4.21. In fact, it is much simpler.

Proposition 4.47 *The quotient stack* $\mathrm{TH}^{ss} / \mathrm{SL}(V)$ *is Deligne-Mumford.* ∎

We put $\widehat{\mathrm{TH}}^{ss} := \mathrm{TH}^{ss} \times_Q \widehat{Q}$, where we put $\widehat{Q} := Q^{ss}(\widehat{\boldsymbol{y}}, [L], \alpha_*, \delta)$. We put $\widehat{M} := \widehat{\mathrm{TH}}^{ss} / \mathrm{GL}(V)$, which is called the master space in the oriented case. We have the G_m-action $\overline{\rho}$ as in the enhanced case. From Proposition 4.47, we obtain the following.

Proposition 4.48 $\widehat{M}$ *is Deligne-Mumford and proper.* ∎

We put $\widehat{\mathrm{TH}}_i^{ss} := \mathcal{B}_i^{ss} \times_{\mathcal{A}} \widehat{\mathrm{TH}}^{ss}$, and we put $\widehat{M}_i := \widehat{\mathrm{TH}}_i^{ss} / \mathrm{GL}(V)$. We put $\widehat{M}^* := \widehat{M} - \widehat{M}_1 \cup \widehat{M}_2$. It is easy to observe that $\widehat{M}^*$ is an open substack of $\mathcal{M}(m, \widehat{\boldsymbol{y}}, L)$.

Due to Lemma 4.14, $\widehat{M}_1$ and $\widehat{M}_2$ are isomorphic to $\mathcal{M}^{ss}(\widehat{\boldsymbol{y}}, [L], \alpha_*, \delta_+)$ and $\mathcal{M}^{ss}(\widehat{\boldsymbol{y}}, [L], \alpha_*, \delta_-)$ respectively. They give the obvious fixed point sets of $\widehat{M}$ with respect to $\overline{\rho}$.

Let us see the other components of the fixed point set. A decomposition type is defined to be $\mathfrak{I} := (\boldsymbol{y}_1, \boldsymbol{y}_2) \in \mathcal{T}ype^2$ satisfying $\boldsymbol{y}_1 + \boldsymbol{y}_2 = \boldsymbol{y}$ and $P^{\alpha_*,\delta}_{\boldsymbol{y}} = P^{\alpha_*,\delta}_{\boldsymbol{y}_1} = P^{\alpha_*}_{\boldsymbol{y}_2}$. For such a decomposition type $\mathfrak{I} := (\boldsymbol{y}_1, \boldsymbol{y}_2)$, we consider the moduli stack $\widehat{M}^{G_m}(\mathfrak{I})$ of objects $\big(E_*^{(1)}, \phi, E_*^{(2)}, \rho\big)$ as follows:

- $(E_*^{(1)}, \phi)$ is δ-stable L-Bradlow pair of type $\boldsymbol{y}_1$.
- $E_*^{(2)}$ is stable parabolic sheaf of type $\boldsymbol{y}_2$.
- ρ is an orientation of $E^{(1)} \oplus E^{(2)}$.

Note that the 2-stability condition for $(\boldsymbol{y}, L, \alpha_*, \delta)$ implies the 1-stability conditions for $(\boldsymbol{y}_1, L, \alpha_*, \delta)$ and $(\boldsymbol{y}_2, \alpha_*)$ if $\mathcal{M}^{ss}(\boldsymbol{y}_1, L, \alpha_*, \delta) \times \mathcal{M}^{ss}(\boldsymbol{y}_2, \alpha_*) \neq \emptyset$. We have the naturally defined morphism $\widehat{M}^{G_m}(\mathfrak{I}) \longrightarrow \widehat{M}$ as in the subsubsection 4.5.3.

Let $S(\boldsymbol{y}, \alpha_*, \delta)$ denote the set of decomposition types. Then we can show the following by the same argument as the proof of Proposition 4.42.

Proposition 4.49 $\widehat{M}_1 \sqcup \widehat{M}_2 \sqcup \coprod_{\mathfrak{I} \in S(\boldsymbol{y}, \alpha_*, \delta)} M^{G_m}(\mathfrak{I})$ *is the stack theoretic fixed point set of* $\widehat{M}$ *with respect to* $\overline{\rho}$. ∎

We naturally have the oriented reduced L-Bradlow pair $(\widehat{E}_*^{\widehat{M}}, [\phi^{\widehat{M}}], \rho^{\widehat{M}})$ on $\widehat{M} \times X$, as in the subsubsection 4.5.1. The restriction of $(\widehat{E}_*^{\widehat{M}}, [\phi^{\widehat{M}}], \rho^{\widehat{M}})$ to $\widehat{M}_i \times X$ has the universal property with respect to the moduli theoretic meaning above. Let $\varphi : \widehat{M} \longrightarrow \mathcal{M}(m, \widehat{\boldsymbol{y}}, [L])$ denote the naturally defined morphism. Then the restriction of $\varphi^* \mathcal{O}_{\mathrm{rel}}(1)$ to $\widehat{M}^*$ is canonically trivialized. Therefore, the restriction of $[\phi^{\widehat{M}}]$ to $\widehat{M}^*$ gives the L-section, which we denote by $\phi^{\widehat{M}}$. Then, the restriction of $(\widehat{E}^{\widehat{M}}, \phi^{\widehat{M}}, \rho^{\widehat{M}})$ to $\widehat{M}^{G_m}(\mathfrak{I})$ has the universal property with respect to the moduli theoretic meaning above. Correspondingly, we have the decomposition $\widehat{E}^{\widehat{M}}_{|\widehat{M}^{G_m}(\mathfrak{I})} = E_1^{\widehat{M}} \oplus E_2^{\widehat{M}}$.

It is convenient to decompose $\widehat{M}^{G_m}(\mathfrak{I})$ into the product of two moduli stacks up to etale finite morphisms. By the same argument as the proof of Proposition 4.45, we obtain the following description of $\widehat{M}^{G_m}(\mathfrak{I})$ up to etale finite morphisms.

Proposition 4.50 *We put* $r_i := \operatorname{rank} \boldsymbol{y}_i$. *There exists the algebraic stack* $\mathcal{S}$ *with the following properties:*

- *There exists the following diagram:*

$$\widehat{M}^{G_m}(\mathfrak{I}) \xleftarrow{F} \mathcal{S} \xrightarrow{G} \mathcal{M}^{ss}\big(\widehat{\boldsymbol{y}}_1, [L], \alpha_*, \delta\big) \times \mathcal{M}^{ss}\big(\widehat{\boldsymbol{y}}_2, \alpha_*\big)$$

  The morphisms F *and* G *are etale and finite of degree* $(r_1 \cdot r_2)^{-1}$ *and* r_2^{-1}, *respectively. We also have the following diagram:*

$$\widehat{M}^{G_m}(\mathfrak{I}) \xleftarrow{F} \mathcal{S} \xrightarrow{G'} \mathcal{M}^{ss}\big(\boldsymbol{y}_1, L, \alpha_*, \delta\big) \times \mathcal{M}^{ss}\big(\widehat{\boldsymbol{y}}_2, \alpha_*\big) \tag{82}$$

  Here, G' *is etale and finite of degree* $(r_1 r_2)^{-1}$.

- *Let* $\mathcal{O}_{1,\mathrm{rel}}(1)$ *denote the tautological line bundle of* $\mathcal{M}^{ss}(\widehat{\boldsymbol{y}}_1, [L], \alpha_*, \delta)$. *We use the same notation to denote the pull back via an appropriate morphism. Then, there exists the line bundle* $\mathcal{O}_{1,\mathrm{rel}}(1/r_2)$ *on* $\mathcal{S}$ *such that* $\mathcal{O}_{1,\mathrm{rel}}(1/r_2)^{r_2} = G^* \mathcal{O}_{1,\mathrm{rel}}(1)$, *and we have the following relation:*

$$F^* E_1^{\widehat{M}} \simeq G^* \widehat{E}_1^u \otimes \mathcal{O}_{1,\mathrm{rel}}(1), \quad F^* E_2^{\widehat{M}} \simeq G^* \widehat{E}_2^u \otimes \mathcal{O}_{1,\mathrm{rel}}(-r_1/r_2) \tag{83}$$

  Here $\widehat{E}_1^u$ *and* $\widehat{E}_2^u$ *are induced by the universal sheaves over* $\mathcal{M}^{ss}(\widehat{\boldsymbol{y}}_1, [L], \alpha_*, \delta) \times X$ *and* $\mathcal{M}^{ss}(\widehat{\boldsymbol{y}}_2, \alpha_*) \times X$, *respectively. We also have the following relation:*

$$F^* E_1^{\widehat{M}} \simeq G'^* E_1^u, \quad F^* E_2^{\widehat{M}} \simeq G'^* \widehat{E}_2^u \otimes \mathcal{O}r(E_1^u)^{-1/r_2}$$

  Here, E_1^u *denotes the pull back of the universal sheaf over* $\mathcal{M}^{ss}(\boldsymbol{y}_1, L, \alpha_*, \delta) \times X$, *and* $\mathcal{O}r(E_1^u)^{-1/r_2}$ *denotes* $\mathcal{O}_{1,\mathrm{rel}}(-r_1/r_2)$.

- *The weights of the* G_m*-action* $\overline{\rho}$ *on* $E_1^{\widehat{M}}$ *and* $E_2^{\widehat{M}}$ *are* -1 *and* r_1/r_2, *respectively.* ∎

4.7.2 The case of oriented reduced $\boldsymbol{L}$-Bradlow pairs

Let $\boldsymbol{L}=(L_1,L_2)$ be a pair of line bundles on X. Let $\boldsymbol{\delta}=(\delta_1,\delta_2)$ be an element of $\big(\mathcal{P}^{\mathrm{br}}\big)^2$. We can discuss the transition of the moduli stacks $\mathcal{M}^{ss}(\boldsymbol{y},\boldsymbol{L},\alpha_*,\boldsymbol{\delta})$, when δ_1 is moved. For simplicity, we restrict ourselves to the case where both of δ_i are sufficiently small. Recall the results in the subsubsection 3.5.3. Then, the 2-stability condition is always satisfied, and the problem is simple as in the subsubsection 4.7.1. We give only an indication without proof.

Let $\boldsymbol{\delta}=(\delta_1,\delta_2)$ be an element of $\mathcal{P}^{\mathrm{br}\,2}$ such that δ_i are sufficiently small. Assume that the 1-stability condition does not hold for $(\boldsymbol{y},\boldsymbol{L},\alpha_*,\boldsymbol{\delta})$. In that case, we have the positive integers r_i $(i=1,2)$ satisfying the following:

$$r_1+r_2=r,\quad \frac{\delta_1}{r_1}=\frac{\delta_2}{r_2}$$

We take $\delta_{1,-}$ and $\delta_{1,+}$ such that $\delta_{1,-}<\delta_1<\delta_{1,+}$, which are sufficiently close to δ_1. We put $\boldsymbol{\delta}_\kappa:=(\delta_{1,\kappa},\delta_2)$ for $\kappa=\pm$. We would like to compare the moduli stacks $\mathcal{M}^{ss}(\boldsymbol{y},\boldsymbol{L},\alpha_*,\boldsymbol{\delta}_-)$ and $\mathcal{M}^{ss}(\boldsymbol{y},\boldsymbol{L},\alpha_*,\boldsymbol{\delta}_+)$.

We use the notation in the subsubsection 4.1.1. We put $\mathcal{A}:=\mathcal{A}_m(\boldsymbol{y},[\boldsymbol{L}])$ and $\mathcal{L}:=\mathcal{L}_{\boldsymbol{y},\boldsymbol{L}}\big(P_{\boldsymbol{y}}^{\alpha_*,\boldsymbol{\delta}}(m),\epsilon_*,\boldsymbol{\delta}(m)\big)$, which gives a $\mathrm{GL}(V)$-polarization on $\mathcal{A}$. We put $\mathcal{L}_\gamma:=\mathcal{L}\otimes\mathcal{O}_{\mathbb{P}^{(1)}_m}(\gamma)$ for rational number γ. Let $\mathcal{A}^{ss}(\mathcal{L}_\gamma)$ denote the set of the semistable points of $\mathcal{A}$ with respect to $\mathcal{L}_\gamma$.

We put $Q:=Q^{ss}(m,\boldsymbol{y},[\boldsymbol{L}],\alpha_*,\boldsymbol{\delta})$. We have the $\mathrm{GL}(V)$-actions on $\mathcal{A}$ and Q. We also have the equivariant morphism $\Psi_m:Q\longrightarrow\mathcal{A}$. The $\boldsymbol{\delta}_\kappa$-semistability condition determines the open subset Q^{ss}_κ. The following lemma can be shown by the same argument as the proof of Lemma 4.14.

Lemma 4.51 *Assume that the absolute value of $\gamma\neq 0$ is sufficiently small.*

- *Then, we have $\Psi_m^{-1}\big(\mathcal{A}^{ss}(\mathcal{L}_\gamma)\big)=Q^{ss}_{\mathrm{sign}(\gamma)}$.*
- *The induced morphism $\Psi_m:Q^{ss}_{\mathrm{sign}(\gamma)}\longrightarrow\mathcal{A}^{ss}(\mathcal{L}_\gamma)$ is a closed immersion. Moreover, the image is contained in $\mathcal{A}^s(\mathcal{L}_\gamma)$.* ∎

We take a positive rational number γ_1 and a negative rational number γ_2 such that $|\gamma_i|$ are sufficiently small. We take large number k' such that $k'(\gamma_1-\gamma_2)=1$. We put $\mathcal{L}_i:=\mathcal{L}_{\gamma_i}^{\otimes k'}$. We have $\mathcal{L}_2=\mathcal{L}_1\otimes\mathcal{O}_{\mathbb{P}^{(1)}_m}(-1)$.

Let us consider $\mathcal{B}:=\mathbb{P}\big(\mathcal{O}_{\mathbb{P}^{(1)}_m}(0)\oplus\mathcal{O}_{\mathbb{P}^{(1)}_m}(1)\big)$ over $\mathcal{A}$. We put $\mathcal{B}_1=\mathbb{P}\big(\mathcal{O}_{\mathbb{P}^{(1)}_m}(0)\big)$ and $\mathcal{B}_2=\mathbb{P}\big(\mathcal{O}_{\mathbb{P}^{(1)}_m}(1)\big)$, which are naturally regarded as the closed subscheme of $\mathcal{B}$. We have the tautological line bundle $\mathcal{O}_{\mathbb{P}}(1)$, and we put $\mathcal{O}_{\mathcal{B}}(1):=\mathcal{O}_{\mathbb{P}}(1)\otimes\mathcal{L}_1$. We have the natural $\mathrm{SL}(V)$-action on $\mathcal{B}$, and $\mathcal{O}_{\mathcal{B}}(1)$ gives the equivariant polarization. Let $\mathcal{B}^{ss}$ denote the set of the semistable points of $\mathcal{B}$ with respect to $\mathcal{O}_{\mathcal{B}}(1)$. We put $\mathrm{TH}^{ss}:=Q\times_{\mathcal{A}}\mathcal{B}^{ss}$. We have the natural $\mathrm{SL}(V)$-action on TH^{ss}. As in the case of Proposition 4.47, the following proposition can be shown easily.

Proposition 4.52 *The quotient stack $\mathrm{TH}^{ss}\,/\,\mathrm{SL}(V)$ is Deligne-Mumford.* ∎

We put $\widehat{\mathrm{TH}}^{ss}:=\mathrm{TH}^{ss}\times_Q\widehat{Q}$, where we put $\widehat{Q}:=Q(\widehat{\boldsymbol{y}},[\boldsymbol{L}],\alpha_*,\boldsymbol{\delta})$. We put $\widehat{M}:=\widehat{\mathrm{TH}}^{ss}\,/\,\mathrm{GL}(V)$, which is called the master space. We have the G_m-action $\overline{\rho}$ as usual. From Proposition 4.52, we obtain the following.

Proposition 4.53 *$\widehat{M}$ is Deligne-Mumford and proper.* ∎

We put $\widehat{\mathrm{TH}}_i^{ss}:=\mathcal{B}_i^{ss}\times_{\mathcal{A}}\widehat{\mathrm{TH}}^{ss}$, and we put $\widehat{M}_i:=\widehat{\mathrm{TH}}_i^{ss}\,/\,\mathrm{GL}(V)$. We put $\widehat{M}^*:=\widehat{M}-\widehat{M}_1\cup\widehat{M}_2$. It is easy to observe that $\widehat{M}^*$ is an open substack of the moduli stack $\mathcal{M}(m,\widehat{\boldsymbol{y}},L_1,[L_2])$ of the tuple $(E_*,\rho,\phi_1,[\phi_2])$ as follow:

- E_* is a parabolic sheaf with an orientation ρ satisfying the condition O_m.
- ϕ_1 is an L_1-section of E_* such that $\phi_1\neq 0$.
- $[\phi_2]$ is a reduced L_2-section of E_* such that $[\phi_2]\neq 0$.

From Lemma 4.51, $\widehat{M}_1$ and $\widehat{M}_2$ are isomorphic to $\mathcal{M}^{ss}(\widehat{\boldsymbol{y}},[\boldsymbol{L}],\alpha_*,\boldsymbol{\delta}_+)$ and $\mathcal{M}^{ss}(\widehat{\boldsymbol{y}},[\boldsymbol{L}],\alpha_*,\boldsymbol{\delta}_-)$ respectively. They give the obvious fixed point sets of $\widehat{M}$ with respect to $\overline{\rho}$.

Let us see the other components of the fixed point set. A decomposition type is defined to be a datum $\mathfrak{I}:=(\boldsymbol{y}_1,\boldsymbol{y}_2)\in\mathcal{T}ype^2$ satisfying the following:

$$\boldsymbol{y}_1+\boldsymbol{y}_2=\boldsymbol{y},\quad P^{\alpha_*}_{\boldsymbol{y}}=P^{\alpha_*}_{\boldsymbol{y}_1}=P^{\alpha_*}_{\boldsymbol{y}_2},\quad r_i=\mathrm{rank}(\boldsymbol{y}_i)$$

For a decomposition type $\mathfrak{I}:=(\boldsymbol{y}_1,\boldsymbol{y}_2)$, let $\widehat{M}^{G_m}(\mathfrak{I})$ be the moduli stack of the objects $\bigl(E^{(1)}_*,\phi_1,E^{(2)}_*,[\phi_2],\rho\bigr)$ as follows:

- $(E^{(1)}_*,\phi_1)$ is δ_1-stable L_1-Bradlow pair of type $\boldsymbol{y}_1$.
- $(E^{(2)}_*,[\phi_2])$ is δ_2-stable reduced L_2-Bradlow pair of type $\boldsymbol{y}_2$.
- ρ is an orientation of $E^{(1)}\oplus E^{(2)}$.

Note that δ_i are sufficiently small, and hence the 1-stability conditions for $(\boldsymbol{y}_1,L_1,\alpha_*,\delta_1)$ and $(\boldsymbol{y}_2,L_2,\alpha_*,\delta_2)$ are satisfied. We have the naturally defined morphism $\widehat{M}^{G_m}(\mathfrak{I})\longrightarrow\widehat{M}$, as in the subsubsection 4.5.3.

Let $S(\boldsymbol{y},\alpha_*,\boldsymbol{\delta})$ denote the set of decomposition type. Then, we can show the following by the same argument as the proof of Proposition 4.42.

Proposition 4.54 $\widehat{M}_1\sqcup\widehat{M}_2\sqcup\coprod_{\mathfrak{I}\in S(\boldsymbol{y},\alpha_*,\boldsymbol{\delta})}M^{G_m}(\mathfrak{I})$ *is the stack theoretic fixed point set of* $\widehat{M}$ *with respect to* $\overline{\rho}$. ∎

We naturally have the oriented reduced $\boldsymbol{L}$-Bradlow pair $(\widehat{E}^{\widehat{M}}_*,[\phi^{\widehat{M}}_1],[\phi^{\widehat{M}}_2],\rho)$ on $\widehat{M}\times X$, as in the subsubsection 4.5.1. The restriction of $(\widehat{E}^{\widehat{M}}_*,[\phi^{\widehat{M}}_1],[\phi^{\widehat{M}}_2],\rho)$ to $\widehat{M}_i\times X$ has the universal property with respect to the moduli theoretic meaning of $\widehat{M}_i$.

Let $\mathcal{O}^{(i)}_{\mathrm{rel}}(1)$ denote the line bundle on $\mathcal{M}(m,\widehat{\boldsymbol{y}},[\boldsymbol{L}])$ which is the pull back of the relatively tautological line bundle on $\mathcal{M}(m,\widehat{\boldsymbol{y}},[L_i])$ via the natural morphism $\mathcal{M}(m,\widehat{\boldsymbol{y}},[\boldsymbol{L}])\longrightarrow\mathcal{M}(m,\widehat{\boldsymbol{y}},[L_i])$. Let $\varphi:\widehat{M}\longrightarrow\mathcal{M}(m,\widehat{\boldsymbol{y}},[\boldsymbol{L}])$ denote the naturally defined morphism. The restriction of $\varphi^*\mathcal{O}^{(1)}_{\mathrm{rel}}(1)$ to $\widehat{M}^*$ is canonically trivialized. Thus, the restriction $[\phi^{\widehat{M}}_1]_{|\widehat{M}^*}$ induces the L_1-section of $\widehat{E}^{\widehat{M}}$, which we denote by $\phi^{\widehat{M}}_1$. We put $\mathcal{I}^{(2)}:=\varphi^*\mathcal{O}^{(2)}_{\mathrm{rel}}(-1)$. Then, $[\phi^{\widehat{M}}_2]$ gives the morphism $\mathcal{I}^{(2)}\otimes L_2\longrightarrow\widehat{E}^{\widehat{M}}$. The restriction of $(\widehat{E}^{\widehat{M}},\phi^{\widehat{M}}_1,[\phi^{\widehat{M}}_2],\rho)$ to $\widehat{M}^{G_m}(\mathfrak{I})$ has the universal property with respect to the moduli theoretic meaning above. Correspondingly, we have the decomposition $\widehat{E}^{\widehat{M}}_{|\widehat{M}^{G_m}(\mathfrak{I})}=E^{\widehat{M}}_1\oplus E^{\widehat{M}}_2$.

It is convenient to decompose $\widehat{M}^{G_m}(\mathfrak{I})$ into the product of two moduli stacks up to etale finite morphisms. By the same argument as the proof of Proposition 4.45, we obtain the following description of $\widehat{M}^{G_m}(\mathfrak{I})$ up to etale finite morphisms.

Proposition 4.55 *We put* $r_i:=\mathrm{rank}\,\boldsymbol{y}_i$. *There exists the algebraic stack* $\mathcal{S}$ *with the following properties:*

- *There exist the following diagram:*

$$\widehat{M}^{G_m}(\mathfrak{I})\xleftarrow{\;F\;}\mathcal{S}\xrightarrow{\;G\;}\mathcal{M}^{ss}\bigl(\widehat{\boldsymbol{y}}_1,[L_1],\alpha_*,\delta_1\bigr)\times\mathcal{M}^{ss}\bigl(\widehat{\boldsymbol{y}}_2,[L_2],\alpha_*,\delta_2\bigr)\tag{84}$$

  The morphisms F *and* G *are etale and finite of degree* $(r_1\cdot r_2)^{-1}$ *and* r_2^{-1}, *respectively. We also have the following diagram:*

$$\widehat{M}^{G_m}(\mathfrak{I})\xleftarrow{\;F\;}\mathcal{S}\xrightarrow{\;G'\;}\mathcal{M}^{ss}\bigl(\boldsymbol{y}_1,L_1,\alpha_*,\delta_1\bigr)\times\mathcal{M}^{ss}\bigl(\widehat{\boldsymbol{y}}_2,[L_2],\alpha_*,\delta_2\bigr)\tag{85}$$

  Here G' *is etale and finite of degree* $(r_1r_2)^{-1}$.

- *Let $\mathcal{O}_{i,\mathrm{rel}}(1)$ denote the pull back of the relative tautological line bundle of $\mathcal{M}^{ss}(\boldsymbol{y}_i,[L_i],\alpha_*,\delta)$. There exists the line bundle $\mathcal{O}_{1,\mathrm{rel}}(1/r_2)$ on $\mathcal{S}$ such that $\mathcal{O}_{1,\mathrm{rel}}(1/r_2)^{r_2}\simeq G^*\mathcal{O}_{1,\mathrm{rel}}(1)$, and we have the following relation:*
$$F^*E_1^{\widehat{M}}\simeq G^*\widehat{E}_1^u\otimes\mathcal{O}_{1,\mathrm{rel}}(1),\quad F^*E_2^{\widehat{M}}\simeq G^*\widehat{E}_2^u\otimes\mathcal{O}_{1,\mathrm{rel}}(-r_1/r_2),\quad F^*\mathcal{I}^{(2)}\simeq G^*\mathcal{O}_{2,\mathrm{rel}}(-1)\otimes\mathcal{O}_{1,\mathrm{rel}}(-r_1/r_2). \tag{86}$$
Here $\widehat{E}_i^u$ denotes the pull back of the universal sheaf over $\mathcal{M}^{ss}(\widehat{\boldsymbol{y}}_i,[L_i],\alpha_*,\delta_i)\times X$. We also have the following:
$$F^*E_1^{\widehat{M}}\simeq G'^*E_1^u,\quad F^*E_2^{\widehat{M}}\simeq G'^*\widehat{E}_2^u\otimes\mathcal{O}r(E_1^u)^{-1/r_2}$$
Here, E_1^u denotes the pull back of the universal sheaf over $\mathcal{M}^{ss}(\boldsymbol{y}_1,L_1,\alpha_*,\delta_1)\times X$, and $\mathcal{O}r(E_1^u)^{-1/r_2}$ denotes $\mathcal{O}_{1,\mathrm{rel}}(-r_1/r_2)$.
- *The weights of $\overline{\rho}$ on $E_1^{\widehat{M}}$, $E_2^{\widehat{M}}$ and $\mathcal{I}^{(2)}_{|\widehat{M}^{G_m}(\mathfrak{I})}$ are -1, r_1/r_2 and r_1/r_2, respectively.* ∎

5 Obstruction Theory of the Moduli Stacks and the Master spaces

In this section, we discuss the obstruction theory of the moduli stack of parabolic reduced Bradlow pairs on a smooth projective *surface* X. We also assume the smoothness of the divisor D, which is the support of the parabolic structure. The naive strategy is explained in the subsubsection 2.4.2. We will also discuss the obstruction theory for the master spaces.

Notation Let S be a scheme, Z be a stack over S, and G be a smooth group scheme over S. When G acts on Z, the quotient stack is denoted by Z_G.

5.1 Deformation of Torsion-free Sheaves

5.1.1 Construction of the basic complex

Let U be any algebraic stack over k, and let E be a torsion-free U-coherent sheaf defined over $U\times X$. Assume that we have a locally free resolution $V_\cdot=(V_{-1}\to V_0)$ of E on U, i.e., V_i are locally free sheaves of finite ranks, and we have the surjection $V_0\longrightarrow E$ whose kernel is V_{-1}. The inclusion $V_{-1}\subset V_0$ is denoted by f. We put as follows (see the subsubsection 2.1.3):
$$\mathfrak{g}(V_\cdot):=\mathcal{H}om(V_\cdot,V_\cdot)^\vee[-1]$$

Let W_{-1} and W_0 be vector spaces over k such that $\operatorname{rank}W_i=\operatorname{rank}V_i$. We denote $W_i\otimes\mathcal{O}_X$ by $W_{i\,X}$. We put $\mathrm{GL}(W_\cdot):=\mathrm{GL}(W_{-1})\times\mathrm{GL}(W_0)$. We have the natural right $\mathrm{GL}(W_\cdot)$-action on the vector bundle $N(W_{-1\,X},W_{0\,X})$ given by $(g_{-1},g_0)\cdot f=g_0^{-1}\circ f\circ g_{-1}$. Here g_i denotes an element of $GL(W_i)$, and f denotes an element of $N(W_{-1\,X},W_{0\,X})$. The quotient stack is denoted by $Y(W_\cdot)$.

Then, we have the classifying map $\Phi(V_\cdot):U\times X\longrightarrow Y(W_\cdot)$ over X, and thus $\Phi(V_\cdot)^*L_{Y(W_\cdot)/X}\longrightarrow L_{U\times X/X}$. As explained in Example 2.21, $\Phi(V_\cdot)^*L_{Y(W_\cdot)/X}$ is represented by the complex $\mathfrak{g}(V_\cdot)_{\leq 1}$. We have the naturally defined morphism $\mathfrak{g}(V_\cdot)\longrightarrow\mathfrak{g}(V_\cdot)_{\leq 1}$, and hence $\mathfrak{g}(V_\cdot)\longrightarrow L_{U\times X/X}$.

Let ω_X denote the dualizing complex of X, and we put as follows:
$$\mathrm{Ob}(V_\cdot):=Rp_{X\,*}\big(\mathfrak{g}(V_\cdot)\otimes\omega_X\big)$$
Then, we have the naturally defined morphism $\mathrm{ob}(V_\cdot):\mathrm{Ob}(V_\cdot)\longrightarrow L_U$.

Lemma 5.1 *The object $\mathrm{Ob}(V_\cdot)$ and the morphism $\mathrm{ob}(V_\cdot)$ depends only on E in the derived category $D(U)$, in the sense that it is independent of the choice of a resolution $V_\cdot$.*

Proof Let $V_\cdot^{(1)}$ be another resolution. We would like to compare the two morphisms $\mathrm{ob}(V_\cdot^{(1)})$ and $\mathrm{ob}(V_\cdot)$ in the derived category $D(U)$. We put $V_0^{(2)}:=V_0\oplus V_0^{(1)}$ and $V_{-1}^{(2)}:=\ker(V_0^{(2)}\longrightarrow E)$. We have the morphism $V_\cdot\longrightarrow V_\cdot^{(2)}$ and $V_\cdot\longrightarrow V_\cdot^{(1)}$. Therefore, we have only to compare the morphisms $\mathrm{ob}(V_\cdot^{(2)})$ and $\mathrm{ob}(V_\cdot)$. Note that V_i is a subbundle of $V_i^{(2)}$, i.e., we have the filtration $V. \subset V.^{(2)}$. Let $\mathcal{H}om'\big(V_i^{(2)}, V_j^{(2)}\big)$ $(i,j = 0,-1)$ denote the sheaf of $\mathcal{O}_X$-morphisms $V_i^{(2)} \longrightarrow V_j^{(2)}$ preserving the filtrations. They naturally form the complex of sheaves $\mathcal{H}om'(V.^{(2)}, V.^{(2)})$. We put $\mathfrak{g}(V., V.^{(2)}) := \mathcal{H}om'(V.^{(2)}, V.^{(2)})^{\vee}[-1]$. We have the naturally defined quasi isomorphisms $\mathcal{H}om'(V.^{(2)}, V.^{(2)}) \longrightarrow \mathcal{H}om(V., V.)$ and $\mathcal{H}om'(V.^{(2)}, V.^{(2)}) \longrightarrow \mathcal{H}om(V.^{(2)}, V.^{(2)})$. They induce the quasi isomorphisms $\gamma_1 : \mathfrak{g}(V.) \longrightarrow \mathfrak{g}(V., V.^{(2)})$ and $\gamma_2 : \mathfrak{g}(V^{(2)}) \longrightarrow \mathfrak{g}(V., V.^{(2)})$.

Let $W_i^{(2)}$ be vector spaces such that $\operatorname{rank} W_i^{(2)} = \operatorname{rank} V_i^{(2)}$ $(i = 1,2)$. We fix inclusions $W_i \subset W_i^{(2)}$. We denote $W_i^{(2)} \otimes \mathcal{O}_X$ by $W_{i\,X}^{(2)}$. We have the filtration $W_{i\,X} \subset W_{i\,X}^{(2)}$. Let $\mathcal{H}om'(W_{i\,X}^{(2)}, W_{j\,X}^{(2)})$ be the sheaf of $\mathcal{O}_X$-morphisms $W_{i\,X}^{(2)} \longrightarrow W_{j\,X}^{(2)}$ preserving the filtration. The corresponding vector bundle is denoted by $N'(W_{i\,X}^{(2)}, W_{j\,X}^{(2)})$. We have the natural morphisms $N'\big(W_{i\,X}^{(2)}, W_{j\,X}^{(2)}\big) \longrightarrow N(W_{i\,X}, W_{j\,X})$ and $N'\big(W_{i\,X}^{(2)}, W_{j\,X}^{(2)}\big) \longrightarrow N\big(W_{i\,X}^{(2)}, W_{j\,X}^{(2)}\big)$. Let $\mathrm{GL}'(W_i^{(2)})$ be the subgroup of $\mathrm{GL}(W_i^{(2)})$, which consists of the elements of $\mathrm{GL}(W_i^{(2)})$ preserving the filtration. Then we have the natural right $\mathrm{GL}'(W_1^{(2)}) \times \mathrm{GL}'(W_2^{(2)})$-action on $N'(W_j^{(2)}, W_l^{(2)})$. The quotient stack is denoted by $Y'(W.^{(2)})$. We have the homomorphisms $\mathrm{GL}'(W_i^{(2)}) \longrightarrow \mathrm{GL}(W_i^{(2)})$ and $\mathrm{GL}'(W_i^{(2)}) \longrightarrow \mathrm{GL}(W_i)$. Thus we have the morphisms $Y'(W^{(2)}) \longrightarrow Y(W.)$ and $Y'(W^{(2)}) \longrightarrow Y(W.^{(2)})$.

From the tuple $\big(V., V.^{(2)}\big)$, we obtain the morphism $\Phi\big(V., V.^{(2)}\big) : U \times X \longrightarrow Y'(W.^{(2)})$. Then, it can be shown by the argument in the subsubsection 2.3.2, that the complex $\Phi\big(V., V.^{(2)}\big)$ is represented by $\mathfrak{g}(V., V.^{(2)})$. We also have the following commutative diagram:

$$\begin{array}{ccccc}
\Phi(V.)^* L_{Y(W.)/X} & \longrightarrow & \Phi(V., V.^{(2)})^* L_{Y'(W.^{(2)})/X} & \longleftarrow & \Phi(V.^{(2)})^* L_{Y(W.^{(2)})/X} \\
\uparrow & & \uparrow & & \uparrow \\
\mathfrak{g}(V.)_{\leq 1} & \xrightarrow{\gamma_1} & \mathfrak{g}(V., V.^{(2)})_{\leq 1} & \xleftarrow{\gamma_2} & \mathfrak{g}(V.^{(2)})_{\leq 1}
\end{array} \tag{87}$$

Then, we obtain the following diagram:

$$\begin{array}{ccccc}
L_{U\times X/X} & \xrightarrow{=} & L_{U\times X/X} & \xleftarrow{=} & L_{U\times X/X} \\
\uparrow & & \uparrow & & \uparrow \\
\Phi(V.)^* L_{Y(W.)/X} & \longrightarrow & \Phi(V., V.^{(2)})^* L_{Y'(W.^{(2)})/X} & \longleftarrow & \Phi(V.^{(2)})^* L_{Y(W.^{(2)})/X} \\
\uparrow & & \uparrow & & \uparrow \\
\mathfrak{g}(V.) & \xrightarrow{\simeq} & \mathfrak{g}(V., V.^{(2)}) & \xleftarrow{\simeq} & \mathfrak{g}(V.^{(2)}).
\end{array}$$

We put $\mathrm{Ob}\big(V.^{(2)}, V.\big) := Rp_{X\,*}\big(\mathfrak{g}(V.^{(2)}, V.) \otimes \omega_X\big)$. Then we obtain the following diagram in $D(U)$:

$$\begin{array}{ccccc}
L_U & \xrightarrow{=} & L_U & \xleftarrow{=} & L_U \\
\uparrow & & \uparrow & & \uparrow \\
\mathrm{Ob}(V.) & \xrightarrow{\simeq} & \mathrm{Ob}\big(V.^{(2)}, V.\big) & \xleftarrow{\simeq} & \mathrm{Ob}(V.^{(2)})
\end{array}$$

Thus we are done. ∎

If E is a vector bundle of rank R, we may take $V_0 = E$ and $V_{-1} = 0$. In this case, the construction can be reworded as follows: We have the classifying map $\Phi(E) : U \times X \longrightarrow X_{\mathrm{GL}(R)}$. It induces the morphism $\Phi(E)^* L_{X_{\mathrm{GL}(R)}/X} \longrightarrow L_{U\times X/X}$. We have the expression $\Phi(E)^* L_{X_{\mathrm{GL}(R)}/X} \simeq \mathcal{H}om(E,E)$. We put $\mathrm{Ob}(E) := Rp_{X\,*}\big(\mathcal{H}om(E,E) \otimes \omega_X\big)$, and we obtain the morphism $\mathrm{ob}(E) : \mathrm{Ob}(E) \longrightarrow L_{U/k}$.

5.1.2 The trace-free part and the diagonal part

We have the homomorphism $\mathrm{GL}(W.) \longrightarrow G_m$ given by $(f_{-1}, f_0) \longmapsto \det(f_{-1})^{-1} \cdot \det(f_0)$. It induces the morphism $\mathfrak{w} : Y(W.) \longrightarrow X_{G_m}$. The composite $\mathfrak{w} \circ \Phi(V.) : U \times X \longrightarrow X_{G_m}$ is same as the classifying map of the determinant bundle $\det(E) \simeq \det(V_0) \otimes \det(V_{-1})^{-1}$, which is denoted by $\Phi(\det(E))$. Then, we obtain the following commutative diagram:

$$\begin{tikzcd}
\Phi\big(\det(E)\big)^* L_{X_{G_m}/X} \arrow[r] & \Phi(V_\cdot)^* L_{Y(W_\cdot)/X} \arrow[r] & L_{U\times X/X} \\
\mathcal{O}[-1] \arrow[u, "\simeq"] \arrow[r, "i"] & \mathfrak{g}(V_\cdot) \arrow[u] &
\end{tikzcd}$$

Here, the map $i : \mathcal{O}[-1] \longrightarrow \mathfrak{g}(V_\cdot)$ is given as follows:

$$\mathcal{O} \longrightarrow \mathcal{H}om(V_0, V_0) \oplus \mathcal{H}om(V_{-1}, V_{-1}), \quad f \longmapsto (f \cdot \mathrm{id}_{V_0}, -f \cdot \mathrm{id}_{V_{-1}})$$

On the other hand, we have the trace map $\mathrm{tr} : \mathfrak{g}(V_\cdot) \longrightarrow \mathcal{O}[-1]$:

$$\mathcal{H}om(V_0, V_0) \oplus \mathcal{H}om(V_{-1}, V_{-1}) \longrightarrow \mathcal{O}, \quad (f_0, f_{-1}) \longmapsto \mathrm{tr}(f_0) + \mathrm{tr}(f_{-1})$$

We put $\mathrm{Ker}(\mathrm{tr}) := \mathfrak{g}^\circ(V_\cdot)$, and $\mathfrak{g}^d(V_\cdot) := \mathrm{Im}(i)$. We have the decomposition $\mathfrak{g}(V_\cdot) = \mathfrak{g}^\circ(V_\cdot) \oplus \mathfrak{g}^d(V_\cdot)$, which induces the decomposition $\mathrm{Ob}(V_\cdot) = \mathrm{Ob}^\circ(V_\cdot) \oplus \mathrm{Ob}^d(V_\cdot)$. The complexes $\mathfrak{g}^\circ(V_\cdot)$ and $\mathrm{Ob}^\circ(V_\cdot)$ (resp. $\mathfrak{g}^d(V_\cdot)$ and $\mathrm{Ob}^d(V_\cdot)$) are called the trace-free part (resp. the diagonal part).

The determinant bundle induces the morphism $\widetilde{\det}_E : U \longrightarrow \mathcal{M}(1)$. We also have the following commutative diagram:

$$\begin{tikzcd}
U \times X \arrow[r, "\Phi(V_\cdot)"] \arrow[d, "\widetilde{\det}_{E,X}"'] & Y(W_\cdot) \arrow[d, "\mathfrak{w}"'] \\
\mathcal{M}(1) \times X \arrow[r] & X_{G_m}
\end{tikzcd}$$

The composite $\mathfrak{w} \circ \Phi(V_\cdot)$ is same as the classifying map $\Phi(\det(E))$ of $\det(E)$. Thus we obtain the following commutative diagram:

$$\begin{tikzcd}
L_{U\times X/X} & \Phi(V_\cdot)^* L_{Y(W_\cdot)/X} \arrow[l] & \mathfrak{g}(V_\cdot) \arrow[l] \\
\widetilde{\det}^*_{E,X} L_{\mathcal{M}(1)\times X/X} \arrow[u] & \Phi(\det(E))^* L_{X_{G_m}/X} \arrow[l] \arrow[u] \arrow[r, equal] & \mathfrak{g}^d(V_\cdot) \arrow[u]
\end{tikzcd}$$

Therefore, we obtain the following commutative diagram:

$$\begin{tikzcd}
L_U & \mathrm{Ob}(V_\cdot) \arrow[l] \\
\widetilde{\det}^*_E L_{\mathcal{M}(1)} \arrow[u] & \mathrm{Ob}^d(V_\cdot) \arrow[l] \arrow[u]
\end{tikzcd} \tag{88}$$

5.1.3 Preparation for Master space

We also put $A(W_\cdot) := X_{\mathrm{GL}(W_0)}$. We have the natural morphism $\Gamma : Y(W_\cdot) \longrightarrow A(W_\cdot)$. We put $\Psi(V_\cdot) := \Gamma \circ \Phi(V_\cdot)$. Then $\Psi(V_\cdot)^* L_{A(W_\cdot)/X}$ is represented by $\mathcal{H}om(V_0, V_0)^\vee[-1]$. For these representatives, the natural morphism $\Psi(V_\cdot)^* L_{A(W_\cdot)/X} \longrightarrow \Phi(V_\cdot)^* L_{Y(W_\cdot)/X}$ is expressed by the obvious inclusion of $\mathcal{H}om(V_0, V_0)$ to $\mathcal{H}om(V_0, V_0) \oplus \mathcal{H}om(V_{-1}, V_{-1})$. We put $\mathfrak{h}(V_\cdot) := \mathcal{H}om(V_0, V_\cdot)^\vee[-1]$ and $\mathrm{Ob}^G(V_\cdot) := Rp_{X\,*}\big(\mathfrak{h}(V_\cdot) \otimes \omega_X\big)$. Then we obtain the following diagram:

$$\begin{tikzcd}
\mathfrak{h}(V_\cdot) \arrow[r] \arrow[d] & \mathfrak{g}(V_\cdot) \arrow[d] & \\
\Psi(V_\cdot)^* L_{A(W_\cdot)/X} \arrow[r] & \Phi(V_\cdot)^* L_{Y(W_\cdot)/X} \arrow[r] & L_{U\times X/X}
\end{tikzcd}$$

Therefore, we obtain the morphisms $\mathrm{Ob}^G(V_\cdot) \longrightarrow \mathrm{Ob}(V_\cdot) \longrightarrow L_{U/k}$.

Now, we assume the following condition (C) for E and $V_\cdot$:

(C1) For any point $u \in U$, the higher cohomology groups $H^i\big(X, E_{|\{u\}\times X}\big)$ vanish.

(C2) We put $V' := p_{X\,*}(E)$. Then we have $V_0 = p_X^* V'$.

We put $B(W.) := \mathrm{Spec}(k)_{\mathrm{GL}(W_0)}$. We remark $A(W.) = X \times B(W.)$. Because of $V_0 = p_X^* V'$, we have $\operatorname{rank} W_0 = \operatorname{rank} V'$, and hence we have the classifying map $\Psi(V') : U \longrightarrow B(W.)$ of V'. We have the following commutative diagram:

$$\begin{array}{ccccc} U \times X & \xrightarrow{\Phi(V.)} & Y(W.) & \longrightarrow & X \\ {\scriptstyle \Psi(V')_X}\big\downarrow & & {\scriptstyle \Gamma}\big\downarrow & & \big\downarrow \\ B(W.) \times X & \xrightarrow{=} & A(W.) & \longrightarrow & X \end{array} \tag{89}$$

Thus we obtain the following diagram on $U \times X$:

$$\begin{array}{ccccc} L_{U\times X/X} & \longleftarrow & \Phi(V.)^* L_{Y(W.)/X} & \longleftarrow & \mathfrak{g}(V.) \\ \big\uparrow & & \big\uparrow & & \big\uparrow \\ \Psi(V')_X^* L_{B(W.)\times X/X} & \xleftarrow{\simeq} & \Psi(V.)^* L_{A(W.)/X} & \longleftarrow & \mathfrak{h}(V.) \end{array} \tag{90}$$

Hence, we obtain the following diagram on U:

$$\begin{array}{ccc} L_U & \longleftarrow & \mathrm{Ob}(V.) \\ \big\uparrow & & \big\uparrow \\ \Psi(V')^* L_{B(W.)} & \xleftarrow{\tau_1} & \mathrm{Ob}^G(V.). \end{array} \tag{91}$$

It is easy to see that both of $\mathrm{Ob}^G(V.)$ and $\Psi(V')^* L_{B(W.)}$ are isomorphic to $\mathcal{H}om(V', V')[-1]$ under the condition (C).

Lemma 5.2 *The morphism τ_1 in (91) is isomorphic.*

Proof The composite of the following naturally defined morphisms is isomorphic:

$$Rp_{X\,*}\Big(\big(\mathcal{H}om(V_0, V_0) \to \mathcal{H}om(V_{-1}, V_0)\big) \otimes \omega_X\Big) \longrightarrow Rp_{X\,*}\big(\mathcal{H}om(V_0, V_0) \otimes \omega_X\big) \longrightarrow \mathcal{H}om(V', V')$$

Then, the claim of the lemma immediately follows. ∎

5.1.4 Basic complex on the moduli stack $\mathcal{M}(m, y)$

Let $y \in H^*(X)$ be a Chern character of a coherent sheaf on X. Let H denote the polynomial associated to y. We take an $H(m)$-dimensional vector space V_m. We have the scheme $Q^\circ(m, y)$. (See the subsubsection 3.6.2.) We consider the universal quotient $q^u : p_{Q^\circ(m,y)}^* V_{m,X} \longrightarrow E^u(m)$ defined over $Q^\circ(m, y) \times X$. We put $V_0^u := p_{Q^\circ(m,y)}^* V_{m,X}$ and $V_{-1}^u := \ker\big(V_0^u \longrightarrow E^u(m)\big)$. The inclusion $V_{-1}^u \longrightarrow V_0^u$ is denoted by f^u. We put $V' := V_m \otimes \mathcal{O}_{Q^\circ(m,y)} = p_{X\,*} V_0^u$. We have the morphism $\pi : Q^\circ(m, y) \longrightarrow \mathcal{M}(m, y) := Q^\circ(m, y)/GL(V_m)$. The latter is an open subset of the moduli stack of torsion-free sheaves of type y, determined by the condition O_m. The descent of E^u, $V.$ and V' with respect to the $\mathrm{GL}(V_m)$-action are denoted by $\mathcal{E}^u$, $\mathcal{V}.$ and $\mathcal{V}'$. The sheaf $\mathcal{E}^u$ is the universal sheaf.

We put $W_0 := V_m$, and we take a vector space W_{-1} such that $\dim W_{-1} = H(m) - \mathrm{rank}(y)$. Applying the result in the subsubsection 5.1.1, we obtain the complex $\mathrm{Ob}(\mathcal{V}.)$ and the morphism $\mathrm{ob}(\mathcal{V}.) : \mathrm{Ob}(\mathcal{V}.) \longrightarrow L_{\mathcal{M}(m,y)/k}$. We obviously have $\pi^* \mathrm{Ob}(\mathcal{V}.) = \mathrm{Ob}(V.^u)$.

Lemma 5.3 *We have the following morphism of the distinguished triangles on $Q^\circ(m, y)$:*

$$\begin{array}{ccccccc} \pi^* \mathrm{Ob}(\mathcal{V}.) & \longrightarrow & \mathrm{Ob}(V_{-1}^u, f^u) & \longrightarrow & \mathcal{H}om(V', V') & \longrightarrow & \pi^* \mathrm{Ob}(\mathcal{V}.)[1] \\ \big\downarrow & & \big\downarrow & & \simeq\big\downarrow & & \big\downarrow \\ \pi^* L_{\mathcal{M}(m,y)} & \longrightarrow & L_{Q^\circ(m,y)} & \longrightarrow & L_{Q^\circ(m,y)/\mathcal{M}(m,y)} & \longrightarrow & \pi^* L_{\mathcal{M}(m,y)}[1]. \end{array}$$

Proof We use the notation in the subsubsection 2.4.3. We have the following commutative diagram:

$$\begin{tikzcd}
Q^{\circ}(m,y) \arrow[r, "\Phi(V^u_{-1},f^u)"] \arrow[d] & Y(W.)_{quo} \arrow[r] \arrow[d, "\pi_0"'] & X \arrow[d] \\
\mathcal{M}(m,y) \arrow[r] & Y(W.) \arrow[r] & X
\end{tikzcd}$$

The composite $\pi_0 \circ \Phi(V^u_{-1}, f^u)$ is same as $\Phi(V^u_{\cdot})$. Thus we obtain the following morphism of distinguished triangles on $Q^{\circ}(m,y) \times X$:

$$\begin{tikzcd}
\Phi(V^u_{\cdot})^* L_{Y(W.)/X} \arrow[r, "a"] \arrow[d] & \Phi(V^u_{-1}, f^u)^* L_{Y_{\mathrm{quo}}(W.)/X} \arrow[r] \arrow[d] & \Phi(V^u_{-1}, f^u)^* L_{Y_{\mathrm{quo}}(W.)/Y(W.)} \arrow[d, "\simeq"'] \\
\pi^* L_{\mathcal{M}(m,y)\times X/X} \arrow[r] & L_{Q^{\circ}(m,y)\times X/X} \arrow[r] & L_{Q^{\circ}(m,y)\times X/\mathcal{M}(m,y)\times X}.
\end{tikzcd}$$

Recall that $\Phi(V^u_{\cdot})^* L_{Y(W.)/X}$ and $\Phi(V^u_{-1}, f^u)^* L_{Y_{\mathrm{quo}}(W.)/X}$ are expressed by $\mathfrak{g}(V^u_{\cdot})_{\leq 1}$ and $\mathfrak{g}(V^u_{-1}, f^u)_{\leq 1}$, respectively. It is easy to see that the morphism a is expressed by the naturally defined morphism $\mathfrak{g}(V^u_{\cdot})_{\leq 1} \longrightarrow \mathfrak{g}(V^u_{-1}, f^u)_{\leq 1}$ We put $\mathfrak{k}(V^u_{\cdot}) := \mathrm{Cone}\big(\mathfrak{g}(V^u_{\cdot}) \longrightarrow \mathfrak{g}(V^u_{-1}, f^u)\big)$. Then, obtain the following morphism of the distinguished triangles on $Q^{\circ}(m,y) \times X$:

$$\begin{tikzcd}
\mathfrak{g}(V^u_{\cdot}) \arrow[r] \arrow[d] & \mathfrak{g}(V^u_{-1}, f^u) \arrow[r] \arrow[d] & \mathfrak{k}(V^u_{\cdot}) \arrow[r] \arrow[d] & \mathfrak{g}(V^u_{\cdot})[1] \arrow[d] \\
\pi^* L_{\mathcal{M}\times X/X} \arrow[r] & L_{Q^{\circ}(m,y)\times X/X} \arrow[r] & L_{Q^{\circ}(m,y)\times X/\mathcal{M}(m,y)\times X} \arrow[r] & \pi^* L_{\mathcal{M}(m,y)\times X/X}[1].
\end{tikzcd}$$

Hence we obtain the following morphism of distinguished triangles on $Q^{\circ}(m,y)$:

$$\begin{tikzcd}
\mathrm{Ob}(V.) \arrow[r] \arrow[d] & \mathrm{Ob}(V_{-1}, f) \arrow[r] \arrow[d] & Rp_{X*}\big(\mathfrak{k}(V.) \otimes \omega_X\big) \arrow[r] \arrow[d, "\varphi"'] & \mathrm{Ob}(V.)[1] \arrow[d] \\
\pi^* L_{\mathcal{M}} \arrow[r] & L_{Q^{\circ}(m,y)} \arrow[r] & L_{Q^{\circ}(m,y)/\mathcal{M}(m,y)} \arrow[r] & \pi^* L_{\mathcal{M}(m,y)}[1].
\end{tikzcd}$$

Recall $\mathfrak{g}(V^u_{\cdot}) = \mathcal{H}om\big(V^u_{\cdot}, V^u_{\cdot}\big)^{\vee}[-1]$ and $\mathfrak{g}(V^u_{-1}, f^u) = \mathcal{H}om\big(V^u_{-1}[1], V^u_{\cdot}\big)^{\vee}[-1]$. It is easy to observe that $\mathfrak{k}(V^u_{\cdot})$ is expressed by the complex $\mathcal{H}om(V^u_0, V^u_0) \to \mathcal{H}om(V^u_{-1}, V^u_0)$, where the first term stands at the degree 0. Under the identification, the morphism φ is given by the identity of $\mathcal{H}om(V^u_0, V^u_0)$. Then, it is easy to check that $Rp_{X*}(\mathfrak{k}(V.))$ and $L_{Q^{\circ}(m,y)/\mathcal{M}(m,y)}$ are quasi isomorphic to their 0-th cohomology sheaves $\mathcal{H}om(V', V')$, and that the morphism φ in the diagram is isomorphic, as in Lemma 5.2. ∎

Corollary 5.4 *The morphism* $\mathrm{ob}(\mathcal{V}.)$ *gives an obstruction theory for* $\mathcal{M}(m,y)$. ∎

We also use the notation $\mathrm{Ob}(m,y)$ and $\mathrm{ob}(m,y)$ to denote $\mathrm{Ob}(\mathcal{V}.)$ and $\mathrm{ob}(\mathcal{V}.)$.

5.1.5 The case of the moduli of line bundles

Let $\mathcal{P}oin$ denote the Poincaré bundle on $\mathrm{Pic} \times X$. Then we have the classifying map $\Phi(\mathcal{P}oin) : \mathrm{Pic} \times X \longrightarrow X_{G_m}$. We put $\mathfrak{g}(\mathcal{P}oin) := \Phi(\mathcal{P}oin)^* L_{X_{G_m}/X}$ and $\mathrm{Ob}(\mathcal{P}oin) := Rp_{X\,*}\big(\mathfrak{g}(\mathcal{P}oin) \otimes \omega_X\big)$. Then we have the morphism $\mathrm{ob}(\mathcal{P}oin) : \mathrm{Ob}(\mathcal{P}oin) \longrightarrow L_{\mathrm{Pic}}$ on Pic.

Since $\mathfrak{g}(\mathcal{P}oin) \simeq \mathcal{O}[-1]$, we have an isomorphism in the derived category $D(\mathrm{Pic})$:

$$\mathrm{Ob}(\mathcal{P}oin) \simeq \Big(H^0(X,\mathcal{O})^{\vee} \otimes \mathcal{O}_{\mathrm{Pic}}[-1]\Big) \oplus \Big(H^1(X,\mathcal{O})^{\vee} \otimes \mathcal{O}_{\mathrm{Pic}}[0]\Big) \oplus \Big(H^2(X,\mathcal{O})^{\vee} \otimes \mathcal{O}_{\mathrm{Pic}}[1]\Big)$$

We have to be careful that the decomposition is not canonical, but the morphism $\mathrm{Ob}(\mathcal{P}oin) \longrightarrow L_{\mathrm{Pic}} \longrightarrow H^1(X,\mathcal{O})^{\vee} \otimes \mathcal{O}_{\mathrm{Pic}}[0] \simeq \mathcal{H}^1\big(\mathrm{Ob}(\mathcal{P}oin)\big)$ induces the canonical decomposition:

$$\tau_{\geq 0}\,\mathrm{Ob}(\mathcal{P}oin) \simeq \Big(H^1(X,\mathcal{O})^{\vee} \otimes \mathcal{O}_{\mathrm{Pic}}[0]\Big) \oplus \Big(H^0(X,\mathcal{O})^{\vee} \otimes \mathcal{O}_{\mathrm{Pic}}[-1]\Big)$$

We also remark that the composite $\tau_{\leq -1}\,\mathrm{Ob}(\mathcal{P}oin) \longrightarrow L_{\mathrm{Pic}}$ is trivial.

5.2 Relative Obstruction Theory for Orientations

5.2.1 Construction of the complex

We use the notation in the subsubsections 5.1.1–5.1.2. Let U_1 be an Artin stack. Let $F_1 : U_1 \longrightarrow U$ be a morphism. Assume that we have an orientation ρ of the sheaf $F_{1X}^*(E)$ over $U_1 \times X$. We have the morphism $\det_E : U_1 \longrightarrow \mathrm{Pic}$ induced by $\det(E)$. We denote $\det_E \times \mathrm{id}_X : U_1 \times X \longrightarrow \mathrm{Pic} \times X$ by $\det_{E,X}$. We have $\det_{E,X}^* \mathcal{P}oin \simeq \det(E)$. Then we have the following commutative diagram:

$$\begin{tikzcd} U_1 \times X \arrow[r, "\Phi(V_\cdot)"] \arrow[d, "\det_{E,X}"'] & Y(W_\cdot) \arrow[d, "\mathfrak{w}"'] \\ \mathrm{Pic} \times X \arrow[r, "\Phi(\mathcal{P}oin)"] & X_{G_m} \end{tikzcd}$$

It induce the following commutative diagram:

$$\begin{tikzcd} L_{U_1 \times X/X} & \Phi(V_\cdot)^* L_{Y(W_\cdot)/X} \arrow[l] & \mathfrak{g}(V_\cdot) \arrow[l] \\ \det_{E,X}^* L_{\mathrm{Pic} \times X/X} \arrow[u] & \Phi(\det(E))^* L_{X_{G_m}/X} \arrow[l] \arrow[u] & \Phi(\det(E))^* L_{X_{G_m}/X} \arrow[l] \arrow[u] \end{tikzcd}$$

Since we have $\Phi(\det(E))^* L_{X_{G_m}/X} = \mathfrak{g}^d(V_\cdot)$, we obtain the following:

$$\begin{tikzcd} L_{U_1} & \mathrm{Ob}(V_\cdot) \arrow[l] \\ \det_E^* L_{\mathrm{Pic}} \arrow[u] & \mathrm{Ob}^d(V_\cdot) \arrow[l] \arrow[u] \end{tikzcd}$$

We put $\mathrm{Ob}_{\mathrm{rel}}(V_\cdot, \rho) := \mathrm{Cone}\big(\mathrm{Ob}^d(V_\cdot) \longrightarrow \det_E^* L_{\mathrm{Pic}}\big)$. We have the morphisms $\mathrm{Ob}_{\mathrm{rel}}(V_\cdot, \rho)[-1] \longrightarrow \mathrm{Ob}^d(V_\cdot) \longrightarrow \mathrm{Ob}(V_\cdot)$.

Let $\mathcal{M}(1)$ denote the moduli of line bundles, i.e., $\mathcal{M}(1) = \mathrm{Pic}/G_m$. Let π denote the projection $\mathrm{Pic} \longrightarrow \mathcal{M}(1)$. We have the following commutative diagrams induced by $\det(E)$:

$$\begin{tikzcd} U_1 \arrow[r, "\det_E"] \arrow[d, "F_1"'] & \mathrm{Pic} \arrow[d, "\pi"'] \\ U \arrow[r, "\widetilde{\det}_E"] & \mathcal{M}(1) \end{tikzcd} \qquad \begin{tikzcd} U_1 \times X \arrow[r, "\det_{E,X}"] \arrow[d, "F_{1X}"'] & \mathrm{Pic} \times X \arrow[d, "\pi_X"'] & \\ U \times X \arrow[r] & \mathcal{M}(1) \times X \arrow[r] & X_{G_m} \end{tikzcd}$$

Hence, we have the following diagram on $U_1 \times X$:

$$\begin{tikzcd} L_{U_1 \times X/X} & \det_{E,X}^* L_{\mathrm{Pic} \times X/X} \arrow[l] & \\ F_{1X}^* L_{U \times X/X} \arrow[u] & \det_{E,X}^* \pi_X^* L_{\mathcal{M}(1) \times X/X} \arrow[l] \arrow[u] & \Phi(\det(E))^* L_{X_{G_m}/X} \arrow[l] \end{tikzcd} \tag{92}$$

Therefore, we obtain the following diagram on U_1:

$$\begin{tikzcd} L_{U_1} & \det_E^* L_{\mathrm{Pic}} \arrow[l] & \\ F_1^* L_U \arrow[u] & \det_E^* \pi^* L_{\mathcal{M}(1)} \arrow[l] \arrow[u] & \mathrm{Ob}^d(V_\cdot) \arrow[l] \end{tikzcd} \tag{93}$$

Thus, we obtain the following commutative diagram:

$$\begin{tikzcd} \mathrm{Ob}_{\mathrm{rel}}(V_\cdot, \rho)[-1] \arrow[r] \arrow[d] & \mathrm{Ob}(V_\cdot) \arrow[d] \\ L_{U_1/U}[-1] \arrow[r] & F_1^* L_U \end{tikzcd} \tag{94}$$

The following lemma can be shown by an argument similar to Lemma 5.1.

Lemma 5.5 *The diagram* (94) *depends only on* (E,ρ) *in the derived category* $D(U_1)$. ∎

The following lemma is easy to see by construction and the argument in the subsubsection 5.1.5.

Lemma 5.6 $\mathrm{Ob}_{\mathrm{rel}}(V.,\rho)$ *is isomorphic to* $\Big(H^0(X,\mathcal{O})^\vee\otimes\mathcal{O}_{U_1}[0]\Big)\oplus\Big(H^2(X,\mathcal{O})^\vee\otimes\mathcal{O}_{U_1}[2]\Big)$. *The composite of the morphisms* $\big(\tau_{\leq -2}\,\mathrm{Ob}_{\mathrm{rel}}(V.,\rho)\big)[-1]\longrightarrow\mathrm{Ob}(V.)\longrightarrow L_{U_1}$ *is trivial.* ∎

5.2.2 Relative obstruction property

For any U-scheme $g:T\longrightarrow U$, let $F_1(T)$ denote the set of orientations of g^*E. Then, we obtain the functor F_1 of the category of U-schemes to the category of sets. The functor F_1 is representable by the scheme $\mathcal{O}r(E)^*$. Let π denote the projection $\mathcal{O}r(E)^*\longrightarrow U$. On $\mathcal{O}r(E)^*\times X$, we have the universal orientation ρ^u of π^*E. From the resolution $V.$ and the orientation ρ^u, we obtain the morphism:

$$\mathrm{ob}_{\mathrm{rel}}(V.,\rho^u):\mathrm{Ob}_{\mathrm{rel}}(V.,\rho^u)\longrightarrow L_{\mathcal{O}r(E)^*/U}$$

Lemma 5.7 *The morphism* $\mathrm{ob}_{\mathrm{rel}}(V.,\rho^u)$ *gives the relative obstruction theory for* $\mathcal{O}r(E)^*$ *over* U.

Proof We have only to show that $\mathcal{H}^0(\mathrm{ob}_{\mathrm{rel}}(V.,\rho^u))$ is isomorphic. From the diagram (93), we obtain the following morphisms:

$$\mathrm{Ob}_{\mathrm{rel}}(V.,\rho^u)\xrightarrow{\varphi_1}\det_E^*\mathrm{Cone}\big(\pi^*L_{\mathcal{M}(1)}\longrightarrow L_{\mathrm{Pic}}\big)\xrightarrow{\varphi_2}L_{\mathcal{O}r(E)^*/U}$$

Since we have the isomorphism $\mathcal{O}r(E)^*\simeq U\times_{\mathcal{M}(1)}\mathrm{Pic}$, the morphism φ_2 is isomorphic. We have the following commutative diagram:

$$\begin{CD}
\mathcal{H}^0(\mathrm{Ob}^d) @>>> \det_E^*\Omega_{\mathrm{Pic}} @>>> \mathcal{H}^0\big(\mathrm{Ob}_{\mathrm{rel}}(V.,\rho^u)\big) @>>> \mathcal{H}^1(\mathrm{Ob}^d)\\
@V{a_1}VV @V{\simeq}VV @V{a_2}VV @V{a_3}VV\\
\mathcal{H}^0(\det_E^*\pi^*L_{\mathcal{M}(1)}) @>>> \det_E^*\Omega_{\mathrm{Pic}} @>>> \det_E^*L_{\mathrm{Pic}/\mathcal{M}(1)} @>>> \det_E^*\pi^*\mathcal{H}^1(L_{\mathcal{M}(1)})
\end{CD}$$

We also have $\mathcal{H}^1(\det_E^*L_{\mathrm{Pic}})=0$. The morphisms a_i $(i=1,3)$ are isomorphic, by applying Corollary 5.4 to $\mathcal{M}(1)$. Thus a_2 is isomorphic. Then, we obtain the claim of the lemma. ∎

5.3 Relative Obstruction Theory for L-Section

5.3.1 Construction of the complex

We use the notation in the subsubsection 5.1.1. Let L be a line bundle on X. Let $P.:=\big(P_{-1}\xrightarrow{\partial_L}P_0\big)\simeq L$ be a locally free resolution, where P_0 stands in the degree 0. We have the natural right $\mathrm{GL}(W.)$-action on $N\big(P_i,W_j\big)$ given by $(g_{-1},g_0)\cdot f=g_j^{-1}\circ f$. It induces the $\mathrm{GL}(W.)$-actions on the varieties $N\big(P_{-1},W_{0\,X}\big)$, X and $N(W_{-1\,X},W_{0\,X})\times_X N(P_0,W_{0\,X})\times_X N(P_{-1},W_{-1\,X})$. The quotient stacks are denoted by $Y_0(W.,P.)$, $Y_1(W.,P.)$, and $Y_2(W.,P.)$ respectively.

We have the equivariant map $h:N(W_{-1\,X},W_{0\,X})\times_X N(P_{-1},W_{-1\,X})\times_X N(P_0,W_{0\,X})\longrightarrow N(P_{-1},W_{0\,X})$ given by $h(f,a_{-1},a_0)=f\circ a_{-1}-a_0\circ\partial_L$. Since $\partial_{L|x}$ is injective for any point $x\in X$, the map h is smooth. It induces the smooth morphism $Y_2(W.,P.)\longrightarrow Y_0(W.,P.)$. We also have the morphism $Y_1(W.,P.)\longrightarrow Y_0(W.,P.)$ induced by the 0-section $X\longrightarrow N(P_{-1},W_{0,X})$. We denote the fiber product $Y_1\times_{Y_0}Y_2$ by $Y(W.,P.)$.

Let U_2 be an algebraic stack with a morphism $F_2:U_2\longrightarrow U$ and an L-section $\phi:p_{U_2}^*(L)\longrightarrow F_{2\,X}^*(E)$. We assume that we have a morphism of complexes $\widetilde{\phi}.=(\widetilde{\phi}_{-1},\widetilde{\phi}_0):p_{U_2}^*(P.)\longrightarrow F_{2\,X}^*(V.)$ which gives ϕ in the cohomology level. Such a $\widetilde{\phi}$ is called a lift of ϕ. We put as follows:

$$\mathfrak{g}_{\mathrm{rel}}(V.,\widetilde{\phi})=\mathcal{H}om\big(p_{U_2}^*P.,\,F_{2,X}^*V.\big)^\vee$$

We have the naturally induced morphism $\gamma(\widetilde{\phi}.) : \mathfrak{g}_{\mathrm{rel}}(V., \widetilde{\phi})[-1] \longrightarrow \mathfrak{g}(V.)$ and $\gamma(\widetilde{\phi}.)_{\leq 1} : \mathfrak{g}_{\mathrm{rel}}(V., \widetilde{\phi})[-1]_{\leq 1} \longrightarrow \mathfrak{g}(V.)_{\leq 1}$.

We have the classifying map $\Phi(V., \widetilde{\phi}) : U_2 \times X \longrightarrow Y(W., P.)$ which induces the maps $\Phi_i(V., \widetilde{\phi}) : U_2 \times X \longrightarrow Y_i(W., P.)$.

Lemma 5.8 *$\Phi(V., \widetilde{\phi})^* L_{Y(W.,P.)/X}$ is expressed by the complex* $\mathrm{Cone}\big(\gamma(\widetilde{\phi}.)_{\leq 1}\big)$.

Proof We have the induced morphisms $\kappa_i : \Phi_0(V., \widetilde{\phi})^* L_{Y_0(W.,P.)/X} \longrightarrow \Phi_i(V., \widetilde{\phi})^* L_{Y_i(W.,P.)/X}$ $(i = 1, 2)$. Since $Y_2(W., P.) \longrightarrow Y_0(W., P.)$ is smooth, $\Phi(V., \widetilde{\phi})^* L_{Y(W.,P.)/X}$ is isomorphic to the cone of the induced morphism $\Phi_0(V., \widetilde{\phi})^* L_{Y_0(W.,P.)/X} \longrightarrow \bigoplus_{i=1,2} \Phi_i(V., \widetilde{\phi})^* L_{Y_i(W.,P.)/X}$.

Let us see $\Phi_i(V., \widetilde{\phi})^* L_{Y_i(W.,P.)/X}$. We use the argument explained in the subsubsection 2.3.2. We will omit to denote $p_{U_2}^*$ and $F_{2\,X}^*$. In the case $i = 2$, it is expressed by the following complex:

$$\begin{gathered} \mathcal{H}om(V_0, V_{-1}) \oplus \mathcal{H}om(V_{-1}, P_{-1}) \oplus \mathcal{H}om(V_0, P_0) \longrightarrow \mathcal{H}om(V_0, V_0) \oplus \mathcal{H}om(V_{-1}, V_{-1}) \\ (b, c_{-1}, c_0) \longmapsto \big(f \circ b + \widetilde{\phi}_0 \circ c_0,\ \ -b \circ f + \widetilde{\phi}_{-1} \circ c_{-1}\big) \end{gathered} \tag{95}$$

Here, the first term stands in the degree 0. In the case $i = 0$, it is expressed by the following complex:

$$\mathcal{H}om(V_0, P_{-1}) \longmapsto \mathcal{H}om\big(V_0, V_0\big) \oplus \mathcal{H}om\big(V_{-1}, V_{-1}\big), \quad c_2 \longmapsto \big(0, 0\big) \tag{96}$$

Again, the first term stands in the degree 0. In the case $i = 1$, it is expressed by the following:

$$0 \longrightarrow \mathcal{H}om(V_0, V_0) \oplus \mathcal{H}om(V_{-1}, V_{-1}) \tag{97}$$

Here the term 0 stands in the degree 0. For the description (95) and (96), the degree 0-part of κ_2 is given as follows:

$$\mathcal{H}om(V_0, P_{-1}) \longrightarrow \mathcal{H}om(V_0, V_{-1}) \oplus \mathcal{H}om(V_{-1}, P_{-1}) \oplus \mathcal{H}om(V_0, P_0), \quad a \longmapsto \big(\phi_{-1} \circ a,\ a \circ f,\ -\partial_L \circ a\big).$$

The degree 1-part of κ_2 is given by the identity. On the other hand, the morphism κ_1 is the obvious one for the descriptions (96) and (97). Then the claim of the lemma can be checked directly. ∎

We put $\mathfrak{g}(V., \widetilde{\phi}.) := \mathrm{Cone}\big(\gamma(\widetilde{\phi}.)\big)$. Then, we obtain the following commutative diagram:

$$\begin{array}{ccccc} \mathfrak{g}(V.) & \longrightarrow & \Phi\big(F_{2\,X}^* V.\big)^* L_{Y(W.)/X} & \longrightarrow & F_{2,X}^* L_{U \times X/X} \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ \mathfrak{g}(V., \widetilde{\phi}.) & \longrightarrow & \Phi(V., \widetilde{\phi}.)^* L_{Y(W.,P.)/X} & \longrightarrow & L_{U_2 \times X/X} \end{array}$$

We obtain the following morphism of the distinguished triangles on $U_2 \times X$:

$$\begin{array}{ccccccc} \mathfrak{g}(V.) & \longrightarrow & \mathfrak{g}(V., \widetilde{\phi}.) & \longrightarrow & \mathfrak{g}_{\mathrm{rel}}(V., \widetilde{\phi}.) & \longrightarrow & \mathfrak{g}(V.)[1] \\ \big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow \\ F_{2\,X}^* L_{U \times X/X} & \longrightarrow & L_{U_2 \times X/X} & \longrightarrow & L_{U_2 \times X/U \times X} & \longrightarrow & F_{2\,X}^* L_{U \times X/X}[1]. \end{array}$$

We put $\mathrm{Ob}_{\mathrm{rel}}\big(V., \widetilde{\phi}.\big) := Rp_{X*}\big(\mathfrak{g}_{\mathrm{rel}}(V., \widetilde{\phi}.) \otimes \omega_X\big)$. Then we obtain the following diagram on U_2:

$$\begin{array}{ccc} \mathrm{Ob}_{\mathrm{rel}}\big(V., \widetilde{\phi}.\big)[-1] & \longrightarrow & \mathrm{Ob}(V.) \\ \big\downarrow & & \big\downarrow \\ L_{U_2/U}[-1] & \longrightarrow & F_{2\,X}^* L_{U/k} \end{array} \tag{98}$$

Lemma 5.9 *The diagram* (98) *depends only on (E, ϕ) in the derived category $D(U_2)$* .

Proof Let $\big(V.^{(1)}, P.^{(1)}, \widetilde{\phi}^{(1)}\big)$ be another choice. We take the resolution $V.^{(2)}$ of E as in the proof of Lemma 5.1. We put $P_0^{(2)} = P_0^{(1)} \oplus P_0$ and $P_{-1}^{(2)} = \ker(P_0^{(2)} \longrightarrow L)$. Then the lift $\widetilde{\phi}^{(2)} : P.^{(2)} \longrightarrow F_{2,X}^* V.^{(2)}$ is naturally obtained from the lifts $\widetilde{\phi}^{(1)}$ and $\widetilde{\phi}$. We have the compatible inclusions $P. \longrightarrow P.^{(2)}$ and $V. \longrightarrow V.^{(2)}$. Then we can show the claim of the lemma by using the filtered objects as in Lemma 5.1. ∎

5.3.2 Relative obstruction property

For any U-scheme $g: T \longrightarrow U$, let $F(T)$ denote the set of L-sections of g_X^*E. Thus we obtain the functor of the category of U-schemes to the category of sets. The functor F is representable by a scheme $M(L)$. Let $M(L) \longrightarrow U$ denote the projection. On $M(L)\times X$, we have the universal L-section ϕ of $\pi_X^* E$. Assume that we have a locally free resolution $P.$ of L for which we have a lift $\widetilde{\phi}: P. \longrightarrow V.$ of ϕ. Then we obtain the morphism:

$$\mathrm{ob}_{\mathrm{rel}}(V., \widetilde{\phi}) : \mathrm{Ob}_{\mathrm{rel}}(V., \widetilde{\phi}) \longrightarrow L_{M(L)/U}$$

Lemma 5.10 $\mathrm{ob}_{\mathrm{rel}}(V., \widetilde{\phi})$ *gives the relative obstruction theory for $M(L)$ over U.*

Proof We have only to show the claim on any sufficiently small open subsets $\mathcal{U}$ of $M(L)$. Recall that $\mathrm{ob}_{\mathrm{rel}}(V., \widetilde{\phi})$ is independent of the choice of $P.$ and $\widetilde{\phi}$. Therefore, we may assume $H^i\big(X, P_0^{-1}\otimes V_{l\,|\,\{u\}\times X}\big) = 0$ for $i = 0,1$, $l = 0,-1$ and for any $u \in \mathcal{U}$.

For simplicity of the notation, we put $C_0 := \mathfrak{g}_{\mathrm{rel}}(V., \widetilde{\phi})$ and $C_1 := \mathfrak{g}_{\mathrm{rel}}(V., \widetilde{\phi})_{\leq 0}$. We have the obvious map $C_0 \longrightarrow C_1$. By the argument in the subsubsection 2.3.2, the complex C_1 expresses $\Psi(V., \widetilde{\phi})^* L_{Y(W.,P.)/Y(W.)}$. Thus we have the natural morphism $C_1 \longrightarrow L_{\mathcal{U}\times X/U\times X}$. We put $\mathrm{Ob}_1 := Rp_{X\,*}\big(C_1\otimes\omega_X\big)$. Then we have the induced morphisms $\mathrm{Ob}_{\mathrm{rel}}(V., \widetilde{\phi}) \longrightarrow \mathrm{Ob}_1$ and $\mathrm{ob}_1 : \mathrm{Ob}_1 \longrightarrow L_{\mathcal{U}/U}$. It is easy to see that the composite of the morphisms is same as $\mathrm{ob}_{\mathrm{rel}}(V., \widetilde{\phi})$. We use the following lemma.

Lemma 5.11 *The morphism $\mathcal{H}^0\big(\mathrm{Ob}_{\mathrm{rel}}(V., \widetilde{\phi})\big) \longrightarrow \mathcal{H}^0\big(\mathrm{Ob}_1\big)$ is isomorphic, and $\mathcal{H}^1\big(\mathrm{Ob}_{\mathrm{rel}}(V., \widetilde{\phi})\big) \longrightarrow \mathcal{H}^1\big(\mathrm{Ob}_1\big)$ is surjective.*

Proof We have the exact sequence of the complexes $0 \longrightarrow \mathcal{H}om\big(V_{-1}, P_0\big)[-1] \longrightarrow C_0 \longrightarrow C_1 \longrightarrow 0$. Due to our choice of $P.$, we have $H^i\big(X, P_0^{-1}\otimes V_{-1\,|\,\{u\}\times X}\big) = 0$ for any $u\in\mathcal{U}$. Then the claim can be easily shown. ∎

Therefore, we have only to show that ob_1 gives the obstruction theory for $\mathcal{U}$ over U. For that purpose, we have only to check the conditions (A1) and (A2) in Proposition 2.28. We use the same argument as that in the proof of Lemma 2.35. Let T be a scheme with a morphism $h: T \longrightarrow \mathcal{U}$. Then, we have the following commutative diagram, for any coherent sheaf J on T:

$$\begin{array}{ccc}
\mathrm{Ext}^i\big(h^* L_{\mathcal{U}/U}, J\big) & \xrightarrow{\ \phi\ } & \mathrm{Ext}^i\big(h^*\,\mathrm{Ob}_1, J\big) \\
\big\downarrow & & \simeq\big\uparrow \\
\mathrm{Ext}^i\big(h_X^* L_{\mathcal{U}\times X/U\times X}, J_X\big) & \longrightarrow & \mathrm{Ext}^i\big(h_X^* C_1,\, J_X\big)
\end{array}$$

Let $\overline{T}$ be a scheme such that T is a closed subscheme of $\overline{T}$ whose corresponding ideal sheaf J is square 0. Let h' be a morphism $\overline{T} \longrightarrow U$ such that the restriction $h'_{|T}$ is same as $\pi\circ h$, where π denotes the projection $\mathcal{U} \longrightarrow U$. We have the obstruction class $o(h,h') \in \mathrm{Ext}^1\big(h^* L_{\mathcal{U}/U},\, J\big)$. We put $\widehat{h}_X := \Phi(V., \widetilde{\phi}.)\circ h_X : T\times X \longrightarrow Y(W., \widetilde{\phi})$ and $\widehat{h}'_X := \Phi(V.)\circ h'_X : \overline{T}\times X \longrightarrow Y(W.)$. Since the complex C_1 expresses $\Psi(V., \widetilde{\phi})^* L_{Y(W.,P.)/Y(W.)}$, we have the obstruction class $o\big(\widehat{h}_X, \widehat{h}'_X\big) \in \mathrm{Ext}^1\big(h^* C_1, J_X\big)$. By the functoriality, $o(h,h')$ is mapped to $o\big(\widehat{h}_X, \widehat{h}'_X\big)$ in the diagram. Hence, $\phi\big(o(h,h')\big) = 0$ implies that $\widehat{h}_X$ can be extended to a morphism $\widehat{h}_{1,X} : \overline{T} \longrightarrow Y(W., P.)$, which is a lift of $\widehat{h}'_X$. Hence, we obtain an L-section of $h_X'^* E$. Then we obtain an extension of h to a morphism $h_1 : \overline{T} \longrightarrow \mathcal{U}$ which is a lift of h', due to the universal property of $\mathcal{M}(L)$. Thus the condition (A1) is checked. The condition (A2) can also be checked easily, and the proof of Lemma 5.10 is finished. ∎

5.3.3 Preparation for the obstruction theory of Master space

We will use the notation in the subsubsections 5.1.3 and 5.3.1. We have the naturally defined right $\mathrm{GL}(W_0)$-action on $N\big(\mathcal{O}_X, W_{0,X}\big)$. The quotient stack is denoted by $A(W., P.)$. Assume that we have a morphism $\widetilde{\iota}: \mathcal{O}_X \longrightarrow P_0$. Then we have the induced morphism $Y_1(W., P.) \longrightarrow A(W., P.)$, and thus $\Gamma_L : Y(W., P.) \longrightarrow A(W., P.)$. From $(F_{2X}^* V., \widetilde{\phi})$ on $U_2\times X$ and $\widetilde{\iota}$, we obtain the morphism $\overline{\phi} : p_{U_2}^*\mathcal{O}_X \longrightarrow F_{2X}^* V_0$. We put $\mathfrak{h}_{\mathrm{rel}}(V., \widetilde{\phi}) := \mathcal{H}om\big(p_{U_2}^*\mathcal{O}_X,\, F_{2X}^* V.\big)^{\vee}$. Then, we have the induced maps $\gamma(\overline{\phi}) : \mathfrak{h}_{\mathrm{rel}}(V., \widetilde{\phi})[-1] \longrightarrow \mathfrak{h}(V.)$.

We put $\Psi(V_{\cdot},\widetilde{\phi}_{\cdot}):=\Gamma_L\circ\Phi(V_{\cdot},\widetilde{\phi})$, which is the classifying map of $\bigl(F_{2\,X}^{*}V_0,\overline{\phi}\bigr)$. Then, $\Psi(V_{\cdot},\widetilde{\phi}_{\cdot})^{*}L_{A(W_{\cdot},P_{\cdot})/X}$ is expressed by $\mathrm{Cone}\bigl(\gamma(\overline{\phi})_{\leq 1}\bigr)$. The induced morphism $\Psi(V_{\cdot},\widetilde{\phi}_{\cdot})^{*}L_{A(W_{\cdot},P_{\cdot})/X}\longrightarrow\Phi(V_{\cdot},\widetilde{\phi}_{\cdot})^{*}L_{Y(W_{\cdot},P_{\cdot})/X}$ is expressed by the naturally given morphism $\mathrm{Cone}\bigl(\gamma(\overline{\phi})_{\leq 1}\bigr)\longrightarrow\mathrm{Cone}\bigl(\gamma(\widetilde{\phi}_{\cdot})_{\leq 1}\bigr)$.

We put $\mathfrak{h}(V_{\cdot},\widetilde{\phi}):=\mathrm{Cone}\bigl(\gamma(\overline{\phi})\bigr)$. Then, we obtain the following commutative diagram:

$$\begin{array}{ccccc}
\mathfrak{h}(V_{\cdot},\widetilde{\phi}) & \longrightarrow & \mathfrak{g}(V_{\cdot},\widetilde{\phi}) & \longrightarrow & L_{U_2\times X/X}\\
\uparrow & & \uparrow & & \uparrow\\
\mathfrak{h}(V_{\cdot}) & \longrightarrow & \mathfrak{g}(V_{\cdot}) & \longrightarrow & F_{2\,X}^{*}L_{U\times X/X}
\end{array}$$

We put $\mathrm{Ob}^{G}_{\mathrm{rel}}(V_{\cdot},\widetilde{\phi}):=Rp_{X\,*}\bigl(\mathfrak{h}_{\mathrm{rel}}(V_{\cdot},\widetilde{\phi})\otimes\omega_X\bigr)$. Then, we have the following diagram on U_2:

$$\begin{array}{ccccc}
L_{U_2/U}[-1] & \longleftarrow & \mathrm{Ob}_{\mathrm{rel}}(V_{\cdot},\widetilde{\phi})[-1] & \longleftarrow & \mathrm{Ob}^{G}_{\mathrm{rel}}(V_{\cdot},\widetilde{\phi})[-1]\\
\downarrow & & \downarrow & & \downarrow\\
F_2^{*}L_{U/k} & \longleftarrow & \mathrm{Ob}(V_{\cdot}) & \longleftarrow & \mathrm{Ob}^{G}(V_{\cdot})
\end{array}\tag{99}$$

Now, we assume the condition (C) in the subsubsection 5.1.3. We have the natural right $\mathrm{GL}(W_0)$-action on $N(k,W_0)$. The quotient stack is denoted by $B(W_{\cdot},P_{\cdot})$. We have the natural isomorphism $J_L:B(W_{\cdot},P_{\cdot})\times X\longrightarrow A(W_{\cdot},P_{\cdot})$. From $\overline{\phi}:p_{U_2}^{*}\mathcal{O}_X\longrightarrow V_0$, we obtain the morphism $\Xi(V_{\cdot},\overline{\phi}):U_2\longrightarrow B(W_{\cdot},P_{\cdot})$. Note that the composite of $\Xi(V_{\cdot},\overline{\phi})_X$ and J_L is same as $\Psi(V_{\cdot},\overline{\phi})$. Therefore, we have the following commutative diagram:

$$\begin{array}{ccccc}
U_2\times X & \longrightarrow & B(W_{\cdot},P_{\cdot})\times X & \longrightarrow & A(W_{\cdot},P_{\cdot})\\
\downarrow & & \downarrow & & \downarrow\\
U\times X & \longrightarrow & B(W_{\cdot})\times X & \longrightarrow & A(W_{\cdot})
\end{array}$$

Thus, we obtain the following commutative diagram:

$$\begin{array}{ccccc}
F_2^{*}L_{U/k} & \longleftarrow & \Phi(V_{\cdot})^{*}L_{B(W_{\cdot})/k} & \xleftarrow[\simeq]{\tau_1} & \mathrm{Ob}^{G}(V_{\cdot})\\
\uparrow & & \uparrow & & \uparrow\\
L_{U_2/U}[-1] & \longleftarrow & \Xi(V_{\cdot},\overline{\phi})^{*}L_{B(W_{\cdot},P_{\cdot})/B(W_{\cdot})}[-1] & \xleftarrow{\tau_2} & \mathrm{Ob}^{G}_{\mathrm{rel}}(V_{\cdot},\widetilde{\phi})
\end{array}$$

Lemma 5.12 *The morphism τ_2 is isomorphic.*

Proof The complex $\mathfrak{h}_{\mathrm{rel}}(V_{\cdot},\widetilde{\phi})$ is quasi isomorphic to $\bigl(\mathcal{H}om(V_0,\mathcal{O})\to\mathcal{H}om(V_{-1},\mathcal{O})\bigr)$, where the first term stands at the degree 0. And the degree 0-part of the morphism $\mathfrak{h}_{\mathrm{rel}}(V_{\cdot},\widetilde{\phi})_{\leq 1}\longrightarrow\Xi(V,\overline{\phi})_X^{*}L_{B(W_{\cdot},P_{\cdot})/B(W_{\cdot})}\simeq\mathcal{H}om(V_0,\mathcal{O})$ is given by the identity of $\mathcal{H}om(V_0,\mathcal{O})$. Hence, the claim can be shown as in Lemma 5.2. ∎

5.3.4 Preparation for Proposition 6.23

We have the morphism of the complexes $\mathfrak{f}:\mathfrak{g}_{\mathrm{rel}}(V_{\cdot},\phi)\longrightarrow\mathcal{O}_{U_2\times X}$ given as follows:

$$\mathcal{H}om(F_{2X}^{*}V_0,p_{U_2}^{*}P_0)\oplus\mathcal{H}om(F_{2X}^{*}V_{-1},p_{U_2}^{*}P_{-1})\longrightarrow\mathcal{O}_{U_2\times X},\quad(a_0,a_{-1})\longmapsto\mathrm{tr}\bigl(\widetilde{\phi}_0\circ a_0\bigr)+\mathrm{tr}\bigl(\widetilde{\phi}_{-1}\circ a_{-1}\bigr)$$

It is easy to check that the following diagram is commutative:

$$\begin{array}{ccc}
\mathfrak{g}_{\mathrm{rel}}(V_{\cdot},\widetilde{\phi})[-1] & \longrightarrow & \mathfrak{g}(V_{\cdot})\\
{\scriptstyle\mathfrak{f}}\downarrow & & {\scriptstyle\mathrm{tr}}\downarrow\\
\mathcal{O}_{U_2\times X}[-1] & \xrightarrow{\mathrm{id}} & \mathcal{O}_{U_2\times X}[-1]
\end{array}$$

It induces the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Ob}_{\mathrm{rel}}(V.,\widetilde{\phi})[-1] & \longrightarrow & \mathrm{Ob}(V.) \\ \big\downarrow & & \big\downarrow \\ \mathcal{O}_{U_2\times X}[-1] & \longrightarrow & \mathcal{O}_{U_2\times X}[-1] \end{array} \tag{100}$$

5.4 Relative Obstruction Theory for Reduced L-Section

5.4.1 Construction of the complex

We use the notation in the subsubsection 5.3.1. The weight (-1)-action of G_m on P_i induces the G_m-action on $Y_i(W.,P.)$ $(i=0,1,2)$ and $Y(W.,P.)$. The quotient stacks are denoted by $Y_i(W.,[P.])$ $(i=0,1,2)$ and $Y(W.,[P.])$. We have the morphism $\pi_3 : Y(W.,[P.]) \longrightarrow X_{G_m}$ induced by $Y(W.,P.) \longrightarrow X$.

Let U_3 be a stack with a morphism $F_3 : U_3 \longrightarrow U$ and a reduced L-section $[\phi] : p_{U_3}^*(L)\otimes p_X^*(M) \longrightarrow F_{3\,X}^*(E)$, where M denotes a line bundle on U_3. Let $\Phi_M : U_3 \longrightarrow \mathrm{Spec}(k)_{G_m}$ denote the classifying map for M.

Notation 5.13 *When we have a map $g : U_3 \longrightarrow T$, then Φ_M induces the morphism $U_3 \longrightarrow T\times \mathrm{Spec}(k)_{G_m}$, which is denoted by g_M in the following argument.* ▮

We assume that we have a morphism of complexes $[\widetilde{\phi}.] = \big([\widetilde{\phi}_{-1}],[\widetilde{\phi}_0]\big) : p_{U_3}^*(P.)\otimes p_X^*M \longrightarrow F_{3\,X}^*(V.)$ which gives $[\phi]$ in the cohomology level. Such a map $[\widetilde{\phi}]$ is called a lift of $[\phi]$. We put as follows:

$$\mathfrak{g}'_{\mathrm{rel}}(V.,[\widetilde{\phi}]) = \mathcal{H}om\big(p_U^*P.\otimes p_X^*M,\ F_{3,X}^*V.\big)^\vee$$

We have the naturally induced morphism $\gamma[\widetilde{\phi}.] : \mathfrak{g}'_{\mathrm{rel}}(V.,[\widetilde{\phi}])[-1] \longrightarrow \mathfrak{g}'(V.)$ and $\gamma[\widetilde{\phi}.]_{\leq 1} : \mathfrak{g}'(V.,[\widetilde{\phi}])[-1]_{\leq 1} \longrightarrow \mathfrak{g}'(V.)_{\leq 1}$.

We have the classifying map $\Phi\big(V.,[\widetilde{\phi}]\big) : U_3\times X \longrightarrow Y(W.,[P.])$. We consider the trivial G_m-actions on $U\times X$, $Y(W.)$ and X, and the quotient stacks are denoted by $(U\times X)_{G_m}$, $Y(W.)_{G_m}$ and X_{G_m} respectively. Then, we have the following commutative diagram:

$$\begin{array}{ccccc} U_3\times X & \xrightarrow{\Phi(V.,[\widetilde{\phi}.])} & Y(W.,[P.]) & \longrightarrow & X_{G_m} \\ {\scriptstyle F_{3XM}}\big\downarrow & & {\scriptstyle \pi_3}\big\downarrow & & \big\downarrow \\ (U\times X)_{G_m} & \longrightarrow & Y(W.)_{G_m} & \longrightarrow & X_{G_m} \end{array}$$

The composite $\pi_3\circ\Phi(V.,[\widetilde{\phi}.])$ is same as $\Phi(F_{3,X}^*V.)_M$. The following lemma can be shown by the same argument as the proof of Lemma 5.8.

Lemma 5.14 $\Phi(V.,[\widetilde{\phi}.])^*L_{Y(W.,[P.])/X_{G_m}}$ *is expressed by the complex* $\mathrm{Cone}\big(\gamma[\widetilde{\phi}.]_{\leq 1}\big)$. ▮

We put $\mathfrak{g}'(V.,[\widetilde{\phi}.]) := \mathrm{Cone}\big(\gamma[\widetilde{\phi}.]\big)$. Then we obtain the following commutative diagram:

$$\begin{array}{ccccc} \mathfrak{g}(V.) & \longrightarrow & \Phi\big(F_3^*V.\big)_M^* L_{Y(W.)_{G_m}/X_{G_m}} & \longrightarrow & F_{3,X,M}^*L_{(U\times X)_{G_m}/X_{G_m}} \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ \mathfrak{g}'(V.,[\widetilde{\phi}.]) & \longrightarrow & \Phi(V.,\widetilde{\phi}.)^*L_{Y(W.,[P.])/X_{G_m}} & \longrightarrow & L_{U_3\times X/X_{G_m}} \end{array}$$

We remark $F_{3,X,M}^*L_{(U\times X)_{G_m}/X_{G_m}}$ and $\Phi\big(F_{3\,X}^*V.\big)_M^*L_{Y(W.)_{G_m}/X_{G_m}}$ are naturally isomorphic to $F_{3\,X,M}^*L_{U\times X/X}$ and $\Phi\big(F_{3\,X}^*V.\big)^*L_{Y(W.)/X}$, respectively. Then, we obtain the following morphism of the distinguished triangles on $U_3\times X$:

$$\begin{array}{ccccccc} \mathfrak{g}(V.) & \longrightarrow & \mathfrak{g}'(V.,[\widetilde{\phi}.]) & \longrightarrow & \mathfrak{g}'_{\mathrm{rel}}(V.,[\widetilde{\phi}.]) & \longrightarrow & \mathfrak{g}(V.)[1] \\ \big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow \\ F_{3\,X}^*L_{(U\times X)_{G_m}/X_{G_m}} & \longrightarrow & L_{U_3\times X/X_{G_m}} & \longrightarrow & L_{U_3\times X/(U\times X)_{G_m}} & \longrightarrow & F_{3\,X}^*L_{(U\times X)_{G_m}/X_{G_m}}[1]. \end{array}$$

We put $\mathrm{Ob}'_{\mathrm{rel}}\big(V.,[\widetilde{\phi}]\big) := Rp_{X\,*}\big(\mathfrak{g}'_{\mathrm{rel}}(V.,[\widetilde{\phi}])\big)$. Then we obtain the following diagram on U_3:

$$\begin{tikzcd}
\mathrm{Ob}'_{\mathrm{rel}}\big(V.,[\widetilde{\phi}]\big)[-1] \arrow[r] \arrow[d] & L_{U_3/U_{G_m}}[-1] \arrow[d] & \\
\mathrm{Ob}(V.) \arrow[r] & F_{3,M}^{*}L_{U_{G_m}/k} \arrow[r] & F_{3,M}^{*}L_{U_{G_m}/U}
\end{tikzcd} \tag{101}$$

We obtain the morphism $\mathrm{Ob}'_{\mathrm{rel}}\big(V.,[\widetilde{\phi}.]\big)[-1] \longrightarrow F_{3,M}^{*}L_{U_{G_m}/U}$, by taking the composite of the morphisms in the diagram above. We put as follows:

$$\mathrm{Ob}_{\mathrm{rel}}(V.,[\widetilde{\phi}]) := \mathrm{Cone}\big(\mathrm{Ob}'_{\mathrm{rel}}(V.,[\widetilde{\phi}]) \longrightarrow F_{3,M}^{*}L_{U_{G_m}/U}[1]\big)[-1].$$

We obtain the following morphism of the distinguished triangles:

$$\begin{tikzcd}
F_{3,M}^{*}L_{U_{G_m}/U} \arrow[r] \arrow[d] & \mathrm{Ob}_{\mathrm{rel}}(V.,[\widetilde{\phi}]) \arrow[r] \arrow[d] & \mathrm{Ob}'_{\mathrm{rel}}(V.,[\widetilde{\phi}]) \arrow[r] \arrow[d] & F_{3,M}^{*}L_{U_{G_m}/U}[1] \arrow[d] \\
F_{3,M}^{*}L_{U_{G_m}/U} \arrow[r] & L_{U_3/U} \arrow[r] & L_{U_3/U_{G_m}} \arrow[r] & F_{3,M}^{*}L_{U_{G_m}/U}[1]
\end{tikzcd}$$

Therefore, we obtain the following commutative diagram on U_3:

$$\begin{tikzcd}
\mathrm{Ob}_{\mathrm{rel}}\big(V.,[\widetilde{\phi}]\big)[-1] \arrow[r] \arrow[d, "\mathrm{ob}_{\mathrm{rel}}(V.,[\widetilde{\phi}])"'] & \mathrm{Ob}(V.) \arrow[d] \\
L_{U_3/U}[-1] \arrow[r] & F_3^{*}L_U.
\end{tikzcd} \tag{102}$$

Lemma 5.15 *The diagram* (102) *depends only on* $(E,[\phi])$.

Proof By an argument similar to the proof of Lemma 5.9, we can show that the diagram (101) is independent of a choice of $(V.,[P.],[\widetilde{\phi}])$. Then the diagram (102) is also independent. ∎

5.4.2 Relative obstruction property

For any U-scheme $g : T \longrightarrow U$, let $F(T)$ denote the set of the reduced L-sections of $g_X^{*}E$. Thus, we obtain the functor of the category of U-schemes to the category of sets. The functor is representable by a scheme $M[L]$. Let $\pi : M[L] \longrightarrow U$ denote the projection. We have the line bundle $\mathcal{I}$ on $M[L]$, and the universal reduced L-section $[\phi^u] : p_X^{*}\mathcal{I}\otimes p_{M[L]}^{*}L \longrightarrow \pi_X^{*}E$ over $M[L]\times X$. Assume that we have a locally free resolution $P.$ of L for which we have a lift $[\widetilde{\phi}] : p_X^{*}\mathcal{I}\otimes p_{M[L]}^{*}P. \longrightarrow \pi_X^{*}V.$ of $[\phi]$. Then, we obtain the morphism $\mathrm{ob}_{\mathrm{rel}}\big(V.,[\widetilde{\phi}.]\big) : \mathrm{Ob}_{\mathrm{rel}}\big(V.,[\widetilde{\phi}.]\big) \longrightarrow L_{M[L]/U}$. We also have the complex $\mathrm{Ob}'_{\mathrm{rel}}\big(V.,[\widetilde{\phi}.]\big)$ and the morphism $\mathrm{Ob}'_{\mathrm{rel}}\big(V.,[\widetilde{\phi}]\big) \longrightarrow L_{M[L]/U_{G_m}}$.

Let $M(L)$ be as in the subsubsection 5.3.2. It is easy to observe that $M(L)$ is isomorphic to $\mathcal{I}^{*}$. We have the smooth projection $\pi_2 : M(L) \longrightarrow M[L]$. The pull back of $[\widetilde{\phi}]$ via $\pi_{2\,X}$ is denoted by $\widetilde{\phi}$.

Lemma 5.16 *We have the following commutative diagram:*

$$\begin{tikzcd}
\pi_2^{*}\,\mathrm{Ob}'_{\mathrm{rel}}(V,[\widetilde{\phi}]) \arrow[r, "\simeq"] \arrow[d] & \mathrm{Ob}_{\mathrm{rel}}(V,\widetilde{\phi}) \arrow[d] \\
\pi_2^{*}L_{M[L]/U_{G_m}} \arrow[r, "\simeq"] & L_{M(L)/U}
\end{tikzcd} \tag{103}$$

Proof We have the following commutative diagram:

$$\begin{tikzcd}
M(L)\times X \arrow[r, "\Phi(V.,\widetilde{\phi})"] \arrow[d] & Y(W.,P.) \arrow[r, "\widetilde{\psi}"] \arrow[d] & Y(W.) \arrow[r] \arrow[d] & X \arrow[d] \\
M[L]\times X \arrow[r, "\Phi(V.,[\widetilde{\phi}])"] & Y(W.,[P.]) \arrow[r, "\psi"] & Y(W.)_{G_m} \arrow[r] & X_{G_m}
\end{tikzcd}$$

Therefore, we obtain the following:

$$\begin{array}{ccccc}
L_{M(L)\times X/U\times X} & \longleftarrow & \Phi(\phi)^* L_{Y(W.,P.)/Y(W.)} & \longleftarrow & \mathfrak{g}_{\mathrm{rel}}(V.,\widetilde{\phi}) \\
\simeq\uparrow & & \simeq\uparrow & & \simeq\uparrow \\
\pi_{2,X}^* L_{M[L]\times X/U\times X_{G_m}} & \longleftarrow & \Phi([\phi])^* L_{Y(W.,[P.])/Y(W.)_{G_m}} & \longleftarrow & \pi_{2,X}^* \mathfrak{g}'_{\mathrm{rel}}(V.,[\widetilde{\phi}])
\end{array}$$

Then the claim is clear. ∎

We have the morphisms $\mathrm{Ob}_{\mathrm{rel}}(V.,\widetilde{\phi}) \longrightarrow L_{M(L)/U} \longrightarrow L_{M(L)/M[L]}$, where the latter is the projection.

Lemma 5.17 *We have the commutative diagram:*

$$\begin{array}{ccccccc}
\pi_2^*\,\mathrm{Ob}_{\mathrm{rel}}\big(V.,[\widetilde{\phi}]\big) & \longrightarrow & \mathrm{Ob}_{\mathrm{rel}}(V.,\widetilde{\phi}) & \longrightarrow & L_{M(L)/M[L]} & \longrightarrow & \pi_2^*\,\mathrm{Ob}_{\mathrm{rel}}(V.,[\widetilde{\phi}])[1] \\
\downarrow & & \downarrow & & \simeq\downarrow & & \downarrow \\
\pi_2^* L_{M[L]/U} & \longrightarrow & L_{M(L)/U} & \longrightarrow & L_{M(L)/M[L]} & \longrightarrow & \pi_2^* L_{M[L]/U}[1].
\end{array} \tag{104}$$

In particular, $\pi_2^*\,\mathrm{Ob}_{\mathrm{rel}}(V.,[\widetilde{\phi}]) \simeq \mathrm{Cone}\big(\mathrm{Ob}_{\mathrm{rel}}(V.,[\widetilde{\phi}]) \longrightarrow L_{M(L)/M[L]}\big)$.

Proof Let $F_{3,\mathcal{I}}$ denote the morphism $M[L] \longrightarrow U_{G_m}$, induced by $\mathcal{I}$. Then we have the following isomorphism of the distinguished triangles on $M(L)$:

$$\begin{array}{ccccccc}
\pi_2^* L_{M[L]/U} & \longrightarrow & \pi_2^* L_{M[L]/U_{G_m}} & \longrightarrow & \pi_2^* F_{3,\mathcal{I}}^* L_{U_{G_m}/U}[1] & \longrightarrow & \pi_2^* L_{M[L]/U}[1] \\
=\downarrow & & \simeq\downarrow & & \simeq\downarrow & & =\downarrow \\
\pi_2^* L_{M[L]/U} & \longrightarrow & L_{M(L)/U} & \longrightarrow & L_{M(L)/M[L]} & \longrightarrow & \pi_2^* L_{M[L]/U}[1]
\end{array} \tag{105}$$

We obtain (104) from (103) and (105). ∎

Lemma 5.18 $\mathrm{ob}_{\mathrm{rel}}\big(V.,[\widetilde{\phi}.]\big)$ *gives a relative obstruction theory for* $M[L]$ *over* U.

Proof The complex $L_{M(L)/M[L]}$ is quasi isomorphic to the 0-th cohomology sheaf, and we know that $\mathrm{ob}_{\mathrm{rel}}(V.,\widetilde{\phi})$ gives the obstruction theory for $M(L)$ over U (Lemma 5.10). Then, the claim of the lemma follows from the diagram (104). ∎

5.4.3 Preparation for the obstruction theory of Master space

We will use the notation in the subsubsections 5.4.1 and 5.1.3. We have the weight (-1)-action of G_m on $\mathcal{O}_X$. It induces the G_m-action on $A(W.,P.)$. The quotient stack is denoted by $A(W.,[P.])$. We have the induced morphism $\Gamma_{[L]} : Y(W.,[P.]) \longrightarrow A(W.,[P.])$. From $(F_{3,X}^* V.,[\widetilde{\phi}])$ on $U_3 \times X$ and $[\widetilde{\imath}]$, we obtain the morphism $[\overline{\phi}] : p_{U_3}^*\mathcal{O}_X \otimes p_X^* M \longrightarrow F_{3,X}^* V_0$. We put $\mathfrak{h}'_{\mathrm{rel}}(V.,[\widetilde{\phi}]) := \mathcal{H}om\big(p_{U_3}^*\mathcal{O}_X \otimes p_X^* M, F_{3X}^* V.\big)^\vee$. We have the induced map $\gamma([\overline{\phi}]) : \mathfrak{h}'_{\mathrm{rel}}(V.,[\widetilde{\phi}])[-1] \longrightarrow \mathfrak{h}(V.)$, and $\gamma([\overline{\phi}])_{\leq 1} : \mathfrak{h}'_{\mathrm{rel}}(V.,[\widetilde{\phi}])[-1]_{\leq 1} \longrightarrow \mathfrak{h}(V.)_{\leq 1}$.

We put $\Psi(V.,[\widetilde{\phi}.]) := \Gamma_{[L]} \circ \Phi(V.,[\widetilde{\phi}])$, which give the classifying map of the tuple $(F_{X,D}^* V_0, [\overline{\phi}])$. Then, $\Psi(V.,[\widetilde{\phi}.])^* L_{A(W.,[P.])/X_{G_m}}$ is expressed by the complex $\mathrm{Cone}\big(\gamma([\overline{\phi}])_{\leq 1}\big)$, and the naturally defined morphism $\mathrm{Cone}\big(\gamma([\overline{\phi}])_{\leq 1}\big) \longrightarrow \mathrm{Cone}\big(\gamma([\widetilde{\phi}.])_{\leq 1}\big)$ expresses $\Psi(V.,[\widetilde{\phi}.])^* L_{A(W.,[P.])/X_{G_m}} \longrightarrow \Phi(V.,[\widetilde{\phi}.])^* L_{Y(W.,[P.])/X_{G_m}}$.

We put $\mathfrak{h}'(V.,\widetilde{\phi}) := \mathrm{Cone}\big(\gamma([\overline{\phi}])\big)$. Then, we obtain the following commutative diagram:

$$\begin{array}{ccccc}
\mathfrak{h}'(V.,[\widetilde{\phi}]) & \longrightarrow & \mathfrak{g}'(V.,[\widetilde{\phi}]) & \longrightarrow & L_{U_3\times X/X_{G_m}} \\
\uparrow & & \uparrow & & \uparrow \\
\mathfrak{h}(V.) & \longrightarrow & \mathfrak{g}(V.) & \longrightarrow & F_{3\,X}^* L_{U\times X/X}
\end{array}$$

We put $\mathrm{Ob}'^{G}_{\mathrm{rel}}(V_\cdot,[\widetilde{\phi}]) := Rp_{X\,*}\big(\mathfrak{h}'_{\mathrm{rel}}(V_\cdot,[\widetilde{\phi}])\otimes\omega_X\big)$. Then, we have the following diagram on U_3:

$$\begin{array}{ccccc}
L_{U_3/U_{G_m}}[-1] & \longleftarrow & \mathrm{Ob}'_{\mathrm{rel}}(V_\cdot,[\widetilde{\phi}])[-1] & \longleftarrow & \mathrm{Ob}'^{G}_{\mathrm{rel}}(V_\cdot,[\widetilde{\phi}])[-1] \\
\Big\downarrow & & \Big\downarrow & & \Big\downarrow \\
F_3^* L_{U/k} & \longleftarrow & \mathrm{Ob}(V_\cdot) & \longleftarrow & \mathrm{Ob}^G(V_\cdot)
\end{array} \tag{106}$$

We obtain the composite $\mathrm{Ob}'^{G}_{\mathrm{rel}}(V,[\widetilde{\phi}]) \longrightarrow L_{U_3/U_{G_m}} \longrightarrow F^*_{3,M}L_{U_{G_m}/U}[1]$. Let $\mathrm{Ob}^G_{\mathrm{rel}}(V,[\widetilde{\phi}])$ denote the cone of the morphism. Then, we obtain the following diagram:

$$\begin{array}{ccccc}
L_{U_3/U}[-1] & \longleftarrow & \mathrm{Ob}_{\mathrm{rel}}(V_\cdot,[\widetilde{\phi}])[-1] & \longleftarrow & \mathrm{Ob}^{G}_{\mathrm{rel}}(V_\cdot,[\widetilde{\phi}])[-1] \\
\Big\downarrow & & \Big\downarrow & & \Big\downarrow \\
F_3^* L_{U/k} & \longleftarrow & \mathrm{Ob}(V_\cdot) & \longleftarrow & \mathrm{Ob}^G(V_\cdot)
\end{array} \tag{107}$$

Now, we assume the condition (C) in the subsubsection 5.1.3. We have the weight (-1)-action of G_m on the one dimensional vector space k. It induces the G_m-action on $B(W_\cdot,P_\cdot)$. The quotient stack is denoted by $B(W_\cdot,[P_\cdot])$. Similarly, we obtain $A(W_\cdot,[P_\cdot])$ from $A(W_\cdot,P_\cdot)$. We have the natural isomorphism $J_{[L]}$: $B(W_\cdot,[P_\cdot])\times X \longrightarrow A(W_\cdot,[P_\cdot])$. From $[\overline{\phi}] : p_X^*M\otimes p_{U_3}^*\mathcal{O}_X \longrightarrow V_0$, we obtain the morphism $\Xi(V_\cdot,[\overline{\phi}]) : U_3 \longrightarrow B(W_\cdot,[P_\cdot])$. Note that the composite of $\Xi(V_\cdot,[\overline{\phi}])_X$ and $J_{[L]}$ is same as $\Psi(V_\cdot,[\overline{\phi}])$. Therefore, we have the following commutative diagram:

$$\begin{array}{ccccc}
U_3\times X & \longrightarrow & B(W_\cdot,[P_\cdot])\times X & \longrightarrow & A(W_\cdot,[P_\cdot]) \\
\Big\downarrow & & \Big\downarrow & & \Big\downarrow \\
(U\times X)_{G_m} & \longrightarrow & \big(B(W_\cdot)\times X\big)_{G_m} & \longrightarrow & A(W_\cdot)_{G_m}
\end{array}$$

Therefore, we obtain the following commutative diagram:

$$\begin{array}{ccccc}
F^*_{3,M}L_{U_{G_m}/k_{G_m}} & \longleftarrow & \Phi(V_\cdot)^*_M L_{B(W_\cdot)_{G_m}/k_{G_m}} & \longleftarrow & \mathrm{Ob}^G(V_\cdot) \\
\Big\uparrow & & \Big\uparrow & & \Big\uparrow \\
L_{U_3/U_{G_m}}[-1] & \longleftarrow & \Xi(V_\cdot,[\overline{\phi}])^*L_{B(W_\cdot,[P_\cdot])/B(W_\cdot)_{G_m}}[-1] & \xleftarrow{\tau_3'} & \mathrm{Ob}'^{G}_{\mathrm{rel}}(V_\cdot,[\widetilde{\phi}])
\end{array}$$

The following lemma is similar to Lemma 5.12.

Lemma 5.19 *The morphism τ_3' is isomorphic.* ∎

By the standard modification, we obtain the following commutative diagram:

$$\begin{array}{ccccc}
F_3^* L_{U/k} & \longleftarrow & \Phi(V_\cdot)^* L_{B(W_\cdot)/k} & \longleftarrow & \mathrm{Ob}^G(V_\cdot) \\
\Big\uparrow & & \Big\uparrow & & \Big\uparrow \\
L_{U_3/U}[-1] & \longleftarrow & \Xi(V_\cdot,[\overline{\phi}])^*L_{B(W_\cdot,[P_\cdot])/B(W_\cdot)}[-1] & \xleftarrow{\tau_3} & \mathrm{Ob}^{G}_{\mathrm{rel}}(V_\cdot,[\widetilde{\phi}])[-1]
\end{array} \tag{108}$$

The following lemma immediately follows from the previous lemma.

Lemma 5.20 *The morphism τ_3 is isomorphic.* ∎

5.4.4 Preparation for Proposition 6.23

We use the notation in the subsubsection 5.4.1. As a preparation for the proof of Proposition 6.23, let us see the morphism $\varphi_1 : \mathrm{Ob}'_{\mathrm{rel}}(V_\cdot,[\widetilde{\phi}]) \longrightarrow F_3^*L_{U_{G_m}/U}$ on U_3, more closely. We recall $\Phi(V,[\overline{\phi}])^*L_{Y(W_\cdot,[P_\cdot])/Y(W_\cdot)_{G_m}} \simeq \mathfrak{g}'_{\mathrm{rel}}(V_\cdot,[\widetilde{\phi}])_{\leq 0}$.

Lemma 5.21 *We have the following commutative diagram:*

$$\begin{tikzcd}
\Phi(V,[\widetilde{\phi}])^{*}L_{Y(W.,[P.])/Y(W.)_{G_m}} \arrow[r] & \Phi(F_{3,X}^{*}V.)^{*}_{M}L_{Y(W.)_{G_m}/Y(W.)}[1] \\
\mathfrak{g}'_{\mathrm{rel}}(V.,[\widetilde{\phi}]) \arrow[u] \arrow[r, "\mathfrak{f}'"] & \mathcal{O} \arrow[u, "\simeq"']
\end{tikzcd}$$

Here, the morphism $\mathfrak{f}'$ is given as follows:

$$\mathcal{H}om\big(F_{3,X}^{*}V_0,\ p_U^{*}P_0\otimes p_X^{*}M\big)\oplus\mathcal{H}om\big(F_{3,X}^{*}V_0,\ p_U^{*}P_0\otimes p_X^{*}M\big)\longrightarrow\mathcal{O},\quad (a_0,a_{-1})\longmapsto \mathrm{tr}\big([\widetilde{\phi}_0]\circ a_0\big)+\mathrm{tr}\big([\widetilde{\phi}_{-1}]\circ a_{-1}\big).$$

Proof It follows from Corollary 2.15. ∎

The morphism $\mathfrak{f}'$ induces the following morphisms:

$$\mathrm{Ob}'_{\mathrm{rel}}(V.,[\widetilde{\phi}])=Rp_{X\,*}\big(\mathfrak{g}_{\mathrm{rel}}(V.,[\widetilde{\phi}])\otimes\omega_X\big)\longrightarrow Rp_{X\,*}\big(p_{U_3}^{*}\omega_X\big)\longrightarrow\mathcal{O}_{U_3}.$$

It is easy to check that the composite is same as φ_1.

5.5 Relative Obstruction Theory for Parabolic Structure

5.5.1 Construction of the complex

We use the notation in the subsubsection 5.1.1. Let $F_4:U_4\longrightarrow U$ be a morphism, and let F_* be a quasi-parabolic structure of $F_{4\,X}^{*}E$ at D. We denote the kernel of $F_{4,X}^{*}V_{0\,|\,D}\longrightarrow \mathrm{Cok}_{h-1}(E)$ by $V_D^{(h)}$. Since the smoothness of D is assumed, $V_D^{(h)}$ are locally free. The filtered vector bundle $V_D^{(1)}\supset V_D^{(2)}\supset\cdots\supset V_D^{(l+1)}$ is denoted by V_D^{*}.

We put $W^{(1)}:=W_0$ and $W^{(l+1)}:=W_{-1}$. Let $W^{(h)}$ $(h=2,\dots,l)$ be vector spaces over k such that $\dim W^{(h)}=\mathrm{rank}\,V_D^{(h)}$. Let D be a smooth divisor of X. We denote $W^{(h)}\otimes\mathcal{O}_D$ and $W_i\otimes\mathcal{O}_D$ by $W_{i\,D}$ and $W_D^{(h)}$ respectively. We have the natural right $\mathrm{GL}(W.)$-action on $N(W_{-1,D},W_{0,D})$ given by $(g_0,g_{-1})\cdot f=g_0^{-1}\circ f\circ g_{-1}$. The quotient stack is denoted by $Y_D(W.)$. Similarly, we have the natural right $\mathrm{GL}(W^{(h)})\times\mathrm{GL}(W^{(h+1)})$-action on $N(W_D^{(h+1)},W_D^{(h)})$ given by $(g^{(h)},g^{(h+1)})\cdot f=g^{(h)\,-1}\circ f\circ g^{(h+1)}$. Thus we obtain the right $\prod_{h=1}^{l+1}\mathrm{GL}(W^{(h)})$-action on $\prod_{h=1}^{l}N\big(W_D^{(h+1)},W_D^{(h)}\big)$, where the latter fiber product is taken over D. The quotient stack is denoted by $Y_D(W.,W^{*})$. The composition of the morphisms induce the map $\prod_{h=1}^{l}N\big(W_D^{(h+1)},W_D^{(h)}\big)\longrightarrow N(W_{-1\,D},W_{0\,D})$. It induce the morphism $Y_D(W.,W^{*})\longrightarrow Y_D(W.)$.

We have the classifying map $\Phi_D(V.,F_*):U_4\times D\longrightarrow Y_D(W.,W^{*})$ over D obtained from the tuple V_D^{*}. We also have the classifying map $\Phi(V._{|D}):U\times D\longrightarrow Y_D(W.)$. Thus, we obtain the following diagram on $U_4\times D$:

$$\begin{tikzcd}
L_{U_4\times D/D} & \Phi_D(V.,F_*)^{*}L_{Y_D(W.,W^{*})/D} \arrow[l] \\
F_{4,D}^{*}L_{U\times D/D} \arrow[u] & \Phi\big(F_{4\,D}^{*}(V._{|D})\big)^{*}L_{Y_D(W.)/D}. \arrow[l] \arrow[u]
\end{tikzcd}$$

We use the notation in the subsubsection 2.1.5. We put $\mathfrak{g}_D(V.,F_*):=C_1(V_D^{*},V_D^{*})^{\vee}[-1]$ and $\mathfrak{g}_{\mathrm{rel}}(V.,F_*):=C_2(V_D^{*},V_D^{*})^{\vee}[-1]$. We also put $\mathfrak{g}(V._{|D}):=\mathcal{H}om(V._{|D},V._{|D})^{\vee}[-1]$. We have the morphism $\gamma_D:\mathfrak{g}(V._{|D})\longrightarrow\mathfrak{g}_D(V.,F_*)$ induced by φ given in 19. It is easy to see that $\mathfrak{g}_{\mathrm{rel}}(V.,F_*)$ is quasi isomorphic to $\mathrm{Cone}(\gamma_D)$.

By an argument explained in the subsubsection 2.3.2, we can show that $\Phi_D(V.,F_*)^{*}L_{Y_D(W.,W^{*})/D}$ and $\Phi(V._{|D})^{*}L_{Y_D(W.)/D}$ are expressed by $\mathfrak{g}_D(V.,F_*)_{\leq 1}$ and $\mathfrak{g}(V._{|D})_{\leq 1}$. Under the identification, the natural morphism $\Phi_D(V.,F_*)^{*}L_{Y_D(W.,W^{*})/D}\longrightarrow\Phi(V._{\,|\,D})^{*}L_{Y_D(W.)/D}$ is given by the restriction $\gamma_{D\leq 1}$. Then we obtain the following commutative diagram:

$$\begin{tikzcd}
\mathfrak{g}(V._{|D}) \arrow[r] \arrow[d] & \mathfrak{g}(V._{|D})_{\leq 1} \arrow[r] \arrow[d] & F_4^{*}L_{U/k} \arrow[d] \\
\mathfrak{g}_D(V.,F_*) \arrow[r] & \mathfrak{g}_D(V.,F_*)_{\leq 1} \arrow[r] & L_{U_4/k}
\end{tikzcd}$$

Then we obtain the following morphism of the distinguished triangles on U_4:

$$\begin{array}{ccccccc}
\mathfrak{g}(V_{\cdot|D}) & \longrightarrow & \mathfrak{g}_D(V.,F_*) & \longrightarrow & \mathfrak{g}_{\mathrm{rel}}(V_{\cdot|D},F_*) & \longrightarrow & \mathfrak{g}(V_{\cdot|D})[1] \\
\Big\downarrow & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow \\
F_{4\,X}^{*}L_{U\times X/X} & \longrightarrow & L_{U_4\times X/X} & \longrightarrow & L_{U_4\times X/U\times X} & \longrightarrow & F_{4\,X}^{*}L_{U\times X/X}
\end{array}$$

Let ω_D denote the dualizing complex of D. We put $\mathrm{Ob}_{\mathrm{rel}}(V.,F_*) := Rp_{D\,*}\big(\mathfrak{g}_{\mathrm{rel}}\big(V.,F_*\big)\otimes\omega_D\big)$ and $\mathrm{Ob}(V._{\,|\,D}) := Rp_{D\,*}\big(\mathfrak{g}(V._{\,|\,D})\otimes\omega_D\big)$. Then, we obtain the following commutative diagram:

$$\begin{array}{ccc}
\mathrm{Ob}_{\mathrm{rel}}\big(V.,F_*\big)[-1] & \longrightarrow & L_{U_4/U}[-1] \\
\Big\downarrow & & \Big\downarrow \\
\mathrm{Ob}(V_{\cdot|D}) & \xrightarrow{\mathrm{ob}(V_{\cdot|D})} & F_4^{*}L_{U/k}.
\end{array} \tag{109}$$

We have the following exact sequence of complexes:

$$0 \longrightarrow \mathfrak{g}(V.)\otimes\omega \longrightarrow \mathfrak{g}(V.)\otimes\omega(D) \longrightarrow \mathfrak{g}(V_{\cdot|D})\otimes\omega_D[1] \longrightarrow 0.$$

Thus we obtain the morphism $Rp_{D\,*}\big(\mathfrak{g}(V_{\cdot|D})\otimes\omega_D\big) \longrightarrow Rp_{X\,*}\big(\mathfrak{g}(V.)\otimes\omega_X\big)$, that is, $\eta : \mathrm{Ob}(V_{\cdot|D}) \longrightarrow \mathrm{Ob}(V.)$. We have the morphism $\mathrm{ob}(V.) : \mathrm{Ob}(V.) \longrightarrow L_U$, and hence the composite $\mathrm{ob}(V.)\circ\eta : \mathrm{Ob}(V_{\cdot|D}) \longrightarrow L_U$. On the other hand, we have the morphism $\mathrm{ob}(V_{\cdot|D})$ in the diagram (109).

Lemma 5.22 *Two morphisms* $\mathrm{ob}(V.)\circ\eta$ *and* $\mathrm{ob}(V._{\,|\,D})$ *are same in the derived category.*

Proof It is easy to observe that $\mathfrak{g}(V_{\cdot|D}) \longrightarrow \Phi(V_{\cdot|D})^{*}L_{U\times D/D}$ is the restriction of $\mathfrak{g}(V.) \longrightarrow \Phi(V.)^{*}L_{U\times X/X}$ to D. Then the coincidence $\mathrm{ob}(V.)\circ\eta = \mathrm{ob}(V_{D}.)$ follows from the compatibility of of the traces for ω_X and ω_D. (See [1], for example). ∎

Thus, we obtain the following commutative diagram:

$$\begin{array}{ccccc}
\mathrm{Ob}_{\mathrm{rel}}\big(V.,F_*\big)[-1] & \longrightarrow & \mathrm{Ob}(V_{\cdot|D}) & \longrightarrow & \mathrm{Ob}(V.) \\
{\scriptstyle \mathrm{ob}_{\mathrm{rel}}(V.,F_*)}\Big\downarrow & & \Big\downarrow & & \Big\downarrow \\
L_{U_4/U}[-1] & \longrightarrow & F_4^{*}L_{U/k} & \longrightarrow & F_4^{*}L_{U/k}.
\end{array} \tag{110}$$

The following lemma can be shown by an argument to use the filtered objects as in the proof of Lemma 5.1.

Lemma 5.23 *The diagram* (110) *depends only on* $\big(E,F_*(E)\big)$. ∎

5.5.2 Relative obstruction property

For any U-scheme $g : T \longrightarrow U$, let $F(T)$ denote the set of the parabolic structure of $g_X^{*}E$ of a fixed type. Thus, we obtain the functor of the category of U-schemes to the category of sets. It is easy to see that the functor F is representable by a scheme M via the method of the quot schemes.

Let $\pi : M \longrightarrow U$ denote the projection. On $M\times X$, we have the universal parabolic structure F_*^u of $\pi_X^{*}E$. From the resolution $V.$ and F_*^u, we obtain the complex $\mathrm{Ob}_{\mathrm{rel}}(V.,F_*^u)$ and the morphism $\mathrm{ob}_{\mathrm{rel}}(V.,F_*^u) : \mathrm{Ob}_{\mathrm{rel}}(V.,F_*^u) \longrightarrow L_{M/U}$.

Lemma 5.24 $\mathrm{ob}_{\mathrm{rel}}(V.,F_*^u)$ *gives an obstruction theory of* M *over* U.

Proof It follows from Lemma 2.41. Note that $\mathfrak{g}_{\mathrm{rel}}(V.,F_*^u)$ is naturally isomorphic as the complex considered there. ∎

5.5.3 The decomposition into the trace-free part and the diagonal part

Lemma 5.25 *The morphism* $\mathrm{Ob}_{\mathrm{rel}}(V, F_*)[-1] \longrightarrow \mathrm{Ob}(V.)$ *factors through the trace free part* $\mathrm{Ob}^{\circ}(V.)$.

Proof We have the trace map $\mathrm{tr} : \mathfrak{g}_D(V., F_*) \longrightarrow \mathcal{O}_D[-1]$ given as follows:

$$\bigoplus_{i=1}^{l} \mathcal{H}om(V_D^{(i)}, V_D^{(i)}) \longrightarrow \mathcal{O}, \quad (f_i) \longmapsto \sum \mathrm{tr}(f_i)$$

We also have the map $i : \mathcal{O}_D[-1] \longrightarrow \mathfrak{g}_D(V, F_*)$ given as follows:

$$\mathcal{O} \longrightarrow \bigoplus_{i=1}^{l} \mathcal{H}om(V_D^{(i)}, V_D^{(i)}), \quad t \longmapsto (t \cdot \mathrm{id}_{V^{(1)}}, 0, \ldots, 0, -t \cdot \mathrm{id}_{V^{(l+1)}}).$$

We put $\mathfrak{g}_D^{\circ}(V., F_*) := \mathrm{Ker}(\mathrm{tr})$ and $\mathfrak{g}_D^{d}(V., F_*) := \mathrm{Im}(i)$. We also have the decomposition $\mathfrak{g}(V._{|D}) = \mathfrak{g}^{\circ}(V._{|D}) \oplus \mathfrak{g}^{d}(V._{|D})$ as in the subsubsection 5.1.2. We have the following commutative diagrams:

$$\begin{array}{ccc} \mathcal{O}[-1] & \longrightarrow & \mathcal{O}[-1] \\ {\scriptstyle i}\downarrow & & {\scriptstyle i}\downarrow \\ \mathfrak{g}(V._{|D}) & \longrightarrow & \mathfrak{g}_D(V., F_*) \end{array} \qquad \begin{array}{ccc} \mathfrak{g}(V._{|D}) & \longrightarrow & \mathfrak{g}_D(V., F_*) \\ {\scriptstyle \mathrm{tr}}\downarrow & & {\scriptstyle \mathrm{tr}}\downarrow \\ \mathcal{O}[-1] & \longrightarrow & \mathcal{O}[-1] \end{array}$$

Therefore, the decomposition is compatible with $\mathfrak{g}(V._{|D}) \longrightarrow \mathfrak{g}_D(V., F_*)$. It follows from $\mathfrak{g}_{\mathrm{rel}}(V., F_*)[-1] \longrightarrow \mathfrak{g}(V._{|D})$ factors through $\mathfrak{g}^{\circ}(V._{|D})$. Therefore, $\mathrm{Ob}_{\mathrm{rel}}(V, F_*)[-1] \longrightarrow \mathrm{Ob}(V._{|D})$ factors through $\mathrm{Ob}^{\circ}(V._{|D})$. It is easy to see that the morphism $\mathrm{Ob}(V._{|D}) \longrightarrow \mathrm{Ob}(V.)$ is compatible with the decomposition into the trace-free part and the diagonal part. Thus we are done. ∎

As an immediate corollary, we obtain the decomposition of the cone:

$$\mathrm{Cone}\Big(\mathrm{Ob}_{\mathrm{rel}}(V., F_*) \longrightarrow \mathrm{Ob}(V.)\Big) \simeq \mathrm{Cone}\Big(\mathrm{Ob}_{\mathrm{rel}}(V., F_*) \longrightarrow \mathrm{Ob}^{\circ}(V.)\Big) \oplus \mathrm{Ob}_{\mathrm{rel}}^{d}(V.) \tag{111}$$

5.6 Obstruction Theory for Moduli Stacks of Stable Objects

5.6.1 Relative complexes

Let y be an element of $H^*(X)$. Let $\boldsymbol{y}$ be an element of $\mathcal{T}ype$ whose $H^*(X)$-component is y. Let $\mathcal{M}(m, \boldsymbol{y})$ be the open subset of $\mathcal{M}(\boldsymbol{y})$ determined by the condition O_m. We have the natural morphism $\mathfrak{p}_1 : \mathcal{M}(m, \boldsymbol{y}) \longrightarrow \mathcal{M}(m, y)$. On $\mathcal{M}(m, \boldsymbol{y}) \times X$, we have the universal sheaf $\mathfrak{p}_1^*{}_X \mathcal{E}^u$ over $\mathcal{M}(m, \boldsymbol{y}) \times X$ with the parabolic structure F_*^u at D. From the resolution $\mathcal{V}.$ of $\mathcal{E}^u(m)$ and the parabolic structure F, we obtain the complex $\mathrm{Ob}_{\mathrm{rel}}(\mathcal{V}., F_*)$ and the morphism:

$$\mathrm{ob}_{\mathrm{rel}}(m, \boldsymbol{y}) : \mathrm{Ob}_{\mathrm{rel}}(m, \boldsymbol{y}) \longrightarrow L_{\mathcal{M}(m,\boldsymbol{y})/\mathcal{M}(m,y)}$$

Let $\mathcal{M}(m, y, L)$ denote the open subset of $\mathcal{M}(y, L)$ determined by the condition O_m. The natural morphism $\mathcal{M}(m, y, L) \longrightarrow \mathcal{M}(m, y)$ is denoted by $\mathfrak{p}_2$. We have the universal L-section ϕ^u of $\mathfrak{p}_2^*{}_X \mathcal{E}^u$. It induces the $L(m)$-section of $\mathfrak{p}_2^*{}_X \mathcal{E}^u(m)$, which is also denoted by ϕ^u. We fix an inclusion $\iota : \mathcal{O}(-m) \longrightarrow L$. If m is sufficiently large, we may assume that $L(m)$ has a locally free resolution $P. = (P_{-1} \to P_0)$ such that P_0 is a direct sum of some $\mathcal{O}_X$. Since we have the isomorphism $p_{X\,*}\big(\mathcal{E}^u(m)\big) \simeq p_{X\,*}(\mathcal{V}_0)$, the $L(m)$-section ϕ^u is canonically lifted to the morphism $\widetilde{\phi}^u_{\cdot} : p^*_{\mathcal{M}(m,y,L)} P. \longrightarrow \mathfrak{p}_2^*{}_X \mathcal{V}..$ Then we obtain the morphism:

$$\mathrm{ob}_{\mathrm{rel}}(m, y, L) : \mathrm{Ob}_{\mathrm{rel}}(m, y, L) \longrightarrow L_{\mathcal{M}(m,y,L)/\mathcal{M}(m,y)}$$

Let $\mathcal{M}(m,y,[L])$ denote the open subset of $\mathcal{M}(y,[L])$ determined by the condition O_m. The natural morphism $\mathcal{M}(m,y,[L]) \longrightarrow \mathcal{M}(m,y)$ is denoted by $\mathfrak{p}_3$. Then we have the universal reduced $[L]$-section $[\phi^u]$ of $\mathfrak{p}_{3\,X}^*\mathcal{E}^u$ over $\mathcal{M}(m,y,[L]) \times X$. As before, we obtain the morphism:

$$\mathrm{ob}_{\mathrm{rel}}\big(m,y,[L]\big) : \mathrm{Ob}_{\mathrm{rel}}\big(m,y,[L]\big) \longrightarrow L_{\mathcal{M}(m,y,[L])/\mathcal{M}(m,y)}.$$

Let $\mathcal{M}(m,\widehat{y})$ denote the open subset of $\mathcal{M}(\widehat{y})$ determined by the condition O_m. The natural morphism $\mathcal{M}(m,\widehat{y}) \longrightarrow \mathcal{M}(m,y)$ is denoted by $\mathfrak{p}_4$. Then we have the universal orientation ρ^u of $\mathfrak{p}_{4\,X}^*\mathcal{E}^u$ over $\mathcal{M}(m,\widehat{y})\times X$. From the resolution $\mathcal{V}.$ and the orientation ρ^u, we obtain the morphism:

$$\mathrm{ob}_{\mathrm{rel}}(m,\widehat{y}) : \mathrm{Ob}_{\mathrm{rel}}(m,\widehat{y}) \longrightarrow L_{\mathcal{M}(m,\widehat{y})/\mathcal{M}(m,y)}.$$

5.6.2 Construction of the complexes and the morphisms

We have the naturally defined morphisms of $\mathcal{M}(m,\widehat{\boldsymbol{y}},[L])$ to $\mathcal{M}(m,y)$, $\mathcal{M}(m,\boldsymbol{y})$, $\mathcal{M}(m,y,[L])$ and $\mathcal{M}(m,\widehat{y})$. The pull back of the complexes $\mathrm{Ob}(m,y)$, $\mathrm{Ob}_{\mathrm{rel}}\big(m,\boldsymbol{y}\big)$, $\mathrm{Ob}_{\mathrm{rel}}\big(m,y,[L]\big)$ and $\mathrm{Ob}_{\mathrm{rel}}\big(m,\widehat{y}\big)$ are denoted by the same notation. Then, we put as follows on $\mathcal{M}(m,\widehat{\boldsymbol{y}},[L])$:

$$\mathrm{Ob}(m,\widehat{\boldsymbol{y}},[L]) := \mathrm{Cone}\Big(\mathrm{Ob}_{\mathrm{rel}}\big(m,\widehat{y}\big)[-1] \oplus \mathrm{Ob}_{\mathrm{rel}}\big(m,y,[L]\big)[-1] \oplus \mathrm{Ob}_{\mathrm{rel}}\big(m,\boldsymbol{y}\big)[-1] \longrightarrow \mathrm{Ob}\big(m,y\big)\Big)$$

Proposition 5.26 *We have the naturally defined morphism* $\mathrm{ob}(m,\widehat{\boldsymbol{y}},[L]) : \mathrm{Ob}(m,\widehat{\boldsymbol{y}},[L]) \longrightarrow L_{\mathcal{M}(m,\widehat{\boldsymbol{y}},[L])}$.

Proof We put $C := \mathrm{Ob}_{\mathrm{rel}}\big(m,\widehat{y}\big) \oplus \mathrm{Ob}_{\mathrm{rel}}\big(m,y,[L]\big) \oplus \mathrm{Ob}_{\mathrm{rel}}\big(m,\boldsymbol{y}\big)$. Due to the diagrams (94), (102) and (110), we have the following commutative diagram:

$$\begin{array}{ccc}
C[-1] & \longrightarrow & \mathrm{Ob}(\mathcal{V}.) \\
\big\downarrow & & \big\downarrow \\
L_{\mathcal{M}(m,\widehat{\boldsymbol{y}},[L])/\mathcal{M}(m,y)}[-1] & \longrightarrow & L_{\mathcal{M}(m,y)/k}
\end{array}$$

It induces the desired morphism. ∎

Similarly, on the moduli stack $\mathcal{M}(m,\boldsymbol{y},L)$, we put as follows:

$$\mathrm{Ob}\big(m,\boldsymbol{y},L\big) := \mathrm{Cone}\Big(\mathrm{Ob}_{\mathrm{rel}}(m,y,L)[-1] \oplus \mathrm{Ob}_{\mathrm{rel}}\big(m,\boldsymbol{y}\big)[-1] \longrightarrow \mathrm{Ob}\big(m,y\big)\Big).$$

By using the sequence of the morphisms, $\mathcal{M}(m,\boldsymbol{y},L) \longrightarrow \mathcal{M}(m,\boldsymbol{y}) \longrightarrow \mathcal{M}(m,y)$, we obtain the morphism: $\mathrm{ob}\big(m,\boldsymbol{y},L\big) : \mathrm{Ob}\big(m,\boldsymbol{y},L\big) \longrightarrow L_{\mathcal{M}(m,\boldsymbol{y},L)/k}$.

We put as follows on the moduli $\mathcal{M}(m,\widehat{\boldsymbol{y}})$:

$$\mathrm{Ob}\big(m,\widehat{\boldsymbol{y}}\big) := \mathrm{Cone}\Big(\mathrm{Ob}_{\mathrm{rel}}(m,\widehat{y})[-1] \oplus \mathrm{Ob}_{\mathrm{rel}}(m,\boldsymbol{y})[-1] \longrightarrow \mathrm{Ob}(m,y)\Big).$$

Then we obtain the morphism $\mathrm{ob}\big(m,\widehat{\boldsymbol{y}}\big) : \mathrm{Ob}\big(m,\widehat{\boldsymbol{y}}\big) \longrightarrow L_{\mathcal{M}(m,\widehat{\boldsymbol{y}})/k}$.

Let $\boldsymbol{L} = (L_1, L_2)$ be a tuple of line bundles on X. We put as follows, on $\mathcal{M}(m,\widehat{\boldsymbol{y}},[\boldsymbol{L}])$:

$$\mathrm{Ob}\big(m,\widehat{\boldsymbol{y}},[\boldsymbol{L}]\big) := \mathrm{Cone}\Big(\mathrm{Ob}_{\mathrm{rel}}(m,\widehat{y})[-1] \oplus \mathrm{Ob}_{\mathrm{rel}}(m,\boldsymbol{y})[-1] \oplus \bigoplus_{i=1,2} \mathrm{Ob}_{\mathrm{rel}}(m,y,[L_i])[-1]. \longrightarrow \mathrm{Ob}(m,y)\Big).$$

Then we obtain the morphism $\mathrm{ob}\big(m,\widehat{\boldsymbol{y}},[\boldsymbol{L}]\big) : \mathrm{Ob}\big(m,\widehat{\boldsymbol{y}},[\boldsymbol{L}]\big) \longrightarrow L_{\mathcal{M}(m,\widehat{\boldsymbol{y}},[\boldsymbol{L}])/k}$.

5.6.3 Obstruction theory of the quot scheme and the moduli stacks

Let $Q^{\circ}(m,\widehat{\boldsymbol{y}},[L])$ be as in the subsubsection 3.6.7. Let $\pi : Q^{\circ}(m,\widehat{\boldsymbol{y}},[L]) \longrightarrow \mathcal{M}(m,\widehat{\boldsymbol{y}},[L])$ denote the projection. On $Q^{\circ}(m,\widehat{\boldsymbol{y}},[L])$, we put as follows:

$$\mathrm{Ob}^{Q}(m,\widehat{\boldsymbol{y}},[L]) := \mathrm{Cone}\Big(\pi^{*}\,\mathrm{Ob}_{\mathrm{rel}}(m,\widehat{y}) \oplus \pi^{*}\,\mathrm{Ob}_{\mathrm{rel}}(m,y,[L]) \oplus \pi^{*}\,\mathrm{Ob}_{\mathrm{rel}}(m,\boldsymbol{y}) \longrightarrow \mathrm{Ob}(V_{-1},f)\Big)$$

Then we obtain the morphism $\mathrm{ob}^{Q}(m,\widehat{\boldsymbol{y}},[L]) : \mathrm{Ob}^{Q}(m,\widehat{\boldsymbol{y}},[L]) \longrightarrow L_{Q^{\circ}(m,\widehat{\boldsymbol{y}},[L])/k}$ by an argument as in the subsubsection 5.6.2.

Proposition 5.27 *The morphism* $\mathrm{ob}^{Q}(m,\widehat{\boldsymbol{y}},[L])$ *gives an obstruction theory of* $Q^{\circ}(m,\widehat{\boldsymbol{y}},[L])$.

Proof It follows from Proposition 2.38, Remark 2.40, Lemma 5.24, 5.18, and Lemma 5.7. ∎

Proposition 5.28 *The morphism* $\mathrm{ob}(m,\widehat{\boldsymbol{y}},[L])$ *gives an obstruction theory of* $\mathcal{M}(m,\widehat{\boldsymbol{y}},[L])$ *over* k.

Proof We have $\pi^{*}\mathcal{H}^{i}\big(\mathrm{Ob}(m,\widehat{\boldsymbol{y}},[L])\big) \simeq \mathcal{H}^{i}\big(\mathrm{Ob}^{Q}(m,\widehat{\boldsymbol{y}},[L])\big)$ for $i<0$. We also have the following diagram from Lemma 5.3 and the construction of the complexes:

$$\begin{tikzcd}
\pi^{*}\mathcal{H}^{0}\big(\mathrm{Ob}(m,\widehat{\boldsymbol{y}},[L])\big) \arrow[r] \arrow[d] & \mathcal{H}^{0}\big(\mathrm{Ob}^{Q}(m,\widehat{\boldsymbol{y}},[L])\big) \arrow[r] \arrow[d, "\simeq"'] & \mathcal{H}om(V',V') \arrow[r] \arrow[d, "\simeq"'] & \pi^{*}\mathcal{H}^{1}\big(\mathrm{Ob}(m,\widehat{\boldsymbol{y}},[L])\big) \arrow[d] \\
\pi^{*}\mathcal{H}^{0}(L_{\mathcal{M}(m,\widehat{\boldsymbol{y}},[L])/k}) \arrow[r] & \mathcal{H}^{0}(L_{Q^{\circ}(m,\widehat{\boldsymbol{y}},[L])/k}) \arrow[r] & \mathcal{H}om(V',V') \arrow[r] & \pi^{*}\mathcal{H}^{1}(L_{\mathcal{M}(m,\widehat{\boldsymbol{y}},[L])/k})
\end{tikzcd}$$

We also remark that $\mathcal{H}^{1}\big(\mathrm{Ob}^{Q}(m,\boldsymbol{y},[L])\big) = \mathcal{H}^{1}\big(L_{Q^{\circ}(m,\boldsymbol{y},[L])/k}\big) = 0$ and $\mathcal{H}^{-1}(L_{Q(m)/\mathcal{M}(m)}) = 0$. Therefore, the claim follows from Proposition 5.27. ∎

By a similar argument, we obtain the following:

Proposition 5.29 $\mathrm{ob}(m,\boldsymbol{y},L)$, $\mathrm{ob}(m,\widehat{\boldsymbol{y}})$ *and* $\mathrm{ob}(m,\widehat{\boldsymbol{y}},[\boldsymbol{L}])$ *give obstruction theories of* $\mathcal{M}(m,\boldsymbol{y},L)$, $\mathcal{M}(m,\widehat{\boldsymbol{y}})$ *and* $\mathcal{M}(m,\widehat{\boldsymbol{y}},[\boldsymbol{L}])$ *respectively.* ∎

5.6.4 Obstruction theory of the moduli stacks of stable objects

Let α_{*} be a system of weight, and let δ be an element of $\mathcal{P}^{\mathrm{br}}$. Take a sufficiently large integer m. Then $\mathcal{M}^{s}(\widehat{\boldsymbol{y}},[L],\alpha_{*},\delta)$ is the open substack of $\mathcal{M}(m,\widehat{\boldsymbol{y}},[L])$.

Proposition 5.30 *The restriction of* $\mathrm{ob}(m,\widehat{\boldsymbol{y}},[L])$ *gives an obstruction theory of* $\mathcal{M}^{s}(\widehat{\boldsymbol{y}},[L],\delta,\alpha_{*})$. *It is independent of a choice of* m.

Proof The first claim follows from Proposition 5.28. The second claim follows from Lemma 5.5, Lemma 5.15 and Lemma 5.23. ∎

By the same argument, we can show the following proposition:

Proposition 5.31 *Let* $\boldsymbol{y}$ *be an element of* $\mathcal{T}ype$. *Let* α_{*} *be a system of weights. We take a large integer* m *appropriately in the following claims.*

- *The morphism* $\mathrm{ob}(m,\widehat{\boldsymbol{y}}) : \mathrm{Ob}(m,\widehat{\boldsymbol{y}}) \longrightarrow L_{\mathcal{M}^{s}(\widehat{\boldsymbol{y}},\alpha_{*})/k}$ *gives an obstruction theory.*
- *Let* δ *be an element of* $\mathcal{P}^{\mathrm{br}}$. *The morphism* $ob\big(m,\boldsymbol{y},L\big) : \mathrm{Ob}(m,\boldsymbol{y},L) \longrightarrow L_{\mathcal{M}^{s}(\boldsymbol{y},L,\alpha_{*},\delta)}$ *gives an obstruction theory.*
- *Let* $\boldsymbol{\delta} = (\delta_1,\delta_2)$ *be an element of* $\mathcal{P}^{\mathrm{br}}$. *The morphism* $\mathrm{ob}\big(m,\boldsymbol{y},[\boldsymbol{L}]\big) : \mathrm{Ob}(m,\boldsymbol{y},[\boldsymbol{L}]) \longrightarrow L_{\mathcal{M}^{s}(\widehat{\boldsymbol{y}},[\boldsymbol{L}],\alpha_{*},\boldsymbol{\delta})/k}$ *gives an obstruction theory.* ∎

5.7 Obstruction Theory for the Enhanced Master Space and the Related Stacks

5.7.1 The enhanced master space

Take a sufficiently large number m. We put $N := H_y(m)$. We take an N-dimensional vector space V_m. We put $\mathbb{P}_m := \mathbb{P}(V_m^\vee)$. We put $Z_1 := \mathbb{P}\big(\mathcal{O}_{\mathbb{P}_m}(0) \oplus \mathcal{O}_{\mathbb{P}_m}(1)\big)$ over $\mathbb{P}_m$. We have the natural right $\mathrm{GL}(V_m)$-action on $Z_2 := Z_1 \times \mathrm{Flag}(V_m, \underline{N})$, where $\mathrm{Flag}(V_m, \underline{N})$ denotes the full flag variety of V_m as in (54). The quotient stack is denoted by $\widetilde{\mathcal{Q}}$. The quotient stack $(\mathbb{P}_m)_{\mathrm{GL}(V_m)}$ is same as $B(W_\cdot, [P_\cdot])$ in the subsubsection 5.4.3, when $W_0 = V_m$. So we use the notation.

Let us fix an inclusion $\iota : \mathcal{O}(-m) \longrightarrow L$. Since reduced L-sections induce $\mathcal{O}(-m)$-sections, we obtain the morphism $\Xi(\mathcal{V}_\cdot, [\overline{\phi}]) : \mathcal{M}(m, \widehat{\boldsymbol{y}}, [L]) \longrightarrow B(W_\cdot, [P_\cdot])$. For simplicity of the notation, we use Ψ_1 instead of $\Xi(\mathcal{V}_\cdot, [\overline{\phi}])$. We use the following lemma in the construction of the deformation theory of the enhanced master space.

Lemma 5.32 *We have the morphism $\varphi : \Psi_1^* L_{B(W_\cdot,[P_\cdot])/k} \longrightarrow \mathrm{Ob}\big(m, \widehat{\boldsymbol{y}}, [L]\big)$ such that the composite of φ and $\mathrm{ob}\big(m, \widehat{\boldsymbol{y}}, [L]\big)$ is same as the naturally defined morphism $\Psi_1^* L_{B(W_\cdot,[P_\cdot])/k} \longrightarrow L_{\mathcal{M}(m,\widehat{\boldsymbol{y}},[L])/k}$.*

Proof Recall the complexes $\mathrm{Ob}^G(\mathcal{V}_\cdot)$ and $\mathrm{Ob}^G_{\mathrm{rel}}(\mathcal{V}_\cdot, [\widetilde{\phi}])$ constructed in the subsubsection 5.1.3 and the subsubsection 5.4.3 respectively. We put $\mathrm{Ob}^G(\mathcal{V}_\cdot, [\widetilde{\phi}]) = \mathrm{Cone}\big(\mathrm{Ob}^G_{\mathrm{rel}}(\mathcal{V}_\cdot, [\widetilde{\phi}])[-1] \longrightarrow \mathrm{Ob}^G(\mathcal{V})\big)$. Then we obtain the following commutative diagram from (107) and (108):

$$
\begin{array}{ccc}
\mathrm{Ob}^G(\mathcal{V}_\cdot, [\widetilde{\phi}]) & \longrightarrow & \mathrm{Ob}\big(m, \widehat{\boldsymbol{y}}, [L]\big) \\
{\scriptstyle \varphi_1}\big\downarrow & & \big\downarrow \\
\Psi_1^* L_{B(W_\cdot,[P_\cdot])/k} & \longrightarrow & L_{\mathcal{M}(m,\widehat{\boldsymbol{y}},[L])/k}.
\end{array}
$$

It is easy to show that φ_1 is isomorphic due to Lemma 5.2 and Lemma 5.20. ∎

The fiber product $\mathcal{M}(m, \widehat{\boldsymbol{y}}, [L]) \times_{B(W_\cdot,[P_\cdot])} \widetilde{\mathcal{Q}}$ is denoted by $\mathcal{N}$. By construction, the enhanced master space $\widehat{M}$ is an open subset of $\mathcal{N}$. The induced morphism $\mathcal{N} \longrightarrow \widetilde{\mathcal{Q}}$ is denoted by Ψ_2. Let $\mathfrak{p}$ denote the naturally defined morphism of $\widehat{M}$ to $\mathcal{M}(m, \widehat{\boldsymbol{y}}, [L])$. Then we obtain the following morphism of the distinguished triangles on $\widehat{M}$:

$$
\begin{array}{ccccccc}
\Psi_2^* L_{\widetilde{\mathcal{Q}}/B(W_\cdot,[P_\cdot])}[-1] & \longrightarrow & \mathfrak{p}^* \Psi_1^* L_{B(W_\cdot,[P_\cdot])/k} & \longrightarrow & \Psi_2^* L_{\widetilde{\mathcal{Q}}/k} & \longrightarrow & \Psi_2^* L_{\widetilde{\mathcal{Q}}/B(W_\cdot,[P_\cdot])} \\
{\scriptstyle \simeq}\big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow \\
L_{\widehat{M}/\mathcal{M}(m,\widehat{\boldsymbol{y}},[L])}[-1] & \longrightarrow & \mathfrak{p}^* L_{\mathcal{M}(m,\widehat{\boldsymbol{y}},[L])/k} & \longrightarrow & L_{\widehat{M}/k} & \longrightarrow & L_{\widehat{M}/\mathcal{M}(m,\widehat{\boldsymbol{y}},[L])}
\end{array}
\tag{112}
$$

From Lemma 5.32 and the diagram (112), we obtain the morphism $\Psi_2^* L_{\widetilde{\mathcal{Q}}/B(W_\cdot,[P_\cdot])}[-1] \longrightarrow \mathfrak{p}^* \mathrm{Ob}(m, \boldsymbol{y}, [L])$. The cone is denoted by $\mathrm{Ob}(\widehat{M})$. We have the induced morphism $\mathrm{ob}(\widehat{M}) : \mathrm{Ob}(\widehat{M}) \longrightarrow L_{\widehat{M}/k}$.

Proposition 5.33 *The morphism $\mathrm{ob}(\widehat{M})$ gives an obstruction theory of the master space $\widehat{M}$.*

Proof We put $\mathcal{M} := \mathcal{M}(m, \widehat{\boldsymbol{y}}, [L])$. By construction, we have the following morphism of distinguished triangles:

$$
\begin{array}{ccccccc}
L_{\widehat{M}/\mathcal{M}}[-1] & \longrightarrow & \mathfrak{p}^* \mathrm{Ob}(m, \widehat{\boldsymbol{y}}, [L]) & \longrightarrow & \mathrm{Ob}(\widehat{M}) & \longrightarrow & L_{\widehat{M}/\mathcal{M}} \\
{\scriptstyle \simeq}\big\downarrow & & \big\downarrow & & \big\downarrow & & {\scriptstyle \simeq}\big\downarrow \\
L_{\widehat{M}/\mathcal{M}}[-1] & \longrightarrow & \mathfrak{p}^* L_{\mathcal{M}/k} & \longrightarrow & L_{\widehat{M}/k} & \longrightarrow & L_{\widehat{M}/\mathcal{M}}
\end{array}
\tag{113}
$$

Then the claim follows from Proposition 5.28. ∎

5.7.2 The substack $\widehat{M}^*$

We have the natural $\mathrm{GL}(V_m)$-action on $V_m^* \times \mathrm{Flag}(V_m, \underline{N})$. The quotient stack is denoted by $\widetilde{\mathcal{Q}}^*$. We put $B(W_\cdot, P_\cdot) := (V_m^*)_{\mathrm{GL}(V_m)}$. We have the natural morphism $\widetilde{\mathcal{Q}}^* \longrightarrow B(W_\cdot, P_\cdot)$ which is a full flag bundle. Since V_m^* naturally gives the open subset $Z_1 - \mathbb{P}\big(\mathcal{O}_{\mathbb{P}}(0)\big) \cup \mathbb{P}\big(\mathcal{O}_{\mathbb{P}}(1)\big)$ of Z_1, the stack $\widetilde{\mathcal{Q}}^*$ naturally gives the open subset of $\widetilde{\mathcal{Q}}$.

Recall that we put $\widehat{M}^* := \widehat{M} - \big(\widehat{M}_1 \cup \widehat{M}_2\big)$ (the subsection 4.3). The stack $\widehat{M}^*$ is an open subset of $\mathcal{M}(m, \widehat{\boldsymbol{y}}, L) \times_{B(W_\cdot, P_\cdot)} \widetilde{\mathcal{Q}}^*$ by construction. We have the commutative diagrams:

$$
\begin{tikzcd}
\widehat{M}^* \arrow[r, "\Psi_4"] \arrow[d, "\mathfrak{p}_2"'] & \widetilde{\mathcal{Q}}^* \arrow[d] \\
\mathcal{M}(m, \widehat{\boldsymbol{y}}, L) \arrow[r, "\Psi_3"] & B(W_\cdot, P_\cdot)
\end{tikzcd}
\qquad
\begin{tikzcd}
\Psi_4^* L_{\widetilde{\mathcal{Q}}^*/B(W_\cdot, P_\cdot)}[-1] \arrow[r] \arrow[d, "\simeq"'] & \mathfrak{p}_2^* \Psi_3^* L_{B(W_\cdot, P_\cdot)} \arrow[d, "\varphi"'] \\
L_{\widehat{M}^*/\mathcal{M}(m, \widehat{\boldsymbol{y}}, L)}[-1] \arrow[r] & \mathfrak{p}_2^* L_{\mathcal{M}(m, \widehat{\boldsymbol{y}}, L)}
\end{tikzcd}
$$

Lemma 5.34 *φ factors through $\mathfrak{p}_2^* \mathrm{Ob}(m, \widehat{\boldsymbol{y}}, L)$.*

Proof Recall the complexes $\mathrm{Ob}^G(\mathcal{V}_\cdot)$ and $\mathrm{Ob}^G_{\mathrm{rel}}(\mathcal{V}_\cdot, \widetilde{\phi})$ constructed in the subsubsection 5.1.3 and the subsubsection 5.3.3, respectively. We put $\mathrm{Ob}^G(\mathcal{V}_\cdot, \widetilde{\phi}) := \mathrm{Cone}\big(\mathrm{Ob}^G_{\mathrm{rel}}(\mathcal{V}_\cdot, \widetilde{\phi})[-1] \longrightarrow \mathrm{Ob}^G(\mathcal{V}_\cdot)\big)$. We obtain the following commutative diagram on $\mathcal{M}(m, \boldsymbol{y}, L)$:

$$
\begin{tikzcd}
\mathrm{Ob}^G(\mathcal{V}_\cdot, \widetilde{\phi}) \arrow[r] \arrow[d, "\varphi_2"'] & \mathrm{Ob}(m, \boldsymbol{y}, L) \arrow[d] \\
\Psi_3^* L_{B(W,P)/k} \arrow[r] & L_{\mathcal{M}(m, \boldsymbol{y}, L)}
\end{tikzcd}
$$

It is easy to show φ_2 is isomorphic by using Lemma 5.2 and Lemma 5.12. ∎

Since φ factors through $\mathrm{Ob}(m, \widehat{\boldsymbol{y}}, L)$, we obtain the morphism $\Psi_4^* L_{\widetilde{\mathfrak{Q}}/B(W,P)}[-1] \longrightarrow \mathrm{Ob}(m, \widehat{\boldsymbol{y}}, L)$. The cone is denoted by $\mathrm{Ob}(\widehat{M}^*)$. Then we obtain the morphism $\mathrm{ob}(\widehat{M}^*) : \mathrm{Ob}(\widehat{M}^*) \longrightarrow L_{\widehat{M}^*}$.

Lemma 5.35 *There exists the quasi isomorphism $u : \mathrm{Ob}(\widehat{M})_{|\widehat{M}^*} \longrightarrow \mathrm{Ob}(\widehat{M}^*)$ such that $\mathrm{ob}(\widehat{M}^*) \circ u = \mathrm{ob}(\widehat{M})$.*

Proof Let π_1 denote the natural morphism $\mathcal{M}(m, \widehat{\boldsymbol{y}}, L) \longrightarrow \mathcal{M}(m, \widehat{\boldsymbol{y}}, [L])$. By construction of $\mathrm{Ob}^G(\mathcal{V}_\cdot, [\widetilde{\phi}])$ and $\mathrm{Ob}^G(\mathcal{V}_\cdot, \widetilde{\phi})$, we have the following commutative diagram on $\mathcal{M}(m, \widehat{\boldsymbol{y}}, L)$:

$$
\begin{tikzcd}
\Phi_2^* L_{B(W,P)/k} & \mathrm{Ob}^G(\mathcal{V}, \widetilde{\phi}) \arrow[l, "\simeq"'] \arrow[r] & \mathrm{Ob}(m, \widehat{\boldsymbol{y}}, L) \\
\pi_1^* \Phi_1^* L_{B(W,[P])/k} \arrow[u, "\psi_1"] & \pi_1^* \mathrm{Ob}^G(\mathcal{V}, [\widetilde{\phi}]) \arrow[l, "\simeq"'] \arrow[r] \arrow[u, "\psi_2"] & \pi_1^* \mathrm{Ob}(m, \widehat{\boldsymbol{y}}, [L]) \arrow[u, "\psi_3"]
\end{tikzcd}
\tag{114}
$$

Moreover, the induced morphisms $\mathrm{Cone}(\psi_2) \longrightarrow \mathrm{Cone}(\psi_i)$ $(i = 1, 3)$ are isomorphisms. Hence, we obtain the following commutative diagram on $\widehat{M}^*$:

$$
\begin{tikzcd}
\Psi_4^* L_{\widetilde{\mathcal{Q}}^*/B(W,P)}[-1] \arrow[r] & \mathfrak{p}_2^* \Psi_3^* L_{B(W,P)} \arrow[r] & \mathfrak{p}_2^* \mathrm{Ob}(m, \widehat{\boldsymbol{y}}, L) \arrow[r] & \mathfrak{p}_2^* L_{\mathcal{M}(m, \widehat{\boldsymbol{y}}, L)} \\
\Psi_4^* L_{\widetilde{\mathcal{Q}}^*/B(W,[P])}[-1] \arrow[r] \arrow[u, "\mu_1"] & \mathfrak{p}^* \Psi_1^* L_{B(W,[P])} \arrow[r] \arrow[u, "\mu_2"] & \mathfrak{p}^* \mathrm{Ob}(m, \widehat{\boldsymbol{y}}, [L]) \arrow[r] \arrow[u, "\mu_3"] & \mathfrak{p}^* L_{\mathcal{M}(m, \widehat{\boldsymbol{y}}, [L])} \arrow[u]
\end{tikzcd}
$$

Moreover, the induced morphisms $\mathrm{Cone}(\mu_1) \longrightarrow \mathrm{Cone}(\mu_2) \longrightarrow \mathrm{Cone}(\mu_3)$ are quasi isomorphic. Then the claim of the lemma is clear. ∎

We have another description of $\mathrm{Ob}(\widehat{M}^*)$. When $V_m = W_0$, we have $B(W.) := \mathrm{Spec}(k)_{\mathrm{GL}(V_m)}$. We put $\overline{F} := \mathrm{Flag}(V_m, \underline{N})_{\mathrm{GL}(V_m)}$. We have the following commutative diagram:

$$\begin{tikzcd}
\widehat{M}^* \arrow[r, "\Psi_4"] \arrow[d, "\mathfrak{p}_2"'] & \widetilde{Q}^* \arrow[r, "\Gamma_1"] \arrow[d] & \overline{F} \arrow[d] \\
\mathcal{M}(m, \widehat{\boldsymbol{y}}, L) \arrow[r, "\Psi_2"] & B(W., P.) \arrow[r, "\Gamma_2"] & B(W.)
\end{tikzcd} \tag{115}$$

Then, we have the isomorphism $\Psi_4^* L_{\widetilde{Q}^*/B(W.,P.)} \simeq \Psi_4^* \Gamma_1^* L_{\overline{F}/B(W.)}$. We also obtain the following morphisms:

$$\Psi_2^* \Gamma_2^* L_{B(W.)} \longrightarrow \Psi_2^* L_{B(W.,P.)} \longrightarrow \mathrm{Ob}(m, \widehat{\boldsymbol{y}}, L)$$

Therefore, we obtain the morphism $\alpha : \Psi_4^* \Gamma_1^* L_{\overline{F}/B(W.)}[-1] \longrightarrow \mathfrak{p}_2^* \mathrm{Ob}(m, \widehat{\boldsymbol{y}}, L)$. We naturally obtain the following quasi-isomorphism:

$$\mathrm{Cone}(\alpha) \simeq \mathrm{Ob}(\widehat{M}^*) \tag{116}$$

5.7.3 The moduli stack $\widetilde{\mathcal{M}}(m, \widehat{\boldsymbol{y}}, [L])$

Let $\widetilde{\mathcal{M}}(m, \widehat{\boldsymbol{y}}, [L])$ be the moduli stack of the tuple $(E_*, \rho, [\phi], \mathcal{F})$ as follows:

- $(E_*, \rho, [\phi])$ is an oriented reduced L-Bradlow pair of type $\boldsymbol{y}$, satisfying the condition O_m.
- $\mathcal{F}$ is a full flag of $H^0(X, E(m))$.

By an argument in the subsubsection 5.7.1, we can obtain the obstruction theory of $\widetilde{\mathcal{M}}(m, \widehat{\boldsymbol{y}}, [L])$. We also use the notation $\widetilde{\mathcal{M}}$ and $\mathcal{M}$ to denote $\widetilde{\mathcal{M}}(m, \widehat{\boldsymbol{y}}, [L])$ and $\mathcal{M}(m, \widehat{\boldsymbol{y}}, [L])$, respectively. When $V_m = W_0$, we have $B(W.) := \mathrm{Spec}(k)_{\mathrm{GL}(V_m)}$. We also put $\overline{F} := \mathrm{Flag}(V_m)_{\mathrm{GL}(V_m)}$. The following diagram is Cartesian:

$$\begin{tikzcd}
\widetilde{\mathcal{M}}(m, \widehat{\boldsymbol{y}}, [L]) \arrow[r, "\Psi_{11}"] \arrow[d, "\mathfrak{p}_1"'] & \overline{F} \arrow[d] \\
\mathcal{M}(m, \widehat{\boldsymbol{y}}, [L]) \arrow[r, "\Psi_{12}"] & B(W.)
\end{tikzcd}$$

Then, we obtain the following morphism of distinguished triangles on $\widetilde{\mathcal{M}}(m, \widehat{\boldsymbol{y}}, [L])$:

$$\begin{tikzcd}
\Psi_{11}^* L_{\overline{F}/B(W.)}[-1] \arrow[r] \arrow[d, "\simeq"'] & \mathfrak{p}_1^* \Psi_{12}^* L_{B(W.)/k} \arrow[r] \arrow[d, "\varphi"'] & \Psi_{11}^* L_{\overline{F}/k} \arrow[r] \arrow[d] & \Psi_{11}^* L_{\overline{F}/B(W.)} \arrow[d, "\simeq"'] \\
L_{\widetilde{\mathcal{M}}/\mathcal{M}}[-1] \arrow[r] & \mathfrak{p}_1^* L_{\mathcal{M}/k} \arrow[r] & L_{\widetilde{\mathcal{M}}/k} \arrow[r] & L_{\widetilde{\mathcal{M}}/\mathcal{M}}
\end{tikzcd}$$

Lemma 5.36 *φ factors through $\mathfrak{p}_1^* \mathrm{Ob}(m, \widehat{\boldsymbol{y}}, [L])$*

Proof It can be shown by using the complex $\mathrm{Ob}^G(V.)$ and the argument in the proof of Lemma 5.32. ∎

Then, we obtain the morphism $\Psi_{11}^* L_{\overline{F}/B(W.)}[-1] \longrightarrow \mathfrak{p}_1^* \mathrm{Ob}(m, \widehat{\boldsymbol{y}}, [L])$. The cone is denoted by $\widetilde{\mathrm{Ob}}(m, \widehat{\boldsymbol{y}}, [L])$. We obtain the natural morphism:

$$\widetilde{\mathrm{ob}}\big(m, \widehat{\boldsymbol{y}}, [L]\big) : \widetilde{\mathrm{Ob}}(m, \widehat{\boldsymbol{y}}, [L]) \longrightarrow L_{\widetilde{\mathcal{M}}(m, \widehat{\boldsymbol{y}}, [L])/k}$$

By the same argument as the proof of Proposition 5.33, we can show the following.

Proposition 5.37 $\widetilde{\mathrm{ob}}\big(m, \widehat{\boldsymbol{y}}, [L]\big)$ *gives an obstruction theory for* $\widetilde{\mathcal{M}}\big(m, \boldsymbol{y}, [L]\big)$. ∎

We have the equivalent obstruction theory. Let $\mathcal{Q}_1$ denote the quotient stack of $\mathbb{P}_m \times \mathrm{Flag}(V, \underline{N})$ via the natural $\mathrm{GL}(V_m)$-action. Then, the following diagram is Cartesian:

$$\begin{array}{ccc}
\widetilde{\mathcal{M}}(m, \widehat{\boldsymbol{y}}, [L]) & \xrightarrow{\Psi_{13}} & \mathcal{Q}_1 \\
{\scriptstyle \mathfrak{p}_1}\downarrow & & \downarrow \\
\mathcal{M}(m, \widehat{\boldsymbol{y}}, [L]) & \xrightarrow{\Psi_{14}} & B(W, [P])
\end{array}$$

Then, we have the following diagram on $\widetilde{\mathcal{M}}(m, \widehat{\boldsymbol{y}}, [L])$:

$$\begin{array}{ccccccc}
\Psi_{13}^* L_{\mathcal{Q}_1/B(W,[P])}[-1] & \longrightarrow & \mathfrak{p}_1^* \Psi_{14}^* L_{B(W,[P])} & \longrightarrow & \Psi_{13}^* L_{\mathcal{Q}_1/k} & \longrightarrow & \Psi_{13}^* L_{\mathcal{Q}_1/B(W,[P])} \\
\downarrow & & {\scriptstyle \varphi_1}\downarrow & & \downarrow & & \downarrow \\
L_{\widetilde{\mathcal{M}}/\mathcal{M}}[-1] & \longrightarrow & \mathfrak{p}_1^* L_{\mathcal{M}/k} & \longrightarrow & L_{\widetilde{\mathcal{M}}/k} & \longrightarrow & L_{\widetilde{\mathcal{M}}/\mathcal{M}}
\end{array}$$

By an argument similar to the proof of Lemma 5.32, we can show that φ_1 factors through $\mathfrak{p}_1^* \mathrm{Ob}(m, \widehat{\boldsymbol{y}}, [L])$. Let $\widetilde{\mathrm{Ob}}_2(m, \widehat{\boldsymbol{y}}, [L])$ denote the cone of $\Psi_{13}^* L_{\mathcal{Q}_1/B(W,[P])}[-1] \longrightarrow \mathfrak{p}_1^* \mathrm{Ob}(m, \widehat{\boldsymbol{y}}, [L])$, and then we have the naturally defined morphism:

$$\widetilde{\mathrm{ob}}_2(m, \widehat{\boldsymbol{y}}, [L]) : \widetilde{\mathrm{Ob}}_2(m, \widehat{\boldsymbol{y}}, [L]) \longrightarrow L_{\widetilde{\mathcal{M}}(m, \widehat{\boldsymbol{y}}, [L])}$$

Lemma 5.38 *We have the natural quasi isomorphism* $\psi : \mathrm{Ob}(m, \widehat{\boldsymbol{y}}, [L]) \longrightarrow \mathrm{Ob}_2(m, \widehat{\boldsymbol{y}}, [L])$ *such that the composite* $\mathrm{ob}_2(m, \widehat{\boldsymbol{y}}, [L]) \circ \psi$ *is same as* $\mathrm{ob}(m, \widehat{\boldsymbol{y}}, [L])$.

Proof Let $\mathcal{V}.$ be the canonical resolution of $\mathcal{E}^u(m)$ over $\mathcal{M}(m, \widehat{\boldsymbol{y}}, [L])$. We obtain the complexes $\mathrm{Ob}^G(\mathcal{V}.)$ and $\mathrm{Ob}^G_{\mathrm{rel}}(\mathcal{V}., [\widetilde{\phi}])$ by the constructions in the subsubsection 5.1.3 and the subsubsection 5.4.3. Let $\mathrm{Ob}^G(\mathcal{V}., [\widetilde{\phi}])$ denote the cone of the morphism $\mathrm{Ob}^G_{\mathrm{rel}}(\mathcal{V}., [\widetilde{\phi}])[-1] \longrightarrow \mathrm{Ob}^G(\mathcal{V}.)$. We have the following commutative diagram on $\mathcal{M}(m, \widehat{\boldsymbol{y}}, [L])$:

$$\begin{array}{ccccc}
\Psi_{12}^* L_{B(W)} & \longrightarrow & \mathrm{Ob}^G(\mathcal{V}.) & \longrightarrow & \mathrm{Ob}(m, \widehat{\boldsymbol{y}}) \\
\downarrow & & \downarrow & & \downarrow \\
\Psi_{14}^* L_{B(W,[P])} & \longrightarrow & \mathrm{Ob}^G(\mathcal{V}., [\widetilde{\phi}]) & \longrightarrow & \mathrm{Ob}(m, \widehat{\boldsymbol{y}}, [L])
\end{array}$$

We have the commutative diagram:

$$\begin{array}{ccccc}
\widetilde{\mathcal{M}}(m, \widehat{\boldsymbol{y}}, [L]) & \longrightarrow & \mathcal{Q}_1 & \longrightarrow & \overline{F} \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{M}(m, \widehat{\boldsymbol{y}}, [L]) & \longrightarrow & B(W, [P]) & \longrightarrow & B(W)
\end{array}$$

We obtain the following diagram on $\widetilde{\mathcal{M}}(m, \widehat{\boldsymbol{y}}, [L])$:

$$\begin{array}{ccccccc}
\Psi_{13}^* L_{\mathcal{Q}_1/B(W,[P])}[-1] & \longrightarrow & \mathfrak{p}_1^* \Psi_{14}^* L_{B(W,[P])} & \longrightarrow & \mathfrak{p}_1^* \mathrm{Ob}(m, \widehat{\boldsymbol{y}}, [L]) & \longrightarrow & \mathfrak{p}_1^* L_{\mathcal{M}(m, \widehat{\boldsymbol{y}}, [L])} \\
{\scriptstyle \simeq}\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\Psi_{11}^* L_{\overline{F}/B(W)}[-1] & \longrightarrow & \mathfrak{p}_1^* \Psi_{12}^* L_{B(W)} & \longrightarrow & \mathfrak{p}_1^* \mathrm{Ob}(m, \widehat{\boldsymbol{y}}, [L]) & \longrightarrow & \mathfrak{p}_1^* L_{\mathcal{M}(m, \widehat{\boldsymbol{y}}, [L])}
\end{array}$$

Then, the claim is clear. ∎

Recall that the moduli stack $\widetilde{\mathcal{M}}^s(\widehat{\boldsymbol{y}}, [L], \alpha_*, (\delta, \ell))$ of (δ, ℓ)-stable objects is the open substack of $\widetilde{\mathcal{M}}(m, \widehat{\boldsymbol{y}}, [L])$. By restricting $\widetilde{\mathrm{ob}}(m, \widehat{\boldsymbol{y}}, [L])$, we obtain the obstruction theory of $\widetilde{\mathcal{M}}^s(\widehat{\boldsymbol{y}}, [L], \alpha_*, (\delta, \ell))$.

5.7.4 The moduli stack $\widetilde{\mathcal{M}}(m, \boldsymbol{y}, L)$

Let $\widetilde{\mathcal{M}}(m, \boldsymbol{y}, L)$ be the moduli stack of the tuple $(E_*, \phi, \mathcal{F})$ as follows:

- (E_*, ϕ) is an L-Bradlow pair of type $\boldsymbol{y}$, satisfying the condition O_m.
- $\mathcal{F}$ is a full flag of $H^0(X, E(m))$.

We use the notation in the subsubsection 5.7.3. We have the following Cartesian diagram:

$$\begin{array}{ccc} \widetilde{\mathcal{M}}(m, \boldsymbol{y}, L) & \xrightarrow{\Psi_{11}} & \overline{F} \\ {\scriptstyle \mathfrak{p}_1}\big\downarrow & & \big\downarrow \\ \mathcal{M}(m, \boldsymbol{y}, L) & \xrightarrow{\Psi_{12}} & B(W.) \end{array}$$

By using the construction in the subsubsection 5.7.3, we obtain the obstruction theory:

$$\widetilde{\mathrm{ob}}(m, \boldsymbol{y}, L) : \widetilde{\mathrm{Ob}}(m, \boldsymbol{y}, L) \longrightarrow L_{\widetilde{\mathcal{M}}(m, \boldsymbol{y}, L)}$$

The moduli stack $\widetilde{\mathcal{M}}^{ss}\big(\boldsymbol{y}, L, \alpha_*, (\delta, \ell)\big)$ (the subsubsection 3.3.3) is the open substack of $\widetilde{\mathcal{M}}(m, \boldsymbol{y}, L)$. By restricting $\widetilde{\mathrm{ob}}(m, \boldsymbol{y}, L)$, we obtain the obstruction theory of $\widetilde{\mathcal{M}}^{ss}\big(\boldsymbol{y}, L, \alpha_*, (\delta, \ell)\big)$.

5.7.5 The moduli stack $\widetilde{\mathcal{M}}^s(\widehat{\boldsymbol{y}}, \alpha_*, +)$

Let $\widetilde{\mathcal{M}}(m, \widehat{\boldsymbol{y}})$ denote the moduli stack of the objects $(E_*, \mathcal{F})$ as follows:

- E_* is a parabolic torsion-free sheaf satisfying the condition O_m.
- $\mathcal{F}$ is a full flag of $H^0(X, E(m))$.

We use the same notation in the subsubsection 5.7.3. In this case, we have the following Cartesian diagram:

$$\begin{array}{ccc} \widetilde{\mathcal{M}}(m, \widehat{\boldsymbol{y}}) & \xrightarrow{\Psi_{11}} & \overline{F} \\ \big\downarrow & & \big\downarrow \\ \mathcal{M}(m, \widehat{\boldsymbol{y}}) & \xrightarrow{\Psi_{12}} & B(W.). \end{array}$$

By using the construction in the subsubsection 5.7.3, we obtain the obstruction theory:

$$\widetilde{\mathrm{ob}}(m, \widehat{\boldsymbol{y}}) : \widetilde{\mathrm{Ob}}(m, \widehat{\boldsymbol{y}}) \longrightarrow L_{\widetilde{\mathcal{M}}(m, \widehat{\boldsymbol{y}})}$$

Recall $\widetilde{\mathcal{M}}^s(\widehat{\boldsymbol{y}}, \alpha_*, +)$ denotes the moduli stack of the objects $(E_*, \mathcal{F})$ as follows (the subsubsection 4.6.1):

- E_* is a parabolic torsion-free sheaf of type $\boldsymbol{y}$ with weight α_*.
- $\mathcal{F}$ is a full flag of $H^0(X, E(m))$.
- $(E_*, \mathcal{F}_{\min})$ is ϵ-semistable reduced $\mathcal{O}(-m)$-Bradlow pair, where ϵ denotes any sufficiently small positive number.

Since $\widetilde{\mathcal{M}}^s(\widehat{\boldsymbol{y}}, \alpha_*, +)$ is the open substack of $\widetilde{\mathcal{M}}(m, \widehat{\boldsymbol{y}}, \alpha_*)$, we obtain the obstruction theory of $\widetilde{\mathcal{M}}^s(\widehat{\boldsymbol{y}}, \alpha_*, +)$ by restricting $\widetilde{\mathrm{ob}}(m, \widehat{\boldsymbol{y}})$.

5.7.6 The case where the 2-stability condition is satisfied

Let us construct the obstruction theory of the master space in the case where the 2-stability condition is satisfied for $(\boldsymbol{y}, L, \alpha_*, \delta)$. It can be done by the way given in the subsubsection 5.7.1, so we give only an indication. We use the notation in the subsubsections 4.7.1 and 5.7.1.

We have the natural $\mathrm{GL}(V)$-action on the $\mathbb{P}^1$-bundle $\mathbb{P}\bigl(\mathcal{O}_{\mathbb{P}_m}(0) \oplus \mathcal{O}_{\mathbb{P}_m}(1)\bigr)$ over $\mathbb{P}_m$. The quotient stack is denoted by $\mathcal{Q}$. We have the map $\Psi_2 : \mathcal{M}(m, \widehat{\boldsymbol{y}}, [L]) \longrightarrow B(W., [P])$, and $\widehat{M}$ is an open substack of $\mathcal{M}(m, \widehat{\boldsymbol{y}}, [L]) \times_{B(W,[P])} \mathcal{Q}$.

$$\begin{tikzcd} \widehat{M} \arrow[r, "\Psi_1"] \arrow[d, "\mathfrak{p}"'] & \mathcal{Q} \arrow[d] \\ \mathcal{M}(m, \widehat{\boldsymbol{y}}, [L]) \arrow[r, "\Psi_2"] & B(W, [P]) \end{tikzcd}$$

We obtain the following morphism of distinguished triangles on $\widehat{M}$:

$$\begin{tikzcd} \Psi_1^* L_{\mathcal{Q}/B(W,[P])}[-1] \arrow[r] \arrow[d] & \mathfrak{p}^*\Psi_2^* L_{B(W,[P])} \arrow[r] \arrow[d, "\varphi"'] & \Psi_1^* L_{\mathcal{Q}} \arrow[r] \arrow[d] & \Psi_1^* L_{\mathcal{Q}/B(W,[P])} \arrow[d] \\ L_{\widehat{M}/\mathcal{M}(m,\widehat{\boldsymbol{y}},[L])}[-1] \arrow[r] & \mathfrak{p}^* L_{\mathcal{M}(m,\widehat{\boldsymbol{y}},[L])} \arrow[r] & L_{\widehat{M}} \arrow[r] & L_{\widehat{M}/\mathcal{M}(m,\widehat{\boldsymbol{y}},[L])} \end{tikzcd}$$

Since φ factors through $\mathfrak{p}^* \operatorname{Ob}(m, \widehat{\boldsymbol{y}}, [L])$ (Lemma 5.32), the morphism $\Psi_1^* L_{\mathcal{Q}/B(W,[P])}[-1] \longrightarrow \operatorname{Ob}(m, \widehat{\boldsymbol{y}}, [L])$ is obtained. The cone is denoted by $\operatorname{Ob}(\widehat{M})$. We have the naturally defined morphism $\operatorname{ob}(\widehat{M}) : \operatorname{Ob}(\widehat{M}) \longrightarrow L_{\widehat{M}}$. By an argument similar to the proof of Proposition 5.33, we can show that $\operatorname{ob}(\widehat{M})$ gives an obstruction theory of $\widehat{M}$.

Recall we put $\widehat{M}^* := \widehat{M} - \widehat{M}_1 \cup \widehat{M}_2$. It is an open substack of $\mathcal{M}(m, \widehat{\boldsymbol{y}}, L)$. We put $\operatorname{Ob}(\widehat{M}^*) := \operatorname{ob}(m, \widehat{\boldsymbol{y}}, L)$, and then we have the obstruction theory $\operatorname{ob}(\widehat{M}^*) : \operatorname{Ob}(\widehat{M}^*) \longrightarrow L_{\widehat{M}^*}$.

Lemma 5.39 *There exists the quasi isomorphism* $u : \operatorname{Ob}(\widehat{M})_{|\widehat{M}^*} \longrightarrow \operatorname{Ob}(\widehat{M}^*)$ *such that* $\operatorname{ob}(\widehat{M}^*) \circ u = \operatorname{ob}(\widehat{M})$.

Proof We have the commutative diagram:

$$\begin{tikzcd} \widehat{M}^* \arrow[r, "\Psi_1"] \arrow[d, "\pi"'] & B(W, P) \arrow[d] \\ \mathcal{M}(m, \widehat{\boldsymbol{y}}, [L]) \arrow[r, "\Psi_2"] & B(W, [P]) \end{tikzcd}$$

From the diagram (114), we obtain the following diagram:

$$\begin{tikzcd} \Psi_1^* L_{B(W,P)} \arrow[r] & \operatorname{Ob}(m, \widehat{\boldsymbol{y}}, L) \arrow[r] & L_{\widehat{M}^*} \\ \pi^*\Psi_2^* L_{B(W,[P])} \arrow[r] \arrow[u, "\mu_1"] & \pi^* \operatorname{Ob}(m, \widehat{\boldsymbol{y}}, [L]) \arrow[r] \arrow[u, "\mu_2"] & \pi^* L_{\mathcal{M}(m,\widehat{\boldsymbol{y}},[L])} \arrow[u, "\mu_3"] \end{tikzcd}$$

Moreover, the induced morphisms $\operatorname{Cone}(\mu_1) \longrightarrow \operatorname{Cone}(\mu_2) \longrightarrow \operatorname{Cone}(\mu_3)$ are quasi isomorphic. Then the claim of the lemma is clear. ∎

5.7.7 The case of oriented reduced $\boldsymbol{L}$-Bradlow pairs

Let $\boldsymbol{L} = (L_1, L_2)$ be a pair of line bundles over X. Let us construct the obstruction theory of the master space for the moduli stacks of the oriented reduced $\boldsymbol{L}$-Bradlow pairs, under the setting in the subsubsection 4.7.2. We give only an indication. We use the notation in the subsubsection 5.7.6.

We construct the master space $\widehat{M}$ as in the subsubsection 4.7.2. Let us take an inclusion $\iota_1 : \mathcal{O}(-m) \longrightarrow L_1$. Then the universal reduced L_1-section $[\phi_1^u]$ induces the reduced $\mathcal{O}_X$-section $[\overline{\phi}_1]$. Therefore, we obtain the morphism $\Psi_2 : \mathcal{M}(m, \widehat{\boldsymbol{y}}, [\boldsymbol{L}]) \longrightarrow B(W, [P])$. By construction, $\widehat{M}$ is an open subset of $\mathcal{M}(m, \widehat{\boldsymbol{y}}, [\boldsymbol{L}]) \times_{B(W,[P])} \mathcal{Q}$.

$$\begin{array}{ccc}
\widehat{M} & \xrightarrow{\Psi_1} & \mathcal{Q} \\
\mathfrak{p}\big\downarrow & & \big\downarrow \\
\mathcal{M}(m, \widehat{\boldsymbol{y}}, [\boldsymbol{L}]) & \xrightarrow{\Psi_2} & B(W, [P])
\end{array}$$

We obtain the following morphism of distinguished triangles on $\widehat{M}$:

$$\begin{array}{ccccccc}
\Psi_1^* L_{\mathcal{Q}/B(W,[P])}[-1] & \longrightarrow & \mathfrak{p}^* \Psi_2^* L_{B(W,[P])} & \longrightarrow & \Psi_1^* \mathcal{Q} & \longrightarrow & \Psi_1^* L_{\mathcal{Q}/B(W,[P])} \\
\big\downarrow & & \varphi\big\downarrow & & \big\downarrow & & \big\downarrow \\
L_{\widehat{M}/\mathcal{M}(m,\widehat{\boldsymbol{y}},[\boldsymbol{L}])}[-1] & \longrightarrow & \mathfrak{p}^* L_{\mathcal{M}(m,\widehat{\boldsymbol{y}},[\boldsymbol{L}])} & \longrightarrow & L_{\widehat{M}} & \longrightarrow & L_{\widehat{M}/\mathcal{M}(m,\widehat{\boldsymbol{y}},[\boldsymbol{L}])}
\end{array}$$

By an argument similar to the proof of Lemma 5.32, it can be shown that φ factors through $\mathfrak{p}^* \operatorname{Ob}(m, \widehat{\boldsymbol{y}}, [\boldsymbol{L}])$. Hence we obtain the morphism $\Psi_1^* L_{\mathcal{Q}/B(W,[P])}[-1] \longrightarrow \operatorname{Ob}(m, \widehat{\boldsymbol{y}}, [\boldsymbol{L}])$. The cone is denoted by $\operatorname{Ob}(\widehat{M})$. We have the naturally defined morphism $\operatorname{ob}(\widehat{M}) : \operatorname{Ob}(\widehat{M}) \longrightarrow L_{\widehat{M}}$. By an argument similar to the proof of Proposition 5.33, we can show that $\operatorname{ob}(\widehat{M})$ gives an obstruction theory of $\widehat{M}$.

Recall we put $\widehat{M}^* := \widehat{M} - \widehat{M}_1 \cup \widehat{M}_2$. It is an open substack of the moduli stack $\mathcal{M}(m, \widehat{\boldsymbol{y}}, L_1, [L_2])$. (See the subsubsection 4.7.2 for $\mathcal{M}(m, \widehat{\boldsymbol{y}}, L_1, [L_2])$.) On $\mathcal{M}(m, \widehat{\boldsymbol{y}}, L_1, [L_2])$, we have the following morphism:

$$\operatorname{Ob}_{\mathrm{rel}}(m, \boldsymbol{y})[-1] \oplus \operatorname{Ob}_{\mathrm{rel}}(m, y, L_1)[-1] \oplus \operatorname{Ob}_{\mathrm{rel}}(m, y, [L_2])[-1] \oplus \operatorname{Ob}_{\mathrm{rel}}(m, \widehat{y})[-1] \longrightarrow \operatorname{Ob}(m, y)$$

The cone is denoted by $\operatorname{Ob}(m, \widehat{\boldsymbol{y}}, L_1, [L_2])$. As in the subsubsection 5.6.2, we can naturally construct the morphism $\operatorname{ob}(m, \widehat{\boldsymbol{y}}, L_1, [L_2]) : \operatorname{Ob}(m, \widehat{\boldsymbol{y}}, L_1, [L_2]) \longrightarrow L_{\mathcal{M}(m,\widehat{\boldsymbol{y}},L_1,[L_2])}$. By an argument similar to the proof of Proposition 5.30, it can be shown that $\operatorname{ob}(m, \widehat{\boldsymbol{y}}, L_1, [L_2])$ gives an obstruction theory.

Lemma 5.40 *There exists the quasi isomorphism* $u : \operatorname{Ob}(\widehat{M})_{|\widehat{M}^*} \longrightarrow \operatorname{Ob}(\widehat{M}^*)$ *such that* $\operatorname{ob}(\widehat{M}^*) \circ u = \operatorname{ob}(\widehat{M})$.

Proof It can be shown by the same argument as the proof of Lemma 5.39. ∎

5.8 Moduli Theoretic Obstruction Theory of the Fixed Point Set

5.8.1 Statement

Let $\mathfrak{I} = (\boldsymbol{y}_1, \boldsymbol{y}_2, I_1, I_2)$ be a decomposition type as in Definition 4.33. We use the notation in the subsubsection 4.6.1. We put $\widetilde{\mathcal{M}}_{\mathrm{split}} := \widetilde{\mathcal{M}}^{ss}\big(\boldsymbol{y}_1, L, \alpha_*, (\delta, k_0)\big) \times \widetilde{\mathcal{M}}^{ss}(\widehat{\boldsymbol{y}}_2, \alpha_*, +)$. Recall that we constructed the obstruction theory $\widetilde{\operatorname{Ob}}(m, \boldsymbol{y}_1, L)$ of $\widetilde{\mathcal{M}}^{ss}\big(\boldsymbol{y}_1, L, \alpha_*, (\delta, k_0)\big)$ (the subsubsection 5.7.4) and the obstruction theory $\widetilde{\operatorname{Ob}}(m, \boldsymbol{y}_2)$ of $\widetilde{\mathcal{M}}^{ss}(\widehat{\boldsymbol{y}}_2, \alpha_*, +)$ (the subsubsection 5.7.5). The direct sum $\operatorname{Ob}(\widetilde{\mathcal{M}}_{\mathrm{split}})$ gives the obstruction theory of $\widetilde{\mathcal{M}}_{\mathrm{split}}$. The affine line $\operatorname{Spec} k[t]$ is denoted by A^1.

Proposition 5.41 *We have the obstruction theory* $\operatorname{ob}(\widehat{M}^{G_m}(\mathfrak{I})) : \operatorname{Ob}(\widehat{M}^{G_m}(\mathfrak{I})) \longrightarrow L_{\widehat{M}^{G_m}(\mathfrak{I})}$ *and the deformation* $\widetilde{\operatorname{ob}}(\widehat{M}^{G_m}(\mathfrak{I})) : \widetilde{\operatorname{Ob}}(\widehat{M}^{G_m}(\mathfrak{I})) \longrightarrow L_{\widehat{M}^{G_m}(\mathfrak{I}) \times A^1/A^1}$ *with the following property:*

- *We have the following commutative diagram:*

$$\begin{array}{ccc}
\varphi_{\mathfrak{I}}^* \operatorname{Ob}(\widehat{M}) & \longrightarrow & \varphi_{\mathfrak{I}}^* L_{\widehat{M}} \\
\big\downarrow & & \big\downarrow \\
\operatorname{Ob}(\widehat{M}^{G_m}(\mathfrak{I})) & \longrightarrow & L_{\widehat{M}^{G_m}(\mathfrak{I})}
\end{array} \tag{117}$$

- *Let* $\widetilde{\mathrm{ob}}_a(\widehat{M}^{G_m}(\mathfrak{I})) : \widetilde{\mathrm{Ob}}_a(\widehat{M}^{G_m}(\mathfrak{I})) \longrightarrow L_{\widehat{M}^{G_m}(\mathfrak{I})}$ *denote the specialization of* $\widetilde{\mathrm{ob}}(\widehat{M}^{G_m}(\mathfrak{I}))$ *at* $t=a$*. At* $t=1$*, we have* $\widetilde{\mathrm{ob}}_1(\widehat{M}^{G_m}(\mathfrak{I})) = \mathrm{ob}(\widehat{M}^{G_m}(\mathfrak{I}))$*. At* $t=0$*, we have the following commutative diagram in the diagram (79):*
$$\begin{array}{ccc} F^*\widetilde{\mathrm{Ob}}_0(\widehat{M}^{G_m}(\mathfrak{I})) & \longrightarrow & F^*L_{\widehat{M}^{G_m}(\mathfrak{I})} \\ \simeq\big\downarrow & & \simeq\big\downarrow \\ G'^*\,\mathrm{Ob}(\widetilde{\mathcal{M}}_{\mathrm{split}}) & \longrightarrow & G'^*L_{\widetilde{\mathcal{M}}_{\mathrm{split}}} \end{array} \tag{118}$$
On each a*, we have the following distinguished triangle:*
$$G'^*\widetilde{\mathrm{Ob}}(m,\boldsymbol{y}_1,L) \longrightarrow F^*\widetilde{\mathrm{Ob}}_a(\widehat{M}^{G_m}(\mathfrak{I})) \longrightarrow G'^*\widetilde{\mathrm{Ob}}(m,\widehat{\boldsymbol{y}}_2) \longrightarrow G'^*\widetilde{\mathrm{Ob}}(m,\boldsymbol{y}_1,L)[1] \tag{119}$$

We will prove Proposition 5.41 in the subsubsection 5.8.6, after some preparation.

5.8.2 The moduli stack of split objects with an orientation

We put $\mathcal{M}_1 := \mathcal{M}(m,\boldsymbol{y}_1,L)$ and $\mathcal{M}_2 := \mathcal{M}(m,\boldsymbol{y}_2)$. We put $\mathcal{M}_3 := \mathcal{M}_1 \times \mathcal{M}_2$.

Let us consider the moduli stack $\widehat{\mathcal{M}}_3$ of the objects $(E_1, F_{1\,*}, \phi, E_2, F_{2\,*}, \rho)$ as follows:

- $(E_1, F_{1\,*}, \phi) \in \mathcal{M}_1$ and $(E_2, F_{2\,*}) \in \mathcal{M}_2$.
- ρ denotes an orientation of $E_1 \oplus E_2$.

We have the obstruction theory $\mathrm{Ob}(\mathcal{M}_3) := \mathrm{Ob}(m,\boldsymbol{y}_1,L) \oplus \mathrm{Ob}(m,\boldsymbol{y}_2)$ of $\mathcal{M}_3$. The relative obstruction theory of $\widehat{\mathcal{M}}_3$ over $\mathcal{M}_3$ is constructed in the standard manner, which we explain in the following. Let $\pi : \widehat{\mathcal{M}}_3 \longrightarrow \mathcal{M}_3$ denote the projection. We have the universal objects $(\mathcal{E}_1^u, F_{1*}^u, \phi^u)$ over $\mathcal{M}_1 \times X$ and $(\mathcal{E}_2^u, F_{2*}^u)$ over $\mathcal{M}_2 \times X$. We also have the canonical resolutions $\mathcal{V}_{\cdot}^{(i)}$ of $\mathcal{E}_i^u(m)$. We denote the induced objects over $\widehat{\mathcal{M}}_3 \times X$ by the same notation. Then we have the orientation of $\mathcal{E}_1^u \oplus \mathcal{E}_2^u$ over $\widehat{\mathcal{M}}_3$. Therefore, we obtain the following diagram:
$$\begin{array}{ccc} \mathfrak{g}^d(\mathcal{V}_{\cdot}^{(1)} \oplus \mathcal{V}_{\cdot}^{(2)}) & \longrightarrow & \mathfrak{g}(\mathcal{V}_{\cdot}^{(1)}) \oplus \mathfrak{g}(\mathcal{V}_{\cdot}^{(2)}) \\ \big\downarrow & & \big\downarrow \\ \det^*_{\mathcal{E}_1^u \oplus \mathcal{E}_2^u, X} L_{\mathrm{Pic}\times X/X} & \longrightarrow & L_{\widehat{\mathcal{M}}_3 \times X/X} \end{array}$$
Therefore, we obtain the following:
$$\begin{array}{ccc} \mathrm{Ob}^d(\mathcal{V}_{\cdot}^{(1)} \oplus \mathcal{V}_{\cdot}^{(2)}) & \longrightarrow & \mathrm{Ob}(m,\boldsymbol{y}_1,L) \oplus \mathrm{Ob}(m,\boldsymbol{y}_2) \\ \big\downarrow & & \big\downarrow \\ \det^*_{\mathcal{E}_1^u \oplus \mathcal{E}_2^u} L_{\mathrm{Pic}} & \longrightarrow & L_{\widehat{\mathcal{M}}_3} \end{array}$$
We put $\mathrm{Ob}_{\mathrm{rel}}(\widehat{\mathcal{M}}_3/\mathcal{M}_3) := \mathrm{Cone}\big(\mathrm{Ob}^d(\mathcal{V}_{\cdot}^{(1)} \oplus \mathcal{V}_{\cdot}^{(2)}) \longrightarrow \det^*_{\mathcal{E}_1^u \oplus \mathcal{E}_2^u} L_{\mathrm{Pic}}\big)$. We have the natural morphism:
$$\gamma : \mathrm{Ob}_{\mathrm{rel}}(\widehat{\mathcal{M}}_3/\mathcal{M}_3)[-1] \longrightarrow \mathrm{Ob}(m,\boldsymbol{y}_1,L) \oplus \mathrm{Ob}(m,\boldsymbol{y}_2) = \pi^*\,\mathrm{Ob}(\mathcal{M}_3)$$
The cone of γ is denoted by $\mathrm{Ob}(\widehat{\mathcal{M}}_3)$. Then we have the natural morphism $\mathrm{ob}(\widehat{\mathcal{M}}_3) : \mathrm{Ob}(\widehat{\mathcal{M}}_3) \longrightarrow L_{\widehat{\mathcal{M}}_3}$.

We have the following commutative diagram as in the subsubsection 5.2.1:
$$\begin{array}{ccccc} L_{\widehat{\mathcal{M}}_3} & \longleftarrow & \det^*_{\mathcal{E}_1^u \oplus \mathcal{E}_2^u} L_{\mathrm{Pic}} & & \\ \big\uparrow & & \big\uparrow & & \\ \pi^*L_{\mathcal{M}_3} & \longleftarrow & \Phi\big(\det(\mathcal{E}_1^u \oplus \mathcal{E}_2^u)\big)^*\pi_1^*L_{\mathcal{M}(1)} & \longleftarrow & \mathrm{Ob}^d(\mathcal{V}_{\cdot}^{(1)} \oplus \mathcal{V}_{\cdot}^{(2)}) \end{array}$$

Here, π_1 denotes the projection $\mathrm{Pic} \longrightarrow \mathcal{M}(1)$. Therefore, we have the following commutative diagram:

$$\begin{array}{ccc}
\mathrm{Ob}_{\mathrm{rel}}(\widehat{\mathcal{M}}_3/\mathcal{M}_3)[-1] & \xrightarrow{\gamma} & \pi^*\,\mathrm{Ob}(\mathcal{M}_3) \\
{\scriptstyle \mathrm{ob}_{\mathrm{rel}}(\widehat{\mathcal{M}}_3/\mathcal{M}_3)}\big\downarrow & & \big\downarrow \\
L_{\widehat{\mathcal{M}}_3/\mathcal{M}_3}[-1] & \longrightarrow & \pi^* L_{\mathcal{M}_3}
\end{array}$$

By the same argument as the proof of Lemma 5.7, it can be shown that $\mathrm{ob}_{\mathrm{rel}}(\widehat{\mathcal{M}}_3/\mathcal{M}_3)$ gives a relative obstruction theory of $\widehat{\mathcal{M}}_3$ over $\mathcal{M}_3$. Therefore, $\mathrm{Ob}(\widehat{\mathcal{M}}_3)$ gives an obstruction theory of $\widehat{\mathcal{M}}_3$.

5.8.3 The embedding into the moduli stack of non-split objects

We put $\mathcal{M}_0 := \mathcal{M}(m,\boldsymbol{y},L)$ and $\widehat{\mathcal{M}}_0 := \mathcal{M}(m,\widehat{\boldsymbol{y}},L)$. The projection $\widehat{\mathcal{M}}_0 \longrightarrow \mathcal{M}_0$ is denoted by π_0 Let $\mathcal{V}_\cdot$ denote the canonical resolution of $p_{X\,*}\mathcal{E}^u(m)$ on $\mathcal{M}_0\times X$. We put $\mathcal{V}_0' := p_{X\,*}\mathcal{V}_0 = p_{X\,*}\mathcal{E}^u(m)$. We have the naturally defined morphism $\mathfrak{f}:\mathcal{M}_3 \longrightarrow \mathcal{M}_0$. We have the decomposition $\mathfrak{f}_X^*\mathcal{E}^u = \mathcal{E}_1^u\oplus\mathcal{E}_2^u$, $\mathfrak{f}_X^*\mathcal{V}_\cdot = \mathcal{V}_\cdot^{(1)}\oplus\mathcal{V}_\cdot^{(2)}$ and $\mathfrak{f}^*\mathcal{V}_0' = \mathcal{V}_0'^{(1)}\oplus\mathcal{V}_0'^{(2)}$. We have the naturally defined projections:

$$\mathfrak{f}_X^*\mathfrak{g}(\mathcal{V}_\cdot) \longrightarrow \mathfrak{g}(\mathcal{V}_\cdot^{(1)})\oplus\mathfrak{g}(\mathcal{V}_\cdot^{(2)}),\quad \mathfrak{f}_X^*\mathfrak{g}_{\mathrm{rel}}(\mathcal{V}_\cdot,\widetilde{\phi}) \longrightarrow \mathfrak{g}_{\mathrm{rel}}(\mathcal{V}_\cdot^{(1)},\widetilde{\phi}),$$
$$\mathfrak{f}_X^*\mathfrak{g}(\mathcal{V}_{\cdot|D}) \longrightarrow \mathfrak{g}(\mathcal{V}_{\cdot|D}^{(1)})\oplus\mathfrak{g}(\mathcal{V}_{\cdot|D}^{(2)}),\quad \mathfrak{f}_X^*\mathfrak{g}_{\mathrm{rel}}(\mathcal{V}_\cdot,F_*^u) \longrightarrow \mathfrak{g}_{\mathrm{rel}}(\mathcal{V}_\cdot^{(1)},F_*^{u\,(1)})\oplus\mathfrak{g}_{\mathrm{rel}}(\mathcal{V}_\cdot^{(2)},F_*^{u\,(2)}).$$

They induce the following morphisms:

$$\mathfrak{f}^*\,\mathrm{Ob}(\mathcal{V}_\cdot) \longrightarrow \mathrm{Ob}(\mathcal{V}_\cdot^{(1)})\oplus\mathrm{Ob}(\mathcal{V}_\cdot^{(2)}),\quad \mathfrak{f}^*\,\mathrm{Ob}_{\mathrm{rel}}(\mathcal{V}_\cdot,\widetilde{\phi}) \longrightarrow \mathrm{Ob}_{\mathrm{rel}}(\mathcal{V}^{(1)},\widetilde{\phi})$$
$$\mathfrak{f}^*\,\mathrm{Ob}(\mathcal{V}_{\cdot|D}) \longrightarrow \mathrm{Ob}(\mathcal{V}_{\cdot|D}^{(1)})\oplus\mathrm{Ob}(\mathcal{V}_{\cdot|D}^{(2)}),\quad \mathfrak{f}^*\,\mathrm{Ob}(\mathcal{V}_\cdot,F_*^u) \longrightarrow \mathrm{Ob}_{\mathrm{rel}}(\mathcal{V}_\cdot^{(1)},F_*^{u\,(1)})\oplus\mathrm{Ob}_{\mathrm{rel}}(\mathcal{V}_\cdot^{(2)},F_*^{u\,(2)})$$

Therefore, we obtain the morphism:

$$\mu_1:\mathfrak{f}^*\,\mathrm{Ob}(m,\boldsymbol{y},L) \longrightarrow \mathrm{Ob}(m,\boldsymbol{y}_1,L)\oplus\mathrm{Ob}(m,\boldsymbol{y}_2)$$

Lemma 5.42 *The following diagram is commutative:*

$$\begin{array}{ccc}
\mathfrak{f}^*\,\mathrm{Ob}(m,\boldsymbol{y},L) & \longrightarrow & \mathfrak{f}^*L_{\mathcal{M}_0} \\
\big\downarrow & & \big\downarrow \\
\mathrm{Ob}(m,\boldsymbol{y}_1,L)\oplus\mathrm{Ob}(m,\boldsymbol{y}_2) & \longrightarrow & L_{\mathcal{M}_3}
\end{array}\tag{120}$$

Proof We take an $H_{\boldsymbol{y}}(m)$-dimensional vector space $W_0 = V_m$ with a decomposition $W_0 = W_0^{(1)}\oplus W_0^{(2)}$, where $\dim W_0^{(i)} = H_{\boldsymbol{y}_i}(m)$. We also take a $\big(H_{\boldsymbol{y}}(m)-\mathrm{rank}(\boldsymbol{y})\big)$-dimensional vector space W_{-1} with a decomposition $W_{-1} = W_{-1}^{(1)}\oplus W_{-1}^{(2)}$, where $\dim W_{-1}^{(i)} = H_{\boldsymbol{y}_i}(m)-\mathrm{rank}(\boldsymbol{y}_i)$. We put $Y(W_\cdot^{(1)},W_\cdot^{(2)}) := Y(W_\cdot^{(1)})\times Y(W_\cdot^{(2)})$. We have the naturally defined morphism $Y(W_\cdot^{(1)},W_\cdot^{(2)}) \longrightarrow Y(W_\cdot)$. By considering the classifying map of $\mathcal{V}_\cdot$ and $\mathcal{V}_\cdot^{(1)}\oplus\mathcal{V}_\cdot^{(2)}$, we obtain the following commutative diagram:

$$\begin{array}{ccc}
\mathcal{M}_0\times X & \xrightarrow{\Phi(\mathcal{V}_\cdot)} & Y(W_\cdot) \\
{\scriptstyle \mathfrak{f}_X}\big\uparrow & & \big\uparrow \\
\mathcal{M}_3\times X & \xrightarrow{\Phi(\mathcal{V}_\cdot^{(1)},\mathcal{V}_\cdot^{(2)})} & Y(W_\cdot^{(1)},W_\cdot^{(2)})
\end{array}$$

By the argument in the subsubsection 2.3.2, we can show that $\mathfrak{f}_X^*\mathfrak{g}(\mathcal{V}_\cdot)_{\leq 1} \longrightarrow \mathfrak{g}(\mathcal{V}_\cdot^{(1)})_{\leq 1}\oplus\mathfrak{g}(\mathcal{V}_\cdot^{(2)})_{\leq 1}$ expresses the morphism $\mathfrak{f}_X^*\Phi(\mathcal{V}_\cdot)^*L_{Y(W_\cdot)/X} \longrightarrow \Phi(\mathcal{V}_\cdot^{(1)},\mathcal{V}_\cdot^{(2)})^*L_{Y(W_\cdot^{(1)},W_\cdot^{(2)})/X}$. Therefore, we obtain the following commutative diagram:

$$\begin{array}{ccccc}
\mathfrak{f}_X^*L_{\mathcal{M}_0\times X/X} & \longleftarrow & \mathfrak{f}_X^*\Phi(\mathcal{V}_\cdot)^*L_{Y(W_\cdot)/X} & \longleftarrow & \mathfrak{g}(\mathcal{V}_\cdot) \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
L_{\mathcal{M}_3\times X/X} & \longleftarrow & \Phi(\mathcal{V}_\cdot^{(1)},\mathcal{V}_\cdot^{(2)})^*L_{Y(W_\cdot^{(1)},W_\cdot^{(2)})/X} & \longleftarrow & \mathfrak{g}(\mathcal{V}_\cdot^{(1)})\oplus\mathfrak{g}(\mathcal{V}_\cdot^{(2)})
\end{array}$$

Therefore, we obtain the following commutative diagram:

$$\begin{tikzcd}
\mathfrak{f}^* L_{\mathcal{M}_0} & \mathfrak{f}^* \operatorname{Ob}(m,y) \arrow[l] \arrow[d] \\
L_{\mathcal{M}_3} \arrow[from=u, d] & \operatorname{Ob}(m,y_1) \oplus \operatorname{Ob}(m,y_2). \arrow[l]
\end{tikzcd} \tag{121}$$

Let $P_\cdot$ be a locally free resolution of $L(m)$, as in the subsubsection 5.6.1. We put $Y(W_\cdot^{(1)}, W_\cdot^{(2)}, P_\cdot) := Y(W_\cdot^{(1)}, P_\cdot) \times Y(W_\cdot^{(2)})$. Then, we have the naturally defined morphism $Y(W_\cdot^{(1)}, W_\cdot^{(2)}, P_\cdot) \longrightarrow Y(W_\cdot, P_\cdot)$. Then we have the following diagram:

$$\begin{tikzcd}
\mathcal{M}_0 \times X \arrow[r, "\Phi(\mathcal{V}_\cdot, \widetilde{\phi})"] & Y(W_\cdot, P_\cdot) \\
\mathcal{M}_3 \times X \arrow[u, "\mathfrak{f}_X"] \arrow[r, "\Phi(\mathcal{V}_\cdot^{(1)}, \mathcal{V}_\cdot^{(2)}, \widetilde{\phi})"] & Y(W_\cdot^{(1)}, W_\cdot^{(2)}, P_\cdot) \arrow[u]
\end{tikzcd}$$

Then, we obtain the following commutative diagram:

$$\begin{tikzcd}
\mathfrak{f}_X^* L_{\mathcal{M}_0 \times X/X} \arrow[d] & \mathfrak{f}_X^* \Phi(\mathcal{V}_\cdot, \widetilde{\phi})^* L_{Y(W_\cdot, P_\cdot)/X} \arrow[l] \arrow[d] & \mathfrak{f}_X^* \mathfrak{g}(\mathcal{V}_\cdot, \widetilde{\phi}) \arrow[l] \arrow[d] \\
L_{\mathcal{M}_3 \times X/X} & \Phi(\mathcal{V}_\cdot^{(1)}, \mathcal{V}_\cdot^{(2)}, \widetilde{\phi})^* L_{Y(W_\cdot^{(1)}, W_\cdot^{(2)}, P_\cdot)} \arrow[l] & \mathfrak{g}(\mathcal{V}_\cdot^{(1)}, \widetilde{\phi}) \oplus \mathfrak{g}(\mathcal{V}_\cdot^{(2)}) \arrow[l]
\end{tikzcd}$$

It is easy to observe that $Rp_{X\,*}\mathfrak{g}(\mathcal{V}_\cdot, \widetilde{\phi}) \otimes \omega_X$ is naturally isomorphic to $\operatorname{Ob}(m,y,L)$. Then, we obtain the following diagram:

$$\begin{tikzcd}
\mathfrak{f}^* L_{\mathcal{M}_0} \arrow[d] & \mathfrak{f}^* \operatorname{Ob}(m,y,L) \arrow[l] \arrow[d] \\
L_{\mathcal{M}_3} & \operatorname{Ob}(m,y_1,L) \oplus \operatorname{Ob}(m,y_2) \arrow[l]
\end{tikzcd} \tag{122}$$

We have the natural morphism $\operatorname{Ob}(m,y) \longrightarrow \operatorname{Ob}(m,y,L)$ and $\operatorname{Ob}(m,y_1) \oplus \operatorname{Ob}(m,y_2) \longrightarrow \operatorname{Ob}(m,y_1,L) \oplus \operatorname{Ob}(m,y_2)$. The diagrams (121) and (122) are compatible for the natural morphisms in the sense of the subsubsection 2.1.4.

We put $\mathcal{V}_D^{(i)(j)} := \operatorname{Ker}(\mathcal{V}_{0|D}^{(i)}) \longrightarrow \operatorname{Cok}_{j-1}^{(i)}$ which are locally free sheaves on $\mathcal{M}_3 \times D$. Let $\mathcal{V}_D^{(i)*}$ denote the vector bundle $\mathcal{V}_D^{(i)}$ with the filtration $\mathcal{V}_D^{(i)(1)} \supset \mathcal{V}_D^{(i)(2)} \cdots \supset \mathcal{V}_D^{(i)(l+1)}$. Similarly, we have the filtered vector bundle $\mathcal{V}_D^*$ on $\mathcal{M}_0 \times D$.

We put $W^{(i)(1)} := W_0^{(i)}$ and $W^{(i)(l+1)} = W_{-1}^{(i)}$. We take vector spaces $W^{(i)}$ ($i = 2, \ldots, l$) with decompositions $W^{(i)} = W^{(i)(1)} \oplus W^{(i)(2)}$, where $\operatorname{rank} W^{(i)} = \operatorname{rank} \mathcal{V}_D^{(i)}$ and $\operatorname{rank} W^{(i)(j)} = \operatorname{rank} \mathcal{V}_D^{(i)(j)}$. We use the notation in the subsubsection 5.5.1. We put $Y_D(W_\cdot, W^{(1)\,*}, W^{(2)}) := Y_D(W_\cdot^{(1)}, W^{(1)*}) \times Y_D(W_\cdot^{(2)}, W^{(2)*})$, and $Y_D(W_\cdot^{(1)}, W_\cdot^{(2)}) := Y_D(W_\cdot^{(1)}) \times Y_D(W_\cdot^{(2)})$. We have the naturally defined commutative diagram:

$$\begin{tikzcd}
Y_D(W_\cdot, W^*) \arrow[r] & Y_D(W_\cdot) \\
Y_D(W_\cdot, W^{(1)*}, W^{(2)*}) \arrow[u] \arrow[r] & Y_D(W_\cdot^{(1)}, W_\cdot^{(2)}) \arrow[u]
\end{tikzcd}$$

By considering the classifying maps of $\mathcal{V}_D^*$ and $(\mathcal{V}_D^{(1)*}, \mathcal{V}_D^{(2)*})$, we obtain the following commutative diagram:

$$\begin{tikzcd}
\mathcal{M}_0 \times D \arrow[r, "\Phi_D(\mathcal{V}_\cdot^*, F_*)"] & Y_D(W_\cdot, W^*) \arrow[r] & Y_D(W_\cdot) \\
\mathcal{M}_3 \times D \arrow[u] \arrow[r] & \Phi_1 Y_D(W_\cdot, W^{(1)*}, W^{(2)*}) \arrow[u] \arrow[r] & Y_D(W_\cdot^{(1)}, W_\cdot^{(2)}) \arrow[u]
\end{tikzcd}$$

Therefore, we obtain the following:

$$\begin{array}{ccccc}
\mathfrak{f}_D^* L_{\mathcal{M}_0\times D/D} & \longleftarrow & \mathfrak{f}_D^*\Phi_D(\mathcal{V}_\cdot, F_*^u)^* L_{Y_D(W_\cdot,W^*)/D} & \longleftarrow & \mathfrak{f}_D^*\Phi(\mathcal{V}_{\cdot|D})^* L_{Y_D(W_\cdot)/D} \\
\downarrow & & \downarrow & & \downarrow \\
L_{\mathcal{M}_3\times D/D} & \longleftarrow & \Phi_1^* L_{Y_D(W_\cdot,W^{(1)*},W^{(2)*})/D} & \longleftarrow & \Phi(\mathcal{V}^{(1)}_{\cdot|D}\oplus\mathcal{V}^{(2)}_{\cdot|D})^* L_{Y_D(W_\cdot^{(1)},W_\cdot^{(2)})/D}
\end{array}$$

Then, we obtain the following:

$$\begin{array}{ccccc}
\mathfrak{f}_D^* L_{\mathcal{M}_0\times D/D} & \longleftarrow & \mathfrak{f}_D^*\mathfrak{g}(\mathcal{V}_\cdot, F_*^u) & \longleftarrow & \mathfrak{f}_D^*\mathfrak{g}(\mathcal{V}_{\cdot|D}) \\
\downarrow & & \downarrow & & \downarrow \\
L_{\mathcal{M}_3\times D/D} & \longleftarrow & \mathfrak{g}_D(\mathcal{V}_\cdot^{(1)}, F_*^{u(1)})\oplus\mathfrak{g}_D(\mathcal{V}_\cdot^{(2)}, F_*^{u(2)}) & \longleftarrow & \mathfrak{g}(\mathcal{V}^{(1)}_{\cdot|D})\oplus\mathfrak{g}(\mathcal{V}^{(2)}_{\cdot|D})
\end{array}$$

We put $\mathrm{Ob}_D(\mathcal{V}, F_*^u) := Rp_{D\,*}\big(\mathfrak{g}(\mathcal{V}_\cdot, F_*^u)\otimes\omega_D\big)$. Similarly, we obtain $\mathrm{Ob}_D(\mathcal{V}^{(i)}, F_*^{u(i)})$. Then, we obtain the following commutative diagram:

$$\begin{array}{ccccc}
\mathfrak{f}^* L_{\mathcal{M}_0} & \longleftarrow & \mathfrak{f}^*\,\mathrm{Ob}_D(\mathcal{V}_\cdot, F_*^u) & \longleftarrow & \mathfrak{f}^*\,\mathrm{Ob}(\mathcal{V}_{\cdot|D}) \\
\downarrow & & \downarrow & & \downarrow \\
L_{\mathcal{M}_3} & \longleftarrow & \mathrm{Ob}_D(\mathcal{V}_\cdot^{(1)}, F_*^{u(1)})\oplus\mathrm{Ob}_D(\mathcal{V}_\cdot^{(2)}, F_*^{u(2)}) & \longleftarrow & \mathrm{Ob}(\mathcal{V}^{(1)}_{\cdot|D})\oplus\mathrm{Ob}(\mathcal{V}^{(2)}_{\cdot|D})
\end{array}$$

We remark that the cone of $\mathrm{Ob}(\mathcal{V}_{\cdot|D})\longrightarrow\mathrm{Ob}(\mathcal{V}_\cdot)\oplus\mathrm{Ob}_D(\mathcal{V}, F_*)$ is naturally isomorphic to $\mathrm{Ob}(m,\boldsymbol{y})$. Thus, we obtain the following commutative diagram, which is compatible with (121):

$$\begin{array}{ccc}
\mathfrak{f}^* L_{\mathcal{M}_0} & \longleftarrow & \mathfrak{f}^*\,\mathrm{Ob}(m,\boldsymbol{y}) \\
\downarrow & & \downarrow \\
L_{\mathcal{M}_3} & \longleftarrow & \mathrm{Ob}(m,\boldsymbol{y}_1)\oplus\mathrm{Ob}(m,\boldsymbol{y}_2)
\end{array}\tag{123}$$

From (121), (122) and (123), we obtain the desired diagram (120). Thus the proof of Lemma 5.42 is finished. ∎

We have the naturally defined morphism $\widehat{\mathfrak{f}}:\widehat{\mathcal{M}}_3\longrightarrow\widehat{\mathcal{M}}_0$. By construction of the obstruction theories, we have the following commutative diagram:

$$\begin{array}{ccc}
\widehat{\mathfrak{f}}^*\,\mathrm{Ob}_{\mathrm{rel}}(m,\widehat{y})[-1] & \longrightarrow & \widehat{\mathfrak{f}}^*\pi_0^*\,\mathrm{Ob}(m,\boldsymbol{y},L) \\
\simeq\downarrow & & \mu_1\downarrow \\
\mathrm{Ob}_{\mathrm{rel}}(\widehat{\mathcal{M}}_3/\mathcal{M}_3)[-1] & \longrightarrow & \pi^*\,\mathrm{Ob}(m,\boldsymbol{y}_1,L)\oplus\mathrm{Ob}(m,\boldsymbol{y}_2)
\end{array}$$

Therefore, we obtain the following commutative diagram:

$$\begin{array}{ccc}
\widehat{\mathfrak{f}}^*\,\mathrm{Ob}(m,\widehat{\boldsymbol{y}},L) & \xrightarrow{\mu_2} & \mathrm{Ob}(\widehat{\mathcal{M}}_3) \\
\downarrow & & \downarrow \\
\widehat{\mathfrak{f}}^* L_{\widehat{\mathcal{M}}_0} & \longrightarrow & L_{\widehat{\mathcal{M}}_3}
\end{array}$$

5.8.4 Some compatibility

We take $H_{\boldsymbol{y}_i}(m)$-dimensional vector spaces $W_0^{(i)}$. We put $B(W_0^{(1)},W_0^{(2)}) := k_{\mathrm{GL}(W_0^{(1)})\times\mathrm{GL}(W_0^{(2)})}$. We put $\mathcal{V}_0'^{(i)} := p_{X\,*}\mathcal{V}_0^{(i)}$. Then we have the classifying map $\Phi(\mathcal{V}_0'^{(1)}\oplus\mathcal{V}_0'^{(2)}):\mathcal{M}_3\longrightarrow B(W_0^{(1)},W_0^{(2)})$. Therefore, we have the morphism $\varphi:\Phi(\mathcal{V}_0'^{(1)}\oplus\mathcal{V}_0'^{(2)})^* L_{B(W_0^{(1)},W_0^{(2)})}\longrightarrow L_{\mathcal{M}_3}$.

Lemma 5.43 *The morphism φ factors through $\mathrm{Ob}(m, y_1, L) \oplus \mathrm{Ob}(m, y_2)$. In particular, it factors through $\mathrm{Ob}(m, \boldsymbol{y}_1, L) \oplus \mathrm{Ob}(m, \boldsymbol{y}_2)$.*

Proof We have the following commutative diagram:

$$\begin{array}{ccc}
\mathfrak{g}(\mathcal{V}_\cdot^{(1)}) \oplus \mathfrak{g}(\mathcal{V}_\cdot^{(2)}) & \longleftarrow & \mathfrak{h}(\mathcal{V}_\cdot^{(1)}) \oplus \mathfrak{h}(\mathcal{V}_\cdot^{(2)}) \\
\big\downarrow & & \big\downarrow \\
L_{\mathcal{M}_3 \times X} & \longleftarrow & \Phi(\mathcal{V}_0'^{(1)}, \mathcal{V}_0'^{(2)})^* L_{B(W_0^{(1)}, W_0^{(2)}) \times X/X}
\end{array}$$

It induces the following diagram:

$$\begin{array}{ccc}
\mathrm{Ob}(m, y_1) \oplus \mathrm{Ob}(m, y_2) & \longleftarrow & \mathrm{Ob}^G(\mathcal{V}_\cdot^{(1)}) \oplus \mathrm{Ob}^G(\mathcal{V}_\cdot^{(2)}) \\
\big\downarrow & & \varphi_1 \big\downarrow \\
L_{\mathcal{M}_3} & \longleftarrow & \Phi(\mathcal{V}_0'^{(1)}, \mathcal{V}_0'^{(2)})^* L_{B(W_0^{(1)}, W_0^{(2)})}
\end{array}$$

It is easy to check φ_1 is isomorphic. Then the claim of the lemma is clear. ∎

We put $W_0 := W_0^{(1)} \oplus W_0^{(2)}$ and $B(W_0) := k_{\mathrm{GL}(W_0)}$. By considering the classifying maps of the vector bundles $\mathcal{V}_0$ and $\mathcal{V}_0^{(i)}$, we obtain the following commutative diagram:

$$\begin{array}{ccccc}
\widehat{\mathcal{M}}_3 & \xrightarrow{\pi} & \mathcal{M}_3 & \xrightarrow{\Phi(\mathcal{V}_0^{(1)}, \mathcal{V}_0^{(2)})} & B(W_0^{(1)}, W_0^{(2)}) \\
\widehat{\mathfrak{f}} \big\downarrow & & \mathfrak{f} \big\downarrow & & \big\downarrow \\
\widehat{\mathcal{M}}_0 & \xrightarrow{\pi_2} & \mathcal{M}_0 & \xrightarrow{\Phi(\mathcal{V}_0)} & B(W_0)
\end{array} \tag{124}$$

Lemma 5.44 *We have the following commutative diagram:*

$$\begin{array}{ccccc}
L_{\widehat{\mathcal{M}}_3} & \longleftarrow & \mathrm{Ob}(\widehat{\mathcal{M}}_3) & \longleftarrow & \pi^* \Phi(\mathcal{V}_0^{(1)}, \mathcal{V}_0^{(2)})^* L_{B(W_0^{(1)}, W_0^{(2)})} \\
\big\uparrow & & \mu_2 \big\uparrow & & \big\uparrow \\
\widehat{\mathfrak{f}}^* L_{\widehat{\mathcal{M}}_0} & \longleftarrow & \widehat{\mathfrak{f}}^* \mathrm{Ob}(m, \widehat{\boldsymbol{y}}, L) & \longleftarrow & \widehat{\mathfrak{f}}^* \pi_2^* \Phi(\mathcal{V}_0)^* L_{B(W_0)}
\end{array} \tag{125}$$

Here, the composite of the horizontal arrows are the naturally defined morphisms from the diagram (124)*.*

Proof We have the following commutative diagram:

$$\begin{array}{ccc}
\mathfrak{f}_X^* \mathfrak{h}(\mathcal{V}_\cdot) & \longrightarrow & \mathfrak{f}_X^* \mathfrak{g}(\mathcal{V}_\cdot) \\
\big\downarrow & & \big\downarrow \\
\mathfrak{h}(\mathcal{V}_\cdot^{(1)}) \oplus \mathfrak{h}(\mathcal{V}_\cdot^{(2)}) & \longrightarrow & \mathfrak{g}(\mathcal{V}_\cdot^{(1)}) \oplus \mathfrak{g}(\mathcal{V}_\cdot^{(2)})
\end{array}$$

Therefore, we obtain the following:

$$\begin{array}{ccc}
\mathfrak{f}^* \mathrm{Ob}^G(\mathcal{V}_\cdot) & \longrightarrow & \mathfrak{f}^* \mathrm{Ob}(m, y) \\
\big\downarrow & & \big\downarrow \\
\mathrm{Ob}^G(\mathcal{V}_\cdot^{(1)}) \oplus \mathrm{Ob}^G(\mathcal{V}_\cdot^{(2)}) & \longrightarrow & \mathrm{Ob}(m, y_1) \oplus \mathrm{Ob}(m, y_2)
\end{array}$$

On the other hand, we have the following commutative diagram:

$$\begin{array}{ccc}
\mathfrak{f}_X^* \mathfrak{h}(\mathcal{V}_\cdot) & \longrightarrow & \mathfrak{f}_X^* \Phi(\mathcal{V}_0)_X^* L_{B(W_0) \times X/X} \\
\big\downarrow & & \big\downarrow \\
\mathfrak{h}(\mathcal{V}_\cdot^{(1)}) \oplus \mathfrak{h}(\mathcal{V}_\cdot^{(2)}) & \longrightarrow & \Phi(\mathcal{V}_0'^{(1)}, \mathcal{V}_0'^{(2)})_X^* L_{B(V_m^{(1)}, V_m^{(2)}) \times X/X}
\end{array}$$

Therefore, we obtain the following:

$$\begin{array}{ccc}
\mathfrak{f}^*\operatorname{Ob}^G(\mathcal{V}.) & \xrightarrow{\tau_1} & \mathfrak{f}^*\Phi(\mathcal{V}_0)^* L_{B(W_0)} \\
\downarrow & & \downarrow \\
\operatorname{Ob}^G(\mathcal{V}.^{(1)}) \oplus \operatorname{Ob}^G(\mathcal{V}.^{(2)}) & \xrightarrow{\tau_2} & \Phi(\mathcal{V}_0'^{(1)}, \mathcal{V}_0'^{(2)})^* L_{B(W_0^{(1)}, W_0^{(2)})}
\end{array}$$

The morphisms τ_i are isomorphic. Thus we obtain the claim of the lemma. ∎

5.8.5 Deformation

As explained in the subsubsection 5.5.3, we have the decomposition $\operatorname{Ob}(m, \boldsymbol{y}_2) = \operatorname{Ob}^\circ(m, \boldsymbol{y}_2) \oplus \operatorname{Ob}^d(m, \boldsymbol{y}_2)$. We put $\overline{\operatorname{Ob}}(m, \boldsymbol{y}_2) := \operatorname{Ob}^\circ(m, \boldsymbol{y}_2) \oplus \tau_{\leq 0} \operatorname{Ob}^d(m, \boldsymbol{y}_2)$. We have the following commutative diagram:

$$\begin{array}{ccc}
\operatorname{Ob}_{\mathrm{rel}}(\mathcal{V}.^{(1)} \oplus \mathcal{V}.^{(2)})[-1] & \longrightarrow & \operatorname{Ob}(m, \boldsymbol{y}_1, L) \oplus \operatorname{Ob}(m, \boldsymbol{y}_2) \\
\uparrow & & \uparrow \\
\tau_{\leq -1} \operatorname{Ob}^d(\mathcal{V}.^{(1)} \oplus \mathcal{V}.^{(2)}) & \xrightarrow{\lambda_1} & \operatorname{Ob}(m, \boldsymbol{y}_1, L) \oplus \overline{\operatorname{Ob}}(m, \boldsymbol{y}_2)
\end{array}$$

We put $\operatorname{Ob}_1(\widehat{\mathcal{M}}_3) := \operatorname{Cone}(\lambda_1)$. Then we have the morphisms $\operatorname{Ob}_1(\widehat{\mathcal{M}}_3) \longrightarrow \operatorname{Ob}(\widehat{\mathcal{M}}_3) \longrightarrow L_{\widehat{\mathcal{M}}_3}$. Since the first morphism is quasi isomorphic, the composite $\operatorname{ob}_1(\widehat{\mathcal{M}}_3)$ of the morphisms gives an obstruction theory of $\widehat{\mathcal{M}}_3$. We remark we have the following commutative diagram (We put $C_1 := \tau_{\leq -1} \operatorname{Ob}^d(\mathcal{V}.^{(1)} \oplus \mathcal{V}.^{(2)})$ and $C_2 := \operatorname{Ob}(m, \boldsymbol{y}_1, L) \oplus \overline{\operatorname{Ob}}(m, \boldsymbol{y}_2)$ in the diagram, to save the space.):

$$\begin{array}{ccccccc}
\mathcal{H}^{-1}(C_1) & \longrightarrow & \mathcal{H}^{-1}(C_2) & \longrightarrow & \mathcal{H}^{-1}\big(\operatorname{Ob}(\widehat{\mathcal{M}}_3)\big) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{H}^{-1}(\pi^* L_{\mathcal{M}_3}) & \longrightarrow & \mathcal{H}^{-1}\big(L_{\widehat{\mathcal{M}}_3}\big) & \longrightarrow & 0
\end{array} \tag{126}$$

$$\begin{array}{ccccccccc}
\mathcal{H}^0(C_2) & \xrightarrow{\simeq} & \mathcal{H}^0\big(\operatorname{Ob}_1(\widehat{\mathcal{M}}_3)\big) & \longrightarrow & 0 & \longrightarrow & \mathcal{H}^1(C_2) & \longrightarrow & \mathcal{H}^1(\operatorname{Ob}(\widehat{\mathcal{M}}_3)) \\
\downarrow & & \simeq\downarrow & & \downarrow & & \downarrow & & \simeq\downarrow \\
\mathcal{H}^0(\pi^* L_{\mathcal{M}_3}) & \longrightarrow & \mathcal{H}^0(L_{\widehat{\mathcal{M}}_3}) & \longrightarrow & \mathcal{H}^0(L_{\widehat{\mathcal{M}}_3/\mathcal{M}_3}) & \xrightarrow{\simeq} & \mathcal{H}^1(\pi^* L_{\mathcal{M}_3}) & \longrightarrow & \mathcal{H}^1(L_{\widehat{\mathcal{M}}_3})
\end{array} \tag{127}$$

We would like to deform $\operatorname{ob}_1(\widehat{\mathcal{M}}_3)$. Let i_1, i_2 and η denote the following naturally defined morphisms:

$$i_1 : \tau_{\leq -1} \operatorname{Ob}^d(\mathcal{V}_1) \longrightarrow \operatorname{Ob}(m, \boldsymbol{y}_1, L), \quad i_2 : \tau_{\leq -1} \operatorname{Ob}^d(\mathcal{V}_2) \longrightarrow \overline{\operatorname{Ob}}(m, \boldsymbol{y}_2),$$

$$\eta : \tau_{\leq -1} \operatorname{Ob}^d(\mathcal{V}_1 \oplus \mathcal{V}_2) \longrightarrow \tau_{\leq -1} \operatorname{Ob}^d(\mathcal{V}_1) \oplus \tau_{\leq -1} \operatorname{Ob}^d(\mathcal{V}_2)$$

The following is a special case of Lemma 5.6.

Lemma 5.45 *The composite $i_1 \circ \operatorname{ob}(m, \boldsymbol{y}_1, L)$ and $i_2 \circ \operatorname{ob}(m, \boldsymbol{y}_2)$ are trivial.* ∎

For any $a \in k$, let $\varphi_a : \tau_{\leq -1} \operatorname{Ob}^d(\mathcal{V}.^{(1)} \oplus \mathcal{V}.^{(2)}) \longrightarrow \operatorname{Ob}(m, \boldsymbol{y}_1, L) \oplus \overline{\operatorname{Ob}}(m, \boldsymbol{y}_2)$ be the morphism given by $\varphi_a := \big(a \cdot i_1, i_2\big) \circ \eta$. Then the following diagram is commutative for any a, due to Lemma 5.45:

$$\begin{array}{ccc}
\tau_{\leq -1} \operatorname{Ob}^d(\mathcal{V}.^{(1)} \oplus \mathcal{V}.^{(2)}) & \xrightarrow{\varphi_a} & \operatorname{Ob}(m, \boldsymbol{y}_1, L) \oplus \overline{\operatorname{Ob}}(m, \boldsymbol{y}_2) \\
\downarrow & & \downarrow \\
L_{\widehat{\mathcal{M}}_3/\mathcal{M}_3}[-1] & \longrightarrow & \pi^* L_{\mathcal{M}_3}
\end{array} \tag{128}$$

We put $\widetilde{\operatorname{Ob}}_a(\widehat{\mathcal{M}}_3) := \operatorname{Cone}(\varphi_a)$. From the commutativity of the diagram (128), we obtain the morphism $\widetilde{\operatorname{ob}}_a(\widehat{\mathcal{M}}_3) : \widetilde{\operatorname{Ob}}_a(\widehat{\mathcal{M}}_3) \longrightarrow L_{\widehat{\mathcal{M}}_3}$ for any a. The choice of a does not have any effect on the diagram (127). Hence, it is easy to observe that $\big\{\widetilde{\operatorname{ob}}_a(\widehat{\mathcal{M}}_3) \,\big|\, a \in k\big\}$ gives an obstruction theory of $\widetilde{\operatorname{ob}}(\widehat{\mathcal{M}}_3) : \widetilde{\operatorname{Ob}}(\widetilde{\mathcal{M}}_3) \longrightarrow L_{\widehat{\mathcal{M}}_3 \times A^1/A^1}$ of $\widehat{\mathcal{M}}_3 \times A^1$ over A^1. At $a = 1$, $\widetilde{\operatorname{Ob}}_1(\widehat{\mathcal{M}}_3)$ is same as $\operatorname{Ob}(\widehat{\mathcal{M}}_3)$ in the derived category.

Lemma 5.46 *We put $\widehat{\mathcal{M}}_2 := \mathcal{M}(m, \widehat{\boldsymbol{y}}_2)$. There exists an algebraic stack $\mathcal{S}'$ with the following diagram:*

$$\begin{array}{ccccc}
\widehat{\mathcal{M}}_3 & \xleftarrow{F_1} & \mathcal{S}' & \xrightarrow{G_1'} & \mathcal{M}_1 \times \widehat{\mathcal{M}}_2 \\
\uparrow & & \uparrow & & \uparrow \\
\widehat{M}^{G_m}(\mathfrak{I}) & \xleftarrow{F} & \mathcal{S} & \xrightarrow{G'} & \widetilde{\mathcal{M}}_{\mathrm{split}}
\end{array}$$

Here the bottom horizontal arrows are given in (79). *The morphisms F_1 and G_1' are etale and finite of degree $(r_1 \cdot r_2)^{-1}$.*

Proof Similar to Proposition 4.45 and Corollary 4.46. ▮

Lemma 5.47 *For each a, we have the following distinguished triangle:*

$$G_1'^* \operatorname{Ob}(m, \boldsymbol{y}_1, L) \longrightarrow F_1^* \widetilde{\operatorname{Ob}}_a(\widehat{\mathcal{M}}_3) \longrightarrow G_1'^* \operatorname{Ob}(m, \widehat{\boldsymbol{y}}_2) \longrightarrow G_1'^* \operatorname{Ob}(m, \boldsymbol{y}_1, L)[1] \tag{129}$$

We also have $F_1^* \widetilde{\operatorname{Ob}}_0(\widehat{\mathcal{M}}_3) = G_1'^* \operatorname{Ob}(m, \boldsymbol{y}_1, L) \oplus G_1'^* \operatorname{Ob}(m, \widehat{\boldsymbol{y}}_2)$.

Proof We have the following naturally defined distinguished triangle on $\widehat{\mathcal{M}}_3$:

$$\operatorname{Ob}(m, \boldsymbol{y}_1, L) \longrightarrow \widetilde{\operatorname{Ob}}_a(\widehat{\mathcal{M}}_3) \longrightarrow \operatorname{Cone}(i_2) \longrightarrow \operatorname{Ob}(m, \boldsymbol{y}_1, L)[1] \tag{130}$$

In the case $a = 0$, it splits. It is easy to see $F_1^* \operatorname{Ob}(m, \boldsymbol{y}_1, L) \simeq G_1'^* \operatorname{Ob}(m, \boldsymbol{y}_1, L)$ and $F_1^* \operatorname{Cone}(i_2) \simeq \operatorname{Ob}(m, \widehat{\boldsymbol{y}}_2)$. Hence we obtain (129) from (130). ▮

Lemma 5.48 *The composite of the following morphisms is independent of the choice of a, and it is same as the naturally defined one:*

$$\pi^* \Phi(\mathcal{V}_0^{(1)} \oplus \mathcal{V}_0^{(2)})^* L_{B(V_m^{(1)}, V_m^{(2)})} \longrightarrow \pi^* \big(\operatorname{Ob}(m, \boldsymbol{y}_1, L) \oplus \operatorname{Ob}(m, \boldsymbol{y}_2)\big) \longrightarrow \operatorname{Ob}_a(\widehat{\mathcal{M}}_3) \xrightarrow{\varphi_a} L_{\widehat{\mathcal{M}}_3} \tag{131}$$

Proof It is clear from the construction. ▮

5.8.6 Proof of Proposition 5.41

Let us construct an obstruction theory of $\widehat{M}^{G_m}(\mathfrak{I})$. We take $H_{\boldsymbol{y}_i}(m)$-dimensional vector spaces $V_m^{(i)}$. Let F_i denote the full flag variety of $V_m^{(i)}$. We put $\overline{F}_i := F_{i\ \mathrm{GL}(V_m^{(i)})}$. Then, $\widehat{M}^{G_m}(\mathfrak{I})$ is an open subset of the fiber product of $\overline{F}_1 \times \overline{F}_2$ and $\widehat{\mathcal{M}}_3$ over $B(W_0^{(1)}, W_0^{(2)})$. Hence we have the following commutative diagram:

$$\begin{array}{ccc}
\widehat{M}^{G_m}(\mathfrak{I}) & \xrightarrow{g} & \overline{F}_1 \times \overline{F}_2 \\
{\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle q} \\
\widehat{\mathcal{M}}_3 & \longrightarrow & B(W_0^{(1)}, W_0^{(2)})
\end{array} \tag{132}$$

Then we have the isomorphism $g^* L_{\overline{F}_1 \times \overline{F}_2 / B(W_0^{(1)}, W_0^{(2)})} \simeq L_{\widehat{M}^{G_m}(\mathfrak{I}) / \widehat{\mathcal{M}}_3}$. Due to Lemma 5.43, we have the following morphisms:

$$g^* L_{\overline{F}_1 \times \overline{F}_2 / B(W_0^{(1)}, W_0^{(2)})}[-1] \xrightarrow{\varphi_1} g^* q^* L_{B(W_0^{(1)}, W_0^{(2)})} \xrightarrow{\varphi_2} \pi_1^* \operatorname{Ob}(\widehat{\mathcal{M}}_3)$$

We put $\operatorname{Ob}\big(\widehat{M}^{G_m}(\mathfrak{I})\big) := \operatorname{Cone}(\varphi_2 \circ \varphi_1)$. Then we obtain the morphism $\operatorname{ob}\big(\widehat{M}^{G_m}(\mathfrak{I})\big) : \operatorname{Ob}\big(\widehat{M}^{G_m}(\mathfrak{I})\big) \longrightarrow L_{\widehat{M}^{G_m}(\mathfrak{I})}$. By the same argument as that in the subsubsection 5.7.1, it can be shown that $\operatorname{ob}\big(\widehat{M}^{G_m}(\mathfrak{I})\big)$ gives an obstruction theory for $\widehat{M}^{G_m}(\mathfrak{I})$.

We put $N := H_y(m)$. Let $\operatorname{Flag}(V_m^{(i)}, I_i)$ denote the moduli of filtrations $\mathcal{F}_*^{(i)}$ of $V_m^{(i)}$ as follows:

$$\mathcal{F}_1^{(i)} \subset \mathcal{F}_2^{(i)} \subset \cdots \subset \mathcal{F}_N^{(i)} = V_m^{(i)}, \quad \dim \mathcal{F}_j^{(i)} / \mathcal{F}_{j-1}^{(i)} = \begin{cases} 1 & (j \in I_i) \\ 0 & (j \notin I_i) \end{cases}$$

We put $V_m := V_m^{(1)} \oplus V_m^{(2)}$. Recall that $\mathrm{Flag}(V_m, \underline{N})$ denotes the full flag variety of V_m. We have the naturally defined inclusion $\mathrm{Flag}(V_m^{(1)}, I_1) \times \mathrm{Flag}(V_m^{(2)}, I_2) \longrightarrow \mathrm{Flag}(V_m, \underline{N})$. Clearly, $\mathrm{Flag}(V_m^{(i)}, I_i)$ is naturally isomorphic F_i. We use the notation in the subsubsection 5.7.2. Then, the diagram (132) is compatible with the following diagram, which is given by (115):

$$\begin{array}{ccc} \widehat{M}^* & \longrightarrow & \overline{F} \\ \big\downarrow & & \big\downarrow \\ \widehat{\mathcal{M}}_0 & \longrightarrow & B(W_0) \end{array}$$

By using the description (116) of $\mathrm{Ob}(\widehat{M}^*)$, we also obtain (117) from (125).

We would like to construct $\widetilde{\mathrm{Ob}}\big(\widehat{M}^{G_m}(\mathfrak{I})\big)$. Corresponding to the equivalence $\widetilde{\mathrm{Ob}}_1(\widehat{\mathcal{M}}_3) \simeq \mathrm{Ob}(\widehat{\mathcal{M}}_3)$, we have the equivalent obstruction theory $\widetilde{\mathrm{Ob}}_1\big(\widehat{M}^{G_m}(\mathfrak{I})\big) \simeq \mathrm{Ob}\big(\widehat{M}^{G_m}(\mathfrak{I})\big)$. Due to Lemma 5.48, we obtain the deformation $\widetilde{\mathrm{ob}}\big(\widehat{M}^{G_m}(\mathfrak{I})\big) : \widetilde{\mathrm{Ob}}\big(\widehat{M}^{G_m}(\mathfrak{I})\big) \longrightarrow L_{\widehat{M}^{G_m}(\mathfrak{I})\times A^1/A^1}$ from $\widetilde{\mathrm{ob}}(\widehat{\mathcal{M}}_3) : \widetilde{\mathrm{Ob}}(\widehat{\mathcal{M}}_3) \longrightarrow L_{\widehat{\mathcal{M}}_3\times A^1/A^1}$. We also obtain the distinguished triangle (119) and the splitting (118) at $t = 0$ from Lemma 5.47. Thus the proof of Proposition 5.41 is finished. ∎

5.8.7 The case where the 2-stability condition is satisfied

Let us describe the obstruction theory of the fixed point set of the master space, when the 2-stability condition is satisfied. We use the notation in the subsubsection 4.7.1. We give only the statement. We put $\mathcal{M}_{\mathrm{split}} := \mathcal{M}^s(\boldsymbol{y}_1, L, \alpha_*, \delta) \times \mathcal{M}^s(\widehat{\boldsymbol{y}}_2, \alpha_*)$.

Proposition 5.49 *We have the obstruction theory* $\mathrm{ob}(\widehat{M}^{G_m}(\mathfrak{I})) : \mathrm{Ob}(\widehat{M}^{G_m}(\mathfrak{I})) \longrightarrow L_{\widehat{M}^{G_m}(\mathfrak{I})}$ *and the deformation* $\widetilde{\mathrm{ob}}(\widehat{M}^{G_m}(\mathfrak{I})) : \widetilde{\mathrm{Ob}}(\widehat{M}^{G_m}(\mathfrak{I})) \longrightarrow L_{\widehat{M}^{G_m}(\mathfrak{I})\times A^1/A^1}$ *with the following property:*

- *We have the following commutative diagram:*

$$\begin{array}{ccc} \varphi_{\mathfrak{I}}^*\,\mathrm{Ob}(\widehat{M}) & \longrightarrow & \varphi_{\mathfrak{I}}^* L_{\widehat{M}} \\ \big\downarrow & & \big\downarrow \\ \mathrm{Ob}(\widehat{M}^{G_m}(\mathfrak{I})) & \longrightarrow & L_{\widehat{M}^{G_m}(\mathfrak{I})} \end{array} \tag{133}$$

- *Let* $\widetilde{\mathrm{ob}}_a(\widehat{M}^{G_m}(\mathfrak{I})) : \widetilde{\mathrm{Ob}}_a(\widehat{M}^{G_m}(\mathfrak{I})) \longrightarrow L_{\widehat{M}^{G_m}(\mathfrak{I})}$ *denote the specialization of* $\widetilde{\mathrm{ob}}(\widehat{M}^{G_m}(\mathfrak{I}))$ *at* $t = a$. *At* $t = 1$, *we have* $\widetilde{\mathrm{ob}}_1(\widehat{M}^{G_m}(\mathfrak{I})) = \mathrm{ob}(\widehat{M}^{G_m}(\mathfrak{I}))$. *At* $t = 0$, *we have the following commutative diagram in the diagram* (82)*:*

$$\begin{array}{ccc} F^*\widetilde{\mathrm{Ob}}_0(\widehat{M}^{G_m}(\mathfrak{I})) & \longrightarrow & F^* L_{\widehat{M}^{G_m}(\mathfrak{I})} \\ \simeq\big\downarrow & & \simeq\big\downarrow \\ G'^*\,\mathrm{Ob}(\mathcal{M}_{\mathrm{split}}) & \longrightarrow & G'^* L_{\mathcal{M}_{\mathrm{split}}} \end{array} \tag{134}$$

  On each a, *we have the following distinguished triangle:*

$$G'^*\,\mathrm{Ob}(m, \boldsymbol{y}_1, L) \longrightarrow F^*\widetilde{\mathrm{Ob}}_a(\widehat{M}^{G_m}(\mathfrak{I})) \longrightarrow G'^*\,\mathrm{Ob}(m, \widehat{\boldsymbol{y}}_2) \longrightarrow G'^*\,\mathrm{Ob}(m, \boldsymbol{y}_1, L)[1] \tag{135}$$

Proof It can be shown by an argument similar to the proof of Proposition 5.41. In this case, $\widehat{M}^{G_m}(\mathfrak{I})$ is an open substack of $\widehat{\mathcal{M}}_3$. The obstruction theory $\mathrm{ob}(\widehat{\mathcal{M}}_3) : \mathrm{Ob}(\widehat{\mathcal{M}}_3) \longrightarrow \widehat{M}^{G_m}(\mathfrak{I})$ and the deformation $\widetilde{\mathrm{ob}}(\widehat{\mathcal{M}}_3) : \widetilde{\mathrm{Ob}}(\widehat{\mathcal{M}}_3) \longrightarrow L_{\widehat{M}^{G_m}(\mathfrak{I})\times A^1/A^1}$ gives the desired objects. ∎

5.8.8 The case of oriented reduced $\boldsymbol{L}$-Bradlow pair

Let $\boldsymbol{L} = (L_1, L_2)$ be a pair of line bundles over X. Let us describe the obstruction theory of the fixed point set of the master space for the moduli stack of the oriented $\boldsymbol{L}$-Bradlow pairs, under the setting in the subsubsection 4.7.2. We give only the statement. We put $\mathcal{M}_{\mathrm{split}} := \mathcal{M}^s(\boldsymbol{y}_1, L_1, \alpha_*, \delta_1) \times \mathcal{M}^s(\widehat{\boldsymbol{y}}_2, [L_2], \alpha_*, \delta_2)$.

Proposition 5.50 *We have the obstruction theory* $\mathrm{ob}(\widehat{M}^{G_m}(\mathfrak{I})) : \mathrm{Ob}(\widehat{M}^{G_m}(\mathfrak{I})) \longrightarrow L_{\widehat{M}^{G_m}(\mathfrak{I})}$ *and the deformation* $\widetilde{\mathrm{ob}}(\widehat{M}^{G_m}(\mathfrak{I})) : \widetilde{\mathrm{Ob}}(\widehat{M}^{G_m}(\mathfrak{I})) \longrightarrow L_{\widehat{M}^{G_m}(\mathfrak{I})\times A^1/A^1}$ *with the following property:*

- *We have the following commutative diagram:*
$$\begin{array}{ccc} \varphi_{\mathfrak{I}}^* \mathrm{Ob}(\widehat{M}) & \longrightarrow & \varphi_{\mathfrak{I}}^* L_{\widehat{M}} \\ \Big\downarrow & & \Big\downarrow \\ \mathrm{Ob}(\widehat{M}^{G_m}(\mathfrak{I})) & \longrightarrow & L_{\widehat{M}^{G_m}(\mathfrak{I})} \end{array} \tag{136}$$
- *Let* $\widetilde{\mathrm{ob}}_a(\widehat{M}^{G_m}(\mathfrak{I})) : \widetilde{\mathrm{Ob}}_a(\widehat{M}^{G_m}(\mathfrak{I})) \longrightarrow L_{\widehat{M}^{G_m}(\mathfrak{I})}$ *denote the specialization of* $\widetilde{\mathrm{ob}}(\widehat{M}^{G_m}(\mathfrak{I}))$ *at* $t = a$. *At* $t = 1$, *we have* $\widetilde{\mathrm{ob}}_1(\widehat{M}^{G_m}(\mathfrak{I})) = \mathrm{ob}(\widehat{M}^{G_m}(\mathfrak{I}))$. *At* $t = 0$, *we have the following commutative diagram in the diagram* (82)*:*
$$\begin{array}{ccc} F^*\widetilde{\mathrm{Ob}}_0(\widehat{M}^{G_m}(\mathfrak{I})) & \longrightarrow & F^* L_{\widehat{M}^{G_m}(\mathfrak{I})} \\ \simeq\Big\downarrow & & \simeq\Big\downarrow \\ G'^* \mathrm{Ob}(\mathcal{M}_{\mathrm{split}}) & \longrightarrow & G'^* L_{\mathcal{M}_{\mathrm{split}}} \end{array} \tag{137}$$
On each a, *we have the following distinguished triangle:*
$$G'^* \mathrm{Ob}(m, \boldsymbol{y}_1, L_1) \longrightarrow F^*\widetilde{\mathrm{Ob}}_a(\widehat{M}^{G_m}(\mathfrak{I})) \longrightarrow G'^* \mathrm{Ob}(m, \widehat{\boldsymbol{y}}_2, [L_2]) \longrightarrow G'^* \mathrm{Ob}(m, \boldsymbol{y}_1, L_1)[1] \tag{138}$$

Proof We put $\mathcal{M}_1 := \mathcal{M}(m, \boldsymbol{y}_1, L_1)$, $\mathcal{M}_2 := \mathcal{M}(m, \widehat{\boldsymbol{y}}_2, [L_2])$ and $\mathcal{M}_3 = \mathcal{M}_1 \times \mathcal{M}_2$. We consider the moduli stack $\widehat{\mathcal{M}}_3$ of the objects $(E_1, F_{1\,*}, \phi_1, E_2, F_{2\,*}, [\phi_2], \rho)$ as follows:

- $(E_1, F_{1\,*}, \phi_1) \in \mathcal{M}_1$ and $(E_2, F_{2\,*}, [\phi_2]) \in \mathcal{M}_2$.
- ρ denotes an orientation of $E_1 \oplus E_2$.

Then $\widehat{M}^{G_m}(\mathfrak{I})$ is the open substack of $\widehat{\mathcal{M}}_3$.

We can construct the obstruction theory $\mathrm{ob}(\widehat{\mathcal{M}}_3) : \mathrm{Ob}(\widehat{\mathcal{M}}_3) \longrightarrow L_{\widehat{\mathcal{M}}_3}$ by the argument in the subsubsection 5.8.2. We can also construct the deformation $\widetilde{\mathrm{ob}}(\widehat{\mathcal{M}}_3) : \widetilde{\mathrm{Ob}}(\widehat{\mathcal{M}}_3) \longrightarrow L_{\widehat{M}^{G_m}(\mathfrak{I})\times A^1/A^1}$ by the argument in the subsubsection 5.8.5. It can be checked that they give the desired objects by an argument similar to the proof of Proposition 5.41. ∎

5.9 Equivariant Obstruction Theory of the Master Space

5.9.1 Statements

In this subsection, we would like to explain the following claim, first.

Proposition 5.51 *We have the* G_m*-equivariant lift of the obstruction theory* $\mathrm{ob}(\widehat{M})$ *of the enhanced master space* $\widehat{M}$.

The construction is explained in the subsubsection 5.9.2–5.9.3. We explain that $\mathrm{Ob}(\widehat{M}^*)$ can also be lifted equivariantly, in the subsubsection 5.9.4.

We will later apply the localization formula of Graber and Pandharipande ([24]). For that purpose, we have to see the induced obstruction theory and the virtual normal bundle at the fixed point set. Let $\iota_i : \widehat{M}_i \longrightarrow \widehat{M}$

denote the inclusion ($i = 0,1$). We have the decomposition of $\iota_i^* \operatorname{Ob}(\widehat{M})$ into the invariant part $\iota_i^* \operatorname{Ob}(\widehat{M})^{\mathrm{inv}}$ and the moving part $\iota_i^* \operatorname{Ob}(\widehat{M})^{\mathrm{mov}}$. We put $\mathfrak{N}(\widehat{M}_i) := \bigl(\iota_i^* \operatorname{Ob}(\widehat{M})^{\mathrm{mov}}\bigr)^{\vee}[1]$, which is called the virtual normal bundle. We have the induced obstruction theory $\operatorname{ob}_1(\widehat{M}_i) : \iota_i^* \operatorname{Ob}(\widehat{M})^{\mathrm{inv}} \longrightarrow L_{\widehat{M}_i}$ ([24]). We will prove the following proposition in the subsubsection 5.9.5

Proposition 5.52 *The induced obstruction theory $\operatorname{ob}_1(\widehat{M}_i)$ is equivalent to the moduli theoretic obstruction theory $\operatorname{ob}(\widehat{M}_i) : \operatorname{Ob}(\widehat{M}_i) \longrightarrow L_{\widehat{M}_i}$ given in the subsubsection 6.3.3. The virtual normal bundle $\mathfrak{N}(\widehat{M}_i)$ is isomorphic to $\mathcal{O}_{\mathrm{rel}}\bigl((-1)^{i-1}\bigr)$, and the weight of the induced G_m-action is $(-1)^i$.*

Let $\mathfrak{I} = (\boldsymbol{y}_1, \boldsymbol{y}_2, I_1, I_2)$ be a decomposition type. Let $\varphi_{\mathfrak{I}} : \widehat{M}^{G_m}(\mathfrak{I}) \longrightarrow \widehat{M}$ denote the inclusion. Similarly, we have the decomposition of $\varphi_{\mathfrak{I}}^* \operatorname{Ob}(\widehat{M})$ into the invariant part $\varphi_{\mathfrak{I}}^* \operatorname{Ob}(\widehat{M})^{\mathrm{inv}}$ and the moving part $\varphi_{\mathfrak{I}}^* \operatorname{Ob}(\widehat{M})^{\mathrm{mov}}$. We obtain the virtual normal bundle $\mathfrak{N}(\widehat{M}^{G_m}(\mathfrak{I})) := \bigl(\varphi_{\mathfrak{I}}^* \operatorname{Ob}(\widehat{M})^{\mathrm{mov}}\bigr)^{\vee}[1]$, and the obstruction theory $\operatorname{ob}_1(\widehat{M}^{G_m}(\mathfrak{I})) : \varphi_{\mathfrak{I}}^* \operatorname{Ob}(\widehat{M})^{\mathrm{inv}} \longrightarrow L_{\widehat{M}^{G_m}(\mathfrak{I})}$.

To describe $\mathfrak{N}(\widehat{M}^{G_m}(\mathfrak{I}))$, we prepare some notation. In the K-theory of coherent sheaves on $\widehat{M}^{G_m}(\mathfrak{I})$, we put as follows:

$$\mathfrak{N}(E_i^{\widehat{M}}, E_j^{\widehat{M}}) := -\sum_{i=0,1,2} (-1)^i R^i p_{X*} \mathcal{RH}om(E_i^{\widehat{M}}, E_j^{\widehat{M}}) \tag{139}$$

$$\mathfrak{N}(L, E_2^{\widehat{M}}) := \sum_{i=0,1,2} (-1)^i R^i p_{X\,*} \mathcal{H}om(L, E_2^{\widehat{M}}) \tag{140}$$

$$\mathfrak{N}_D(E_{i\,*}^{\widehat{M}}, E_{j\,*}^{\widehat{M}}) := -\sum_{i=0,1} (-1)^i R^i p_{D\,*} \mathcal{RH}om_2'\bigl(E_{i|D\,*}^{\widehat{M}}, E_{j|D\,*}^{\widehat{M}}\bigr) \tag{141}$$

Here $E_{i|D\,*}^{\widehat{M}}$ denotes the restriction $E_i^{\widehat{M}} \otimes \mathcal{O}_{\widehat{M}^{G_m}(\mathfrak{I})\times D}$ with the induced filtration. (See the subsubsection 2.1.5 for $\mathcal{RH}om_2'$.) Let $\widehat{\mathcal{M}}_0$ and $\widehat{\mathcal{M}}_3$ be as in the subsubsections 5.8.2–5.8.3. It is easy to observe that $\widehat{M}^{G_m}(\mathfrak{I}) \longrightarrow \widehat{M} \times_{\widehat{\mathcal{M}}_0} \widehat{\mathcal{M}}_3$ is a regular embedding. The normal bundle is denoted by N_0.

We will prove the following proposition in the subsubsection 5.9.6.

Proposition 5.53

- *The induced obstruction theory $\varphi_{\mathfrak{I}}^* \operatorname{Ob}(\widehat{M}^*)^{\mathrm{inv}}$ is naturally isomorphic to $\operatorname{Ob}(\widehat{M}^{G_m}(\mathfrak{I}))$, and the induced obstruction theory $\varphi_{\mathfrak{I}}^* \operatorname{Ob}(\widehat{M}^*)^{\mathrm{inv}} \longrightarrow L_{\widehat{M}^{G_m}(\mathfrak{I})}$ is equivalent to $\operatorname{ob}(\widehat{M}^{G_m}(\mathfrak{I})) : \operatorname{Ob}(\widehat{M}^{G_m}(\mathfrak{I})) \longrightarrow L_{\widehat{M}^{G_m}(\mathfrak{I})}$.*
- *The virtual normal bundle $\mathfrak{N}(\widehat{M}^{G_m}(\mathfrak{I}))$ is K-theoretically given as follows:*

$$\begin{aligned}\mathfrak{N}(E_1^{\widehat{M}}, E_2^{\widehat{M}}) \otimes I_{1+r_1/r_2} + \mathfrak{N}(E_2^{\widehat{M}}, E_1^{\widehat{M}}) \otimes I_{-1-r_1/r_2} + \mathfrak{N}(L, E_2^{\widehat{M}}) \otimes I_{1+r_1/r_2} \\ + \mathfrak{N}_D(E_{1*}^{\widehat{M}}, E_{2*}^{\widehat{M}}) \otimes I_{1+r_1/r_2} + \mathfrak{N}_D(E_{2*}^{\widehat{M}}, E_{1*}^{\widehat{M}}) \otimes I_{-1-r_1/r_2} + N_0\end{aligned} \tag{142}$$

Here, I_n denote the trivial line bundle on $\widehat{M}^{G_m}(\mathfrak{I})$ with the G_m-action of weight n.

Before going into the proof, we give a remark. Recall the diagram (77). We put $A = 1 + r_1/r_2$. Let $\mathcal{O}_{1,\mathrm{rel}}(1)$ denote the tautological line bundle obtained from $\widetilde{\mathcal{M}}\bigl(\widehat{\boldsymbol{y}}_1, [L], \alpha_*, (\delta, k)\bigr)$. On $\mathcal{S}$, we have the following:

$$F^*\mathfrak{N}(E_1^{\widehat{M}}, E_2^{\widehat{M}}) = G^*\mathfrak{N}(\widehat{E}_1^u, \widehat{E}_2^u) \otimes \mathcal{O}_{1,\mathrm{rel}}(-A), \quad F^*\mathfrak{N}(E_2^{\widehat{M}}, E_1^{\widehat{M}}) = G^*\mathfrak{N}(\widehat{E}_2^u, \widehat{E}_1^u) \otimes \mathcal{O}_{1,\mathrm{rel}}(A) \tag{143}$$

$$F^*\mathfrak{N}(L, E_2^{\widehat{M}}) = G^*\mathfrak{N}(L, \widehat{E}_2^u) \otimes \mathcal{O}_{1,\mathrm{rel}}(-r_1/r_2) \tag{144}$$

$$F^*\mathfrak{N}_D(E_{1*}^{\widehat{M}}, E_{2*}^{\widehat{M}}) = G^*\bigl(\mathfrak{N}_D(\widehat{E}_{1*}^u, \widehat{E}_{2*}^u) \otimes \mathcal{O}_{1,\mathrm{rel}}(-A)\bigr), \quad F^*\mathfrak{N}_D(E_{2*}^{\widehat{M}}, E_{1*}^{\widehat{M}}) = G^*\bigl(\mathfrak{N}_D(\widehat{E}_{2*}^u, \widehat{E}_{1*}^u) \otimes \mathcal{O}_{1,\mathrm{rel}}(A)\bigr). \tag{145}$$

Here, $\mathfrak{N}(\widehat{E}_a^u, \widehat{E}_b^u)$, $\mathfrak{N}(L, \widehat{E}_2^u)$ and $\mathfrak{N}_D(\widehat{E}_{a*}^u, \widehat{E}_{b*}^u)$ are elements of the $K\bigl(\widetilde{\mathcal{M}}\bigl(\widehat{\boldsymbol{y}}_1, [L], \alpha_*, (\delta, \ell)\bigr) \times \widetilde{\mathcal{M}}(\widehat{\boldsymbol{y}}_2, \alpha_*)\bigr)$, given as follows:

$$\mathfrak{N}(\widehat{E}_i^u, \widehat{E}_j^u) := -\sum_{i=0,1,2} (-1)^i R^i p_{X*} \mathcal{RH}om(\widehat{E}_i^u, \widehat{E}_j^u) \tag{146}$$

$$\mathfrak{N}(L, \widehat{E}_2^u) := \sum_{i=0,1,2} (-1)^i R^i p_{X*} \mathcal{H}om(L, \widehat{E}_2^u) \tag{147}$$

$$\mathfrak{N}_D(\widehat{E}_{i*}^u, \widehat{E}_{j*}^u) := -\sum_{i=0,1} (-1)^i R^i p_{D*} \mathcal{RH}om_2'\big(\widehat{E}_{i|D*}^u, \widehat{E}_{j|D*}^u\big) \tag{148}$$

Here $\widehat{E}_{i|D*}^u$ denotes the restriction $\widehat{E}_i^u \otimes \mathcal{O}_{\widehat{M}^{G_m}(\mathfrak{I})\times D}$ with the induced filtration.

5.9.2 G_m-equivariant lift of $\mathrm{Ob}(\widehat{M})$ and $\mathrm{ob}(\widehat{M})$

We would like to obtain the G_m-equivairant structure of $\mathrm{Ob}(\widehat{M})$. We use the notation in the subsubsection 5.7.1. We have the following commutative diagram:

$$\begin{array}{ccc}
\widehat{M} & \xrightarrow{\Psi_2} & \widetilde{\mathcal{Q}} \\
{\scriptstyle \mathfrak{p}}\big\downarrow & & {\scriptstyle \mathfrak{p}'}\big\downarrow \\
\mathcal{M}(m, \widehat{\boldsymbol{y}}, [L]) & \xrightarrow{\Psi_1} & B(W_\cdot, [P_\cdot])
\end{array}$$

We have the G_m-action on Z_1 given by $t[u_0 : u_1] = [t{\cdot}u_0 : u_1]$. It induces the G_m-action on $\widetilde{\mathcal{Q}}$. We remark that Ψ_2 is G_m-equivariant map. We have the natural G_m-equivariant structure of $\mathfrak{p}^* \mathrm{Ob}(m, \widehat{\boldsymbol{y}}, [L])$ and $\Psi_2^* L_{\widetilde{\mathcal{Q}}/B(W_\cdot,[P_\cdot])}$.

Let C_i $(i = 1, 2)$ be a bounded G_m-complex on $\widehat{M}$. We have the induced G_m-action on $\mathrm{Ext}^0(C_1, C_2)$, where $\mathrm{Ext}^0(C_1, C_2)$ denotes the vector space of the homomorphisms of C_1 to C_2 in the derived category $D(\widehat{M})$. In the following, we say that a morphism $\varphi : C_1 \longrightarrow C_2$ is contained in the G_m-invariant part of the Ext^0-group, if φ is contained in the G_m-invariant part of $\mathrm{Ext}^0(C_1, C_2)$.

Lemma 5.54 *The morphism $\Psi_2^* L_{\widetilde{\mathcal{Q}}/B(W_\cdot,[P_\cdot])}[-1] \longrightarrow \mathfrak{p}^* \mathrm{Ob}(m, \widehat{\boldsymbol{y}}, [L])$ is contained in the G_m-invariant part of the Ext^0-group.*

Proof By using Remark 2.18, we obtain the natural G_m-equivariant representatives $C'(\widetilde{\mathcal{Q}})$ and $C'(B(W_\cdot, [P_\cdot]))$, C_3 of $\Psi_2^* L_{\widetilde{\mathcal{Q}}}$, $\Psi_2^* \mathfrak{p}'^* L_{B(W_\cdot,[P_\cdot])}$ and $\Psi_2^* L_{\widetilde{\mathcal{Q}}/B(W_\cdot,[P_\cdot])}$ respectively. We regard them as G_m-equivariant complex. We have the natural G_m-equivariant morphism $\alpha : C(B(W_\cdot, [P_\cdot])) \longrightarrow C(\widetilde{\mathcal{Q}})$. We put $C'(\widetilde{\mathcal{Q}}/B(W_\cdot, [P_\cdot])) := \mathrm{Cone}(\alpha)$. Then, we obtain the G_m-equivariant quasi isomorphism $C_3 \simeq C'(\widetilde{\mathcal{Q}}/B(W_\cdot, [P_\cdot]))$. Then the morphism $\Psi_2^* L_{\widetilde{\mathcal{Q}}/B(W_\cdot,[P_\cdot])}[-1] \longrightarrow \Psi_2^* \mathfrak{p}'^* L_{B(W_\cdot,[P_\cdot])}$ can be expressed by the G_m-equivariant morphism:

$$C'(\widetilde{\mathcal{Q}}/B(W_\cdot, [P_\cdot]))[-1] \longrightarrow C'(B(W_\cdot, [P_\cdot]))$$

On the other hand, the morphism $\mathfrak{p}^* u : \mathfrak{p}^* \Psi_2^* L_{B(W_\cdot,[P_\cdot])} \longrightarrow \mathfrak{p}^* \mathrm{Ob}(m, \widehat{\boldsymbol{y}}, [L])$ is contained in the G_m-invariant part of the Ext^0-group. Then the claim of the lemma is clear. ∎

By applying the general non-sense in the subsubsection 2.5.2, we obtain G_m-equivariant representative $C(\widehat{M})$ of $\mathrm{Ob}(\widehat{M})$.

Recall that we need only the (-1)-truncated cotangent complexes for the construction of the virtual classes, and the complex $C(\widehat{M})$ above are used for the localization theory by Graber and Pandharipande ([24]). Since $L_{\widehat{M}/\mathcal{M}(m,\widehat{\boldsymbol{y}},[L])}$ is isomorphic to the 0-th cohomology sheaf, we have the distinguished triangle:

$$\tau_{\geq -1} \mathfrak{p}^* L_{\mathcal{M}(m,\widehat{\boldsymbol{y}},[L])} \longrightarrow \tau_{\geq -1} L_{\widehat{M}} \longrightarrow L_{\widehat{M}/\mathcal{M}(m,\widehat{\boldsymbol{y}},[L])} \longrightarrow \tau_{\geq -1} \mathfrak{p}^* L_{\mathcal{M}(m,\widehat{\boldsymbol{y}},[L])}[1]$$

Since the morphism $\mathfrak{p}^* \mathrm{Ob}(m, \widehat{\boldsymbol{y}}, [L]) \longrightarrow \tau_{\geq -1} \mathfrak{p}^* L_{\mathcal{M}(m,\widehat{\boldsymbol{y}},[L])}$ is contained in the G_m-invariant part of the Ext^0-group, the morphism $L_{\widehat{M}/\mathcal{M}(m,\widehat{\boldsymbol{y}},[L])}[-1] \longrightarrow \mathfrak{p}^* L_{\mathcal{M}(m,\widehat{\boldsymbol{y}},[L])}$ is also contained in the G_m-invariant part of the Ext^0-group, due to Lemma 5.54. Therefore, we also obtain the G_m-equivariant structure of $\tau_{\geq -1} L_{\widehat{M}}$.

By construction, $\mathrm{ob}(\widehat{M}) : \mathrm{Ob}(\widehat{M}) \longrightarrow \tau_{\geq -1} L_{\widehat{M}}$ is contained in the G_m-invariant part of the Ext^0-group. Therefore, we can take the G_m-equivariant representatives of $\mathrm{Ob}(\widehat{M})$, $\tau_{\geq -1} L_{\widehat{M}}$ and $\mathrm{ob}(\widehat{M})$.

5.9.3 Equivalent G_m-equivariant structure of $\tau_{\geq -1}L_{\widehat{M}}$

On the other hand, we have the G_m-equivariant structure of $\tau_{\geq -1}L_{\widehat{M}}$, obtained from the G_m-equivariant embedding into a smooth Deligne-Mumford stack. We use the notation in the subsubsection 4.3.1.

Let $\widehat{Z}_m$ be the vector bundle as in (38). Then we put $\widehat{\widetilde{\mathcal{B}}} := \widetilde{\mathcal{B}} \times_{Z_m} \widehat{Z}_m$. We have the natural morphism $\widehat{\mathrm{TH}}^{ss} \longrightarrow \widehat{\widetilde{\mathcal{B}}}$ which is $\mathrm{GL}(V_m) \times G_m$-equivariant. Therefore, we obtain the G_m-equivariant immersion $\iota : \widehat{M} \longrightarrow \widehat{\widetilde{\mathcal{B}}}_{\mathrm{GL}(V_m)}$. Since $\widehat{M}$ is Deligne-Mumford, we can take a smooth Deligne-Mumford open substack P of $\widehat{\widetilde{\mathcal{B}}}_{\mathrm{GL}(V_m)}$, which contains $\widehat{M}$.

Let I denote the ideal sheaf of P corresponding to $\widehat{M}$. We put $C(\widehat{M}) := \mathrm{Cone}\big(I/I^2 \longrightarrow \iota^* C(P)\big)$ on $\widehat{M}$, where we put $C(P) := \Omega_{P/k}$. It is naturally G_m-complex, and it is the representatives of the (-1)-truncated cotangent complex $\tau_{\geq -1}L_{\widehat{M}/k}$.

Lemma 5.55 *The above two G_m-equivariant structures of $\tau_{\geq -1}L_{\widehat{M}}$ are equivalent.*

Proof We have the complex $C'(\widehat{\widetilde{\mathcal{B}}}) := \mathrm{Cone}(\Omega_{\widehat{\widetilde{\mathcal{B}}}} \longrightarrow \Omega_{\widehat{\widetilde{\mathcal{B}}}/P})[-1]$ on $\widehat{\widetilde{\mathcal{B}}}$. It is naturally $\mathrm{GL}(V_m) \times G_m$-equivariant, and thus it induces G_m-equivariant complex $C'(P)$ on P. We have the natural G_m-equivariant quasi isomorphism $C(P) \longrightarrow C'(P)$ on P. We put $C'(\widehat{M}) := \mathrm{Cone}\big(I/I^2 \longrightarrow \iota^* C'(P)\big)$ on $\widehat{M}$, then we have the natural G_m-equivariant quasi isomorphism $C(\widehat{M}) \longrightarrow C'(\widehat{M})$.

We put $\widehat{\mathcal{A}} := \mathcal{A} \times_{Z_m} \widehat{Z}_m$. We have the natural $\mathrm{GL}(V_m)$-action on $\widehat{\mathcal{A}}$. The quotient stack is denoted by $\mathfrak{Q}$. We have the $\mathrm{GL}(V_m)$-equivariant map $Q^\circ(m, \widehat{\boldsymbol{y}}, [L]) \longrightarrow \widehat{\mathcal{A}}$, and hence $\mathcal{M}(m, \widehat{\boldsymbol{y}}, [L]) \longrightarrow \mathfrak{Q}$.

Let $\mathfrak{p}_3 : \widehat{\widetilde{\mathcal{B}}} \longrightarrow \widehat{\mathcal{A}}$ denote the projection. We put $C'(\widehat{\mathcal{A}}) := \mathrm{Cone}\big(\mathfrak{p}_3^* \Omega_{\widehat{\mathcal{A}}} \longrightarrow \mathfrak{p}_3^* \Omega_{\widehat{\mathcal{A}}/\mathfrak{Q}}\big)[-1]$. The complex is provided with the natural $\mathrm{GL}(V_m)$-action. Hence, it induces the complex $C'(\mathfrak{Q})$ on P. We have the natural morphism $C'(\mathfrak{Q}) \longrightarrow C'(P)$.

It is clear that the morphism $I/I^2 \longrightarrow \iota^* C(P)$ factors through $\iota^* C'(\mathfrak{Q})$. We put $C'(\mathcal{M}) = \mathrm{Cone}\big(I/I^2 \longrightarrow \iota^* C'(\mathfrak{Q})\big)$. We have the exact sequences of the G_m-equivariant complexes:

$$0 \longrightarrow C'(\mathfrak{Q}) \longrightarrow C'(P) \longrightarrow \Omega_{P/\mathfrak{Q}} \longrightarrow 0 \qquad \text{on } P$$

$$0 \longrightarrow C'(\mathcal{M}) \longrightarrow C'(\widehat{M}) \longrightarrow \Omega_{\widehat{M}/\mathcal{M}} \longrightarrow 0 \quad \text{on } \widehat{M}$$

We put $C'(P/\mathfrak{Q}) := \mathrm{Cone}\big(C'(\mathfrak{Q}) \longrightarrow C'(P)\big)$. We have the G_m-equivariant morphism $C'(P/\mathfrak{Q})[-1] \longrightarrow C'(\mathfrak{Q})$ on P. We have the natural G_m-equivariant morphism $\iota^* C'(P/\mathfrak{Q})[-1] \longrightarrow \iota^* C'(\mathfrak{Q}) \longrightarrow C'(\mathcal{M})$ of the G_m-complexes on $\widehat{M}$. We put $C_1(\widehat{M}) := \mathrm{Cone}\big(\iota^* C(P/\mathfrak{Q})[-1] \longrightarrow C'(\mathcal{M})\big)$.

Let us show that $C_1(\widehat{M})$ is G_m-equivariant quasi-isomorphic to $C'(\widehat{M})$. We put $C_0 := \mathrm{Cone}(I/I^2 \longrightarrow I/I^2)$. We have the composite of the morphisms $C'(\widehat{M}) \longrightarrow I/I^2[1] \longrightarrow C_0[1]$. We have the naturally defined G_m-equivariant quasi isomorphism $\mathrm{Cone}\big(C'(\widehat{M}) \to C_0[1]\big)[-1] \longrightarrow C'(\widehat{M})$. We have the morphism $I/I^2 \longrightarrow \iota^* C'(\mathfrak{Q})$. It induces the G_m-equivariant quasi isomorphism $\mathrm{Cone}\big(C'(\widehat{M}) \to C_0[1]\big)[-1] \longrightarrow C_1(\widehat{M})$.

We have the following commutative diagram:

$$\begin{array}{ccccc}
\widehat{M} & \longrightarrow & P & \longrightarrow & \widetilde{\mathcal{Q}} \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{M} & \longrightarrow & \mathfrak{Q} & \longrightarrow & B(W_\cdot, [P_\cdot])
\end{array}$$

It induces the following commutative diagram:

$$\begin{array}{ccc}
C'(\widetilde{\mathcal{Q}}) & \longrightarrow & C'(P) \\
\uparrow & & \uparrow \\
C'(B(W_\cdot, [P_\cdot])) & \longrightarrow & C'(\mathfrak{Q})
\end{array}$$

Hence, we obtain the isomorphism $C'(\widetilde{\mathcal{Q}}/B(W_\cdot, [P_\cdot])) \simeq C'(P/\mathfrak{Q})$. We also have $C'(\mathcal{M}) \simeq \mathfrak{p}^* \tau_{\leq -1} L_{\mathcal{M}}$. Therefore, two G_m-equivariant structures are equivalent. ∎

5.9.4 G_m-equivariant structure of $\mathrm{Ob}(\widehat{M}^*)$

Since we have the quasi isomorphism $\mathrm{Ob}(\widehat{M})_{|\widehat{M}^*} \simeq \mathrm{Ob}(\widehat{M}^*)$, we have already obtained the G_m-equivariant structure of $\mathrm{Ob}(\widehat{M}^*)$. We give another description of the G_m-equivariant structure. We use the notation in the subsubsection 5.7.2.

We have the natural G_m-equivariant structure on the sheaves $\widehat{E}^{\widehat{M}}$ over $\widehat{M}\times X$. It induces the G_m-equivariant structure on $\mathfrak{p}_2^* \mathrm{Ob}(m, \widehat{\boldsymbol{y}}, L)$. On the other hand, the morphisms Ψ_3 and $\mathfrak{p}_2$ are G_m-equivariant. Therefore, we have the G_m-equivariant representative of $L_{B(W.,P.)}$.

Lemma 5.56 *The morphism $\mathfrak{p}_2^*\Psi_3^* L_{B(W.,P.)} \longrightarrow \mathfrak{p}_2^* \mathrm{Ob}(m, \widehat{\boldsymbol{y}}, L)$ is G_m-equivariant.*

Proof Let $\mathcal{V}.$ denote the canonical resolution of $\widehat{E}^{\widehat{M}}(m)$. By the remark 2.18, $\mathfrak{h}(\mathcal{V}., \widetilde{\phi})_{\leq 1}$ gives the G_m-equivariant representative of $\Phi_{3X}^* L_{Y(W.,P.)/X}$, where the G_m-equivariant structure of $\mathfrak{h}(\mathcal{V}., \widetilde{\phi})$ is induced by the G_m-equivariant structure of $\widehat{E}^{\widehat{M}}$. Therefore, the G_m-equivariant representative of $\Phi_X^* L_{Y(W.,P.)/X}$ is given by $\mathrm{Ob}^G(\mathcal{V}., \widetilde{\phi})$. On the other hand, the morphism $\mathrm{Ob}^G(\mathcal{V}., \widetilde{\phi}) \longrightarrow \mathrm{Ob}(m, \widehat{\boldsymbol{y}}, L)$ is G_m-equivariant, because their G_m-equivariant structures are induced by that of $\widehat{E}^{\widehat{M}}$. Thus we are done. ∎

Then, the morphism $\Psi_4^* L_{\widetilde{\mathcal{Q}}^*/B(W.,P.)}[-1] \longrightarrow \mathfrak{p}_2^* \mathrm{Ob}(m, \widehat{\boldsymbol{y}}, L)$ is contained in the G_m-invariant part of the Ext^0-group. Therefore, we can take the G_m-equivariant representative of $\mathrm{Ob}(\widehat{M}^*)$.

Lemma 5.57 *Two G_m-equivariant structure of $\mathrm{Ob}(\widehat{M}^*)$ are equivalent.*

Proof We have the following diagram:

$$\begin{array}{ccc}
\Psi_4^* L_{\widetilde{\mathcal{Q}}^*/Y(W.,P.)}[-1] & \longrightarrow & \mathrm{Ob}(m, \widehat{\boldsymbol{y}}, L) \\
{\scriptstyle a}\big\uparrow & & {\scriptstyle b}\big\uparrow \\
\Psi_4^* L_{\widetilde{\mathcal{Q}}^*/Y(W.,[P.])}[-1] & \longrightarrow & \mathrm{Ob}(m, \widehat{\boldsymbol{y}}, [L])
\end{array}$$

The morphisms are contained in the G_m-invariant part of the Ext^0-groups. Therefore, the induced G_m-equivariant structures on $\mathrm{Cone}(a)$ and $\mathrm{Cone}(b)$ are equivalent. ∎

5.9.5 Proof of Proposition 5.52

We use the notation in the subsection 4.5 and the subsection 5.7. We put $\overline{F} := \mathrm{Flag}(V_m, \underline{N})_{\mathrm{GL}(V_m)}$ and $\overline{Z}_1 := Z_{1\ \mathrm{GL}(V_m)}$. Since we have $\widetilde{\mathcal{Q}} = \overline{Z}_1 \times_{B(W.)} \overline{F}$, we have $L_{\widetilde{\mathcal{Q}}/B(W,[P])} = L_{\overline{Z}_1/B(W,[P])} \oplus L_{\overline{F}/B(W)}$.

Recall $\mathrm{Ob}(\widehat{M})$ is given by $\mathrm{Cone}\bigl(\Psi_2^* L_{\widetilde{\mathcal{Q}}/B(W,[P])} \longrightarrow \mathfrak{p}^* \mathrm{Ob}(m, \widehat{\boldsymbol{y}}, [L])\bigr)$. The moving part $\iota_i^* \mathrm{Ob}(\widehat{M})^{\mathrm{mov}}$ is given by the following:

$$\iota_i^* \Psi_2^* L_{\overline{Z}_1/B(W,[P])} \simeq L_{\widehat{M}_i/\widehat{M}} \simeq \mathcal{O}_{\mathrm{rel}}\bigl((-1)^i\bigr)[1]$$

Then, it is easy to see that the virtual normal bundle $\mathfrak{N}_{\widehat{M}}(\widehat{M}_i)$ is given by the line bundle $\mathcal{O}_{\mathrm{rel}}\bigl((-1)^{i-1}\bigr)$, and that the weight of the induced G_m-action is $(-1)^i$. By construction, we have the following:

$$\iota_i^* \mathrm{Ob}(\widehat{M})^{\mathrm{inv}} := \mathrm{Cone}\bigl(\Psi_{13}^* L_{\overline{F}/B(W)}[-1] \longrightarrow \mathrm{Ob}(m, \widehat{\boldsymbol{y}}, [L])\bigr) \simeq \mathrm{Ob}(\widehat{M}_i)$$

It is easy to observe that we have the following commutative diagram:

$$\begin{array}{ccc}
\iota_i^* \mathrm{Ob}(\widehat{M}) & \xrightarrow{\varphi_1} & \mathrm{Ob}(\widehat{M}_i) \\
{\scriptstyle \iota_i^* \mathrm{ob}(\widehat{M})}\big\downarrow & & {\scriptstyle \mathrm{ob}(\widehat{M}_i)}\big\downarrow \\
\iota_i^* L_{\widehat{M}} & \xrightarrow{\varphi_2} & L_{\widehat{M}_i}
\end{array}$$

Here, φ_2 is the naturally defined one, and φ_1 is the projection onto the invariant part. Thus the induced obstruction theory is given by $\mathrm{ob}(\widehat{M}_i)$.

5.9.6 Proof of Proposition 5.53

We use the notation in the subsubsections 5.7.2 and 5.8.4. Recall the expression of $\mathrm{Ob}(\widehat{M}^*)$ as in (116). Let us describe the decomposition of $\Psi_4^*\Gamma^* L_{\overline{F}/B(W_0)}$. We take a decomposition $W_0 = W_0^{(1)} \oplus W_0^{(2)}$ as in the subsubsection 5.8.4. We put $\overline{F}_i := \mathrm{Flag}(W_0^{(i)}, I_i)_{\mathrm{GL}(W_0^{(i)})}$. We also put $\overline{F}' := \mathrm{Flag}(W_0, \underline{N})_{\mathrm{GL}(W_0^{(1)})\times \mathrm{GL}(W_0^{(2)})}$. Then we have the regular immersion $\overline{F}_1 \times \overline{F}_2 \longrightarrow \overline{F}'$. We have the following commutative diagram:

$$\begin{array}{ccccc}
\widehat{M}^{G_m}(\mathfrak{I}) & \xrightarrow{\Psi_{21}} & \overline{F}_1 \times \overline{F}_2 & \longrightarrow & \overline{F} \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
\widehat{\mathcal{M}}_3 & \longrightarrow & B(W_0^{(1)}, W_0^{(2)}) & \longrightarrow & B(W_0)
\end{array}$$

Therefore, we have the isomorphism:

$$\Psi_4^*\Gamma^* L_{\overline{F}/B(W.)} \simeq \Psi_{21}^* L_{\overline{F}'/B(W_0^{(1)}, W_0^{(2)})}$$

The invariant part of $\Psi_4^*\Gamma^* L_{\overline{F}/B(W.)}$ is isomorphic to the pull back of the relative cotangent bundle of $\overline{F}_1 \times \overline{F}_2$ over $B(W_0^{(1)}, W_0^{(2)})$. The moving part is same as the pull back of the conormal bundle of $\overline{F}_1 \times \overline{F}_2$ in $\overline{F}'$, which is naturally isomorphic to $N_0^\vee[1]$.

Let us see the decomposition of $\mathrm{Ob}(m, \boldsymbol{y}, L)$. Corresponding to the decomposition $\varphi_{\mathfrak{J}}^* \widehat{E}^{\widehat{M}} = E_1^{\widehat{M}} \oplus E_2^{\widehat{M}}$, we obtain the decomposition of the resolution $\mathcal{V}. = \mathcal{V}_1. \oplus \mathcal{V}_2.$. It induces the following decompositions:

$$\mathfrak{g}(\mathcal{V}.)^{\mathrm{inv}} = \mathfrak{g}(\mathcal{V}.^{(1)}) \oplus \mathfrak{g}(\mathcal{V}.^{(2)}), \quad \mathfrak{g}(\mathcal{V}.)^{\mathrm{mov}} = \mathcal{H}om\big(\mathcal{V}.^{(1)}, \mathcal{V}.^{(2)}\big)^\vee[-1] \oplus \mathcal{H}om\big(\mathcal{V}.^{(2)}, \mathcal{V}.^{(1)}\big)^\vee[-1]$$

$$\mathfrak{g}_{\mathrm{rel}}(\mathcal{V}., \phi)^{\mathrm{inv}} = \mathfrak{g}_{\mathrm{rel}}(\mathcal{V}.^{(1)}, \phi), \quad \mathfrak{g}_{\mathrm{rel}}(\mathcal{V}., \phi)^{\mathrm{mov}} = \mathcal{H}om\big(P., \mathcal{V}.^{(2)}\big)^\vee$$

$$\mathfrak{g}_D(\mathcal{V}, F_*)^{\mathrm{inv}} = \mathfrak{g}_D(\mathcal{V}^{(1)}, F_*^{(1)}) \oplus \mathfrak{g}_D(\mathcal{V}^{(2)}, F_*^{(2)}), \quad \mathfrak{g}_D(\mathcal{V}, F_*)^{\mathrm{mov}} = C_1\big(\mathcal{V}_D^{(1)*}, \mathcal{V}_D^{(2)*}\big)^\vee[-1] \oplus C_1\big(\mathcal{V}_D^{(2)*}, \mathcal{V}_D^{(1)*}\big)^\vee[-1]$$

(See the subsubsection 2.1.5 for C_1.)

$$\mathfrak{g}(\mathcal{V}._{|D})^{\mathrm{inv}} = \mathfrak{g}(\mathcal{V}^{(1)}_{\cdot|D}) \oplus \mathfrak{g}(\mathcal{V}^{(2)}_{\cdot|D}), \quad \mathfrak{g}(\mathcal{V}._{|D})^{\mathrm{mov}} = \mathcal{H}om_{\mathcal{O}_D}(\mathcal{V}^{(1)}_{\cdot|D}, \mathcal{V}^{(2)}_{\cdot|D})^\vee[-1] \oplus \mathcal{H}om_{\mathcal{O}_D}(\mathcal{V}^{(2)}_{\cdot|D}, \mathcal{V}^{(1)}_{\cdot|D})^\vee[-1]$$

We also have $\mathfrak{g}^d(\mathcal{V}.)^{\mathrm{inv}} = \mathfrak{g}^d(\mathcal{V}.)$. The contribution to the virtual normal bundle can be calculated formally. We can also easily observe that $\mathrm{Ob}(m, \boldsymbol{y}, L)^{\mathrm{inv}}$ is naturally isomorphic to $\mathrm{Ob}(\widehat{\mathcal{M}}_3)$ given in the subsubsection 5.8.2. Then $\iota^* \mathrm{Ob}(\widehat{M}^*)^{\mathrm{inv}}$ is obtained as the cone of the composite of the following morphisms:

$$L_{\overline{F}_1 \times \overline{F}_2 / B(W^{(1)}, W^{(2)})}[-1] \longrightarrow L_{B(W^{(1)}, W^{(2)})} \longrightarrow \mathrm{Ob}(\widehat{\mathcal{M}}_3)$$

Namely, it is same as $\mathrm{Ob}(\widehat{M}^{G_m}(\mathfrak{I}))$. (See the subsubsection 5.8.6.) We also have the diagram (117). Then it is easy to observe that the induced obstruction theory is given by $\mathrm{ob}(\widehat{M}^{G_m}(\mathfrak{I}))$.

5.9.7 The case where the 2-stability condition is satisfied

We give only the statement about the G_m-equivariant obstruction theory of the master space in the case where the 2-stability condition is satisfied. The proof is similar to those of Proposition 5.51, Proposition 5.52 and Proposition 5.53.

Proposition 5.58 *Under the setting in the subsubsection* 4.7.1*, the following claims hold:*

- *We have the equivariant obstruction theory of the master space* $\widehat{M}$.

- *The induced obstruction theory of the fixed point sets $\widehat{M}_i$ and $\widehat{M}^{G_m}(\mathfrak{I})$ are equivalent to the moduli theoretic obstruction theory.*
- *The virtual normal bundle $\mathfrak{N}(\widehat{M}_i)$ of $\widehat{M}_i$ in $\widehat{M}$ is given by $\mathcal{O}_{\mathrm{rel}}\big((-1)^{i-1}\big)$ with the G_m-action of the weight $(-1)^i$.*
- *The virtual normal bundle $\mathfrak{N}(\widehat{M}^{G_m}(\mathfrak{I}))$ of $\widehat{M}^{G_m}(\mathfrak{I})$ in $\widehat{M}$ is given by the following:*

$$\mathfrak{N}(E_1^{\widehat{M}},E_2^{\widehat{M}})\otimes I_{1+r_1/r_2}+\mathfrak{N}(E_2^{\widehat{M}},E_1^{\widehat{M}})\otimes I_{-1-r_1/r_2}+\mathfrak{N}(L,E_2^{\widehat{M}})\otimes I_{1+r_1/r_2} \\ +\mathfrak{N}_D(E_{1*}^{\widehat{M}},E_{2*}^{\widehat{M}})\otimes I_{1+r_1/r_2}+\mathfrak{N}_D(E_{2*}^{\widehat{M}},E_{1*}^{\widehat{M}})\otimes I_{-1-r_1/r_2} \quad (149)$$

Here, I_w denotes the trivial line bundle with the G_m-action of weight w, and the terms are as in (139), (140) *and* (141). *(See also* (143), (144) *and* (145).*)* ∎

5.9.8 The case of oriented reduced $\boldsymbol{L}$-Bradlow pairs

We give only the statement about the G_m-equivariant obstruction theory of the master space in the case of the oriented reduced $\boldsymbol{L}$-Bradlow pairs, under the setting of the subsubsection 4.7.2. We prepare some notation. For any decomposition type $\mathfrak{I}$, the elements $\mathfrak{N}(E_i^{\widehat{M}},E_j^{\widehat{M}})$, $\mathfrak{N}_D(E_{i*}^{\widehat{M}},E_{j*}^{\widehat{M}})$ and $\mathfrak{N}(L_1,E_2^{\widehat{M}})$ of $K^{G_m}(\widehat{M}(\mathfrak{I}))$ are given as in (139), (140) and (141). (See also (143), (144) and (145).) Let $\varphi:\widehat{M}\longrightarrow\mathcal{M}(m,\widehat{\boldsymbol{y}},[\boldsymbol{L}])$ denote the naturally defined morphism. We put $\mathcal{I}^{(2)}:=\varphi^*\mathcal{O}_{\mathrm{rel}}^{(2)}(-1)$, which is naturally provided with the G_m-action. We have the following element of $K^{G_m}(\widehat{M}^{G_m}(\mathfrak{I}))$:

$$\mathfrak{N}(L_2\otimes\mathcal{I}^{(2)},E_1^{\widehat{M}}):=\sum_{i=0,1,2}(-1)^iR^ip_{X\,*}\mathcal{H}om\big(L_2\otimes\mathcal{I}^{(2)},E_1^{\widehat{M}}\big)$$

Lemma 5.59 *Under the setting in the subsubsection* 4.7.1*, the following holds:*

- *We have the equivariant obstruction theory of the master space $\widehat{M}$.*
- *The induced obstruction theory of the fixed point sets $\widehat{M}_i$ and $\widehat{M}^{G_m}(\mathfrak{I})$ are equivalent to the moduli theoretic obstruction theory.*
- *The virtual normal bundle $\mathfrak{N}(\widehat{M}_i)$ of $\widehat{M}_i$ in $\widehat{M}$ is given by $\mathcal{O}_{\mathrm{rel}}^{(1)}\big((-1)^{i-1}\big)$ with the G_m-action of the weight $(-1)^i$.*
- *The virtual normal bundle $\mathfrak{N}\big(\widehat{M}^{G_m}(\mathfrak{I})\big)$ of $\widehat{M}^{G_m}(\mathfrak{I})$ in $\widehat{M}$ is given by the following:*

$$\mathfrak{N}(E_1^{\widehat{M}},E_2^{\widehat{M}})\otimes I_{1+r_1/r_2}+\mathfrak{N}(E_2^{\widehat{M}},E_1^{\widehat{M}})\otimes I_{-1-r_1/r_2}+\mathfrak{N}(L_1,E_2^{\widehat{M}})\otimes I_{1+r_1/r_2}+\mathfrak{N}(L_2\otimes\mathcal{I}^{(2)},E_1^{\widehat{M}})\otimes I_{-1-r_1/r_2} \\ +\mathfrak{N}_D(E_{1*}^{\widehat{M}},E_{2*}^{\widehat{M}})\otimes I_{1+r_1/r_2}+\mathfrak{N}_D(E_{2*}^{\widehat{M}},E_{1*}^{\widehat{M}})\otimes I_{-1-r_1/r_2} \quad (150)$$

- *We have the following equality on $\mathcal{S}$:*

$$F^*\mathfrak{N}(L_2\otimes\mathcal{I}^{(2)},E_1^{\widehat{M}})=G^*\big(\mathfrak{N}(L_2,E_1^u)\otimes\mathcal{O}_{1,\mathrm{rel}}(1+r_1/r_2)\otimes\mathcal{O}_{2,\mathrm{rel}}(1)\big)$$

Here, we put $\mathfrak{N}(L_2,E_1^u):=Rp_{X\,*}\mathcal{H}om(L_2,E_1^u)$. ∎

6 Virtual fundamental Classes

6.1 Perfectness of the Obstruction Theories for some Stacks

6.1.1 The moduli stacks

Let $\boldsymbol{y}$ be an element of $\mathcal{T}ype$, and let α_* be a system of weights. Let L be a line bundle on X. We use the notation in the subsection 5.6. The proof of the following propositions will be given in the subsubsection 6.1.4 after the preparation in the subsubsection 6.1.3. The expected dimensions can be calculated formally. We give the results in the subsubsection 6.1.5.

Proposition 6.1 *Let m be a sufficiently large integer.*

- *The obstruction theory* $\mathrm{Ob}(m,\widehat{\boldsymbol{y}})$ *of* $\mathcal{M}^{ss}(\widehat{\boldsymbol{y}},\alpha_*)$ *is perfect in the sense of Definition* 2.29*.*
- *Let δ be an element of $\mathcal{P}^{\mathrm{br}}$. The obstruction theory* $\mathrm{Ob}(m,\boldsymbol{y},L)$ *of* $\mathcal{M}^{ss}(\boldsymbol{y},L,\alpha_*,\delta)$ *is perfect.*

Proposition 6.2 *Assume* $\mathrm{rank}(\boldsymbol{y})>1$*. Let δ be an element of $\mathcal{P}^{\mathrm{br}}$. Let m be a sufficiently large integer.*

- *The obstruction theory* $\mathrm{Ob}(m,\widehat{\boldsymbol{y}},L)$ *of* $\mathcal{M}^{ss}(\widehat{\boldsymbol{y}},L,\alpha_*,\delta)$ *is perfect.*
- *The obstruction theory* $\mathrm{Ob}(m,\widehat{\boldsymbol{y}},[L])$ *of* $\mathcal{M}^{ss}(\widehat{\boldsymbol{y}},[L],\alpha_*,\delta)$ *is perfect.*

Proposition 6.3 *Assume* $\mathrm{rank}(\boldsymbol{y})=1$*. Let m be a sufficiently large integer.*

- *We have the vanishing* $\mathcal{H}^i\bigl(\mathrm{Ob}(m,\widehat{\boldsymbol{y}},[L])\bigr)$ *unless $i=0$. In particular, the moduli* $\mathcal{M}(\widehat{\boldsymbol{y}})$ *is smooth.*
- *If the 2-vanishing condition is satisfied for $(\boldsymbol{y},L)$, then the obstruction theory* $\mathrm{Ob}(m,\widehat{\boldsymbol{y}},L)$ *of* $\mathcal{M}(\widehat{\boldsymbol{y}},L)$ *is perfect, and the obstruction theory* $\mathrm{Ob}(m,\widehat{\boldsymbol{y}},[L])$ *of* $\mathcal{M}(\widehat{\boldsymbol{y}},[L])$ *is perfect.*

We remark $\mathrm{Ob}(m,\boldsymbol{y},L)$ *is always perfect as in Proposition* 6.1*.*

Proposition 6.4 *Let $\boldsymbol{L}=(L_1,L_2)$ be a pair of line bundles on X. Let $\boldsymbol{\delta}=(\delta_1,\delta_2)$ be a pair of sufficiently small parameters δ_i as in Lemma* 3.63*. Let α_* be a system of weights. Assume that the 2-vanishing condition holds for $(\boldsymbol{y},L_2,\alpha_*)$. Then, the obstruction theory* $\mathrm{Ob}(m,\widehat{\boldsymbol{y}},[\boldsymbol{L}])$ *is perfect on* $\mathcal{M}^{ss}(\widehat{\boldsymbol{y}},[\boldsymbol{L}],\alpha_*,\boldsymbol{\delta})$*.*

Notation 6.5 *Due to Proposition* 6.1–6.4*, we obtain the perfect obstruction theories of the moduli stacks $\mathcal{M}$ of the corresponding stable objects, which induces the virtual fundamental classes due to Proposition* 2.30*. They are denoted by $[\mathcal{M}]$. We use the notation $\int_{\mathcal{M}}\Phi$ for the evaluation of a cohomology class via $[\mathcal{M}]$. (See the subsection* 7.1*.)* ∎

6.1.2 The master space and the related stacks

We also obtain the following propositions.

Proposition 6.6 *The obstruction theory* $\mathrm{Ob}(\widehat{M})$ *of* $\widehat{M}$ *is perfect. (See the subsubsections* 5.7.1*,* 5.7.6*,* 5.7.7 *for* $\mathrm{Ob}(\widehat{M})$*.)*

Proof We consider the obstruction theory for the enhanced master space $\widehat{M}$ given in the subsubsection 5.7.1. We have the naturally defined smooth morphism $\mathfrak{p}:\widehat{M}\longrightarrow\mathcal{M}(m,\widehat{\boldsymbol{y}},[L])$. We remark that the image of $\mathfrak{p}$ is contained in the open substack $\mathcal{M}:=\mathcal{M}^{ss}(\widehat{\boldsymbol{y}},[L],\alpha_*,\delta)$. Then, the claim immediately follows from the diagram (113) and Proposition 6.2.

We obtain the perfectness of the obstruction theories $\mathrm{Ob}(\widehat{M})$ (subsubsection 5.7.6) by the the same argument. We obtain the perfectness of the obstruction theories $\mathrm{Ob}(\widehat{M})$ in the subsubsection 5.7.7 by using the same argument and Proposition 6.4. ∎

The following claims can be shown by a similar argument.

Proposition 6.7

- *The obstruction theory* $\widetilde{\mathrm{ob}}(m,\widehat{\boldsymbol{y}})$ *of* $\widetilde{\mathcal{M}}^{s}(\widehat{\boldsymbol{y}},\alpha_*,+)$ *is perfect.*
- *The obstruction theory* $\widetilde{\mathrm{ob}}\bigl(m,\boldsymbol{y},L\bigr)$ *of the moduli stack* $\widetilde{\mathcal{M}}^{s}\bigl(\widehat{\boldsymbol{y}},L,\alpha_*,(\delta,\ell)\bigr)$ *is perfect.*
- *Assume one of the following:*
  - $\mathrm{rank}(\boldsymbol{y})>1$.
  - $\mathrm{rank}(\boldsymbol{y})=1$ *and the 2-vanishing condition holds for* $(\boldsymbol{y},L,\alpha_*,\delta)$*.*

  The obstruction theory $\mathrm{ob}\bigl(m,\widehat{\boldsymbol{y}},[L]\bigr)$ *of the moduli stack* $\widetilde{\mathcal{M}}^{s}\bigl(\widehat{\boldsymbol{y}},[L],\alpha_*,(\delta,\ell)\bigr)$ *is perfect.* ∎

Recall that we obtained obstruction theory $\widetilde{\mathrm{ob}}(\widehat{M}^{G_m}(\mathfrak{I}))$ of $\widehat{M}^{G_m}(\mathfrak{I}) \times A^1$ over A^1 in Proposition 5.41. The specialization at $t = a$ is denoted by $\widetilde{\mathrm{ob}}_a(\widehat{M}^{G_m}(\mathfrak{I}))$.

Proposition 6.8 *The obstruction theories $\widetilde{\mathrm{Ob}}_a(\widehat{M}^{G_m}(\mathfrak{I}))$ are perfect for any $a \in k$.*

Proof Due to the distinguished triangle (119), we have only to show that $\widetilde{\mathrm{Ob}}(m, \boldsymbol{y}_1, L)$ and $\widetilde{\mathrm{Ob}}(m, \widehat{\boldsymbol{y}}_2)$ gives the obstruction theories of $\widetilde{\mathcal{M}}^s\big(\boldsymbol{y}_1, L, \alpha_*, (\delta, k_0)\big)$ and $\widetilde{\mathcal{M}}^s(\widehat{\boldsymbol{y}}_2, \alpha_*, +)$ as in Proposition 6.7. ∎

Notation 6.9 *We obtain the virtual fundamental classes of $M^{G_m}(\mathfrak{I})$ with respect to the obstruction theories $\mathrm{Ob}_a(M^{G_m}(\mathfrak{I}))$ for each $a \in k$. It is independent of a choice of a (See Proposition 7.2 of [5].) Therefore, we denote it by $[M^{G_m}(\mathfrak{I})]$.* ∎

We use the notation in the subsubsection 4.6.1. We put $\widetilde{\mathcal{M}}_{\mathrm{split}} := \widetilde{\mathcal{M}}^{ss}\big(\boldsymbol{y}_1, L, \alpha_*, (\delta, k_0)\big) \times \widetilde{\mathcal{M}}^{ss}(\widehat{\boldsymbol{y}}_2, \alpha_*, +)$.

Proposition 6.10 *In the diagram (79), we have the relation $F^*\big([\widehat{M}^{G_m}(\mathfrak{I})]\big) = G'^*\big([\widetilde{\mathcal{M}}_{\mathrm{split}}]\big)$.*

Proof It follows from the diagram (118). ∎

We can show the following propositions by the same argument.

Proposition 6.11 *Assume that the 2-stability condition holds for $(\boldsymbol{y}, L, \alpha_*, \delta)$.*

- *We have the perfectness of the obstruction theories $\widetilde{\mathrm{ob}}_a(\widehat{M}^{G_m}(\mathfrak{I}))$ in Proposition 5.49.*
- *They give the virtual fundamental class $[\widehat{M}^{G_m}(\mathfrak{I})]$.*
- *We have the relation $F^*\big([\widehat{M}^{G_m}(\mathfrak{I})]\big) = G'^*\big([\mathcal{M}^s(\boldsymbol{y}_1, L, \alpha_*, \delta) \times \mathcal{M}^s(\widehat{\boldsymbol{y}}_2, \alpha_*)]\big)$ in the diagram (82).* ∎

Proposition 6.12 *Under the situation of the subsubsection 5.8.8, the following claims hold:*

- *We have the perfectness of the obstruction theories $\widetilde{\mathrm{ob}}_a(\widehat{M}^{G_m}(\mathfrak{I}))$ in Proposition 5.50.*
- *They give the virtual fundamental class $[\widehat{M}^{G_m}(\mathfrak{I})]$.*
- *We have the relation $F^*\big([\widehat{M}^{G_m}(\mathfrak{I})]\big) = G'^*\big([\mathcal{M}^s(\boldsymbol{y}_1, L, \alpha_*, \delta) \times \mathcal{M}^s(\widehat{\boldsymbol{y}}_2, \alpha_*)]\big)$ in the diagram (85).* ∎

6.1.3 Vanishing of some cohomology groups

We use the notation in the section 5. Let E be a torsion-free sheaf on X. Let $V_\cdot = (V_{-1} \to V_0)$ be a locally free resolution of E. Then we put $C(E) := \mathfrak{g}^\vee(V_\cdot)$, where $\mathfrak{g}^\vee(V_\cdot)$ denotes the dual of $\mathfrak{g}^\vee(V_\cdot)$ as $\mathcal{O}_X$-complexes. We also put $C^\circ(E) := \mathfrak{g}^{\circ\,\vee}(V_\cdot)$ and $C^d(E) := \mathfrak{g}^{d\,\vee}(V_\cdot)$.

Let F_* be a quasi-parabolic structure of E at D. We have $\mathfrak{g}(V_{\cdot|D})$ and $\mathfrak{g}_D(V_\cdot, F_*)$ with the natural morphism $\mathfrak{g}_D(V_\cdot, F_*) \longrightarrow \mathfrak{g}(V_{\cdot|D})$ on D. (See the subsubsection 5.5.1.) We have the dual complexes $\mathfrak{g}^\vee(V_{\cdot|D})$ and $\mathfrak{g}^\vee_D(V_\cdot, F_*)$ as $\mathcal{O}_D$-complexes. We have the natural morphism $C(E) \longrightarrow \mathfrak{g}^\vee(V_\cdot)_{|D} = \mathfrak{g}^\vee(V_{\cdot|D})$. Thus we have the morphism $\alpha : C(E) \oplus \mathfrak{g}^\vee_D(V_\cdot, F_*) \longrightarrow \mathfrak{g}^\vee(V_{\cdot|D})$. We put $C(E, F_*) := \mathrm{Cone}(\alpha)[-1]$.

Recall we have the decompositions $\mathfrak{g}(V_{\cdot|D}) = \mathfrak{g}^\circ(V_{\cdot|D}) \oplus \mathfrak{g}^d(V_{\cdot|D})$ and $\mathfrak{g}_D(V_\cdot, F_*) = \mathfrak{g}^\circ_D(V_\cdot, F_*) \oplus \mathfrak{g}^d_D(V_\cdot, F_*)$. (See the subsubsection 5.5.3.) Thus, we obtain the complexes $C^\circ(E, F_*)$ and $C^d(E, F_*)$. It is easy to see $C^d(E, F_*) = C^d(E)$.

Lemma 6.13 *Let (E, F_*) be as above. The hyper-cohomology group $\mathbb{H}^i\big(X, C^\circ(E, F_*)\big)$ vanishes unless $i = -1, 0, 1$. If (E, F_*) is stable with respect to some system of weights α_*, we also have $\mathbb{H}^{-1}(X, C^\circ(E, F_*)) = 0$.*

Proof The i-th cohomology sheaf $\mathcal{H}^i\big(C^\circ(V_\cdot, F_*)\big)$ vanishes unless $-2 \leq i \leq 1$ by construction. It is easy to see that the morphisms $C^\circ(E)^{-2} \longrightarrow C^\circ(E)^{-1}$ and $\mathfrak{g}^{\circ\vee}_D(V_\cdot, F_*)^{-2} \longrightarrow \mathfrak{g}^{\circ\vee}_D(V_\cdot, F_*)^{-1}$ are injective. Therefore, we have $\mathcal{H}^{-2}\big(C^\circ(E, F_*)\big) = 0$. Since the morphism $C^\circ(E)^1 \longrightarrow \mathfrak{g}^{\circ\vee}(V_\cdot)^1$ is surjective, we have $\mathcal{H}^1\big(C^\circ(E, F_*)\big) = 0$. Let $P \notin D$ be a point where E is locally free. Then we have $\mathcal{H}^0\big(C^\circ(E, F_*)\big) = 0$ around P. Thus the support of the sheaf $\mathcal{H}^0\big(C^\circ(E, F_*)\big)$ is 1-dimensional.

Then, we obtain the vanishing $\mathbb{H}^i\big(X, C^\circ(E,F_*)\big)$ unless $i=-1,0,1$ by using the spectral sequence. It is easy to see that $\mathbb{H}^{-1}\big(X, C^\circ(E,F_*)\big)$ is the set of endomorphisms of (E,F_*) whose trace is 0. If (E,F_*) is assumed to be stable with respect to some weight α_*, we obtain $\mathbb{H}^{-1}\big(X, C^\circ(E,F_*)\big)=0$. ∎

In the rank one case, we have the following result.

Lemma 6.14 *Let (E,F_*) be as above, and we assume* $\mathrm{rank}(E)=1$. *Then, we have $\mathbb{H}^i\big(X, C^\circ(E,F_*)\big)=0$ unless $i=0$.*

Proof Let x be a point of X with one of the following:

- x is contained in $X-D$, and E is locally free around x.
- x is contained in D, E is locally free around x as $\mathcal{O}_X$-module, and $\mathrm{Cok}_i(E)$ are locally free around x as $\mathcal{O}_D$-module.

Around such a point, we can compute the cohomology sheaves of $C^\circ(E,F_*)$ in the case $V_0=E$ and $V_{-1}=0$. Hence it is easy to check $C^\circ(E,F_*)\simeq 0$ around such a point.

We know $\mathcal{H}^i\big(C^\circ(E,F_*)\big)=0$ unless $i=-1,0$, in general. By the above consideration, we know that the support of $\mathcal{H}^i\big(C^\circ(E,F_*)\big)$ is 0-dimensional. Therefore, we obtain that $\mathbb{H}^i\big(X, C^\circ(E,F_*)\big)=0$ unless $i=-1,0$. Since (E,F_*) is always stable in the case $\mathrm{rank}(E)=1$. we also obtain the vanishing $\mathbb{H}^{-1}\big(X, C^\circ(E,F_*)\big)=0$. Thus we are done. ∎

Let L be a line bundle on X. Let ϕ be an L-section of E. We take a locally free resolution $P_\cdot=(P_{-1}\longrightarrow P_0)$ of the line bundle L so that we have a lift $\widetilde{\phi}: P_\cdot\longrightarrow V_\cdot$ of ϕ. We have the complex $\mathfrak{g}_{\mathrm{rel}}(V_\cdot,\widetilde{\phi})=\mathcal{H}om(P_\cdot,V_\cdot)^\vee$ and the dual complex $\mathfrak{g}^\vee_{\mathrm{rel}}(V_\cdot,\widetilde{\phi})$. We have the natural morphism $\mathfrak{g}_{\mathrm{rel}}(V_\cdot,\widetilde{\phi})[-1]\longrightarrow\mathfrak{g}(V)$. Thus we have the morphism $\gamma_L: C(E)\longrightarrow\mathfrak{g}^\vee_{\mathrm{rel}}(V_\cdot,\widetilde{\phi})[1]$. It induces the morphisms $\alpha_L: C(E,F_*)\longrightarrow\mathfrak{g}^\vee_{\mathrm{rel}}(V_\cdot,\widetilde{\phi})[1]$ and $\alpha^\circ_L: C^\circ(E,F_*)\longrightarrow\mathfrak{g}^\vee_{\mathrm{rel}}(V_\cdot,\widetilde{\phi})[1]$. We put $C(E,F_*,\phi):=\mathrm{Cone}(\alpha_L)[-1]$ and $C^\circ(E,F_*,\phi):=\mathrm{Cone}(\alpha^\circ_L)[-1]$. It is easy to see that $C(E,F_*,\phi)$ and $C^\circ(E,F_*,\phi)$ are well defined in $D(X)$.

Lemma 6.15 *Let (E,F_*,ϕ) be as above. Assume $\phi\neq 0$. Then the hyper-cohomology groups $\mathbb{H}^i\big(X, C(E,F_*,\phi)\big)$ vanish unless $i=-1,0,1$. If (E,F_*,ϕ) is (α_*,δ)-stable for some $\delta\in\mathcal{P}^{\mathrm{br}}$ and some weight α_*, we have $\mathbb{H}^{-1}\big(X, C(E,F_*,\phi)\big)=0$.*

Proof We have $\mathcal{H}^i\big(C(E,F_*,\phi)\big)=0$ unless $-2\leq i\leq 1$ by construction. We also have $\mathcal{H}^{-2}\big(C(E,F_*,\phi)\big)=0$ as in the proof of Lemma 6.13. Since the morphism $\mathcal{H}om(P_0,V_0)\longrightarrow\mathcal{H}om(P_{-1},V_0)$ is surjective, we obtain the vanishing $\mathcal{H}^1\big(C(E,F_*,\phi)\big)=0$. Let x be a point of $X-D$ with $\phi(x)\neq 0$ such that E is locally free around x. Then, it is easy to show the vanishing of $\mathcal{H}^0\big(C(E,F_*,\phi)\big)$ around x. Therefore, the support of $\mathcal{H}^0\big(C(E,F_*,\phi)\big)$ is one dimensional. Then we obtain $\mathbb{H}^i\big(X, C(E,F_*,\phi)\big)=0$ unless $i=-1,0,1$ by using the spectral sequence. Since $\mathbb{H}^{-1}\big(X, C(E,F_*,\phi)\big)$ is the set of endomorphisms of (E,F_*,ϕ), we obtain $\mathbb{H}^{-1}\big(X, C(E,F_*,\phi)\big)=0$ from the stability assumption of (E,F_*,ϕ). ∎

Lemma 6.16 *Let (E,F_*,ϕ) be as above. We assume $\phi\neq 0$ and* $\mathrm{rank}(E)>1$. *Then, the hyper-cohomology groups $\mathbb{H}^i\big(X, C^\circ(E,F_*,\phi)\big)$ vanish unless $i=-1,0,1$.*

Proof The i-th cohomology sheaf $\mathcal{H}^i\big(C^\circ(E,F_*,\phi)\big)$ vanish unless $-2\leq i\leq 1$ by construction. We also have the vanishings of $\mathcal{H}^j\big(C^\circ(E,F_*,\phi)\big)$ $j=-2,1$, as in the proof of Lemma 6.15. Let x be a point of $X-D$ with $\phi(x)\neq 0$ such that E is locally free around x. Under our assumption $\mathrm{rank}(E)>1$, we can easily check the vanishing $\mathcal{H}^0\big(C^\circ(E,F_*,\phi)\big)=0$ around such a point x. Thus the claim can be shown by using the spectral sequence. ∎

In the rank one case, we have the following:

Lemma 6.17 *Let (E,F_*,ϕ) be as above. We assume $\phi\neq 0$ and* $\mathrm{rank}(E)=1$. *Moreover, we assume $H^2\big(X, L^{-1}\otimes E\big)=0$. Then, the hyper-cohomology groups $\mathbb{H}^i(X, C^\circ(E,F_*,\phi))$ vanish unless $i=0,1$.*

Proof We have the distinguished triangle $\mathfrak{g}_{\mathrm{rel}}^{\vee}(V,\widetilde{\phi}) \longrightarrow C^{\circ}(E,F_*,\phi) \longrightarrow C^{\circ}(E,F_*) \longrightarrow \mathfrak{g}_{\mathrm{rel}}^{\vee}(V,\widetilde{\phi})[1]$. Hence we obtain the long exact sequence:

$$\cdots \longrightarrow H^i\bigl(X,L^{-1}\otimes E\bigr) \longrightarrow \mathbb{H}^i\bigl(X,C^{\circ}(E,F_*,\phi)\bigr) \longrightarrow \mathbb{H}^i\bigl(X,C(E,F_*)\bigr) \longrightarrow H^{i+1}\bigl(X,L^{-1}\otimes E\bigr) \longrightarrow \cdots$$

Then the claim follows from the assumption and Lemma 6.14. ∎

Let $\boldsymbol{L}=(L_1,L_2)$ be a pair of line bundles on X. Let ϕ_i be L_i-sections of E. By taking appropriate resolutions $P^{(i)}_{\cdot}$ of L_i and lifts $\widetilde{\phi}_i : P^{(i)}_{\cdot} \longrightarrow V_{\cdot}$ of ϕ_i, we obtain the complexes $C(E,F_*,\widetilde{\phi}_1,\widetilde{\phi}_2)$ and $C^{\circ}(E,F_*,\widetilde{\phi}_1,\widetilde{\phi}_2)$ as above. They are well defined in $D(X)$.

Lemma 6.18 *Assume $H^2\bigl(X,L_2^{-1}\otimes E\bigr)=0$ and $\phi_j\neq 0$ for $j=1,2$. We also assume $\mathrm{rank}(E)>1$. Then, we have $\mathbb{H}^i\bigl(X,C^{\circ}(E,F_*,\phi_1,\phi_2)\bigr)=0$ unless $i=-1,0,1$. If (E,F_*,ϕ_1,ϕ_2) is $(\alpha_*,\delta_1,\delta_2)$-stable, then we also have $\mathbb{H}^{-1}\bigl(X,C^{\circ}(E,F_*,\phi_1,\phi_2)\bigr)=0$.*

Proof We have the exact sequence: $0\longrightarrow \mathcal{H}om\bigl(P^{(2)}_{\cdot},V_{\cdot}\bigr)\longrightarrow C(E,F_*,\phi_1,\phi_2)\longrightarrow C(E,F_*,\phi_1)\longrightarrow 0$. Then the claims can be reduced to Lemma 6.15 and Lemma 6.16. ∎

6.1.4 Proof of the propositions in the subsubsection 6.1.1

Proposition 6.1 immediately follows from the following lemma.

Lemma 6.19 *The obstruction theory $\mathrm{Ob}(m,\widehat{\boldsymbol{y}})$ of $\mathcal{M}(m,\widehat{\boldsymbol{y}})$ is perfect in the sense of Definition 2.29. The obstruction theory $\mathrm{Ob}(m,\boldsymbol{y},L)$ of $\mathcal{M}(m,\boldsymbol{y},L)$ is perfect.*

Proof Let us discuss $\mathrm{Ob}(m,\widehat{\boldsymbol{y}})$. We would like to show $\mathrm{Ob}(m,\widehat{\boldsymbol{y}})$ is quasi-isomorphic to a complex $E^{-1}\to E^0\to E^1$ of locally free sheaves on $\mathcal{M}(m,\widehat{\boldsymbol{y}})$. Since $\mathrm{Ob}(m,\widehat{\boldsymbol{y}})$ is obtained from the push-forward of perfect complexes on $\mathcal{M}(m,\widehat{\boldsymbol{y}})\times X$ and $\mathcal{M}(m,\widehat{\boldsymbol{y}})\times D$, it is easy to show that $\mathrm{Ob}(m,\widehat{\boldsymbol{y}})$ is quasi isomorphic to a bounded complex of locally free sheaves by using the projectivity of X and D.

Therefore, we have only to check $H^i\bigl(i_z^*\,\mathrm{Ob}(m,\widehat{\boldsymbol{y}})\bigr)=0$ unless $i=-1,0,1$ for any point $z\in\mathcal{M}(m,\widehat{\boldsymbol{y}})$, where i_z denotes the inclusion of z to $\mathcal{M}(m,\widehat{\boldsymbol{y}})$. Let (E,F_*,ρ) be the parabolic oriented torsion-free sheaf corresponding to z. Then the dual of $\mathcal{H}^i\bigl(i_z^*\,\mathrm{Ob}(m,\widehat{\boldsymbol{y}})\bigr)$ is isomorphic to $\mathbb{H}^{-i}\bigl(X,C^{\mathrm{or}}(E,F_*)\bigr)\oplus H^1(X,\mathcal{O})[0]$. Therefore, the claim follows from Lemma 6.13.

The perfectness of $\mathrm{Ob}(m,\boldsymbol{y},L)$ can be shown similarly, by using Lemma 6.15. ∎

Lemma 6.20 *Assume $\mathrm{rank}(\boldsymbol{y})>1$. The obstruction theory $\mathrm{Ob}(m,\widehat{\boldsymbol{y}},L)$ of $\mathcal{M}(m,\widehat{\boldsymbol{y}},L)$ is perfect. The obstruction theory $\mathrm{Ob}(m,\widehat{\boldsymbol{y}},[L])$ of $\mathcal{M}(m,\widehat{\boldsymbol{y}},[L])$ is perfect.*

Proof We have the following commutative diagram:

$$\begin{array}{ccccc}
\mathrm{Ob}_{\mathrm{rel}}(m,\widehat{y})[-1] & \longrightarrow & \mathrm{Ob}_{\mathrm{rel}}(m,y,L)\oplus\mathrm{Ob}_{\mathrm{rel}}(m,\widehat{y})[-1] & \longrightarrow & \mathrm{Ob}_{\mathrm{rel}}(m,y,L)[-1]\\
\big\downarrow & & \big\downarrow & & \big\downarrow\\
\mathrm{Ob}^d(m,\boldsymbol{y}) & \longrightarrow & \mathrm{Ob}(m,\boldsymbol{y}) & \longrightarrow & \mathrm{Ob}^{\circ}(m,\boldsymbol{y})
\end{array}$$

We put $C_1:=\mathrm{Cone}\bigl(\mathrm{Ob}_{\mathrm{rel}}(m,\widehat{y})[-1]\longrightarrow\mathrm{Ob}^d(m,\boldsymbol{y})\bigr)$ and $C_2:=\mathrm{Cone}\bigl(\mathrm{Ob}_{\mathrm{rel}}(m,y,L)[-1]\longrightarrow\mathrm{Ob}^{\circ}(m,\boldsymbol{y})\bigr)$. We have only to show that C_1 and C_2 are perfect of amplitude in $[-1,1]$. It is easy to see that C_1 is isomorphic to $H^1(X,\mathcal{O})^{\vee}\otimes\mathcal{O}[0]$. To check the claim for C_2, we have only to see $i_z^*C_2$ as in the proof of Lemma 6.19. Then the dual of $\mathcal{H}^i\bigl(i_z^*C_2\bigr)$ are isomorphic to $\mathbb{H}^{-i}\bigl(X,C^{\circ}(E,F_*,\phi)\bigr)$. Thus the claim for C_2 follows from Lemma 6.16. Therefore, we obtain the first claim of the lemma.

Let us show the second claim. We have only to show the vanishing of $\mathcal{H}^i\bigl(\mathrm{Ob}(m,\widehat{\boldsymbol{y}},[L])\bigr)=0$ for $i<-1$. Let $\pi:\mathcal{M}^s(\widehat{\boldsymbol{y}},L,\alpha_*,\delta)\longrightarrow\mathcal{M}^s(\widehat{\boldsymbol{y}},[L],\alpha_*,\delta)$ be the natural morphism, which is smooth. Hence, we have only to show $\mathcal{H}^i\bigl(\pi^*\,\mathrm{Ob}(m,\widehat{\boldsymbol{y}},[L])\bigr)=0$ for $i<-1$. Since we have $\mathcal{H}^i\bigl(\pi^*\,\mathrm{Ob}(m,\widehat{\boldsymbol{y}},[L])\bigr)\simeq\mathcal{H}^i\bigl(\mathrm{Ob}(m,\widehat{\boldsymbol{y}},L)\bigr)$ for $i<-1$, the claim follows from Lemma 6.19. ∎

Let us see Proposition 6.3. The first claim can be shown by using Lemma 6.14 and the argument in the proof of Lemma 6.19. The second and third claims can be shown by using Lemma 6.17 and the argument in the proof of Lemma 6.20.

Proposition 6.4 can be shown by using Lemma 6.18 and the argument in the proof of Lemma 6.20. ∎

6.1.5 Expected dimension

We put $p_g := \dim H^2(X,\mathcal{O})$ and $\chi(\mathcal{O}_X) := 1 - \dim H^1(X,\mathcal{O}_X) + \dim H^2(X,\mathcal{O}_X)$. It is easy to obtain the formulas of the expected dimension in the non-parabolic case. Let y be an element of $H^{\mathrm{ev}}(X)$. The degree $2i$-part of y is denoted by y_i. We put as follows:

$$d(y) := \bigl(y_1^2 - 2y_0 \cdot y_2\bigr) \cap [X] - y_0^2 \cdot \chi(\mathcal{O}_X)$$

For a line bundle L, we put as follows:

$$d_{\mathrm{rel}}(y,L) := \bigl(\mathrm{ch}(L^{-1}) \cdot y \cdot \mathrm{Td}(X)\bigr) \cap [X]$$

Proposition 6.21 *For the moduli of the non-parabolic objects, the expected dimensions are as follows:*

- $\dim^f \mathcal{M}(\widehat{y}) = d(y) + 1 + p_g$.
- $\dim^f \mathcal{M}(y,L,\delta) = d(y) + d_{\mathrm{rel}}(y,L)$.
- $\dim^f \mathcal{M}(\widehat{y},L,\delta) = d(y) + 1 + p_g + d_{\mathrm{rel}}(y,L)$.
- $\dim^f \mathcal{M}(\widehat{y},[L],\delta) = d(y) + p_g + d_{\mathrm{rel}}(y,L)$.

Proof Let us consider the first case. For any $(E,\rho,F_*) \in \mathcal{M}(\widehat{y})$, the virtual tangent space is K-theoretically given by $\sum_i (-1)^i \mathbb{H}^i\bigl(X, C^{\circ}(E)\bigr) + H^1(X,\mathcal{O}_X)$. The Euler number can be easily calculated, and it is given by $d(y) + \chi(\mathcal{O}_X) + \dim H^1(X,\mathcal{O}_X) = d(y) + 1 + p_g$. The second and third cases can be discussed similarly.

Let $\mathfrak{p}$ denote the natural morphism $\mathcal{M}(\widehat{y},L,\delta) \longrightarrow \mathcal{M}(\widehat{y},[L],\delta)$. We have the distinguished triangle on $\mathcal{M}(m,y,L)$.

$$L_{\mathcal{M}(m,\widehat{y},L)/\mathcal{M}(m,\widehat{y},[L])}[-1] \longrightarrow \mathfrak{p}^* \mathrm{Ob}(m,\widehat{y},[L]) \longrightarrow \mathrm{Ob}(m,\widehat{y},L) \longrightarrow L_{\mathcal{M}(m,\widehat{y},L)/\mathcal{M}(m,\widehat{y},[L])}$$

Then the fourth claim is obtained. ∎

The contribution of the parabolic structure can also be calculated formally. Let $\boldsymbol{y} = (y, y_1, y_2, \ldots, y_l)$ be an element of $\mathcal{T}ype$. In this case, y_i can be regarded as elements of $H^{\mathrm{ev}}(D)$. We put $s_i + w_i := \sum_{j \leq i} y_j$, where s_i (resp. w_i) denotes the element of $H^0(D)$ (resp. $H^2(D)$). Let g denote the genus of D. Let $d_{\mathrm{rel}}(y,\boldsymbol{y})$ denote the Euler number of the complex $\mathfrak{g}_{\mathrm{rel}}(V_\cdot, F_*)$ for (E,F_*) of type $\boldsymbol{y}$. The result is as follows:

$$d_{\mathrm{rel}}(y,\boldsymbol{y}) := 2g - 2\sum_{i=1}^{l} s_i(s_i - s_{i+1}) + \sum_{i=1}^{l-1} \int_D \bigl(s_{i+1} \cdot w_i - s_i \cdot w_{i+1}\bigr).$$

Proposition 6.22

- $\dim^f \mathcal{M}(\widehat{\boldsymbol{y}},\alpha_*) = d(y) + 1 + p_g + d_{\mathrm{rel}}(y,\boldsymbol{y})$
- $\dim^f \mathcal{M}(\widehat{\boldsymbol{y}},[L],\alpha_*,\delta) = d(y) + p_g + d_{\mathrm{rel}}(y,\boldsymbol{y}) + d_{\mathrm{rel}}(y,L)$. ∎

6.2 Comparison of the Oriented Reduced Case and the Unoriented Unreduced Case

6.2.1 Statements

We have the natural morphism $\kappa : \mathcal{M}(m,\widehat{\boldsymbol{y}},[L]) \longrightarrow \mathcal{M}(m,\boldsymbol{y},L)$ which is etale and finite of degree $\mathrm{rank}(\boldsymbol{y})^{-1}$. But the obstruction theories are not same, in the case $p_g := \dim H^2(X,\mathcal{O}) > 0$. We would like to compare them.

Proposition 6.23 *We have the following commutative diagram in the derived category $D(\mathcal{M}(m,\widehat{\boldsymbol{y}},[L]))$:*

$$\begin{array}{ccc}\kappa^*\operatorname{Ob}(m,\boldsymbol{y},L) & \longrightarrow & \operatorname{Ob}(m,\widehat{\boldsymbol{y}},[L]) \\ \Big\downarrow & & \Big\downarrow \\ \kappa^* L_{\mathcal{M}(m,\boldsymbol{y},L)} & \xrightarrow{\simeq} & L_{\mathcal{M}(m,\widehat{\boldsymbol{y}},[L])}\end{array}$$

We also have the following distinguished triangle:

$$\kappa^*\operatorname{Ob}(m,\boldsymbol{y},L) \longrightarrow \operatorname{Ob}(m,\widehat{\boldsymbol{y}},[L]) \longrightarrow H^2(X,\mathcal{O})^\vee\otimes\mathcal{O}_{\mathcal{M}(m,\widehat{\boldsymbol{y}},[L])}[1] \longrightarrow \kappa^*\operatorname{Ob}(m,\boldsymbol{y},L)[1]$$

A proof will be given in the next subsubsection. Before going into the proof, we give some consequences.

We have the natural morphism $\mathcal{M}^s(\widehat{\boldsymbol{y}},[L],\alpha_*,\delta)\longrightarrow\mathcal{M}^s(\boldsymbol{y},L,\alpha_*,\delta)$ which is etale and finite of degree $\operatorname{rank}(\boldsymbol{y})^{-1}$. It is also denoted by κ.

Proposition 6.24 *Assume* $\operatorname{rank}(\boldsymbol{y})>1$. *In the case $p_g>0$, we have the vanishing $\big[\mathcal{M}^s(\boldsymbol{y},L,\alpha_*,\delta)\big]=0$. In the case $p_g=0$, we have the following relation:*

$$\kappa^*\big(\big[\mathcal{M}^s(\boldsymbol{y},L,\alpha_*,\delta)\big]\big)=\big[\mathcal{M}^s(\widehat{\boldsymbol{y}},[L],\alpha_*,\delta)\big]$$

Proof Under the assumption $\operatorname{rank}(\boldsymbol{y})>1$, the obstruction theory $\operatorname{ob}(m,\widehat{\boldsymbol{y}},[L])$ of $\mathcal{M}^s(\widehat{\boldsymbol{y}},[L],\alpha_*,\delta)$ is perfect. From Proposition 6.23, we have the relation:

$$\operatorname{Eu}\big(H^2(X,\mathcal{O})\otimes\mathcal{O}\big)\cap\big[\mathcal{M}^s(\widehat{\boldsymbol{y}},[L],\alpha_*,\delta)\big]=\big[\mathcal{M}^s(\boldsymbol{y},L,\alpha_*,\delta)\big]$$

Then the claim is clear. ∎

When $\operatorname{rank}(\boldsymbol{y})=1$, the obstruction theory of $\mathcal{M}(\widehat{\boldsymbol{y}},[L])$ is not perfect, and hence, a similar vanishing result does not hold, in general. But, we obtain the following proposition by the same argument. Since the stability condition is trivial, we omit to denote "s", α_* and δ.

Proposition 6.25 *Let $\boldsymbol{y}$ be an element of $\mathcal{T}ype$ such that* $\operatorname{rank}(\boldsymbol{y})=1$.

- *In the case $p_g>0$, we assume $H^2\big(X,L^{-1}\otimes E\big)=0$ for any L-Bradlow pair $(E_*,\phi)\in\mathcal{M}(\boldsymbol{y},L)$. Then we have $\big[\mathcal{M}(\boldsymbol{y},L)\big]=0$.*
- *In the case $p_g=0$, we have $\kappa^*[\mathcal{M}(\boldsymbol{y},L)]=[\mathcal{M}\big(\widehat{\boldsymbol{y}},[L]\big)]$.*

We remark that the assumption in the first claim always holds in the case $p_g=0$. ∎

Similarly, we have the natural morphism $\kappa:\widetilde{\mathcal{M}}^s(\widehat{\boldsymbol{y}},[L],\alpha_*,\delta,\ell)\longrightarrow\widetilde{\mathcal{M}}^s(\boldsymbol{y},L,\alpha_*,\delta,\ell)$, which is etale and finite of degree $1/\operatorname{rank}(\boldsymbol{y})$. By the same argument, we obtain the following propositions.

Proposition 6.26 *Assume* $\operatorname{rank}(\boldsymbol{y})>1$. *In the case $p_g>0$, we have the vanishing $[\widetilde{\mathcal{M}}^s(\boldsymbol{y},L,\alpha_*,\delta,\ell)]=0$. In the case $p_g=0$, we have the relation $\kappa^*\big[\widetilde{\mathcal{M}}^s(\boldsymbol{y},L,\alpha_*,\delta,\ell)\big]=\big[\widetilde{\mathcal{M}}^s(\widehat{\boldsymbol{y}},[L],\alpha_*,\delta,\ell)\big]$.* ∎

Proposition 6.27 *Assume* $\operatorname{rank}(\boldsymbol{y})=1$.

- *In the case $p_g>0$, we assume $H^2\big(X,L^{-1}\otimes E\big)=0$ for any $(E_*,\phi)\in\mathcal{M}(\boldsymbol{y},L)$. Then, we have the vanishing $[\widetilde{\mathcal{M}}^s(\boldsymbol{y},L,\alpha_*,\delta,\ell)]=0$.*
- *If $p_g=0$, we have the relation $\kappa^*\big[\widetilde{\mathcal{M}}^s(\boldsymbol{y},L,\alpha_*,\delta,\ell)\big]=\big[\widetilde{\mathcal{M}}^s(\widehat{\boldsymbol{y}},[L],\alpha_*,\delta,\ell)\big]$.* ∎

We obtain the following vanishing result for the virtual fundamental class of the fixed point set of the master space.

Proposition 6.28 *Let $\mathfrak{I}=(\boldsymbol{y}_1,\boldsymbol{y}_2,I_1,I_2)$ be a decomposition type (Definition* 4.33*). Assume $p_g>0$.*

- *In the case* $\operatorname{rank}(\boldsymbol{y}_1)>1$, *we have $[M^{G_m}(\mathfrak{I})]=0$.*
- *In the case* $\operatorname{rank}(\boldsymbol{y}_1)=1$, *we assume $H^2\big(X,L^{-1}\otimes E\big)=0$ for any torsion-free sheaf E of type y, moreover. Then we have $[M^{G_m}(\mathfrak{I})]=0$.*

Proof It follows from the propositions 6.10, 6.26 and 6.27. ∎

6.2.2 Proof of Proposition 6.23

The canonical resolutions of the universal sheaves on $\mathcal{M}(m,\boldsymbol{y},[L])\times X$, $\mathcal{M}(m,\boldsymbol{y},L)\times X$ and $\mathcal{M}(m,\widehat{\boldsymbol{y}},[L])\times X$ are denoted by $\mathcal{V}[L]$, $\mathcal{V}(L)$ and $\widehat{\mathcal{V}}[L]$, respectively.

Let us see $\mathrm{Ob}(m,\boldsymbol{y},L)$. From (100), we obtain the following diagram:

$$\begin{tikzcd} \mathrm{Ob}_{\mathrm{rel}}(\mathcal{V}(L),\widetilde{\phi})[-1] \arrow[r] \arrow[d, "\mathfrak{f}_1"'] & \mathrm{Ob}(m,\boldsymbol{y}) \arrow[d] \\ \mathcal{O}[-1] \arrow[r] & \mathcal{O}[-1] \end{tikzcd} \tag{151}$$

We put as follows:

$$\overline{\mathrm{Ob}}_{\mathrm{rel}}(\mathcal{V}(L),\widetilde{\phi}) := \mathrm{Cone}\big(\mathrm{Ob}_{\mathrm{rel}}(\mathcal{V}(L),\widetilde{\phi}) \longrightarrow \mathcal{O}\big)[-1], \quad \overline{\mathrm{Ob}}(m,\boldsymbol{y}) := \mathrm{Cone}\big(\mathrm{Ob}(m,\boldsymbol{y}) \longrightarrow \mathcal{O}[-1]\big)[-1]$$

Then we obtain the following commutative diagram:

$$\begin{tikzcd} \overline{\mathrm{Ob}}_{\mathrm{rel}}(\mathcal{V}(L),\widetilde{\phi})[-1] \arrow[r] \arrow[d] & \mathrm{Ob}_{\mathrm{rel}}(\mathcal{V}(L),\widetilde{\phi})[-1] \arrow[r] \arrow[d] & L_{\mathcal{M}(m,\boldsymbol{y},L)/\mathcal{M}(m,\boldsymbol{y})}[-1] \arrow[d] \\ \overline{\mathrm{Ob}}(m,\boldsymbol{y}) \arrow[r] & \mathrm{Ob}(m,\boldsymbol{y}) \arrow[r] & L_{\mathcal{M}(m,\boldsymbol{y})} \end{tikzcd}$$

Thus we put as follows:

$$\overline{\mathrm{Ob}}(m,\boldsymbol{y},L) := \mathrm{Cone}\big(\overline{\mathrm{Ob}}_{\mathrm{rel}}(\mathcal{V}(L),\phi)[-1] \longrightarrow \overline{\mathrm{Ob}}(m,\boldsymbol{y})\big)$$

Then we obtain the morphisms $\overline{\mathrm{Ob}}(m,\boldsymbol{y},L) \longrightarrow \mathrm{Ob}(m,\boldsymbol{y},L) \longrightarrow L_{\mathcal{M}(m,\boldsymbol{y},L)}$. Since the first morphism is quasi isomorphic, the composite gives the equivalent obstruction theory.

On the other hand, let $\pi_2 : \mathcal{M}(m,\boldsymbol{y},L) \longrightarrow \mathcal{M}(m,\boldsymbol{y},[L])$ denote the natural projection. Due to Lemma 5.21, we have the following commutative diagram on $\mathcal{M}(m,\boldsymbol{y},L)$ from the diagram (151):

$$\begin{tikzcd} \pi_2^*\,\mathrm{Ob}'_{\mathrm{rel}}(\mathcal{V}[L],[\widetilde{\phi}])[-1] \arrow[r, "\simeq"] \arrow[d] & \mathrm{Ob}_{\mathrm{rel}}(\mathcal{V}(L),\widetilde{\phi})[-1] \arrow[r] \arrow[d] & \mathrm{Ob}(m,\boldsymbol{y}) \arrow[d] \\ L_{\mathcal{M}(m,\boldsymbol{y})_{G_m}/\mathcal{M}(m,\boldsymbol{y})} \arrow[r, "\simeq"] & \mathcal{O}[-1] \arrow[r] & \mathcal{O}[-1] \end{tikzcd}$$

In particular, we have the isomorphism

$$\pi_2^*\,\mathrm{Ob}_{\mathrm{rel}}(\mathcal{V}[L],[\widetilde{\phi}]) \simeq \overline{\mathrm{Ob}}(\mathcal{V}(L),\widetilde{\phi}) \tag{152}$$

On $\mathcal{M}(m,\widehat{\boldsymbol{y}},[L])$, we have $\mathrm{Ob}_{\mathrm{rel}}(\widehat{\mathcal{V}}[L],\rho^u) := \mathrm{Cone}\Big(\mathrm{Ob}^d(\widehat{\mathcal{V}}[L]) \longrightarrow L_{\mathrm{Pic}}\Big)$. The trace map induces the following commutative diagram:

$$\begin{tikzcd} \mathrm{Ob}_{\mathrm{rel}}(\widehat{\mathcal{V}}[L],\rho^u)[-1] \arrow[r] \arrow[d] & \mathrm{Ob}(m,\boldsymbol{y}) \arrow[d, "\mathrm{tr}"'] \\ \mathcal{O}[-1] \arrow[r] & \mathcal{O}[-1] \end{tikzcd}$$

We put $\overline{\mathrm{Ob}}_{\mathrm{rel}}(\widehat{\mathcal{V}}[L],\rho^u) := \mathrm{Cone}\Big(\mathrm{Ob}_{\mathrm{rel}}(\widehat{\mathcal{V}}[L],\rho^u) \longrightarrow \mathcal{O}\Big)[-1]$. Then we have the following commutative diagram:

$$\begin{tikzcd} \kappa^*\pi^*\overline{\mathrm{Ob}}_{\mathrm{rel}}(\mathcal{V}[L],[\widetilde{\phi}]) \oplus \overline{\mathrm{Ob}}_{\mathrm{rel}}(\widehat{\mathcal{V}}[L],\rho^u) \arrow[r] \arrow[d] & \overline{\mathrm{Ob}}(m,\boldsymbol{y}) \arrow[d] \\ \mathrm{Ob}_{\mathrm{rel}}(\widehat{\mathcal{V}}[L],[\widetilde{\phi}]) \oplus \mathrm{Ob}_{\mathrm{rel}}(\widehat{\mathcal{V}}[L],\rho^u) \arrow[r] & \mathrm{Ob}(m,\boldsymbol{y}) \end{tikzcd}$$

We put as follows:

$$\overline{\mathrm{Ob}}(m,\widehat{\boldsymbol{y}},[L]) := \mathrm{Cone}\Big(\kappa^*\pi^*\overline{\mathrm{Ob}}_{\mathrm{rel}}(\mathcal{V}[L],[\widetilde{\phi}]) \oplus \overline{\mathrm{Ob}}_{\mathrm{rel}}(\widehat{\mathcal{V}}[L],\rho^u) \longrightarrow \overline{\mathrm{Ob}}(m,\boldsymbol{y})\Big)$$

Then we have the morphisms $\overline{\mathrm{Ob}}(m,\widehat{\boldsymbol{y}},[L]) \longrightarrow \mathrm{Ob}(m,\widehat{\boldsymbol{y}},[L]) \longrightarrow L_{\mathcal{M}(m,\widehat{\boldsymbol{y}},[L])}$. Since the first morphism is quasi isomorphic, the composite gives the equivalent obstruction theory.

By the construction, we have the following isomorphism:

$$\overline{\mathrm{Ob}}_{\mathrm{rel}}(\widehat{\mathcal{V}}[L],\rho^u) \simeq H^2(X,\mathcal{O})^\vee \otimes \mathcal{O}[2]$$

Then the claim of the proposition immediately follows. ∎

6.3 Rank One Case

6.3.1 The moduli of L-abelian pairs

Let L be a line bundle on X. An L-Bradlow pair (E,ϕ) is called an L-abelian pair, if E is a line bundle. Similarly, a reduced L-Bradlow pair $(E,[\phi])$ is called a reduced L-abelian pair, if E is a line bundle.

Let c be an element of $H^2(X)$. We denote by $M(c,L)$ the moduli of L-abelian pairs (E,ϕ) such that $c_1(E)=c$. We denote by $M(\widehat{c},[L])$ the moduli of oriented reduced L-abelian pairs $(E,[\phi],\rho)$ such that $c_1(E)=c$. We have the isomorphism $M(\widehat{c},[L]) \simeq M(c,L)$ of schemes. We have the projection $M(c,L) \longrightarrow \mathrm{Pic}(c)$, which is a projectivization of a cone over $\mathrm{Pic}(c)$.

The universal object on $M(\widehat{c},[L])\times X$ is denoted by $(\widehat{\mathcal{L}}^u,[\phi^u])$. The line bundle $\widehat{\mathcal{L}}^u$ is the pull back of the Poincaré bundle on $\mathrm{Pic}(c)\times X$. On the other hand, the universal object on $M(c,L)\times X$ is denoted by $(\mathcal{L}^u,\phi^u)$. We have the relation $\mathcal{L}^u = \widehat{\mathcal{L}}^u \otimes \mathcal{O}_{\mathrm{rel}}(1)$.

The obstruction theories of $M(c,L)$ and $M(\widehat{c},[L])$ are denoted by $\mathrm{Ob}\big(M(c,L)\big)$ and $\mathrm{Ob}\big(M(\widehat{c},[L])\big)$ respectively. They are not equivalent in general. In fact, $\mathrm{Ob}\big(M(\widehat{c},[L])\big)$ is not perfect in the case $p_g>0$ in general, unless $H^2(X,\mathcal{L})=0$ for any $\mathcal{L}$ such that $c_1(\mathcal{L})=c$.

In the case $H^1(X,\mathcal{O}_X)=0$, we have a simple description of the moduli of abelian pairs. We work on the complex number fields. Let c be an element of $H^2(X,\mathbb{Z})$. Let $\mathcal{L}$ be a line bundle on X such that $c_1(\mathcal{L})=c$. Due to the assumption $H^1(X,\mathcal{O}_X)=0$, any line bundle $\mathcal{L}'$ with $c_1(\mathcal{L}')$ is isomorphic to $\mathcal{L}$. Hence, the moduli $M(c,L)$ is isomorphic to the projective space $\mathbb{P}\big(H^0(X,L^{-1}\otimes\mathcal{L})^\vee\big)$.

Since the moduli is smooth, the obstruction theory of $M(c,L)$ is given by the obstruction bundle $O(c)$, and the virtual fundamental class is the Euler class of $O(c)$. The vector bundle $O(c)$ is obtained as follows: We have the universal sheaf $\mathcal{L}^u=\mathcal{L}\otimes\mathcal{O}_{\mathrm{rel}}(1)$ and the universal L-section $\phi: p_{M(c,L)}^*L \longrightarrow \mathcal{L}^u$ over $M(c,L)\times X$. We have the cokernel sheaf $\mathrm{Cok}:=\mathrm{Cok}(\phi)$. Then the sheaves $p_{X*}(\mathrm{Cok})$ and $R^1p_{X*}(\mathrm{Cok})$ are locally free $\mathcal{O}_{M(c,L)}$-modules. The vector bundle $p_{X\,*}(\mathrm{Cok})$ gives the tangent bundle of the moduli, and $R^1\pi_{X\,*}(\mathrm{Cok})$ gives the obstruction bundle $O(c)$. The virtual fundamental class is given by the Euler class $Eu(O(c))\cap[M(c,L)]_0$, where $[M(c,L)]_0$ is the ordinary fundamental class of the smooth variety $M(c,L)$.

Let $d(c,L)$ denote the dimension of the smooth variety $M(c,L)$, which is same as $\dim(H^0(X,L^{-1}\otimes\mathcal{L}))-1$. We put $\chi(L^{-1}\otimes\mathcal{L}) := \dim H^0(X,L^{-1}\otimes\mathcal{L}) - \dim H^1(X,L^{-1}\otimes\mathcal{L}) + \dim H^2(X,L^{-1}\otimes\mathcal{L})$.

Proposition 6.29 *Assume $H^1(X,\mathcal{O}_X)=0$ and $p_g=\dim H^2(X,\mathcal{O}_X)>0$. Moreover, we assume that the virtual fundamental class of $M(c,L)$ is not 0. Let $\mathcal{L}$ be a line bundle such that $c_1(\mathcal{L})=c$. Then, we have $\chi(L^{-1}\otimes\mathcal{L})=\chi(\mathcal{O}_X)$, and the expected dimension of $M(c,L)$ is 0.*

The total Chern class of the obstruction bundle $O(c)$ is $\big(1+c_1(\mathcal{O}_{\mathrm{rel}}(1))\big)^{d(c,L)-p_g}$, and the virtual fundamental class is as follows:

$$\frac{\prod_{i=1}^{d(c,L)}(i-p_g)}{d(c,L)!}\cdot[p]$$

Here $[p]$ denotes the cohomology class of a point $M(c,L)$. We also have the inequalities $d(c,L)<p_g$ and $\dim H^1(X,L^{-1}\otimes\mathcal{L}) < \dim H^2(X,L^{-1}\otimes\mathcal{L})$.

Proof Since we have assumed $H^1(X,\mathcal{O}_X)=0$, the obstruction bundle $O(c)$ can be decomposed on $M(c,L)$ as follows:

$$0 \longrightarrow H^1(X,L^{-1}\otimes\mathcal{L})\otimes\mathcal{O}_{\mathrm{rel}}(1) \longrightarrow O(c) \longrightarrow H^2(X,\mathcal{O})\otimes\mathcal{O}_{M(c,L)} \longrightarrow H^2(X,L^{-1}\otimes\mathcal{L})\otimes\mathcal{O}_{\mathrm{rel}}(1) \longrightarrow 0.$$

Thus the total Chern class of the vector bundle $O(c)$ is $\bigl(1+c_1(\mathcal{O}_{\mathrm{rel}}(1))\bigr)^{\dim(H^1(X,L^{-1}\otimes\mathcal{L}))-\dim(H^2(X,L^{-1}\otimes\mathcal{L}))}$. On the other hand, the rank of $O(c)$ is $\dim\bigl(H^1(X,L^{-1}\otimes\mathcal{L})\bigr)-\dim\bigl(H^2(X,L^{-1}\otimes\mathcal{L})\bigr)+p_g$. Thus, if $\dim(H^1(X,L^{-1}\otimes\mathcal{L}))-\dim(H^2(X,L^{-1}\otimes\mathcal{L}))\geq 0$, the Euler class of $O(c)$ is 0 due to our assumption $p_g>0$. It contradicts with our assumption that the virtual fundamental class is not 0. Therefore, we have $-e=\dim(H^1(X,L^{-1}\otimes\mathcal{L}))-\dim(H^2(X,L^{-1}\otimes\mathcal{L}))<0$. Then, the coefficient of $c_1(\mathcal{O}_{\mathrm{rel}}(1))^{d(c,L)}$ in the polynomial $\bigl(1+c_1(\mathcal{O})_{\mathrm{rel}}\bigr)^{-e}$ is not 0. Thus, the rank of the $O\bigl(c\bigr)$ must be $d(c,L)=\dim(H^0(X,L^{-1}\otimes\mathcal{L}))-1$. Therefore, we obtain $\chi(\mathcal{O}_X)=\chi(L^{-1}\otimes\mathcal{L})$, and the expected dimension of the moduli is 0. By a formal calculation, we obtain the virtual fundamental class. ∎

Remark 6.30 *In particular, we have the vanishing of the virtual fundamental class in the case $H^2(X,L^{-1}\otimes\mathcal{L})=0$. We can derive it also from Proposition* 6.25. ∎

Let us consider the virtual fundamental class of $M(\widehat{c},[L])$ when $H^2(X,L^{-1}\otimes\mathcal{L})=0$ holds for any line bundle $\mathcal{L}$ such that $c_1(\mathcal{L})=c$. We have $M(\widehat{c},[L])\simeq M(c,L)\simeq\mathbb{P}\bigl(H^0(X,L^{-1}\otimes\mathcal{L})^{\vee}\bigr)$ for such a line bundle $\mathcal{L}$. The obstruction theory of $M(\widehat{c},[L])$ is perfect (Proposition 6.3). Since the moduli is smooth, we have the obstruction bundle $\widehat{O}(c)$, and the virtual fundamental class is the Euler class of $\widehat{O}(c)$. The following proposition can be checked directly from the construction of the obstruction theory.

Proposition 6.31 *We assume $H^1(X,\mathcal{O})=0$ and $H^2(X,L^{-1}\otimes\mathcal{L})=0$ for any line bundle $\mathcal{L}$ such that $c_1(\mathcal{L})=c$. Let $\mathcal{L}$ be a line bundle such that $c_1(\mathcal{L})=c$. The expected dimension of $M(\widehat{c},[L])$ is $\chi(L^{-1}\otimes\mathcal{L})$. The obstruction bundle $\widehat{O}(c)$ is given by $H^1(X,\mathcal{L})\otimes\mathcal{O}_{\mathrm{rel}}(1)$. The total Chern class of $\widehat{O}(c)$ is $\bigl(1+c_1(\mathcal{O}_{\mathrm{rel}}(1))\bigr)^{\dim H^1(X,\mathcal{L})}$. In particular, the virtual fundamental class is given by $c_1(\mathcal{O}_{\mathrm{rel}}(1))^{\dim H^1(X,\mathcal{L})}\cap[M(\widehat{c},[L])]_0$, where $[M(\widehat{c},[L])]$ denotes the ordinary fundamental class.* ∎

6.3.2 Parabolic Hilbert scheme

Let $\boldsymbol{y}$ be an element of $\mathcal{T}ype$ such that $\mathrm{rank}(\boldsymbol{y})=1$. We assume that the first Chern class of y is trivial. Then, let $X^{[\boldsymbol{y}]}$ denote the moduli of the oriented parabolic sheaves E_* of rank one such that $\det(E)=\mathcal{O}_X$. In other words, $X^{[\boldsymbol{y}]}$ denotes the moduli space of ideal sheaves of 0-dimensional schemes with parabolic structure of an appropriate type. We call $X^{[\boldsymbol{y}]}$ the parabolic Hilbert scheme of type $\boldsymbol{y}$. The universal sheaf $\mathcal{I}^u$ over $X^{[\boldsymbol{y}]}\times X$ can be regarded as the ideal sheaf of the relatively 0-dimensional scheme $\mathcal{Z}(\boldsymbol{y})$. The relative length is given by $-y_2$, where y_2 denotes the $H^4(X)$-component of y.

Proposition 6.32 *When D is smooth, the parabolic Hilbert scheme is smooth. The expected dimension is same as the ordinary dimension.*

Proof $X^{[\boldsymbol{y}]}$ is the fiber of the smooth morphism $\mathcal{M}(\widehat{\boldsymbol{y}})\longrightarrow\mathrm{Pic}$. Hence the claim follows from the first claim of Proposition 6.3. ∎

Due to Proposition 6.32, the obstruction theory of $X^{[\boldsymbol{y}]}$ is obvious. But, we give another expression of the obstruction theory of $X^{[\boldsymbol{y}]}$ for later use.

Let $V_\cdot=(V_{-1}\to V_0)$ be a locally free resolution of the universal sheaf E over $X^{[\boldsymbol{y}]}\times X$. We take vector spaces W_i such that $\mathrm{rank}\,W_i=\mathrm{rank}\,V_i$. We put $\mathrm{SGL}(W_\cdot):=\bigl\{(g_{-1},g_0)\in\mathrm{GL}(W_\cdot),\bigm|\det(g_{-1})\det(g_0)=1\bigr\}$. We put $\overline{Y}(W):=N(W_{-1\,X},W_{0\,X})_{\mathrm{SGL}(W_\cdot)}$. Then, we have the classifying map $\overline{\Phi}(E):X^{[\boldsymbol{y}]}\times X\longrightarrow\overline{Y}(W_\cdot)$. Thus, we obtain the morphism $\overline{\Phi}(E)^*L_{\overline{Y}(W_\cdot)/X}\longrightarrow L_{X^{[\boldsymbol{y}]}\times X/X}$. We can show that $\overline{\Phi}(E)^*L_{\overline{Y}(W_\cdot)/X}$ is expressed by $\mathfrak{g}^{\circ}(V_\cdot)_{\leq 1}$. Therefore, we obtain the morphism $\mathfrak{g}^{\circ}(V_\cdot)\longrightarrow L_{X^{[\boldsymbol{y}]}\times X/X}$, and it induces $\mathrm{Ob}^{\circ}(V_\cdot)\longrightarrow L_{X^{[\boldsymbol{y}]}}$. Since it is uniquely determined in the derived category $D(X^{[\boldsymbol{y}]})$, we denote it by $\mathrm{Ob}^{\circ}(y)$.

We put $\overline{Y}_D(W.) := N(W_{-1\,D}, W_{0\,D})_{\mathrm{SGL}(W.)}$. We have the naturally defined right action of $\prod_{i=2}^{l}\mathrm{GL}(W^{(i)}) \times \mathrm{SGL}(W.)$ on $\prod_{i=1}^{l} N\big(W^{(i+1)}, W^{(i)}\big)$. The quotient stack is denoted by $\overline{Y}_D(W., W^*)$. We have the naturally defined morphism $\pi : \overline{Y}_D(W., W^*) \longrightarrow \overline{Y}_D(W.)$.

We have the morphisms $\overline{\Phi}(V._{|D}) : X^{[\boldsymbol{y}]} \times D \longrightarrow \overline{Y}_D(W.)$ and $\overline{\Phi}_D(V._{|D}, V^*_D) : X^{[\boldsymbol{y}]} \times D \longrightarrow \overline{Y}(W., W^*)$. We have $\pi \circ \overline{\Phi}_D(V._{|D}, V^*) = \overline{\Phi}(V._{|D})$. Thus, we obtain the morphisms:

$$L_{X^{[\boldsymbol{y}]}\times D/D} \longleftarrow \overline{\Phi}_D(V._{|D}, V^*_D)^* L_{\overline{Y}_D(W., W^*)/D} \longleftarrow \overline{\Phi}(V._{|D})^* L_{\overline{Y}_D(W.)/D}$$

It is easy to see that $\overline{\Phi}_D(V._{|D}, V^*_D)^* L_{\overline{Y}_D(W., W^*)/D}$ and $\overline{\Phi}(V._{|D})^* L_{\overline{Y}_D(W.)/D}$ are expressed by $\mathfrak{g}^\circ_D(V., F_*)_{\leq 1}$ and $\mathfrak{g}^\circ(V._{|D})_{\leq 1}$. Thus, we obtain the morphisms:

$$L_{X^{[\boldsymbol{y}]}\times D/D} \longleftarrow \mathfrak{g}^\circ_D(V., F_*) \longleftarrow \mathfrak{g}^\circ(V._{|D})$$

We put $\mathrm{Ob}^\circ_D(\boldsymbol{y}) := Rp_{D\,*}\big(\mathfrak{g}^\circ_D(V., F_*) \otimes \omega_D\big)$. We also put $\mathrm{Ob}^\circ_D(y) := Rp_{D\,*}\mathfrak{g}^\circ(V._{|D}) \otimes \omega_D$. They are independent of the choice of $V.$ in the derived category. Then, we obtain the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Ob}^\circ_D(y) \arrow[r, "i_1"] \arrow[d, "i_2"'] & \mathrm{Ob}^\circ(y) \arrow[d] \\
\mathrm{Ob}^\circ_D(\boldsymbol{y}) \arrow[r] & L_{X^{[\boldsymbol{y}]}}
\end{tikzcd}$$

The cone of $(i_1, -i_2) : \mathrm{Ob}^\circ_D(y) \longrightarrow \mathrm{Ob}(y) \oplus \mathrm{Ob}^\circ_D(\boldsymbol{y})$ is denoted by $\mathrm{Ob}^\circ(\boldsymbol{y})$. We obtain the morphism $\mathrm{ob}^\circ(\boldsymbol{y}) : \mathrm{Ob}^\circ(\boldsymbol{y}) \longrightarrow L_{X^{[\boldsymbol{y}]}}$.

Lemma 6.33 *The morphism* $\mathrm{ob}^\circ(\boldsymbol{y})$ *is quasi-isomorphic.*

Proof We have the inclusion $X^{[\boldsymbol{y}]} \longrightarrow \mathcal{M}(\widehat{\boldsymbol{y}})$. We take a locally free resolution of the universal sheaf on $\mathcal{M}(\widehat{\boldsymbol{y}}) \times X$. The restriction to $X^{[\boldsymbol{y}]} \times X$ gives the locally free resolution of the universal sheaf on $X^{[\boldsymbol{y}]} \times X$. Then, we obtain the following naturally defined commutative diagram:

$$\begin{tikzcd}
X^{[\boldsymbol{y}]} \times X \arrow[r, "j_1"] \arrow[d] & \mathcal{M}(\widehat{\boldsymbol{y}}) \times X \arrow[r, "j_2"] \arrow[d] & \mathrm{Pic} \times X \arrow[d] \\
\overline{Y}(W.) \arrow[r] & Y(W.) \arrow[r] & X_{G_m}
\end{tikzcd}$$

Then, we can obtain the following commutative diagram on $X^{[\boldsymbol{y}]}$:

$$\begin{tikzcd}
L_{X^{[\boldsymbol{y}]}} & j_1^* L_{\mathcal{M}(\widehat{\boldsymbol{y}})} \arrow[l] & j_1^* j_2^* L_{\mathrm{Pic}} \arrow[l] \\
\mathrm{Ob}^\circ(y) \arrow[u] & \mathrm{Ob}(y) \arrow[l] \arrow[u] & \mathrm{Ob}^d(y) \arrow[l] \arrow[u]
\end{tikzcd}$$

By the construction of $\mathrm{Ob}(\widehat{y})$, we can obtain the following:

$$\begin{tikzcd}
L_{X^{[\boldsymbol{y}]}} & j_1^* L_{\mathcal{M}(\widehat{\boldsymbol{y}})} \arrow[l] & j_1^* j_2^* L_{\mathrm{Pic}} \arrow[l] \\
\mathrm{Ob}^\circ(y) \arrow[u] & \mathrm{Ob}(\widehat{y}) \arrow[l] \arrow[u] & j_1^* j_2^* L_{\mathrm{Pic}} \arrow[l] \arrow[u]
\end{tikzcd}$$

It is easy to observe that the both of the horizontal rows are distinguished.

We also have the following diagram:

$$\begin{tikzcd}
X^{[\boldsymbol{y}]} \times X \arrow[r] \arrow[d] & \overline{Y}_D(W., W^*) \arrow[r] \arrow[d] & \overline{Y}_D(W.) \arrow[d] \\
\mathcal{M}(\widehat{\boldsymbol{y}}) \arrow[r] & Y_D(W., W^*) \arrow[r] & Y_D(W.)
\end{tikzcd}$$

We remark $\mathrm{Ob}(\widehat{\boldsymbol{y}}) \simeq \mathrm{Cone}\big(\mathrm{Ob}_D(y) \longrightarrow \mathrm{Ob}_D(\boldsymbol{y}) \oplus \mathrm{Ob}(\widehat{y})\big)$. Then, we can derive the following commutative diagram by construction of the complexes:

$$\begin{array}{ccccc}
L_{X^{[\boldsymbol{y}]}} & \longleftarrow & \mathrm{Ob}^{\circ}_D(\boldsymbol{y}) & \longleftarrow & \mathrm{Ob}^{\circ}_D(y) \\
\uparrow & & \uparrow & & \uparrow \\
j_1^* L_{\mathcal{M}(\widehat{\boldsymbol{y}})} & \longleftarrow & \mathrm{Ob}_D(\boldsymbol{y}) & \longleftarrow & \mathrm{Ob}_D(y)
\end{array}$$

Here, $\mathrm{Ob}_D(\boldsymbol{y}) := Rp_{D\,*}\mathfrak{g}_D(V, F_*) \otimes \omega_D$ and $\mathrm{Ob}_D(y) := Rp_{D\,*}\mathfrak{g}(V_{|D}) \otimes \omega_D$ on $\mathcal{M}(\widehat{\boldsymbol{y}})$. Then, we obtain the following morphism of the distinguished triangles:

$$\begin{array}{ccccc}
L_{X^{[\boldsymbol{y}]}} & \longleftarrow & j_1^* L_{\mathcal{M}(\widehat{\boldsymbol{y}})} & \longleftarrow & j_1^* j_2^* L_{\mathrm{Pic}} \\
\uparrow & & {\scriptstyle a_1}\uparrow & & {\scriptstyle a_2}\uparrow \\
\mathrm{Ob}^{\circ}(\boldsymbol{y}) & \longleftarrow & \mathrm{Ob}(\widehat{\boldsymbol{y}}) & \longleftarrow & j_1^* j_2^* L_{\mathrm{Pic}}
\end{array}$$

The morphisms a_i $(i = 1, 2)$ are isomorphic. Therefore, the claim of the lemma follows. ∎

6.3.3 Splitting into the moduli of the abelian pairs and the parabolic Hilbert schemes

Let $\boldsymbol{y}$ be an element of $\mathcal{T}ype$ such that $\mathrm{rank}(\boldsymbol{y}) = 1$. We have the following description of $\mathcal{M}\big(\boldsymbol{y}, L\big)$. Let c denote the $H^2(X)$-part of y, and $\boldsymbol{y}(-c) := \boldsymbol{y} \cdot \exp(-c)$. We have the universal line bundle $\mathcal{L}_c$ on $M(c, L) \times X$. We also have the structure sheaf $\mathcal{O}_{\mathcal{Z}(\boldsymbol{y}(-c))}$ of the subscheme $\mathcal{Z}(\boldsymbol{y}(-c)) \subset X^{[\boldsymbol{y}(-c)]} \times X$. The pull back of them via the projection $M(c, L) \times X^{[\boldsymbol{y}(-c)]} \times X$ onto $M(c, L) \times X$ and $X^{[\boldsymbol{y}(-c)]} \times X$ are denoted by the same notation. We put as follows:

$$\mathcal{K} := \mathcal{L}_c^u \otimes L^{-1} \otimes \mathcal{O}_{\mathcal{Z}(\boldsymbol{y}(-c))}.$$

It is easy to see that $\mathfrak{V} := p_{X\,*}\mathcal{K}$ is a locally free sheaf on $M(c, L) \times X^{[\boldsymbol{y}(-c)]}$. We have the natural morphism $p_{X\,*}\mathcal{L}_c^u \otimes L^{-1} \longrightarrow \mathfrak{V}$. We also have the natural section of $p_{X\,*}(L^{-1} \otimes \mathcal{L}_c^u)$ induced by the universal section ϕ^u for $M(c, L)$. Therefore, we obtain the section $\psi : M(c, L) \times X^{[\boldsymbol{y}(-c)]} \longrightarrow \mathfrak{V}$. It is easy to observe that $\psi^{-1}(0)$ is isomorphic to the moduli $\mathcal{M}(\boldsymbol{y}, L)$.

We have the following Cartesian diagram:

$$\begin{array}{ccc}
\mathcal{M}(\boldsymbol{y}, L) & \xrightarrow{F} & M(c, L) \times X^{[\boldsymbol{y}(-c)]} \\
{\scriptstyle j}\big\downarrow & & \big\downarrow{\scriptstyle \psi} \\
M(c, L) \times X^{[\boldsymbol{y}(-c)]} & \xrightarrow{i} & \mathfrak{V}
\end{array} \tag{153}$$

Here i denotes the 0-section. Since i is a regular embedding, we can define the Gysin map $i^!$.

Proposition 6.34

- *We have the relation* $i^!\big([M(c, L)] \times [X^{[\boldsymbol{y}(-c)]}]\big) = [\mathcal{M}(\boldsymbol{y}, L)]$ *among the virtual fundamental classes. In particular, we have* $i_*[\mathcal{M}(\boldsymbol{y}, L)] = \mathrm{Eu}(\mathfrak{V}) \cap \big([M(c, L)] \times [X^{[\boldsymbol{y}(-c)]}]\big)$*, where* $\mathrm{Eu}(\mathfrak{V})$ *denotes the Euler class of* $\mathfrak{V}$*.*
- *Assume* $H^2\big(X, \mathcal{L}\big) = 0$ *for any* $\mathcal{L} \in \mathrm{Pic}(c)$ *. Then, we have the relation* $i^!\big([M(\widehat{c}, L)] \times [X^{[\boldsymbol{y}(-c)]}]\big) = \big[\mathcal{M}(\widehat{\boldsymbol{y}}, [L])\big]$ *and* $i_*[\mathcal{M}(\widehat{\boldsymbol{y}}, [L])] = \mathrm{Eu}(\mathfrak{V}) \cap \big([M(\widehat{c}, [L])] \times [X^{[\boldsymbol{y}(-c)]}]\big)$*.*

The proof will be given in the next subsubsections 6.3.4–6.3.7.

Before going into the proof, we remark that the study of $\big[\mathcal{M}(\boldsymbol{y}, L)\big]$ is reduced to the study of $X^{[\boldsymbol{y}(-c)]}$ by Proposition 6.29 and Proposition 6.34, when we assume $H^1(X, \mathcal{O}) = 0$ and $p_g > 0$. We are mainly interested in the cap product of some cohomology class Φ and the fundamental class $\big[\mathcal{M}(\boldsymbol{y}, L)\big]$. In the interesting cases, the cohomology class Φ is defined on $M(c, L) \times X^{[\boldsymbol{y}(-c)]}$, and thus we have only to consider the product of Φ and $i_*\big[\mathcal{M}(\boldsymbol{y}, L)\big]$. Let $\mathcal{L}$ be any line bundle on X such that $c_1(\mathcal{L}) = c$. We put $\widetilde{\mathcal{K}} := \mathcal{O}_{\mathcal{Z}(\widehat{\boldsymbol{y}}(-c))} \otimes \mathcal{L} \otimes L^{-1}$. We put $\widetilde{\mathfrak{V}} := p_{X\,*}\widetilde{\mathcal{K}}$ which is the vector bundle on $X^{[\boldsymbol{y}(-c)]}$.

Proposition 6.35 *Assume $H^1(X,\mathcal{O})=0$ and $p_g>0$. We have the following formula in the cohomology ring $H^*\bigl(M(c,L)\times X^{[\boldsymbol{y}(-c)]}\bigr)\simeq H^*\bigl(M(c,L)\bigr)\otimes H^*(X^{[\boldsymbol{y}(-c)]})$:*

$$i_*\bigl[\mathcal{M}(\boldsymbol{y},L)\bigr]=\frac{\prod_{i=1}^{d(c,L)}(i-p_g)}{d(c,L)!}\cdot[p]\times\bigl(\mathrm{Eu}(\widetilde{\mathfrak{V}})\cap[X^{[\boldsymbol{y}(-c)]}]\bigr) \tag{154}$$

Here $[p]$ denotes the cohomology class of a point of $M(c,L)$.

Proof When $[\mathcal{M}(\boldsymbol{y},L)]\neq 0$, the expected dimension of $M(c,L)$ is 0, and $[M(c,L)]$ is same as $\bigl(d(c,L)!\bigr)^{-1}\cdot\prod_{i=1}^{d(c,L)}(i-p_g)$ points, due to Proposition 6.29. Therefore, we have only to consider $\mathrm{Eu}(\mathfrak{V})\cap\bigl([p]\times[X^{[\boldsymbol{y}(-c)]}]\bigr)$. Moreover, we can replace $\mathfrak{V}$ with $\mathfrak{V}_{|\{p\}\times X^{[\boldsymbol{y}(-c)]}}=\widetilde{\mathfrak{V}}_{|\{p\}\times X^{[\boldsymbol{y}(-c)]}}$. Then we obtain the formula (154). ∎

6.3.4 The morphism to the moduli of abelian pairs and the obstruction theory

We use the notation in the subsubsection 5.3.1. Recall the diagram (88). We would like to obtain a similar diagram from an L-Bradlow pair in the case $\mathrm{rank}(E)=1$. Since E is contained in $\det(E)$, we have the naturally induced section $\phi':L\longrightarrow\det(E)$. Assume that U is connected, for simplicity. Hence $c=c_1\bigl(\det(E)_{|\{u\}\times X}\bigr)$ is independent of the choice of $u\in U$. Therefore, we obtain the morphism $\det_{E,\phi}:U_2\longrightarrow M(c,L)$.

Proposition 6.36 *We have the following commutative diagram:*

$$\begin{array}{ccc} L_{U_2} & \longleftarrow & \mathrm{Ob}(V_\cdot,\widetilde{\phi})\\ \uparrow & & \uparrow\\ \det_{E,\phi}^*L_{M(c,L)} & \longleftarrow & \det_{E,\phi}^*\mathrm{Ob}\bigl(M(c,L)\bigr)\end{array} \tag{155}$$

Here, we put $\mathrm{Ob}(V_\cdot,\widetilde{\phi}):=Rp_{X\,*}\bigl(\mathfrak{g}(V_\cdot,\widetilde{\phi})\otimes\omega_X\bigr)$, and $\mathrm{Ob}\bigl(M(c,L)\bigr)$ denotes the obstruction theory of $M(c,L)$.

Proof Let us begin with a general non-sense. Let $\mathfrak{V}_0$ and $\mathfrak{V}_{-1}$ be locally free sheaves on a stack Z with a morphism $\mathfrak{f}:\mathfrak{V}_{-1}\longrightarrow\mathfrak{V}_0$. Assume $\mathrm{rank}\,\mathfrak{V}_0-\mathrm{rank}\,\mathfrak{V}_{-1}=1$. Let $v_1,\ldots,v_l$ be a local frame of $\mathfrak{V}_{-1}$, and let u be a local section of $\mathfrak{V}_0$. We put $\omega:=v_1\wedge\cdots\wedge v_l$. Then we put $\Lambda_{\mathfrak{f}}(u):=\bigl(u\wedge\mathfrak{f}(\omega)\bigr)\otimes\omega^{-1}$. It is easy to see $\Lambda_{\mathfrak{f}}(u)$ is independent of a choice of the frame. Therefore, we have the morphism $\Lambda_{\mathfrak{f}}:\mathfrak{V}_0\longrightarrow\det(\mathfrak{V}_\cdot)$. It is easy to see that $\Lambda_{\mathfrak{f}}\circ f=0$. Hence, we have the morphism of the complexes $\mathrm{Cone}(f)\longrightarrow\det(\mathfrak{V}_\cdot)$.

Applying the above construction, we obtain the morphism $\Lambda_f:V_\cdot\longrightarrow\det(V_\cdot)$. We also obtain the morphism $\Lambda_f\circ\widetilde{\phi}_0:P_0\longrightarrow\det(V_\cdot)\simeq\det(E)$. Since we have $\Lambda_f\circ\widetilde{\phi}_0(P_{-1})=0$, the morphisms ϕ' and $\Lambda_f\circ\widetilde{\phi}_0$ are essentially same. We put $\mathfrak{g}_{\mathrm{rel}}:=\mathcal{H}om\bigl(P_\cdot,\det(V_\cdot)\bigr)^\vee$. The morphism $\Lambda_f\circ\widetilde{\phi}_0$ naturally induces the morphism $\mathfrak{g}_{\mathrm{rel}}[-1]\longrightarrow\mathcal{O}[-1]$. The cone is denoted by $\mathfrak{g}\bigl(\det(V_\cdot),\phi'\bigr)$.

It is convenient to make a minor change in the construction of $\mathrm{Ob}(V_\cdot,\widetilde{\phi})$. We have the natural right $\mathrm{GL}(W_\cdot)$-action on $N(W_{-1\,X},W_{0\,X})\times N(P_{-1},W_{0\,X})$. The quotient stack is denoted by $\widetilde{Y}_0(W_\cdot,P_\cdot)$. We put $\widetilde{Y}_1(W_\cdot,P_\cdot):=Y(W_\cdot)$ and $\widetilde{Y}_2(W_\cdot,P_\cdot)=Y_2(W_\cdot,P_\cdot)$. We put $\widetilde{Y}(W_\cdot,P_\cdot):=\widetilde{Y}_1(W_\cdot,P_\cdot)\times_{\widetilde{Y}_0(W_\cdot,P_\cdot)}\widetilde{Y}_2(W_\cdot,P_\cdot)$. We have the natural morphisms $\widetilde{Y}_i(W_\cdot,P_\cdot)\longrightarrow Y_i(W_\cdot,P_\cdot)$. The induced morphism $\widetilde{Y}(W_\cdot,P_\cdot)\longrightarrow Y(W_\cdot,P_\cdot)$ is isomorphic.

We have the classifying map $\widetilde{\Phi}:U_2\times X\longrightarrow\widetilde{Y}(W_\cdot,P_\cdot)$ and the induced maps $\widetilde{\Phi}_i:U_2\times X\longrightarrow\widetilde{Y}_i(W_\cdot,P_\cdot)$. In the construction of the subsubsection 5.3.1, we can replace $\Phi(V_\cdot,\widetilde{\phi})^*L_{Y(W_\cdot,P_\cdot)/X}$ with $\widetilde{\Phi}^*L_{\widetilde{Y}(W_\cdot,P_\cdot)/X}$.

We have the weight 1-action of G_m on $\det(W_\cdot)$. It induces the G_m-action on $N\bigl(P_i,\det(W_{\cdot\,X})\bigr)$ $(i=0,1)$. We put as follows:

$$Z_0(W_\cdot,P_\cdot):=N\bigl(P_{-1},\det(W_{\cdot\,X})\bigr)_{G_m},\quad Z_1(W_\cdot,P_\cdot):=X_{G_m},\quad Z_2(W_\cdot,P_\cdot):=N\bigl(P_0,\det(W_{\cdot\,X})\bigr)_{G_m}$$

The fiber product $Z_1(W_\cdot,P_\cdot)\times_{Z_0(W_\cdot,P_\cdot)}Z_2(W_\cdot,P_\cdot)$ is isomorphic to $N(\mathcal{O}_X,\det(W_{\cdot\,X}))_{G_m}$, and it is denoted by $Z(W_\cdot,P_\cdot)$. From the section $\Lambda_f\circ\widetilde{\phi}_0:P_0\longrightarrow\det(V_\cdot)$, we obtain the classifying map $\Psi:U_2\times X\longrightarrow Z(W_\cdot,P_\cdot)$ and the induced maps $\Psi_i:U_2\times X\longrightarrow Z_i(W_\cdot,P_\cdot)$ $(i=0,1,2)$.

We have the morphism $\Gamma_2' : N(W_{-1}, W_0) \times N(P_{-1}, W_{-1}) \times N(P_0, W_0) \longrightarrow N(P_0, \det(W_{\cdot X}))$ given by $\Gamma_2'(e, a_{-1}, a_0) := \Lambda_e \circ a_0$. We have the morphism $\mathrm{GL}(W.) \longrightarrow G_m$ given by $(g_{-1}, g_0) \longmapsto \det(g_{-1})^{-1} \cdot \det(g_0)$. Then Γ_2' is equivariant with respect to the actions of $\mathrm{GL}(W.)$ and G_m. Therefore, we obtain the morphism $\Gamma_2 : \widetilde{Y}_2(W., P.) \longrightarrow Z_2(W., P.)$. Similarly, we have the equivariant morphism $\Gamma_0' : N(P_{-1}, W_0) \longrightarrow N\bigl(P_{-1}, \det(W_{\cdot X})\bigr)$ given by $\Gamma_0'(e, a) = \Lambda_e \circ a$, which induces $\Gamma_0 : \widetilde{Y}_0(W., P.) \longrightarrow Z_0(W., P.)$. We also have the obvious map $\Gamma_1 : \widetilde{Y}_1(W., P.) \longrightarrow Z_1(W., P.)$. We have $\Gamma_i \circ \widetilde{\Phi}_i = \Psi_i$ for $i = 0, 1, 2$. We have the induced map $\Gamma : \widetilde{Y}(W., P.) \longrightarrow Z(W., P.)$ and the relation $\Gamma \circ \widetilde{\Phi} = \Psi$.

We have the universal L-abelian pair $\bigl(\mathcal{L}^u, \phi^u\bigr)$ over $M(c, L) \times X$. We have the classifying map $\Phi_i(\mathcal{L}^u, \phi^u) : M(c, L) \times X \longrightarrow Z_i(W., P.)$. We obtain the following commutative diagram:

$$\begin{CD}
U_2 \times X @>{\widetilde{\Phi}_i}>> \widetilde{Y}_i(W., P.) \\
@V{\det_{E,\phi,X}}VV @VV{\Gamma_i}V \\
M(c, L) \times X @>{\Phi_i(\mathcal{L}^u, \phi^u)}>> Z_i(W., P.)
\end{CD} \tag{156}$$

We obtain the following morphism on $U_2 \times X$:

$$\varphi : \mathrm{Cone}\Bigl(\Psi_0^* L_{Z_0(W.,P.)/X} \to \bigoplus_{i=1,2} \Psi_i^* L_{Z_i(W.,P.)/X}\Bigr) \longrightarrow \mathrm{Cone}\Bigl(\widetilde{\Phi}_0^* L_{\widetilde{Y}_0(W.,P.)/X} \to \bigoplus_{i=1,2} \widetilde{\Phi}_i^* L_{\widetilde{Y}_i(W.,P.)/X}\Bigr)$$

By the argument in the subsubsection 2.3.2, we can show that φ is expressed by the morphism of the complexes $\mathfrak{g}\bigl(\det(V.), \phi'\bigr) \longrightarrow \mathrm{Cone}\bigl(\gamma(\widetilde{\phi})_{\leq 1}\bigr)$.

Lemma 6.37 *The morphism φ naturally factors through $\mathfrak{g}(\det(V.), \phi') \longrightarrow \mathfrak{g}(V., \widetilde{\phi}) \longrightarrow \mathrm{Cone}\bigl(\gamma(\widetilde{\phi})_{\leq 1}\bigr)$.*

Proof We give only an indication. The following diagram is commutative:

$$\begin{CD}
\mathcal{O} @>>> \mathcal{H}om(V_0, V_0) \oplus \mathcal{H}om(V_{-1}, V_{-1}) \\
@VVV @VVV \\
0 @>>> \mathcal{H}om(V_{-1}, V_0)
\end{CD}$$

In fact, it is a part of the morphism of the complexes $\mathcal{O} \longrightarrow \mathcal{H}om(V., V.)^\vee$. The following diagram is commutative:

$$\begin{CD}
\mathcal{H}om\bigl(\det(V.), P_0\bigr) @>>> \mathcal{H}om(V_{-1}, P_{-1}) \oplus \mathcal{H}om(V_0, P_0) \\
@VVV @VVV \\
0 @>>> \mathcal{H}om(V_{-1}, P_0)
\end{CD}$$

In fact, it is a part of the morphism of the complexes $\mathcal{H}om\bigl(P., \det(V.)\bigr)^\vee \longrightarrow \mathcal{H}om\bigl(P., V.\bigr)^\vee$. Then the claim of the lemma can be checked easily. ∎

Let us finish the proof of Proposition 6.36. We obtain the following commutative diagram from (156) and Lemma 6.37:

$$\begin{CD}
\mathfrak{g}(V., \widetilde{\phi}) @<<< \mathfrak{g}\bigl(\det(V.), \phi'\bigr) \\
@VVV @VVV \\
L_{U_2 \times X/X} @<<< \det_{E,\phi,X}^* L_{M(c,L) \times X/X}
\end{CD}$$

It is easy to observe $\mathrm{Ob}\bigl(M(c, L)\bigr) = Rp_{X\,*}\bigl(\mathfrak{g}\bigl(\det(V.), \phi'\bigr) \otimes \omega_X\bigr)$. Thus, we obtain the desired commutative diagram (155). ∎

By a similar argument, we obtain a similar commutative diagram in the reduced case. We use the notation in the subsubsection 5.4.1. Assume $\mathrm{rank}(E) = 1$. Then the reduced L-pair $\bigl(\det(E), [\phi']\bigr)$ is induced. Therefore, we have the morphism $\det_{E,[\phi]} : U_3 \longrightarrow M(c, [L])$.

Lemma 6.38 *We have the following commutative diagram:*

$$\begin{array}{ccc} L_{U_3} & \longleftarrow & \mathrm{Ob}(V_\cdot,[\widetilde{\phi}]) \\ \uparrow & & \uparrow \\ \det^*_{E,[\phi]} L_{M(c,[L])} & \longleftarrow & \mathrm{Ob}\big(M(c,[L])\big) \end{array} \tag{157}$$

Here, we put $\mathrm{Ob}(V_\cdot,[\widetilde{\phi}]) := \mathrm{Cone}\big(\mathrm{Ob}_{\mathrm{rel}}(V_\cdot,[\widetilde{\phi}]) \longrightarrow \mathrm{Ob}(V_\cdot)\big)$*, and* $\mathrm{Ob}\big(M(c,[L])\big)$ *denotes the obstruction theory of* $M(c,[L])$*.*

Proof We indicate only an outline. From the weight (-1)-actions of G_m on $P_\cdot$ and $\mathcal{O}_X$, we obtain the G_m-actions on $\widetilde{Y}_i(W_\cdot,P_\cdot)$ and $Z_i(W_\cdot,P_\cdot)$. The quotient stacks are denoted by $\widetilde{Y}_i(W_\cdot,[P_\cdot])$ and $Z_i(W_\cdot,[P_\cdot])$. The natural morphisms $\Gamma_i : \widetilde{Y}_i(W_\cdot,[P_\cdot]) \longrightarrow Z_i(W_\cdot,[P_\cdot])$ are induced.

We have the induced map $\widetilde{\Phi}'_i : U_3 \times X \longrightarrow \widetilde{Y}_i(W_\cdot,[P_\cdot])$. We put $\Psi'_i := \Gamma_i \circ \widetilde{\Phi}_i$. We also have the classifying maps $\Phi_i(\mathcal{L}^u,[\phi^u]) : M(c,[L]) \times X \longrightarrow Z_i(W_\cdot,[P_\cdot])$ for the universal objects on $M(c,[L]) \times X$. We have the following commutative diagram:

$$\begin{array}{ccccc} U_3 \times X & \xrightarrow{\widetilde{\Phi}'_i} & Y_i(W_\cdot,[P_\cdot]) & \longrightarrow & X_{G_m} \\ {\scriptstyle \det_{E,[\phi],X}}\downarrow & & {\scriptstyle \Gamma_i}\downarrow & & {\scriptstyle =}\downarrow \\ M(c,[L]) \times X & \xrightarrow{\Phi(\mathcal{L}^u,[\phi^u])} & Z_i(W_\cdot,[P_\cdot]) & \longrightarrow & X_{G_m} \end{array}$$

We have the following induced morphism on $U_3 \times X$:

$$\varphi' : \mathrm{Cone}\Big(\Psi_0'^{*} L_{Z_0(W_\cdot,[P_\cdot])/X_{G_m}} \to \bigoplus_{i=1,2} \Psi_i'^{*} L_{Z_i(W_\cdot,[P_\cdot])/X_{G_m}}\Big) \longrightarrow$$
$$\mathrm{Cone}\Big(\widetilde{\Phi}_0'^{*} L_{Y_0(W_\cdot,[P_\cdot])/X_{G_m}} \to \bigoplus_{i=1,2} \widetilde{\Phi}_i'^{*} L_{Y_i(W_\cdot,[P_\cdot])/X_{G_m}}\Big) \tag{158}$$

We can show that φ' is expressed by the morphism of the complexes $\mathfrak{g}(\det(V_\cdot),[\phi']) \longrightarrow \mathrm{Cone}(\gamma[\widetilde{\phi}]_{\leq 0}[-1])$, where $\mathfrak{g}\big(\det(V_\cdot),[\phi']\big)$ is given by $\mathcal{H}om\big(\det(V_\cdot),[P_{-1}]\big) \longrightarrow \mathcal{H}om\big(\det(V_\cdot),[P_0]\big) \longrightarrow \mathcal{O}$, as in the proof of Proposition 6.36. It factors through $\mathfrak{g}'(V_\cdot,[\widetilde{\phi}])$. Therefore, we obtain the following diagram:

$$\begin{array}{ccc} \mathfrak{g}'(V_\cdot,[\widetilde{\phi}]) & \longleftarrow & \mathfrak{g}\big(\det(V_\cdot),[\phi']\big) \\ \downarrow & & \downarrow \\ L_{U_3 \times X/X_{G_m}} & \longleftarrow & \det^*_{E,[\phi],X} L_{M(c,[L]) \times X/X_{G_m}} \end{array}$$

Then, we obtain the following:

$$\begin{array}{ccccc} Rp_{X\,*}\big(\mathfrak{g}'(V_\cdot,[\widetilde{\phi}]) \otimes \omega_X\big) & \xrightarrow{\varphi_1} & L_{U_3/k_{G_m}} & \xrightarrow{\varphi_2} & L_{k_{G_m}/k} \\ \uparrow & & \uparrow & & \uparrow \\ Rp_{X\,*}\big(\mathfrak{g}\big(\det(V_\cdot),[\phi']\big) \otimes \omega_X\big) & \xrightarrow{\psi_1} & \det^*_{E,[\phi]} L_{M(c,[L])/k_{G_m}} & \xrightarrow{\psi_2} & L_{k_{G_m}/k}[1] \end{array}$$

It is easy to observe $\mathrm{Ob}(V_\cdot,[\widetilde{\phi}]) \simeq \mathrm{Cone}\big(\varphi_2 \circ \varphi_1\big)$ and $\det^*_{E,[\phi]} \mathrm{Ob}\big(M(c,[L])\big) \simeq \mathrm{Cone}\big(\psi_2 \circ \psi_1\big)$. Thus, we obtain the desired diagram (157). ∎

When E has an orientation ρ, we have the morphisms $\mathrm{Ob}_{\mathrm{rel}}(V_\cdot,\rho)[-1] \longrightarrow \mathrm{Ob}(V_\cdot) \longrightarrow \mathrm{Ob}(V_\cdot,[\widetilde{\phi}])$. Let $\mathrm{Ob}(V_\cdot,\rho,[\widetilde{\phi}])$ denote the cone of the composite of the morphisms. On the other hand, we have the morphism $\det_{E,[\phi],\rho} : U_3 \longrightarrow M(\widehat{c},[L])$.

Proposition 6.39 *We have the following commutative diagram:*

$$\begin{array}{ccc} L_{U_3} & \longleftarrow & \mathrm{Ob}(V.,\rho,[\widetilde{\phi}]) \\ \uparrow & & \uparrow \\ \det^*_{E,[\phi],\rho} L_{M(\widehat{c},[L])} & \longleftarrow & \det^*_{E,[\phi],\rho}\mathrm{Ob}\big(M(\widehat{c},[L])\big) \end{array}$$

Here, $\mathrm{Ob}\big(M(\widehat{c},[L])\big)$ *denotes the obstruction theory of* $M\big(\widehat{c},[L]\big)$.

Proof The orientation induces the morphisms $U_3 \longrightarrow M(\widehat{c},[L]) \longrightarrow \mathrm{Pic}(c)$, and we have the following diagram:

$$\begin{array}{ccccc} \mathrm{Ob}^d(V.) & \longrightarrow & \mathrm{Ob}^d(V.) & \longrightarrow & \mathrm{Ob}(V.) \\ \downarrow & & \downarrow & & \downarrow \\ \det^*_{E,\rho} L_{\mathrm{Pic}} & \longrightarrow & \det^*_{E,[\phi],\rho} L_{M(\widehat{c},[L])} & \longrightarrow & L_{U_3} \end{array}$$

Then the claim follows from the construction of the relative obstruction complex for orientations. ∎

6.3.5 The morphism to the parabolic Hilbert scheme and the obstruction theories

Let U be a k-scheme. Let (E,F_*) be a quasi-parabolic torsion free sheaf of type $\boldsymbol{y}$. Assume $\mathrm{rank}(\boldsymbol{y})=1$. Let c be the $H^2(X)$-part of y. Then we put $\boldsymbol{y}(-c):=\boldsymbol{y}\cdot\exp(-c)$. We put $I(E):=\det(E)^{-1}\otimes E$, which has the induced quasi-parabolic structure F_*. The type of $\big(I(E),F_*\big)$ is $\boldsymbol{y}(-c)$. Thus, we have the morphism $\Xi(E):U\longrightarrow X^{[\boldsymbol{y}(-c)]}$.

Lemma 6.40 *We have the following commutative diagram:*

$$\begin{array}{ccc} L_U & \longleftarrow & \mathrm{Ob}(V.,F_*) \\ \uparrow & & \uparrow \\ \Xi(E)^*L_{X^{[\boldsymbol{y}(-c)]}} & \longleftarrow & \Xi(E)^*\,\mathrm{Ob}(X^{[\boldsymbol{y}(-c)]}) \end{array} \tag{159}$$

Here, we put $\mathrm{Ob}(V.,F_*):=\mathrm{Cone}\big(\mathrm{Ob}_{\mathrm{rel}}(V,F_*)[-1]\to\mathrm{Ob}(V.)\big)$, *and* $\mathrm{Ob}(X^{[\boldsymbol{y}(-c)]})$ *denotes the obstruction theory of* $X^{[\boldsymbol{y}(-c)]}$. *Moreover,* φ *factors through* $\mathrm{Ob}^{\circ}(V.,F_*)$.

Proof Let $\mathcal{I}^u$ denote the universal sheaf on $X^{[\boldsymbol{y}(-c)]}\times X$. We take a locally free resolution $\mathcal{V}.$ of $\mathcal{I}^u$. It is easy to observe that $V.:=\Xi(E)^*_X\mathcal{V}.\otimes\det(E)$ is a locally free resolution of E. We have $\mathfrak{g}(V.)=\Xi(E)^*_X\mathfrak{g}(\mathcal{V}.)$, $\mathfrak{g}(V._{|D})=\Xi(E)^*_D\mathfrak{g}(\mathcal{V}._{|D})$ and $\mathfrak{g}(V.,F_*)=\Xi(E)^*_D\mathfrak{g}(\mathcal{V}.,F_*)$.

We take vector spaces W_i $(i=-1,0)$ such that $\mathrm{rank}\,W_i=\mathrm{rank}\,V_i$. We have $\mathrm{rank}\,W_0-\mathrm{rank}\,W_{-1}=1$. In that case, we have the homomorphism $\mathrm{GL}(W.)\longrightarrow\mathrm{SGL}(W.)$, given as follows:

$$\big(g_{-1},g_0\big)\longmapsto\big(\det(g.)^{-1}\cdot g_{-1},\ \det(g.)^{-1}\cdot g_0\big)$$

Here, $\det(g.)$ denotes $\det(g_0)\cdot\det(g_{-1})^{-1}$. Therefore, we have the isomorphism $Y(W.)\simeq\overline{Y}(W.)\times_X X_{G_m}$. In particular, we have the morphism:

$$\mathfrak{w}_1:Y(W.)\longrightarrow\overline{Y}(W.) \tag{160}$$

Then, we obtain the following commutative diagram:

$$\begin{array}{ccc} U\times X & \longrightarrow & Y(W.) \\ \downarrow & & \downarrow \\ X^{[\boldsymbol{y}(-c)]}\times X & \longrightarrow & \overline{Y}(W.) \end{array}$$

Therefore, we obtain the following commutative diagram on $U\times X$:

$$\begin{array}{ccccc} L_{U\times X/X} & \longleftarrow & \Phi(V.)^*L_{Y(W.)} & \longleftarrow & \mathfrak{g}(V.) \\ \uparrow & & \uparrow & & \uparrow \\ \Xi(E)_X^*L_{X^{[\boldsymbol{y}(-c)]}\times X/X} & \longleftarrow & \Xi(E)_X^*\Phi(\mathcal{V}.)^*L_{\overline{Y}(W.)} & \longleftarrow & \Xi(E)^*\mathfrak{g}^{\circ}(\mathcal{V}.) \end{array}$$

Hence we obtain the following commutative diagram:

$$\begin{array}{ccc} L_U & \longleftarrow & \mathrm{Ob}(V.) \\ \uparrow & & \uparrow \\ \Xi(E)^*L_{X^{[\boldsymbol{y}(-c)]}} & \longleftarrow & \Xi(E)^*\,\mathrm{Ob}^{\circ}(\mathcal{V}.) \end{array}$$

We obtain a similar diagram from $V._{|D}$ and $\mathcal{V}._{|D}$. Moreover, we obtain the following diagram by the argument of Lemma 5.22:

$$\begin{array}{ccccc} L_U & \longleftarrow & \mathrm{Ob}(V.) & \longleftarrow & \mathrm{Ob}(V._{|D}) \\ \uparrow & & \uparrow & & \uparrow \\ \Xi(E)^*L_{X^{[\boldsymbol{y}(-c)]}} & \longleftarrow & \Xi(E)^*\,\mathrm{Ob}^{\circ}(\mathcal{V}.) & \longleftarrow & \Xi(E)^*\,\mathrm{Ob}^{\circ}(\mathcal{V}._{|D}) \end{array}$$

We also have the following commutative diagram:

$$\begin{array}{ccccc} L_U & \longleftarrow & Rp_{D\,*}\big(\mathfrak{g}_D(V.,F_*)\otimes\omega_D\big) & \longleftarrow & \mathrm{Ob}(V._{|D}) \\ \uparrow & & \uparrow & & \uparrow \\ \Xi(E)^*L_{X^{[\boldsymbol{y}(-c)]}} & \longleftarrow & \Xi(E)^*Rp_{D\,*}\big(\mathfrak{g}_D^{\circ}(\mathcal{V}.,F_*)\otimes\omega_D\big) & \longleftarrow & \Xi(E)^*\,\mathrm{Ob}^{\circ}(\mathcal{V}._{|D}) \end{array}$$

We remark the following:

$$\mathrm{Ob}(V,F_*)\simeq\mathrm{Cone}\Big(\mathrm{Ob}(V._{|D})\longrightarrow\mathrm{Ob}(V.)\oplus Rp_{D\,*}\big(\mathfrak{g}_D(V.,F_*)\otimes\omega_D\big)\Big)$$

$$\mathrm{Ob}(X^{[\boldsymbol{y}(-c)]})\simeq\mathrm{Cone}\Big(\mathrm{Ob}^{\circ}(\mathcal{V}._{|D})\longrightarrow\mathrm{Ob}^{\circ}(\mathcal{V}.)\oplus Rp_{D\,*}\big(\mathfrak{g}^{\circ}_D(\mathcal{V}.,F_*)\otimes\omega_D\big)\Big)$$

Hence, we obtain the desired commutative diagram (159). ∎

6.3.6 The mixed case

Let (E,F_*,ϕ) be a quasi parabolic L-Bradlow pair of type $\boldsymbol{y}$ over $U\times X$. Assume $\mathrm{rank}(\boldsymbol{y})=1$. Let c denote the $H^2(X)$-part of y, and we put $\boldsymbol{y}(-c):=\boldsymbol{y}\cdot\exp(-c)$. Then, we have the morphisms $\det_{E,\phi}:U\longrightarrow M(c,L)$ and $\Xi(E):U\longrightarrow X^{[\boldsymbol{y}(-c)]}$.

Assume we have a locally free resolution $V.$ of E, a locally free resolution $P.$ of L, and a lift $\widetilde{\phi}:P.\longrightarrow V.$ of ϕ. We have the natural morphism $i_1:\mathrm{Ob}(V.)\longrightarrow\mathrm{Ob}(V.,\widetilde{\phi})$ and $i_2:\mathrm{Ob}(V.)\longrightarrow\mathrm{Ob}(V.,F_*)$. We put as follows:

$$\mathrm{Ob}(V.,F_*,\phi):=\mathrm{Cone}\Big(\mathrm{Ob}(V.)\stackrel{(i_1,-i_2)}{\longrightarrow}\mathrm{Ob}(V.,F_*)\oplus\mathrm{Ob}(V.,\phi)\Big)$$

We have the induced map $\mathrm{Ob}(V.,F_*,\phi)\longrightarrow L_U$.

Lemma 6.41 *We have the following commutative diagram:*

$$\begin{array}{ccc} \det^*_{E,\phi}\mathrm{Ob}\big(M(c,L)\big)\oplus\Xi(E)^*\,\mathrm{Ob}(X^{[\boldsymbol{y}(-c)]}) & \longrightarrow & \mathrm{Ob}(V.,F_*,\phi) \\ \downarrow & & \downarrow \\ \det^*_{E,\phi}L_{M(c,L)}\oplus\Xi(E)^*L_{X^{[\boldsymbol{y}(-c)]}} & \longrightarrow & L_U \end{array}$$

Proof It follows from Proposition 6.36 and Lemma 6.40. ∎

Let $(E, F_*, \rho, [\phi])$ be an oriented quasi-parabolic reduced L-Bradlow pair of type $\boldsymbol{y}$ over $U \times X$. We have the morphism $\det_{E,\rho,[\phi]} : U \longrightarrow M(\widehat{c}, [L])$ and $\Xi(E) : U \longrightarrow X^{[\boldsymbol{y}(-c)]}$.

Assume we have a locally free resolution $V.$ of E, a locally free resolution $P.$ of L, and a lift $[\widetilde{\phi}]$ of $[\phi]$. We have the natural morphisms $i_1 : \mathrm{Ob}(V.) \longrightarrow \mathrm{Ob}(V., \rho, [\widetilde{\phi}])$ and $i_2 : \mathrm{Ob}(V.) \longrightarrow \mathrm{Ob}(V., F_*)$. We put as follows:

$$\mathrm{Ob}(V., F_*, [\widetilde{\phi}], \rho) := \mathrm{Cone}\Big(\mathrm{Ob}(V.) \overset{(i_1,-i_2)}{\longrightarrow} \mathrm{Ob}(V., [\widetilde{\phi}], \rho) \oplus \mathrm{Ob}(V., F_*)\Big)$$

Lemma 6.42 *We have the following commutative diagram:*

$$\begin{CD}
\det^*_{E,\rho,[\phi]} \mathrm{Ob}\big(M(\widehat{c}, [L])\big) \oplus \Xi(E)^* \mathrm{Ob}(X^{[\boldsymbol{y}(-c)]}) @>>> \mathrm{Ob}(V., F_*, [\phi], \rho) \\
@VVV @VVV \\
\det^*_{E,\rho,[\phi]} L_{M(\widehat{c},[L])} \oplus \Xi(E)^* L_{X^{[\boldsymbol{y}(-c)]}} @>>> L_U
\end{CD}$$

Proof It follows from Proposition 6.39 and Lemma 6.40. ∎

6.3.7 Proof of Proposition 6.34

Applying the construction in the subsubsection 6.3.6 to the universal object $(\mathcal{E}^u, \phi)$ on $\mathcal{M}(\boldsymbol{y}, L)$, we obtain the morphism $F : \mathcal{M}(\boldsymbol{y}, L) \longrightarrow M(c, L) \times X^{[\boldsymbol{y}(-c)]}$. It is same as the inclusion given in the subsubsection 6.3.3. Let us denote the obstruction theory of $\mathcal{M}(\boldsymbol{y}, L)$ by $\mathrm{Ob}(\boldsymbol{y}, L)$. We have the following commutative diagram, due to Lemma 6.41:

$$\begin{CD}
F^*\Big(\mathrm{Ob}\big(M(c, L)\big) \oplus \mathrm{Ob}(X^{[\boldsymbol{y}(-c)]})\Big) @>>> \mathrm{Ob}\big(\boldsymbol{y}, L\big) \\
@VVV @VVV \\
F^*\Big(L_{M(c,L)\times X^{[\boldsymbol{y}(-c)]}}\Big) @>>> L_{\mathcal{M}(\boldsymbol{y},L)}
\end{CD} \tag{161}$$

Let $\mathcal{L}_c$ denote the universal line bundle on $M(c, L) \times X$. We have the natural inclusion $\mathcal{E}^u \otimes L^{-1} \longrightarrow F^*\mathcal{L}^u_c \otimes L^{-1}$, and the quotient is isomorphic to $\mathcal{K}_{|\mathcal{M}(\boldsymbol{y},L)\times X}$. Thus, we have the following:

$$\begin{aligned}
\mathrm{Cone}\Big(F^*\big[\mathrm{Ob}\big(X^{[\boldsymbol{y}(-c)]}\big) \oplus \mathrm{Ob}\big(M(c, L)\big)\big] \longrightarrow \mathrm{Ob}(\boldsymbol{y}, L)\Big)& \\
\simeq Rp_{X\,*}\Big(\mathrm{Cone}\big[\mathcal{RH}om\big(F^*\mathcal{L}^u_c, L\big) \to \mathcal{RH}om\big(\mathcal{E}^u, L\big)\big] \otimes \omega_X\Big)& \\
\simeq Rp_{X\,*}\big(\mathcal{K}\big)^\vee_{|\mathcal{M}(\boldsymbol{y},L)}[1] \simeq \mathfrak{V}^\vee[1]_{|\mathcal{M}(\boldsymbol{y},L)} \simeq j^*L_{M(c,L)\times X^{[\boldsymbol{y}(-c)]}/\mathfrak{Y}}&
\end{aligned} \tag{162}$$

From the diagram (161), we obtain the morphism $\nu : j^*L_{M(c,L)\times X^{[\boldsymbol{y}(-c)]}/\mathfrak{Y}} \longrightarrow L_{\mathcal{M}(\boldsymbol{y},L)/M(c,L)}$

Lemma 6.43 *The morphism ν is same as the morphism $j^*L_{M(c,L)\times X^{[\boldsymbol{y}(-c)]}/\mathfrak{Y}} \longrightarrow L_{\mathcal{M}(\boldsymbol{y},L)/M(c,L)}$ obtained from the diagram* (153). *In particular, the obstruction theories of $M(c, L) \times X^{[\boldsymbol{y}(-c)]}$ and $\mathcal{M}(\boldsymbol{y}, L)$ are compatible over the morphism i.*

Proof We take a locally free resolution $V.$ of $\mathcal{E}^u$. We take vector spaces W_i such that $\operatorname{rank} W_i = \operatorname{rank} V_i$. We obtain the following commutative diagram:

$$\begin{CD}
\mathcal{M}(\boldsymbol{y}, L) \times X @>{\Phi(V.,\phi)}>> Y(W., P.) \\
@VVV @VV{\mathfrak{w}_1\times\Gamma}V \\
X^{[\boldsymbol{y}(-c)]} \times M(c, L) @>{\Phi(\det(E)^{-1}\otimes V.)\times\Phi(\mathcal{L},\phi)}>> \overline{Y}(W.) \times Z(W., P.)
\end{CD}$$

It induces the morphism:

$$L_{\mathcal{M}(m,\boldsymbol{y})\times X/M(c,L)\times X^{[\boldsymbol{y}(-c)]}\times X} \longleftarrow \Phi(V., \phi)^* L_{Y(W.,P.)/\overline{Y}(W.)\times Z(W.,P.)}.$$

We use the notation in the subsubsection 2.3.3. We put $\widetilde{V}_i = V_i$ $(i = 0, -1)$ and $\widetilde{V}_1 = \mathcal{L}_c^u$. By the argument in the subsubsection 2.3.2, we can show that $\Phi(V_\cdot, \phi)^* L_{Y(W_\cdot, P_\cdot)/\overline{Y}(W_\cdot)\times Z(W_\cdot, P_\cdot)}$ is expressed by $\mathfrak{k}(V_\cdot, P_\cdot, \phi, \widetilde{\phi})_{\leq 0}$. Thus we obtain the following morphism:

$$\mathfrak{k}(\widetilde{V}_\cdot, P_\cdot, \phi, \widetilde{\phi}) \longrightarrow L_{\mathcal{M}(\boldsymbol{y},L)\times X/M(c,L)\times X^{[\boldsymbol{y}(-c)]}\times X}$$

We put $\mathrm{Ob}^H(V_\cdot, P_\cdot, \phi, \widetilde{\phi}) := Rp_{X*}\big(\mathfrak{k}(\widetilde{V}_\cdot, P_\cdot, \phi, \widetilde{\phi}) \otimes \omega_X\big)$. Then, $\mathrm{Ob}^H(V_\cdot, P_\cdot, \phi, \widetilde{\phi})$ is isomorphic to the cone of the morphism $\mathrm{Ob}(M(c,L)) \oplus \mathrm{Ob}(X^{[\boldsymbol{y}(-c)]}) \longrightarrow \mathrm{Ob}(\mathcal{M}(\boldsymbol{y},L))$. The induced morphism $\mathrm{Ob}^H(V_\cdot, P_\cdot, \phi, \widetilde{\phi}) \longrightarrow L_{\mathcal{M}(\boldsymbol{y},L)/M(c,L)\times X^{[\boldsymbol{y}(-c)]}}$ is same as ν.

We have the following factorization:

$$\begin{array}{ccccc}
\mathcal{M}(\boldsymbol{y},L)\times X & \xrightarrow{\Phi_1} & Z_1(\widetilde{V}, P_\cdot) & \longrightarrow & Y(W_\cdot, P_\cdot) \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
M(c,L)\times X^{[\boldsymbol{y}(-c)]}\times X & \longrightarrow & Z_2(\widetilde{V}, P_\cdot) & \longrightarrow & \overline{Y}(W_\cdot)\times Z(W_\cdot, P_\cdot)
\end{array}$$

Thus, we have the following morphisms:

$$L_{\mathcal{M}(\boldsymbol{y},L)\times X/M(c,L)\times X^{[\boldsymbol{y}(-c)]}\times X} \longleftarrow \Phi_1^* L_{Z_1(\widetilde{V},P_\cdot)/Z_2(\widetilde{V},P_\cdot)} \xleftarrow{\simeq} \Phi(V_\cdot,\phi)^* L_{Y(W_\cdot,P_\cdot)/\overline{Y}(W_\cdot)\times Z(W_\cdot,P_\cdot)}$$

Therefore, the morphism ν is same as $\mathfrak{r}(\widetilde{V}_\cdot, P_\cdot, \phi, \widetilde{\phi})$ in the subsubsection 2.3.3. Then, the claim follows from Proposition 2.25. ∎

Recall that $X^{[\boldsymbol{y}]}$ is smooth. Let m be a sufficiently large integer such that $H^i\big(X, \mathcal{L}(m)\big) = 0$ $(i = 1, 2)$ for any line bundle $\mathcal{L}$ such that $c_1(\mathcal{L}) = c$. Then the moduli stacks $\mathcal{M}\big(\boldsymbol{y}, \mathcal{O}(-m)\big)$ and $M\big(c, \mathcal{O}(-m)\big)$ are smooth. Let $\iota : \mathcal{O}(-m) \longrightarrow L$ be an inclusion. Then we have the inclusions $\mathcal{M}(\boldsymbol{y}, L) \longrightarrow \mathcal{M}\big(\boldsymbol{y}, \mathcal{O}(-m)\big)$ and $M(c,L) \longrightarrow M\big(c, \mathcal{O}(-m)\big)$. On $M\big(c, \mathcal{O}(-m)\big) \times X^{[\boldsymbol{y}(-c)]} \times X$, we put $\mathcal{K}' := \mathcal{L}_c^u \otimes L^{-1} \otimes \mathcal{O}_{\mathcal{Z}(\boldsymbol{y}(-c))}$. Then $\mathfrak{V}' := p_{X*}\mathcal{K}'$ gives the vector bundle over $M\big(c, \mathcal{O}(-m)\big) \times X^{[\boldsymbol{y}(-c)]}$ such that $\mathfrak{V}'_{|M(c,L)\times X^{[\boldsymbol{y}(-c)]}} = \mathfrak{V}$. Therefore, we obtain the following Cartesian diagrams:

$$\begin{array}{ccccc}
\mathcal{M}(\boldsymbol{y},L) & \longrightarrow & M(c,L)\times X^{[\boldsymbol{y}(-c)]} & \longrightarrow & M\big(c,\mathcal{O}(-m)\big) \\
\big\downarrow & & {\scriptstyle i}\big\downarrow & & {\scriptstyle i_1}\big\downarrow \\
M(c,L)\times X^{[\boldsymbol{y}(-c)]} & \longrightarrow & \mathfrak{V} & \longrightarrow & \mathfrak{V}'
\end{array}$$

We have $i^! = i_1^!$. Therefore, we obtain the relation $i^!\big([M(c,L)] \times [X^{[\boldsymbol{y}]}]\big) = [\mathcal{M}(\boldsymbol{y}, L)]$ from (162), due to Proposition 2.32. The relation $i^!\big([M(\widehat{c}, [L])] \times [X^{[\boldsymbol{y}]}]\big) = [\mathcal{M}(\widehat{\boldsymbol{y}}, [L])]$ can be shown by a similar argument. Thus we finish the proof of Proposition 6.34. ∎

6.4 Bradlow Perturbation

6.4.1 Statements

Let L be a line bundle on X. If $\delta \in \mathcal{P}^{\mathrm{br}}$ is sufficiently small, we have the projective morphism:

$$\mathfrak{F} : \mathcal{M}^s(\widehat{\boldsymbol{y}}, [L], \alpha_*, \delta) \longrightarrow \mathcal{M}^{ss}(\widehat{\boldsymbol{y}}, \alpha_*) \tag{163}$$

To discuss $\mathfrak{F}$, let us consider the following condition for $(\boldsymbol{y}, L, \alpha_*)$:

(i-vanishing condition) We have $H^j\big(X, E \otimes L^{-1}\big) = 0$ for any $j \geq i$ and for any $E_* \in \mathcal{M}^{ss}(\boldsymbol{y}, \alpha_*)$.

The 1-vanishing condition obviously implies the 2-vanishing condition.

Proposition 6.44 *Assume the* 1*-vanishing condition holds for* $(\boldsymbol{y}, L, \alpha_*)$.

- *The morphism* $\mathfrak{F} : \mathcal{M}^s(\widehat{\boldsymbol{y}}, [L], \alpha_*, \delta) \longrightarrow \mathcal{M}^{ss}(\widehat{\boldsymbol{y}}, \alpha_*)$ *is smooth.*
- *Assume, moreover, that the* 1*-stability condition holds for* $(\boldsymbol{y}, \alpha_*)$. *Then, we have the following relation:*

$$\mathfrak{F}^*\big([\mathcal{M}^s(\widehat{\boldsymbol{y}}, \alpha_*)]\big) = \big[\mathcal{M}^s(\widehat{\boldsymbol{y}}, [L], \alpha_*, \delta)\big]$$

Proof We give only a remark. The smoothness of $\mathfrak{F}$ is clear. We put $\mathcal{M}_1 := \mathcal{M}^{ss}(\widehat{\boldsymbol{y}}, \alpha_*)$ and $\mathcal{M}_2 := \mathcal{M}^s(\widehat{\boldsymbol{y}}, [L], \alpha_*, \delta)$. It is easy to obtain the following morphism of distinguished triangles:

$$\begin{tikzcd}
L_{\mathcal{M}_2/\mathcal{M}_1}[-1] \arrow[r] & \mathfrak{F}^* L_{\mathcal{M}_1/k} \arrow[r] & L_{\mathcal{M}_2/k} \arrow[r] & L_{\mathcal{M}_2/\mathcal{M}_1} \\
\mathrm{Ob}_{\mathrm{rel}}(m, \boldsymbol{y}, [L])[-1] \arrow[r] \arrow[u, "\simeq"] & \mathfrak{F}^*\big(\mathrm{Ob}(m, \boldsymbol{y})\big) \arrow[r] \arrow[u] & \mathrm{Ob}(m, \boldsymbol{y}, [L]) \arrow[r] \arrow[u] & \mathrm{Ob}_{\mathrm{rel}}(m, \boldsymbol{y}, [L]) \arrow[u, "\simeq"]
\end{tikzcd}$$

Due to Proposition 2.32, we obtain the equality $\Psi^*[\mathcal{M}_1] = [\mathcal{M}_2]$. ∎

Let $\boldsymbol{L} = (L_1, L_2)$ be a pair of line bundles on X. We take $\delta_i \in \mathcal{P}^{\mathrm{br}}$ such that both of δ_i are sufficiently small. If δ_2 is sufficiently smaller than δ_1, we have the projective morphism:

$$\mathfrak{F}_1 : \mathcal{M}^s(\widehat{\boldsymbol{y}}, [\boldsymbol{L}], \alpha_*, \boldsymbol{\delta}) \longrightarrow \mathcal{M}^s(\widehat{\boldsymbol{y}}, [L_1], \alpha_*, \delta_1)$$

Proposition 6.45 *Assume that the* 1*-vanishing condition holds for* $(\boldsymbol{y}, \alpha_*, L_2)$. *The morphism* $\mathfrak{F}_1$ *is smooth, and we have the following relation:*

$$\mathfrak{F}_1^*\big(\big[\mathcal{M}^s(\widehat{\boldsymbol{y}}, [L_1], \alpha_*, \delta_1)\big]\big) = \big[\mathcal{M}^s(\widehat{\boldsymbol{y}}, [\boldsymbol{L}], \alpha_*, \boldsymbol{\delta})\big]$$

Proof It can be shown by an argument similar to the proof of Proposition 6.44. ∎

If the 1-vanishing condition does not hold, $\mathfrak{F}_1$ is not smooth, in general. The following proposition will be proved in the next subsubsections 6.4.2–6.4.4.

Proposition 6.46 *Assume that the* 2*-vanishing condition holds for* $(\boldsymbol{y}, \alpha_*, L_2)$. *Then, there exists a Deligne-Mumford stack* $\mathfrak{B}$ *over* $\mathcal{M}^s(\widehat{\boldsymbol{y}}, [L_1], \alpha_*, \delta_1)$ *with the vector bundle* $\mathfrak{V}$ *and the section* ψ *such that the following holds:*

- *The morphism* $\mathfrak{G} : \mathfrak{B} \longrightarrow \mathcal{M}^s(\widehat{\boldsymbol{y}}, [L_1], \alpha_*, \delta_1)$ *is smooth.*
- $\mathcal{M}^s(\widehat{\boldsymbol{y}}, [\boldsymbol{L}], \alpha_*, \boldsymbol{\delta})$ *is* $\psi^{-1}(0)$.
- *We have the following relation:*

$$\psi^!\big(\mathfrak{G}^*[\mathcal{M}^s(\widehat{\boldsymbol{y}}, [L_1], \alpha_*, \delta_1)]\big) = [\mathcal{M}^s(\widehat{\boldsymbol{y}}, [\boldsymbol{L}], \alpha_*, \boldsymbol{\delta})]$$

Here, $\psi^!$ *denotes the Gysin map for the inclusion* $\mathcal{M}^s(\widehat{\boldsymbol{y}}, [\boldsymbol{L}], \alpha_*, \boldsymbol{\delta}) \longrightarrow \mathfrak{V}$.

Before going into the proof of Proposition 6.46, we give a corollary. Let P be any point of $\mathcal{M}^s(\widehat{\boldsymbol{y}}, [L_1], \alpha_*, \delta_1)$. The fiber $\mathfrak{G}^{-1}(P)$ is smooth, and $\mathfrak{F}_1^{-1}(P)$ is the 0-set of the section $\psi_{|\mathfrak{G}^{-1}(P)}$ of $\mathfrak{V}_{|\mathfrak{G}^{-1}(P)}$. Therefore, we obtain the following.

Corollary 6.47 *For any* k*-valued point* $P \in \mathcal{M}^s(\widehat{\boldsymbol{y}}, [L_1], \alpha_*, \delta)$, *the fiber* $\mathfrak{F}_1^{-1}(P)$ *is provided with the perfect obstruction theory. We also have the following formula:*

$$\int_{\mathcal{M}^s(\widehat{\boldsymbol{y}}, [\boldsymbol{L}], \alpha_*, \boldsymbol{\delta})} \Phi \cdot \Psi = \int_{\mathfrak{F}_1^{-1}(P)} \Psi \cdot \int_{\mathcal{M}^s(\widehat{\boldsymbol{y}}, [L_1], \alpha_*, \delta_1)} \Phi$$

Here, Φ *and* Ψ *are cohomology classes on* $\mathcal{M}^s(\widehat{\boldsymbol{y}}, [L_1], \alpha_*, \delta_1)$ *and* $\mathcal{M}^s(\widehat{\boldsymbol{y}}, [\boldsymbol{L}], \alpha_*, \boldsymbol{\delta})$, *respectively, such that* $\deg(\Phi) = 2\dim^f \mathcal{M}^s(\boldsymbol{y}, \alpha_*)$. *(See the subsection* 7.1 *for the cohomology classes and the evaluation considered in this paper.)* ∎

Remark 6.48 *Similar claims also hold for the morphism* $\mathfrak{F}$ *in* (163), *if the* 2*-vanishing condition holds for* $(\boldsymbol{y}, L, \alpha_*)$. *The proof is similar.* ∎

6.4.2 Construction of $\mathfrak{B}$

Let m be a sufficiently large such that the 1-vanishing condition holds for $(\boldsymbol{y}, \mathcal{O}(-m), \alpha_*)$. We put $L_2' := \mathcal{O}(-m)$ and $\boldsymbol{L}_2' := (L_1, L_2')$. We put $\mathfrak{B} := \mathcal{M}^s(\widehat{\boldsymbol{y}}, [\boldsymbol{L}'], \alpha_*, \boldsymbol{\delta})$. Then, we have the natural smooth morphism $\mathfrak{G} : \mathfrak{B} \longrightarrow \mathcal{M}^s(\widehat{\boldsymbol{y}}, [L_1], \alpha_*, \delta_1)$.

We take an inclusion $\iota : L_2' \longrightarrow L_2$ such that the cokernel L_2/L_2' is a line bundle on some smooth divisor of X. It naturally induces the morphism $L_2^{-1} \longrightarrow L_2'^{-1}$. The cokernel is denoted by Cok. We also obtain the inclusion $\mathcal{M}^s(\widehat{\boldsymbol{y}}, [\boldsymbol{L}], \alpha_*, \boldsymbol{\delta}) \longrightarrow \mathfrak{B}$.

We have the morphisms $\mathcal{M}^s(\widehat{\boldsymbol{y}}, [\boldsymbol{L}], \alpha_*, \boldsymbol{\delta}) \longrightarrow \mathcal{M}(\widehat{\boldsymbol{y}}, [L_i])$ $(i = 1, 2)$. The pull back of the relative tautological line bundles are denoted by $\mathcal{O}_{\mathrm{rel}}^{(i)}(1)$. Similarly, we have the morphisms of $\mathfrak{B}$ to $\mathcal{M}(\widehat{\boldsymbol{y}}, [L_1])$ and $\mathcal{M}(\widehat{\boldsymbol{y}}, [L_2'])$. The pull back of the relative tautological bundles are also denoted by $\mathcal{O}_{\mathrm{rel}}^{(1)}(1)$ and $\mathcal{O}_{\mathrm{rel}}^{(2)}(1)$, respectively.

Let $\widehat{E}'^{\,u}$ denote the universal sheaf over $\mathfrak{B} \times X$. We have the universal reduced $[L_2']$-section $[\phi_2] : \mathcal{O}_{\mathrm{rel}}^{(2)}(-1) \otimes L_2' \longrightarrow \widehat{E}'^{\,u}$. We put $E'^{\,u} := \widehat{E}'^{\,u} \otimes \mathcal{O}_{\mathrm{rel}}^{(2)}(1)$. We put as follows:

$$\mathfrak{V} := p_{X\,*}\bigl(E'^{\,u} \otimes \mathrm{Cok}\bigr).$$

It is easy to see $R^i p_{X\,*}\bigl(E'^{\,u} \otimes \mathrm{Cok}\bigr) = 0$ for $i = 1, 2$, and hence $\mathfrak{V}$ is a locally free sheaf on $\mathfrak{B}$. The universal reduced L_2'-section $[\phi_2^u]$ induces the section ψ of $\mathfrak{V}$ over $\mathfrak{B}$. It is easy to observe that $\psi^{-1}(0)$ is isomorphic to $\mathcal{M}^s(\widehat{\boldsymbol{y}}, [\boldsymbol{L}], \alpha_*, \boldsymbol{\delta})$.

Thus, we obtain the following Cartesian diagram:

$$\begin{tikzcd}
\mathcal{M}^s(\widehat{\boldsymbol{y}}, [\boldsymbol{L}], \alpha_*, \boldsymbol{\delta}) \arrow[r, "i_1"] \arrow[d, "j"'] & \mathfrak{B} \arrow[d, "\psi"] \\
\mathfrak{B} \arrow[r, "i"] & \mathfrak{V}
\end{tikzcd} \tag{164}$$

Here i denotes the 0-section. For the proof of Proposition 6.46, we have only to show $i^![\mathfrak{B}] = [\mathcal{M}^s(\widehat{\boldsymbol{y}}, [\boldsymbol{L}], \alpha_*, \boldsymbol{\delta})]$.

6.4.3 Compatibility of the obstruction theories

Lemma 6.49 *The obstruction theories of $\mathcal{M}^s(\widehat{\boldsymbol{y}}, [\boldsymbol{L}], \alpha_*, \boldsymbol{\delta})$ and $\mathfrak{B}$ are compatible over i in the diagram* (164).

Proof We discuss in some more general situation. Let L be a line bundle on X. We take a sufficiently large integer m. We take an inclusion $\iota : \mathcal{O}(-m) \longrightarrow L$. It naturally induces the morphism $L^{-1} \longrightarrow \mathcal{O}(m)$. We assume that the cokernel Cok is a line bundle on some smooth divisor of X.

We use the notation $\mathcal{M}_1$ and $\mathcal{M}_2$ to denote the moduli stacks $\mathcal{M}(m, \widehat{\boldsymbol{y}}, [L])$ and $\mathcal{M}(m, \widehat{\boldsymbol{y}}, [\mathcal{O}(-m)])$. The inclusion $\mathcal{M}(m, \widehat{\boldsymbol{y}}, [L]) \longrightarrow \mathcal{M}(m, \widehat{\boldsymbol{y}}, [\mathcal{O}(-m)])$ is induced by ι. Let $\widehat{E}_2^u$ denote the universal sheaf over $\mathcal{M}_2 \times X$. We put $E_2^u := \widehat{E}_2^u \otimes \mathcal{O}_{\mathrm{rel}}(1)$. We put $\mathfrak{V} := p_{X\,*}\bigl(\widehat{E}_2^u \otimes \mathrm{Cok}\bigr)$. The universal reduced $\mathcal{O}(-m)$-section of $\widehat{E}_2^u$ induces the section ψ of $\mathfrak{V}$. We have $\mathcal{M}_1 \simeq \psi^{-1}(0)$ and the following commutative diagram:

$$\begin{tikzcd}
\mathcal{M}_1 \arrow[r, "i_1"] \arrow[d, "j"'] & \mathcal{M}_2 \arrow[d, "\psi"'] \\
\mathcal{M}_2 \arrow[r, "i"] & \mathfrak{V}
\end{tikzcd} \tag{165}$$

To prove the claim of the lemma, we have only to show that the obstruction theories of $\mathcal{M}_i$ are compatible over i in the diagram (165).

Let $\widehat{E}_1^u$ denote the universal sheaf on $\mathcal{M}_1$, and let $[\phi_1^u]$ denote the universal reduced L-section. We put $\mathcal{V}_{1,0} := p_X^* p_{X\,*} \widehat{E}_1^u(m)$, and the kernel of the surjection $\mathcal{V}_{1,0} \longrightarrow \widehat{E}_1^u(m)$ is denoted by $\mathcal{V}_{1,-1}$. We take a locally free resolution $P_\cdot$ of $L(m)$ such that P_0 is a direct sum of $\mathcal{O}_X$. We have the canonical lift $[\widetilde{\phi}_1] : p_{\mathcal{M}_1}^* P_\cdot \otimes p_X^* \mathcal{O}_{\mathrm{rel}}(-1) \longrightarrow \mathcal{V}_{1,\cdot}$. Recall $\mathfrak{g}'_{\mathrm{rel}}(\mathcal{V}_{1,\cdot}, [\widetilde{\phi}_1]) := \mathcal{H}om\bigl(p_{\mathcal{M}_1}^* P_\cdot \otimes p_X^* \mathcal{O}_{\mathrm{rel}}(-1),\ \mathcal{V}_{1,\cdot}\bigr)^\vee$ and $\mathfrak{g}(\mathcal{V}_{1,\cdot}) := \mathcal{H}om\bigl(\mathcal{V}_{1,\cdot}, \mathcal{V}_{1,\cdot}\bigr)^\vee[-1]$.

Similarly, we take a locally free resolution $\mathcal{V}_{2\,\cdot}$ of the universal sheaf $\widehat{E}_2$ on $\mathcal{M}_2 \times X$. Let $[\phi_2]$ denote the universal reduced $\mathcal{O}(-m)$-section of $\widehat{E}_2$. We have the canonical lift $[\widetilde{\phi}_2] : p_{\mathcal{M}_2}^* \mathcal{O}_X \otimes p_X^* \mathcal{O}_{\mathrm{rel}}(-1) \longrightarrow \mathcal{V}_{2,0}$ of $[\phi_2]$. In this case, we have $\mathfrak{g}'_{\mathrm{rel}}(\mathcal{V}_{2\,\cdot}, [\widetilde{\phi}_2]) = \mathcal{H}om\bigl(p_{\mathcal{M}_2}^* \mathcal{O} \otimes p_X^* \mathcal{O}_{\mathrm{rel}}(1),\ \mathcal{V}_{2,\cdot}\bigr)^\vee$, and $\mathfrak{g}(\mathcal{V}_{2\,\cdot}) = \mathcal{H}om\bigl(\mathcal{V}_{2\,\cdot}, \mathcal{V}_{2\,\cdot}\bigr)^\vee[-1]$.

The inclusion $\iota : \mathcal{O} \longrightarrow L(m)$ has the canonical lift $\mathcal{O}_X \longrightarrow P.$. Therefore, we have the following commutative diagram on $\mathcal{M}_1$:

$$\begin{array}{ccc} i_1^*\mathfrak{g}'_{\mathrm{rel}}(\mathcal{V}_2., [\widetilde{\phi}_2])[-1] & \longrightarrow & \mathfrak{g}'_{\mathrm{rel}}(\mathcal{V}_1., [\widetilde{\phi}_1])[-1] \\ \downarrow & & \downarrow \\ i_1^*\mathfrak{g}(\mathcal{V}_2.) & \longrightarrow & \mathfrak{g}(\mathcal{V}_1.) \end{array} \tag{166}$$

Lemma 6.50 *The diagram* (166) *is compatible with the morphisms to the cotangent complexes:*

$$\begin{array}{ccc} i_1^*L_{\mathcal{M}_2\times X/(\mathcal{M}(m,\boldsymbol{y})\times X)_{G_m}}[-1] & \longrightarrow & L_{\mathcal{M}_1\times X/(\mathcal{M}(m,\boldsymbol{y})\times X)_{G_m}}[-1] \\ \downarrow & & \downarrow \\ i_1^*\pi_2^*L_{\mathcal{M}(m,\boldsymbol{y})\times X/X_{G_m}} & \longrightarrow & \pi_1^*L_{\mathcal{M}(m,\boldsymbol{y})\times X/X_{G_m}} \end{array}$$

Here, π_i denote the natural morphism of $\mathcal{M}_i$ to $\mathcal{M}(m,\boldsymbol{y})$. (See the subsubsection 2.1.4 *for the compatibility of diagrams.)*

Proof We give only an indication. We use the notation in the subsubsection 5.4.1. We construct the stack $Y(W., [\mathcal{O}_X])$ by replacing $P.$ with the complex $(0 \to \mathcal{O}_X)$. Then we have the following commutative diagram:

$$\begin{array}{ccccc} \mathcal{M}_1 \times X & \longrightarrow & Y(W., [P.]) & \longrightarrow & X_{G_m} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{M}_2 \times X & \longrightarrow & Y(W., [\mathcal{O}_X]) & \longrightarrow & X_{G_m} \\ \downarrow & & \downarrow & & \downarrow \\ (\mathcal{M}(m,\boldsymbol{y}) \times X)_{G_m} & \longrightarrow & Y(W.)_{G_m} & \longrightarrow & X_{G_m} \end{array} \tag{167}$$

Then the desired compatibility follows from the construction of the complexes. ∎

Lemma 6.51 *We have the following commutative diagram:*

$$\begin{array}{ccc} i_1^*\,\mathrm{Ob}\big(m, \boldsymbol{y}, [\mathcal{O}(-m)]\big) & \longrightarrow & \mathrm{Ob}(m, \boldsymbol{y}, [L]) \\ \downarrow & & \downarrow \\ i_1^*L_{\mathcal{M}_2} & \longrightarrow & L_{\mathcal{M}_1} \end{array} \tag{168}$$

Proof We obtain the following commutative diagram from (166):

$$\begin{array}{ccc} i_1^*\,\mathrm{Ob}'_{\mathrm{rel}}\big(\mathcal{V}_2., [\widetilde{\phi}_2]\big)[-1] & \longrightarrow & \mathrm{Ob}'_{\mathrm{rel}}\big(\mathcal{V}_1., [\widetilde{\phi}_1]\big)[-1] \\ \downarrow & & \downarrow \\ i_1^*\,\mathrm{Ob}\big(\mathcal{V}_{2,}.\big) & \longrightarrow & \mathrm{Ob}\big(\mathcal{V}_1.\big) \end{array} \tag{169}$$

It is compatible with the morphisms to the cotangent complexes due to Lemma 6.50:

$$\begin{array}{ccc} i_1^*L_{\mathcal{M}_2/\mathcal{M}(m,\boldsymbol{y})_{G_m}}[-1] & \longrightarrow & L_{\mathcal{M}_1/\mathcal{M}(m,\boldsymbol{y})_{G_m}}[-1] \\ \downarrow & & \downarrow \\ i_1^*\pi_2^*L_{\mathcal{M}(m,\boldsymbol{y})_{G_m}} & \longrightarrow & \pi_1^*L_{\mathcal{M}(m,\boldsymbol{y})_{G_m}} \end{array}$$

By a modification as in the subsubsection 5.4.1, we obtain the commutative diagram from (169):

$$\begin{array}{ccc} i_1^*\,\mathrm{Ob}_{\mathrm{rel}}\big(\mathcal{V}_2., [\widetilde{\phi}_2]\big)[-1] & \longrightarrow & \mathrm{Ob}_{\mathrm{rel}}\big(\mathcal{V}_1., [\widetilde{\phi}_1]\big)[-1] \\ \downarrow & & \downarrow \\ i_1^*\,\mathrm{Ob}\big(\mathcal{V}_{2,}.\big) & \longrightarrow & \mathrm{Ob}\big(\mathcal{V}_1.\big) \end{array}$$

It is compatible with the following commutative diagram:

$$\begin{tikzcd} i_1^*L_{\mathcal{M}_2/\mathcal{M}(m,\boldsymbol{y})}[-1] \arrow[r] \arrow[d] & L_{\mathcal{M}_1/\mathcal{M}(m,\boldsymbol{y})}[-1] \arrow[d] \\ i_1^*\pi_2^*L_{\mathcal{M}(m,\boldsymbol{y})} \arrow[r] & \pi_2^*L_{\mathcal{M}(m,\boldsymbol{y})} \end{tikzcd}$$

Then we obtain the desired commutative diagram (168) by construction. ∎

We put $\widetilde{\mathcal{V}}_{1,\cdot} := \mathcal{V}_{1,\cdot} \otimes \mathcal{O}_{\mathrm{rel}}(1)$. We put as follows:

$$\mathfrak{g}_{\mathrm{rel}} := \mathrm{Cone}\Big(i_1^*\mathcal{H}om\big(p_{\mathcal{M}_2}^*\mathcal{O}_X,\, \widetilde{\mathcal{V}}_{2,\cdot}\big)^\vee \longrightarrow \mathcal{H}om\big(p_{\mathcal{M}_1}^*P_\cdot,\, \widetilde{\mathcal{V}}_{1,\cdot}\big)^\vee\Big), \qquad \mathrm{Ob}_{\mathrm{rel}} := Rp_{X\,*}\big(\mathfrak{g}_{\mathrm{rel}} \otimes \omega_X\big)$$

It is easy to observe that $\mathrm{Ob}_{\mathrm{rel}}$ naturally isomorphic to $\mathrm{Cone}\big(\mathrm{Ob}(m,\widehat{\boldsymbol{y}},[\mathcal{O}(-m)]) \longrightarrow \mathrm{Ob}(m,\widehat{\boldsymbol{y}},[L])\big)$. From the diagram (167), we obtain the following morphism:

$$L_{\mathcal{M}_1\times X/\mathcal{M}_2\times X} \longleftarrow \Phi(\mathcal{V}_\cdot,[\widetilde{\phi}])^*L_{Y(W_\cdot,[P_\cdot])/Y(W_\cdot,[\mathcal{O}_X])} \longleftarrow \mathfrak{g}_{\mathrm{rel}} \tag{170}$$

It is easy to observe that the composite of (170) induces the morphism $\beta_1 : \mathrm{Ob}_{\mathrm{rel}} \longrightarrow L_{\mathcal{M}_1/\mathcal{M}_2}$, which is same as the morphism induced from (168).

We have the naturally defined morphism:

$$h : N\big(\mathcal{O}_{\mathrm{rel}}(-1)\otimes P_{-1}, \mathcal{V}_{1,-1}\big) \times N\big(\mathcal{O}_{\mathrm{rel}}(-1)\otimes P_0, \mathcal{V}_{1,0}\big) \longrightarrow N\big(\mathcal{O}_{\mathrm{rel}}(-1)\otimes P_{-1}, \mathcal{V}_{1,0}\big)$$

We put $Z_1 := h^{-1}(0)$. We also put $Z_2 := N\big(\mathcal{O}_{\mathrm{rel}}(-1), \mathcal{V}_{2,0}\big)$ which is the vector bundle over $\mathcal{M}_2 \times X$. Then, the lifts $[\widetilde{\phi}_i]$ induce the following factorization:

$$\begin{tikzcd} \mathcal{M}_1\times X \arrow[r, "\Phi_1"] \arrow[d] & Z_1 \arrow[r] \arrow[d] & Y(W_\cdot,[P_\cdot]) \arrow[d] \\ \mathcal{M}_2\times X \arrow[r] & Z_2 \arrow[r] & Y(W_\cdot,[\mathcal{O}_X]) \end{tikzcd}$$

We obtain the following factorization:

$$\Phi(\mathcal{V}_\cdot,[\widetilde{\phi}])^*L_{Y(W_\cdot,[P_\cdot])/Y(W_\cdot,[\mathcal{O}_X])} \xrightarrow{\simeq} \Phi_1^*L_{Z_1/Z_2} \longrightarrow L_{\mathcal{M}_1\times X/\mathcal{M}_2\times X}$$

We put $Z_3 := N(p_{\mathcal{M}_2}^*\mathcal{O}_X,\, p_X^*\mathfrak{V})$, which is the vector bundle over $\mathcal{M}_2 \times X$. We have the naturally defined morphism $\mathcal{V}_{2,0}\otimes\mathcal{O}_{\mathrm{rel}}(1) \longrightarrow p_X^*\mathfrak{V}$. Then, we obtain the following diagram:

$$\begin{tikzcd} \mathcal{M}_1\times X \arrow[r, "\Phi_1"] \arrow[d] & Z_1 \arrow[r, "\Phi_2"] \arrow[d] & \mathcal{M}_2\times X \arrow[d] \\ \mathcal{M}_2\times X \arrow[r] & Z_2 \arrow[r] & Z_3 \end{tikzcd}$$

We naturally have $\Phi_1^*\Phi_2^*L_{\mathcal{M}_2\times X/Z_3} \simeq j_X^*L_{\mathcal{M}_2\times X/p_X^*\mathfrak{V}}$. Thus, we obtain the morphism: $\alpha : j_X^*L_{\mathcal{M}_2\times X/p_X^*\mathfrak{V}} \longrightarrow \Phi_1^*L_{Z_1/Z_2}$. It can be checked that α factors through $\mathfrak{g}_{\mathrm{rel}}$. Namely, we have the following morphisms:

$$j_X^*L_{\mathcal{M}_2\times X/p_X^*\mathfrak{V}} \longrightarrow \mathfrak{g}_{\mathrm{rel}} \longrightarrow L_{\mathcal{M}_1\times X/\mathcal{M}_2\times X}.$$

It induces the morphism:

$$j^*L_{\mathcal{M}_2/\mathfrak{V}} \otimes Rp_{X\,*}\omega_X \xrightarrow{\beta_0} \mathrm{Ob}_{\mathrm{rel}} \xrightarrow{\beta_1} L_{\mathcal{M}_1/\mathcal{M}_2}$$

It is easy to observe that β_0 induces the isomorphism $\beta_2 : j^*L_{\mathcal{M}_2/\mathfrak{V}} \longrightarrow \mathrm{Ob}_{\mathrm{rel}}$, and the composite $\beta_1\circ\beta_2$ is same as the morphism obtained from the diagram (165). Thus the proof of Lemma 6.49 is finished. ∎

6.4.4 Ambient smooth stack

Since $\mathfrak{B}$ is not smooth in general, we construct an ambient smooth stack to use Proposition 2.32. Let C denote the support of Cok. Recall that C is smooth and that Cok is isomorphic to a line bundle $\mathcal{L}_C$ on C, due to our choice of ι.

For a k-scheme T, let $F(T)$ denote the set of the quotients $q : p_T^* V_{m,C} \longrightarrow \mathcal{E}$ over $T \times C$, satisfying the following condition:

- $\mathcal{E}$ is flat over T, and the type of $\mathcal{E}$ is $y_{|C} \cdot \mathrm{ch}(\mathcal{L}_C)$.
- For any point $u \in T$, $H^1\bigl(C, \mathcal{E}_{|\{u\}\times C}\bigr) = 0$.

Then, we obtain the functor F of the category of k-schemes to the category of sets. The functor F is representable by a scheme Q_1°.

Lemma 6.52 *The scheme Q_1° is smooth.*

Proof We have the perfect obstruction theory $\mathrm{Ob}\bigl(Q_1^\circ\bigr)$ of Q_1° (Proposition 2.38). Let $z = (q, \mathcal{E})$ be a point of Q_1°, and let i_z denote the inclusion $\{z\} \longrightarrow Q_1^\circ$. Let $\mathcal{K}$ denote the kernel of $V_{m,C} \longrightarrow \mathcal{E}$. We have only to show $\mathcal{H}^{-1}\bigl(i_z^* \mathrm{Ob}(Q_1^\circ)\bigr) = 0$.

The dual $\mathcal{H}^i\bigl(i_z^* \mathrm{Ob}(Q_1^\circ)\bigr)^\vee$ is isomorphic to $\mathrm{Ext}^{-i}\bigl(\mathcal{K}, \mathcal{E}\bigr)$. We have the exact sequence $\mathrm{Ext}^1\bigl(V_{m,C}, \mathcal{E}\bigr) \longrightarrow \mathrm{Ext}^1\bigl(\mathcal{K}, \mathcal{E}\bigr) \longrightarrow \mathrm{Ext}^2\bigl(\mathcal{E}, \mathcal{E}\bigr)$. We have the vanishing of the first term by definition of Q_1°. Since C is a smooth curve, we also have the vanishing of the third term. Therefore, we obtain the desired vanishing, and hence Q_1° is smooth. ∎

We have the universal quotient sheaf $p_{Q_1^\circ} V_{m,C} \longrightarrow \mathcal{C}$ on $Q_1^\circ \times C$. The push-forward $p_{C\,*}\mathcal{C}$ gives the vector bundle on Q_1°. We denote it by $\mathfrak{V}_1$.

We use the notation in the subsubsections 4.1.1. We put $Q^{ss}(m, \widehat{\boldsymbol{y}}, \alpha_*) := Q^{ss}(m, \boldsymbol{y}, \alpha_*) \times_{Q(m,y)} Q(m, \widehat{y})$. Let $\widehat{Z}_m$ be as in (38). We have the $\mathrm{GL}(V_m)$-closed immersion $Q^{ss}(m, \widehat{\boldsymbol{y}}, \alpha_*) \longrightarrow \widehat{Z}_m \times \prod_i G_{m,i}$. (See the subsubsection 3.6.3 for $G_{m,i}$.)

We also have the naturally defined $\mathrm{GL}(V_m)$-equivariant morphism $Q^{ss}(m, \widehat{\boldsymbol{y}}, \alpha_*) \longrightarrow Q_1^\circ$ by the correspondence $(q, \mathcal{E}, F_*, \rho) \longmapsto (q', \mathcal{E}(-m) \otimes \mathrm{Cok})$, where q' denotes the naturally induced map $V_{m,C} \longrightarrow \mathcal{E}(-m) \otimes \mathrm{Cok}$.

We have the natural right $\mathrm{GL}(V_m)$-action on $\widehat{Z}_m \times \prod_i G_{m,i} \times Q_1^\circ \times \mathbb{P}_m^{(1)} \times \mathbb{P}_m^{(2)}$. The quotient stack is denoted by $\widetilde{\mathfrak{B}}'$. The bundle $\mathfrak{V}_1$ induce the $\mathrm{GL}(V_m)$-vector bundle on $\widehat{Z}_m \times \prod_i G_{m,i} \times \mathbb{P}_m \times Q_1^\circ$, and hence the vector bundle $\widetilde{\mathfrak{V}}$ on $\widetilde{\mathfrak{B}}'$.

We have the $\mathrm{GL}(V_m)$-equivariant immersion $Q^\circ(m, \widehat{\boldsymbol{y}}, [L_1]) \times \mathbb{P}_m^{(2)} \longrightarrow \widehat{Z}_m \times \prod_i G_{m,i} \times Q_1^\circ \times \mathbb{P}_m^{(1)} \times \mathbb{P}_m^{(2)}$, which induces the immersion $\mathfrak{B} \longrightarrow \widetilde{\mathfrak{B}}'$. Since $\mathfrak{B}$ is Deligne-Mumford, we can take an open neighbourhood $\widetilde{\mathfrak{B}}$ of $\mathfrak{B}$ in $\widetilde{\mathfrak{B}}'$, which is Deligne-Mumford and smooth. The restriction of $\widetilde{\mathfrak{V}}$ to $\widetilde{\mathfrak{B}}$ is denoted by $\widetilde{\mathfrak{V}}$. By the construction, the restriction of $\widetilde{\mathfrak{V}}$ to $\mathfrak{B}$ is same as $\mathfrak{V}$. We put $\mathcal{M} = \mathcal{M}^{ss}(\widehat{\boldsymbol{y}}, [\boldsymbol{L}], \alpha_*, \boldsymbol{\delta})$. Then, we obtain the following diagram:

$$\begin{tikzcd}
\mathcal{M} \arrow[r] \arrow[d, "i_1"'] & \mathfrak{B} \arrow[r] \arrow[d, "i"'] & \widetilde{\mathfrak{B}} \arrow[d, "i_2"'] \\
\mathfrak{B} \arrow[r, "\psi"] & \mathfrak{V} \arrow[r] & \widetilde{\mathfrak{V}}
\end{tikzcd}$$

Here i and i_2 denote the 0-section. We have $i^! = i_2^!$. The obstruction theories of $\mathcal{M}$ and $\mathfrak{B}$ are compatible (Lemma 6.49). Then, we obtain $i^!\bigl([\mathcal{M}_2]\bigr) = [\mathcal{M}_1]$, due to Proposition 2.32. Thus, the proof of Proposition 6.44 is finished. ∎

6.5 Comparison with Full Flag Bundles

Let $\boldsymbol{y}$ be an element of $\mathcal{T}ype$. Let α_* be a system of weights. Let L be a line bundle on X. We take a sufficiently large integer m. Let $\widetilde{\mathcal{M}}^s(\widehat{\boldsymbol{y}}, [L], \alpha_*, \delta)$ denote the full flag bundle associated to the bundle $p_{X\,*}\widehat{E}^u(m)$. We have the natural smooth morphism:

$$\mathfrak{F}_1 : \widetilde{\mathcal{M}}^s(\widehat{\boldsymbol{y}}, [L], \alpha_*, \delta) \longrightarrow \mathcal{M}^s(\widehat{\boldsymbol{y}}, [L], \alpha_*, \delta)$$

Lemma 6.53 *We have the following relation:*

$$\mathfrak{F}_1^*\big([\mathcal{M}^s(\widehat{\boldsymbol{y}},[L],\alpha_*,\delta)]\big)=[\widetilde{\mathcal{M}}^s(\widehat{\boldsymbol{y}},[L],\alpha_*,\delta)]$$

Proof Let π denote the projection $\widetilde{\mathcal{M}}(m,\widehat{\boldsymbol{y}},[L])\longrightarrow\mathcal{M}(m,\widehat{\boldsymbol{y}},[L])$. By construction in the subsubsection 5.7.3, we have the following morphism of the distinguished triangles on $\widetilde{\mathcal{M}}(m,\widehat{\boldsymbol{y}},[L])$:

$$\begin{array}{ccccccc}
\pi^*\operatorname{Ob}(m,\widehat{\boldsymbol{y}},[L]) & \longrightarrow & \widetilde{\operatorname{Ob}}(m,\widehat{\boldsymbol{y}},[L]) & \longrightarrow & L_{\widetilde{\mathcal{M}}(m,\widehat{\boldsymbol{y}},[L])/\mathcal{M}(m,\widehat{\boldsymbol{y}},[L])} & \longrightarrow & \pi^*\operatorname{Ob}(m,\widehat{\boldsymbol{y}},[L])[1]\\
\big\downarrow & & \big\downarrow & & \simeq\big\downarrow & & \big\downarrow\\
\pi^*L_{\mathcal{M}(m,\widehat{\boldsymbol{y}},[L])} & \longrightarrow & L_{\widetilde{\mathcal{M}}(m,\widehat{\boldsymbol{y}},[L])} & \longrightarrow & L_{\widetilde{\mathcal{M}}(m,\widehat{\boldsymbol{y}},[L])/\mathcal{M}(m,\widehat{\boldsymbol{y}},[L])} & \longrightarrow & \pi^*L_{\mathcal{M}(m,\widehat{\boldsymbol{y}},[L])}[1]
\end{array}$$

Hence we obtain the compatibility of the obstruction theories $\operatorname{Ob}(m,\widehat{\boldsymbol{y}},[L])$ and $\widetilde{\operatorname{Ob}}(m,\widehat{\boldsymbol{y}},[L])$ of $\mathcal{M}^s(\widehat{\boldsymbol{y}},[L],\alpha_*,\delta)$ and $\widetilde{\mathcal{M}}^s(\widehat{\boldsymbol{y}},[L],\alpha_*,\delta)$. Then, the claim follows from Proposition 2.32. ∎

Let $\boldsymbol{y}$ and α_* be as above. We take a sufficiently large integer m. Let $\widetilde{\mathcal{M}}^s(\widehat{\boldsymbol{y}},\alpha_*,+)$ be as in the subsubsection 5.7.5. Let ϵ be sufficiently small positive number. Then, we have the naturally defined morphism:

$$\mathfrak{F}_2:\widetilde{\mathcal{M}}^s(\widehat{\boldsymbol{y}},\alpha_*,+)\longrightarrow\mathcal{M}^s(\widehat{\boldsymbol{y}},[\mathcal{O}(-m)],\alpha_*,\epsilon)$$

Lemma 6.54 *We have the following relation:*

$$\mathfrak{F}_2^*\big([\mathcal{M}^s(\widehat{\boldsymbol{y}},[\mathcal{O}(-m)],\alpha_*,\epsilon)]\big)=[\widetilde{\mathcal{M}}^s(\widehat{\boldsymbol{y}},\alpha_*,+)]\tag{171}$$

Proof In this case, we have the other way of construction for the obstruction theory $\operatorname{Ob}(m,\widehat{\boldsymbol{y}},[\mathcal{O}(-m)])$ on $\mathcal{M}(m,\widehat{\boldsymbol{y}},[\mathcal{O}(-m)])$ by the method in the subsubsection 5.7.1. Let V_m be an $H_y(m)$-dimensional vector space. We put $B(W):=k_{\mathrm{GL}(V_m)}$ and $B(W,[P]):=\mathbb{P}(V_m^\vee)_{\mathrm{GL}(V_m)}$ as in the subsubsections 5.1.3 and 5.4.3. Let $\widehat{E}^u$ denote the universal sheaf over $\mathcal{M}(m,\widehat{\boldsymbol{y}})\times X$. Then, $\mathcal{M}(m,\widehat{\boldsymbol{y}},[\mathcal{O}(-m)])$ is the projectivization of $p_{X\,*}(\widehat{E}^u(m))$. Hence, we have the following naturally defined Cartesian diagram:

$$\begin{array}{ccc}
\mathcal{M}(m,\widehat{\boldsymbol{y}},[\mathcal{O}(-m)]) & \xrightarrow{\ \Psi\ } & B(W,[P])\\
\pi\big\downarrow & & \big\downarrow\\
\mathcal{M}(m,\widehat{\boldsymbol{y}}) & \xrightarrow{\ \varphi\ } & B(W)
\end{array}$$

Since φ factors through $\operatorname{Ob}(m,\widehat{\boldsymbol{y}})$, we obtain the following morphism:

$$\Psi^*L_{B(W,[P])/B(W)}[-1]\longrightarrow\pi^*L_{B(W)}\longrightarrow\pi^*\operatorname{Ob}(m,\widehat{\boldsymbol{y}})$$

The cone of the composite is denoted by $\operatorname{Ob}_2(m,\widehat{\boldsymbol{y}},[\mathcal{O}(-m)])$. Then, we obtain the morphism:

$$\mathrm{ob}_2(m,\widehat{\boldsymbol{y}},[\mathcal{O}(-m)]):\operatorname{Ob}_2(m,\widehat{\boldsymbol{y}},[\mathcal{O}(-m)])\longrightarrow L_{\mathcal{M}(m,\widehat{\boldsymbol{y}},[\mathcal{O}(-m)])}$$

We use the following lemma.

Lemma 6.55 *We have* $\mathrm{ob}_2(m,\widehat{\boldsymbol{y}},[\mathcal{O}(-m)])=\mathrm{ob}(m,\widehat{\boldsymbol{y}},[\mathcal{O}(-m)])$ *in* $D(\mathcal{M}(m,\widehat{\boldsymbol{y}},[\mathcal{O}(-m)]))$.

Proof We use the result in the subsubsection 5.4.3. We take the canonical locally free resolution of $\widehat{E}^u(m)$. Namely, we put $\mathcal{V}_0:=p_X^*\big(p_{X\,*}\widehat{E}^u(m)\big)$, and the $\mathcal{V}_{-1}$ denotes the kernel of the canonical morphism $\mathcal{V}_0\longrightarrow\widehat{E}^u(m)$. The reduced $\mathcal{O}_X$-section $[\phi]$ of $\widehat{E}^u(m)$ is canonically lifted to the reduced $\mathcal{O}_X$-section $[\widetilde{\phi}]$ of $\mathcal{V}_0$. In this case, the diagram (107) is as follows:

$$\begin{array}{ccccc}
L_{\mathcal{M}(m,\widehat{\boldsymbol{y}},[L])/\mathcal{M}(m,\widehat{\boldsymbol{y}})}[-1] & \longleftarrow & \operatorname{Ob}_{\mathrm{rel}}(\mathcal{V}_\cdot,[\widetilde{\phi}])[-1] & \xleftarrow[a_1]{} & \operatorname{Ob}^G_{\mathrm{rel}}(\mathcal{V}_\cdot,[\widetilde{\phi}])[-1]\\
\big\downarrow & & \big\downarrow & & \big\downarrow\\
\pi^*L_{\mathcal{M}(m,\widehat{\boldsymbol{y}})} & \longleftarrow & \operatorname{Ob}(\mathcal{V}_\cdot) & \longleftarrow & \operatorname{Ob}^G(\mathcal{V}_\cdot)
\end{array}\tag{172}$$

The diagram (108) is as follows:

$$\begin{array}{ccccc}
\pi^* L_{\mathcal{M}(m,\widehat{\boldsymbol{y}})} & \longleftarrow & \Phi(\mathcal{V}.)^* L_{B(W)/k} & \xleftarrow{\tau_1} & \mathrm{Ob}^G(V.) \\
\uparrow & & \uparrow & & \uparrow \\
L_{\mathcal{M}(m,\widehat{\boldsymbol{y}},[L])/\mathcal{M}(m,\widehat{\boldsymbol{y}})}[-1] & \longleftarrow & \Psi^* L_{B(W,[P])/B(W)}[-1] & \xleftarrow{\tau_3} & \mathrm{Ob}^G_{\mathrm{rel}}(\mathcal{V}.,[\widetilde{\phi}])[-1]
\end{array} \tag{173}$$

The morphisms τ_1 and τ_3 are isomorphic, in general (Lemma 5.2 and Lemma 5.20). The morphism a_1 is also isomorphic in this case. Then, the claim of Lemma 6.55 immediately follows. ∎

It is easy to see the compatibility of the obstruction theories $\widetilde{\mathrm{ob}}(m,\widehat{\boldsymbol{y}})$ and $\widetilde{\mathrm{ob}}_2(m,\widehat{\boldsymbol{y}},[\mathcal{O}(-m)])$ on the moduli stacks $\widetilde{\mathcal{M}}^s(\widehat{\boldsymbol{y}},\alpha_*,+)$ and $\mathcal{M}^s(\widehat{\boldsymbol{y}},[\mathcal{O}(-m)],\alpha_*,\epsilon)$. Hence, we obtain (171) due to Proposition 2.32. Thus the proof of Lemma 6.54 is finished. ∎

6.6 Parabolic Perturbation

Since we do not use the result in this subsection later, the reader can skip here.

6.6.1 Statement

Let y be an element of $H^*(X)$. Let $\boldsymbol{y}$ be an element of $\mathcal{T}ype$ whose $H^*(X)$-component is y, and let α_* denote a system of weights. Let us discuss the relation of the obstruction theories and the virtual fundamental classes of the moduli stacks $\mathcal{M}^s(\widehat{y})$ and $\mathcal{M}^s(\widehat{\boldsymbol{y}},\alpha_*)$.

We assume that any semistable torsion-free sheaf of type y is also μ-stable, and that α_i are sufficiently close to 1. Then we have the morphism $\mathfrak{F}:\mathcal{M}^s(\widehat{\boldsymbol{y}},\alpha_*)\longrightarrow\mathcal{M}^s(\widehat{y})$.

Proposition 6.56 *There exists a Deligne-Mumford stack $\mathfrak{B}(\widehat{\boldsymbol{y}},\alpha_*)$ over $\mathcal{M}^s(\widehat{y})$ with a vector bundle $\mathfrak{V}$ and the section ψ, such that the following holds:*

- *The morphism $\mathfrak{G}:\mathfrak{B}(\widehat{\boldsymbol{y}},\alpha_*)\longrightarrow\mathcal{M}^s(\widehat{y})$ is smooth.*
- *$\mathcal{M}^s(\widehat{\boldsymbol{y}},\alpha_*)$ is isomorphic to $\psi^{-1}(0)$.*
- *We have the following relation:*
$$\psi^!\big(\mathfrak{G}^*[\mathcal{M}^s(\widehat{y})]\big)=\big[\mathcal{M}^s(\widehat{\boldsymbol{y}},\alpha_*)\big] \tag{174}$$
Here $\psi^!$ denote the Gysin map for the inclusion $\mathcal{M}^s(\widehat{\boldsymbol{y}},\alpha_*)\longrightarrow\mathfrak{V}$.

The proof will be given in the next subsubsections. Before going into the proof, we give some remarks.

Corollary 6.57 *Let P be any point of $\mathcal{M}^s(\widehat{y})$. The fiber $\mathfrak{G}^{-1}(P)$ is smooth, and $\mathfrak{F}^{-1}(P)$ is the 0-set of the section $\psi_{|\mathfrak{G}^{-1}(P)}$ of $\mathfrak{V}_{|\mathfrak{G}^{-1}(P)}$ in the situation of Proposition 6.56. Therefore, we obtain the perfect obstruction theory of $\mathfrak{F}^{-1}(P)$.* ∎

We are mainly interested in the cap product of some cohomology classes and the virtual fundamental classes.

Corollary 6.58 *Let Φ be a cohomology class on $\mathcal{M}^s(\widehat{y})$, and let Ψ be a cohomology class on $\mathcal{M}^s(\widehat{\boldsymbol{y}},\alpha_*)$. Assume $\deg(\Phi)=2\dim^f\mathcal{M}^s(\widehat{y})$. Proposition 6.56 implies the following relation for any k-valued point P of $\mathcal{M}^s(\widehat{y})$:*

$$\int_{\mathcal{M}^s(\widehat{\boldsymbol{y}},\alpha_*)}\Phi\cdot\Psi=\int_{\mathfrak{F}^{-1}(P)}\Psi\times\int_{\mathcal{M}^s(\widehat{y})}\Phi \tag{175}$$

(See the subsection 7.1 for the cohomology classes and the evaluation considered in this paper.) ∎

Let L be a line bundle on X, and let δ be an element of $\mathcal{P}^{\mathrm{br}}$. We can discuss a similar relation for $\mathcal{M}^s(\widehat{y},[L],\delta)$ and $\mathcal{M}^s(\widehat{\boldsymbol{y}},[L],\alpha_*,\delta)$. We assume that any δ-semistable L-Bradlow pair is μ-δ-stable, and that α_i are sufficiently close to 1. Then we have the morphism $\mathfrak{F}_L:\mathcal{M}^s(\widehat{\boldsymbol{y}},[L],\alpha_*,\delta)\longrightarrow\mathcal{M}^s(\widehat{y},[L],\delta)$.

Proposition 6.59 *There exists a $\mathcal{M}^s(\widehat{y},[L],\delta)$-scheme $\mathfrak{B}(\widehat{\boldsymbol{y}},[L],\alpha_*,\delta)$ with a vector bundle $\mathfrak{V}$ and its section ψ, such that the following holds:*

- *The morphism $\mathfrak{G}_L : \mathfrak{B}(\widehat{\boldsymbol{y}},[L],\alpha_*,\delta) \longrightarrow \mathcal{M}^s(\widehat{y},[L],\delta)$ is smooth.*
- *$\mathcal{M}^s(\widehat{\boldsymbol{y}},[L],\alpha_*,\delta)$ is isomorphic to $\psi^{-1}(0)$.*
- *We have the following relation:*

$$\psi^!\mathfrak{G}_L^*\big[\mathcal{M}^s(\widehat{y},[L],\delta)\big] = \big[\mathcal{M}^s(\widehat{\boldsymbol{y}},[L],\alpha_*,\delta)\big]$$

Here ψ denotes the Gysin map for $\mathcal{M}^s(\widehat{\boldsymbol{y}},[L],\alpha_*,\delta) \longrightarrow \mathfrak{V}$.

As a result, the following formula holds, for any k-valued point P of $\mathcal{M}^s(\widehat{y},[L],\delta)$:

$$\int_{\mathcal{M}^s(\widehat{\boldsymbol{y}},[L],\alpha_*,\delta)} \Phi\cdot\Psi = \int_{\mathfrak{G}_L^{-1}(P)} \Psi \cdot \int_{\mathcal{M}^s(\widehat{y},[L],\delta)} \Phi$$

Here Φ and Ψ denote cohomology classes on $\mathcal{M}^s(\widehat{y},L,\delta)$ and $\mathcal{M}^s(\widehat{\boldsymbol{y}},L,\alpha_*,\delta)$ respectively, and we assume $\deg(\Phi) = 2\dim^f \mathcal{M}^s(\widehat{y},[L],\delta)$. ∎

We will give the proof of Proposition 6.56 in the next subsubsections. The proof of Proposition 6.59 is similar, and hence we omit to give it.

6.6.2 The construction of a stack $\mathfrak{B}$ and the obstruction theory

Let m be a sufficiently large integer. Let $\mathcal{E}^u$ denote the universal sheaf on $\mathcal{M}(m,\widehat{y})$. We put $\mathcal{V}_0 := p_X^* p_{X\,*}\mathcal{E}^u(m)$, and the kernel of the natural morphism $\mathcal{V}_0 \longrightarrow \mathcal{E}^u(m)$ is denoted by $\mathcal{V}_{-1}$. We obtain the vector bundle $\mathcal{V}_{0\,|\,D}$ on $\mathcal{M}(m,\widehat{y})\times D$.

Let $g : T \longrightarrow \mathcal{M}(m,\widehat{y})$ be a morphism. Let $F(T)$ denote the set of the sequence of the quotients $g_D^*\mathcal{V}_{0\,|\,D} = \mathcal{C}_{l+1} \to \mathcal{C}_l \to \mathcal{C}_{l-1} \to \cdots \to \mathcal{C}_2 \to \mathcal{C}_1$ satisfying the following conditions:

- $\mathcal{C}_i$ are flat over T.
- For any point $u \in T$, the induced morphisms $H^0\big(D,\mathcal{C}_{l+1\,|\,\{u\}\times D}\big) \longrightarrow H^0\big(D,\mathcal{C}_{i\,|\,\{u\}\times D}\big)$ are surjective $(i = 1,\ldots,l)$.
- $H^1\big(D,\mathcal{C}_{i\,|\,\{u\}\times D}\big) = 0$ for any $u\in T$ and for any $i = 1,\ldots,l$.
- The type of C_i is same as $\sum_{j\le i} y_j(m)$.

Then, we obtain the functor F of the category of $\mathcal{M}(m,\widehat{y})$-schemes to the category of sets. The functor is representable by the $\mathcal{M}(m,\widehat{y})$-scheme, which we denote by $\mathfrak{B}$. Let $\pi : \mathfrak{B} \longrightarrow \mathcal{M}(m,\widehat{y})$ denote the natural projection.

Let us discuss the obstruction theory of $\mathfrak{B}$. We put $\mathcal{V}_D^{(1)} := \pi_D^*\mathcal{V}_{0|D}$ on $\mathfrak{B}\times D$. We have the universal quotients $\mathcal{V}_D^{(1)} \longrightarrow \mathcal{C}_i^u$ $(i = 1,\ldots,l)$. We put $\mathcal{V}_D^{(i)} := \mathrm{Ker}\big(\mathcal{V}_D^{(1)} \longrightarrow \mathcal{C}_{i-1}^u\big)$ for $i = 2,\ldots,l+1$. We also have the locally free sheaf $\mathcal{V}_{-1\,|D}$. By changing slightly the construction in the subsubsection 2.1.5, we consider the following complex $\overline{C}(\mathcal{V}_{\cdot|D},\mathcal{V}_D^*)$

$$\mathcal{H}om(\mathcal{V}_{0|D},\mathcal{V}_{-1|D}) \xrightarrow{d^{-1}} \bigoplus_{i=1}^{l+1}\mathcal{H}om(\mathcal{V}_D^{(i)},\mathcal{V}_D^{(i)})\oplus\mathcal{H}om(\mathcal{V}_{-1|D},\mathcal{V}_{-1|D}) \xrightarrow{d^0} \bigoplus_{i=1}^{l}\mathcal{H}om(\mathcal{V}_D^{(i+1)},\mathcal{V}_D^{(i)})\oplus\mathcal{H}om(V_{-1|D},V_{0|D})$$

The first term stands in the degree 0. The morphism d^{-1} is the composite of the following morphisms:

$$\mathcal{H}om(\mathcal{V}_{0|D},\mathcal{V}_{-1|D}) \xrightarrow{a_1} \mathcal{H}om(\mathcal{V}_{0|D},\mathcal{V}_{0|D})\oplus\mathcal{H}om(\mathcal{V}_{-1|D},\mathcal{V}_{-1|D}) \xrightarrow{a_2} \bigoplus_{i=1}^{l+1}\mathcal{H}om(\mathcal{V}_D^{(i)},\mathcal{V}_D^{(i)})\oplus\mathcal{H}om(\mathcal{V}_{-1|D},\mathcal{V}_{-1|D})$$

Here a_1 is the differential of the complex $\mathcal{H}om(\mathcal{V}_{\cdot|D},\mathcal{V}_{\cdot|D})$, and a_2 is the inclusion via $\mathcal{V}_{0|D}=\mathcal{V}_D^{(1)}$. The morphism d^0 is made of the following maps b_i $(i=1,2)$:

$$b_1:\bigoplus_{i=1}^{l+1}\mathcal{H}om(\mathcal{V}_D^{(i)},\mathcal{V}_D^{(i)})\longrightarrow\bigoplus_{i=1}^{l}\mathcal{H}om(\mathcal{V}_D^{(i+1)},\mathcal{V}_D^{(i)})$$

$$b_2:\mathcal{H}om(\mathcal{V}_D^{(1)},\mathcal{V}_D^{(1)})\oplus\mathcal{H}om(\mathcal{V}_{-1|D},\mathcal{V}_{-1|D})\longrightarrow\mathcal{H}om(\mathcal{V}_{-1|D},\mathcal{V}_{0|D})$$

Here b_1 is given as in (18), and b_2 is the differential of the complex $\mathcal{H}om(\mathcal{V}_{\cdot|D},\mathcal{V}_{\cdot|D})$. We put $\overline{\mathfrak{g}}_D(\mathcal{V}_\cdot,\mathcal{V}_D^*):=\overline{C}(\mathcal{V}_{\cdot|D},\mathcal{V}_D^*)^\vee[-1]$. We have the naturally defined morphism $\overline{C}(\mathcal{V}_{\cdot|D},\mathcal{V}_D^*)\longrightarrow\mathcal{H}om(\mathcal{V}_{\cdot|D},\mathcal{V}_{\cdot|D})$. It induces the morphism $\mathfrak{g}(\mathcal{V}_{\cdot|D})\longrightarrow\overline{\mathfrak{g}}_D(\mathcal{V}_\cdot,\mathcal{V}_D^*)$

We take vector spaces $W^{(i)}$ over k such that $\operatorname{rank}W^{(i)}=\operatorname{rank}\mathcal{V}_D^{(i)}$. We also take a vector space W_{-1} over k such that $\operatorname{rank}W_{-1}=\operatorname{rank}\mathcal{V}_{-1}$, and we put $W_0:=W^{(1)}$. We put $W_D^{(i)}:=W^{(i)}\otimes\mathcal{O}_D$ and $W_{i\,D}:=W_i\otimes\mathcal{O}_D$. We have the naturally defined right $\prod_{i=1}^{l+1}\mathrm{GL}(W^{(i)})$-action on $\prod_{i=1}^{l}N\bigl(W_D^{(i+1)},W_D^{(i)}\bigr)$. We also have natural right action of $\mathrm{GL}(W^{(1)})\times\mathrm{GL}(W_{-1})$ on $N(W_{-1\,D},W_{0\,D})$ by the identification $W_0=W^{(1)}$. Therefore, we have the naturally defined right action of $\prod_{i=1}^{l+1}\mathrm{GL}(W^{(i)})\times\mathrm{GL}(W_{-1})$ on $\prod_{i=1}^{l}N\bigl(W_D^{(i+1)},W_D^{(i)}\bigr)\times N\bigl(W_{-1\,D},W_{0\,D}\bigr)$, where the latter fiber product is taken over D. The quotient stack is denoted by $\overline{Y}_D(W_\cdot,W^*)$. (We remark that we used the notation $\overline{Y}_D(W_\cdot,W^*)$ in a different meaning in the subsection 6.3.) On the other hand, we use the stack $Y_D(W_\cdot)$ introduced in the subsubsection 5.5.1. The morphism $\overline{Y}_D(W_\cdot,W^*)\longrightarrow Y_D(W_\cdot)$ is induced by the natural projections $\prod_{i=1}^{l}N(W_D^{(i+1)},W_D^{(i)})\times N(W_{-1\,D},W_{0\,D})$ and $\prod_{i=1}^{l+1}\mathrm{GL}(W^{(i)})\times\mathrm{GL}(W_{-1})$ onto $N(W_{-1\,D},W_{0\,D})$ and $\mathrm{GL}(W_0)\times\mathrm{GL}(W_{-1})$, respectively.

From $\mathcal{V}_D^{(i+1)}$ and $\mathcal{V}_{-1|D}$, we have the classifying map $\Phi(\mathcal{V}_\cdot,\mathcal{V}_D^*):\mathfrak{B}\times D\longrightarrow\overline{Y}_D(W_\cdot,W^*)$. We also have the classifying map $\Phi(\mathcal{V}_{\cdot|D}):\mathcal{M}(m,\widehat{y})\times D\longrightarrow Y_D(W_\cdot)$. They give the following commutative diagram:

$$\begin{array}{ccc}\mathfrak{B}\times D & \longrightarrow & \overline{Y}_D(W_\cdot,W^*)\\ \big\downarrow & & \big\downarrow\\ \mathcal{M}(m,\widehat{y})\times D & \longrightarrow & Y_D(W_\cdot)\end{array}\tag{176}$$

It can be shown that $\overline{\Phi}(\mathcal{V}_\cdot,\mathcal{V}_D^*)^*L_{\overline{Y}_D(W_\cdot,W^*)}$ is expressed by $\overline{\mathfrak{g}}_D(\mathcal{V}_\cdot,\mathcal{V}_D^*)_{\leq 1}$. Moreover, the diagram (176) induces the following commutative diagram:

$$\begin{array}{ccccc}L_{\mathfrak{B}\times D/D} & \longleftarrow & \Phi(\mathcal{V}_\cdot,\mathcal{V}_D^*)^*L_{\overline{Y}_D(W_\cdot,W^*)} & \longleftarrow & \overline{\mathfrak{g}}_D(\mathcal{V},\mathcal{V}_D^*)\\ \big\uparrow & & \big\uparrow & & \big\uparrow\\ \pi_D^*L_{\mathcal{M}(m,\widehat{y})\times D/D} & \longleftarrow & \Phi(\mathcal{V}_{\cdot|D})^*L_{Y_D(W_\cdot)} & \longleftarrow & \mathfrak{g}(V_\cdot)\end{array}$$

We put $\mathfrak{g}_{\mathrm{rel}}(\mathcal{V}_\cdot,\mathcal{V}_D^*):=\mathrm{Cone}\bigl(\mathfrak{g}(\mathcal{V}_{\cdot|D})\longrightarrow\overline{\mathfrak{g}}_D(\mathcal{V}_\cdot,\mathcal{V}_D^*)\bigr)$. Then, we put as follows:

$$\mathrm{Ob}(\mathcal{V}_{\cdot|D}):=Rp_{D\,*}\bigl(\mathfrak{g}(\mathcal{V}_{\cdot|D})\otimes\omega_D\bigr),\quad \mathrm{Ob}_{\mathrm{rel}}(\mathfrak{B}):=Rp_{D\,*}\bigl(\mathfrak{g}_{\mathrm{rel}}(\mathcal{V}_\cdot,\mathcal{V}_D^*)\otimes\omega_D\bigr)$$

Then, we obtain the following commutative diagram:

$$\begin{array}{ccc}\pi^*L_{\mathcal{M}(m,\widehat{y})} & \longleftarrow & \mathrm{Ob}(\mathcal{V}_{\cdot|D})\\ \big\uparrow & & \big\uparrow\\ L_{\mathfrak{B}/\mathcal{M}(m,\widehat{y})}[-1] & \longleftarrow & \mathrm{Ob}_{\mathrm{rel}}(\mathfrak{B})[-1]\end{array}$$

Therefore, we obtain the following commutative diagram:

$$\begin{array}{ccc}\mathrm{Ob}_{\mathrm{rel}}(\mathfrak{B})[-1] & \longrightarrow & \pi^*\,\mathrm{Ob}(m,\widehat{y})\\ \big\downarrow & & \big\downarrow\\ L_{\mathfrak{B}/\mathcal{M}(m,\widehat{y})}[-1] & \longrightarrow & \pi^*L_{\mathcal{M}(m,\widehat{y})}\end{array}$$

We put $\mathrm{Ob}(\mathfrak{B}):=\mathrm{Cone}\bigl(\mathrm{Ob}_{\mathrm{rel}}(\mathfrak{B})\longrightarrow\pi^*\,\mathrm{Ob}(m,\widehat{y})\bigr)$. Then we obtain the morphism: $\mathrm{Ob}(\mathfrak{B})\longrightarrow L_{\mathfrak{B}}$.

Proposition 6.60 *The morphism* $\mathrm{Ob}_{\mathrm{rel}}(\mathfrak{B}) \longrightarrow L_{\mathfrak{B}/\mathcal{M}(m,\widehat{y})}$ *gives a relative obstruction theory of* $\mathfrak{B}$ *over* $\mathcal{M}(m,\widehat{y})$*. The complex* $\mathrm{Ob}_{\mathrm{rel}}(\mathfrak{B})$ *is quasi-isomorphic to the* 0*-th cohomology sheaf.*

Proof The first claim follows from Lemma 2.41. Let us show the second claim. Due to an argument in the proof of Proposition 6.1, we have only to check $H^i\big(i_z^*\,\mathrm{Ob}_{\mathrm{rel}}(\mathfrak{B})\big)=0$ for $i\neq 0$, where i_z denotes the inclusion of any point z into $\mathfrak{B}$. Let (E,V_D^*) denote the tuple corresponding to z. Then, $H^i\big(i_z^*\,\mathrm{Ob}_{\mathrm{rel}}(\mathfrak{B})\big)$ is the dual of the hyper-cohomology group $\mathbb{H}^{-i}(D,Q)$, where Q is the complex:

$$\bigoplus_{i=1}^{l}\mathcal{H}om(V_D^{(i+1)},V_D^{(i+1)})\longrightarrow\bigoplus_{i=1}^{l}\mathcal{H}om(V_D^{(i+1)},V_D^{(i)})$$

Here, the first term stands in the degree -1. We use the following lemma.

Lemma 6.61 *We have the vanishing* $H^1\Big(D,\mathcal{H}om\big(V_D^{(i+1)},V_D^{(i)}/V_D^{(i+1)}\big)\Big)=0$.

Proof By definition of $\mathfrak{B}$, we have the vanishing $H^1\big(D,V_D^{(1)}/V_D^{(i)}\big)=0$ for any $i=1,\ldots,l+1$. We have the following exact sequence:

$$H^0\big(D,V_D^{(1)}/V_D^{(i+1)}\big)\xrightarrow{\nu_1}H^0\big(D,V_D^{(1)}/V_D^{(i)}\big)\longrightarrow H^1\big(D,V_D^{(i)}/V_D^{(i+1)}\big)\longrightarrow H^1\big(D,V_D^{(1)}/V_D^{(i+1)}\big)=0$$

By definition of $\mathfrak{D}$, we have the surjectivity of ν_1. Then we obtain $H^1\big(D,V_D^{(i)}/V_D^{(i+1)}\big)=0$. From the exact sequence, $V_D^{(i+1)}\longrightarrow V_D^{(1)}\longrightarrow V_D^{(1)}/V_D^{(i+1)}$, we have the following exact sequence:

$$\mathrm{Ext}^1\big(V_D^{(1)},V_D^{(i)}/V_D^{(i+1)}\big)\longrightarrow\mathrm{Ext}^1\big(V_D^{(i+1)},V_D^{(i)}/V_D^{(i+1)}\big)\longrightarrow\mathrm{Ext}^2\big(V_D^{(1)}/V_D^{(i+1)},V_D^{(i)}/V_D^{(i+1)}\big)\tag{177}$$

Recall that $V_D^{(1)}$ is a direct sum of $\mathcal{O}_D$, and hence we have the vanishing of the first term in (177). Since the divisor D is smooth, we have the vanishing of the third term in (177). Therefore, we obtain the desired vanishing. ∎

From Lemma 6.61, we can easily obtain the vanishing of $\mathbb{H}^i\big(D,Q\big)$ unless $i=0$. Thus the proof of Proposition 6.60 is finished. ∎

As a result, $\mathrm{Ob}(\mathfrak{B})\longrightarrow L_{\mathfrak{B}}$ is an obstruction theory of $\mathfrak{B}$, and the morphism $\mathfrak{B}\longrightarrow\mathcal{M}(m,\widehat{y})$ is smooth.

6.6.3 Compatibility of the obstruction theories of $\mathfrak{B}$ and $\mathcal{M}(m,\widehat{\boldsymbol{y}})$

On $\mathfrak{B}\times D$, we have the filtration $\mathcal{V}^{(l+1)}\subset\mathcal{V}^{(l)}\subset\cdots\subset\mathcal{V}^{(1)}$. We put as follows:

$$\mathfrak{V}:=p_{D\,*}\mathcal{H}om\big(\mathcal{V}^{(l+1)},\pi_D^*\mathcal{E}^u(m)_{|D}\big)$$

We have the canonical section ψ, which is given by the composite $\mathcal{V}^{(l+1)}\subset\mathcal{V}^{(1)}\longrightarrow\pi_D^*\mathcal{E}^u(m)_{|D}$.

Lemma 6.62 $\mathfrak{V}$ *is a locally free sheaf on* $\mathfrak{B}$.

Proof Let $z=(E,V_D^*)$ be any point of $\mathfrak{B}$. We have only to check $\mathrm{Ext}^1\big(V_D^{(l+1)},E(m)_{|D}\big)=0$. We have the following exact sequence:

$$\mathrm{Ext}^1\big(V_D^{(1)},E(m)_{|D}\big)\longrightarrow\mathrm{Ext}^1\big(V_D^{(l+1)},E(m)_{|D}\big)\longrightarrow\mathrm{Ext}^2\big(V_D^{(1)}/V_D^{(l+1)},E(m)_{|D}\big)$$

Since $V_D^{(1)}$ is a direct sum of $\mathcal{O}_D$, the first term vanishes. Since D is a smooth curve, the last term vanishes. Therefore, we obtain the desired vanishing. ∎

It is easy to observe $\psi^{-1}(0)=\mathcal{M}(m,\widehat{\boldsymbol{y}})$. Therefore, we have the following Cartesian diagram:

$$\begin{tikzcd}\mathcal{M}(m,\widehat{\boldsymbol{y}}) \arrow[r, "i_1"] \arrow[d, "j"'] & \mathfrak{B} \arrow[d, "\psi"'] \\ \mathfrak{B} \arrow[r, "i"] & \mathfrak{V}\end{tikzcd}\tag{178}$$

Here i is the 0-section.

Let us compare the obstruction theories of $\mathfrak{B}$ and $\mathcal{M}(m,\widehat{\boldsymbol{y}})$. We take an isomorphism $\mathfrak{I}: W^{(l+1)} \simeq W_{-1}$. It induces the morphism $Y_D(W.,W^*) \longrightarrow \overline{Y}_D(W.,W^*)$. Then, we obtain the following commutative diagram:

$$\begin{array}{ccccc} \mathcal{M}(m,\boldsymbol{y}) & \longrightarrow & \mathfrak{B} & \longrightarrow & \mathcal{M}(m,y) \\ \downarrow & & \downarrow & & \downarrow \\ Y_D(W.,W^*) & \longrightarrow & \overline{Y}_D(W.,W^*) & \longrightarrow & Y_D(W.) \end{array}$$

It induces the following diagram on $\mathcal{M}(m,\boldsymbol{y})$:

$$\begin{array}{ccccc} L_{\mathcal{M}(m,\widehat{\boldsymbol{y}})} & \longleftarrow & L_{\mathfrak{B}} & \longleftarrow & L_{\mathcal{M}(m,\widehat{y})} \\ \uparrow & & \uparrow & & \uparrow \\ L_{Y_D(W.,W^*)} & \longleftarrow & L_{\overline{Y}_D(W.,W^*)} & \longleftarrow & L_{Y_D(W.)} \\ \uparrow & & \uparrow & & \uparrow \\ \mathfrak{g}(\mathcal{V}_{\cdot|D},\mathcal{V}_D^*) & \longleftarrow & \overline{\mathfrak{g}}(\mathcal{V}_{\cdot|D},\mathcal{V}_D^*) & \longleftarrow & \mathfrak{g}(\mathcal{V}_{\cdot|D}) \end{array}$$

Therefore, we obtain the following commutative diagram on $\mathcal{M}(m,\boldsymbol{y})$:

$$\begin{array}{ccc} i_1^* \operatorname{Ob}(\mathfrak{B}) & \xrightarrow{\beta} & \operatorname{Ob}(m,\widehat{\boldsymbol{y}}) \\ \downarrow & & \downarrow \\ i_1^* L_{\mathfrak{B}} & \longrightarrow & L_{\mathcal{M}(m,\widehat{\boldsymbol{y}})} \end{array} \tag{179}$$

Lemma 6.63 *The cone of β is isomorphic to $j^* L_{\mathfrak{B}/\mathfrak{Y}}$, and the morphism $\operatorname{Cone}(\beta) \longrightarrow L_{\mathcal{M}(m,\widehat{\boldsymbol{y}})/\mathfrak{B}}$ obtained from* (179) *is same as the morphism obtained from the diagram* (178)*. In particular, the obstruction theories of $\mathcal{M}(m,\boldsymbol{y})$ and $\mathfrak{B}$ are compatible over i.*

Proof We put $\mathfrak{g}_{\mathrm{rel}} := \mathcal{H}om\big(\mathcal{V}_{-1\,|\,D}[1], \mathcal{V}_{\cdot|D}\big)^{\vee}$. Then it is easy to see the following:

$$\operatorname{Cone}\big(\overline{\mathfrak{g}}(\mathcal{V}_{\cdot|D},\mathcal{V}^*) \longrightarrow \mathfrak{g}(\mathcal{V}_{\cdot|D},\mathcal{V}^*)\big) \simeq L_{Y_D(W.,W^*)/\overline{Y}_D(W.,W^*)} \simeq \mathfrak{g}_{\mathrm{rel}}.$$

We put $\operatorname{Ob}_{\mathrm{rel}} := Rp_{D\,*}\big(\mathfrak{g}_{\mathrm{rel}} \otimes \omega_D\big)$. We have the induced morphism $a : \operatorname{Ob}_{\mathrm{rel}} \longrightarrow L_{\mathcal{M}(m,\widehat{\boldsymbol{y}})/\mathfrak{B}}$. We have $\operatorname{Ob}_{\mathrm{rel}} \simeq \operatorname{Cone}\big(i_1^* \operatorname{Ob}(\mathfrak{B}) \longrightarrow \operatorname{Ob}(m,\widehat{\boldsymbol{y}})\big)$, and hence we have the morphism $b : \operatorname{Ob}_{\mathrm{rel}} \longrightarrow L_{\mathcal{M}(m,\widehat{\boldsymbol{y}})/\mathfrak{B}}$ obtained from the diagram (179). It is easy to observe $a = b$.

We have the natural $\mathrm{GL}(W_{-1}) \times \mathrm{GL}(W^{(1)}) \times \mathrm{GL}(W^{(l+1)})$-action on $N(W_D^{(l+1)}, W_D^{(1)}) \times_D N(W_{-1D}, W_D^{(1)})$. The quotient stack is denoted by $\mathfrak{Y}_2$. The isomorphisms $W_{-1} \simeq W^{(l+1)}$ and $W_0 \simeq W^{(1)}$ induce $Y_D(W.) \longrightarrow \mathfrak{Y}_2$. Then, we have the following diagram:

$$\begin{array}{ccccc} \mathcal{M}(m,\boldsymbol{y}) \times D & \longrightarrow & Y_D(W.,W^*) & \xrightarrow{\gamma_1} & Y_D(W.) \\ \downarrow & & \downarrow & & \downarrow \\ \mathfrak{B} \times D & \longrightarrow & \overline{Y}_D(W.,W^*) & \longrightarrow & \mathfrak{Y}_2 \end{array}$$

The induced morphism $\Phi(\mathcal{V}_{\cdot|D},\mathcal{V}_D^*)^* \gamma_1^* L_{Y_D(W.)/\mathfrak{Y}_2} \longrightarrow \Phi(\mathcal{V}_{\cdot|D},\mathcal{V}_D^*)^* L_{Y_D(W.,W^*)/\overline{Y}_D(W.,W^*)}$ is isomorphic.

We have the natural $\mathrm{GL}(W_{-1}) \times \mathrm{GL}(W_0) \times \mathrm{GL}(W^{(l+1)})$-action on $N(W_{-1},W_0) \times N(W^{(l+1)},W_{-1})$. The quotient stack is denoted by $\mathfrak{Y}_1$. The isomorphism $\mathfrak{I}$ induces the following map:

$$N(W_{-1},W_0) \longrightarrow N(W_{-1},W_0) \times N(W^{(l+1)},W_{-1}), \quad f \longmapsto (f,\mathfrak{I}) \tag{180}$$

We also have the homomorphism $\mathrm{GL}(W_{-1}) \longrightarrow \mathrm{GL}(W_0) \times \mathrm{GL}(W^{(l+1)})$ induced by $\mathfrak{I}$. The morphism (180) is equivariant with respect to the actions. Therefore, we obtain the morphism $Y_D(W.) \longrightarrow \mathfrak{Y}_1$. It is easy to observe that the morphism is an open immersion. Hence, we have the following diagram:

$$\begin{array}{ccccc} \mathcal{M}(m,\boldsymbol{y}) \times D & \longrightarrow & Y_D(W.) & \xrightarrow{\text{open}} & \mathfrak{Y}_1 \\ \downarrow & & \downarrow & & \downarrow \\ \mathfrak{B} \times D & \longrightarrow & \mathfrak{Y}_2 & \xrightarrow{=} & \mathfrak{Y}_2 \end{array}$$

We put $\mathcal{W}_i := N(W^{(l+1)}, W_i)$ $(i = -1, 0)$. We have the natural right $\mathrm{GL}(\mathcal{W}_{-1}) \times \mathrm{GL}(\mathcal{W}_0)$-action on $N(\mathcal{W}_{-1}, \mathcal{W}_0) \times N(k, \mathcal{W}_{-1})$ and $N(\mathcal{W}_{-1}, \mathcal{W}_0) \times N(k, \mathcal{W}_0)$. The quotient stacks are denoted by $\overline{Z}_1$ and $\overline{Z}_2$ respectively. We have the naturally induced map $\mathfrak{Y}_i \longrightarrow \overline{Z}_i$, and we obtain the following commutative diagram:

$$\begin{array}{ccccc} \mathcal{M}(m,\boldsymbol{y}) \times D & \xrightarrow{a_1} & \mathfrak{Y}_1 & \xrightarrow{a_2} & \overline{Z}_1 \\ \downarrow & & \downarrow & & \downarrow \\ \mathfrak{B} & \longrightarrow & \mathfrak{Y}_2 & \longrightarrow & \overline{Z}_2 \end{array} \tag{181}$$

Thus, we obtain the following:

$$a_1^* a_2^* L_{\overline{Z}_1/\overline{Z}_2} \xrightarrow{\simeq} a_1^* L_{\mathfrak{Y}_1/\mathfrak{Y}_2} \xrightarrow{\simeq} L_{Y_D(W.,W^*)/\overline{Y}_D(W.,W^*)} \longrightarrow L_{\mathcal{M}(m,\boldsymbol{y})/\mathfrak{B}}$$

We would like to use the result in the latter part of the subsubsection 2.3.3. We have the resolution of $\mathcal{H}om\big(\mathcal{V}^{(l+1)}, \mathcal{E}^u(m)\big)$ given by $\mathcal{H}om\big(\mathcal{V}^{(l+1)}, \mathcal{V}_{-1}\big) \longrightarrow \mathcal{H}om\big(\mathcal{V}^{(l+1)}, \mathcal{V}_0\big)$ on $\mathfrak{B} \times D$. We put as follows:

$$V_0 = \mathcal{H}om\big(\mathcal{V}^{(l+1)}, \mathcal{V}_{-1}\big), \quad V_1 = \mathcal{H}om\big(\mathcal{V}^{(l+1)}, \mathcal{V}_0\big)$$

We have the naturally defined map $\phi : \mathcal{O} \longrightarrow V_1$ on $\mathfrak{B} \times D$, and the lift $\widetilde{\phi} : \mathcal{O} \longrightarrow V_0$ on $\mathcal{M}(m,\boldsymbol{y}) \times D$. The section φ of $\mathcal{H}om\big(\mathcal{V}^{(l+1)}, \mathcal{E}^u(m)\big)$ is naturally induced by ϕ. We put $Z_1 = N(\mathcal{O}, V_0)$ and $Z_2 = N(\mathcal{O}, V_1)$. Then, we have the following commutative diagram:

$$\begin{array}{ccccc} \mathcal{M}(m,\boldsymbol{y}) \times D & \longrightarrow & Z_1 & \longrightarrow & \overline{Z}_1 \\ \downarrow & & \downarrow & & \downarrow \\ \mathfrak{B} \times D & \longrightarrow & Z_2 & \longrightarrow & \overline{Z}_2 \end{array} \tag{182}$$

We have the coincidence of the composite of the horizontal arrows in the diagrams (181) and (182). Therefore, we obtain $\mathrm{Ob}_{\mathrm{rel}} = \mathrm{Ob}^H(V., \varphi)$, and the induced morphism $\mathrm{Ob}_{\mathrm{rel}} \longrightarrow L_{\mathcal{M}(m,\boldsymbol{y})/\mathfrak{B}}$ is same as the morphism obtained from the diagram (178), due to Proposition 2.26. Thus the proof of Lemma 6.63 is finished. ∎

6.6.4 Smooth ambient stack

Let V_m be an $H(m)$-dimensional vector space, where H denotes the Hilbert polynomial associated to y. For any k-scheme T, let $F_1(T)$ denote the set of the sequences of quotients $p_T^* V_{m,D} = \mathcal{C}_{l+1} \to \mathcal{C}_l \to \mathcal{C}_{l-1} \to \cdots \to \mathcal{C}_2 \to \mathcal{C}_1$ satisfying the following conditions:

- $\mathcal{C}_i$ and $\mathcal{F}$ are flat over T.
- For any point $u \in T$, the induced morphisms $H^0\big(D, \mathcal{C}_{l+1\,|\,\{u\}\times D}\big) \longrightarrow H^0\big(D, \mathcal{C}_{i\,|\,\{u\}\times D}\big)$ are surjective $(i = 1, \dots, l)$. The induced morphism $H^0\big(D, \mathcal{C}_{1\,|\,u}\big) \longrightarrow H^0\big(D, \mathcal{F}_{|u}\big)$ is also surjective.
- We have $H^1\big(D, \mathcal{F}_{1\,|\,u}\big) = 0$ and $H^1\big(D, \mathcal{C}_{i\,|\,\{u\}\times D}\big) = 0$ for any $u \in T$ and for any $i = 1, \dots, l$.
- The type of $\mathcal{C}_i$ is same as $\sum_{j \le i} y_j(m)$. The type of $\mathcal{F}$ is same as $\sum_j y_j(m)$.

Let $F_2(T)$ denote the set of the quotients $p_T^* V_{m,D} \longrightarrow \mathcal{F}$ satisfying the following conditions:

- $\mathcal{F}$ is flat over T.
- For any point $u \in T$, the induced morphism $V_m \longrightarrow H^0(D, \mathcal{F}_{|u})$ is surjective.
- $H^1\big(D, \mathcal{F}_{|u}\big) = 0$ for any $u \in T$.
- The type of $\mathcal{F}$ is $\sum_j y_j(m)$.

Then, we obtain the functors F_i $(i = 1, 2)$ of the category of k-schemes to the category of sets. The functors F_i are representable by the k-schemes, which we denote by $\mathfrak{B}_i$.

We have the natural right $\mathrm{GL}(V_m)$-action on $Q^{\circ}(m, \widehat{y}) \times \mathfrak{B}_1$. The quotient stack is isomorphic to $\mathfrak{B}$. By considering the restriction to D, we obtain the natural morphism $Q^{\circ}(m, \widehat{y}) \longrightarrow \mathfrak{B}_2$, which is $\mathrm{GL}(V_m)$-equivariant.

Let $\widehat{Z}_m$ be as in (38). Then, we have the natural right $\mathrm{GL}(V_m)$-action on $\widehat{Z}_m \times \mathfrak{B}_1 \times \mathfrak{B}_2$. The quotient stack is denoted by $\widetilde{\mathfrak{B}}$. We have the $\mathrm{GL}(V_m)$-equivariant morphism $Q^{\circ}(m, \widehat{y}) \times \mathfrak{B}_1 \longrightarrow \widehat{Z}_m \times \mathfrak{B}_1 \times \mathfrak{B}_2$, which is immersion. Therefore, we obtain the immersion $\mathfrak{B} \longrightarrow \widetilde{\mathfrak{B}}$.

We have the universal filtration on $\widehat{Z}_m \times \mathfrak{B}_1 \times \mathfrak{B}_2$:

$$V_D^{(l+1)} \subset V_D^{(l)} \subset \cdots \subset V_D^{(2)} \subset V_D^{(1)} = V_m \otimes \mathcal{O}_{\widehat{Z}_m \times \mathfrak{B}_1 \times \mathfrak{B}_2}$$

We also have the universal subsheaf on $\widehat{Z} \times \mathfrak{B}_1 \times \mathfrak{B}_2$:

$$V_{-1\,D} \subset V_{0\,D} = V_m \otimes \mathcal{O}_{\widehat{Z} \times \mathfrak{B}_1 \times \mathfrak{B}_2}$$

The $\mathrm{GL}(V_m)$-action on $\widehat{Z}_m \times \mathfrak{B}_1 \times \mathfrak{B}_2$ is naturally lifted to the action on them. The descents are denoted by $\mathcal{V}_D^{(i)}$ and $\mathcal{V}_{-1\,D}$. We put as follows:

$$\widetilde{\mathfrak{V}} := p_{D\,*}\Big(\mathcal{H}om\big(\mathcal{V}_D^{(l+1)},\ \mathcal{V}_D^{(1)}/\mathcal{V}_{-1\,D}\big)\Big)$$

By the same argument as the proof of Lemma 6.62, it can be shown that $\widetilde{\mathfrak{V}}$ is locally free. We also have $\widetilde{\mathfrak{V}}_{|\mathfrak{B}} \simeq \mathfrak{V}$. Therefore, we obtain the following diagram:

$$\begin{array}{ccccc}
\mathcal{M}(m, \widehat{\boldsymbol{y}}) & \longrightarrow & \mathfrak{B} & \longrightarrow & \widetilde{\mathfrak{B}} \\
{\scriptstyle i_1}\big\downarrow & & {\scriptstyle i}\big\downarrow & & {\scriptstyle i_2}\big\downarrow \\
\mathfrak{B} & \xrightarrow{\psi} & \mathfrak{V} & \longrightarrow & \widetilde{\mathfrak{V}}
\end{array}$$

Here i and i_2 denote the 0-section.

6.6.5 Proof of Proposition 6.56

Let us finish the proof of Proposition 6.56. We take a sufficiently large integer m such that the condition O_m holds for any $(E, F_*, \rho) \in \mathcal{M}^s(\widehat{\boldsymbol{y}}, \alpha_*)$. Then we have the open immersion $\mathcal{M}^s(\widehat{\boldsymbol{y}}, \alpha_*) \longrightarrow \mathcal{M}(m, \widehat{\boldsymbol{y}})$ and $\mathcal{M}^s(\widehat{y}) \longrightarrow \mathcal{M}(m, y)$. We take the stack $\mathfrak{B}$ as in the subsubsection 6.6.2. We put $\mathfrak{B}(\widehat{\boldsymbol{y}}, \alpha_*) := \mathfrak{B} \times_{\mathcal{M}(m,\widehat{y})} \mathcal{M}^s(\widehat{y})$. Due to Proposition 6.60, it is smooth over $\mathcal{M}^s(\widehat{y})$. The restrictions of $\mathfrak{V}$ and ψ to $\mathfrak{B}(\widehat{\boldsymbol{y}}, \alpha_*)$ are denoted by the same notation. It is clear $\psi^{-1}(0) = \mathcal{M}^s(\widehat{\boldsymbol{y}}, \alpha_*)$.

We have the immersion $\mathfrak{B}(\widehat{\boldsymbol{y}}, \alpha_*) \longrightarrow \widetilde{\mathfrak{B}}$. Since $\mathfrak{B}(\widehat{\boldsymbol{y}}, \alpha_*)$ is Deligne-Mumford, there exists an open neighbourhood $\widetilde{\mathfrak{B}}(\widehat{\boldsymbol{y}}, \alpha_*)$ of $\mathfrak{B}(\widehat{\boldsymbol{y}}, \alpha_*)$ in $\widetilde{\mathfrak{B}}$, which is Deligne-Mumford and smooth. The restriction of $\widetilde{\mathfrak{V}}$ to $\widetilde{\mathfrak{B}}(\widehat{\boldsymbol{y}}, \alpha_*)$ are denoted by the same notation. Then, we obtain the following diagram:

$$\begin{array}{ccccc}
\mathcal{M}^s(\widehat{\boldsymbol{y}}, \alpha_*) & \longrightarrow & \mathfrak{B}(\widehat{\boldsymbol{y}}, \alpha_*) & \longrightarrow & \widetilde{\mathfrak{B}}(\widehat{\boldsymbol{y}}, \alpha_*) \\
\big\downarrow & & \big\downarrow{\scriptstyle i} & & \big\downarrow{\scriptstyle i_2} \\
\mathfrak{B}(\widehat{\boldsymbol{y}}, \alpha_*) & \longrightarrow & \mathfrak{V} & \longrightarrow & \widetilde{\mathfrak{V}}
\end{array}$$

Due to Lemma 6.63, the obstruction theories of $\mathcal{M}^s(\widehat{\boldsymbol{y}}, \alpha_*)$ and $\mathfrak{B}(\widehat{\boldsymbol{y}}, \alpha_*)$ are compatible. Therefore, we obtain the relation $i^!\big([\mathfrak{B}(\widehat{\boldsymbol{y}}, \alpha_*)]\big) = \big[\mathcal{M}^s(\widehat{\boldsymbol{y}}, \alpha_*)\big]$ due to Proposition 2.32. We also have the relation $\mathfrak{G}^![\mathcal{M}^s(\widehat{y})] = [\mathfrak{B}(\widehat{\boldsymbol{y}}, \alpha_*)]$. We also remark that $i^! = \psi^!$. Thus we obtain the relation (174). ∎

7 Invariants

For simplicity, we assume that the ground field k is the complex number field $\mathbb{C}$ in this section. Let $H^*(A)$ and $H_*(A)$ denote the singular cohomology and homology groups of a topological space A with $\mathbb{Q}$-coefficient. Let X be a smooth projective surface over $\mathbb{C}$, and let D be a smooth divisor of X. We denote the Picard variety of X by Pic.

7.1 Preliminary

7.1.1 The ring $\mathcal{R}$

Let $\mathrm{Map}_f\big(\mathbb{Z}^2_{\geq 0}, H^*(X)\big)$ denote the set of the maps $\varphi : \mathbb{Z}^2_{\geq 0} \longrightarrow H^*(X)$ such that $\big\{(n_1, n_2)\,\big|\,\varphi(n_1, n_2) \neq 0\big\}$ is finite. We use the notation $\mathrm{Map}_f\big(\mathbb{Z}^3_{\geq 0}, H^*(D)\big)$ in a similar meaning. The sets $\mathrm{Map}_f\big(\mathbb{Z}^2_{\geq 0}, H^*(X)\big)$ and $\mathrm{Map}_f\big(\mathbb{Z}^3_{\geq 0}, H^*(D)\big)$ are naturally vector spaces over $\mathbb{Q}$. We use the notation $\mathrm{Sym}(V)$ to denote the symmetric product of a vector space V. Then, we put as follows:

$$\mathcal{R} := H^*(\mathrm{Pic}) \otimes \mathcal{R}', \qquad \mathcal{R}' := \mathrm{Sym}\Big(\mathrm{Map}_f\big(\mathbb{Z}^2_{\geq 0}, H^*(X)\big)\Big) \otimes \mathrm{Sym}\Big(\mathrm{Map}_f\big(\mathbb{Z}^3_{\geq 0}, H^*(D)\big)\Big)$$

An element of $\mathcal{R}$ is described as a sum of the elements of the following form:

$$P = \mathfrak{c} \cdot \prod_{i=1}^{m_1}(a_i, \boldsymbol{v}_i) \cdot \prod_{j=1}^{m_2}(b_j, \boldsymbol{u}_j) \tag{183}$$

- $\mathfrak{c}$ is an element of $H^*(\mathrm{Pic})$.
- $a_i \in H^*(X)$ and $\boldsymbol{v}_i = \big(v_i(1), v_i(2)\big) \in \mathbb{Z}^2_{\geq 0}$. We identify $(a_i, \boldsymbol{v}_i)$ with the map $\varphi_i : \mathbb{Z}^2_{\geq 0} \longrightarrow H^*(X)$:
$$\varphi_i(\boldsymbol{v}) = \begin{cases} a_i & (\boldsymbol{v} = \boldsymbol{v}_i) \\ 0 & (\boldsymbol{v} \neq \boldsymbol{v}_i) \end{cases}$$
- $b_j \in H^*(D)$ and $\boldsymbol{u}_j = \big(u_j(1), u_j(2), u_j(3)\big) \in \mathbb{Z}^3_{\geq 0}$. We identify $(b_j, \boldsymbol{u}_j)$ with the map $\psi_j : \mathbb{Z}^3_{\geq 0} \longrightarrow H^*(D)$:
$$\psi_j(\boldsymbol{u}) = \begin{cases} b_j & (\boldsymbol{u} = \boldsymbol{u}_j) \\ 0 & (\boldsymbol{u} \neq \boldsymbol{u}_j) \end{cases}$$

We put $d_1(P) := \sum_{i=1}^{m_1} v_i(1) \cdot v_i(2) + \sum_{j=1}^{m_2} u_j(1) \cdot u_j(2) - 2m_1 - m_2$. When a_i, b_j and $\mathfrak{c}$ are homogeneous, we put $d_2(P) := \sum_i \deg(a_i)/2 + \sum_j \deg(b_j)/2 + \deg(\mathfrak{c})/2$ and $d(P) := d_1(P) + d_2(P)$.

7.1.2 The ring $\mathcal{R}_l$

More generally, we put as follows for any $l \geq 1$:

$$\mathcal{R}_l := H^*(\mathrm{Pic}) \otimes \mathcal{R}'_l, \qquad \mathcal{R}'_l := \mathrm{Sym}\Big(\mathrm{Map}_f\big(\mathbb{Z}^{2l}_{\geq 0},\, H^*(X)\big)\Big) \otimes \mathrm{Sym}\Big(\mathrm{Map}_f\big(\mathbb{Z}^{3l}_{\geq 0},\, H^*(D)\big)\Big)$$

An element of $\mathcal{R}_l$ is described as a sum of the elements of the following form

$$P = \mathfrak{c} \cdot \prod_{i=1}^{m_1}(a_i, \boldsymbol{V}_i) \cdot \prod_{j=1}^{m_2}(b_j, \boldsymbol{U}_j) \tag{184}$$

- $\mathfrak{c}$ is an element of $H^*(\mathrm{Pic})$.
- $a_i \in H^*(X)$ and $\boldsymbol{V}_i = \big(v_i^{(h)}(1), v_i^{(h)}(2)\,\big|\,h = 1, \ldots, l\big) \in \mathbb{Z}^{2l}_{\geq 0}$. We identify $(a_i, \boldsymbol{V}_i)$ with the map $\varphi_i : \mathbb{Z}^{2l}_{\geq 0} \longrightarrow H^*(X)$:
$$\varphi_i(\boldsymbol{V}) = \begin{cases} a_i & (\boldsymbol{V} = \boldsymbol{V}_i) \\ 0 & (\boldsymbol{V} \neq \boldsymbol{V}_i) \end{cases}$$

- $b_j \in H^*(D)$ and $\boldsymbol{U}_j = \big(u_j^{(h)}(1), u_j^{(h)}(2), u_j^{(h)}(3)\big) \in \mathbb{Z}^{3l}_{\geq 0}$. We identify $(b_j, \boldsymbol{U}_j)$ with the map $\psi_j : \mathbb{Z}^{3l}_{\geq 0} \longrightarrow H^*(D)$:
$$\psi_j(\boldsymbol{U}) = \begin{cases} b_j & (\boldsymbol{U} = \boldsymbol{U}_j) \\ 0 & (\boldsymbol{U} \neq \boldsymbol{U}_j) \end{cases}$$

We put $d_1(P) := \sum_h \sum_{i=1}^{m_1} v_i^{(h)}(1) \cdot v_i^{(h)}(2) + \sum_h \sum_{j=1}^{m_2} u_j^{(h)}(1) \cdot u_j^{(h)}(2) - 2m_1 - m_2$. If a_i, b_j and $\mathfrak{c}$ are homogeneous, we put $d_2(P) := \sum_i \deg(a_i)/2 + \sum_j \deg(b_j)/2 + \deg(\mathfrak{c})/2$ and $d(P) := d_1(P) + d_2(P)$.

Let $(a, \boldsymbol{V}) \in H^*(X) \times \mathbb{Z}^{2l}_{\geq 0}$ be as above. We regard $\boldsymbol{V}$ as a tuple $(\boldsymbol{v}^{(1)}, \ldots, \boldsymbol{v}^{(l)}) \in (\mathbb{Z}^2_{\geq 0})^l$. Let Δ^l_X denote the diagonal map $X \longrightarrow X^l$, and $\Delta^l_{X\,*}$ denotes the Gysin map $H^*(X) \longrightarrow H^*(X^l) = H^*(X)^{\otimes\, l}$. We have the expression as follows:
$$\Delta^l_{X\,*}(a) = \sum_h \prod_{i=1}^{l} \alpha_{i,h}$$
Then, we put as follows:
$$\mathfrak{q}'_l\big((a, \boldsymbol{V})\big) = \sum_h \big(\alpha_{1,h}, \boldsymbol{v}^{(1)}\big) \otimes \cdots \otimes \big(\alpha_{l,h}, \boldsymbol{v}^{(l)}\big) \tag{185}$$
Let $(b, \boldsymbol{U}) \in H^*(D) \times \mathbb{Z}^{3l}_{\geq 0}$ be as above. We regard $\boldsymbol{U}$ as a tuple $(\boldsymbol{u}^{(1)}, \ldots, \boldsymbol{u}^{(l)}) \in (\mathbb{Z}^3_{\geq 0})^l$. Let Δ^l_D denote the diagonal map $D \longrightarrow D^l$. We have the expression as follows:
$$\Delta^l_{D\,*}(b) = \sum_h \prod_{i=1}^{l} \beta_{i,h} \tag{186}$$
Then, we put as follows:
$$\mathfrak{q}'_l\big((b, \boldsymbol{U})\big) = \sum_h \big(\beta_{1,h}, \boldsymbol{u}^{(1)}\big) \otimes \cdots \otimes \big(\beta_{l,h}, \boldsymbol{u}^{(l)}\big) \tag{187}$$
They induce the algebra homomorphism $\mathfrak{q}'_l : \mathcal{R}'_l \longrightarrow \mathcal{R}'^{\otimes\, l}_1$. We also have the morphism $\mathfrak{q}_l : H^*(\mathrm{Pic}) \longrightarrow H^*(\mathrm{Pic})^{\otimes\, l}$ induced by the multiplication of Pic. Then, we obtain the algebra homomorphism:
$$\mathfrak{q}_l : \mathcal{R}_l \longrightarrow \mathcal{R}^{\otimes\, l} \tag{188}$$

We have the naturally defined algebra homomorphism $\mathcal{R}'^{\otimes\, l} \longrightarrow \mathcal{R}'$. Hence, $\mathfrak{q}'_l$ induces the algebra homomorphism $\mathfrak{r}_l : \mathcal{R}_l \longrightarrow \mathcal{R}$.

7.1.3 Homomorphisms

Let $\mathcal{Y}$ be an algebraic stack over Pic. When we are given a tuple of parabolic sheaves $\boldsymbol{E}_* = (E_{1\,*}, \ldots, E_{l\,*})$ over $\mathcal{Y} \times (X, D)$, we put $\mathcal{R}(\boldsymbol{E}_*) := \mathcal{R}_l$. In that case, $(a, \boldsymbol{V})$ and $(b, \boldsymbol{U})$ are symbolically denoted as follows:
$$\Big(\prod_{h=1}^{l} \mathrm{ch}^{v^{(h)}(2)}_{v^{(h)}(1)}(E_h)\Big)\Big/a, \quad \Big(\prod_{h=1}^{l} \mathrm{ch}^{u^{(h)}(2)}_{u^{(h)}(1)}\big(\mathrm{Gr}_{u^{(h)}(3)}(E_h)\big)\Big)\Big/b \tag{189}$$
In particular, we use the notation $\mathcal{R}(E_*)$ in the case $l = 1$. We will often omit to denote the parabolic structure if there are no risk of confusion, i.e., $\mathcal{R}(\boldsymbol{E})$ and $\mathcal{R}(E)$ are used instead of $\mathcal{R}(\boldsymbol{E}_*)$ and $\mathcal{R}(E_*)$.

When we are given a direct sum $E_* = E_{1\,*} \oplus E_{2\,*}$, we have the algebra homomorphism $\varphi^{E_{1*}, E_{2*}}_{E_*} : \mathcal{R}(E_*) \longrightarrow \mathcal{R}(E_{1\,*}, E_{2\,*})$ induced by the following correspondence:
$$\varphi^{E_{1*}, E_{2*}}_{E_*}\big(\mathrm{ch}^j_i(E)/a\big) = \sum_{h=0}^{j} \frac{j!}{h!(j-h)!} \big(\mathrm{ch}^h_i(E_1) \cdot \mathrm{ch}^{j-h}_i(E_2)\big)\big/a,$$
$$\varphi^{E_{1*}, E_{2*}}_{E_*}\Big(\mathrm{ch}^j_i\big(\mathrm{Gr}_k(E)\big)/b\Big) = \sum_{h=0}^{j} \frac{j!}{h!(j-h)!} \Big(\mathrm{ch}^h_i\big(\mathrm{Gr}_k(E_1)\big) \cdot \mathrm{ch}^{j-h}_i\big(\mathrm{Gr}_k(E_2)\big)\Big)\Big/b$$

As the composite of $\varphi_{E_*}^{E_{1*},E_{2*}}$ and $\mathfrak{q}_2$, we obtain the algebra morphism $\mathcal{R}(E_*) \longrightarrow \mathcal{R}(E_{1*}) \otimes \mathcal{R}(E_{2\,*})$. For an element $P(E) \in \mathcal{R}(E_*)$, we denote the image by $P(E_1 \oplus E_2) \in \mathcal{R}(E_{1\,*}) \otimes \mathcal{R}(E_{2\,*})$.

The algebra homomorphism $\mathfrak{r}_l$ gives $\mathcal{R}(\overbrace{E_*, E_*, \ldots, E_*}^{l}) \longrightarrow \mathcal{R}(E_*)$, which is also denoted by $\mathfrak{r}_l$. Let t be a formal variable. We have the element $\mathrm{ch}_i^j(E \otimes e^t)/a \in \mathcal{R}(E)[t]$ given as follows:

$$\mathrm{ch}_i^j(E \otimes e^t)/a := \sum_{\sum_{k=0}^i j_k = j} \frac{j!}{\prod_{k=0}^i j_k!} \mathfrak{r}_{i+1}\left(\Big(\prod_{k=0}^i \mathrm{ch}_k^{j_k}(E)\Big)\Big/a\right) \cdot \frac{t^{ji-\sum j_k \cdot k}}{\prod_{k=0}^i \big((i-k)!\big)^{j_k}} \tag{190}$$

Similarly, we have the element $\mathrm{ch}_i^j(\mathrm{Gr}_h(E) \otimes e^t)/b \in \mathcal{R}(E)[t]$ given as follows:

$$\mathrm{ch}_i^j\big(\mathrm{Gr}_h(E) \otimes e^t\big)/b := \sum_{\sum_{k=0}^i j_k = j} \frac{j!}{\prod_{k=0}^i j_k!} \mathfrak{r}_{i+1}\left(\Big(\prod_{k=0}^i \mathrm{ch}_k^{j_k}\big(\mathrm{Gr}_h(E)\big)\Big)\Big/b\right) \cdot \frac{t^{ji-\sum j_k \cdot k}}{\prod_{k=0}^i \big((i-k)!\big)^{j_k}}$$

By the correspondences $\mathrm{ch}_i^j(E)/a \longmapsto \mathrm{ch}_i^j(E \cdot e^t)/a$ and $\mathrm{ch}_i^j(\mathrm{Gr}_h(E))/b \longmapsto \mathrm{ch}_i^j(\mathrm{Gr}_h(E) \cdot e^t)/b$, we obtain the algebra isomorphism $\mathcal{R}(E)[t] \longrightarrow \mathcal{R}(E)[t]$. The image of $P(E) \in \mathcal{R}(E)$ is denoted by $P(E \cdot e^t)$.

Remark 7.1 *The formula* (190) *is just a formal development of* $\big(\sum_{h=0}^i \mathrm{ch}_h(E) \cdot (i-h)!^{-1} t^{i-h}\big)^j$. ∎

7.1.4 Twist by line bundle

Let $\mathcal{L}$ be a line bundle on $\mathcal{Y}$. We put $\omega := c_1(\mathcal{L})$. Formally, we often use the notation e^ω to denote $\mathcal{L}$, if there are no risk of confusion. Let E_* be a parabolic sheaf over $\mathcal{Y} \times (X, D)$. Then, we have the natural isomorphism $\mathcal{R}(E_*, \mathcal{Y}) \simeq \mathcal{R}\big(E_* \otimes e^\omega, \mathcal{Y}\big)$ (see Notation 7.3) given by the following correspondence:

$$\mathrm{ch}_i^j(E \otimes e^\omega)/a := \sum_{\sum_{k=0}^i j_k = j} \frac{j!}{\prod_{k=0}^i j_k!} \mathfrak{r}_{i+1}\left(\Big(\prod_{k=0}^i \mathrm{ch}_k^{j_k}(E)\Big)\Big/a\right) \cdot \frac{\omega^{ji-\sum j_k \cdot k}}{\prod_{k=0}^i \big((i-k)!\big)^{j_k}} \tag{191}$$

$$\mathrm{ch}_i^j\big(\mathrm{Gr}_h(E) \otimes e^\omega\big)/b := \sum_{\sum_{k=0}^i j_k = j} \frac{j!}{\prod_{k=0}^i j_k!} \mathfrak{r}_{i+1}\left(\Big(\prod_{k=0}^i \mathrm{ch}_k^{j_k}\big(\mathrm{Gr}_h(E)\big)\Big)\Big/b\right) \cdot \frac{\omega^{ji-\sum j_k \cdot k}}{\prod_{k=0}^i \big((i-k)!\big)^{j_k}} \tag{192}$$

Thus we can naturally regard $P(E \otimes e^\omega) \in \mathcal{R}(E \otimes e^\omega, \mathcal{Y})$ as an element of $\mathcal{R}(E, \mathcal{Y})$.

Remark 7.2 *We will often use "$\cdot$" instead of "$\otimes$" to save the space.* ∎

Let us consider the case $\mathcal{Y} = \mathcal{Y}_1 \times \mathcal{Y}_2$. Assume that L comes from the line bundle on $\mathcal{Y}_1$, and that E comes from the parabolic sheaf on $\mathcal{Y}_2 \times (X, D)$. The formulas (191) and (192) determines the element $P(E \cdot e^\omega) \in A^*(\mathcal{Y}_1) \otimes \mathcal{R}(E)$.

7.1.5 Evaluation

Let $\mathcal{Y}$ be a proper Deligne-Mumford stack over Pic. Assume that we are given a parabolic sheaf E_* over $\mathcal{Y} \times (X, D)$ with a parabolic structure. Let $A_*(\mathcal{Y})$ denote the rational Chow group. Let us take an element $P \in \mathcal{R}(E_*)$ of the following form:

$$P = \mathfrak{c} \cdot \prod_{i=1}^{m_1} \big(\mathrm{ch}_{v_i(1)}^{v_i(2)}(E)/a_i\big) \cdot \prod_{j=1}^{m_2} \Big(\mathrm{ch}_{u_j(1)}^{u_j(2)}\big(\mathrm{Gr}_{u_j(3)}(E)\big)/b_j\Big)$$

We assume that $\mathfrak{c} \in H^*(\mathrm{Pic})$, $a_i \in H^*(X)$ and $b_j \in H^*(D)$ are homogeneous for simplicity. We would like to construct the linear morphism of $A_*(\mathcal{Y})$ to $\mathbb{Q}$. Let $\pi_{X,i}$ denote the projection of $\mathcal{Y} \times X^{m_1} \times D^{m_2}$ onto the product of $\mathcal{Y}$ and the i-th X. Let $\pi_{D,j}$ denote the projection of $\mathcal{Y} \times X^{m_1} \times D^{m_2}$ onto the product of $\mathcal{Y}$ and j-th

D. Let $\mathfrak{p}$ denote the natural morphism $\mathcal{Y}\times X^{m_1}\times D^{m_2}$ to $\mathrm{Pic}\times X^{m_1}\times D^{m_2}$. Let $\mathcal{Z}$ be a d-dimensional algebraic cycle on $\mathcal{Y}$. Then, we obtain the following element of the Chow group $A_*\bigl(\mathrm{Pic}\times X^{m_1}\times D^{m_2}\bigr)$ of $\mathrm{Pic}\times X^{m_1}\times D^{m_2}$ with rational coefficient:

$$\Lambda_P(E_*,\mathcal{Z}):=\mathfrak{p}_*\left(\prod_{i=1}^{m_1}\mathrm{ch}_{v_i(1)}\bigl(\pi_{X,i}^*E\bigr)^{v_i(2)}\cdot\prod_{j=1}^{m_2}\mathrm{ch}_{u_j(1)}\bigl(\pi_{D,j}^*\,\mathrm{Gr}_{u_j(3)}(E)\bigr)^{u_j(2)}\cap[\mathcal{Z}\times X^{m_1}\times D^{m_2}]\right)$$

Thus we obtain the linear map $\Lambda_P(E_*,\cdot):A_d(\mathcal{Y})\longrightarrow A_{d-d_1(P)}(\mathrm{Pic}\times X^{m_1}\times D^{m_2})$. The cycle $\Lambda_P(E_*,\mathcal{Z})$ determines the homology class $\mathrm{cycl}(\Lambda_P(E_*,\mathcal{Z}))$ of $H_{2(d-d_1(P))}(\mathrm{Pic}\times X^{m_1}\times D^{m_2})$. Let π_* denote the push-forward for $\mathrm{Pic}\times X^{m_1}\times D^{m_2}$ to a point pt. Then, we obtain the following:

$$\deg\bigl(P(E_*)\cap[\mathcal{Z}]\bigr):=\pi_*\Bigl(\mathfrak{c}\cdot\prod_{i=1}^{m_1}a_i\cdot\prod_{j=1}^{m_2}b_j\cap\mathrm{cycl}\bigl(\Lambda_P(E_*,\mathcal{Z})\bigr)\Bigr)\in H_{2(d-d(P))}(\mathrm{pt})$$

It is trivial in the case $d\neq d(P)$. We identify $H_0(\mathrm{pt})\simeq\mathbb{Q}$. Thus, we obtain the linear map $\deg\bigl(P(E_*)\cap\cdot\bigr):A_*(\mathcal{Y})\longrightarrow\mathbb{Q}$.

Let $A^*(\mathcal{Y})$ denote the operational Chow ring of $\mathcal{Y}$. Let $\mathcal{Z}$ be an algebraic cycle of $\mathcal{Y}$. Let F be an element of $A^*(\mathcal{Y})$. Then, we obtain the number $\int_{\mathcal{Z}}P(E_*)\cdot F:=\deg\Bigl(P(E_*)\cap F\bigl([\mathcal{Z}]\bigr)\Bigr)$.

Notation 7.3 *Let $\mathcal{Y}$ and E_* be as above. We put $\mathcal{R}(E_*,\mathcal{Y}):=\mathcal{R}(E_*)\otimes A^*(\mathcal{Y})$. We can naturally regard $\mathcal{R}(E_*,\mathcal{Y})$ as an $\bigl(\mathcal{R}(E_*),\,A^*(\mathcal{Y})\bigr)$-bimodule.* ▮

We have the linear morphism $\mathcal{R}(E_*,\mathcal{Y})\otimes A_*(\mathcal{Y})\longrightarrow Hom(A_*,\mathbb{Q})$ by the above construction.

Remark 7.4 *Formally,* $\deg\bigl(P(E_*)\cap[\mathcal{Z}]\bigr)$ *is the following number:*

$$\int_{\mathcal{Z}}\mathfrak{c}\cdot\prod_{i=1}^{m_1}\bigl(\mathrm{ch}_{v_i(1)}(E)^{v_i(2)}/a_i\bigr)\cdot\prod_{j=1}^{m_2}\bigl(\mathrm{ch}_{u_j(1)}\bigl(\mathrm{Gr}_{u_j(3)}(E)\bigr)^{u_j(2)}/b_j\bigr)\qquad(193)$$

The author does now know an appropriate reference for the cohomology and homology theories of Deligne-Mumford stacks with the good cycle maps from the Chow groups, G_m-localization theory and any other expected properties. That is the reason to avoid (193) *as the definition. However, it is easy to observe that the formal argument using* (193) *is valid.* ▮

We are especially interested in the following examples.

Example 7.5 Let $\boldsymbol{y}$ be an element of $\mathcal{T}ype$, and let α_* be a system of weights. We have the universal sheaf $\widehat{E}$ over $\mathcal{M}^{ss}(\widehat{\boldsymbol{y}},\alpha_*)\times X$ with the parabolic structure at $\mathcal{M}^{ss}(\widehat{\boldsymbol{y}},\alpha_*)\times D$. Let P be an element of $\mathcal{R}$. By the identification $\mathcal{R}(\widehat{E}_*)=\mathcal{R}$, we have $\Phi=P(\widehat{E}^u)\in\mathcal{R}(\widehat{E}_*)$. Assume that the 1-stability condition holds for $(\boldsymbol{y},\alpha_*)$. We put as follows:

$$\int_{\mathcal{M}^{ss}(\widehat{\boldsymbol{y}},\alpha_*)}\Phi:=\deg\Bigl(\Phi\cap[\mathcal{M}^{ss}(\widehat{\boldsymbol{y}},\alpha_*)]\Bigr)$$

In other words, we obtain the linear map $\int_{\mathcal{M}^{ss}(\widehat{\boldsymbol{y}},\alpha_*)}:\mathcal{R}\longrightarrow\mathbb{Q}$, under the assumption that the 1-stability condition holds for $(\boldsymbol{y},\alpha_*)$. We will later discuss how to obtain such a morphism in general. ▮

Example 7.6 *Let $\boldsymbol{y}$ and α_* be as above. Let L be a line bundle on X, and let δ be an element of $\mathcal{P}^{\mathrm{br}}$ such that the 1-stability condition holds for $(\boldsymbol{y},L,\alpha_*,\delta)$. We denote by ω the first Chern class of the tautological line bundle $\mathcal{O}_{\mathrm{rel}}(1)$ on $\mathcal{M}^s(\widehat{\boldsymbol{y}},[L],\alpha_*,\delta)$. For any $P\in\mathcal{R}$, we have the element $\Phi=P(\widehat{E}^u)\cdot\omega^k\in\mathcal{R}\bigl(\widehat{E}^u,\mathcal{M}^s(\widehat{\boldsymbol{y}},[L],\alpha_*,\delta)\bigr)$. Thus, we obtain the following number:*

$$\int_{\mathcal{M}^s(\widehat{\boldsymbol{y}},[L],\alpha_*,\delta)}\Phi:=\deg\Bigl(\Phi\cap\bigl[\mathcal{M}^s(\widehat{\boldsymbol{y}},[L],\alpha_*,\delta)\bigr]\Bigr)$$

If the 1-vanishing condition holds for $(\boldsymbol{y},L,\alpha_*,\delta)$, moreover, then we have the relative tangent bundle T_{rel} of the smooth map $\mathcal{M}^s(\widehat{\boldsymbol{y}},[L],\alpha_*,\delta)\longrightarrow\mathcal{M}(\widehat{\boldsymbol{y}},[L])$. Let $\mathrm{Eu}(T_{\mathrm{rel}})$ denote the Euler class of T_{rel}. For any $P\in\mathcal{R}$, we have $\Phi=P(\widehat{E}^u)\cdot\mathrm{Eu}(T_{\mathrm{rel}})\in A^*\bigl(\mathcal{M}^s(\widehat{\boldsymbol{y}},[L],\alpha_*,\delta)\bigr)$. Thus, we obtain the integral $\int_{\mathcal{M}^s(\widehat{\boldsymbol{y}},[L],\alpha_*,\delta)}\Phi$. ▮

Let us consider the case where we are given a tuple of parabolic sheaves $\boldsymbol{E}_*=(E_{1\,*},\ldots,E_{l\,*})$ on $\mathcal{Y}\times(X,D)$. Let us take an element $P\in\mathcal{R}_l(\boldsymbol{E}_*)$ of the following form:

$$P=\mathfrak{c}\cdot\prod_{i=1}^{m_1}\left(\prod_{h=1}^{l}\mathrm{ch}^{v_i^{(h)}(2)}_{v_i^{(h)}(1)}(E)/a_i\right)\cdot\prod_{j=1}^{m_2}\left(\prod_{h=1}^{l}\mathrm{ch}^{u_j(2)}_{u_j(1)}\big(\mathrm{Gr}_{u_j(3)}(E)\big)/b_j\right)$$

We assume that $\mathfrak{c}\in H^*(\mathrm{Pic})$, $a_i\in H^*(X)$ and $b_j\in H^*(D)$ are homogeneous for simplicity. Let $\mathcal{Z}$ be a d-dimensional algebraic cycle on $\mathcal{Y}$. Then, we obtain the following element of the Chow group $A_*\big(\mathrm{Pic}\times X^{m_1}\times D^{m_2}\big)$ of $\mathrm{Pic}\times X^{m_1}\times D^{m_2}$ with rational coefficient:

$$\begin{aligned}&\Lambda_P(\boldsymbol{E}_*,Z):=\\&\mathfrak{p}_*\left(\prod_{h=1}^{l}\Big(\prod_{i=1}^{m_1}\mathrm{ch}_{v_i^{(h)}(1)}\big(\pi_{X,i}^*E_h\big)^{v_i^{(h)}(2)}\cdot\prod_{j=1}^{m_2}\mathrm{ch}_{u_j^{(h)}(1)}\big(\pi_{D,j}^*\,\mathrm{Gr}_{u_j^{(h)}(3)}(E_h)\big)^{u_j^{(h)}(2)}\Big)\cap[\mathcal{Z}\times X^{m_1}\times D^{m_2}]\right)\end{aligned}\tag{194}$$

Thus we obtain the linear map $\Lambda_P(E_*,\cdot):A_d(\mathcal{Y})\longrightarrow A_{d-d_1(P)}(\mathrm{Pic}\times X^{m_1}\times D^{m_2})$. The cycle $\Lambda_P(E_*,\mathcal{Z})$ determines the homology class $\mathrm{cycl}(\Lambda_P(E_*,\mathcal{Z}))$ of $H_{2(d-d_1(P))}(\mathrm{Pic}\times X^{m_1}\times D^{m_2})$. Let π_* denote the push-forward for $\mathrm{Pic}\times X^{m_1}\times D^{m_2}$ to a point pt. Then, we obtain the following:

$$\deg\big(P(\boldsymbol{E}_*)\cap[\mathcal{Z}]\big):=\pi_*\Big(\mathfrak{c}\cdot\prod_{i=1}^{m_1}a_i\cdot\prod_{j=1}^{m_2}b_j\cap\mathrm{cycl}\big(\Lambda_P(E_*,\mathcal{Z})\big)\Big)\in H_{2(d-d(P))}(\mathrm{pt})$$

We identify $H_0(\mathrm{pt})=\mathbb{Q}$. Thus, we obtain the linear map $\deg\big(P(\boldsymbol{E}_*)\cap\cdot\big):A_*(\mathcal{Y})\longrightarrow\mathbb{Q}$.

Notation 7.7 *We put $\mathcal{R}(\boldsymbol{E}_*,\mathcal{Y}):=\mathcal{R}(\boldsymbol{E}_*)\otimes A^*(\mathcal{Y})$. We obtain the linear map $\mathcal{R}(\boldsymbol{E}_*,\mathcal{Y})\otimes A_*(\mathcal{Y})\longrightarrow\mathbb{Q}$ by the above construction.* ∎

We have the following commutative diagram, which we will use implicitly.

$$\begin{array}{ccc}\mathcal{R}(\overbrace{E_*,\ldots,E_*}^{l},\mathcal{Y}) & \longrightarrow & \mathrm{Hom}(A_*(\mathcal{Y}),\mathbb{Q})\\ \big\downarrow{\scriptstyle\mathfrak{r}_l} & & \big\downarrow{\scriptstyle=}\\ \mathcal{R}(E_*,\mathcal{Y}) & \longrightarrow & \mathrm{Hom}(A_*(\mathcal{Y}),\mathbb{Q})\end{array}$$

Assume that we have the decomposition $\mathcal{Y}=\prod_{i=1}^{l}\mathcal{Y}_i$ such that $E_{i\,*}$ are pull back of the parabolic sheaves over $\mathcal{Y}_i\times(X,D)$ via the natural projection, where $\mathcal{Y}_i$ are the stacks over Pic and the map $\mathcal{Y}\longrightarrow\mathrm{Pic}$ is the composite of $\prod\mathcal{Y}_i\longrightarrow\mathrm{Pic}^l$ and the multiplication of Pic. We have the naturally defined morphism:

$$\Gamma_1:\bigotimes_{i=1}^{l}\mathcal{R}(E_{i\,*},\mathcal{Y}_i)\longrightarrow\mathrm{Hom}_{\mathbb{Q}}\left(\bigotimes_{i=1}^{l}A_*(\mathcal{Y}_i),\,\mathbb{Q}\right)$$

We also have the following morphism:

$$\Gamma_2:\mathcal{R}(\boldsymbol{E}_*)\otimes\bigotimes_{i=1}^{l}A^*(\mathcal{Y}_i)\longrightarrow\mathrm{Hom}_{\mathbb{Q}}\left(\bigotimes_{i=1}^{l}A_*(\mathcal{Y}_i),\,\mathbb{Q}\right)$$

The algebra homomorphism $\mathfrak{q}_l$ in (188) induces $\mathcal{R}(\boldsymbol{E}_*)\otimes\bigotimes_{i=1}^{l}A^*(\mathcal{Y}_i)\longrightarrow\bigotimes_{i=1}^{l}\mathcal{R}(E_{i*},\mathcal{Y}_i)$, which is also denoted by $\mathfrak{q}_l$. We will use the following lemma without mention, which can be checked by a formal calculation.

Lemma 7.8 *We have $\Gamma_1\circ\mathfrak{q}_l=\Gamma_2$.* ∎

7.1.6 Equivariant case

We continue to use the setting in the subsubsection 7.1.5. We recall the equivariant Chow groups for the torus action. See [9] and [24] for more detail. Let T denote an l-dimensional torus G_m^l. Let $R(T) = \mathbb{Q}[t_1, \ldots, t_l]$ denote the representation ring of T. Let us consider the case where $\mathcal{Y}$ is provided with a T-action. Let A^{m+1} denote the $m+1$-dimensional linear space, on which G_m acts by component-wise multiplication. We have the naturally defined T-action on $\bigl(A^{m+1} - \{0\}\bigr)^l \times \mathcal{Y}$. The quotient stack is denoted by $\mathcal{Y}^{(m)}$. The T-equivariant Chow group of $A_d^T(\mathcal{Y})$ is defined to be $A_{d+lm}(\mathcal{Y}^{(m)})$ for sufficiently large m in this case. We take a linear inclusion $\iota : A^{m+1} \longrightarrow A^{m+2}$, which induces the regular embedding $\mathcal{Y}^{(m)} \longrightarrow \mathcal{Y}^{(m+1)}$. Thus we obtain the morphism $A_{d+l(m+1)}(\mathcal{Y}^{(m+1)}) \longrightarrow A_{d+lm}(\mathcal{Y}^{(m)})$, which is independent of the choice of ι. It is isomorphism when m is sufficiently large. Thus $A_d^T(\mathcal{Y})$ is well defined.

We have the naturally defined morphism $\mathcal{Y}^{(m)} \longrightarrow (\mathbb{P}^m)^l$. Let $\mathcal{O}^{(i)}(1)$ denote the tautological line bundle of i-th $\mathbb{P}^m$. We have the action of the first Chern class $c_1\bigl(\mathcal{O}^{(i)}(1)\bigr) : A_{d+lm}(\mathcal{Y}^{(m)}) \longrightarrow A_{d-1+lm}(\mathcal{Y}^{(m)})$, which induces the action $t_i : A_d^T(\mathcal{Y}) \longrightarrow A_{d-1}(\mathcal{Y})$. Thus, we can naturally regard $A_*^T(\mathcal{Y})$ as the $R(T)$-module.

Assume that E_* is provided with a T-action. Let P be an element of $\mathcal{R}(E_*)$. We naturally obtain the parabolic sheaf $E_*^{(m)}$ on $\mathcal{Y}^{(m)}$. Let $\mathcal{Z}$ be an element of $A_d^T(\mathcal{Y})$, and let $\mathcal{Z}^{(m)}$ be the corresponding element of $A_{d+lm}(\mathcal{Y}^{(m)})$. We obtain the following element of $A^T_{d-d_1(P)+lm}\bigl((\mathbb{P}^m)^l \times \mathrm{Pic} \times X^{m_1} \times D^{m_2}\bigr)$:

$$\Lambda_P(E_*^{(m)}, \mathcal{Z}^{(m)}) := \mathfrak{p}_*\left(\prod_{i=1}^{m_1} \mathrm{ch}_{v_i(1)}\bigl(\pi_{X,i}^* E^{(m)}\bigr)^{v_i(2)} \prod_{j=1}^{m_2} \mathrm{ch}_{u_j(1)}\bigl(\pi_{D,j}^* \mathrm{Gr}_{u_j(3)}(E^{(m)})\bigr)^{u_j(2)} \cap [\mathcal{Z}^{(m)} \times X^{m_1} \times D^{m_2}]\right)$$

Then, it is easy to observe that $\Lambda_P(E_*^{(m)}, \mathcal{Z}^{(m)})$ determines the element $\Lambda_P^T(E_*, \mathcal{Z})$ of $A^T_{d-d_1(P)}(\mathrm{Pic} \times X^{m_1} \times D^{m_2})$. Thus, we obtain the $R(T)$-morphism $\Lambda_P^T(E_*, \cdot) : A_*^T(\mathcal{Y}) \longrightarrow A^T_{*-d_1(P)}(\mathrm{Pic} \times X^{m_1} \times D^{m_2})$. Then, we obtain the equivariant homology class $\mathrm{cycl}(\Lambda_P^T(E_*, \cdot)) \in H^T_{2(*-d_1(P))}(\mathrm{Pic} \times X^{m_1} \times D^{m_2})$. Since the T-action on $\mathrm{Pic} \times X^{m_1} \times D^{m_2}$ are trivial, $\mathfrak{c}$, a_i and b_j naturally give the equivariant cohomology class. Let π_*^T denote the equivariant Gysin map for $\mathrm{Pic} \times X^{m_1} \times D^{m_2} \longrightarrow \mathrm{pt}$. Then, we obtain the following:

$$\deg^T\bigl(P(E_*) \cap [\mathcal{Z}]\bigr) := \pi_*\Bigl(\mathfrak{c} \cdot \prod_{i=1}^{m_1} a_i \cdot \prod_{j=1}^{m_2} b_j \cap \mathrm{cycl}\bigl(\Lambda_P^T(E, \mathcal{Z})\bigr)\Bigr) \in H_{2(d-d(P))}(\mathrm{pt})$$

Since $H_*^T(\mathrm{pt})$ is the free $R(T)$-module of rank one with the special base $1 \in H_0^T(\mathrm{pt})$, we identify it with $R(T)$. The above procedure is compatible with the actions of t_i. Hence we obtain the morphism $\deg^T\bigl(P(E_*) \cap \cdot\bigr) : A_*^T(\mathcal{Y}) \longrightarrow R(T)$ of $R(T)$-modules.

Notation 7.9 *Let $A_T^*(\mathcal{Y})$ denote the T-equivariant operational Chow group of $\mathcal{Y}$. We put $\mathcal{R}_T(E_*, \mathcal{Y}) := \mathcal{R}(E_*) \otimes_{\mathbb{Q}} A_T^*(\mathcal{Y})$.* ▮

Due to the above construction, we have the morphism $\mathcal{R}_T(E_*, \mathcal{Y}) \longrightarrow \mathrm{Hom}_{R(T)}\bigl(A_*^T(\mathcal{Y}), R(T)\bigr)$.

We are especially interested in the following example:

Example 7.10 Let $\widehat{M}$ be the master space as in the subsubsections 4.5.1, 4.7.1 or 4.7.2. We have the sheaf $\widehat{E}^{\widehat{M}}$ on $\widehat{M} \times X$ with the parabolic structure at $\widehat{M} \times D$. We have the G_m-action $\overline{\rho}$ on $\widehat{M}$, which is naturally lifted to the action on $\widehat{E}^{\widehat{M}}$. Let $\Phi(\widehat{E}_*^{\widehat{M}})$ be an element of $\mathcal{R}_T(\widehat{E}, \widehat{M})$. As explained in the subsection 5.9, the perfect obstruction theory of $\widehat{M}$ is lifted to the G_m-equivariant obstruction theory. Thus, we obtain $[\widehat{M}] \in A_*^{G_m}(\widehat{M})$. Then, we obtain the following number:

$$\int_{\widehat{M}} \Phi(\widehat{E}_*^{\widehat{M}}) := \deg^{G_m}\Bigl(\Phi(\widehat{E}_*^{\widehat{M}}) \cap [\widehat{M}]\Bigr)$$

We have the isomorphism:

$$A_*^{G_m}(\widehat{M}) \otimes_{\mathbb{Q}[t]} \mathbb{Q}[t, t^{-1}] \simeq \bigoplus_{i=1,2} A_*(\widehat{M}_i) \otimes_{\mathbb{Q}} \mathbb{Q}[t, t^{-1}] \oplus \bigoplus_{\mathfrak{I}} A_*\bigl(\widehat{M}^{G_m}(\mathfrak{I})\bigr) \otimes_{\mathbb{Q}} \mathbb{Q}[t, t^{-1}]$$

Under the isomorphism, we obtain the decomposition in $A_*^{G_m}(\widehat{M})\otimes_{\mathbb{Q}[t]}\mathbb{Q}[t,t^{-1}]$ due to [24]:

$$[\widehat{M}]=\sum_{i=1,2}\mathrm{Eu}\big(\mathfrak{N}(\widehat{M}_i)\big)^{-1}\cap[\widehat{M}_i]+\sum_{\mathfrak{I}}\mathrm{Eu}\big(\mathfrak{N}(\widehat{M}^{G_m}(\mathfrak{I}))\big)^{-1}\cap[\widehat{M}^{G_m}(\mathfrak{I})]$$

Then, we obtain the following equality in $A_*(\mathrm{Pic}\times X^{m_1}\times D^{m_2})[t,t^{-1}]$.

$$\Lambda_P^{G_m}\big(\widehat{E}_*^{\widehat{M}},[\widehat{M}]\big)=\sum_{i=1,2}\Lambda_P^{G_m}\left(\big(\widehat{E}_*^{\widehat{M}}\big)_{|\widehat{M}_i},\frac{[\widehat{M}_i]}{\mathrm{Eu}(\mathfrak{N}(\widehat{M}_i))}\right)+\sum_{\mathfrak{I}}\Lambda_P^{G_m}\left(\big(\widehat{E}_*^{\widehat{M}}\big)_{|\widehat{M}^{G_m}(\mathfrak{I})},\frac{[\widehat{M}^{G_m}(\mathfrak{I})]}{\mathrm{Eu}\big(\mathfrak{N}(\widehat{M}^{G_m}(\mathfrak{I}))\big)}\right)$$

The equality holds in $A_*(\mathrm{Pic}\times X^{m_1}\times D^{m_2})[t]$. Then, we obtain the following equality in $\mathbb{Q}[t]$:

$$\int_{\widehat{M}}\Phi(\widehat{E}_*^{\widehat{M}})=\sum_{i=1,2}\int_{\widehat{M}_i}\frac{\Phi\big((\widehat{E}_*^{\widehat{M}})_{|\widehat{M}_i}\big)}{\mathrm{Eu}(\mathfrak{N}(\widehat{M}_i))}+\sum_{\mathfrak{I}}\int_{\widehat{M}^{G_m}(\mathfrak{I})}\frac{\Phi\big((\widehat{E}_*^{\widehat{M}})_{|\widehat{M}^{G_m}(\mathfrak{I})}\big)}{\mathrm{Eu}(\mathfrak{N}(\widehat{M}^{G_m}(\mathfrak{I})))}$$

∎

7.1.7 The ring $\mathcal{R}_{\mathrm{CH}}$

We put as follows:

$$\mathcal{R}_{\mathrm{CH}}:=\mathrm{Sym}\Big(\mathrm{Map}_f\big(\mathbb{Z}^2_{\geq 0},A^*(X)\big)\Big)\otimes\mathrm{Sym}\Big(\mathrm{Map}_f\big(\mathbb{Z}^3_{\geq 0},A^*(D)\big)\Big)\otimes A^*(\mathrm{Pic})$$

An element of $\mathcal{R}_{\mathrm{CH}}$ is described as in the case of $\mathcal{R}$. When a parabolic sheaf E_* is given, we put $\mathcal{R}_{\mathrm{CH}}(E_*):=\mathcal{R}$. Similarly, $\mathcal{R}_{l,\mathrm{CH}}$ are defined for each $l\geq 1$. When a tuple of parabolic sheaves $\boldsymbol{E}_*=(E_{1\,*},\ldots,E_{l,\,*})$ is given, we put $\mathcal{R}_{\mathrm{CH}}(\boldsymbol{E}_*):=\mathcal{R}_{l,\mathrm{CH}}$. We use a convention (189) to denote elements of $\mathcal{R}_{\mathrm{CH}}(\boldsymbol{E}_*)$. The maps $A^*(X)\longrightarrow H^{2\,*}(X)$ and $A^*(D)\longrightarrow H^{2\,*}(D)$ induce the algebra homomorphisms $\mathfrak{T}:\mathcal{R}_{\mathrm{CH}}(\boldsymbol{E}_*)\longrightarrow\mathcal{R}(\boldsymbol{E}_*)$.

Let $\mathcal{Z}$ be an element of $A_*(\mathcal{Y})$. For $\prod_{h=1}^l\mathrm{ch}_{v^{(h)}(1)}^{v^{(h)}(2)}(E_h)/a$, we put as follows:

$$\mathfrak{Q}\Big(\prod_{h=1}^l\mathrm{ch}_{v^{(h)}(1)}^{v^{(h)}(2)}(E_h)/a\Big)\cap[\mathcal{Z}]:=p_{X\,*}\Big(\prod_{h=1}^l\mathrm{ch}_{v^{(h)}(1)}(E_h)^{v^{(h)}(2)}\cdot p_{\mathcal{Y}}^*(a)\cap[\mathcal{Z}\times X]\Big)$$

For $\prod_{h=1}^l\mathrm{ch}_{u^{(h)}(1)}^{u^{(h)}(2)}\big(\mathrm{Gr}_{u^{(h)}(3)}(E_h)\big)/b$, we put as follows:

$$\mathfrak{Q}\Big(\prod_{h=1}^l\mathrm{ch}_{u^{(h)}(1)}^{u^{(h)}(2)}\big(\mathrm{Gr}_{u^{(h)}(3)}(E_h)\big)/b\Big)\cap[\mathcal{Z}]:=p_{D\,*}\Big(\prod_{h=1}^l\mathrm{ch}_{u^{(h)}(1)}\big(\mathrm{Gr}_{u^{(h)}(3)}(E_h)\big)^{u^{(h)}(2)}\cdot p_{\mathcal{Y}}^*(b)\cap[\mathcal{Z}\times D]\Big)$$

They induce the algebra homomorphism $\mathfrak{Q}:\mathcal{R}_{\mathrm{CH}}(\boldsymbol{E}_*)\longrightarrow A^*(\mathcal{Y})$. The following lemma can be checked by a formal calculation. We will use it without mention.

Lemma 7.11 *Let Q be an element of $\mathcal{R}_{\mathrm{CH}}(\boldsymbol{E}_*)$. Let P be an element of $\mathcal{R}(\boldsymbol{E}_*,\mathcal{Y})$. Let $\mathcal{Z}$ be an element of $A_*(\mathcal{Y})$. Then, we have the following equality:*

$$\deg\big(\mathfrak{T}(Q)\cdot P\cap[\mathcal{Z}]\big)=\deg\big(P\cap\mathfrak{Q}(\mathcal{Z})\big)$$

∎

7.1.8 The equivariant Euler class

For an algebra R, let $R[[t^{-1},t]$ denote the algebra of series $\sum a_j\cdot t^j$ such that the set $\big\{j>0|a_j\neq 0\big\}$ is finite. We put $\mathfrak{R}(t):=\mathbb{Q}[[t^{-1},t]$. Inductively, we put as follows:

$$\mathfrak{R}(t_1,t_2,\ldots,t_k):=\mathfrak{R}(t_2,\ldots,t_k)[[t_1^{-1},t_1]$$

Let $E_{i\,*}$ $(i=1,2)$ be parabolic sheaves over $\mathcal{Y}\times(X,D)$. Assume that we are given G_m-actions on $E_{i\,*}$. Then, we obtain the following complexes of G_m-equivariant sheaves on $\mathcal{Y}$:

$$\mathcal{F}_1 := Rp_{X\,*}\Big(R\mathcal{H}om\big(E_1,E_2\big)\Big),\quad \mathcal{F}_2 := Rp_{D\,*}\Big(R\mathcal{H}om_2'\big(E_{1|D\,*},E_{2|D\,*}\big)\Big) \tag{195}$$

(See the subsubsection 2.1.5 for the notation $R\mathcal{H}om_2'$.) We have the G_m-equivariant Euler class $\mathrm{Eu}(\mathcal{F}_a)\in A^*_{G_m}(\mathcal{Y})\otimes_{\mathbb{Q}[t]}\mathbb{Q}[t,t^{-1}]$ of $\mathcal{F}_a$ $(a=1,2)$. They are formally $\sum_{i\leq d} c_{d(a)-i}(\mathcal{F}_a)\cdot\big(w\cdot t\big)^i\in A^*(\mathcal{Y})\otimes\mathfrak{R}(t)$, where $d(a)$ denotes the expected rank of $\mathcal{F}_a$. Due to the Grothendieck-Riemann-Roch theorem, we have the element $\mathrm{Eu}'(\mathcal{F}_a)\in\mathcal{R}_{\mathrm{CH}}(E_{1\,*},E_{2\,*})\otimes\mathfrak{R}(t)$ such that $\mathfrak{Q}\big(\mathrm{Eu}'(\mathcal{F}_a)\big)=\mathrm{Eu}(\mathcal{F}_a)$. For abbreviation, we use the notation $\mathrm{Eu}(\mathcal{F}_a)$ to denote $\mathrm{Eu}'(\mathcal{F}_a)$ and the image of $\mathrm{Eu}'(\mathcal{F}_a)$ via the composite of the morphisms:

$$\mathcal{R}_{\mathrm{CH}}(E_{1\,*},E_{2\,*})\otimes\mathfrak{R}(t)\xrightarrow{\ \mathfrak{T}\ }\mathcal{R}(E_{1\,*},E_{2\,*})\otimes\mathfrak{R}(t)\xrightarrow{\ \mathfrak{q}_2\ }\mathcal{R}(E_{1\,*})\otimes\mathcal{R}(E_{2\,*})\otimes\mathfrak{R}(t) \tag{196}$$

Let T denote the k-dimensional torus. If $E_{i\,*}$ are provided with T-actions, then we have the T-action on $\mathcal{F}_a$. Therefore, we obtain the T-equivariant Euler classes $\mathrm{Eu}(\mathcal{F}_a)$. Due to the Grothendieck-Riemann-Roch theorem, we have the element $\mathrm{Eu}'(\mathcal{F}_a)$ of $\mathcal{R}_{\mathrm{CH}}(E_{1\,*},E_{2\,*})\otimes\mathfrak{R}(t_1,\ldots,t_k)$ such that $\mathfrak{Q}\big(\mathrm{Eu}(\mathcal{F}_a)'\big)=\mathrm{Eu}(\mathcal{F}_a)$. For abbreviation, we use the notation $\mathrm{Eu}(\mathcal{F}_a)$ to denote $\mathrm{Eu}(\mathcal{F}_a)'$ and the image of it via the composite of the morphisms in (196).

7.2 Transition Formulas in the Simple Cases

7.2.1 Basic case

Let $\boldsymbol{y}$ be an element of $\mathcal{T}ype$, and let α_* be a system of weights. Let L be a line bundle on X. Let δ denote an element of $\mathcal{P}^{\mathrm{br}}$. Let P be an element of $\mathcal{R}$, and let k be a non-negative integer. Let ω denote the first Chern class of $\mathcal{O}_{\mathrm{rel}}(1)$ on $\mathcal{M}^s(\widehat{\boldsymbol{y}},[L],\alpha_*,\delta)$. We obtain the element $\Phi:=P(\widehat{E}^u)\cdot\omega^k$ of $\mathcal{R}\big(\widehat{E}^u,\mathcal{M}^s(\widehat{\boldsymbol{y}},[L],\alpha_*,\delta)\big)$. In the case where δ is not critical, we obtain the following number by the procedure explained in the subsubsection 7.1.5:

$$\Phi(\widehat{\boldsymbol{y}},[L],\alpha_*,\delta):=\int_{\mathcal{M}(\widehat{\boldsymbol{y}},[L],\alpha_*,\delta)}\Phi$$

Let δ be a critical parameter. We take parameters $\delta_-<\delta<\delta_+$ such that δ_κ $(\kappa=\pm)$ are sufficiently close to δ. We would like to describe $\Phi(\widehat{\boldsymbol{y}},[L],\alpha_*,\delta_+)-\Phi(\widehat{\boldsymbol{y}},[L],\alpha_*,\delta_-)$ as the sum of the integrals over the products of the moduli stacks of the objects with lower ranks. Such a description is called the transition formula.

For that purpose, we prepare some notation. Let $S(\boldsymbol{y},\alpha_*,\delta)$ denote the set of the decomposition types:

$$S(\boldsymbol{y},\alpha_*,\delta):=\big\{\mathfrak{I}=(\boldsymbol{y}_1,\boldsymbol{y}_2)\in\mathcal{T}ype^2\,\big|\,\boldsymbol{y}_1+\boldsymbol{y}_2=\boldsymbol{y},\ \ P^{\alpha_*,\delta}_{\boldsymbol{y}_1}=P^{\alpha_*}_{\boldsymbol{y}_2}=P^{\alpha_*,\delta}_{\boldsymbol{y}}\big\}$$

For a given $(\boldsymbol{y}_1,\boldsymbol{y}_2)\in S(\boldsymbol{y},\alpha_*,\delta)$, we put $r_i=\operatorname{rank}\boldsymbol{y}_i$. We also put as follows:

$$\mathcal{M}(\boldsymbol{y}_1,\widehat{\boldsymbol{y}}_2,L,\alpha_*,\delta):=\mathcal{M}^{ss}(\boldsymbol{y}_1,L,\alpha_*,\delta)\times\mathcal{M}^{ss}(\widehat{\boldsymbol{y}}_2,\alpha_*).$$

We remark that the 2-stability condition for $(\boldsymbol{y},L,\alpha_*,\delta)$ implies the 1-stability conditions for $(\boldsymbol{y}_1,L,\alpha_*,\delta)$ and $(\boldsymbol{y}_2,\alpha_*)$. Let E_1^u denote the sheaf on $\mathcal{M}(\boldsymbol{y}_1,\widehat{\boldsymbol{y}}_2,L,\alpha_*,\delta)\times X$ which is the pull back of the universal sheaf over $\mathcal{M}^s(\boldsymbol{y}_1,L,\alpha_*,\delta)\times X$ via the natural morphism. We use the notation $\widehat{E}_2^u$ in a similar meaning. We put $\omega_1:=c_1(\mathcal{O}r(E_1^u))/\operatorname{rank}r_1$, and $e^{w\cdot\omega_1}$ denotes $\mathcal{O}r(E_1^u)^{w/r_1}$ formally.

Let G_m be the one dimensional torus. Let $e^{w\cdot t}$ denote the trivial bundle on $\mathcal{M}(\boldsymbol{y}_1,\widehat{\boldsymbol{y}}_2,L,\alpha_*,\delta)$ with G_m-action of weight w. We have the following element of the K-group of the G_m-equivariant coherent sheaves on $\mathcal{M}(\boldsymbol{y}_1,\widehat{\boldsymbol{y}}_2,L,\alpha_*,\delta)$:

$$\begin{aligned}\mathfrak{N}_0(\boldsymbol{y}_1,\boldsymbol{y}_2)=&-Rp_{X\,*}\Big(R\mathcal{H}om\big(E_1^u\cdot e^{-t},\,\widehat{E}_2^u\cdot e^{r_1(t-\omega_1)/r_2}\big)\Big)-Rp_{X\,*}\Big(R\mathcal{H}om\big(\widehat{E}_2^u\cdot e^{r_1(t-\omega_1)/r_2},\,E_1^u\cdot e^{-t}\big)\Big)\\&-Rp_{D\,*}\Big(R\mathcal{H}om_2'\big(E^u_{1|D\,*}\cdot e^{-t},\widehat{E}^u_{2|D\,*}\cdot e^{r_1(t-\omega_1)/r_2}\big)\Big)-Rp_{D\,*}\Big(R\mathcal{H}om_2'\big(E^u_{2|D\,*}\cdot e^{r_1(t-\omega_1)/r_2},\,\widehat{E}^u_{1|D\,*}\cdot e^{-t}\big)\Big)\\&+Rp_{X\,*}\Big(\mathcal{H}om\big(L\cdot e^{-t},\,\widehat{E}_2^u\cdot e^{r_1(t-\omega_1)/r_2}\big)\Big)\end{aligned} \tag{197}$$

(See the subsubsection 2.1.5 for the notation $\mathcal{RH}om'_2$.) We have the equivariant Euler class $\mathrm{Eu}\big(\mathfrak{N}_0(\boldsymbol{y}_1,\boldsymbol{y}_2)\big) \in \mathcal{R}\big(E_1^u, \mathcal{M}^s(\boldsymbol{y}_1, L, \alpha_*, \delta)\big) \otimes \mathcal{R}(\widehat{E}_2)[[t^{-1},t]$. (See the subsubsection 7.1.8 for our convention on the equivariant Euler class.)

By the homomorphisms in the subsubsection 7.1.3 and the twist in the subsubsection 7.1.4, we have the element $P\big(E_1^u \cdot e^{-t} \oplus \widehat{E}_2^u \cdot e^{r_1(t-\omega_1)/r_2}\big)$ of $\mathcal{R}\big(E_1^u, \mathcal{M}^s(\boldsymbol{y}_1, L, \alpha_*, \delta)\big) \otimes \mathcal{R}(\widehat{E}_2^u)[t]$. Thus, we obtain the following element of $\mathcal{R}\big(E_1^u, \mathcal{M}^s(\boldsymbol{y}_1, L, \alpha_*, \delta)\big) \otimes \mathcal{R}(\widehat{E}_2^u)[[t^{-1},t]$:

$$\frac{P\big(E_1^u \cdot e^{-t} \oplus \widehat{E}_2^u \cdot e^{r_1(t-\omega_1)/r_2}\big) \cdot t^k}{\mathrm{Eu}\big(\mathfrak{N}_0(\boldsymbol{y}_1,\boldsymbol{y}_2)\big)}$$

By taking the residue with respect to t, we obtain the following element of $\mathcal{R}\big(E_1^u, \mathcal{M}^s(\boldsymbol{y}_1, L, \alpha_*, \delta)\big) \otimes \mathcal{R}(\widehat{E}_2^u)$:

$$\Psi(\boldsymbol{y}_1,\boldsymbol{y}_2) = \operatorname*{Res}_{t=0}\left(\frac{P\big(E_1^u \cdot e^{-t} \oplus \widehat{E}_2^u \cdot e^{r_1(t-\omega_1)/r_2}\big) \cdot t^k}{\mathrm{Eu}\big(\mathfrak{N}_0(\boldsymbol{y}_1,\boldsymbol{y}_2)\big)}\right) \tag{198}$$

Theorem 7.12 *Assume that the* 2*-stability condition holds for* $(\boldsymbol{y}, L, \alpha_*, \delta)$*. Then, we have the following equality:*

$$\Phi(\widehat{\boldsymbol{y}}, [L], \alpha_*, \delta_+) - \Phi(\widehat{\boldsymbol{y}}, [L], \alpha_*, \delta_-) = \sum_{(\boldsymbol{y}_1,\boldsymbol{y}_2)\in S(\boldsymbol{y},\alpha_*,\delta)} \int_{\mathcal{M}(\boldsymbol{y}_1,\widehat{\boldsymbol{y}}_2,L,\alpha_*,\delta)} \Psi(\boldsymbol{y}_1,\boldsymbol{y}_2) \tag{199}$$

The elements $\Psi(\boldsymbol{y}_1,\boldsymbol{y}_2) \in \mathcal{R}\big(E_1^u, \mathcal{M}^s(\boldsymbol{y}_1, L, \alpha_*, \delta)\big) \otimes \mathcal{R}(\widehat{E}_2^u)$ *are given as in* (198).

The contribution of $(\boldsymbol{y}_1,\boldsymbol{y}_2) \in S(\boldsymbol{y},\alpha_*,\delta)$ *vanishes in the case* $p_g > 0$ *and* $\mathrm{rank}(\boldsymbol{y}_1) \neq 1$,

Proof Let $\widehat{M}$ denote the master space connecting $\mathcal{M}^s(\widehat{\boldsymbol{y}}, [L], \alpha_*, \delta_+)$ and $\mathcal{M}^s(\widehat{\boldsymbol{y}}, [L], \alpha_*, \delta_-)$ as in the subsubsection 4.7.1. Let $\varphi : \widehat{M} \longrightarrow \mathcal{M}(m, \widehat{\boldsymbol{y}}, [L])$ be the naturally defined morphism. Let $\mathcal{T}(1)$ denote the trivial line bundle on $\mathcal{M}(m, \widehat{\boldsymbol{y}}, [L])$ with the G_m-action of weight 1. We have the natural G_m-action on $\widehat{E}^{\widehat{M}}$ and $\varphi^*\mathcal{O}_{\mathrm{rel}}(1)$. We consider the following elements of $\mathcal{R}_{G_m}\big(\widehat{E}^{\widehat{M}}, \widehat{M}\big)$:

$$\Phi_t := P(\widehat{E}^{\widehat{M}}) \cdot c_1\big(\varphi^*\mathcal{O}_{\mathrm{rel}}(1)\big)^k, \quad \widetilde{\Phi}_t := \Phi_t \cdot c_1\big(\varphi^*\mathcal{T}(1)\big).$$

We use Proposition 5.58, then we obtain the polynomial $\int_{\widehat{M}} \widetilde{\Phi}_t$ of t, as explained in Example 7.10. When we forget the G_m-action, we have $c_1\big(\varphi^*\mathcal{T}(1)\big) = 0$. Hence we have $\int_{\widehat{M}} \widetilde{\Phi}_{t|t=0} = 0$. On the other hand, we have the following equality in $\mathbb{Q}[t^{-1},t]$, due to the localization of the virtual fundamental classes ([24]):

$$\int_{\widehat{M}} \widetilde{\Phi}_t = \sum_{i=1,2} \int_{\widehat{M}_i} \frac{\widetilde{\Phi}_t}{\mathrm{Eu}\big(\mathfrak{N}(\widehat{M}_i)\big)} + \sum_{\mathfrak{I}\in S(\boldsymbol{y},\alpha_*,\delta)} \int_{\widehat{M}^{G_m}(\mathfrak{I})} \frac{\widetilde{\Phi}_t}{\mathrm{Eu}\big(\mathfrak{N}(\widehat{M}^{G_m}(\mathfrak{I}))\big)}$$

Here, $\mathfrak{N}(\widehat{M}_i)$ and $\mathfrak{N}(\widehat{M}^{G_m}(\mathfrak{I}))$ denote the virtual normal bundles with the G_m-action given in Proposition 5.58. We have $c_1\big(\varphi^*\mathcal{T}(1)\big)_{|\widehat{M}_i} = t$ and $c_1\big(\varphi^*\mathcal{T}(1)\big)_{|\widehat{M}^{G_m}(\mathfrak{I})} = t$. Therefore, we obtain the following equality in $\mathbb{Q}[t,t^{-1}]$

$$\sum_{i=1,2} \int_{\widehat{M}_i} \operatorname*{Res}_{t=0}\left(\frac{\Phi_t}{\mathrm{Eu}\big(\mathfrak{N}(\widehat{M}_i)\big)}\right) + \sum_{\mathfrak{I}} \int_{\widehat{M}^{G_m}(\mathfrak{I})} \operatorname*{Res}_{t=0}\left(\frac{\Phi_t}{\mathrm{Eu}\big(\mathfrak{N}(\widehat{M}^{G_m}(\mathfrak{I}))\big)}\right) = 0. \tag{200}$$

Let us see the contributions from the components $\widehat{M}_i$. We have $\iota_i^*\widehat{E}^{\widehat{M}} = \widehat{E}^u$ and $\iota_i^*\varphi^*\mathcal{O}_{\mathrm{rel}}(1) = \mathcal{O}_{\mathrm{rel}}(1)$ with the trivial G_m-action. Due to Proposition 5.58, we have the following equality:

$$\mathrm{Eu}\big(\mathfrak{N}(\widehat{M}_i)\big)^{-1} = (-1)^i \cdot (t-\omega)^{-1} = (-1)^i \cdot \frac{1}{t} \cdot \sum_{j=0}^{\infty} \left(\frac{\omega}{t}\right)^j$$

Therefore, we obtain the following:

$$\sum_{i=1,2} \int_{\widehat{M}_i} \operatorname*{Res}_{t=0}\left(\frac{\Phi_t}{\mathrm{Eu}\big(\mathfrak{N}(\widehat{M}_i)\big)}\right) = \sum_{i=1,2} (-1)^i \cdot \int_{\widehat{M}_i} \Phi = -\Phi(\widehat{\boldsymbol{y}}, [L], \alpha_*, \delta_+) + \Phi(\widehat{\boldsymbol{y}}, [L], \alpha_*, \delta_-) \tag{201}$$

Let us calculate the contribution from the component $\widehat{M}^{G_m}(\mathfrak{I})$. We remark that $\varphi^*\mathcal{O}_{\rm rel}(1)_{|\widehat{M}^*}$ and $\varphi^*\mathcal{T}(1)_{|\widehat{M}^*}$ are naturally isomorphic as G_m-equivariant line bundles. We use the relation of $E_i^{\widehat{M}}$, E_1^u and $\widehat{E}_2^u$ in Proposition 4.50. Then, we obtain the following equality in $\mathcal{R}\bigl(G'^*E_1^u, G'^*\widehat{E}_2^u, \mathcal{S}\bigr)[t]$:

$$F^*\Phi_t = G'^*P\bigl(E_1^u\cdot e^{-t}\oplus\widehat{E}_2^u\cdot e^{r_1(t-\omega_1)/r_2}\bigr)\cdot t^k$$

We also have the following equality in $\mathcal{R}\bigl(G'^*E_1^u\bigr)\otimes\mathcal{R}\bigl(G'^*\widehat{E}_2\bigr)[[t^{-1},t]$:

$$F^*\operatorname{Eu}\bigl(\mathfrak{N}(\widehat{M}^{G_m}(\mathfrak{I}))\bigr) = G'^*\operatorname{Eu}\bigl(\mathfrak{N}_0(\boldsymbol{y}_1,\boldsymbol{y}_2)\bigr)$$

We have the equality of the virtual fundamental classes in Proposition 6.11. Thus, we obtain the following equality:

$$\int_{\widehat{M}^{G_m}(\mathfrak{I})}\operatorname*{Res}_{t=0}\left(\frac{\Phi_t}{\operatorname{Eu}\bigl(\mathfrak{N}_0(\boldsymbol{y}_1,\boldsymbol{y}_2)\bigr)}\right) = \int_{\mathcal{M}(\boldsymbol{y}_1,\widehat{\boldsymbol{y}}_2,L,\alpha_*,\delta)}\operatorname*{Res}_{t=0}\left(\frac{P\bigl(E_1^u\cdot e^{-t}\oplus\widehat{E}_2^u\cdot e^{r_1(t-\omega_1)/r_2}\bigr)\cdot t^k}{\operatorname{Eu}\bigl(\mathfrak{N}_0(\boldsymbol{y}_1,\boldsymbol{y}_2)\bigr)}\right) \tag{202}$$

The desired equality (199) follows from (200), (201) and (202).

The second claim of the theorem immediately follows from Proposition 6.24 and the first claim. ∎

Corollary 7.13 *Assume $p_g>0$. Assume that the 2-stability condition holds for $(\boldsymbol{y},L,\alpha_*,\delta)$, and that the 2-vanishing condition holds for $(\boldsymbol{y},L,\alpha_*,\delta)$. Then we have $\Phi(\widehat{\boldsymbol{y}},[L],\alpha_*,\delta_+)=\Phi(\widehat{\boldsymbol{y}},[L],\alpha_*,\delta_-)$.*

Proof It immediately follows from Theorem 7.12 and Proposition 6.3. ∎

7.2.2 The case of the Euler class of the relative tangent bundle

Assume that the 1-vanishing condition holds for $(\boldsymbol{y},L,\alpha_*,\delta)$. Let $T_{\rm rel}$ denote the relative tangent bundle of the smooth morphism $\mathcal{M}^s(\widehat{\boldsymbol{y}},[L],\alpha_*,\delta)\longrightarrow\mathcal{M}(\widehat{\boldsymbol{y}})$. We put as follows:

$$N_L(y) := \int_X \operatorname{Td}(X)\cdot y\cdot\operatorname{ch}(L).$$

We will be interested in the integral of the following element of $\mathcal{R}\bigl(\widehat{E}^u,\mathcal{M}^s(\widehat{\boldsymbol{y}},[L],\alpha_*,\delta)\bigr)$:

$$\Phi = \frac{\operatorname{Eu}(T_{\rm rel})}{N_L(y)}\cdot P(\widehat{E}^u)$$

We take parameters $\delta_-<\delta<\delta_+$ such that δ_κ $(\kappa=\pm)$ are sufficiently close to δ. The transition formula for $\Phi(\widehat{\boldsymbol{y}},[L],\alpha_*,\delta)$ is rather simple, if the 1-vanishing condition and the 2-stability condition hold for $(\widehat{\boldsymbol{y}},L,\alpha_*,\delta)$. Strictly speaking, we do not need it, because we prove a more general formula later. However, we give it as an explanation of the argument. In the case $p_g>0$, the problem is easy.

Proposition 7.14 *Assume $p_g>0$. Assume that the 1-vanishing condition and the 2-stability condition hold for $(\boldsymbol{y},L,\alpha_*,\delta)$. Then, we have the equality $\Phi(\widehat{\boldsymbol{y}},[L],\alpha,\delta_-)=\Phi(\widehat{\boldsymbol{y}},[L],\alpha,\delta_+)$.*

Proof By the same argument as the proof of Theorem 7.12, we can express $\Phi(\widehat{\boldsymbol{y}},[L],\alpha,\delta_+)-\Phi(\widehat{\boldsymbol{y}},[L],\alpha,\delta_-)$ as the sum of the integrals over the fixed point set $\widehat{M}^{G_m}(\mathfrak{I})$. Under the assumption of the proposition, we have $[\widehat{M}^{G_m}(\mathfrak{I})]=0$ due to Proposition 6.11, Proposition 6.24 and Proposition 6.25. Thus we are done. ∎

Let us discuss the case $p_g=0$. For a decomposition type $(\boldsymbol{y}_1,\boldsymbol{y}_2)\in S(\boldsymbol{y},L,\alpha_*)$, we put as follows:

$$\mathcal{M}(\widehat{\boldsymbol{y}}_1,\widehat{\boldsymbol{y}}_2,[L],\alpha_*,\delta) := \mathcal{M}^s(\widehat{\boldsymbol{y}}_1,[L],\alpha_*,\delta)\times\mathcal{M}^s(\widehat{\boldsymbol{y}}_2,\alpha_*).$$

Let $e^{w\cdot s}$ denote the trivial line bundle on $\mathcal{M}(\widehat{\boldsymbol{y}}_1,\widehat{\boldsymbol{y}}_2,[L],\alpha_*,\delta)$ with the G_m-action of weight w. We have the following element of the K-group $K^{G_m}(\mathcal{M}(\widehat{\boldsymbol{y}}_1,\widehat{\boldsymbol{y}}_2,[L],\alpha_*,\delta))$ of the G_m-equivariant coherent sheaves on $\mathcal{M}(\widehat{\boldsymbol{y}}_1,\widehat{\boldsymbol{y}}_2,[L],\alpha_*,\delta)$ (see the subsubsection 5.9.1 for the notation):

$$\mathfrak{N}(\widehat{E}_1^u,\widehat{E}_2^u)\cdot e^{s/r_1+s/r_2}+\mathfrak{N}(\widehat{E}_2^u,\widehat{E}_1^u)\cdot e^{-s/r_1-s/r_2}+\mathfrak{N}_D(\widehat{E}_{1*}^u,\widehat{E}_{2*}^u)\cdot e^{s/r_1+s/r_2}+\mathfrak{N}_D(\widehat{E}_{2*}^u,\widehat{E}_{1*}^u)\cdot e^{-s/r_1-s/r_2} \tag{203}$$

Let $Q(\widehat{E}_1\cdot e^{-s/r_1},\widehat{E}_2\cdot e^{s/r_2})\in\mathcal{R}(\widehat{E}_1)\otimes\mathcal{R}(\widehat{E}_2)[[s^{-1},s]$ denote the equivariant Euler class of (203).

Theorem 7.15 *Assume that the 2-stability condition and the 1-vanishing condition hold for $(\boldsymbol{y}, L, \alpha_*, \delta)$. In the case $p_g = 0$, we have the following equality:*

$$\Phi(\widehat{\boldsymbol{y}}, [L], \alpha_*, \delta_+) - \Phi(\widehat{\boldsymbol{y}}, [L], \alpha_*, \delta_-) = \sum_{(\boldsymbol{y}_1, \boldsymbol{y}_2) \in S(\boldsymbol{y}, \alpha_*, \delta)} \int_{\mathcal{M}(\widehat{\boldsymbol{y}}_1, \widehat{\boldsymbol{y}}_2, [L], \alpha_*, \delta)} \Psi(\boldsymbol{y}_1, \boldsymbol{y}_2) \tag{204}$$

The elements $\Psi(\boldsymbol{y}_1, \boldsymbol{y}_2) \in \mathcal{R}\big(\widehat{E}^u_1, \mathcal{M}^s(\widehat{\boldsymbol{y}}_1, [L], \alpha_*, \delta)\big) \otimes \mathcal{R}\big(\widehat{E}^u_2\big)$ are given as follows:

$$\Psi(\boldsymbol{y}_1, \boldsymbol{y}_2) = \frac{N_L(y_1)}{N_L(y)} \cdot \operatorname*{Res}_{s=0} \left(\frac{P\big(\widehat{E}^u_1 \cdot e^{-s/r_1} \oplus \widehat{E}^u_2 \cdot e^{s/r_2}\big)}{Q\big(\widehat{E}^u_1 \cdot e^{-s/r_1},\ \widehat{E}^u_2 \cdot e^{s/r_2}\big)} \right) \cdot \frac{\mathrm{Eu}(T_{1,\mathrm{rel}})}{N_L(y_1)} \tag{205}$$

Here $T_{1\ \mathrm{rel}}$ denote the vector bundle on $\mathcal{M}(\widehat{\boldsymbol{y}}_1, \widehat{\boldsymbol{y}}_2, [L], \alpha_*, \delta)$ induced by the relative tangent bundle of the smooth morphism $\mathcal{M}^s(\widehat{\boldsymbol{y}}_1, [L], \alpha_*, \delta) \longrightarrow \mathcal{M}(\widehat{\boldsymbol{y}}_1)$.

Proof By using the same argument as the proof of Theorem 7.12, we can reduce the problem to the calculation of the contributions from $\widehat{M}^{G_m}(\mathfrak{I})$ for $\mathfrak{I} = (\boldsymbol{y}_1, \boldsymbol{y}_2) \in S(\boldsymbol{y}, \alpha_*, \delta)$. We use the notation in the subsubsection 4.7.1. Let $\varphi_{\mathfrak{I}}$ denote the inclusion $\widehat{M}^{G_m}(\mathfrak{I}) \longrightarrow \widehat{M}$. We remark the relation of the virtual fundamental classes in the case $p_g = 0$ given in Proposition 6.24. Thus, we put $[\mathcal{S}] := F^*\big([\widehat{M}^{G_m}(\mathfrak{I})]\big) = G^*\big[\mathcal{M}(\widehat{\boldsymbol{y}}_1, \widehat{\boldsymbol{y}}_2, [L], \alpha_*, \delta)\big]$. Let I_w denote the trivial line bundle with the action of G_m of weight w. On $\mathcal{S}$, we have the following decomposition of the equivariant vector bundles:

$$F^* \varphi_{\mathfrak{I}}^* T_{\mathrm{rel}} = G^* T_{1\ \mathrm{rel}} \oplus G^* p_{X\,*} \mathcal{H}om\big(L, \widehat{E}^u_2\big) \otimes \mathcal{O}_{1,\mathrm{rel}}(-r_1/r_2) \otimes I_{1+r_1/r_2}$$

Therefore, we have the following equality in $A^*_{G_m}(\mathcal{S})$:

$$F^* \varphi_{\mathfrak{I}}^* \operatorname{Eu}\big(T_{\mathrm{rel}}\big) = G^* \operatorname{Eu}(T_{1\ \mathrm{rel}}) \cdot G^* \operatorname{Eu}\Big(p_{X\,*} \mathcal{H}om\big(L, \widehat{E}^u_2\big) \cdot e^{(1+r_1/r_2)\cdot t - r_1 \omega_1/r_2}\Big)$$

The second term in the right hand side also appears in $Eu(\mathfrak{N}_0(\boldsymbol{y}_1, \boldsymbol{y}_2))$, and hence they are cancelled out in the evaluation of $[\mathcal{S}]$. Then, we obtain the following equality in $\mathbb{Q}[t^{-1}, t]$:

$$\int_{\mathcal{S}} F^* \left(\frac{\varphi_{\mathfrak{I}}^*\big(P(\widehat{E}^{\widehat{M}}) \cdot \operatorname{Eu}(T_{\mathrm{rel}})\big)}{\operatorname{Eu}\big(\mathfrak{N}(\widehat{M}^{G_m}(\mathfrak{I}))\big)} \right) = \int_{\mathcal{S}} G^* \left(\frac{P\big(\widehat{E}^u_1 \cdot e^{-t+\omega_1} \oplus \widehat{E}^u_2 \cdot e^{r_1(t-\omega_1)/r_2}\big) \cdot \operatorname{Eu}(T_{1\ \mathrm{rel}})}{Q\big(\widehat{E}^u_1 \cdot e^{-t+\omega_1}, \widehat{E}^u_2 \cdot e^{r_1(t-\omega_1)/r_2}\big)} \right) \tag{206}$$

Here, $Q\big(\widehat{E}^u_1 \cdot e^{-t+\omega_1},\ \widehat{E}^u_2 \cdot e^{r_1(t-\omega_1)/r_2}\big)$ is the Euler class of the following element of $K^{G_m}\big(\mathcal{M}(\widehat{\boldsymbol{y}}_1, \widehat{\boldsymbol{y}}_2, [L], \alpha_*, \delta)\big)$, for $A = 1 + r_1/r_2$:

$$\mathfrak{N}(\widehat{E}^u_1, \widehat{E}^u_2) \cdot e^{A(t-\omega_1)} + \mathfrak{N}(\widehat{E}^u_2, \widehat{E}^u_1) \cdot e^{-A(t-\omega_1)} + \mathfrak{N}_D(\widehat{E}^u_{1\,*}, \widehat{E}^u_{2\,*}) \cdot e^{A(t-\omega_1)} + \mathfrak{N}_D(\widehat{E}^u_{2\,*}, \widehat{E}^u_{1\,*}) \cdot e^{-A(t-\omega_1)}$$

We remark that the integrand of the right hand side of (206) is of the form $\sum_j A_j \cdot (t-\omega_1)^j$, where $A_j \in \mathcal{R}\big(\widehat{E}^u_1, \mathcal{M}(\widehat{\boldsymbol{y}}_1, [L], \alpha_*, \delta)\big) \otimes \mathcal{R}\big(\widehat{E}^u_2\big)$. By a direct calculation, we can check the following:

$$\operatorname*{Res}_{t=0} (t-\omega_1)^j = \begin{cases} 1 & (j=-1) \\ 0 & (j \neq -1) \end{cases}$$

Hence, we have $\mathrm{Res}_{t=0}(t-\omega_1)^j = \mathrm{Res}_{t=0}\, t^j$ for any j. In particular, we have the following equality:

$$\operatorname*{Res}_{t=0}\Big(\sum_j A_j \cdot (t-\omega_1)^j\Big) = \operatorname*{Res}_{t=0}\Big(\sum_j A_j \cdot t^j\Big)$$

Thus, we obtain the following equality:

$$\begin{aligned} &\int_{\mathcal{S}} \operatorname*{Res}_t G^* \left(\frac{P\big(\widehat{E}^u_1 \cdot e^{-t+\omega_1} \oplus \widehat{E}^u_2 \cdot e^{r_1(t-\omega_1)/r_2}\big) \cdot \operatorname{Eu}(T_{1\ \mathrm{rel}})}{Q\big(\widehat{E}^u_1 \cdot e^{-t+\omega_1}, \widehat{E}^u_2 \cdot e^{r_1(t-\omega_1)/r_2}\big)} \right) \\ &\qquad = \int_{\mathcal{S}} \operatorname*{Res}_t G^* \left(\frac{P\big(\widehat{E}^u_1 \cdot e^{-t} \oplus \widehat{E}^u_2 \cdot e^{r_1 t/r_2}\big) \cdot \operatorname{Eu}(T_{1\ \mathrm{rel}})}{Q\big(\widehat{E}^u_1 \cdot e^{-t}, \widehat{E}^u_2 \cdot e^{r_1 t/r_2}\big)} \right) \end{aligned} \tag{207}$$

Therefore, we obtain the following:

$$\int_{\widehat{M}^{G_m}(\mathfrak{I})} \operatorname*{Res}_{t=0}\left(\frac{P(\widehat{E}^{\widehat{M}})\cdot \mathrm{Eu}(T_{\mathrm{rel}})}{\mathrm{Eu}(\mathfrak{N}(\widehat{M}^{G_m}(\mathfrak{I}))}\right) = r_1\cdot \int_{\mathcal{M}(\widehat{\boldsymbol{y}}_1,\widehat{\boldsymbol{y}}_2,[L],\alpha_*,\delta)} \operatorname*{Res}_{t=0}\left(\frac{P\big(\widehat{E}_1^u\cdot e^{-t}\oplus \widehat{E}_2^u\cdot e^{r_1t/r_2}\big)\cdot \mathrm{Eu}(T_{1,\mathrm{rel}})}{Q\big(\widehat{E}_1^u\cdot e^{-t},\ \widehat{E}_2^u\cdot e^{r_1t/r_2}\big)}\right) \quad (208)$$

By putting $r_1t=s$, we obtain the desired formula (205). ∎

7.2.3 The $\boldsymbol{L}$ case

Let $\boldsymbol{L}=(L_1,L_2)$ be a pair of line bundles on X. We assume that the 1-vanishing condition holds for $(\boldsymbol{y},L_1,\alpha_*)$, and that the 2-vanishing condition holds for $(\boldsymbol{y},L_2,\alpha_*)$, in the sense of the subsubsection 6.4.1. We discuss the transition formula under the situation of the subsubsection 4.7.2.

Let $\boldsymbol{\delta}=(\delta_1,\delta_2)$ be an element of $\mathcal{P}^{\mathrm{br}\,2}$. Assume that both of δ_i are sufficiently small as in the subsubsection 4.7.2. Recall that the 1-stability condition does not hold for $(\boldsymbol{y},\boldsymbol{L},\alpha_*,\boldsymbol{\delta})$, if and only if the following conditions hold:

- $\delta_1/r_1=\delta_2/r_2$ holds for some pair of positive integers (r_1,r_2) such that r_1+r_2
- There exists a decomposition $\boldsymbol{y}_1+\boldsymbol{y}_2=\boldsymbol{y}$ such that $\operatorname{rank}\boldsymbol{y}_i=r_i$ and $P^{\alpha_*}_{\boldsymbol{y}_1}=P^{\alpha_*}_{\boldsymbol{y}_2}$.

Assume that the 1-stability condition does not hold for $(\boldsymbol{y},\boldsymbol{L},\alpha_*,\boldsymbol{\delta})$. We take elements $\delta_-,\delta_+\in\mathcal{P}^{\mathrm{br}}$ such that $\delta_-<\delta_1<\delta_+$ and that $|\delta_\kappa-\delta_1|$ $(\kappa=\pm)$ are sufficiently small. We put $\boldsymbol{\delta}_\kappa=(\delta_\kappa,\delta_2)$ for $\kappa=\pm$. Let $T^{(1)}_{\mathrm{rel}}$ denote the relative tangent bundle of the smooth morphism $\mathcal{M}^s(\widehat{\boldsymbol{y}},[\boldsymbol{L}],\alpha_*,\boldsymbol{\delta}_\kappa)\longrightarrow\mathcal{M}(\widehat{\boldsymbol{y}},[L_2])$. Let $\mathcal{O}^{(i)}_{\mathrm{rel}}(1)$ denote the pull back of the tautological line bundle on $\mathcal{M}(\widehat{\boldsymbol{y}},[L_i])$ via the morphism $\mathcal{M}^{ss}(\widehat{\boldsymbol{y}},[\boldsymbol{L}],\alpha_*,\delta)\longrightarrow\mathcal{M}(\widehat{\boldsymbol{y}},[L_i])$. We put $\omega^{(i)}:=c_1\big(\mathcal{O}^{(i)}_{\mathrm{rel}}(1)\big)$. We consider the following element of $\mathcal{R}\big(\widehat{E},\mathcal{M}^s(\widehat{\boldsymbol{y}},[\boldsymbol{L}],\alpha_*,\boldsymbol{\delta}_\kappa)\big)$:

$$\Phi=\frac{\mathrm{Eu}(T^{(1)}_{\mathrm{rel}})}{N_{L_1}(y)}\cdot P(\widehat{E}^u)\cdot\omega^{(2)\,k}$$

We put as follows, for $\kappa=\pm$:

$$\Phi(\widehat{\boldsymbol{y}},[\boldsymbol{L}],\alpha_*,\boldsymbol{\delta}_\kappa):=\int_{\mathcal{M}^s(\widehat{\boldsymbol{y}},[\boldsymbol{L}],\alpha_*,\boldsymbol{\delta}_\kappa)}\Phi$$

We would like to discuss the transition formula between $\Phi(\widehat{\boldsymbol{y}},[\boldsymbol{L}],\alpha_*,\boldsymbol{\delta}_+)$ and $\Phi(\widehat{\boldsymbol{y}},[\boldsymbol{L}],\alpha_*,\boldsymbol{\delta}_-)$.

Proposition 7.16 *In the case $p_g>0$, we have the equality:*

$$\Phi(\widehat{\boldsymbol{y}},[\boldsymbol{L}],\alpha_*,\boldsymbol{\delta}_+)=\Phi(\widehat{\boldsymbol{y}},[\boldsymbol{L}],\alpha_*,\boldsymbol{\delta}_-)$$

Proof Let $\varphi:\widehat{M}\longrightarrow\mathcal{M}(m,\widehat{\boldsymbol{y}},[\boldsymbol{L}])$ denote the naturally defined morphism. Let $\mathcal{T}(1)$ denote the trivial line bundle on $\mathcal{M}(m,\widehat{\boldsymbol{y}},[\boldsymbol{L}])$ with the G_m-action of weight 1. We put $\mathcal{I}_i:=\varphi^*\mathcal{O}^{(i)}_{\mathrm{rel}}(-1)$. Let us consider the G_m-equivariant cohomology class:

$$\Phi_t:=P(\varphi^*\widehat{E}^u)\cdot\frac{\mathrm{Eu}\big(\varphi^*T^{(1)}_{\mathrm{rel}}\big)}{N_{L_1}(y)}\cdot c_1(\mathcal{I}_2^{-1})^k,\quad \widetilde{\Phi}_t:=\Phi_t\cdot c_1\big(\varphi^*\mathcal{T}(1)\big)$$

By applying the argument as in the proof of Theorem 7.12 to $\int_{\widehat{M}}\widetilde{\Phi}_t$, we can obtain the description to express $\Phi(\widehat{\boldsymbol{y}},[\boldsymbol{L}],\alpha_*,\boldsymbol{\delta}_+)-\Phi(\widehat{\boldsymbol{y}},[\boldsymbol{L}],\alpha_*,\boldsymbol{\delta}_-)$ as the sum of the integrals over the fixed point sets $\widehat{M}^{G_m}(\mathfrak{I})$. Under the assumption of the proposition, we have $[\widehat{M}^{G_m}(\mathfrak{I})]=0$, due to Proposition 6.12, Proposition 6.24 and Proposition 6.25. Thus we are done. ∎

To discuss the case $p_g=0$, we prepare some notation. We put as follows:

$$S(\boldsymbol{y},\alpha_*,\boldsymbol{\delta}):=\big\{(\boldsymbol{y}_1,\boldsymbol{y}_2)\in\mathcal{T}ype^2\,\big|\,P^{\alpha_*}_{\boldsymbol{y}_1}=P^{\alpha_*}_{\boldsymbol{y}_2},\ \ \delta_1/r_1=\delta_2/r_2\big\}$$

For any $(\boldsymbol{y}_1,\boldsymbol{y}_2)\in S(\boldsymbol{y},\alpha_*,\boldsymbol{\delta})$, we put as follows:

$$\mathcal{M}(\widehat{\boldsymbol{y}}_1,\widehat{\boldsymbol{y}}_2,[\boldsymbol{L}],\alpha_*,\boldsymbol{\delta}):=\mathcal{M}^s(\widehat{\boldsymbol{y}}_1,[L_1],\alpha_*,\delta_1)\times\mathcal{M}^s(\widehat{\boldsymbol{y}}_2,[L_2],\alpha_*,\delta_2)$$

Let $e^{w\cdot s}$ denote the trivial line bundle on $\mathcal{M}(\widehat{\boldsymbol{y}}_1,\widehat{\boldsymbol{y}}_2,[\boldsymbol{L}],\alpha_*,\boldsymbol{\delta})$ with the G_m-action of weight w. We have the following element of the K-group $K^{G_m}\big(\mathcal{M}(\widehat{\boldsymbol{y}}_1,\widehat{\boldsymbol{y}}_2,[\boldsymbol{L}],\alpha_*,\boldsymbol{\delta})\big)$ of the G_m-equivariant coherent sheaves on $\mathcal{M}(\widehat{\boldsymbol{y}}_1,\widehat{\boldsymbol{y}}_2,[\boldsymbol{L}],\alpha_*,\boldsymbol{\delta})$:

$$\mathfrak{N}(\widehat{E}^u_1,\widehat{E}^u_2)\cdot e^{s/r_1+s/r_2}+\mathfrak{N}(\widehat{E}^u_2,\widehat{E}^u_1)\cdot e^{-s/r_1-s/r_2}+\mathfrak{N}_D(\widehat{E}^u_{1*},\widehat{E}^u_{2*})\cdot e^{s/r_1+s/r_2}+\mathfrak{N}_D(\widehat{E}^u_{2*},\widehat{E}^u_{1*})\cdot e^{-s/r_1-s/r_2}$$

The equivariant Euler class is denoted by $Q(\widehat{E}^u_1\cdot e^{s/r_1},\widehat{E}^u_2\cdot e^{s/r_2})\in\mathcal{R}(\widehat{E}_1)\otimes\mathcal{R}(\widehat{E}_2)[[s^{-1},s]$. Let $\mathcal{O}_{2,\mathrm{rel}}(1)$ denote the tautological line bundle on $\mathcal{M}(\widehat{\boldsymbol{y}}_2,[L_2])$. The pull back is also denoted by the same notation. We put $\omega_2:=c_1\big(\mathcal{O}_{2,\mathrm{rel}}(1)\big)$, and we use the notation $e^{w\cdot\omega_2}$ to denote $\mathcal{O}_{2,\mathrm{rel}}(w)$ formally. We also have the following element of $K^{G_m}\big(\mathcal{M}(\widehat{\boldsymbol{y}}_1,\widehat{\boldsymbol{y}}_2,[\boldsymbol{L}],\alpha_*,\delta)\big)$:

$$Rp_{X\,*}\mathcal{H}om(L_2,E^u_1)\cdot e^{-s/r_1-s/r_2+\omega_2}$$

The equivariant Euler class is denoted by $R(L_2\cdot e^{-\omega_2+s/r_2},\ \widehat{E}^u_1\cdot e^{-s/r_1})\in\mathcal{R}(\widehat{E}_1)\otimes A^*\big(\mathcal{M}^s(\widehat{\boldsymbol{y}}_2,[L_2],\alpha_*,\delta)\big)[[s^{-1},s]$.

Proposition 7.17 *The following equality holds in the case $p_g=0$:*

$$\Phi(\widehat{\boldsymbol{y}},[\boldsymbol{L}],\alpha_*,\boldsymbol{\delta}_+)-\Phi(\widehat{\boldsymbol{y}},[\boldsymbol{L}],\alpha_*,\boldsymbol{\delta}_-)=\sum_{(\boldsymbol{y}_1,\boldsymbol{y}_2)\in S(\boldsymbol{y},\alpha_*,\boldsymbol{\delta})}\int_{\mathcal{M}(\widehat{\boldsymbol{y}}_1,\widehat{\boldsymbol{y}}_2,[\boldsymbol{L}],\alpha_*,\delta)}\Psi(\boldsymbol{y}_1,\boldsymbol{y}_2)$$

The elements $\Psi(\boldsymbol{y}_1,\boldsymbol{y}_2)\in\mathcal{R}\big(\widehat{E}_1,\mathcal{M}^s(\widehat{\boldsymbol{y}}_1,[L_1],\alpha_*,\delta_1)\big)\otimes\mathcal{R}\big(\widehat{E}_2,\mathcal{M}^s(\widehat{\boldsymbol{y}}_2,[L_2],\alpha_*,\delta_2)\big)$ are given as follows:

$$\Psi(\boldsymbol{y}_1,\boldsymbol{y}_2):=\frac{N_{L_1}(y_1)}{N_{L_1}(y)}\cdot\operatorname*{Res}_{s=0}\left(\frac{P\big(\widehat{E}^u_1\cdot e^{-s/r_1}\oplus\widehat{E}^u_2\cdot e^{s/r_2}\big)\cdot(\omega_2-s/r_2)^k}{Q\big(\widehat{E}^u_1\cdot e^{-s/r_1},\widehat{E}^u_2\cdot e^{s/r_2}\big)\cdot R\big(L_2\cdot e^{-\omega_2+s/r_2},\widehat{E}^u_1\cdot e^{-s/r_1}\big)}\right)\cdot\frac{\mathrm{Eu}(T_{1,\mathrm{rel}})}{N_{L_1}(y_1)}\qquad(209)$$

Here, $T_{1,\mathrm{rel}}$ denotes the bundle on $\mathcal{M}(\widehat{\boldsymbol{y}}_1,\widehat{\boldsymbol{y}}_2,[\boldsymbol{L}],\alpha_*,\delta)$ induced by the relative tangent bundle of the smooth morphism $\mathcal{M}^s(\widehat{\boldsymbol{y}}_1,[L_1],\alpha_*,\delta)\longrightarrow\mathcal{M}(m,\widehat{\boldsymbol{y}}_1)$.

Proof The argument is essentially same as the proof of Theorem 7.15. Applying the same argument as the proof of Theorem 7.12 to $\int_{\widehat{M}}\widetilde{\Phi}_t$ in the proof of Proposition 7.16, we obtain the following expression of $\Phi(\widehat{\boldsymbol{y}},[\boldsymbol{L}],\alpha_*,\boldsymbol{\delta}_+)-\Phi(\widehat{\boldsymbol{y}},[\boldsymbol{L}],\alpha_*,\boldsymbol{\delta}_-)$:

$$\Phi(\widehat{\boldsymbol{y}},[\boldsymbol{L}],\alpha_*,\boldsymbol{\delta}_+)-\Phi(\widehat{\boldsymbol{y}},[\boldsymbol{L}],\alpha_*,\boldsymbol{\delta}_-)=\sum_{\mathfrak{I}\in S(\boldsymbol{y},\alpha_*,\boldsymbol{\delta})}\int_{\widehat{M}^{G_m}(\mathfrak{I})}\operatorname*{Res}_{t=0}\left(\frac{\Phi_t}{\mathrm{Eu}\big(\mathfrak{N}(\widehat{M}^{G_m}(\mathfrak{I}))\big)}\right)$$

We have the G_m-equivariant isomorphism $F^*\varphi^*_{\mathfrak{I}}\mathcal{I}_2^{-1}\simeq G^*\big(\mathcal{O}_{2,\mathrm{rel}}(1)\big)\otimes\mathcal{O}_{1,\mathrm{rel}}(r_1/r_2)\otimes e^{-r_1\cdot t/r_2}$. Therefore, we have the equality $F^*c_1\big(\mathcal{I}_2^{-1}\big)=G^*\big(\omega_2+r_1\cdot(\omega_1-t)/r_2\big)$. Hence we have the following equality in $\mathcal{R}\big(G^*\widehat{E}_1,G^*\widehat{E}_2,\mathcal{S}\big)[t]$:

$$\begin{aligned}F^*\varphi^*_{\mathfrak{I}}(\Phi_t)=\frac{1}{N_{L_1}(y)}\cdot G^*P\big(\widehat{E}^u_1\cdot e^{\omega_1-t}\oplus\widehat{E}^u_2\cdot e^{-r_1(\omega_1-t)/r_2}\big)\cdot G^*\big(\omega_2+r_1(\omega_1-t)/r_2\big)^k\\ \times G^*\,\mathrm{Eu}(T_{1,\mathrm{rel}})\cdot G^*R\big(L_1\cdot e^{-t},\widehat{E}^u_2\cdot e^{-r_1(\omega_1-t)/r_2}\big)\end{aligned}\qquad(210)$$

Here, $R\big(L_1\cdot e^{-t},\widehat{E}^u_2\cdot e^{-r_1(\omega_1-t)/r_2}\big)$ denotes the equivariant Euler class of the following G_m-vector bundle:

$$p_{X*}\mathcal{H}om(L_1,\widehat{E}^u_2)\cdot e^{-r_1\omega_1/r_2+(1+r_2/r_1)t}$$

We also have the following equality in $\mathcal{R}(G^*\widehat{E}^u_1,G^*\widehat{E}^u_2,\mathcal{S})[[t^{-1},t]$:

$$\begin{aligned}F^*\Big(\mathrm{Eu}\big(\mathfrak{N}(\widehat{M}^{G_m}(\mathfrak{I}))\big)\Big)=G^*Q\big(\widehat{E}^u_1\cdot e^{\omega_1-t},\ \widehat{E}^u_2\cdot e^{-r_1(\omega_1-t)/r_2}\big)\\ \times G^*R\big(L_1\otimes e^{-t},\widehat{E}^u_2\cdot e^{-r_1(\omega_1-t)/r_2}\big)\cdot G^*R\big(L_2\otimes e^{-\omega_2-r_1(\omega_1-t)/r_2},\ \widehat{E}^u_1\cdot e^{-r_1(\omega_1-t)/r_2}\big)\end{aligned}\qquad(211)$$

We put $t-\omega_1=\overline{t}$. Therefore, we obtain the following:

$$\int_{\widehat{M}^{G_m}(\mathfrak{I})}\operatorname*{Res}_{t=0}\left(\frac{\Phi_t}{\mathrm{Eu}\big(\mathfrak{N}(\widehat{M}^{G_m}(\mathfrak{I}))\big)}\right)=$$
$$\frac{r_1}{N_{L_1}(y)}\int_{\mathcal{M}(\widehat{\boldsymbol{y}}_1,\widehat{\boldsymbol{y}}_2,[\boldsymbol{L}],\alpha_*,\boldsymbol{\delta})}\operatorname*{Res}_{t=0}\left(\frac{P\big(\widehat{E}_1^u\cdot e^{-\overline{t}}\oplus\widehat{E}_2^u\cdot e^{r_1\overline{t}/r_2}\big)\cdot\big(\omega_2-r_1\overline{t}/r_2\big)^k}{Q\big(E_1^u\cdot e^{-\overline{t}},\widehat{E}_2^u\cdot e^{r_1\overline{t}/r_2}\big)\cdot R\big(L_2\cdot e^{-\omega_2+r_1\overline{t}/r_2},\widehat{E}_1^u\cdot e^{-\overline{t}}\big)}\right)\cdot\mathrm{Eu}(T_{1,\mathrm{rel}}^{(1)})\quad(212)$$

By an argument in the proof of Theorem 7.15, we can replace $\overline{t}$ with t in (212). We put $s=r_1\cdot t$, and then we can show that the right hand side of (212) is same as the integral of $\Psi(\boldsymbol{y}_1,\boldsymbol{y}_2)$ over $\mathcal{M}(\widehat{\boldsymbol{y}}_1,\widehat{\boldsymbol{y}}_2,[\boldsymbol{L}],\alpha_*,\boldsymbol{\delta})$. ▮

Let us consider the integral of the following element of $\mathcal{R}\big(\widehat{E},\mathcal{M}^s(\widehat{\boldsymbol{y}},[\boldsymbol{L}],\alpha_*,\boldsymbol{\delta})\big)$, assuming that the 1-vanishing condition holds for $(\boldsymbol{y},L_2,\alpha_*)$:

$$\Phi=\frac{\mathrm{Eu}(T_{\mathrm{rel}}^{(1)})}{N_{L_1}(y)}\cdot\frac{\mathrm{Eu}(T_{\mathrm{rel}}^{(2)})}{N_{L_2}(y)}\cdot P(\widehat{E}^u)\qquad(213)$$

Lemma 7.18 *Assume that the 1-vanishing condition holds for $(\boldsymbol{y},L_2,\alpha_*)$. Let Φ be as in (213). Then, the element $\Psi(\boldsymbol{y}_1,\boldsymbol{y}_2)\in\mathcal{R}\big(\widehat{E}_1^u,\mathcal{M}^s(\widehat{\boldsymbol{y}}_1,[L_1],\alpha_*,\delta_1)\big)\otimes\mathcal{R}\big(\widehat{E}_2^u,\mathcal{M}^s(\widehat{\boldsymbol{y}}_2,[L_2],\alpha_*,\delta_2)\big)$ is given as follows:*

$$\frac{N_{L_1}(y_1)\cdot N_{L_2}(y_2)}{N_L(y_1)\cdot N_L(y_2)}\cdot\operatorname*{Res}_{s=0}\left(\frac{P\big(\widehat{E}_1^u\cdot e^{-s/r_1}\oplus\widehat{E}_2^u\cdot e^{s/r_2}\big)}{Q\big(\widehat{E}_1^u\cdot e^{-s/r_1},\widehat{E}_2^u\cdot e^{s/r_2}\big)}\right)\cdot\frac{\mathrm{Eu}(T_{1,\mathrm{rel}})}{N_{L_1}(y_1)}\cdot\frac{\mathrm{Eu}(T_{2,\mathrm{rel}})}{N_{L_2}(y_2)}$$

Here $T_{2,\mathrm{rel}}$ denote vector bundle on $\mathcal{M}(\widehat{\boldsymbol{y}}_1,\widehat{\boldsymbol{y}}_2,[\boldsymbol{L}],\alpha_*,\delta)$ obtained from the relative tangent bundle of the smooth morphism $\mathcal{M}^s(\widehat{\boldsymbol{y}}_2,[L_2],\alpha_*,\delta_2)\longrightarrow\mathcal{M}(\widehat{\boldsymbol{y}}_2)$.

Proof We put $\overline{t}=t-\omega_1$. Then, we formally have the following decomposition:

$$F^*\varphi_{\mathfrak{I}}^*\,\mathrm{Eu}(T_{\mathrm{rel}}^{(2)})=G^*\,\mathrm{Eu}(T_{2,\mathrm{rel}})\cdot G^*R\big(L_2\cdot e^{-\omega_2+r_1\cdot\overline{t}/r_2},\widehat{E}_1^u\cdot e^{-\overline{t}}\big)$$

We also have $G^*R\big(L_2\cdot e^{-\omega_2+r_1\cdot\overline{t}/r_2},\widehat{E}_1^u\cdot e^{-\overline{t}}\big)$ in the decomposition (211) of $F^*\big(\mathrm{Eu}\big(\mathfrak{N}(\widehat{M}^{G_m}(\mathfrak{I}))\big)\big)$, which are cancelled out in the evaluation of $\mathcal{S}$. Then, the claim can be easily obtained. ▮

7.3 Invariants

7.3.1 Construction

Let $\boldsymbol{y}$ be an element of $\mathcal{T}ype$, and let α_* be a system of weights. Let P be an element of $\mathcal{R}$. When the 1-stability condition holds for $(\boldsymbol{y},\alpha_*)$, we obtain the number $\int_{\mathcal{M}^s(\widehat{\boldsymbol{y}},\alpha_*)}P(\widehat{E}^u)$. We would like to obtain such a number even in the case where the 1-stability condition does not hold.

We take a line bundle L on X such that the 1-vanishing condition holds for $(\boldsymbol{y},L,\alpha_*)$ in the sense of the subsubsection 6.4.1. Let δ be a sufficiently small element of $\mathcal{P}^{\mathrm{br}}$ such that there are no critical value smaller than δ. Then the 1-stability condition holds for $(\boldsymbol{y},L,\alpha_*,\delta)$. Let T_{rel} denote the relative tangent bundle of $\mathcal{M}^s(\widehat{\boldsymbol{y}},[L],\alpha_*,\delta)\longrightarrow\mathcal{M}(\widehat{\boldsymbol{y}})$. Then, we obtain the following number:

$$\Phi(L):=\int_{\mathcal{M}^s(\widehat{\boldsymbol{y}},[L],\alpha_*,\delta)}P(\widehat{E}^u)\cdot\frac{\mathrm{Eu}(T_{\mathrm{rel}})}{N_L(y)}$$

Proposition 7.19

- *In the case $p_g>0$, the number $\Phi(L)$ is independent of the choice of L.*
- *In general, let L' and L be line bundles on X. We assume that L^{-1} is ample. Then there exists the limit $\lim_{m\to\infty}\Phi(L'\otimes L^m)$, and it is independent of the choice of L' and L.*
- *Assume that the equality $P_{E_1}=P_{E_2}$ holds for any $E_{1\,*}\oplus E_{2\,*}\in\mathcal{M}^{ss}(\boldsymbol{y},\alpha_*)$. Then, $\Phi\big(\mathcal{O}(-m)\big)$ is independent of the choice of any sufficiently large m.*

Proof Let L_1', L_1 and L_2 be line bundles on X. Assume that L_1^{-1} is ample, and that the 1-vanishing condition holds for $(\boldsymbol{y}, L_2, \alpha_*)$. We put $L_1^{(m)} := L_1' \otimes L_1^m$. If we take a sufficiently large integer m, the 1-vanishing condition holds for $(\boldsymbol{y}, L_1^{(m)}, \alpha_*)$. We put $\boldsymbol{L}^{(m)} := (L_1^{(m)}, L_2)$. We consider the following number:

$$g(L_1^{(m)}, L_2, \delta_1, \delta_2) := \int_{\mathcal{M}^s(\widehat{\boldsymbol{y}}, [\boldsymbol{L}^{(m)}], \alpha_*, \boldsymbol{\delta})} P(\widehat{E}^u) \cdot \frac{\mathrm{Eu}(T_{\mathrm{rel}}^{(1)})}{N_{L_1^{(m)}}(y)} \frac{\mathrm{Eu}(T_{\mathrm{rel}}^{(2)})}{N_{L_2}(y)}$$

Here, $T_{\mathrm{rel}}^{(1)}$ denotes the relative tangent bundle of the smooth map $\mathcal{M}^s(\widehat{\boldsymbol{y}}, [\boldsymbol{L}^{(m)}], \alpha_*, \boldsymbol{\delta}) \longrightarrow \mathcal{M}(\widehat{\boldsymbol{y}}, [L_2])$. We use the notation $T_{\mathrm{rel}}^{(2)}$ in a similar meaning. We assume that both of δ_i are sufficiently small. When δ_1 is sufficiently smaller than δ_2, we have the following equality due to Proposition 6.45:

$$g(L_1^{(m)}, L_2, \delta_1, \delta_2) = \Phi(L_2) \tag{214}$$

Similarly, when δ_2 is sufficiently smaller than δ_1, we have the following equality:

$$g(L_1^{(m)}, L_2, \delta_1, \delta_2) = \Phi(L_1^{(m)}) \tag{215}$$

We fix δ_2, and we move δ_1. The transition of the values $g(L_1^{(m)}, L_2, \delta_1, \delta_2)$ occurs if the following holds:

- $\delta_1 = \delta_2 \cdot r_1 / r_2$ holds for some pair of positive integers (r_1, r_2).
- There exists a decomposition $\boldsymbol{y}_1 + \boldsymbol{y}_2 = \boldsymbol{y}$ such that $\operatorname{rank} \boldsymbol{y}_i = r_i$ and $P_{\boldsymbol{y}_1}^{\alpha_*} = P_{\boldsymbol{y}_2}^{\alpha_*}$.

We put as follows:

$$S(\boldsymbol{y}, \alpha_*) := \bigl\{ (\boldsymbol{y}_1, \boldsymbol{y}_2) \bigm| P_{\boldsymbol{y}_1}^{\alpha_*} = P_{\boldsymbol{y}_2}^{\alpha_*}, \quad \boldsymbol{y}_1 + \boldsymbol{y}_2 = \boldsymbol{y} \bigr\}$$

Therefore, we have the expression of $\Phi(L_1^{(m)}) - \Phi(L_2)$ from (214) and (215):

$$\Phi(L_1^{(m)}) - \Phi(L_2) = \sum_{S(\boldsymbol{y}, \alpha_*)} \mathfrak{G}(\boldsymbol{y}_1, \boldsymbol{y}_2) \tag{216}$$

Due to Proposition 7.16, the contributions $\mathfrak{G}(\boldsymbol{y}_1, \boldsymbol{y}_2)$ are trivial in the case $p_g > 0$. Thus we obtain the first claim. Let us show the second claim. We use an induction on $\operatorname{rank}(\boldsymbol{y})$. Due to Proposition 7.17 and Lemma 7.18:

$$\begin{aligned} \mathfrak{G}(\boldsymbol{y}_1, \boldsymbol{y}_2) = & \frac{N_{L_1^{(m)}}(y_1) \cdot N_{L_2}(y_2)}{N_{L_1^{(m)}}(y) \cdot N_{L_2}(y)} \times \\ & \int_{\mathcal{M}(\widehat{\boldsymbol{y}}_1, \widehat{\boldsymbol{y}}_2, [\boldsymbol{L}^{(m)}], \alpha_*, \boldsymbol{\delta}')} \operatorname*{Res}_{s=0} \left(\frac{P(\widehat{E}_1^u \cdot e^{-s/r_1} \oplus \widehat{E}_2^u \cdot e^{s/r_2})}{Q(\widehat{E}_1^u \cdot e^{-s/r_1}, \widehat{E}_2^u \cdot e^{s/r_2})} \right) \cdot \frac{\mathrm{Eu}(T_{1,\mathrm{rel}})}{N_{L_1^{(m)}}(y_1)} \cdot \frac{\mathrm{Eu}(T_{2,\mathrm{rel}})}{N_{L_2}(y_2)} \end{aligned} \tag{217}$$

Here $\boldsymbol{\delta}' = (\delta_1', \delta_2')$ is any element of $\mathcal{P}^{\mathrm{br}\ 2}$ such that δ_i' are sufficiently small. We have $\operatorname{rank}(\boldsymbol{y}_1) < \operatorname{rank}(\boldsymbol{y})$. Recall $\mathcal{M}(\widehat{\boldsymbol{y}}_1, \widehat{\boldsymbol{y}}_2, [\boldsymbol{L}^{(m)}], \alpha_*, \boldsymbol{\delta}) = \mathcal{M}^s(\widehat{\boldsymbol{y}}_1, [L_1^{(m)}], \alpha_*, \delta_1) \times \mathcal{M}^s(\widehat{\boldsymbol{y}}_2, [L_2], \alpha_*, \delta_2)$. By applying the hypothesis of the induction, we have the following limit:

$$\lim_{m \to \infty} \int_{\mathcal{M}(\widehat{\boldsymbol{y}}_1, \widehat{\boldsymbol{y}}_2, [\boldsymbol{L}^{(m)}], \alpha_*, \boldsymbol{\delta}')} \operatorname*{Res}_{s=0} \left(\frac{P(\widehat{E}_1^u \cdot e^{-s/r_1} \oplus \widehat{E}_2^u \cdot e^{s/r_2})}{Q(\widehat{E}_1^u \cdot e^{-s/r_1}, \widehat{E}_2^u \cdot e^{s/r_2})} \right) \cdot \frac{\mathrm{Eu}(T_{1,\mathrm{rel}})}{N_{L_1^{(m)}}(y_1)} \cdot \frac{\mathrm{Eu}(T_{2,\mathrm{rel}})}{N_{L_2}(y_2)}$$

Moreover, it is independent of the choices of L_1 and L_1'. We obviously have the limit:

$$\lim_{m \to \infty} \frac{N_{L_1^{(m)}}(y_1) \cdot N_{L_2}(y_2)}{N_{L_1^{(m)}}(y) \cdot N_{L_2}(y)} = \frac{r_1 \cdot N_{L_2}(y_2)}{r \cdot N_{L_2}(y)}$$

Therefore, we obtain the existence of the limit of the sequence $\bigl\{ \Phi(L_1^{(m)}) \bigr\}$, and it is independent of the choice of L_1. Hence, the second claim is shown.

Let us show the third claim. We use an induction on $\operatorname{rank} \boldsymbol{y}$. We put $L_1' = \mathcal{O}_X$ and $L_1 = \mathcal{O}_X(-1)$. By the hypothesis of the induction, the following terms are independent of the choice of m:

$$\int_{\mathcal{M}(\widehat{\boldsymbol{y}}_1, \widehat{\boldsymbol{y}}_2, [\boldsymbol{L}^{(m)}], \alpha_*, \boldsymbol{\delta}')} \operatorname*{Res}_{s=0}\left(\frac{P(\widehat{E}_1^u \cdot e^{-s/r_1} \oplus \widehat{E}_2^u \cdot e^{s/r_2})}{Q(\widehat{E}_1^u \cdot e^{-s/r_1}, \widehat{E}_2^u \cdot e^{s/r_2})}\right) \cdot \frac{\operatorname{Eu}(T_{1,\mathrm{rel}})}{H_{y_1}(m)} \cdot \frac{\operatorname{Eu}(T_{2,\mathrm{rel}})}{N_{L_2}(y_2)}$$

We also have the following equality, due to the assumption $P_{y_1} = P_y$:

$$\frac{H_{y_1}(m) \cdot N_{L_2}(y_2)}{H_y(m) \cdot N_{L_2}(y)} = \frac{r_1 \cdot N_{L_2}(y_2)}{r \cdot N_{L_2}(y)}$$

Therefore, we obtain the desired independence. ∎

Definition 7.20 *Let P be an element of $\mathcal{R}$. We take a line bundle L such that L^{-1} is ample, and we take a sufficiently small $\delta \in \mathcal{P}^{\mathrm{br}}$. Then, we put as follows:*

$$\int_{\mathcal{M}^{ss}(\widehat{\boldsymbol{y}}, \alpha_*)} P(\widehat{E}^u) := \lim_{m \to \infty} \int_{\mathcal{M}^{s}(\widehat{\boldsymbol{y}}, [L^m], \alpha_*, \delta)} P(\widehat{E}^u) \cdot \frac{\operatorname{Eu}(T_{\mathrm{rel}})}{N_{L^m}(y)} \tag{218}$$

It is well defined due to Proposition 7.19. ∎

Thus, we obtain the linear map $\int_{\mathcal{M}^{ss}(\widehat{\boldsymbol{y}}, \alpha_*)} : \mathcal{R} \longrightarrow \mathbb{Q}$.

7.3.2 Easy properties

Lemma 7.21 *Assume $p_g > 0$. We take a line bundle L such that the* 1*-vanishing condition holds for $(\boldsymbol{y}, L, \alpha_*)$. Then, the following equality holds:*

$$\int_{\mathcal{M}^{ss}(\widehat{\boldsymbol{y}}, \alpha_*)} P(\widehat{E}^u) = \int_{\mathcal{M}^{s}(\widehat{\boldsymbol{y}}, [L], \alpha_*, \delta)} P(\widehat{E}^u) \cdot \frac{\operatorname{Eu}(T_{\mathrm{rel}})}{N_L(y)} \tag{219}$$

Proof It follows from Proposition 7.19. ∎

The following lemma is clear from the construction and Proposition 6.44.

Lemma 7.22 *When the* 1*-stability condition holds for $(\boldsymbol{y}, \alpha_*)$, Definition* 7.20 *is compatible with the ordinary definition.* ∎

Proposition 7.23 *Assume that the equality $P_{E_1} = P_{E_2}$ holds for any $E_{1\,*} \oplus E_{2\,*} \in \mathcal{M}^{ss}(\boldsymbol{y}, \alpha_*)$. Then, we have the following equality, for any sufficiently large m:*

$$\int_{\mathcal{M}^{ss}(\widehat{\boldsymbol{y}}, \alpha_*)} P(\widehat{E}^u) = \int_{\mathcal{M}^{s}(\widehat{\boldsymbol{y}}, [\mathcal{O}(-m)], \alpha_*, \delta)} P(\widehat{E}^u) \cdot \frac{\operatorname{Eu}(T_{\mathrm{rel}})}{H_y(m)}$$

Proof It immediately follows from the third claim of Proposition 7.19. ∎

We say a system of weights α_* is not critical, if $\mathcal{M}^{ss}(\boldsymbol{y}, \alpha_*) = \mathcal{M}^{ss}(\boldsymbol{y}, \alpha'_*)$ for any α'_* such that $|\alpha_i - \alpha'_i|$ are sufficiently small.

Corollary 7.24 *Assume one of the following:*

- *α_* is not critical.*
- *The parabolic part of $\boldsymbol{y}$ is trivial.*

Then, we have the following equality for any sufficiently large m:

$$\int_{\mathcal{M}^{ss}(\widehat{\boldsymbol{y}}, \alpha_*)} P(\widehat{E}^u) = \int_{\mathcal{M}^{s}(\widehat{\boldsymbol{y}}, [\mathcal{O}(-m)], \alpha_*, \delta)} P(\widehat{E}^u) \cdot \frac{\operatorname{Eu}(T_{\mathrm{rel}})}{H_y(m)}$$

Proof We have only to check the condition in Proposition 7.23. In the case where the parabolic part of $\boldsymbol{y}$ is trivial, the condition is trivial. Let us show the second claim. Assume that α_* is not critical, and take $E_{1\,*} \oplus E_{2\,*} \in \mathcal{M}^{ss}(\boldsymbol{y}, \alpha_*)$. Let $\boldsymbol{y}_i$ be the types of $E_{i\,*}$. Then we have $P^{\alpha'_*}_{\boldsymbol{y}_1} = P^{\alpha'_*}_{\boldsymbol{y}_2}$ for any α'_* which are sufficiently close to α_*. It implies $P_{y_1} = P_{y_2}$, i.e., $P_{E_1} = P_{E_2}$. Thus we are done. ∎

7.3.3 The integral over $\widetilde{\mathcal{M}}^{ss}(\widehat{\boldsymbol{y}},\alpha_*,+)$

Let $\widetilde{\mathcal{M}}^{ss}(\widehat{\boldsymbol{y}},\alpha_*,+)$ be as in the subsubsection 4.6.1. Let $\widetilde{T}_{\mathrm{rel}}$ denote the relative tangent bundle of the smooth morphism $\widetilde{\mathcal{M}}^{ss}(\widehat{\boldsymbol{y}},\alpha_*,+)\longrightarrow\mathcal{M}(\widehat{\boldsymbol{y}})$. Recall the description of $\widetilde{\mathcal{M}}^{ss}(\widehat{\boldsymbol{y}},\alpha_*,+)$ as the full flag bundle over $\mathcal{M}^s(\widehat{\boldsymbol{y}},[\mathcal{O}(-m)],\alpha_*,\epsilon)$ for any sufficiently small positive number ϵ. We also remark the equality of the virtual fundamental classes in Lemma 6.54. Then, we can easily derive the following equality:

$$\int_{\widetilde{\mathcal{M}}^{ss}(\widehat{\boldsymbol{y}},\alpha_*,+)}P(\widehat{E}^u)\cdot\frac{\mathrm{Eu}(\widetilde{T}_{\mathrm{rel}})}{H_y(m)!}=\int_{\mathcal{M}^s(\widehat{\boldsymbol{y}},[\mathcal{O}(-m)],\alpha_*,\epsilon)}P(\widehat{E}^u)\cdot\frac{\mathrm{Eu}(T_{\mathrm{rel}})}{H_y(m)}\tag{220}$$

Here T_{rel} denotes the relative tangent bundle of the smooth morphism $\mathcal{M}^s(\widehat{\boldsymbol{y}},[\mathcal{O}(-m)],\alpha_*,\epsilon)\longrightarrow\mathcal{M}(m,\widehat{\boldsymbol{y}})$.

Lemma 7.25 *Assume one of the following:*

- $p_g>0$
- *The condition in Proposition* 7.23 *is satisfied.*

Then, we have the following equality:

$$\int_{\widetilde{\mathcal{M}}^{ss}(\widehat{\boldsymbol{y}},\alpha_*,+)}P(\widehat{E}^u)\cdot\frac{\mathrm{Eu}(\widetilde{T}_{\mathrm{rel}})}{H_y(m)!}=\int_{\mathcal{M}^{ss}(\widehat{\boldsymbol{y}},\alpha_*)}P(\widehat{E}^u)\tag{221}$$

In particular, if one of the conditions in Corollary 7.24 *is satisfied, the equality* (221) *holds.* ∎

7.3.4 Another expression

In the case $p_g>0$, we have another way to express the integral (218).

Lemma 7.26 *Assume that the* 2*-vanishing condition holds for* $(\boldsymbol{y},L,\alpha_*)$*. We also assume* $p_g>0$ *and* $d:=\chi\big(y\cdot\mathrm{ch}(L)^{-1}\big)-1\geq 0$*. Let* P *be an element of* $\mathcal{R}$*. The following equality holds, for any sufficiently small* $\delta\in\mathcal{P}^{\mathrm{br}}$*:*

$$\int_{\mathcal{M}^s(\widehat{\boldsymbol{y}},[L],\alpha_*,\delta)}P(\widehat{E}^u)\cdot\omega^d=\int_{\mathcal{M}^{ss}(\widehat{\boldsymbol{y}},\alpha_*)}P(\widehat{E}^u)$$

Proof We use the argument in the proof of Proposition 7.19. We also use the notation in the subsubsection 7.2.3. We take a line bundle L_1 on X such that the 1-vanishing condition holds for $(\boldsymbol{y},L_1,\alpha_*)$. The pair (L_1,L) is denoted by $\boldsymbol{L}$. We put as follows:

$$\overline{g}(L_1,L,\delta_1,\delta):=\int_{\mathcal{M}^s(\widehat{\boldsymbol{y}},[\boldsymbol{L}],\alpha_*,\boldsymbol{\delta})}P(\widehat{E}^u)\cdot\frac{\mathrm{Eu}(T^{(1)}_{\mathrm{rel}})}{N_{L_1}(y)}\cdot\omega^{(2)\,d}\tag{222}$$

When δ_1 is sufficiently smaller than δ, we have the following:

$$\overline{g}(L_1,L,\delta_1,\delta)=\int_{\mathcal{M}^s(\widehat{\boldsymbol{y}},[L],\alpha_*,\delta)}P(\widehat{E}^u)\cdot\omega^{(2)\,d}$$

When δ is sufficiently smaller than δ_1, we have the following equality, due to Corollary 6.47:

$$\overline{g}(L_1,L,\delta_1,\delta)=\int_{\mathcal{M}^{ss}(\widehat{\boldsymbol{y}},\alpha_*)}P(\widehat{E}^u)$$

We move δ_1, and then the transitions are trivial due to Proposition 7.16. Thus, we obtain the desired equality (222). ∎

We recall that the 2-vanishing condition can be controlled numerically, in general. We give it for later use.

Lemma 7.27 *Let* y *be an element of* $\mathcal{T}ype^{\circ}$*. Assume* $P_y(t)>P_K(t)$ *for any sufficiently large* t*, where* K *denotes the canonical line bundle of* X*. Then, the* 2*-vanishing condition holds for* $(y,\mathcal{O})$*.*

Proof Take $E_*\in\mathcal{M}^{ss}(y)$. If $H^2(X,E)\neq 0$, we have a non-trivial morphism $E\longrightarrow K_X$. It implies $P_y(t)\leq P_K(t)$, which contradicts the assumption. ∎

7.4 Rank 2 Case

7.4.1 Reduction to the integrals over the products of Hilbert schemes

In this subsubsection, we assume $H^1(X,\mathcal{O})=0$ for simplicity. Let y be an element of $\mathcal{T}ype_2^{\circ}$. In the following, the $H^2(X)$-part of y is denoted by a, and the $H^4(X)$-part of y is denoted by b. The second Chern class corresponding to y is denoted by n. We have the relation $b=a^2/2-n$. We assume $P_y>P_K$ and $\chi(y)-1\geq 0$. We would like to give the expression of $\int_{\mathcal{M}(\widehat{y})}P(\widehat{E}^u)$ as the sum of the integrals over the products of Hilbert schemes for any $P\in\mathcal{R}$.

Let $NS^1(X)$ denote the subgroup of $H^2(X,\mathbb{Z})$ generated by the 1-cycles on X. For any element $a_1\in NS^1(X)$, we put $a_2:=a-a_1$. Let e^{a_i} denote the holomorphic line bundle whose first Chern class is a_i. Since we have assumed $H^1(X,\mathcal{O})=0$, it is uniquely determined up to isomorphisms. Let $\mathcal{I}_i^u$ denote the universal ideal sheaves over $X^{[n_i]}\times X$. Let $\mathcal{Z}_i$ denote the universal 0-scheme over $X^{[n_i]}\times X$. Let Ξ_i denote $p_{X\,*}\big(\mathcal{O}_{\mathcal{Z}_i}\otimes e^{a_i}\big)$. We use the same notation to denote the pull back of them via appropriate morphisms. Let G_m denote the one dimensional torus. Let $e^{w\cdot s}$ denote the trivial line bundle with the G_m-action of weight w. Then, we have the following element of the K-groups of G_m-equivariant sheaves on $X^{[n_1]}\times X^{[n_2]}$

$$-Rp_{X\,*}\mathcal{RH}om\big(\mathcal{I}_1^u\cdot e^{a_1-s},\,\mathcal{I}_2^u\cdot e^{a_2+s}\big)-Rp_{X\,*}\mathcal{RH}om\big(\mathcal{I}_2^u\cdot e^{a_2+s},\,\mathcal{I}_1^u\cdot e^{a_1-s}\big)$$

The equivariant Euler class is denoted by $Q\big(\mathcal{I}_1^u\cdot e^{a_1-s},\mathcal{I}_2^u\cdot e^{a_2+s}\big)\in\mathcal{R}(\mathcal{I}_1^u\cdot e^{a_1})\otimes\mathcal{R}(\mathcal{I}_2^u\cdot e^{a_2})[[s^{-1},s]$. We have the element $P\big(\mathcal{I}_1^u\cdot e^{a_1-s},\,\mathcal{I}_2^u\cdot e^{a_2+s}\big)\in\mathcal{R}(\mathcal{I}_1^u\cdot e^{a_1})\otimes\mathcal{R}(\mathcal{I}_2^u\cdot e^{a_2})[s]$, which is induced as explained in the subsubsection 7.1.3. We also have the equivariant Euler class $\mathrm{Eu}(\Xi_2\cdot e^{2s})\in A^*_{G_m}(X^{[n_1]}\times X^{[n_2]})$. Thus, we obtain the following element of $\mathcal{R}(\mathcal{I}_1^u\cdot e^{a_1})\otimes\mathcal{R}(\mathcal{I}_2^u\cdot e^{a_2})$:

$$\operatorname*{Res}_{s=0}\left(\frac{P\big(\mathcal{I}_1^u\cdot e^{a_1-s}\oplus\mathcal{I}_2^u\cdot e^{a_2+s}\big)}{Q\big(\mathcal{I}_1^u\cdot e^{a_1-s},\mathcal{I}_2^u\cdot e^{a_2+s}\big)}\cdot\frac{\mathrm{Eu}(\Xi_1)\cdot\mathrm{Eu}(\Xi_2\cdot e^{2s})}{(2s)^{n_1+n_2-p_g}}\right)$$

In the case $\big(c_1(\mathcal{O}(1)),a_1\big)<\big(c_1(\mathcal{O}(1)),a_2\big)$, we put as follows:

$$\mathcal{A}(a_1,y):=\sum_{n_1+n_2=n-a_1\cdot a_2}\int_{X^{[n_1]}\times X^{[n_2]}}\operatorname*{Res}_{s=0}\left(\frac{P\big(\mathcal{I}_1^u\cdot e^{a_1-s}\oplus\mathcal{I}_2^u\cdot e^{a_2+s}\big)}{Q\big(\mathcal{I}_1^u\cdot e^{a_1-s},\mathcal{I}_2^u\cdot e^{a_2+s}\big)}\cdot\frac{\mathrm{Eu}(\Xi_1)\cdot\mathrm{Eu}(\Xi_2\cdot e^{2s})}{(2s)^{n_1+n_2-p_g}}\right)$$

In the case $\big(c_1(\mathcal{O}(1)),a_1\big)=\big(c_1(\mathcal{O}(1)),a_2\big)$, we put as follows:

$$\mathcal{A}(a_1,y):=\sum_{\substack{n_1+n_2=n-a_1\cdot a_2\\ n_1>n_2}}\int_{X^{[n_1]}\times X^{[n_2]}}\operatorname*{Res}_{s=0}\left(\frac{P\big(\mathcal{I}_1^u\cdot e^{a_1-s}\oplus\mathcal{I}_2^u\cdot e^{a_2+s}\big)}{Q\big(\mathcal{I}_1^u\cdot e^{a_1-s},\mathcal{I}_2^u\cdot e^{a_2+s}\big)}\cdot\frac{\mathrm{Eu}(\Xi_1)\cdot\mathrm{Eu}(\Xi_2\cdot e^{2s})}{(2s)^{n_1+n_2-p_g}}\right)$$

We put as follows:

$$\mathcal{SW}(X,y):=\big\{a_1\in NS^1(X)\,\big|\,\big[\mathcal{M}(e^{a_1},\mathcal{O})\big]\neq 0,\ \big(a_1,c_1(\mathcal{O}_X(1))\big)\leq\big(a,c_1(\mathcal{O}_X(1))\big)/2\big\}$$

Recall that the expected dimension of $\mathcal{M}(e^{a_1},\mathcal{O})$ is 0, if $[\mathcal{M}(e^{a_1},\mathcal{O})]\neq 0$ (Proposition 6.29). Therefore, we can regard $[\mathcal{M}(e^{a_1},\mathcal{O})]$ as the number, and we denote it by $\mathrm{SW}(a_1)$.

Theorem 7.28 *Assume $p_g>0$ and $H^1(X,\mathcal{O})=0$. Assume $P_y>P_K$ and $\chi(y)-1\geq 0$. We have the following equality:*

$$\int_{\mathcal{M}^{ss}(\widehat{y})}P(\widehat{E}^u)+\sum_{a_1\in\mathcal{SW}(X,y)}\mathrm{SW}(a_1)\cdot 2^{1-\chi(y)}\cdot\mathcal{A}(a_1,y)=0.$$

Proof Let δ be an element of $\mathcal{P}^{\mathrm{br}}$ which is not critical. Let ω denote the first Chern class of the relative tautological line bundle on $\mathcal{M}^s(\widehat{y},[\mathcal{O}],\delta)$. Then we put as follows:

$$\Phi(\delta):=\int_{\mathcal{M}^s(\widehat{y},[\mathcal{O}],\delta)}P(\widehat{E}^u)\cdot\omega^d$$

When $\delta \in \mathcal{P}^{\mathrm{br}}$ is critical, we put as follows:

$$S(y,\delta) := \big\{(y_1,y_2) \in \mathcal{T}ype \,\big|\, y_1+y_2=y,\ H_{y_1}+\delta = H_{y_2}\big\}$$

We take parameters $\delta_- < \delta < \delta_+$ which are sufficiently close to δ. Let E_1^u denote the universal sheaf over $\mathcal{M}(y_1,\mathcal{O}) \times X$, and $\widehat{E}_1^u$ denote the universal sheaf over $\mathcal{M}(\widehat{y}_1,[\mathcal{O}]) \times X$. Recall we have the isomorphism $\mathcal{M}(y_1,\mathcal{O}) \simeq \mathcal{M}(\widehat{y}_1,[\mathcal{O}])$, although it does not preserve the virtual fundamental class. We identify the moduli spaces via the isomorphism. Then, we have the relation $E_1^u = \widehat{E}_1^u \cdot e^{\omega_1}$, where ω_1 denotes the first Chern class of the relative tautological line bundle of $\mathcal{M}(y_1,\mathcal{O}) \simeq \mathcal{M}(\widehat{y}_1,[\mathcal{O}])$. Due to Theorem 7.12, we have the following equality:

$$\Phi(\delta_+) - \Phi(\delta_-) = \sum_{(y_1,y_2)\in S(y,\delta)} \int_{\mathcal{M}(y_1,\mathcal{O})\times\mathcal{M}(\widehat{y}_2)} \operatorname*{Res}_{s=0}\left(\frac{P\big(\widehat{E}_1^u\cdot e^{\omega_1-s}\oplus \widehat{E}_2^u\cdot e^{s-\omega_1}\big)\cdot s^d}{Q\big(\widehat{E}_1^u\cdot e^{\omega_1-s},\ \widehat{E}_2^u\cdot e^{s-\omega_1}\big)\cdot \mathrm{Eu}\big(Rp_{X\,*}\widehat{E}_2^u\cdot e^{2s-\omega_1}\big)}\right) \tag{223}$$

We put as follows:

$$S(y) = \big\{(y_1,y_2)\in \mathcal{T}ype^2 \,\big|\, y_1+y_2=y,\ P_{y_1} < P_{y_2}\big\}$$

Recall $\mathcal{M}^{ss}(\widehat{y},[\mathcal{O}],\delta)=\emptyset$ for any sufficiently large δ (Proposition 3.38). On the other hand, we have $\Phi(\delta) = \int_{\mathcal{M}(\widehat{y})} P(\widehat{E}^u)$ for any sufficiently small δ (Lemma 7.26). Therefore, we obtain the following equality:

$$\int_{\mathcal{M}(\widehat{y})} P(\widehat{E}^u) + \sum_{(y_1,y_2)\in S(y)} \int_{\mathcal{M}(y_1,\mathcal{O})\times\mathcal{M}(\widehat{y}_2)} \operatorname*{Res}_{s=0}\left(\frac{P\big(\widehat{E}_1^u\cdot e^{\omega_1-s}\oplus \widehat{E}_2^u\cdot e^{s-\omega_1}\big)\cdot s^d}{Q\big(\widehat{E}_1^u\cdot e^{\omega_1-s},\ \widehat{E}_2^u\cdot e^{s-\omega_1}\big)\cdot \mathrm{Eu}\big(Rp_{X\,*}\widehat{E}_2^u\cdot e^{2s-\omega_1}\big)}\right) = 0 \tag{224}$$

If $[\mathcal{M}(y_1,\mathcal{O})]\neq 0$, the expected dimension of $\mathcal{M}(e^{a_1},\mathcal{O})$ is 0 (Proposition 6.29). Hence, we can omit ω_1 in the right hand side of (224). By using Proposition 6.34 and $\widehat{E}_i^u = \mathcal{I}_i^u\cdot e^{a_i}$, we obtain the following:

$$\begin{aligned}
&\int_{\mathcal{M}(y_1,\mathcal{O})\times\mathcal{M}(\widehat{y}_2)} \operatorname*{Res}_{s=0}\left(\frac{P\big(\widehat{E}_1^u\cdot e^{\omega_1-s}\oplus \widehat{E}_2^u\cdot e^{s-\omega_1}\big)\cdot s^d}{Q\big(\widehat{E}_1^u\cdot e^{\omega_1-s},\ \widehat{E}_2^u\cdot e^{s-\omega_1}\big)\cdot \mathrm{Eu}\big(Rp_{X\,*}\widehat{E}_2^u\cdot e^{2s-\omega_1}\big)}\right) = \\
&\qquad \mathrm{SW}(a_1)\cdot \int_{X^{[n_1]}\times X^{[n_2]}} \operatorname*{Res}_{s=0}\left(\frac{P\big(\mathcal{I}_1^u\cdot e^{a_1-s}\oplus \mathcal{I}_2^u\cdot e^{a_2+s}\big)\cdot s^d}{Q\big(\mathcal{I}_1^u\cdot e^{a_1-s},\ \mathcal{I}_2\cdot e^{a_2+s}\big)\cdot \mathrm{Eu}\big(Rp_{X\,*}(\mathcal{I}_2^u\cdot e^{a_2+2s})\big)}\right)\cdot \mathrm{Eu}(\Xi_1)
\end{aligned} \tag{225}$$

By a formal calculation, we obtain the following:

$$\frac{1}{\mathrm{Eu}\big(Rp_{X\,*}(\mathcal{I}_2^u\cdot e^{a_2+2s})\big)} = \frac{\mathrm{Eu}(\Xi_2\cdot e^{2s})}{(2s)^{\chi(a_2)}}$$

Since $[\mathcal{M}(e^{a_1},\mathcal{O})]\neq 0$, we have $\chi(a_1) = 1+p_g$ (Proposition 6.29). Hence, we have the following:

$$\chi(a_2) = \chi(y_2)+n_2 = \chi(y)-\chi(y_1)+n_2 = \chi(y)-1-p_g+n_1+n_2.$$

Then, the desired equality can be obtained by a direct calculation. ∎

7.4.2 Dependence on the polarization

Let y be an element of $\mathcal{T}ype_2^{\circ}$. We use a similar convention as in the subsubsection 7.4.1. To distinguish the dependence on the polarization H, we use the notation $\mathcal{M}_H(\widehat{y})$ to denote the moduli stack of torsion-free sheaves of type y which are semistable with respect to H. Let ξ be an element of $NS(X)$ such that $\xi+a$ is divisible by 2 in $NS(X)$. We also assume $a^2-4n \le \xi^2 < 0$. We put $W^{\xi} := \big\{c\in NS(X)\otimes\mathbb{R}\,\big|\,(c,\xi)=0\big\}$, which is called the wall determined by ξ. It is well known that the ample cone is divided into the chambers by such walls, and the moduli $\mathcal{M}_H(\widehat{y})$ depends only on the chambers to which H belongs. For $\Phi := P(\widehat{E}^u)\in \mathcal{F}\big(\mathcal{M}_H(\widehat{y})\big)$, we put as follows:

$$\Phi_H(\widehat{y}) := \int_{\mathcal{M}_H(\widehat{y})} \Phi$$

We would like to discuss how $\Phi_H(\widehat{y})$ changes when the polarizations vary across the wall W^ξ. We put as follows:

$$S(y,\xi)=\big\{(y_0,y_1)\in \mathcal{T}ype(X)^2\,\big|\, y_0+y_1=y,\ a_0-a_1=m\cdot\xi\ (m>0)\text{ in }H^2(X)\big\}$$

For each $(y_0,y_1)\in S(y,\xi)$, we put $\mathcal{M}(\widehat{y}_0,\widehat{y}_1):=\mathcal{M}(\widehat{y}_0)\times\mathcal{M}(\widehat{y}_1)$. Let $\widehat{E}_i$ denote the sheaf over $\mathcal{M}(\widehat{y}_0,\widehat{y}_1)\times X$ which is the pull back of the universal sheaves over $\mathcal{M}(\widehat{y}_i)\times X$ via the appropriate projection. Let $e^{w\cdot s}$ denote the trivial line bundle with the G_m-action of weight w. We have the following element of the K-group of the G_m-equivariant sheaves on $\mathcal{M}(\widehat{y}_0,\widehat{y}_1)$:

$$-Rp_{X\,*}\mathcal{RH}om\big(\widehat{E}_0\cdot e^{-s},\ \widehat{E}_1\cdot e^{s}\big)-Rp_{X\,*}\mathcal{RH}om\big(\widehat{E}_1\cdot e^{s},\ \widehat{E}_0\cdot e^{-s}\big)$$

The equivariant Euler class is denoted by $Q\big(\widehat{E}_0\cdot e^{-s},\widehat{E}_1\cdot e^{s}\big)\in\mathcal{R}(\widehat{E}_0)\otimes\mathcal{R}(\widehat{E}_1)[[s^{-1},s]$. By the homomorphisms in the subsubsection 7.1.3, we have $P\big(\widehat{E}_0\cdot e^{-s}\oplus\widehat{E}_1\cdot e^{s}\big)\in\mathcal{R}(\widehat{E}_0)\otimes\mathcal{R}(\widehat{E}_1)[s]$.

Theorem 7.29 *Let C_+ and C_- be chambers which are divided by the wall W^ξ. Let H_+ and H_- be ample line bundles contained in C_+ and C_-, respectively. We assume $(H_-,\xi)<0<(H_+,\xi)$.*

- *In the case $p_g>0$, we have $\Phi_{H_+}(\widehat{y})=\Phi_{H_-}(\widehat{y})$. Namely, the invariant does not depend on the choice of generic polarization.*
- *In the case $p_g=0$, we have the following equality:*

$$\Phi_{H_+}(\widehat{y})-\Phi_{H_-}(\widehat{y})=\sum_{(y_0,y_1)\in S(y,\xi)}\int_{\mathcal{M}(\widehat{y}_0,\widehat{y}_1)}\operatorname*{Res}_{s=0}\left(\frac{P\big(\widehat{E}_0^u\cdot e^{-s}\oplus\widehat{E}_1^u\cdot e^{s}\big)}{Q\big(\widehat{E}_0\cdot e^{-s},\ \widehat{E}_1\cdot e^{s}\big)}\right)\tag{226}$$

We give two arguments to prove Theorem 7.29. Both of them are based on the following observation.

Lemma 7.30 ([10], [42]) *Let H be an ample line bundle contained in W^ξ. We can take $H_\kappa\in C_\kappa$ $(\kappa=\pm)$ satisfying the following:*

- *There exists a very ample curve C such that $H_+=H\otimes\mathcal{O}(C)$ and $H_-=H\otimes\mathcal{O}(-C)$.*
- *The following holds for a torsion-free sheaf E of type y:*
  - *E is H_+-semistable if and only if $E(C)$ is H-semistable.*
  - *E is H_- -semistable if and only if $E(-C)$ is H-semistable.* ∎

Ellingsrud and Göttsche proved the formula (226) for the Donaldson invariant under the assumption that the wall W^ξ is *good*. They used the parabolic structure $E(-C)\subset E(C)$ with weight α for torsion-free sheaves E. Let $\mathcal{M}^{ss}(\widehat{y},\alpha)$ denote the moduli stack of torsion-free sheaves with the parabolic structure as above, which are semistable with respect to the polarization H. Due to Lemma 7.30, we have $\mathcal{M}^{ss}(\widehat{y},1)=\mathcal{M}_{H_+}(\widehat{y})$ and $\mathcal{M}^{ss}(\widehat{y},0)=\mathcal{M}_{H_-}(\widehat{y})$. We say that α is critical, if $\mathcal{M}^{ss}(\widehat{y},\alpha)\neq\mathcal{M}^{ss}(\widehat{y},\alpha')$ for any sufficiently close $\alpha'\neq\alpha$. We can easily show that there are only finitely many critical values, by using some boundedness result. By investigating the transition of the invariants at critical parabolic weights, they obtained the formula (226). Using their framework and our transition formula (204), we will show the formula (226) without the assumption that the wall is good, in the subsubsection 7.4.3.

We will give another argument for the proof of Theorem 7.29 in the subsubsection 7.4.4. Perhaps, it may be a little more suitable when we discuss a similar problem in the higher rank case.

We give some preliminary for the argument. In the rest of this subsection, we use the polarization H. Namely, the μ-semistability condition and the semistability condition are considered with respect to H. Let $\mathcal{S}$ denote the family of μ-semistable torsion-free sheaves of type y. Let $\overline{\mathcal{S}}$ denote the family of torsion-free sheaves E' of rank one with the following property:

- $\mu(E')=\mu(y)$.
- There is a member E of $\mathcal{S}$, such that E' is a saturated subsheaf of E.

The families $\mathcal{S}$ and $\overline{\mathcal{S}}$ are bounded. We take a sufficiently large integer m such that the family $\mathcal{S}$ satisfies the condition O_m. In the rest of this subsection, δ denotes a polynomial of degree 0.

7.4.3 The proof of Theorem 7.29 (I)

For a torsion-free sheaf E, we denote by $F^{(\alpha)}$ the parabolic structure $\bigl(E(-C)\subset E(C),\alpha\bigr)$. The following lemma is clear.

Lemma 7.31 *If $(E,F^{(\alpha)},\phi)$ is a δ-semistable parabolic L-Bradlow pair, E is μ-semistable. (Recall δ is assumed to be a polynomial of degree 0.)* ∎

Let $\mathcal{M}^{ss}\bigl(\widehat{y},[L],\alpha,\delta\bigr)$ denote the moduli stack of the oriented parabolic reduced L-Bradlow pairs of type y with weight α, where the parabolic structure is given as above.

Let α be any real number, and let E' be any member of $\overline{\mathcal{S}}$. We take $E\in\mathcal{S}$ such that E' is a saturated subsheaf of E. We put $E'':=E/E'$. Then, we put $F_\alpha(E'):=P^{\alpha}_{E'}-P^{\alpha}_{E''}$. The number is determined by $\mathrm{ch}(E')$, and hence it is independent of the choice of E. When we fix α, the function $F_\alpha:\overline{\mathcal{S}}\longrightarrow\mathbb{R}$ has finitely many values, due to the boundedness of $\overline{\mathcal{S}}$. It is clear that α is critical if and only if there exists a member $E'\in\overline{\mathcal{S}}$ such that $F_\alpha(E')=0$.

Let α_0 be critical, and let ϵ be any sufficiently small positive number. We can take a small positive number $\eta>0$ such that the following holds for any α' with $|\alpha'-\alpha_0|<\eta$:

- $F_{\alpha_0}(E')>0\Longleftrightarrow F_{\alpha'}(E')>\epsilon$
- $F_{\alpha_0}(E')<0\Longleftrightarrow F_{\alpha'}(E')<-\epsilon$
- $F_{\alpha_0}(E')=0\Longleftrightarrow |F_{\alpha'}(E)|<\epsilon$

Lemma 7.32 *We have $\mathcal{M}^{ss}\bigl(\widehat{y},[L],\alpha_0,\epsilon\bigr)\simeq\mathcal{M}^{ss}\bigl(\widehat{y},[L],\alpha',\epsilon\bigr)$ for any α' such that $|\alpha_0-\alpha'|<\eta$. Moreover, we have $\mathcal{M}^{ss}(\widehat{y},[L],\alpha_0,\epsilon)=\mathcal{M}^{s}(\widehat{y},[L],\alpha_0,\epsilon)$.*

Proof Let $(E,\rho,F^{(\alpha')},[\phi])$ be an oriented parabolic reduced L-Bradlow pair, such that E is μ-semistable. Let $E'\subset E$ be a member of $\overline{\mathcal{S}}$. We put $E'':=E/E'$. We have $P^{\alpha_0}_{E'}+\epsilon<P^{\alpha_0}_{E''}$ if and only if $P^{\alpha'}_{E'}+\epsilon<P^{\alpha'}_{E''}$. We have $P^{\alpha_0}_{E'}<P^{\alpha_0}_{E''}+\epsilon$ if and only if $P^{\alpha'}_{E'}<P^{\alpha'}_{E''}+\epsilon$. Then, the first claim of the lemma is clear. The second claim is also easy to see. ∎

To compare $\Phi_{H_+}(\widehat{y})$ and $\Phi_{H_-}(\widehat{y})$, we would like to consider the invariants obtained from the moduli stacks $\mathcal{M}^s\bigl(\widehat{y},[\mathcal{O}(-m)],\alpha,\delta\bigr)$. We remark that the divisor of the parabolic structure is not reduced in this case, contrast to that we assumed the smoothness of the divisor in the section 5. We can deal with the point by the following two arguments:

1. We do not have to think the contribution of the parabolic structure to the obstruction theory, because the filtration is canonically determined by the sheaf. In fact, $\mathcal{M}^s(\widehat{y},[L],\alpha,\delta)$ is an open substack of $\mathcal{M}(m,\widehat{y},[L])$ for a sufficiently large m, in this case. Thus we obtain the perfect obstruction theory $\mathrm{Ob}(m,\widehat{y},[L])$ and the virtual fundamental class.
2. In the case $\alpha>1/2$, we consider the parabolic structure $F_1^{(\alpha)}$ given by $(E\subset E(C),2\alpha-1)$ for a torsion-free sheaf E. Then, $(E,F^{(\alpha)},[\phi])$ is δ-semistable if and only if $(E,F_1^{(\alpha)},[\phi])$ is δ-semistable. In the case $\alpha<1/2$, we consider the parabolic structure $F_2^{(\alpha)}$ given by $(E(-C)\subset E,2\alpha)$ for a torsion-free sheaf E. Then, $(E,F^{(\alpha)},[\phi])$ is δ-semistable if and only if $(E,F_1^{(\alpha)},[\phi])$ is δ-semistable.

Therefore, we can freely apply our previous results.

Let T_{rel} denote the relative tangent bundle of the smooth morphism $\mathcal{M}(\widehat{y},[\mathcal{O}(-m)],\alpha,\delta)\longrightarrow\mathcal{M}(m,\widehat{y})$. If δ is not critical with respect to $(y,\mathcal{O}(-m),\alpha)$, we obtain the following number:

$$\widetilde{\Phi}(\alpha,\delta):=\int_{\mathcal{M}^s(\widehat{y},[\mathcal{O}(-m)],\alpha,\delta)}\widetilde{\Phi}.$$

Here, we put as follows:

$$\widetilde{\Phi}:=\Phi\cdot\frac{\mathrm{Eu}(T_{\mathrm{rel}})}{H_y(m)}\in\mathcal{R}\bigl(\widehat{E},\mathcal{M}(\widehat{y},[\mathcal{O}(-m)],\alpha,\delta)\bigr)$$

The following lemma is the special case of Corollary 7.24.

Lemma 7.33 *Assume that α is not critical. Then, $\widetilde{\Phi}(\alpha,\delta)$ is independent of the choice of m, if δ is sufficiently small. The number is denoted by $\Phi(\alpha)$.* ∎

We have $\Phi(1)=\Phi_{H_+}(y)$ and $\Phi(0)=\Phi_{H_-}(y)$. Therefore, we have only to see the transition of the invariants at critical weights. For a critical α, we put as follows:

$$S(y,\xi,\alpha):=\bigl\{(y_0,y_1)\in S(y,\xi)\,\big|\,P^{\alpha}_{y_0}=P^{\alpha}_{y_1}\bigr\}$$

Proposition 7.34 *We take real numbers $\alpha_+>\alpha>\alpha_-$ such that $|\alpha_\kappa-\alpha|<\eta$. In the case $p_g>0$, we have $\Phi(\alpha_+)-\Phi(\alpha_-)=0$. In the case $p_g=0$, we have the following formula:*

$$\Phi(\alpha_+)-\Phi(\alpha_-)=\sum_{(y_0,y_1)\in S(y,\xi,\alpha)}\int_{\mathcal{M}(\widehat{y}_0,\widehat{y}_1)}\operatorname*{Res}_{s=0}\left(\frac{P\bigl(\widehat{E}^u_0\cdot e^{-s}\oplus\widehat{E}^u_1\cdot e^{s}\bigr)}{Q(\widehat{E}^u_0\cdot e^{-s},\widehat{E}^u_1\cdot e^{s})}\right)\tag{227}$$

Proof We have $\mathcal{M}^s(\widehat{y},[\mathcal{O}(-m)],\alpha_+,\epsilon)\simeq\mathcal{M}^s(\widehat{y},[\mathcal{O}(-m)],\alpha_-,\epsilon)$ due to Lemma 7.32. Therefore, we obtain the following:

$$\Phi(\alpha_+)-\Phi(\alpha_-)=\bigl(\Phi(\alpha_-,\epsilon)-\Phi(\alpha_-)\bigr)-\bigl(\Phi(\alpha_+,\epsilon)-\Phi(\alpha_+)\bigr)\tag{228}$$

Let us see the first term in the right hand side of (228). We see the transition of $\Phi(\alpha_+,\delta)$ when we move δ from 0 to ϵ. We use Theorem 7.15. The transition occurs when $P^{\alpha-}_{y_0}+\delta=P^{\alpha-}_{y_1}$ holds for some $(y_0,y_1)\in S(y,\xi,\alpha)$. In the case $p_g>0$, the transitions are trivial. Hence we obtain $\Phi(\alpha_-,\epsilon)-\Phi(\alpha_-)=0$. For any $(y_0,y_1)\in S(y,\xi,\alpha)$, we put $\mathcal{M}(\widehat{y}_0,\widehat{y}_1,[\mathcal{O}(-m)]):=\mathcal{M}\bigl(\widehat{y}_0,[\mathcal{O}(-m)]\bigr)\times\mathcal{M}\bigl(\widehat{y}_1\bigr)$. In the case $p_g=0$, we obtain the following equality:

$$\begin{aligned}\Phi(\alpha_-,\epsilon)-\Phi(\alpha_-)&=\sum_{(y_0,y_1)\in S(y,\xi,\alpha)}\frac{H_{y_0}(m)}{H_y(m)}\cdot\int_{\mathcal{M}(\widehat{y}_0,\widehat{y}_1,[\mathcal{O}(-m)])}\operatorname*{Res}_{s=0}\left(\frac{P\bigl(\widehat{E}^u_0\cdot e^{-s}\oplus\widehat{E}^u_1\cdot e^{s}\bigr)}{Q(\widehat{E}^u_0\cdot e^{-s},\widehat{E}^u_1\cdot e^{s})}\right)\cdot\frac{\mathrm{Eu}(T_{0,\mathrm{rel}})}{H_{y_0}(m)}\\&=\sum_{(y_0,y_1)\in S(y,\xi,\alpha)}\frac{H_{y_0}(m)}{H_y(m)}\cdot\int_{\mathcal{M}(\widehat{y}_0,\widehat{y}_1)}\operatorname*{Res}_{s=0}\left(\frac{P\bigl(\widehat{E}^u_0\cdot e^{-s}\oplus\widehat{E}^u_1\cdot e^{s}\bigr)}{Q(\widehat{E}^u_0\cdot e^{-s},\widehat{E}^u_1\cdot e^{s})}\right)\end{aligned}\tag{229}$$

Similarly, we have $\Phi(\alpha_+,\epsilon)-\Phi(\alpha_+)=0$ in the case $p_g>0$, and we have the following equality in the case $p_g=0$:

$$\begin{aligned}\Phi(\alpha_+,\epsilon)-\Phi(\alpha_+)&=\sum_{(y_0,y_1)\in S(y,\xi,\alpha)}\frac{H_{y_1}(m)}{H_y(m)}\cdot\int_{\mathcal{M}(\widehat{y}_1,\widehat{y}_0)}\operatorname*{Res}_{s=0}\left(\frac{P\bigl(\widehat{E}^u_1\cdot e^{-s}\oplus\widehat{E}^u_0\cdot e^{s}\bigr)}{Q(\widehat{E}^u_1\cdot e^{-s},\widehat{E}^u_0\cdot e^{s})}\right)\\&=\sum_{(y_0,y_1)\in S(y,\xi,\alpha)}-\frac{H_{y_1}(m)}{H_y(m)}\cdot\int_{\mathcal{M}(\widehat{y}_0,\widehat{y}_1)}\operatorname*{Res}_{s=0}\left(\frac{P\bigl(\widehat{E}^u_0\cdot e^{-s}\oplus\widehat{E}^u_1\cdot e^{s}\bigr)}{Q(\widehat{E}^u_0\cdot e^{-s},\widehat{E}^u_1\cdot e^{s})}\right)\end{aligned}\tag{230}$$

Therefore, we obtain the following:

$$\begin{aligned}\Phi(\alpha_+)-\Phi(\alpha_-)&=\sum_{(y_0,y_1)\in S(y,\xi,\alpha)}\left(\frac{H_{y_0}(m)}{H_y(m)}+\frac{H_{y_1}(m)}{H_y(m)}\right)\cdot\int_{\mathcal{M}(\widehat{y}_0,\widehat{y}_1)}\operatorname*{Res}_{s=0}\left(\frac{P\bigl(\widehat{E}^u_0\cdot e^{-s}\oplus\widehat{E}^u_1\cdot e^{s}\bigr)}{Q(\widehat{E}^u_0\cdot e^{-s},\widehat{E}^u_1\cdot e^{s})}\right)\\&=\sum_{(y_0,y_1)\in S(y,\xi,\alpha)}\int_{\mathcal{M}(\widehat{y}_0,\widehat{y}_1)}\operatorname*{Res}_{s=0}\left(\frac{P\bigl(\widehat{E}^u_0\cdot e^{-s}\oplus\widehat{E}^u_1\cdot e^{s}\bigr)}{Q(\widehat{E}^u_0\cdot e^{-s},\widehat{E}^u_1\cdot e^{s})}\right)\end{aligned}\tag{231}$$

Thus the proof of Proposition 7.34 is finished. ∎

The first claim of Theorem 7.29 obviously follows from the first claim of Proposition 7.34. We have $S(y,\xi)=\bigcup_{0<\alpha<1}S(y,\xi,\alpha)$ since the intersection pairing (C,ξ) is sufficiently large. Then, the formula (226) immediately follows from (227). Thus, the first proof of Theorem 7.29 is finished. ∎

7.4.4 The proof of Theorem 7.29 (II)

We put $y(C) := y \cdot \mathrm{ch}(\mathcal{O}(C))$. We use the notation $y(-C)$ in a similar meaning. We regard them as the element of $\mathcal{T}ype$ whose parabolic parts are trivial. We put $\mathcal{O}(-m,C) := \mathcal{O}(-m)\otimes\mathcal{O}(C)$ and $\mathcal{O}(-m,-C) := \mathcal{O}(-m)\otimes\mathcal{O}(-C)$. Let $(E,[\phi])$ be a reduced $\mathcal{O}(-m)$-Bradlow pair such that $\phi \neq 0$. We naturally obtain the reduced $\mathcal{O}(-m,C)$-Bradlow pair $(E(C),[\phi_C])$. Similarly, we obtain the reduced $\mathcal{O}(-m,-C)$-Bradlow pair $(E(-C),[\phi_{-C}])$. Let T_{rel} denote the relative tangent bundle of the smooth map $\mathcal{M}^s\bigl(\widehat{y}(C),[\mathcal{O}(-m,C)],\delta\bigr) \longrightarrow \mathcal{M}^s(m,\widehat{y}(C))$. When δ is not critical, we put as follows:

$$\widetilde{\Phi}_C(\delta) := \int_{\mathcal{M}^s(\widehat{y}(C),[\mathcal{O}(-m,C)],\delta)} \Phi\cdot\frac{\mathrm{Eu}(T_{\mathrm{rel}})}{H_y(m)}, \qquad \widetilde{\Phi}_{-C}(\delta) := \int_{\mathcal{M}^s(\widehat{y}(-C),[\mathcal{O}(-m,-C)],\delta)} \Phi\cdot\frac{\mathrm{Eu}(T_{\mathrm{rel}})}{H_y(m)}. \tag{232}$$

When δ is sufficiently small, we have $\widetilde{\Phi}_C(\delta)=\Phi_{H_+}(y)$ and $\widetilde{\Phi}_C(\delta)=\Phi_{H_-}(y)$.

Let $\mathcal{T}ype(\overline{\mathcal{S}})$ denote the set $\bigl\{\mathrm{ch}(E')\in H^*(X)\,\big|\,E'\in\overline{\mathcal{S}}\bigr\}$. For any $y_0\in\overline{\mathcal{S}}$, we put $y_1 := y-y_0$, and then y_1 is also an element of $\mathcal{T}ype(\overline{\mathcal{S}})$. We put $y_i(C) := y_i\cdot\mathrm{ch}\bigl(\mathcal{O}(C)\bigr)$. We use the notation $y_i(-C)$ in a similar meaning. We remark that $P_{y_0(C)}-P_{y_1(C)}$ and $P_{y_0(-C)}-P_{y_1(-C)}$ are the polynomials of degree 0.

Let a_i, b_i and n_i denote the first Chern class, the second Chern character and the second Chern class corresponding to y_i. We have $(a_0,H)=(a_1,H)$, and H is a generic element of W^{ξ}. Therefore, we have $a_0-a_1 = A\cdot\xi$ for some $A\in\mathbb{Q}$ in $H^2(X)$. Since (C,ξ) is assumed to be sufficiently large, we have $P_{y_0(C)}-P_{y_1(C)}\neq 0$ unless $y_0=y_1$. We also have $P_{y_0(-C)}-P_{y_1(-C)}\neq 0$ unless $y_0=y_1$.

We put as follows:

$$S(y,C) := \bigl\{(y_0,y_1)\,\big|\,y_0+y_1=y,\ y_i\in\mathcal{T}ype(\overline{\mathcal{S}}),\ P_{y_0(C)}<P_{y_1(C)}\bigr\}$$

$$S(y,-C) := \bigl\{(y_0,y_1)\,\big|\,y_0+y_1=y,\ y_i\in\mathcal{T}ype(\overline{\mathcal{S}}),\ P_{y_0(-C)}<P_{y_1(-C)}\bigr\}.$$

We take a positive constant δ_0 satisfying the following inequalities:

$$\delta_0>\max\bigl\{|P_{y_0(C)}-P_{y_1(C)}|\,\big|\,(y_0,y_1)\in S(y,C)\bigr\},\quad \delta_0>\max\bigl\{|P_{y_0(-C)}-P_{y_1(-C)}|\,\big|\,(y_0,y_1)\in S(y,-C)\bigr\}.$$

Lemma 7.35 *$(E(C),\phi_C)$ is δ_0-semistable, if and only if the following conditions hold:*

- *E is μ-semistable.*
- *For any subobject $(E',\phi')\subset(E,\phi)$ such that $\phi'\neq 0$, we have $\mu(E')<\mu(E)$.*

Moreover, $(E(C),\phi_C)$ is δ_0-stable, if it is δ_0-semistable.

Proof Assume that $(E(C),\phi_C)$ is δ_0-semistable. Since δ_0 is a polynomial of degree 0, it is easy to see that $E(C)$ is μ-semistable. Hence, the first condition holds. Let $(E',\phi')\subset(E,\phi)$ be a subobject such that $\phi'\neq 0$. We put $E''=E/E'$. Assume $\mu(E')=\mu(E)$. Then E' is a member of $\overline{\mathcal{S}}$, and hence we have $|P_{E'(C)}(t)-P_{E''(C)}(t)|<\delta_0$, due to our choice of δ_0. Therefore, we obtain $P^{\delta_0}_{E'(C)}(t)>P_{E''(C)}(t)$, which contradicts the δ_0-semistability of $\bigl(E(C),\phi_C\bigr)$. Hence the second condition holds.

Let us assume that the two conditions are satisfied. Let $(E',\phi')\subset(E,\phi)$ be a subobject such that $\phi'\neq 0$. Since we have $\mu(E')<\mu(E)$, the inequality $P^{\delta_0}_{(E',\phi')}(t)<P^{\delta}_{(E,\phi)}$ holds for any sufficiently large t. Take a subobject $E'\subset(E,\phi)$. We have the inequality $\mu(E')\leq\mu(E)$. When the strict inequality holds, we obviously have $P^{\delta_0}_{(E',\phi')}(t)<P^{\delta}_{(E,\phi)}$ for any sufficiently large t. Assume $\mu(E')=\mu(E)$. Then E' is a member of $\overline{\mathcal{S}}$. We put $E''=E/E'$. Then, we have $P_{E'(C)}(t)<P^{\delta_0}_{E''(C)}$ due to our choice of δ_0. Thus we obtain the semistability of $(E(C),\phi_C)$.

From the above argument, we also obtain that $P_{(E'(C),\phi'_C)}=P_{(E(C),\phi_C)}$ cannot hold. Therefore, we obtain the second claim. ∎

Lemma 7.36 *$(E(C),\phi_C)$ is δ_0-semistable, if and only if $(E(-C),\phi_{-C})$ is δ_0-semistable. Moreover, the* 1*-stability condition holds for $(y(C),\mathcal{O}(-m,C),\delta_0)$ and $(y(-C),\mathcal{O}(-m,-C),\delta_0)$.*

Proof By the same argument as the proof of Lemma 7.35, we can show that $(E(-C), \phi_{-C})$ is δ_0-semistable if and only if the two conditions in Lemma 7.35 hold. Then the first claim immediately follows. The second claim can be shown similarly. ∎

Due to Lemma 7.36, we obtain the equality $\widetilde{\Phi}_C(\delta_0) = \widetilde{\Phi}_{-C}(\delta_0)$. Therefore, we obtain the following equality:

$$\Phi_{H_+}(\widehat{y}) - \Phi_{H_-}(\widehat{y}) = \bigl(\widetilde{\Phi}_{-C}(\delta_0) - \Phi_{H_-}(\widehat{y})\bigr) - \bigl(\widetilde{\Phi}_C(\delta_0) - \Phi_{H_+}(\widehat{y})\bigr) \tag{233}$$

Let us see the first term in the right hand side of (233). We see the transition of $\widetilde{\Phi}_{-C}(\delta)$ when we move δ from 0 to δ_0. The transition occurs when $P_{y_0(-C)} + \delta = P_{y_1(-C)}$ holds for some $(y_0, y_1) \in S(y, -C)$. In the case $p_g > 0$, the transitions are trivial. Hence we obtain $\widetilde{\Phi}_{-C}(\delta_0) - \Phi_{H_-}(\widehat{y}) = 0$. In the case $p_g = 0$, we obtain the following equality, as in the equality (229):

$$\widetilde{\Phi}_{-C}(\delta_0) - \Phi_{H_-}(\widehat{y}) = \sum_{(y_0,y_1)\in S(y,-C)} \frac{H_{y_0}(m)}{H_y(m)} \cdot \int_{\mathcal{M}(\widehat{y}_0,\widehat{y}_1)} \operatorname*{Res}_{s=0} \left(\frac{P(\widehat{E}_0^u \cdot e^{-s} \oplus \widehat{E}_1^u \cdot e^s)}{Q(\widehat{E}_0^u \cdot e^{-s}, \widehat{E}_1^u \cdot e^s)} \right) \tag{234}$$

Similarly, we obtain $\widetilde{\Phi}_C(\delta_0) - \Phi_{H_+}(\widehat{y}) = 0$ in the case $p_g > 0$, and we have the following equality in the case $p_g = 0$.

$$\widetilde{\Phi}_C(\delta_0) - \Phi_{H_+}(\widehat{y}) = \sum_{(y_0,y_1)\in S(y,C)} \frac{H_{y_0}(m)}{H_y(m)} \cdot \int_{\mathcal{M}(\widehat{y}_0,\widehat{y}_1)} \operatorname*{Res}_{s=0} \left(\frac{P(\widehat{E}_0^u \cdot e^{-s} \oplus \widehat{E}_1^u \cdot e^s)}{Q(\widehat{E}_0^u \cdot e^{-s}, \widehat{E}_1^u \cdot e^s)} \right) \tag{235}$$

Now, we have already obtained $\Phi_{H_+}(\widehat{y}) - \Phi_{H_-}(\widehat{y}) = 0$ in the case $p_g > 0$. Namely the first claim of Theorem 7.29 is proved. To show the claim in the case $p_g = 0$, we see the sets $S(y, C)$ and $S(y, -C)$ more closely. We put as follows:

$$S_1 := \bigl\{(y_0, y_1) \bigm| y_0 + y_1 = y,\ y_i \in \mathcal{T}ype(\overline{\mathcal{S}}),\ r_0 = r_1 = 1,\ a_0 = a_1,\ b_0 < b_1\bigr\}$$

We also put $S'(y, \xi) := \bigl\{(y_0, y_1) \bigm| (y_1, y_0) \in S(y, \xi)\bigr\}$. Then, it is easy to observe $S(y, -C) = S(y, \xi) \sqcup S_1$ and $S(y, C) = S'(y, \xi) \sqcup S_1$. We remark the equality (230). Therefore, we obtain the following equalities:

$$\begin{aligned}
\Phi_{H_+}(\widehat{y}) - \Phi_{H_-}(\widehat{y}) = &\sum_{(y_0,y_1)\in S(y,\xi)} \frac{H_{y_0}(m)}{H_y(m)} \cdot \int_{\mathcal{M}(\widehat{y}_0,\widehat{y}_1)} \operatorname*{Res}_{s=0} \left(\frac{P(\widehat{E}_0^u \cdot e^{-s} \oplus \widehat{E}_1^u \cdot e^s)}{Q(\widehat{E}_0^u \cdot e^{-s}, \widehat{E}_1^u \cdot e^s)} \right) \\
&+ \sum_{(y_0,y_1)\in S_1} \frac{H_{y_0}(m)}{H_y(m)} \cdot \int_{\mathcal{M}(\widehat{y}_0,\widehat{y}_1)} \operatorname*{Res}_{s=0} \left(\frac{P(\widehat{E}_0^u \cdot e^{-s} \oplus \widehat{E}_1^u \cdot e^s)}{Q(\widehat{E}_0^u \cdot e^{-s}, \widehat{E}_1^u \cdot e^s)} \right) \\
&+ \sum_{(y_0,y_1)\in S(y,\xi)} \frac{H_{y_1}(m)}{H_y(m)} \cdot \int_{\mathcal{M}(\widehat{y}_0,\widehat{y}_1)} \operatorname*{Res}_{s=0} \left(\frac{P(\widehat{E}_0^u \cdot e^{-s} \oplus \widehat{E}_1^u \cdot e^s)}{Q(\widehat{E}_0^u \cdot e^{-s}, \widehat{E}_1^u \cdot e^s)} \right) \\
&- \sum_{(y_0,y_1)\in S_1} \frac{H_{y_0}(m)}{H_y(m)} \cdot \int_{\mathcal{M}(\widehat{y}_0,\widehat{y}_1)} \operatorname*{Res}_{s=0} \left(\frac{P(\widehat{E}_0^u \cdot e^{-s} \oplus \widehat{E}_1^u \cdot e^s)}{Q(\widehat{E}_0^u \cdot e^{-s}, \widehat{E}_1^u \cdot e^s)} \right) \\
&= \sum_{(y_0,y_1)\in S(y,\xi)} \int_{\mathcal{M}(\widehat{y}_0,\widehat{y}_1)} \operatorname*{Res}_{s=0} \left(\frac{P(\widehat{E}_0^u \cdot e^{-s} \oplus \widehat{E}_1^u \cdot e^s)}{Q(\widehat{E}_0^u \cdot e^{-s}, \widehat{E}_1^u \cdot e^s)} \right)
\end{aligned} \tag{236}$$

Thus we are done. ∎

7.5 Higher Rank Case ($p_g > 0$)

7.5.1 Transition formula in the case $p_g > 0$

Let $\boldsymbol{y}$ be an element of $\mathcal{T}ype$, and let α_* be a system of weights. Let L be a line bundle on X. Let $\delta \in \mathcal{P}^{\mathrm{br}}$ be a non-critical parameter. We denote by ω the first Chern class of $\mathcal{O}_{\mathrm{rel}}(1)$ on $\mathcal{M}^s(\widehat{\boldsymbol{y}}, [L], \alpha_*, \delta)$. Let P be an element of $\mathcal{R}$. When the 1-stability condition holds for $(\boldsymbol{y}, L, \alpha_*, \delta)$, we have $P(\widehat{E}^u)\cdot\omega^k \in \mathcal{R}\big(\widehat{E}^u, \mathcal{M}^s(\widehat{\boldsymbol{y}}, [L], \alpha_*, \delta)\big)$, and we put as follows:

$$\Phi(\widehat{\boldsymbol{y}}, [L], \alpha_*, \delta) := \int_{\mathcal{M}^s(\widehat{\boldsymbol{y}},[L],\alpha_*,\delta)} P(\widehat{E}^u)\cdot\omega^k$$

Let us discuss the transition formula, when the 2-stability condition does not hold for $(\boldsymbol{y}, L, \alpha_*, \delta)$. In the case $p_g > 0$, the problem is simpler. Actually, it can be shown that the same formula in Theorem 7.12 holds. We use the notation in the subsubsection 7.2.1. We put as follows:

$$S_1(\boldsymbol{y}, \alpha_*, \delta) := \big\{(\boldsymbol{y}_1, \boldsymbol{y}_2) \in S(\boldsymbol{y}, \alpha_*, \delta) \,\big|\, \operatorname{rank} \boldsymbol{y}_1 = 1\big\}$$

For $(\boldsymbol{y}_1, \boldsymbol{y}_2) \in S_1(\boldsymbol{y}, \alpha_*, \delta)$, we have $\mathcal{M}(\boldsymbol{y}_1, L) = \mathcal{M}^s(\boldsymbol{y}_1, L, \alpha_*, \delta)$.

Theorem 7.37 *The following equality holds:*

$$\Phi(\widehat{\boldsymbol{y}}, [L], \alpha_*, \delta_+) - \Phi(\widehat{\boldsymbol{y}}, [L], \alpha_*, \delta_-) = \sum_{(\boldsymbol{y}_1,\boldsymbol{y}_2)\in S_1(\boldsymbol{y},\alpha_*,\delta)} \int_{\mathcal{M}(\boldsymbol{y}_1,\widehat{\boldsymbol{y}}_2,L,\alpha_*,\delta)} \Psi(\boldsymbol{y}_1, \boldsymbol{y}_2) \tag{237}$$

The elements $\Psi(\boldsymbol{y}_1, \boldsymbol{y}_2) \in \mathcal{R}\big(E_1^u, \mathcal{M}(\boldsymbol{y}_1, L)\big) \otimes \mathcal{R}(\widehat{E}_2^u)$ *are given as in* (198)*:*

$$\Psi(\boldsymbol{y}_1, \boldsymbol{y}_2) = \operatorname*{Res}_{t=0}\left(\frac{P\big(E_1^u\cdot e^{-t} \oplus \widehat{E}_2^u\cdot e^{(t-\omega_1)/(r-1)}\big)\cdot t^k}{\operatorname{Eu}\big(\mathfrak{N}_0(\boldsymbol{y}_1, \boldsymbol{y}_2)\big)}\right)$$

Here, we put $\omega_1 := c_1\big(\mathcal{O}r(E_1^u)\big)$.

Proof We put $\widetilde{\mathcal{M}}(\delta_\kappa) := \widetilde{\mathcal{M}}^{ss}(\widehat{\boldsymbol{y}}, [L], \alpha_*, \delta_\kappa)$. Let $\widetilde{T}_{\mathrm{rel}}$ denote the relative tangent bundle of the smooth morphism of $\widetilde{\mathcal{M}}(m, \widehat{\boldsymbol{y}}, [L])$ to $\mathcal{M}(m, \widehat{\boldsymbol{y}}, [L])$. We have the open embedding $\widetilde{\mathcal{M}}(\delta, \kappa) \subset \widetilde{\mathcal{M}}(m, \widehat{\boldsymbol{y}}, [L])$. The restriction of $\widetilde{T}_{\mathrm{rel}}$ is denoted by the same notation.

Let $\widehat{M}$ be the master space connecting $\widetilde{\mathcal{M}}(\delta_+)$ and $\widetilde{\mathcal{M}}(\delta_-)$ constructed in the subsubsection 4.5.1. Let $\varphi : \widehat{M} \longrightarrow \widetilde{\mathcal{M}}(m, \widehat{\boldsymbol{y}}, [L])$ denote the naturally defined morphism. Let $\mathcal{T}(1)$ denote the trivial line bundle on $\widetilde{\mathcal{M}}(m, \widehat{\boldsymbol{y}}, [L])$ with the G_m-action of weight 1. We consider the following element of $\mathcal{R}_{G_m}\big(\widehat{E}^{\widehat{M}}, \widehat{M}\big)$:

$$\widetilde{\Phi}_t := P\big(\varphi^*\widehat{E}^u\big)\cdot c_1\big(\varphi^*\mathcal{O}_{\mathrm{rel}}(1)\big)^k \cdot \frac{\operatorname{Eu}(\widetilde{T}_{\mathrm{rel}})}{H_y(m)!}, \qquad \overline{\Phi}_t := \widetilde{\Phi}_t \cdot c_1\big(\varphi^*\mathcal{T}(1)\big).$$

We use Proposition 5.51. Then, we obtain the polynomial $\int_{\widehat{M}} \overline{\Phi}_t$ of t. When we forget the G_m-action, we have $c_1\big(\varphi^*\mathcal{T}(1)\big) = 0$. Hence we have $\int_{\widehat{M}} \overline{\Phi}_{t|t=0} = 0$. On the other hand, we have the following equality in $\mathbb{Q}[t^{-1}, t]$, due to the localization of the virtual fundamental classes ([24]):

$$\int_{\widehat{M}} \overline{\Phi}_t = \sum_{i=1,2} \int_{\widehat{M}_i} \frac{\overline{\Phi}_t}{\operatorname{Eu}\big(\mathfrak{N}(\widehat{M}_i)\big)} + \sum_{\mathfrak{I}\in \operatorname{Dec}(m,\boldsymbol{y},\alpha_*,\delta)} \int_{\widehat{M}^{G_m}(\mathfrak{I})} \frac{\overline{\Phi}_t}{\operatorname{Eu}\big(\mathfrak{N}(\widehat{M}^{G_m}(\mathfrak{I}))\big)}$$

Here, $\operatorname{Dec}(m, \boldsymbol{y}, \alpha_*, \delta)$ denotes the set of the decomposition types (Definition 4.33), and $\mathfrak{N}(\widehat{M}_i)$ and $\mathfrak{N}(\widehat{M}^{G_m}(\mathfrak{I}))$ denote the virtual normal bundles with the G_m-action given in Proposition 5.52 and Proposition 5.53. We have $c_1\big(\varphi^*\mathcal{T}(1)\big)_{|\widehat{M}_i} = t$ and $c_1\big(\varphi^*\mathcal{T}(1)\big)_{|\widehat{M}^{G_m}(\mathfrak{I})} = t$. Therefore, we obtain the following equality:

$$\sum_{i=1,2} \int_{\widehat{M}_i} \operatorname*{Res}_{t=0}\left(\frac{\widetilde{\Phi}_t}{\operatorname{Eu}\big(\mathfrak{N}(\widehat{M}_i)\big)}\right) + \sum_{\mathfrak{I}} \int_{\widehat{M}^{G_m}(\mathfrak{I})} \operatorname*{Res}_{t=0}\left(\frac{\widetilde{\Phi}_t}{\operatorname{Eu}\big(\mathfrak{N}(\widehat{M}^{G_m}(\mathfrak{I}))\big)}\right) = 0. \tag{238}$$

As in the proof of Theorem 7.12, the first term of the left hand side of (238) can be rewritten as follows:

$$-\int_{\widehat{M}_1} \widetilde{\Phi} + \int_{\widehat{M}_2} \widetilde{\Phi} = -\int_{\widetilde{\mathcal{M}}(\delta_+)} \widetilde{\Phi} + \int_{\widetilde{\mathcal{M}}(\delta_-)} \widetilde{\Phi} \tag{239}$$

Recall $\widetilde{\mathcal{M}}(\delta_\kappa)$ is the full flag bundle over $\mathcal{M}^{ss}\big(\widehat{\boldsymbol{y}},[L],\alpha_*,\delta_\kappa\big)$, and we have the equality of the virtual fundamental classes as in Lemma 6.53. Hence, we obtain the following equality:

$$\sum_{i=1,2}\int_{\widehat{M}_i}\operatorname*{Res}_{t=0}\left(\frac{\widetilde{\Phi}_t}{\operatorname{Eu}\big(\mathfrak{N}(\widehat{M}_i)\big)}\right)=-\Phi(\widehat{\boldsymbol{y}},[L],\alpha_*,\delta_+)+\Phi(\widehat{\boldsymbol{y}},[L],\alpha_*,\delta_-)\tag{240}$$

The contributions from $\widehat{M}^{G_m}(\mathfrak{I})$ can be calculated by the arguments in the proof of Theorem 7.12 and Theorem 7.15. For any decomposition type $\mathfrak{I}=(\boldsymbol{y}_1,\boldsymbol{y}_2,I_1,I_2)\in\operatorname{Dec}(m,\boldsymbol{y},\alpha_*,\delta)$, we put as follows:

$$\widetilde{\mathcal{M}}(\mathfrak{I}):=\widetilde{\mathcal{M}}^{ss}\big(\boldsymbol{y}_1,L,\alpha_*,\delta,\mathfrak{k}(\mathfrak{I})\big)\times\widetilde{\mathcal{M}}^{ss}(\widehat{\boldsymbol{y}}_2,\alpha_*,+)$$

Let $\widetilde{T}_{1,\mathrm{rel}}$ denote the vector bundle over $\widetilde{\mathcal{M}}(\mathfrak{I})$ induced by the relative tangent bundle of the smooth map $\widetilde{\mathcal{M}}^{ss}(\boldsymbol{y}_1,L,\alpha_*,\delta,\mathfrak{k}(\mathfrak{I}))\longrightarrow\mathcal{M}(m,\boldsymbol{y}_1,L)$. Let $\widetilde{T}_{2,\mathrm{rel}}$ denote the vector bundle over $\widetilde{\mathcal{M}}(\mathfrak{I})$ induced by the relative tangent bundle of the smooth map $\widetilde{\mathcal{M}}^{ss}(\widehat{\boldsymbol{y}}_2,\alpha_*,+)\longrightarrow\mathcal{M}(m,\widehat{\boldsymbol{y}}_2)$. Let N_0 be as in the subsubsection 5.9.1. Then, we have the following decomposition of the vector bundles:

$$\varphi_{\mathfrak{I}}^*\widetilde{T}_{\mathrm{rel}}=\widetilde{T}_{1,\mathrm{rel}}\oplus\widetilde{T}_{2,\mathrm{rel}}\oplus N_0.$$

We remark that $\varphi^*\mathcal{O}_{\mathrm{rel}}(1)_{|\widehat{M}^*}$ and $\varphi^*\mathcal{T}(1)_{|\widehat{M}^*}$ are naturally isomorphic as G_m-equivariant line bundles. We use the relation of $E_i^{\widehat{M}}$, E_1^u and $\widehat{E}_2^u$ in Corollary 4.46. Then, we have the following equality in $\mathcal{R}(G'^*E_1^u,G'^*\widehat{E}_2^u,\mathcal{S})[t]$:

$$F^*\widetilde{\Phi}_t=\frac{H_{y_1}(m)!H_{y_2}(m)!}{H_y(m)!}\cdot G'^*P\big(E_1^u\cdot e^{-t}\oplus\widehat{E}_2^u\cdot e^{r_1(t-\omega_1)/r_2}\big)\cdot t^k\cdot\frac{\operatorname{Eu}(\widetilde{T}_{1,\mathrm{rel}})}{H_{y_1}(m)!}\frac{\operatorname{Eu}(\widetilde{T}_{2,\mathrm{rel}})}{H_{y_2}(m)!}\cdot\operatorname{Eu}(N_0)$$

We also have the following equality of the equivariant Euler classes in $A^*(\mathcal{S})[[t^{-1},t]$:

$$F^*\operatorname{Eu}\big(\mathfrak{N}(\widehat{M}^{G_m}(\mathfrak{I}))\big)=G'^*\operatorname{Eu}\big(\mathfrak{N}_0(\boldsymbol{y}_1,\boldsymbol{y}_2)\big)\cdot\operatorname{Eu}(N_0)$$

Recall that we have the equality of the virtual fundamental classes in Proposition 6.10. Therefore, the contribution from $\widehat{M}^{G_m}(\mathfrak{I})$ is as follows:

$$\frac{H_{y_1}(m)!\cdot H_{y_2}(m)!}{H_y(m)!}\int_{\widetilde{\mathcal{M}}(\mathfrak{I})}\operatorname*{Res}_{t=0}\left(\frac{P\big(E_1^u\cdot e^{-t}\oplus\widehat{E}_2^u\cdot e^{r_1(t-\omega_1)/r_2}\big)\cdot t^k}{\operatorname{Eu}\big(\mathfrak{N}_0(\boldsymbol{y}_1,\boldsymbol{y}_2)\big)}\right)\frac{\operatorname{Eu}(\widetilde{T}_{1,\mathrm{rel}})}{H_{y_1}(m)!}\frac{\operatorname{Eu}(\widetilde{T}_{2,\mathrm{rel}})}{H_{y_2}(m)!}\tag{241}$$

We remark that the virtual fundamental class of $\widetilde{\mathcal{M}}(\mathfrak{I})$ is 0, and hence (241) vanishes, if the conditions $\operatorname{rank}(\boldsymbol{y}_1)>1$ and $p_g>0$ are satisfied (Proposition 6.24). In the case $\operatorname{rank}(\boldsymbol{y}_1)=1$, the (δ,ℓ)-semistability condition is trivial. We also remark that the integrand of (241) is the element of $\mathcal{R}\big(E_1^u,\widetilde{\mathcal{M}}^{ss}(\boldsymbol{y}_1,L,\alpha_*,\delta,\mathfrak{k}(\mathfrak{I}))\big)\otimes\mathcal{R}\big(\widehat{E}_2^u,\widetilde{\mathcal{M}}^{ss}(\widehat{\boldsymbol{y}}_2,\alpha_*,+)\big)$, and hence we have only to consider the component-wise integration. By using Lemma 7.25, we can rewrite (241) as follows:

$$\frac{H_{y_1}(m)!\cdot H_{y_2}(m)!}{H_y(m)!}\int_{\mathcal{M}(\boldsymbol{y}_1,\widehat{\boldsymbol{y}}_2,L,\alpha_*,\delta)}\operatorname*{Res}_{t=0}\left(\frac{P\big(E_1^u\cdot e^{-t}\oplus\widehat{E}_2^u\cdot e^{(t-\omega_1)/(r-1)}\big)\cdot t^k}{\operatorname{Eu}\big(\mathfrak{N}_0(\boldsymbol{y}_1,\boldsymbol{y}_2)\big)}\right)\tag{242}$$

The number of the decompositions (I_1,I_2) of $\{1,\dots,H_y(m)\}$ satisfying $|I_i|=H_{y_i}(m)$ is $H_y(m)!\cdot H_{y_1}(m)!^{-1}\cdot H_{y_2}(m)!^{-1}$. Therefore, the second term in the left hand side of (238) is same as the left hand side of (237). Thus, we obtain the desired formula. ∎

7.5.2 Reduction to the integrals over the products of Hilbert schemes

We assume $p_g>0$ and $\dim H^1(X,\mathcal{O})=0$. Let y be an element of $\mathcal{T}ype_r^{\circ}$. Assume $P_y(t)>P_K(t)$ for any sufficiently large t, where K denotes the canonical line bundle of X. We also assume $\chi(y)-1\geq(r-1)\cdot(1+p_g)$. We give a straightforward generalization of Theorem 7.28. For a given element $y_i\in\mathcal{T}ype$, we use the notation

r_i, a_i, b_i, and n_i to denote the rank, the first Chern class, the second Chern character and the second Chern class, in the following argument.

We put as follows:

$$T_i = \sum_{j<i} \frac{t_j}{r-j} - t_i, \ \ (i = 1, \dots, r-1), \quad T_r = \sum_{j<r} \frac{t_j}{r-j}$$

We put as follows:

$$S(y) = \Big\{ (y_1, y_2, \dots, y_r) \in \mathcal{T}ype_1^{\circ\, r} \ \Big| \ \sum y_i = y, \ H_{y_i} < (r-i)^{-1} \sum_{j>i} H_{y_j}, \ \chi(a_i) = 1 + p_g \ (i < r) \Big\}$$

Let $(y_1, \dots, y_r)$ be an element of $S(y)$. We put $X^{[\boldsymbol{n}]} := \prod_{i=1}^r X^{[n_i]}$. The universal ideal sheaves over $X^{[n_i]} \times X$ are denoted by $\mathcal{I}_i^u$. Let $\mathcal{Z}_i$ denote the universal scheme over $X^{[n_i]} \times X$ of length n_i. We also use the same notation to denote the pull back of $\mathcal{I}_i^u$ and $\mathcal{Z}_i$ via the projection $X^{[\boldsymbol{n}]} \times X \longrightarrow X^{[n_i]} \times X$. Let e^{a_i} denote the holomorphic line bundle corresponding to a_i. Since we have assumed $H^1(X, \mathcal{O}_X) = 0$ in this subsubsection, such a holomorphic line bundle is uniquely determined up to isomorphisms. Let G denote the $(r-1)$-dimensional torus $\operatorname{Spec} k[\tau_1, \tau_1^{-1}, \dots, \tau_{r-1}, \tau_{r-1}^{-1}]$. Let $e^{w \cdot t_i}$ denote the trivial line bundle with the G-action, which is induced by the $\operatorname{Spec} k[\tau_i, \tau_i^{-1}]$-action of weight w. Thus, we obtain the G-equivariant sheaf $\mathcal{I}_i^u \cdot e^{a_i + T_i}$. We use the notation $Q(\mathcal{I}_i^u \cdot e^{a_i+T_i}, \mathcal{I}_j^u \cdot e^{a_j+T_j})$ to denote the G-equivariant Euler class of the following:

$$-Rp_{X\,*}\Big(\mathcal{RH}om\big(\mathcal{I}_i^u \cdot e^{a_i+T_i}, \ \mathcal{I}_j^u \cdot e^{a_j+T_j} \big) \Big) - Rp_{X\,*}\Big(\mathcal{RH}om\big(\mathcal{I}_j^u \cdot e^{a_j+T_j}, \ \mathcal{I}_i^u \cdot e^{a_i+T_i} \big) \Big)$$

We regard it as the element of $\bigotimes_{i=1}^r \mathcal{R}\big(\mathcal{I}_i^u \cdot e^{a_i}\big) \otimes_{\mathbb{Q}} \mathfrak{R}(t_1, \dots, t_{r-1})$. (See the subsubsection 7.1.8 for the ring $\mathfrak{R}(t_1, \dots, t_{r-1})$.) We put as follows:

$$Q\big(\mathcal{I}_1^u \cdot e^{a_1+T_1}, \mathcal{I}_2^u \cdot e^{a_2+T_2}, \dots \mathcal{I}_r^u \cdot e^{a_r+T_r} \big) = \prod_{i<j} Q\big(\mathcal{I}_i^u \cdot e^{a_i+T_i}, \ \mathcal{I}_j^u \cdot e^{a_j+T_j} \big)$$

We also have the element $P\big(\bigoplus_{i=1}^r \mathcal{I}_i^u \cdot e^{a_i+T_i}\big)$ of $\bigotimes_{i=1}^r \mathcal{R}(\mathcal{I}_i^u \cdot e^{a_i})[t_1, \dots, t_{r-1}]$ by the homomorphisms in the subsubsection 7.1.3. Let Ξ_i denote the vector bundle $p_{X\,*}\big(\mathcal{O}_{\mathcal{Z}_i} \otimes e^{a_i}\big)$.

Theorem 7.38 *Assume $p_g > 0$ and $H^1(X, \mathcal{O}) = 0$. We also assume $\chi(y) - 1 \geq (r-1) \cdot (1 + p_g)$ and $P_y > P_K$. We have the following formula:*

$$\int_{\mathcal{M}(\widehat{y})} P(\widehat{E}^u) = (-1)^{r-1} \sum_{(y_1, \dots, y_r) \in S(y)} \prod_{i=1}^{r-1} \mathrm{SW}(a_i) \cdot \int_{X^{[\boldsymbol{n}]}} \Psi(y_1, \dots, y_r)$$

The elements $\Psi(y_1, \dots, y_r) \in \bigotimes_{i=1}^r \mathcal{R}\big(\mathcal{I}_i^u \cdot e^{a_i}\big)$ are given as follows:

$$\Psi(y_1, \dots, y_r) =$$

$$\operatorname*{Res}_{t_{r-1}} \cdots \operatorname*{Res}_{t_1} \left(\frac{P\big(\bigoplus_{i=1}^r \mathcal{I}_i^u \cdot e^{a_i+T_i}\big)}{Q\big(\mathcal{I}_1^u \cdot e^{a_1+T_1}, \dots, \mathcal{I}_r^u \cdot e^{a_r+T_r}\big)} \prod_{i<j} \frac{\mathrm{Eu}(\Xi_j \cdot e^{T_j - T_i})}{(T_j - T_i)^{\chi(a_j)}} \cdot \prod_{i=1}^{r-1} t_i^{\sum_{j \geq i} \chi(y_j) - 1} \right) \cdot \prod_{i=1}^{r-1} \mathrm{Eu}(\Xi_i) \tag{243}$$

Proof We put as follows:

$$S_1(y) := \big\{ (y_1, y_2) \in \mathcal{T}ype_1 \times \mathcal{T}ype_{r-1} \, \big| \, y_1 + y_2 = y, \ P_{y_1} < P_{y_2} \big\}$$

By using the transition formula (237) and the same argument as the proof of Theorem 7.28, we obtain the following equality:

$$\int_{\mathcal{M}(\widehat{y})} P(\widehat{E}^u) = -\sum_{(y_1, y_2) \in S_1(y)} \mathrm{SW}(a_1) \cdot \int_{X^{[n_1]} \times \mathcal{M}(\widehat{y}_2)} \operatorname*{Res}_{t=0} \left(\frac{P\big(\mathcal{I}_1^u \cdot e^{a_1 - t} \oplus \widehat{E}_2^u \cdot e^{t/(r-1)}\big) \cdot t^{\chi(y)-1}}{Q(\mathcal{I}_1^u \cdot e^{-t}, \ \widehat{E}_2^u \cdot e^{t/(r-1)}) \cdot R\big(\mathcal{O}_X \cdot e^{-t}, \ \widehat{E}_2^u \cdot e^{t/(r-1)}\big)} \right) \tag{244}$$

Here, $R\big(\mathcal{O}_X\cdot e^{-t},\, \widehat{E}_2^u\cdot e^{t/(r-1)}\big)$ denotes the equivariant Euler class of $p_{X\,*}\big(\mathcal{H}om(\mathcal{O}_X\cdot e^{-t},\, \widehat{E}_2^u\cdot e^{t/(r-1)})\big)$.

We remark that $\chi(y_2)-1\geq 0$ and $P_{y_2}>P_y>P_K$ and $\chi(y_2)-1\geq (r-2)(1+p_g)$. We also remark that the integrand of (244) is the element of $\mathcal{R}\big(\mathcal{I}_1\cdot e^{a_1}\big)\otimes\mathcal{R}\big(\widehat{E}_2\big)$, and hence we have only to consider the component-wise integration. Thus, we may use (244) inductively. We put $\overline{T}_i:=\sum_{j<i}t_j/(r-j)$. Then, we obtain the following equality:

$$\int_{\mathcal{M}(\widehat{y})}P(\widehat{E}^u)+(-1)^r\sum_{(y_1,\ldots,y_r)\in S(y)}\prod_{i=1}^{r-1}\mathrm{SW}(a_i)\cdot\int_{X^{[\boldsymbol{n}]}}\Psi_1(y_1,\ldots,y_r)=0 \tag{245}$$

The elements $\Psi_1(y_1,\ldots,y_r)\in\bigotimes_{i=1}^r\mathcal{R}(\mathcal{I}_i^u\cdot e^{a_i})$ are given as follows:

$$\Psi_1(y_1,\ldots,y_r)=\operatorname*{Res}_{t_{r-1}}\cdots\operatorname*{Res}_{t_1}\left(\frac{P\big(\bigoplus\mathcal{I}_i^u\cdot e^{a_i+T_i}\big)\cdot\prod_{i=1}^{r-1}t_i^{\sum_{j\geq i}\chi(y_j)-1}\prod_{i=1}^{r-1}\mathrm{Eu}(\Xi_i)}{\prod_{i<j}Q\big(\mathcal{I}_i^u\cdot e^{a_i-t_i},\mathcal{I}_j^u\cdot e^{a_j+T_j-\overline{T}_i}\big)\cdot R\big(\mathcal{O}_X\cdot e^{-t_i},\,\mathcal{I}_j^u\cdot e^{a_j+T_j-\overline{T}_i}\big)}\right)$$

We have $Q\big(\mathcal{I}_i^u\cdot e^{a_i-t_i},\,\mathcal{I}_j^u\cdot e^{a_j+T_j-\overline{T}_i}\big)=Q\big(\mathcal{I}_i^u\cdot e^{a_i+T_i},\,\mathcal{I}_j^u\cdot e^{a_j+T_j}\big)$. We also obtain the following by a formal calculation:

$$R\big(\mathcal{O}_X\cdot e^{-t_i},\,\mathcal{I}_j^u\cdot e^{T_j-\overline{T}_i}\big)=\frac{(T_j-T_i)^{\chi(a_j)}}{\mathrm{Eu}\big(\Xi_j\cdot e^{T_j-T_i}\big)}$$

Hence, we obtain $\Psi_1(y_1,\ldots,y_r)=\Psi(y_1,\ldots,y_r)$. Thus we are done. ∎

7.5.3 Independence from the polarization in the case $p_g>0$

Let y be an element of $\mathcal{T}ype_r^{\circ}$. We use the notation $\mathcal{M}_H(\widehat{y})$ to denote the moduli stack of torsion-free sheaves of type y, which are semistable with respect to a polarization H. For $\Phi=P(\widehat{E}^u)\in\mathcal{R}\big(\widehat{E},\mathcal{M}_H(\widehat{y})\big)$, we put as follows:

$$\Phi_H(\widehat{y}):=\int_{\mathcal{M}_H(\widehat{y})}\Phi$$

Theorem 7.39 *The invariant $\Phi_H(\widehat{y})$ is independent of the choice of a generic polarization in the case $p_g>0$.*

Proof Let ξ be an element of $NS(X)$ such that $\xi^2<0$. Let W^{ξ} be the wall in the ample cone determined by ξ. It is well known that the ample cone is divided into the chambers by such walls, and $\mathcal{M}_H(\widehat{y})$ depends only on the chamber to which H belongs. Moreover, the set of such walls are locally finite ([42]).

Let C_+ and C_- be chambers which are divided by a wall W^{ξ}. Let H_+ and H_- be ample line bundles contained in H_+ and H_- respectively. We assume $(\xi,H_+)>0>(\xi,H_-)$. For the proof of the theorem, we have only to show $\Phi_{H_+}(\widehat{y})=\Phi_{H_-}(\widehat{y})$. We take an ample line bundle H which is generic in W^{ξ}. As in Lemma 7.30, we may assume the following:

- $H_+=H\otimes\mathcal{O}(C)$ and $H_-=H\otimes\mathcal{O}(-C)$ for some sufficiently ample divisor C.
- A torsion free sheaf E of type y is H_+-semistable if and only if $E(C)$ is H-semistable.
- E is H_- -semistable if and only if $E(-C)$ is H-semistable.

We use the notation and the argument in the subsubsection 7.4.4. Let $\mathcal{S}$ be the family of torsion-free sheaves of type y, which are μ-semistable. Let $\overline{\mathcal{S}}$ be the family of torsion-free sheaves E' with the following properties:

- $\mu(E')=\mu(y)$ and $\mathrm{rank}(E')<r$.
- There exists a member E of $\mathcal{S}$ such that E' is a saturated subsheaf of E.

We take a large integer m such that the condition O_m holds for the family $\mathcal{S}$. As in the subsubsection 7.4.4, we consider the integrals over $\mathcal{M}\big(\widehat{y}(C),[\mathcal{O}(-m,C)],\delta\big)$ and $\mathcal{M}\big(\widehat{y}(-C),\mathcal{O}(-m,-C),\delta\big)$ given by (232). When δ is sufficiently small, we have $\widetilde{\Phi}_C(\delta)=\Phi_{H_+}(\widehat{y})$ and $\widetilde{\Phi}_{-C}(\delta)=\Phi_{H_-}(\widehat{y})$.

Let $\mathcal{T}ype(\overline{\mathcal{S}})$ denote the set of the types of members of $\overline{\mathcal{S}}$. For each $y_0\in\mathcal{T}ype(\overline{\mathcal{S}})$, we have $y_1:=y-y_0\in\mathcal{T}ype(\overline{\mathcal{S}})$. We remark that $P_{y_0(C)}-P_{y_1(C)}$ and $P_{y_0(-C)}-P_{y_1(-C)}$ are polynomials of degree 0.

Let r_i, a_i, b_i and n_i denote the rank, the first Chern class, the second Chern character and the second Chern class corresponding to y_i. We have $\mu(y_0) = \mu(y_1)$, and H is a generic element of W^{ξ}. Therefore, we have $a_0/r_0 - a_1/r_1 = A \cdot \xi$ for some $A \in \mathbb{Q}$ in $H^2(X)$. Since the intersection number (C, ξ) is assumed to be sufficiently large, we have $P_{y_0(C)} - P_{y_1(C)} \neq 0$ unless $y_0/r_0 = y_1/r_1$. We also have $P_{y_0(-C)} - P_{y_1(-C)} \neq 0$ unless $y_0/r_0 = y_1/r_1$.

We put as follows:

$$S(y, C) := \bigl\{(y_0, y_1) \bigm| y_0 + y_1 = y,\ y_i \in \mathcal{T}ype(\overline{\mathcal{S}}),\ P_{y_0(C)} < P_{y_1(C)}\bigr\}$$

$$S(y, -C) := \bigl\{(y_0, y_1) \bigm| y_0 + y_1 = y,\ y_i \in \mathcal{T}ype(\overline{\mathcal{S}}),\ P_{y_0(-C)} < P_{y_1(-C)}\bigr\}.$$

We take a positive constant δ_0 satisfying the following inequalities:

$$\delta_0 > \max\bigl\{|P_{y_0(C)} - P_{y_1(C)}| \bigm| (y_0, y_1) \in S(y, C)\bigr\}, \quad \delta_0 > \max\bigl\{|P_{y_0(-C)} - P_{y_1(-C)}| \bigm| (y_0, y_1) \in S(y, -C)\bigr\}.$$

The following lemma can be shown by the same argument as the proof of Lemma 7.35 and Lemma 7.36.

Lemma 7.40 *$(E(C), \phi_C)$ is δ_0-semistable, if and only if $(E(-C), \phi_{-C})$ is δ_0-semistable. Moreover, the δ_0-semistability implies the δ_0-stability in the both cases.* ∎

Due to the lemma, we obtain the following:

$$\Phi_{H_+}(\widehat{y}) - \Phi_{H_-}(\widehat{y}) = \bigl(\widetilde{\Phi}_{-C}(\delta_0) - \Phi_{H_-}(\widehat{y})\bigr) - \bigl(\widetilde{\Phi}_C(\delta_0) - \Phi_{H_+}(\widehat{y})\bigr) \tag{246}$$

Let us move the parameter δ from 0 to δ_0, and we see the transition of the invariants $\widetilde{\Phi}_{-C}(\delta)$ and $\widetilde{\Phi}_C(\delta)$. We use Theorem 7.37. Since the condition O_m holds for $\mathcal{S}$, it is easy to see that the contributions from the decomposition types (y_0, y_1) are trivial even in the case $\mathrm{rank}(y_0) = 1$ (Proposition 6.25). Hence we obtain $\widetilde{\Phi}_{-C}(\delta_0) - \Phi_{H_-}(\widehat{y}) = 0$ and $\widetilde{\Phi}_C(\delta_0) - \Phi_{H_+}(\widehat{y}) = 0$. Thus, the proof of Theorem 7.39 is finished. ∎

7.6 Transition Formula in the Case $p_g = 0$

7.6.1 Statement

Let us discuss the transition formula when the 2-stability condition does not hold for $(\boldsymbol{y}, L, \alpha_*, \delta)$ in the case $p_g = 0$. Let P be an element of $\mathcal{R}$. We restrict ourselves to the case where the 1-vanishing condition holds for $(\boldsymbol{y}, L, \alpha_*, \delta)$, and we discuss the integral of the element $\mathcal{R}\bigl(\widehat{E}^u, \mathcal{M}^s(\widehat{\boldsymbol{y}}, [L], \alpha_*, \delta)\bigr)$ of the following form:

$$\Phi = P(\widehat{E}^u) \cdot \frac{\mathrm{Eu}(T_{\mathrm{rel}})}{N_L(y)} \tag{247}$$

We put as follows, for non critical parameter δ:

$$\Phi(\delta) := \int_{\mathcal{M}^s(\widehat{\boldsymbol{y}}, [L], \alpha_*, \delta)} \Phi.$$

Let δ be a critical parameter. We take $\delta_- < \delta < \delta_+$ such that δ_κ $(\kappa = \pm)$ are sufficiently close to δ. We would like to obtain the formula $\Phi(\delta_+) - \Phi(\delta_-)$ to express the sum as the integrals over the products of the moduli stacks of the objects with lower ranks. We also impose the following condition to $(\boldsymbol{y}, L, \alpha_*, \delta)$:

Condition 7.41 *For any $(E_{1*}, \phi) \oplus E_{2*} \oplus E_{3*} \in \mathcal{M}^{ss}(\boldsymbol{y}, L, \alpha_*, \delta)$, the equality $P_{E_2} = P_{E_3}$ holds for the reduced Hilbert polynomials of E_2 and E_3.* ∎

For each positive integer k we put as follows:

$$S_k(\boldsymbol{y}, \delta) := \bigl\{\boldsymbol{Y} = (\boldsymbol{y}_1, \ldots, \boldsymbol{y}_k) \in \mathcal{T}ype^k \bigm| P^{\alpha_*}_{\boldsymbol{y}_i} = P^{\alpha_*, \delta}_{\boldsymbol{y}}\ (i = 1, \ldots, k)\bigr\}$$

For each element $\boldsymbol{Y} = (\boldsymbol{y}_1, \ldots, \boldsymbol{y}_k) \in S_k(\boldsymbol{y}, \delta)$, we put $|\boldsymbol{Y}| = \sum_{i=1}^k \boldsymbol{y}_i$. We also put as follows:

$$W(\boldsymbol{Y}) := \prod_{i=1}^k \frac{\mathrm{rank}(\boldsymbol{y}_i)}{\sum_{1 \le j \le i} \mathrm{rank}(\boldsymbol{y}_j)}$$

We put as follows:

$$\overline{S}_k(\boldsymbol{y},\delta):=\big\{(\boldsymbol{y}_0,\boldsymbol{Y})\in \mathcal{T}ype\times S_k(\boldsymbol{y},\delta)\,\big|\,\boldsymbol{y}_0+|\boldsymbol{Y}|=\boldsymbol{y}\big\},\quad \overline{S}(\boldsymbol{y},\delta):=\coprod_k \overline{S}_k(\boldsymbol{y},\delta)$$

For any $(\boldsymbol{y}_0,\boldsymbol{Y})\in\overline{S}_k(\boldsymbol{y},\delta)$, we put as follows:

$$\mathcal{M}(\widehat{\boldsymbol{y}}_0,\widehat{\boldsymbol{Y}},[L]):=\mathcal{M}^s\big(\widehat{\boldsymbol{y}}_0,[L],\alpha_*,\delta_-\big)\times\prod_{i=1}^k\mathcal{M}^{ss}(\widehat{\boldsymbol{y}}_i,\alpha_*)$$

Let $\widehat{E}_0^u$ denote the sheaf over $\mathcal{M}(\widehat{\boldsymbol{y}}_0,\widehat{\boldsymbol{Y}},[L])\times X$ which is obtained as the pull back of the universal sheaf over $\mathcal{M}^s(\widehat{\boldsymbol{y}}_0,[L],\alpha_*,\delta_-)\times X$ via the natural projection. We use the notation $\widehat{E}_i^u$ in a similar meaning.

When $(\boldsymbol{y}_0,\boldsymbol{Y})\in\overline{S}_k(\boldsymbol{y},\delta)$ is given, we put as follows:

$$T_0=-\sum_{j>0}\frac{t_j}{\sum_{0\leq h<j}\operatorname{rank}(\boldsymbol{y}_h)},\quad T_i=-\sum_{j>i}\frac{t_j}{\sum_{0\leq h<j}\operatorname{rank}(\boldsymbol{y}_h)}+\frac{t_i}{\operatorname{rank}(\boldsymbol{y}_i)}$$

Here, $t_1,\ldots,t_k$ are variables. Let G denote the k-dimensional torus $\operatorname{Spec}k[\tau_1,\tau_1^{-1},\ldots,\tau_k,\tau_k^{-1}]$. Let $e^{w\cdot t_i}$ denote the trivial line bundle with the G-action which is induced by the $\operatorname{Spec}k[\tau_i,\tau_i^{-1}]$-action of weight w. We have the following element of the K-group of the G-equivariant coherent sheaves on $\mathcal{M}(\widehat{\boldsymbol{y}}_0,\widehat{\boldsymbol{Y}},[L])$:

$$\begin{aligned}&-Rp_{X\,*}\mathcal{RH}om\big(\widehat{E}_i^u\cdot e^{T_i},\,\widehat{E}_j^u\cdot e^{T_j}\big)-Rp_{X\,*}\mathcal{RH}om\big(\widehat{E}_j^u\cdot e^{T_j},\,\widehat{E}_i^u\cdot e^{T_i}\big)\\&\qquad-Rp_{D\,*}\mathcal{RH}om_2'\big(\widehat{E}^u_{i|D\,*}\cdot e^{T_i},\,\widehat{E}^u_{j|D\,*}\cdot e^{T_j}\big)-Rp_{D\,*}\mathcal{RH}om_2'\big(\widehat{E}^u_{j|D\,*}\cdot e^{T_j},\,\widehat{E}^u_{i|D\,*}\cdot e^{T_i}\big)\end{aligned}\tag{248}$$

(See the subsubsection 2.1.5 for the notation $\mathcal{RH}om_2'$.) The equivariant Euler class is denoted by $Q\big(\widehat{E}_i^u\cdot e^{T_i},\widehat{E}_j^u\cdot e^{T_j}\big)$, which is regarded as the element of $\bigotimes_{i=0}^k\mathcal{R}(\widehat{E}_i^u)\otimes\mathfrak{R}(t_k,t_{k-1},\ldots,t_2,t_1)$. We put as follows:

$$Q\big(\widehat{E}_0^u\cdot e^{T_0},\widehat{E}_1^u\cdot e^{T_1},\ldots,\widehat{E}_k^u\cdot e^{T_i}\big):=\prod_{i<j}Q\big(\widehat{E}_i^u\cdot e^{T_i},\widehat{E}_j^u\cdot e^{T_j}\big)\tag{249}$$

Let $T_{0\,\mathrm{rel}}$ denote the vector bundle over $\mathcal{M}(\widehat{\boldsymbol{y}}_0,\widehat{\boldsymbol{Y}},[L])$ obtained from the relative tangent bundle of the smooth map $\mathcal{M}^s(\widehat{\boldsymbol{y}}_0,[L],\alpha_*,\delta_-)\longrightarrow\mathcal{M}(m,\widehat{\boldsymbol{y}}_0,[L])$. Then, we have the following element:

$$\Psi(\boldsymbol{y}_0,\boldsymbol{Y}):=\operatorname*{Res}_{t_1=0}\cdots\operatorname*{Res}_{t_k=0}\left(\frac{P\big(\bigoplus_{i=0}^k\widehat{E}_i^u\cdot e^{T_i}\big)}{Q\big(\widehat{E}_0^u\cdot e^{T_0},\ldots,\widehat{E}_k^u\cdot e^{T_k}\big)}\right)\cdot\frac{\operatorname{Eu}(T_{0,\mathrm{rel}})}{N_L(y_0)}\in\mathcal{R}\big(\widehat{E}_0,\mathcal{M}^s(\widehat{\boldsymbol{y}}_0,[L],\alpha_*,\delta_-)\big)\otimes\bigotimes_{i=1}^k\mathcal{R}(\widehat{E}_i)$$

Theorem 7.42 *Assume that the* 1*-vanishing and Condition* 7.41 *hold for* $(\boldsymbol{y},L,\alpha_*,\delta)$*. For* Φ *as in* (247)*, we have the following transition formula:*

$$\Phi(\delta_+)-\Phi(\delta_-)=\sum_{(\boldsymbol{y}_0,\boldsymbol{Y})\in\overline{S}(\boldsymbol{y},\delta)}\frac{N_L(y_0)}{N_L(y)}\cdot W(\boldsymbol{Y})\cdot\int_{\mathcal{M}(\widehat{\boldsymbol{y}}_0,\widehat{\boldsymbol{Y}},[L])}\Psi(\boldsymbol{y}_0,\boldsymbol{Y})\tag{250}$$

The proof will be given in the next subsubsections 7.6.2–7.6.4.

7.6.2 Step 1

We put $\widetilde{\mathcal{M}}(\delta,\ell):=\widetilde{\mathcal{M}}^{ss}\big(\boldsymbol{y},[L],\alpha_*,(\delta,\ell)\big)$ for a positive integer ℓ. We put $\widetilde{\mathcal{M}}(\delta,0):=\widetilde{\mathcal{M}}^{ss}(\boldsymbol{y},[L],\alpha_*,\delta_+)$. We put $\widetilde{\mathcal{M}}(\delta_-):=\widetilde{\mathcal{M}}^{ss}(\boldsymbol{y},[L],\alpha_*,\delta_-)$. Let $\widetilde{T}_{\mathrm{rel}}$ denote the relative tangent bundle of the smooth morphism $\widetilde{\mathcal{M}}(m,\widehat{\boldsymbol{y}},[L])\longrightarrow\mathcal{M}(m,\widehat{\boldsymbol{y}},[L])$. We have the open immersion of $\widetilde{\mathcal{M}}(\delta,\ell)$ and $\widetilde{\mathcal{M}}(\delta_-)$ into $\widetilde{\mathcal{M}}(m,\widehat{\boldsymbol{y}},[L])$. We use the same notation to denote the restriction of $\widetilde{T}_{\mathrm{rel}}$ to $\widetilde{\mathcal{M}}(\delta,\ell)$ and $\widetilde{\mathcal{M}}(\delta_-)$. We put as follows:

$$\widetilde{\Phi}:=\Phi\cdot\frac{\operatorname{Eu}(\widetilde{T}_{\mathrm{rel}})}{H_y(m)!},\quad\widetilde{\Phi}(\delta,\ell):=\int_{\widetilde{\mathcal{M}}(\delta,\ell)}\widetilde{\Phi},\quad\widetilde{\Phi}(\delta_\kappa):=\int_{\widetilde{\mathcal{M}}(\delta_\kappa)}\widetilde{\Phi}\tag{251}$$

Then, we have $\widetilde{\Phi}(\delta,0)=\Phi(\delta_+)$ and $\widetilde{\Phi}(\delta_-)=\Phi(\delta_-)$.

Recall that $\mathrm{Dec}(m,\boldsymbol{y},\alpha_*,\delta)$ denotes the set of the decomposition types (Definition 4.33). We put as follows:

$$S(\ell):=\bigl\{\mathfrak{I}=(\boldsymbol{y}_1,\boldsymbol{y}_2,I_1,I_2)\in\mathrm{Dec}(m,\boldsymbol{y},\alpha_*,\delta)\,\big|\,\underline{\ell}\subset I_1\bigr\}$$

For any decomposition type $\mathfrak{I}=(\boldsymbol{y}_1,\boldsymbol{y}_2,I_1,I_2)\in\mathrm{Dec}(m,\boldsymbol{y},\alpha_*,\delta)$, we put as follows:

$$\widetilde{\mathcal{M}}(\widehat{\mathfrak{I}}):=\widetilde{\mathcal{M}}^{ss}\bigl(\widehat{\boldsymbol{y}}_1,[L],\alpha_*,\delta,\mathfrak{k}(\mathfrak{I})\bigr)\times\mathcal{M}^{ss}(\widehat{\boldsymbol{y}}_2,\alpha_*)$$

Let $\widetilde{T}_{1,\mathrm{rel}}$ denote the vector bundle over $\widetilde{\mathcal{M}}(\widehat{\mathfrak{I}})$ induced by the relative tangent bundle of the smooth map $\widetilde{\mathcal{M}}^{ss}\bigl(\widehat{\boldsymbol{y}}_1,[L],\alpha_*,\delta,\mathfrak{k}(\mathfrak{I})\bigr)\longrightarrow\mathcal{M}(m,\widehat{\boldsymbol{y}}_1,[L])$. Let $T_{1,\mathrm{rel}}$ denote the vector bundle over $\widetilde{\mathcal{M}}(\widehat{\mathfrak{I}})$ obtained from the relative tangent bundle of the smooth morphism $\mathcal{M}(m,\widehat{\boldsymbol{y}}_1,[L])\longrightarrow\mathcal{M}(m,\widehat{\boldsymbol{y}}_1)$.

Proposition 7.43 *We have the following equality:*

$$\widetilde{\Phi}(\delta,\ell)-\Phi(\delta_-)=\sum_{\mathfrak{I}\in S(\ell)}\frac{N_L(y_1)}{N_L(y)}\frac{H_{y_1}(m)!\cdot H_{y_2}(m)!}{H_y(m)!}\int_{\widetilde{\mathcal{M}}(\widehat{\mathfrak{I}})}\Psi(\mathfrak{I}).\tag{252}$$

The elements $\Psi(\mathfrak{I})\in\mathcal{R}\bigl(\widehat{E}_1,\widetilde{\mathcal{M}}^{ss}(\widehat{\boldsymbol{y}}_1,[L],\alpha_*,\delta,\mathfrak{k}(\mathfrak{I}))\bigr)\otimes\mathcal{R}(\widehat{E}_2)$ are given as follows:

$$\Psi(\mathfrak{I})=\mathop{\mathrm{Res}}_{s=0}\left(\frac{P\bigl(\widehat{E}_1^u\cdot e^{-s/r_1}\oplus\widehat{E}_2^u\cdot e^{s/r_2}\bigr)}{Q(\widehat{E}_1^u\cdot e^{-s/r_1},\widehat{E}_2^u\cdot e^{s/r_2})}\right)\cdot\frac{\mathrm{Eu}(T_{1,\mathrm{rel}})}{N_L(y_1)}\frac{\mathrm{Eu}(\widetilde{T}_{1,\mathrm{rel}})}{H_{y_1}(m)!}\tag{253}$$

Proof The argument is essentially same as the proof of Theorem 7.37 and Theorem 7.15. Hence, we give only an indication. Let $\widehat{M}$ denote the master space connecting $\widetilde{\mathcal{M}}(\delta,\ell)$ and $\widetilde{\mathcal{M}}(\delta_-)$ constructed in the subsubsection 4.5.1. By using $\widehat{M}$ and the argument in the proof of Theorem 7.37, we obtain the following expression:

$$\widetilde{\Phi}(\delta,\ell)-\Phi(\delta_-)=\sum_{\mathfrak{I}\in S(\ell)}\int_{\widehat{M}^{G_m}(\mathfrak{I})}\frac{\widetilde{\Phi}_t}{\mathrm{Eu}\bigl(\mathfrak{N}(\widehat{M}^{G_m}(\mathfrak{I}))\bigr)}$$

The contributions from $\widehat{M}^{G_m}(\mathfrak{I})$ can be calculated by the arguments in the proof of Theorem 7.37 and Theorem 7.15. We remark that we can use Lemma 7.25, due to Condition 7.41. Then, we arrive at the formulas (252) and (253) ∎

For our later argument, we reword Proposition 7.43. Let I be a finite subset of $\mathbb{Z}_{>0}$ such that $|I|=H_y(m)$. We naturally regard I as the totally ordered set. Let i_0 be any element of I. Let $\widetilde{\mathcal{M}}^{ss}\bigl(\widehat{\boldsymbol{y}},[L],\alpha_*,(\delta,i_0),I\bigr)$ be the moduli stack of the objects $(E_*,[\phi],\rho,\mathcal{F})$ as follows:

- $(E_*,[\phi],\rho)$ is a δ-semistable oriented reduced L-Bradlow pairs.
- $\mathcal{F}$ is a full flag of $H^0(X,E(m))$ indexed by I.
- The tuple $(E_*,[\phi],\rho,\mathcal{F})$ is (δ,i_0)-semistable in following sense. We take a partial Jordan-Hölder filtration of $(E_*,[\phi])$:

  $$E_*^{(1)}\subsetneq E_*^{(2)}\subsetneq\cdots\subsetneq E_*^{(j-1)}\subsetneq(E_*^{(j)},[\phi])\subsetneq(E_*^{(j+1)},[\phi])\subsetneq\cdots\subsetneq(E_*^{(k-1)},[\phi])\subsetneq(E_*^{(k)},[\phi])$$

  Then, $\mathcal{F}_{i_0}\cap H^0\bigl(X,E^{(j-1)}(m)\bigr)=\{0\}$ and $\mathcal{F}_{i_0}\not\subset H^0\bigl(X,E^{(k-1)}(m)\bigr)$.

We have the bijection $\varphi:I\longrightarrow\{1,\ldots,H_y(m)\}$ preserving the order. Then, we have the isomorphism $\widetilde{\mathcal{M}}^{ss}\bigl(\widehat{\boldsymbol{y}},[L],\alpha_*,(\delta,i_0),I\bigr)\simeq\widetilde{\mathcal{M}}^{ss}\bigl(\widehat{\boldsymbol{y}},[L],\alpha_*,(\delta,\varphi(i_0))\bigr)$. We have the open immersion of $\widetilde{\mathcal{M}}^{ss}\bigl(\widehat{\boldsymbol{y}},[L],\alpha_*,(\delta,i_0),I\bigr)$ into $\widetilde{\mathcal{M}}(m,\widehat{\boldsymbol{y}},[L])$. The pull back of $\widetilde{T}_{\mathrm{rel}}$ is denoted by the same notation. Let $\widetilde{\Phi}$ be as in (251). We put as follows:

$$\widetilde{\Phi}(\delta,i_0,I):=\int_{\widetilde{\mathcal{M}}^{ss}(\widehat{\boldsymbol{y}},[L],\alpha_*,(\delta,i_0),I)}\widetilde{\Phi}$$

Let $\mathrm{Dec}(m,\boldsymbol{y},\alpha_*,\delta,I)$ denote the set of the tuples $\mathfrak{I}=(\boldsymbol{y}_1,\boldsymbol{y}_2,I_1,I_2)$ satisfying the following conditions:

$$\boldsymbol{y}_1+\boldsymbol{y}_2=\boldsymbol{y},\quad P^{\alpha_*,\delta}_{\boldsymbol{y}}=P^{\alpha_*,\delta}_{\boldsymbol{y}_1}=P^{\alpha_*}_{\boldsymbol{y}_2},\quad I_1\sqcup I_2=I,\quad |I_i|=H_{y_i}(m)$$

For any $\mathfrak{I}=(\boldsymbol{y}_1,\boldsymbol{y}_2,I_1,I_2)\in\mathrm{Dec}(m,\boldsymbol{y},\alpha_*,\delta,I)$, we put as follows:

$$\mathfrak{k}(\mathfrak{I}):=\max\bigl\{i\in I_1\,\big|\,i<\min(I_2)\bigr\}$$

We also put as follows:

$$\widetilde{\mathcal{M}}(\mathfrak{I}):=\widetilde{\mathcal{M}}^{ss}\bigl(\widehat{\boldsymbol{y}}_1,[L],\alpha_*,(\delta,\mathfrak{k}(\mathfrak{I})),I_1\bigr)\times\mathcal{M}^{ss}(\widehat{\boldsymbol{y}}_2,\alpha_*),$$

We put $S(i_0,I):=\bigl\{\mathfrak{I}\in\mathrm{Dec}(m,\boldsymbol{y},\alpha_*,\delta,I)\,\big|\,\mathfrak{k}(\mathfrak{I})\geq i_0\bigr\}$.

Then, Proposition 7.43 can be reworded as follows.

Proposition 7.44 *We have the following equality:*

$$\widetilde{\Phi}(\delta,i_0,I)-\Phi(\delta_-)=\sum_{\mathfrak{I}\in S(i_0,I)}\frac{N_L(y_1)}{N_L(y)}\frac{H_{y_1}(m)!\cdot H_{y_2}(m)!}{H_y(m)!}\int_{\widetilde{\mathcal{M}}(\mathfrak{I})}\Psi(\mathfrak{I}).\tag{254}$$

Here, $\Psi(\mathfrak{I})$ *are given as in* (253). ∎

7.6.3 Step 2

We define the set $\mathrm{Dec}^{(j)}(m,\boldsymbol{y},\alpha_*,\delta)$ inductively. Put $\mathrm{Dec}^{(1)}(m,\boldsymbol{y},\alpha_*,\delta):=\mathrm{Dec}(m,\boldsymbol{y},\alpha_*,\delta)$. Assume that $\mathrm{Dec}^{(j-1)}(m,\boldsymbol{y},\alpha_*,\delta)$ is already given. Let $\mathrm{Dec}^{(j)}(m,\boldsymbol{y},\alpha_*,\delta)$ be the set of the tuple $\mathfrak{I}^{(j)}=(\boldsymbol{y}_1^{(j)},\boldsymbol{Y}_2^{(j)},I_1^{(j)},\boldsymbol{I}_2^{(j)})$ as follows:

- $\boldsymbol{Y}_2^{(j)}$ denotes an element $(\boldsymbol{y}_2^{(j)},\boldsymbol{y}_2^{(j-1)},\ldots,\boldsymbol{y}_2^{(1)})$ of $S_j(\boldsymbol{y},\delta)$.
- $\boldsymbol{I}_2^{(j)}$ denotes a tuple $(I_2^{(j)},I_2^{(j-1)},\ldots,I_2^{(1)})$ of subsets of $\{1,\ldots,H_y(m)\}$. Assume $\min(I_2^{(i)})>\min(I_2^{(i-1)})$ for $i=2,\ldots,j$
- We assume $\{1,\ldots,H_y(m)\}=I_1^{(j)}\sqcup\coprod_{i=1}^{j}I_2^{(i)}$.
- We put $\boldsymbol{y}_1^{(j-1)}:=\boldsymbol{y}_1^{(j)}+\boldsymbol{y}_2^{(j)}$ and $I_1^{(j-1)}:=I_1^{(j)}\sqcup I_2^{(j)}$. Then, $(\boldsymbol{y}_1^{(j)},\boldsymbol{y}_2^{(j)},I_1^{(j)},I_2^{(j)})$ is an element of $\mathrm{Dec}(m,\boldsymbol{y}_1^{(j-1)},\alpha_*,I_1^{(j-1)})$, in the sense of the subsubsection 7.6.2.
- We put $\boldsymbol{Y}_2^{(j-1)}:=(\boldsymbol{y}_2^{(j-1)},\ldots,\boldsymbol{y}_2^{(1)})$ and $\boldsymbol{I}_2^{(j-1)}=(I_2^{(j-1)},\ldots,I_2^{(1)})$. Then, $(\boldsymbol{y}_1^{(j-1)},\boldsymbol{Y}_2^{(j-1)},I_1^{(j-1)},\boldsymbol{I}_2^{(j-1)})$ is an element of $\mathrm{Dec}^{(j-1)}(m,\boldsymbol{y},\alpha_*,\delta)$.

Let $\mathfrak{I}^{(j)}=(\boldsymbol{y}_1^{(j)},\boldsymbol{Y}_2^{(j)},I_1^{(j)},\boldsymbol{I}_2^{(j)})$ be an element of $\mathrm{Dec}^{(j)}(m,\boldsymbol{y},\alpha_*,\delta)$. We put as follows:

$$\mathfrak{k}(\mathfrak{I}^{(j)}):=\max\bigl\{i\in I_1^{(j)}\,\big|\,i<\min(I_2^{(j)})\bigr\}$$

We also put as follows:

$$\widetilde{\mathcal{M}}(\mathfrak{I}^{(j)}):=\widetilde{\mathcal{M}}^{ss}\bigl(\widehat{\boldsymbol{y}}_1^{(j)},[L],\alpha_*,(\delta,\mathfrak{k}(\mathfrak{I}^{(j)})),I_1^{(j)}\bigr)\times\prod_{i=1}^{j}\mathcal{M}^{ss}(\widehat{\boldsymbol{y}}_2^{(i)},\alpha_*)$$

$$\mathcal{M}_-(\mathfrak{I}^{(j)}):=\mathcal{M}^{ss}\bigl(\widehat{\boldsymbol{y}}_1^{(j)},[L],\alpha_*,\delta_-\bigr)\times\prod_{i=1}^{j}\mathcal{M}^{ss}(\widehat{\boldsymbol{y}}_2^{(i)},\alpha_*)$$

Let $\widehat{E}_1^{(j)}$ denote the universal sheaves on $\widetilde{\mathcal{M}}^{ss}\bigl(\widehat{\boldsymbol{y}}_1^{(j)},[L],\alpha_*,(\delta,\mathfrak{k}(\mathfrak{I}^{(j)})),I_1^{(j)}\bigr)\times X$ and $\mathcal{M}^{ss}\bigl(\widehat{\boldsymbol{y}}_1^{(j)},[L],\alpha_*,\delta_-\bigr)\times X$. The appropriate pull backs are denoted by the same notation. Let $\widehat{E}_2^{(i)}$ $(i=1,\ldots,j)$ denote the universal sheaves on $\mathcal{M}^{ss}\bigl(\widehat{\boldsymbol{y}}_2^{(i)},\alpha_*\bigr)\times X$. The appropriate pull backs are denoted by the same notation.

Let $T^{(j)}_{1,\mathrm{rel}}$ denote the relative tangent bundle of the smooth morphism of $\mathcal{M}\bigl(m,\widehat{\boldsymbol{y}}^{(j)}_1,[L]\bigr)$ to $\mathcal{M}\bigl(m,\widehat{\boldsymbol{y}}^{(j)}_1\bigr)$. Let $\widetilde{T}^{(j)}_{1,\mathrm{rel}}$ denote the relative tangent bundle of $\widetilde{\mathcal{M}}(m,\widehat{\boldsymbol{y}}^{(j)}_1,[L])\longrightarrow\mathcal{M}(m,\widehat{\boldsymbol{y}}^{(j)}_1,[L])$. The appropriate pull backs are denoted by the same notation.

We put as follows for variables $s^{(1)},\ldots,s^{(j)}$:

$$T^{(j)}_1:=-\sum_{h\leq j}\frac{s^{(h)}}{r^{(h)}_1},\quad T^{(i)}_2:=-\sum_{h<i}\frac{s^{(h)}}{r^{(h)}_1},+\frac{s^{(i)}}{r^{(i)}_2},$$

Here, we put as follows:

$$r^{(j)}_1:=\operatorname{rank}\boldsymbol{y}^{(j)}_1,\quad r^{(h)}_1:=r^{(j)}_1+\sum_{h<p\leq j}r^{(p)}_2$$

Let G be the j-dimensional torus $\operatorname{Spec}k[\sigma^{(1)},\ldots,\sigma^{(j)}]$. Let $e^{w\cdot s^{(i)}}$ denote the trivial line bundle with G-action which is induced by the $\operatorname{Spec}k[\sigma^{(i)},\sigma^{(i)\,-1}]$-action of weight w. We have the following element of the K-group of the G-equivariant coherent sheaves on $\mathcal{M}(\widehat{\boldsymbol{y}}_0,\widehat{\boldsymbol{Y}},[L])$:

$$\begin{aligned}&-Rp_{X\,*}\mathcal{RH}om\bigl(\widehat{E}^{(a)}_i\cdot e^{T^{(a)}_i},\,\widehat{E}^{(b)}_l\cdot e^{T^{(b)}_l}\bigr)-Rp_{X\,*}\mathcal{RH}om\bigl(\widehat{E}^{(b)}_l\cdot e^{T^{(b)}_l},\,\widehat{E}^{(a)}_i\cdot e^{T^{(a)}_i}\bigr)\\&\quad-Rp_{D\,*}\mathcal{RH}om'\bigl(\widehat{E}^{(a)}_{i|D\,*}\cdot e^{T^{(a)}_i},\,\widehat{E}^{(b)}_{l|D\,*}\cdot e^{T^{(b)}_l}\bigr)-Rp_{D\,*}\mathcal{RH}om'\bigl(\widehat{E}^{(b)}_{l|D\,*}\cdot e^{T^{(b)}_l},\,\widehat{E}^{(a)}_{i|D\,*}\cdot e^{T^{(a)}_i}\bigr)\end{aligned}\tag{255}$$

The equivariant Euler class is denoted by $Q\bigl(\widehat{E}^{(a)}_i\cdot e^{T^{(a)}_i},\widehat{E}^{(b)}_l\cdot e^{T^{(b)}_l}\bigr)$, which is regarded as the element of $\mathcal{R}(\widehat{E}^{(j)}_1)\otimes\bigotimes_{h=1}^{j}\mathcal{R}(\widehat{E}^{(h)}_2)\otimes\mathfrak{R}\bigl(s^{(1)},s^{(2)},\ldots,s^{(j-1)},s^{(j)}\bigr)$. We put as follows:

$$Q\bigl(\widehat{E}^{(j)}_1e^{T^{(j)}_1},\widehat{E}^{(j)}_2e^{T^{(j)}_2},\ldots,\widehat{E}^{(1)}_2e^{T^{(1)}_2}\bigr):=\prod_{h<i\leq j}Q\bigl(\widehat{E}^{(h)}_2e^{T^{(h)}_2},\widehat{E}^{(i)}_2e^{T^{(i)}_2}\bigr)\times\prod_{h\leq j}Q\bigl(\widehat{E}^{(j)}_1e^{T^{(j)}_1},\widehat{E}^{(h)}_2e^{T^{(h)}_2}\bigr)\tag{256}$$

Then, we put as follows:

$$\Psi^{(j)}(\boldsymbol{y}^{(j)}_1,\boldsymbol{Y}^{(j)}_2):=\operatorname*{Res}_{s^{(j)}}\cdots\operatorname*{Res}_{s^{(1)}}\left(\frac{P\bigl(\widehat{E}^{(j)}_1\cdot e^{T^{(j)}_1}\oplus\bigoplus_{i=0}^{k}\widehat{E}^{(i)}_2\cdot e^{T^{(i)}_2}\bigr)}{Q\bigl(\widehat{E}^{(j)}_1\cdot e^{T^{(j)}_1},\widehat{E}^{(j)}_2\cdot e^{T^{(j)}_2},\ldots,\widehat{E}^{(1)}_2\cdot e^{T^{(1)}_2}\bigr)}\right)\cdot\frac{\mathrm{Eu}(T^{(j)}_{1,\mathrm{rel}})}{N_L(y^{(j)}_1)}$$

We also put as follows:

$$\widetilde{\Psi}^{(j)}(\boldsymbol{y}^{(j)}_1,\boldsymbol{Y}^{(j)}_2):=\Psi^{(j)}(\boldsymbol{y}^{(j)}_1,\boldsymbol{Y}^{(j)}_2)\cdot\frac{\mathrm{Eu}(\widetilde{T}^{(j)}_{1,\mathrm{rel}})}{H_{y^{(j)}_1}(m)!}$$

Lemma 7.45 *For each j, we have the following formula:*

$$\begin{aligned}\Phi(\delta_+)-\Phi(\delta_-)=&\sum_{i<j}\sum_{\mathfrak{I}^{(i)}\in\mathrm{Dec}^{(i)}(m,\boldsymbol{y},\alpha_*,\delta)}\frac{N_L(y^{(i)}_1)}{N_L(y)}\frac{H_{y^{(i)}_1}(m)!\cdot\prod_{h=1}^{i}H_{y^{(h)}_2}(m)!}{H_y(m)!}\int_{\mathcal{M}_-(\mathfrak{I}^{(i)})}\Psi^{(i)}(\boldsymbol{y}^{(i)}_1,\boldsymbol{Y}^{(i)}_2)\\&+\sum_{\mathfrak{I}^{(j)}\in\mathrm{Dec}^{(j)}(m,\boldsymbol{y},\alpha_*,\delta)}\frac{N_L(y^{(j)}_1)}{N_L(y)}\frac{H_{y^{(j)}_1}(m)!\cdot\prod_{h=1}^{j}H_{y^{(h)}_2}(m)!}{H_y(m)!}\int_{\widetilde{\mathcal{M}}(\mathfrak{I}^{(j)})}\widetilde{\Psi}^{(j)}(\boldsymbol{y}^{(j)}_1,\boldsymbol{Y}^{(j)}_2)\end{aligned}\tag{257}$$

Proof We use an induction on j. In the case $j=1$, the claim is the proven in Proposition 7.43 (the case $\ell=0$). Assume that the formula (257) holds for j, and we will prove it for $j+1$. By definition, we have the naturally defined morphism $\pi_j:\mathrm{Dec}^{(j)}\longrightarrow\mathrm{Dec}^{(j-1)}$. Due to Proposition 7.44, we obtain the following equality:

$$\begin{aligned}&\int_{\widetilde{\mathcal{M}}(\mathfrak{I}^{(j)})}\widetilde{\Psi}^{(j)}(\boldsymbol{y}^{(j)}_1,\boldsymbol{Y}^{(j)}_2)-\int_{\mathcal{M}_-(\mathfrak{I}^{(j)})}\Psi^{(j)}(\boldsymbol{y}^{(j)}_1,\boldsymbol{Y}^{(j)}_2)=\\&\qquad\sum_{\substack{\mathfrak{I}^{(j+1)}\in\mathrm{Dec}^{(j+1)}(m,\boldsymbol{y},\alpha_*,\delta)\\\pi_j(\mathfrak{I}^{(j+1)})=\mathfrak{I}^{(j)}}}\frac{N_L(y^{(j+1)}_1)}{N_L(y^{(j)}_1)}\cdot\frac{H_{y^{(j+1)}_1}(m)\cdot H_{y^{(j+1)}_2}(m)}{H_{y^{(j)}_1}(m)!}\int_{\widetilde{\mathcal{M}}(\mathfrak{I}^{(j+1)})}\widetilde{\Psi}'^{\,(j+1)}(\boldsymbol{y}^{(j+1)}_1,\boldsymbol{Y}^{(j+1)}_2)\end{aligned}\tag{258}$$

Here, the elements $\widetilde{\Psi}'^{(j+1)}(\boldsymbol{y}_1^{(j+1)},\boldsymbol{Y}_2^{(j+1)})\in\mathcal{R}(\widehat{E}_1^{(j+1)})\otimes\bigotimes_{h=1}^{j+1}\mathcal{R}(\widehat{E}_2^{(h)})$ are given as follows:

$$\begin{aligned}&\widetilde{\Psi}'^{(j+1)}(\boldsymbol{y}_1^{(j+1)},\boldsymbol{Y}_2^{(j+1)})=\\&\quad\operatorname*{Res}_{s^{(j+1)}}\operatorname*{Res}_{s^{(j)}}\cdots\operatorname*{Res}_{s^{(1)}}\Biggl(\frac{P\bigl(E_1^{(j+1)}e^{T_1^{(j+1)}}\oplus\bigoplus_{i=1}^{j}E_2^{(i)}\cdot e^{T_2^{(i)}}\bigr)}{\prod_{h<j+1}Q\bigl(E_1^{(j+1)}e^{T_1^{(j+1)}},E_2^{(h)}e^{T_2^{(h)}}\bigr)\cdot\prod_{h<i\le j+1}Q\bigl(E_2^{(h)}e^{T_2^{(h)}},E_2^{(i)}e^{T_2^{(i)}}\bigr)}\\&\qquad\qquad\times\frac{1}{Q\bigl(E_1^{(j+1)}e^{-s^{(j+1)}/r_1^{(j+1)}},E_2^{(j+1)}e^{-s^{(j+1)}/r_2^{(j+1)}}\bigr)}\Biggr)\end{aligned}\tag{259}$$

It is easy to observe $\widetilde{\Psi}'^{(j+1)}(\boldsymbol{y}_1^{(j+1)},\boldsymbol{Y}_2^{(j+1)})=\widetilde{\Psi}^{(j+1)}(\boldsymbol{y}_1^{(j+1)},\boldsymbol{Y}_2^{(j+1)})$. Thus the formula (257) holds for $j+1$. ∎

7.6.4 Step 3

When j is sufficiently large, we have $\mathrm{Dec}^{(j)}(m,\boldsymbol{y},\alpha_*,\delta)=\emptyset$. Therefore, we obtain the following:

$$\Phi(\delta_+)-\Phi(\delta_-)=\sum_k\sum_{\mathfrak{I}^{(k)}\in\mathrm{Dec}^{(k)}(m,\boldsymbol{y},\alpha_*,\delta)}\frac{N_L(y_1^{(k)})}{N_L(y)}\frac{H_{y_1^{(k)}}(m)!\cdot\prod_{h=1}^k H_{y_2^{(h)}}(m)!}{H_y(m)!}\int_{\mathcal{M}_-(\mathfrak{I}^{(k)})}\Psi^{(k)}(\boldsymbol{y}_1^{(k)},\boldsymbol{Y}_2^{(k)})\tag{260}$$

We have the map $\rho_k:\mathrm{Dec}^{(k)}(m,\boldsymbol{y},\alpha_*,\delta)\longrightarrow\overline{S}_k(\boldsymbol{y},\delta)$ given as follows:

$$\rho_k(\mathfrak{I}^{(k)})=\bigl(\boldsymbol{y}_1^{(k)},\boldsymbol{y}_2^{(k)},\boldsymbol{y}_2^{(k-1)},\ldots,\boldsymbol{y}_2^{(1)}\bigr)=(\boldsymbol{y}_0,\boldsymbol{y}_1,\boldsymbol{y}_2,\ldots,\boldsymbol{y}_k)$$

Then, (260) can be rewritten as follows:

$$\Phi(\delta_+)-\Phi(\delta_-)=\sum_k\sum_{(\boldsymbol{y}_0,\boldsymbol{Y})\in\overline{S}_k(\boldsymbol{y},\delta)}\frac{N_L(y_0)}{N_L(y)}\frac{\prod_{i=0}^k H_{y_i}(m)!}{H_y(m)!}\cdot\#\rho_k^{-1}(\boldsymbol{y}_0,\boldsymbol{Y})\cdot\int_{\mathcal{M}(\widehat{\boldsymbol{y}}_0,\widehat{\boldsymbol{Y}},[L])}\Psi(\boldsymbol{y}_0,\boldsymbol{Y})\tag{261}$$

Lemma 7.46 *Under Condition* 7.41*, we have the following equality:*

$$\frac{\prod_{i=0}^k H_{y_i}(m)!}{H_y(m)!}\cdot\#\rho_k^{-1}(\boldsymbol{y}_0,\boldsymbol{Y})=W(\boldsymbol{Y})$$

Proof It is easy to observe that $\rho_k^{-1}(\boldsymbol{y}_0,\boldsymbol{Y})$ is bijective to the following set:

$$\Bigl\{(I_0,I_1,\ldots,I_k)\Bigm|\coprod_{i=0}^k I_i=\{1,\ldots,H_y(m)\},\ |I_i|=H_{y_i(m)},\ \min(I_k)<\min(I_{k-1})<\cdots<\min(I_1)\Bigr\}$$

The correspondence is given by $(I_1^{(k)},I_2^{(k)},\ldots,I_2^{(1)})=(I_0,I_1,\ldots,I_k)$.

We put $N=H_y(m)$ and $N_i=H_{y_i}(m)$. We have the following:

$$\begin{aligned}\#\rho_k^{-1}(\boldsymbol{y}_0,\boldsymbol{Y})&=\frac{N!}{N_0!(N-N_0)!}\cdot\frac{(N-N_0-1)!}{(N_k-1)!(N-N_0-N_k)!}\cdot\frac{(N-N_0-N_k-1)!}{(N_{k-1}-1)!(N-N_0-N_k-N_{k-1})!}\cdots\\&=\frac{N!}{N_0!(N-N_0)!}\cdot\prod_{i=1}^k\frac{(N-N_0-\sum_{j>i}N_j-1)!}{(N_i-1)!(N-N_0-\sum_{j\ge i}N_j)!}=\frac{N!}{\prod_{i=0}^k N_i!}\cdot\frac{\prod_{i=1}^k N_i}{\prod_{i=1}^k\sum_{1\le j\le i}N_j}\end{aligned}\tag{262}$$

Under the condition 7.41, we have $N_i/r_i=N_j/r_j$ for $1\le i,j\le k$. Therefore, we obtain the following:

$$\#\rho_k^{-1}(\boldsymbol{y}_0,\boldsymbol{Y})\cdot\frac{\prod_{i=1}^k N_i!}{N!}=\frac{\prod_{i=1}^k N_i}{\prod_{i=1}^k\sum_{1\le j\le i}N_j}=\frac{\prod_{i=1}^k r_i}{\prod_{i=1}^k\sum_{1\le j\le i}r_j}$$

Thus the proof of Lemma 7.46 is finished. ∎

We immediately obtain (250) from (261) and Lemma 7.46. ∎

7.7 Weak Wall Crossing Formula

7.7.1 Statement

Let y be an element of $\mathcal{T}ype_r^{\circ}$. We use the notation $\mathcal{M}_H(\widehat{y})$ to denote the moduli stack of torsion-free sheaves of type y, which are semistable with respect to a polarization H. For $\Phi = P(\widehat{E}^u) \in \mathcal{R}(\widehat{E}^u)$, we put as follows:

$$\Phi_H(\widehat{y}) := \int_{\mathcal{M}_H(\widehat{y})} \Phi \tag{263}$$

Let ξ be an element of $NS(X)$ such that $\xi^2 < 0$. Let W^{ξ} be the wall in the ample cone determined by ξ. It is well known that the ample cone is divided into the chambers by such walls, and $\mathcal{M}_H(\widehat{y})$ depends only on the chamber to which H belongs. Moreover, the set of such walls are locally finite ([42]).

Let C_+ and C_- be chambers which are divided by a wall W^{ξ}. Let H_+ and H_- be ample line bundles contained in H_+ and H_- respectively. We assume $(\xi, H_+) > (\xi, H_-)$. We would like to obtain the description $\Phi_{H_+} - \Phi_{H_-}$ as the sum of the integrals over the products of the moduli stacks of the objects with lower ranks. In the following, r_i, a_i and b_i denote the rank, the first Chern class and the second Chern character of a given $y_i \in \mathcal{T}ype^{\circ}$, respectively. We take a generic $H \in W^{\xi}$.

Let $S_{+,k}$ denote the set of $\mathfrak{y} = (y_0, y_1, \ldots, y_k) \in \mathcal{T}ype^{\circ\,k+1}$ satisfying the following conditions:

- There exist $A_i \geq 0$ $(i = 1, \ldots, k)$ such that $a_i/r_i - a_{i-1}/r_{i-1} = A_i \cdot \xi$ in $H^2(X)$.
- In the case $A_1 = 0$, the inequality $b_0/r_0 < b_1/r_1$ holds.
- In the case $A_i = 0$ for some $i \geq 2$, we have $b_i/r_i \leq b_{i+1}/r_{i+1}$.

Let $S_{-,k}$ denote the set of $\mathfrak{y} = (y_0, y_1, \ldots, y_k) \in \mathcal{T}ype^{\circ\,k+1}$ satisfying the following conditions:

- There exist $A_i \leq 0$ $(i = 1, \ldots, k)$ such that $a_i/r_i - a_{i-1}/r_{i-1} = A_i \cdot \xi$.
- In the case $A_1 = 0$, the inequality $b_0/r_0 < b_1/r_1$ holds.
- In the case $A_i = 0$ for some $i \geq 2$, we have $b_i/r_i \leq b_{i+1}/r_{i+1}$.

Then, we put as follows:

$$S_+ := \bigcup_k S_{+,k}, \quad S_- := \bigcup_k S_{-,k}$$

Let $\mathfrak{y} = (y_0, \ldots, y_k)$ be an element of $S_{+,k}$ or $S_{-,k}$. Then the integers $1 = i(1) < i(2) < \cdots < i(s) \leq k$ are determined by the following condition:

$$P_{y_{i(j)-1}} \neq P_{y_{i(j)}} = \cdots = P_{y_{i(j+1)-1}} \neq P_{y_{i(j+1)}}$$

We put $i(s+1) = k+1$ formally. Then, we put as follows:

$$\overline{B}(\mathfrak{y}) := \frac{r_0}{r} B(\mathfrak{y}), \quad B(\mathfrak{y}) := \frac{\prod_{i=1}^{k} r_i}{\prod_{j=1}^{s} \prod_{h=i(j)}^{i(j+1)-1} \sum_{i(j) \leq l \leq h} r_l} \tag{264}$$

We also put as follows, for $\kappa = \pm$

$$\mathcal{M}_{H_\kappa}(\widehat{\mathfrak{y}}) = \prod_{i=0}^{k} \mathcal{M}_{H_\kappa}(\widehat{y}_i)$$

Let $\widehat{E}_i^u$ denote the sheaf which is the pull back of the universal sheaf over $\mathcal{M}(\widehat{y}_i) \times X$.

When $\mathfrak{y}$ is given, we put as follows for variables $t_1, \ldots, t_k$:

$$T_0 = -\sum_{j>0} \frac{t_j}{\sum_{0 \leq h < j} \operatorname{rank}(\boldsymbol{y}_h)}, \quad T_i = -\sum_{j>i} \frac{t_j}{\sum_{0 \leq h < j} \operatorname{rank}(\boldsymbol{y}_h)} + \frac{t_i}{\operatorname{rank}(\boldsymbol{y}_i)}$$

Let G denote the k-dimensional torus $\operatorname{Spec} k[\tau_1,\tau_1^{-1},\ldots,\tau_k,\tau_k^{-1}]$. Let $e^{w\cdot t_i}$ denote the trivial line bundle with G-action which is induced by the $\operatorname{Spec} k[\tau_i,\tau_i^{-1}]$-action of weight w. We have the following element of the K-group of the G-equivariant coherent sheaves over $\mathcal{M}(\widehat{\mathfrak{y}})$ as in the subsubsection 7.6.1:

$$-Rp_{X\,*}\Bigl(\mathcal{RH}om\bigl(\widehat{E}_i\cdot e^{T_i},\,\widehat{E}_j\cdot e^{T_j}\bigr)\Bigr)-Rp_{X\,*}\Bigl(\mathcal{RH}om\bigl(\widehat{E}_j\cdot e^{T_j},\,\widehat{E}_i\cdot e^{T_i}\bigr)\Bigr)$$

The equivariant Euler class is denoted by $Q\bigl(\widehat{E}_i\cdot e^{T_i},\widehat{E}_j\cdot e^{T_j}\bigr)$. We regarded it as the element of $\bigotimes_{i=0}^{k}\mathcal{R}(\widehat{E}_i)\otimes\mathfrak{R}(t_k,\ldots,t_1)$. We put as follows:

$$Q\bigl(\widehat{E}_0\cdot e^{T_0},\widehat{E}_1\cdot e^{T_1},\ldots,\widehat{E}_k\cdot e^{T_k}\bigr):=\prod_{i<j}Q\bigl(\widehat{E}_i\cdot e^{T_i},\widehat{E}_j\cdot e^{T_j}\bigr)$$

Then, we put as follows:

$$\Psi(\mathfrak{y}):=\operatorname*{Res}_{t_1}\cdots\operatorname*{Res}_{t_k}\left(\frac{P\bigl(\bigoplus_{i=0}^{k}\widehat{E}^u_i\cdot e^{T_i}\bigr)}{Q\bigl(\widehat{E}^u_0\cdot e^{T_0},\ldots,\widehat{E}^u_k\cdot e^{T_k}\bigr)}\right)\tag{265}$$

Theorem 7.47 *We have the following formula:*

$$\Phi_{H_+}(\widehat{y})-\Phi_{H_-}(\widehat{y})=-\sum_{\mathfrak{y}\in S_+}\overline{B}(\mathfrak{y})\cdot\int_{\mathcal{M}_{H_+}(\widehat{\mathfrak{y}})}\Psi(\mathfrak{y})+\sum_{\mathfrak{y}\in S_-}\overline{B}(\mathfrak{y})\cdot\int_{\mathcal{M}_{H_-}(\widehat{\mathfrak{y}})}\Psi(\mathfrak{y})\tag{266}$$

We will write down the formula for the rank 3 case in the subsubsection 7.7.3.

7.7.2 Proof of the theorem

We use the argument and the notation in the proof of Theorem 7.39.

Lemma 7.48 *Let $\mathfrak{y}=(y_0,\ldots,y_k)$ be an element of $\mathcal{T}ype^{\circ k+1}$ such that $\sum y_i=y$ and $\mu(y_i)=\mu(y)$. We also assume that there are μ-semistable torsion-free sheaves of type y_i $(i=0,\ldots,k)$. Then, it is contained in $S_{+,k}$ if and only if the following inequality holds for any sufficiently large t:*

$$P_{y_0(C)}(t)<P_{y_1(C)}(t)\le P_{y_2(C)}(t)\le\cdots\le P_{y_k(C)}(t)\tag{267}$$

Similarly, $\mathfrak{y}$ is contained in $S_{-,k}$ if and only if the following inequality holds for any sufficiently large t:

$$P_{y_0(-C)}(t)<P_{y_1(-C)}(t)\le P_{y_2(-C)}(t)\le\cdots\le P_{y_k(-C)}(t)$$

Proof We remark that finiteness of $\mathfrak{y}$ as in the assumption of the lemma, due to the boundedness of $\overline{\mathcal{S}}$. The condition (267) is equivalent to the following:

$$\frac{b_0+(a_0,C)}{r_0}<\frac{b_1+(a_1,C)}{r_1}\le\frac{b_2+(a_2,C)}{r_2}\le\cdots\le\frac{b_k+(a_k,C)}{r_k}\tag{268}$$

We have $(a_{i+1},H)/r_{i+1}=(a_i,H)/r_i$, and H is generic in W^{ξ}. Hence we have $a_{i+1}/r_{i+1}-a_i/r_i=A_i\cdot\xi$ for some rational numbers A_i, and we have $(a_{i+1},C)/r_{i+1}-(a_i,C)/r_i=A_i\cdot(\xi,C)$. Since (ξ,C) is sufficiently large, the condition (268) is equivalent to $\mathfrak{y}\in S_{+,k}$. Thus the first claim is proved. The second claim can be shown similarly. ∎

For any $\mathfrak{y}=(y_0,\ldots,y_k)\in S_+$, we put as follows:

$$\mathcal{M}\bigl(\widehat{\mathfrak{y}}(C),[\mathcal{O}(-m,C)]\bigr):=\mathcal{M}^s\bigl(\widehat{y}_0(C),[\mathcal{O}(-m,C)],\epsilon\bigr)\times\prod_{i=1}^{k}\mathcal{M}^{ss}\bigl(\widehat{y}_i(C)\bigr)$$

Here, ϵ denotes any sufficiently small positive number. Similarly, we put as follows for any $\mathfrak{y}\in S_-$:

$$\mathcal{M}\bigl(\widehat{\mathfrak{y}}(-C),[\mathcal{O}(-m,-C)]\bigr):=\mathcal{M}^s\bigl(\widehat{y}_0(-C),[\mathcal{O}(-m,-C)],\epsilon\bigr)\times\prod_{i=1}^{k}\mathcal{M}^{ss}\bigl(\widehat{y}_i(-C)\bigr)$$

Let T_{rel} denote the vector bundle over $\mathcal{M}\big(\widehat{\mathfrak{y}}(C),[\mathcal{O}(-m,C)]\big)$ obtained from the relative tangent bundle of the smooth morphism $\mathcal{M}^s\big(\widehat{y}(C),[\mathcal{O}(-m,C)],\epsilon\big)\longrightarrow\mathcal{M}(m,\widehat{y}(C))$. We use the same notation to denote the vector bundle over $\mathcal{M}^s\big(\widehat{\mathfrak{y}}(-C),[\mathcal{O}(-m,-C)],\epsilon\big)$ obtained from the relative tangent bundle of the smooth morphism $\mathcal{M}\big(\widehat{y}(-C),[\mathcal{O}(-m,-C)],\epsilon\big)\longrightarrow\mathcal{M}(m,\widehat{y}(-C))$.

Let δ_0 be as in the proof of Theorem 7.39. To obtain the description for $\widetilde{\Phi}_C(\delta_0)-\Phi(H_+)$ and $\widetilde{\Phi}_{-C}(\delta_0)-\Phi(H_-)$, we use the transition formula (Theorem 7.42) inductively. Due to Lemma 7.48, we obtain the following equality:

$$\widetilde{\Phi}_C(\delta_0)=\Phi_{H_+}(\widehat{y})+\sum_{\mathfrak{y}\in S_+}\frac{H_{y_0}(m)}{H_y(m)}\cdot B(\mathfrak{y})\cdot\int_{\mathcal{M}(\widehat{\mathfrak{y}}(C),[\mathcal{O}(-m,C)])}\Psi(\mathfrak{y})\cdot\frac{\mathrm{Eu}(T_{\mathrm{rel}})}{H_{y_0}(m)}\tag{269}$$

We also obtain the following equality:

$$\widetilde{\Phi}_{-C}(\delta_0)=\Phi_{H_-}(\widehat{y})+\sum_{\mathfrak{y}\in S_-}\frac{H_{y_0}(m)}{H_y(m)}\cdot B(\mathfrak{y})\cdot\int_{\mathcal{M}(\widehat{\mathfrak{y}}(-C),[\mathcal{O}(-m,-C)])}\Psi(\mathfrak{y})\cdot\frac{\mathrm{Eu}(T_{\mathrm{rel}})}{H_{y_0}(m)}\tag{270}$$

From the equality $\widetilde{\Phi}_{-C}(\delta_0)=\widetilde{\Phi}_C(\delta_0)$ (Lemma 7.40), we obtain the following equality:

$$\begin{aligned}\Phi_{H_+}(\widehat{y})&+\sum_{\mathfrak{y}\in S_+}\frac{H_{y_0}(m)}{H_y(m)}\cdot B(\mathfrak{y})\cdot\int_{\mathcal{M}(\widehat{\mathfrak{y}}(C),[\mathcal{O}(-m,C)])}\Psi(\mathfrak{y})\cdot\frac{\mathrm{Eu}(T_{\mathrm{rel}})}{H_{y_0}(m)}\\&=\Phi_{H_-}(\widehat{y})+\sum_{\mathfrak{y}\in S_-}\frac{H_{y_0}(m)}{H_y(m)}\cdot B(\mathfrak{y})\cdot\int_{\mathcal{M}(\widehat{\mathfrak{y}}(-C),[\mathcal{O}(-m,-C)])}\Psi(\mathfrak{y})\cdot\frac{\mathrm{Eu}(T_{\mathrm{rel}})}{H_{y_0}(m)}\end{aligned}\tag{271}$$

By taking the limit $m\to\infty$, we obtain the desired equality (266), due to Proposition 7.19. ∎

7.7.3 Weak wall crossing formula in the rank 3 case

As an example, we write down the formula (266) in the rank 3 case. In the following, a_i and b_i denote the first Chern class and the second Chern character of a given y_i. We put as follows:

$$U_1:=\big\{(y_1,y_2)\in\mathcal{T}ype_1\times\mathcal{T}ype_2\,\big|\,y_1+y_2=y\big\},\quad U_2:=\big\{(y_1,y_2)\in\mathcal{T}ype_2\times\mathcal{T}ype_1\,\big|\,y_1+y_2=y\big\},$$
$$U_3:=\big\{(y_1,y_2,y_3)\in\mathcal{T}ype_1^3\,\big|\,y_1+y_2+y_3=y\big\}$$

We put as follows:

$$\begin{aligned}
&S^{(1)}_{1,+}:=\big\{(y_1,y_2)\in U_1\,\big|\,a_2-2a_1=A\cdot\xi\ (A>0)\big\},\ S^{(2)}_{1,+}:=\big\{(y_1,y_2)\in U_1\,\big|\,2a_1=a_2,\ 2b_1<b_2\big\}\\
&S^{(1)}_{2,+}:=\big\{(y_1,y_2)\in U_2\,\big|\,2a_2-a_1=A\cdot\xi\ (A>0)\big\},\ S^{(2)}_{2,+}:=\big\{(y_1,y_2)\in U_2\big|\,a_1=2a_2,\ b_1<2b_2\big\}\\
&S^{(1)}_{3,+}:=\big\{(y_1,y_2,y_3)\in U_3\,\big|\,a_2-a_1=A\cdot\xi,\ a_3-a_2=B\cdot\xi\ (A,B>0)\big\}\\
&S^{(2)}_{3,+}:=\big\{(y_1,y_2,y_3)\in U_3\,\big|\,a_1+A\cdot\xi=a_2=a_3\ (A>0),\ b_2<b_3\big\}\\
&S^{(3)}_{3,+}:=\big\{(y_1,y_2,y_3)\in U_3\,\big|\,a_1=a_2=a_3-A\cdot\xi\ (A>0),\ b_1<b_2\big\}\\
&S^{(4)}_{3+}:=\big\{(y_1,y_2,y_3)\in U_3\,\big|\,a_1=a_2=a_3,\ b_1<b_2<b_3\big\}\\
&S^{(1)}_{4,+}:=\big\{(y_1,y_2,y_3)\in U_3\,\big|\,a_1+A\cdot\xi=a_2=a_3\ (A>0),\ b_2=b_3\big\}\\
&S^{(2)}_{4,+}:=\big\{(y_1,y_2,y_3)\in U_3\,\big|\,a_1=a_2=a_3,\ b_1<b_2=b_3\big\}
\end{aligned}$$

We also put as follows:

$$\begin{aligned}
&S^{(1)}_{1,-}:=\big\{(y_1,y_2)\in U_1\,\big|\,a_2-2a_1=A\cdot\xi\ (A<0)\big\},\ S^{(2)}_{1,-}:=S'_{1,+}=\big\{(y_1,y_2)\in U_1\,\big|\,2a_1=a_2,\ 2b_1<b_2\big\}\\
&S^{(1)}_{2,-}:=\big\{(y_1,y_2)\in U_2\,\big|\,2a_2-a_1=A\cdot\xi\ (A<0)\big\},\ S^{(2)}_{2,-}:=S^{(2)}_{2,+}=\big\{(y_1,y_2)\in U_2\big|\,a_1=2a_2,\ b_1<2b_2\big\}\\
&S^{(1)}_{3,-}:=\big\{(y_1,y_2,y_3)\in U_3\,\big|\,a_2-a_1=A\cdot\xi,\ a_3-a_2=B\cdot\xi\ (A,B<0)\big\}\\
&S^{(2)}_{3,-}:=\big\{(y_1,y_2,y_3)\in U_3\,\big|\,a_1+A\cdot\xi=a_2=a_3\ (A<0),\ b_2<b_3\big\}\\
&S^{(3)}_{3,-}:=\big\{(y_1,y_2,y_3)\in U_3\,\big|\,a_1=a_2=a_3-A\cdot\xi\ (A<0),\ b_1<b_2\big\}\\
&S^{(4)}_{3,-}:=S^{(4)}_{3,+}=\big\{(y_1,y_2,y_3)\in U_3\,\big|\,a_1=a_2=a_3,\ b_1<b_2<b_3\big\}\\
&S^{(1)}_{4,-}:=\big\{(y_1,y_2,y_3)\in U_3\,\big|\,a_1+A\cdot\xi=a_2=a_3\ (A<0),\ b_2=b_3\big\}\\
&S^{(2)}_{4,-}:=S^{(2)}_{4,+}=\big\{(y_1,y_2,y_3)\in U_3\,\big|\,a_1=a_2=a_3,\ b_1<b_2=b_3\big\}
\end{aligned}$$

We use the notation $\mathcal{M}(\widehat{y}, H_\kappa)$ to denote the moduli stack of oriented torsion-free sheaves of type y which are semistable with respect to H_κ. In the case $\mathrm{rank}(y) = 1$, we omit to denote H_κ. Let (y_0, y_1) be an element of $\mathcal{T}ype^2$. Let G_m be a one dimensional torus. From G_m-equivariant coherent sheaf $\mathcal{E}_i$ $(i = 1, 2)$ on $\mathcal{M}(\widehat{y}_1, H_\kappa) \times \mathcal{M}(\widehat{y}_2, H_\kappa) \times X$, we obtain the following element of the k-group of the G-equivariant coherent sheaves on $\mathcal{M}(\widehat{y}_1, H_\kappa) \times \mathcal{M}(\widehat{y}_2, H_\kappa)$:

$$-Rp_{X\,*}\big(\mathcal{RH}om(\mathcal{E}_1, \mathcal{E}_2)\big) - Rp_{X\,*}\big(\mathcal{RH}om(\mathcal{E}_2, \mathcal{E}_1)\big)$$

The G_m-equivariant Euler class is denoted by $Q(\mathcal{E}_1, \mathcal{E}_2)$. Let P be an element of $\mathcal{R}$. Then, we put as follows:

$$\mathcal{P}(\mathcal{E}_1, \mathcal{E}_2) := \mathcal{P}(\mathcal{E}_1, \mathcal{E}_2) := \frac{P(\mathcal{E}_1 \oplus \mathcal{E}_2)}{Q(\mathcal{E}_1, \mathcal{E}_2)} \in \mathcal{R}(\mathcal{E}_1) \otimes \mathcal{R}(\mathcal{E}_2) \otimes \mathfrak{R}(t)$$

Let (y_1, y_2, y_3) be an element of $\mathcal{T}ype^3$. Let G_m^2 denote the two dimensional torus. Similarly, we put as follows for given G_m^2-equivariant coherent sheaves $\mathcal{E}_i$ $(i = 1, 2, 3)$:

$$\mathcal{P}(\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3) := \frac{P(\mathcal{E}_1 \oplus \mathcal{E}_2 \oplus \mathcal{E}_3)}{\prod_{i<j} Q(\mathcal{E}_i, \mathcal{E}_j)} \in \mathcal{R}(\mathcal{E}_1) \otimes \mathcal{R}(\mathcal{E}_2) \otimes \mathcal{R}(\mathcal{E}_3) \otimes \mathfrak{R}((t_2, t_1)).$$

Then, we put as follows for $\kappa = \pm$:

$$\begin{aligned}
\Lambda(H_\kappa) := \Phi_{H_\kappa}(\widehat{y}) &+ \sum_{i=1,2} \sum_{(y_1,y_2)\in S^{(i)}_{1,\kappa}} \frac{1}{3} \int_{\mathcal{M}(\widehat{y}_1)\times\mathcal{M}(\widehat{y}_2, H_\kappa)} \operatorname*{Res}_t \Big(\mathcal{P}\big(\widehat{E}^u_1 \cdot e^{-t},\, \widehat{E}^u_2 \cdot e^{t/2}\big)\Big) \\
&+ \sum_{i=1,2} \sum_{(y_1,y_2)\in S^{(i)}_{2,\kappa}} \frac{2}{3} \int_{\mathcal{M}(\widehat{y}_1, H_\kappa)\times\mathcal{M}(\widehat{y}_2)} \operatorname*{Res}_t \Big(\mathcal{P}\big(\widehat{E}^u_1 \cdot e^{-t/2},\, \widehat{E}^u_2 \cdot e^{t}\big)\Big) \\
&+ \sum_{i=1}^{4} \sum_{(y_1,y_2,y_3)\in S^{(i)}_{3,\kappa}} \frac{1}{3} \int_{\prod_i \mathcal{M}(\widehat{y}_i)} \operatorname*{Res}_{t_1} \operatorname*{Res}_{t_2} \Big(\mathcal{P}\big(\widehat{E}^u_1 \cdot e^{-t_1-t_2/2},\, \widehat{E}^u_2 \cdot e^{t_1-t_2/2},\, \widehat{E}^u_3 \cdot e^{t_2}\big)\Big) \\
&+ \sum_{i=1,2} \sum_{(y_1,y_2,y_3)\in S^{(i)}_{4,\kappa}} \frac{1}{6} \int_{\prod_i \mathcal{M}(\widehat{y}_i)} \operatorname*{Res}_{t_1} \operatorname*{Res}_{t_2} \Big(\mathcal{P}\big(\widehat{E}^u_1 \cdot e^{-t_1-t_2/2},\, \widehat{E}^u_2 \cdot e^{t_1-t_2/2},\, \widehat{E}^u_3 \cdot e^{t_2}\big)\Big) \qquad (272)
\end{aligned}$$

Then, the formula (266) says $\Lambda(H_+) = \Lambda(H_-)$. The contributions from $S^{(1)}_{3,\kappa}$ and $S^{(2)}_{4,\kappa}$ are cancelled out.

7.7.4 Weak intersection rounding formula in the rank 3 case

The weak wall crossing formula (266) is not so easy to deal with, even in the rank 3 case. We would like to derive a more accessible quantity from our invariants. For that purpose, we consider the "intersection rounding formula". We will show it in the general case, later (the subsection 7.8). However, the general statement is a little complicated. So we give the formula in the rank 3 case, in this subsubsection.

We take an element $\boldsymbol{\xi} = (\xi_1, \xi_2) \in NS(X)^2$ such that ξ_1 and ξ_2 are linearly independent, and we put $W^{\boldsymbol{\xi}} := W^{\xi_1} \cap W^{\xi_2}$. A connected component T of $W^{\boldsymbol{\xi}} \setminus \bigcup_{W \neq W^{\xi_i}} W$ is called a tile. For each tile T, there exists four chambers whose closures contain T. Let C be such a chamber, and let H_C be an ample line bundle contained in C. Then the map $\varphi_C : \{1, 2\} \longrightarrow \{1, -1\}$ is determined by $\varphi_C(i) = \mathrm{sign}(H_C, \xi_i)$. The chambers and the maps correspond bijectively. We denote by $C(\varphi)$ the chamber corresponding to a map $\varphi : \{1, 2\} \longrightarrow \{1, -1\}$. We put $k(\varphi) := \#\{i \,|\, \varphi(i) = -1\}$. We take any ample line bundle $H_\varphi \in C(\varphi)$. Then, we put as follows:

$$\mathcal{D}^T_{\boldsymbol{\xi}} \Phi(\widehat{y}) := \sum_{\varphi} (-1)^{k(\varphi)} \Phi_{H_\varphi}(\widehat{y})$$

We would like to show that $\mathcal{D}^T_{\boldsymbol{\xi}} \Phi(\widehat{y})$ is independent of a choice of T, and we would like to express $\mathcal{D}^T_{\boldsymbol{\xi}} \Phi(\widehat{y})$ as the sum of the integrals over the products of Hilbert schemes.

Let $S(2, 1)$ be the set of $\boldsymbol{a} = (a_0, a_1, a_2) \in NS(X)^3$ with the following property:

- $(a_0+a_1)/2-a_2=A_1\cdot\xi_1$ and $a_0-a_1=A_2\cdot\xi_2$ for some $A_i>0$.

Let $S(1,2)$ be the set of $\boldsymbol{a}=(a_0,a_1,a_2)\in NS(X)^3$ with the following property:

- $a_0-(a_1+a_2)/2=A_1\cdot\xi_1$ and $a_1-a_2=A_2\cdot\xi_2$ for some $A_i>0$.

For each $\boldsymbol{a}$, we put as follows:

$$X(y,\boldsymbol{a}):=\coprod_{n_0+n_1+n_2=N(y,\boldsymbol{a})}\prod_{i=0}^{2}(X^{[n_i]}\times\mathrm{Pic}(a_i))\quad N(y,\boldsymbol{a})=n+\frac{a_0^2+a_1^2+a_2^2-a^2}{2}$$

Here, a and n denote the first Chern class and the second Chern class of y. Let $\widehat{E}_i^u$ denote the sheaf over $\prod(X^{[n_i]}\times\mathrm{Pic}(a_i))\times X$ which is the pull back of the universal ideal sheaf over $X^{[n_i]}\times\mathrm{Pic}(a_i)\times X$.

Let $G=G_m^2$ denote the two dimensional torus. Let $\mathcal{E}_j$ $(j=1,2,3)$ be G-equivariant coherent sheaves on $\prod(X^{[n_i]}\times\mathrm{Pic}(a_i))\times X$. Then, $Q(\mathcal{E}_j,\mathcal{E}_k)$ denotes the G-equivariant Euler class of the following:

$$-Rp_{X\,*}\big(\mathcal{RH}om(\mathcal{E}_j,\mathcal{E}_k)\big)-Rp_{X\,*}\big(\mathcal{RH}om(\mathcal{E}_k,\mathcal{E}_j)\big)$$

We use the following notation:

$$\mathcal{P}(\mathcal{E}_1,\mathcal{E}_2,\mathcal{E}_3):=\frac{P\big(\mathcal{E}_1\oplus\mathcal{E}_2\oplus\mathcal{E}_3\big)}{\prod_{i<j}Q(\mathcal{E}_i,\mathcal{E}_j)}\in\mathcal{R}(\mathcal{E}_1)\otimes\mathcal{R}(\mathcal{E}_2)\otimes\mathcal{R}(\mathcal{E}_3)\otimes\mathfrak{R}(t_1,t_2)$$

Proposition 7.49

- *$\mathcal{D}^T_{\boldsymbol{\xi}}\Phi(\widehat{y})$ is independent of the choice of T. Therefore, we can omit to denote T.*
- *The following equality holds:*

$$\mathcal{D}_{\boldsymbol{\xi}}\Phi(\widehat{y})=\sum_{\boldsymbol{a}\in S(1,2)}\int_{X(y,\boldsymbol{a})}\operatorname*{Res}_{t_2}\operatorname*{Res}_{t_1}\Big(\mathcal{P}(\widehat{E}_0^u e^{-t_1},\ \widehat{E}_1 e^{t_1/2-t_2},\ \widehat{E}_2^u e^{t_1/2+t_2})\Big)$$
$$+\sum_{\boldsymbol{a}\in S(2,1)}\int_{X(y,\boldsymbol{a})}\operatorname*{Res}_{t_2}\operatorname*{Res}_{t_1}\Big(\mathcal{P}\big(\widehat{E}_0^u e^{-t_1/2-t_2},\ \widehat{E}_1^u e^{-t_1/2+t_2},\ \widehat{E}_2^u e^{t_1}\big)\Big)\quad(273)$$

Proof We use the notation in the subsubsection 7.7.3. We identify φ and $(\varphi(1),\varphi(2))$. We have the equality:

$$\Lambda(H_{+,+})-\Lambda(H_{+,-})=\Lambda(H_{+,-})-\Lambda(H_{-,-})\quad(274)$$

We put as follows:

$$S^{(1)}_{1,+}:=\big\{(y_1,y_2)\in U_1\,\big|\,a_2-2a_1=A\cdot\xi_1\ (A>0)\big\},\ \ S^{(2)}_{1,+}:=\big\{(y_1,y_2)\in U_1\,\big|\,2a_1=a_2,\ 2b_1<b_2\big\}$$
$$S^{(1)}_{2,+}:=\big\{(y_1,y_2)\in U_2\,\big|\,2a_2-a_1=A\cdot\xi_1\ (A>0)\big\},\ \ S^{(2)}_{2,+}:=\big\{(y_1,y_2)\in U_2\big|\,a_1=2a_2,\ b_1<2b_2\big\}$$

We also put as follows:

$$S^{(1)}_{1,-}:=\big\{(y_1,y_2)\in U_1\,\big|\,a_2-2a_1=A\cdot\xi_1\ (A<0)\big\},\quad S^{(2)}_{1,-}:=S'_{1,+}$$
$$S^{(1)}_{2,-}:=\big\{(y_1,y_2)\in U_2\,\big|\,2a_2-a_1=A\cdot\xi_1\ (A<0)\big\},\quad S^{(2)}_{2,-}:=S^{(2)}_{2,+}$$

Then, we have the following equality for $\kappa = \pm$:

$$\begin{aligned}\Lambda(H_{\kappa,+}) - \Lambda(H_{\kappa,-}) = \Phi(H_{\kappa,+}) - \Phi(H_{\kappa,-})\\
+\sum_{i=1,2}\sum_{(y_1,y_2)\in S^{(i)}_{1,\kappa}} \frac{1}{3}\int_{\mathcal{M}(\widehat{y}_1)\times\mathcal{M}(\widehat{y}_2,H_{\kappa,+})} \operatorname*{Res}_t\Big(\mathcal{P}\big(\widehat{E}^u_1\cdot e^{-t},\ \widehat{E}^u_2\cdot e^{t/2}\big)\Big)\\
-\sum_{i=1,2}\sum_{(y_1,y_2)\in S^{(i)}_{1,\kappa}} \frac{1}{3}\int_{\mathcal{M}(\widehat{y}_1)\times\mathcal{M}(\widehat{y}_2,H_{\kappa,-})} \operatorname*{Res}_t\Big(\mathcal{P}\big(\widehat{E}^u_1\cdot e^{-t},\ \widehat{E}^u_2\cdot e^{t/2}\big)\Big)\\
+\sum_{i=1,2}\sum_{(y_1,y_2)\in S^{(i)}_{2,\kappa}} \frac{2}{3}\int_{\mathcal{M}(\widehat{y}_1,H_{\kappa,+})\times\mathcal{M}(\widehat{y}_2)} \operatorname*{Res}_t\Big(\mathcal{P}\big(\widehat{E}^u_1\cdot e^{-t/2},\ \widehat{E}^u_2\cdot e^{t}\big)\Big)\\
-\sum_{i=1,2}\sum_{(y_1,y_2)\in S^{(i)}_{2,\kappa}} \frac{2}{3}\int_{\mathcal{M}(\widehat{y}_1,H_{\kappa,-})\times\mathcal{M}(\widehat{y}_2)} \operatorname*{Res}_t\Big(\mathcal{P}\big(\widehat{E}^u_1\cdot e^{-t/2},\ \widehat{E}^u_2\cdot e^{t}\big)\Big)\end{aligned}\tag{275}$$

By a formal calculations using the weak wall crossing formula in the rank 2 (Theorem 7.29), we can show the following equalities:

$$\begin{aligned}\sum_{(y_1,y_2)\in S^{(1)}_{1,-}}\left(\int_{\mathcal{M}(\widehat{y}_1)\times\mathcal{M}(\widehat{y}_2,H_{-+})}\operatorname*{Res}_t\mathcal{P}\big(\widehat{E}^u_1\cdot e^{-t},\ \widehat{E}^u_2\cdot e^{t/2}\big)-\int_{\mathcal{M}(\widehat{y}_1)\times\mathcal{M}(\widehat{y}_2,H_{--})}\operatorname*{Res}_t\mathcal{P}\big(\widehat{E}^u_1\cdot e^{-t},\ \widehat{E}^u_2\cdot e^{t/2}\big)\right)\\
=\sum_{\boldsymbol{a}\in S(1,2)}\operatorname*{Res}_{t_2}\operatorname*{Res}_{t_1}\int_{X(y,\boldsymbol{a})}\mathcal{P}\big(\widehat{E}^u_0\cdot e^{-t_1},\widehat{E}^u_1\cdot e^{t_1/2-t_2},\widehat{E}^u_2\cdot e^{t_1/2+t_2}\big)\end{aligned}\tag{276}$$

$$\begin{aligned}\sum_{(y_1,y_2)\in S^{(1)}_{2,-}}\left(\int_{\mathcal{M}(\widehat{y}_1,H_{-+})\times\mathcal{M}(\widehat{y}_2)}\operatorname*{Res}_t\mathcal{P}\big(\widehat{E}_1\cdot e^{-t/2},\widehat{E}_2\cdot e^{t}\big)-\int_{\mathcal{M}(\widehat{y}_1,H_{--})\times\mathcal{M}(\widehat{y}_2)}\operatorname*{Res}_t\mathcal{P}\big(\widehat{E}_1\cdot e^{-t/2},\widehat{E}_2\cdot e^{t}\big)\right)\\
=\sum_{\boldsymbol{a}\in S(2,1)}\int_{X(y,\boldsymbol{a})}\operatorname*{Res}_{t_2}\operatorname*{Res}_{t_1}\mathcal{P}\big(\widehat{E}_0\cdot e^{-t_1/2-t_2},\widehat{E}_1\cdot e^{-t_1/2+t_2},\widehat{E}_2\cdot e^{t_1}\big)\end{aligned}\tag{277}$$

We also have the following:

$$\begin{aligned}\sum_{(y_1,y_2)\in S^{(1)}_{1,+}}\left(\int_{\mathcal{M}(\widehat{y}_1)\times\mathcal{M}(\widehat{y}_2,H_{++})}\operatorname*{Res}_t\mathcal{P}\big(\widehat{E}^u_1\cdot e^{-t},\widehat{E}^u_2 e^{t/2}\big)-\int_{\mathcal{M}(\widehat{y}_1)\times\mathcal{M}(\widehat{y}_2,H_{+-})}\operatorname*{Res}_t\mathcal{P}\big(\widehat{E}^u_1\cdot e^{-t},\widehat{E}^u_2 e^{t/2}\big)\right)\\
=-\sum_{(y_1,y_2)\in S^{(1)}_{1,+}}\left(\int_{\mathcal{M}(\widehat{y}_2,H_{++})\times\mathcal{M}(\widehat{y}_1)}\operatorname*{Res}_t\mathcal{P}\big(\widehat{E}^u_2 e^{-t/2},\widehat{E}^u_1\cdot e^{t}\big)-\int_{\mathcal{M}(\widehat{y}_2,H_{+-})\times\mathcal{M}(\widehat{y}_1)}\operatorname*{Res}_t\mathcal{P}\big(\widehat{E}^u_2 e^{t/2},\widehat{E}^u_1\cdot e^{-t}\big)\right)\\
=-\sum_{\boldsymbol{a}\in S(2,1)}\int_{X(y,\boldsymbol{a})}\operatorname*{Res}_{t_2}\operatorname*{Res}_{t_1}\mathcal{P}\big(\widehat{E}_0\cdot e^{-t_1/2-t_2},\widehat{E}_1\cdot e^{-t_1/2+t_2},\widehat{E}_2\cdot e^{t_1}\big)\end{aligned}\tag{278}$$

$$\begin{aligned}\sum_{(y_1,y_2)\in S^{(1)}_{2,+}}\left(\int_{\mathcal{M}(\widehat{y}_1,H_{++})\times\mathcal{M}(\widehat{y}_2)}\operatorname*{Res}_t\mathcal{P}\big(\widehat{E}_1\cdot e^{-t/2},\widehat{E}^u_2\cdot e^{t}\big)-\int_{\mathcal{M}(\widehat{y}_1,H_{+-})\times\mathcal{M}(\widehat{y}_2)}\operatorname*{Res}_t\mathcal{P}\big(\widehat{E}_1\cdot e^{-t/2},\widehat{E}^u_2\cdot e^{t}\big)\right)\\
=-\sum_{(y_1,y_2)\in S^{(1)}_{2,+}}\left(\int_{\mathcal{M}(\widehat{y}_2)\times\mathcal{M}(\widehat{y}_1,H_{++})}\operatorname*{Res}_t\mathcal{P}\big(\widehat{E}^u_2\cdot e^{-t},\widehat{E}_1\cdot e^{t/2}\big)-\int_{\mathcal{M}(\widehat{y}_2)\times\mathcal{M}(\widehat{y}_1,H_{+-})}\operatorname*{Res}_t\mathcal{P}\big(\widehat{E}^u_2\cdot e^{t},\widehat{E}_1\cdot e^{-t/2}\big)\right)\\
=-\sum_{\boldsymbol{a}\in S(1,2)}\operatorname*{Res}_{t_2}\operatorname*{Res}_{t_1}\int_{X(y,\boldsymbol{a})}\mathcal{P}\big(\widehat{E}^u_0\cdot e^{-t_1},\widehat{E}^u_1\cdot e^{t_1/2-t_2},\widehat{E}^u_2\cdot e^{t_1/2+t_2}\big)\end{aligned}\tag{279}$$

Recall $S^{(2)}_{1,+}=S^{(2)}_{1,-}$. Then, it is also easy to observe the following equalities by using the weak wall crossing formula in the rank 2 case:

$$\begin{aligned}
&\sum_{(y_1,y_2)\in S^{(2)}_{1,+}} \int_{\mathcal{M}(\widehat{y}_1)\times\mathcal{M}(\widehat{y}_2,H_{+,+})} \operatorname*{Res}_t\Big(\mathcal{P}\big(\widehat{E}^u_1\cdot e^{-t},\,\widehat{E}^u_2\cdot e^{t/2}\big)\Big)\\
&\qquad -\sum_{(y_1,y_2)\in S^{(2)}_{1,+}} \int_{\mathcal{M}(\widehat{y}_1)\times\mathcal{M}(\widehat{y}_2,H_{+,-})} \operatorname*{Res}_t\Big(\mathcal{P}\big(\widehat{E}^u_1\cdot e^{-t},\,\widehat{E}^u_2\cdot e^{t/2}\big)\Big)\\
&\qquad =\sum_{(y_1,y_2)\in S^{(2)}_{1,-}} \int_{\mathcal{M}(\widehat{y}_1)\times\mathcal{M}(\widehat{y}_2,H_{-,+})} \operatorname*{Res}_t\Big(\mathcal{P}\big(\widehat{E}^u_1\cdot e^{-t},\,\widehat{E}^u_2\cdot e^{t/2}\big)\Big)\\
&\qquad\qquad -\sum_{(y_1,y_2)\in S^{(2)}_{1,-}} \int_{\mathcal{M}(\widehat{y}_1)\times\mathcal{M}(\widehat{y}_2,H_{-,-})} \operatorname*{Res}_t\Big(\mathcal{P}\big(\widehat{E}^u_1\cdot e^{-t},\,\widehat{E}^u_2\cdot e^{t/2}\big)\Big)
\end{aligned}\tag{280}$$

$$\begin{aligned}
&\sum_{(y_1,y_2)\in S^{(2)}_{2,+}} \int_{\mathcal{M}(\widehat{y}_1,H_{+,+})\times\mathcal{M}(\widehat{y}_2)} \operatorname*{Res}_t\Big(\mathcal{P}\big(\widehat{E}^u_1\cdot e^{-t/2},\,\widehat{E}^u_2\cdot e^{t}\big)\Big)\\
&\qquad -\sum_{(y_1,y_2)\in S^{(2)}_{2,+}} \int_{\mathcal{M}(\widehat{y}_1,H_{+,-})\times\mathcal{M}(\widehat{y}_2)} \operatorname*{Res}_t\Big(\mathcal{P}\big(\widehat{E}^u_1\cdot e^{-t/2},\,\widehat{E}^u_2\cdot e^{t}\big)\Big)\\
&\qquad =\sum_{(y_1,y_2)\in S^{(2)}_{2,-}} \int_{\mathcal{M}(\widehat{y}_1,H_{-,+})\times\mathcal{M}(\widehat{y}_2)} \operatorname*{Res}_t\Big(\mathcal{P}\big(\widehat{E}^u_1\cdot e^{-t/2},\,\widehat{E}^u_2\cdot e^{t}\big)\Big)\\
&\qquad\qquad -\sum_{(y_1,y_2)\in S^{(2)}_{2,-}} \int_{\mathcal{M}(\widehat{y}_1,H_{-,-})\times\mathcal{M}(\widehat{y}_2)} \operatorname*{Res}_t\Big(\mathcal{P}\big(\widehat{E}^u_1\cdot e^{-t/2},\,\widehat{E}^u_2\cdot e^{t}\big)\Big)
\end{aligned}\tag{281}$$

Then, we obtain (273) by a formal calculation. ∎

7.7.5 Transition for a critical parabolic weight

We give a generalization of Proposition 7.34 in the higher rank case. The argument is essentially same as the proof of Theorem 7.47. Hence we state the result without a proof.

Let α be a critical parabolic weight. We take $\alpha_-<\alpha<\alpha_+$ sufficiently closely. For $\Phi=P(\widehat{E}^u)\in\mathcal{R}(\widehat{E}^u)$, we put as follows:

$$\Phi(\widehat{y},\alpha_\kappa):=\int_{\mathcal{M}(\widehat{y},\alpha_\kappa)}\Phi$$

Let $S_{+,k}(y,\xi,\alpha)$ be the set of $\mathfrak{y}=(y_0,\ldots,y_k)\in NS(X)^{k+1}$ with the following properties:

- $\sum y_i=y$ and $P^\alpha_{y_i}=P^\alpha_y$.
- We have $a_0/r_0-a_1/r_1=m_0\cdot\xi$ for some $m_0<0$.
- We have $a_i/r_i-a_{i+1}/r_{i+1}=m_i\cdot\xi$ for some $m_i\leq 0$. In the case $a_i/r_i=a_{i+1}/r_{i+1}$, we have $y_i/r_i=y_{i+1}/r_{i+1}$.

Here r_i and a_i denote the rank and the first Chern class of y_i, as usual. Let $S_{-,k}(y,\xi,\alpha)$ denote the set of $\mathfrak{y}=(y_0,\ldots,y_k)\in NS(X)^{k+1}$ with the following properties:

- $\sum y_i=y$ and $P^\alpha_{y_i}=P^\alpha_y$.
- We have $a_0/r_0-a_1/r_1=m_0\cdot\xi$ for some $m_0>0$.

- We have $a_i/r_i - a_{i+1}/r_{i+1} = m_i \cdot \xi$ for some $m_i \geq 0$. In the case $a_i/r_i = a_{i+1}/r_{i+1}$, we have $y_i/r_i = y_{i+1}/r_{i+1}$.

Then, we put as follows:

$$S_\kappa(y,\xi,\alpha) := \bigcup_k S_{\kappa,k}(y,\xi,\alpha)$$

For each $\mathfrak{y} = (y_0,\dots,y_k) \in S_{\kappa,k}(y,\xi,\alpha)$, we put as follows:

$$\mathcal{M}(\widehat{\mathfrak{y}},\alpha_\kappa) := \prod_{i=0}^{k} \mathcal{M}(\widehat{y}_i,\alpha_\kappa)$$

Proposition 7.50 *The following equality holds:*

$$\Phi(\widehat{y},\alpha_+) - \Phi(\widehat{y},\alpha_-) = -\sum_{\mathfrak{y}\in S_+(y,\xi,\alpha)} \overline{B}(\mathfrak{y}) \cdot \int_{\mathcal{M}(\widehat{\mathfrak{y}},\alpha_+)} \Psi(\mathfrak{y}) + \sum_{\mathfrak{y}\in S_-(y,\xi,\alpha)} \overline{B}(\mathfrak{y}) \cdot \int_{\mathcal{M}(\widehat{\mathfrak{y}},\alpha_-)} \Psi(\mathfrak{y}) \tag{282}$$

Here, $\Psi(\mathfrak{y})$ denote the elements of $\bigotimes_{i=0}^k \mathcal{R}(\widehat{E}_i^u)$ given by (265)*, and $\overline{B}(\mathfrak{y})$ denote the numbers given by* (264)*.* ∎

As an example, we write down the formula (282) in the case of rank 3. We use the notation in the subsubsection 7.7.3. We put as follows:

$$U_1(\alpha) := \bigl\{(y_1,y_2) \in U_i \bigm| P^\alpha_{y_i} = P^\alpha_y\bigr\} \quad (i=1,2), \quad U_3(\alpha) := \bigl\{(y_1,y_2,y_3) \in U_3 \bigm| P^\alpha_{y_i} = P^\alpha_y\bigr\}$$

We put as follows:

$$\begin{aligned}
S_{1+} &:= \bigl\{(y_1,y_2) \in U_1(\alpha) \bigm| 2a_1 - a_2 = m\cdot\xi \ (m<0)\bigr\}\\
S_{2,+} &:= \bigl\{(y_1,y_2) \in U_2(\alpha) \bigm| a_1 - 2a_2 = m\cdot\xi \ (m<0)\bigr\}\\
S_{3+} &:= \bigl\{(y_1,y_2,y_3) \in U_3(\alpha) \bigm| a_1 - a_2 = m_1\cdot\xi,\ a_2 - a_3 = m_2\cdot\xi \ (m_i<0)\bigr\}\\
S_{4+} &:= \bigl\{(y_1,y_2,y_3) \bigm| a_1 - a_2 = m\cdot\xi \ (m<0),\ \ y_2 = y_3\bigr\}
\end{aligned}$$

Here, a_i denotes the first Chern class of y_i. We also put as follows:

$$\begin{aligned}
S_{1-} &:= \bigl\{(y_1,y_2) \in U_1(\alpha) \bigm| 2a_1 - a_2 = m\cdot\xi \ (m>0)\bigr\}\\
S_{2,-} &:= \bigl\{(y_1,y_2) \in U_2(\alpha) \bigm| a_1 - 2a_2 = m\cdot\xi \ (m>0)\bigr\}\\
S_{3-} &:= \bigl\{(y_1,y_2,y_3) \in U_3(\alpha) \bigm| a_1 - a_2 = m_1\cdot\xi,\ a_2 - a_3 = m_2\cdot\xi \ (m_i>0)\bigr\}\\
S_{4-} &:= \bigl\{(y_1,y_2,y_3) \bigm| a_1 - a_2 = m\cdot\xi \ (m>0),\ \ y_2 = y_3\bigr\}
\end{aligned}$$

We put as follows for $\kappa = \pm$:

$$\begin{aligned}
\Lambda(\alpha_\kappa) := \Phi(\widehat{y},\alpha_\kappa) &+ \sum_{(y_1,y_2)\in S_{1,\kappa}} \frac{1}{3} \int_{\mathcal{M}(\widehat{y}_1)\times\mathcal{M}(\widehat{y}_2,\alpha_\kappa)} \operatorname*{Res}_t \Bigl(\mathcal{P}\bigl(\widehat{E}_1^u \cdot e^{-t},\ \widehat{E}_2^u \cdot e^{t/2}\bigr)\Bigr)\\
&+ \sum_{(y_1,y_2)\in S_{2,\kappa}} \frac{2}{3} \int_{\mathcal{M}(\widehat{y}_1,\alpha_\kappa)\times\mathcal{M}(\widehat{y}_2)} \operatorname*{Res}_t \Bigl(\mathcal{P}\bigl(\widehat{E}_1^u \cdot e^{-t/2},\ \widehat{E}_2^u \cdot e^{t}\bigr)\Bigr)\\
&+ \sum_{(y_1,y_2,y_3)\in S_{3,\kappa}} \frac{1}{3} \int_{\prod_i \mathcal{M}(\widehat{y}_i)} \operatorname*{Res}_{t_1} \operatorname*{Res}_{t_2} \Bigl(\mathcal{P}\bigl(\widehat{E}_1^u \cdot e^{-t_1-t_2/2},\ \widehat{E}_2^u \cdot e^{t_1-t_2/2},\ \widehat{E}_3^u \cdot e^{t_2}\bigr)\Bigr)\\
&+ \sum_{(y_1,y_2,y_3)\in S_{4,\kappa}} \frac{1}{6} \int_{\prod_i \mathcal{M}(\widehat{y}_i)} \operatorname*{Res}_{t_1} \operatorname*{Res}_{t_2} \Bigl(\mathcal{P}\bigl(\widehat{E}_1^u \cdot e^{-t_1-t_2/2},\ \widehat{E}_2^u \cdot e^{t_1-t_2/2},\ \widehat{E}_3^u \cdot e^{t_2}\bigr)\Bigr)
\end{aligned} \tag{283}$$

Then, Proposition 7.50 says $\Lambda(\alpha_+) = \Lambda(\alpha_-)$.

7.8 Weak Intersection Rounding Formula

7.8.1 Preliminary

We prepare some terminology to state the theorem. An oriented tree is a tree provided with an orientation for each edge. For an oriented tree R, let $V(R)$ denote the set of the vertices of R. We have the natural order $\leq_R$ on $V(R)$. Let $V_{\max}(R)$ denote the set of the maximal vertices of R, and we put $V^{\circ}(R) := V(R) - V_{\max}(R)$. An oriented tree R is called an indexed rooted oriented binary plane tree of rank r, if the following conditions are satisfied:

- R is embedded in $\mathbb{R} \times \mathbb{R}_{\geq 0}$, and it intersects with $\mathbb{R} \times \{0\}$ transversally. The intersection $R \cap (\mathbb{R} \times \{0\})$ consists of the maximal points.
- There exists the unique minimal vertex o with respect to the order induced by the orientation. The vertex is called the root.
- The maximal vertices are $\{(0,0),(1,0),\ldots,(r-1,0)\}$.
- For any $v \in V(R) - \big(\{o\} \cup V_{\max}(R)\big)$, there are three edges which contain v.
- A bijective map $\varphi : V^{\circ}(R) \longrightarrow \{1,\ldots,r-1\}$ is provided, which preserves the orders on the sets $V^{\circ}(R)$ and $\{1,\ldots,r-1\}$.

The notion of isotopy is naturally given. Let $\mathfrak{T}(r)$ denote the set of the isotropy classes of an indexed rooted oriented binary plane trees of rank r. It is easy to see $\#\mathfrak{T}(2) = 1$, $\#\mathfrak{T}(3) = 2$ and $\#\mathfrak{T}(4) = 4$, for example.

Remark 7.51 $\mathfrak{T}(r)$ *parameterizes the composition rules.* ∎

Let R be an indexed rooted oriented binary plane tree of rank r. For any $v \in V(R)$, we put as follows:

$$r(v) := \#\big\{u \in V_{\max}(R) \,\big|\, u \geq_R v\big\}$$

We put $j(v) := \max\big\{j \,\big|\, (j,0) \geq_R v\big\}$.

Take $v \in V^{\circ}(R)$. It is easy to see that there are two edges e_1 and e_2 going out from v. Let v_i denote the vertex of e_i which is not v. We may assume $j(v_1) < j(v_2)$, and we put $v^l = v_1$ and $v^r = v_2$.

If v is not the root, there is the unique vertex v^b of R such that there exists the edge from v^b to v. We put as follows:

$$\operatorname{sign}(v) := \begin{cases} 1 & (v = (v^b)^r) \\ -1 & (v = (v^b)^l) \end{cases}$$

We take $t_1,\ldots,t_{r-1}$ be formal variables, and we put as follows for $i = 0,\ldots,r-1$:

$$T_R^i := \sum_{\substack{v \leq_R (i,0) \\ v \neq o}} \operatorname{sign}(v) \frac{t_{\varphi(v^b)}}{r(v)}$$

We identify $(i,0) \in V_{\max}(R)$ and i. For any $\boldsymbol{a} = (a_0,\ldots,a_{r-1}) \in NS^1(X)$ and each $v \in V(R)$, we put as follows:

$$a_v := \sum_{\substack{u \in V_{\max}(R) \\ u \geq_R v}} a_u$$

Let a be an element of $NS^1(X)$. Let $\boldsymbol{\xi} = (\xi_1,\ldots,\xi_{r-1})$ be an element of $NS^1(X)^{r-1}$ such that ξ_i are linearly independent. We denote by $S(r,a,\boldsymbol{\xi},R)$ the set of $\boldsymbol{a} = (a_0,\ldots,a_{r-1}) \in NS^1(X)$ satisfying $\sum a_i = a$ and the following condition:

- There exists a positive rational number A such that the following equality holds in $NS^1(X) \otimes \mathbb{Q}$ for any $v \in V^{\circ}(R)$:
$$\frac{a_{v^l}}{r(v^l)} - \frac{a_{v^r}}{r(v^r)} = A_v \cdot \xi_{\varphi(v)}$$

Let $\boldsymbol{a}$ be any element of $S(r,a,\boldsymbol{\xi},R)$. Let y be an element of $\mathcal{T}ype$ such that $\mathrm{rank}(y)=r$ and $\det(y)=a$. Let n denote the second Chern class of y. We put as follows:

$$\mathcal{M}(\boldsymbol{a},y):=\coprod_{\sum_{i=0}^{r-1}n_i=N(\boldsymbol{a},y)}\prod_{i=0}^{r-1}\bigl(X^{[n_i]}\times \mathrm{Pic}(a_i)\bigr),\quad N(\boldsymbol{a},y):=n-\frac{a^2}{2}+\sum_{i=1}^{r-1}\frac{a_i^2}{2}$$

On $\mathcal{M}(\boldsymbol{a},y)\times X$, we have the sheaf $\widehat{E}^u_i$ which is the pull back of the universal sheaf on $X^{[n_i]}\times\mathrm{Pic}(a_i)\times X$ via the naturally defined projection. Let G denote the $(r-1)$-dimensional torus $\mathrm{Spec}\, k[\tau_1,\tau_1^{-1},\ldots,\tau_{r-1},\tau_{r-1}^{-1}]$. Let $e^{w\cdot t_i}$ denote the trivial line bundle with the G-action induced by the action of $\mathrm{Spec}\, k[\tau_i,\tau_i^{-1}]$ of weight w. We have the following element of the K-group of the G-equivariant coherent sheaves on $\mathcal{M}(\boldsymbol{a},y)$:

$$-Rp_{X\,*}\mathcal{RH}om\bigl(\widehat{E}^u_i\cdot e^{T_i},\ \widehat{E}^u_j\cdot e^{T_j}\bigr)-Rp_{X\,*}\mathcal{RH}om\bigl(\widehat{E}^u_j\cdot e^{T_j},\ \widehat{E}^u_i\cdot e^{T_i}\bigr)$$

The equivariant Euler class is denoted by $Q\bigl(\widehat{E}^u_i\cdot e^{T_i},\widehat{E}^u_j\cdot e^{T_j}\bigr)$. We regard it as the element of $\bigotimes_{i=0}^{r-1}\mathcal{R}(\widehat{E}^u_i)\otimes\mathfrak{R}(t_1,\ldots,t_{r-1})$. We put as follows:

$$Q\bigl(\widehat{E}^u_0\cdot e^{T_0},\widehat{E}^u_1\cdot e^{T_1},\ldots,\widehat{E}^u_{r-1}\cdot e^{T_{r-1}}\bigr):=\prod_{i<j}Q\bigl(\widehat{E}^u_i\cdot e^{T_i},\ \widehat{E}_j\cdot e^{T_j}\bigr)$$

Let P be an element of $\mathcal{R}$. We obtain $P\bigl(\bigoplus_{i=0}^{r-1}\widehat{E}^u_i\cdot e^{T_i}\bigr)\in\bigotimes_{i=0}^{r-1}\mathcal{R}(\widehat{E}^u_i)[t_1,\ldots,t_{r-1}]$ by the homomorphisms in the subsubsection 7.1.3. Then, we put as follows:

$$\Psi(\boldsymbol{a},R):=\operatorname*{Res}_{t_{r-1}}\operatorname*{Res}_{t_{r-2}}\cdots\operatorname*{Res}_{t_1}\left(\frac{P\bigl(\bigoplus_{i=0}^{r-1}\widehat{E}^u_i e^{T_i}\bigr)}{Q\bigl(\widehat{E}^u_0 e^{T_0},\widehat{E}^u_1 e^{T_1},\ldots,\widehat{E}^u_{r-1}e^{T_{r-1}}\bigr)}\right)\in\bigotimes_{i=0}^{r-1}\mathcal{R}(\widehat{E}^u_i)\tag{284}$$

7.8.2 Statement

We take an element $\boldsymbol{\xi}=(\xi_1,\xi_2,\ldots,\xi_l)\in NS^1(X)^l$ such that $\xi_1,\ldots,\xi_l$ are linearly independent. We put $W^{\boldsymbol{\xi}}:=\bigcap_{i=1}^{l}W^{\xi_i}$. A connected component of $W^{\boldsymbol{\xi}}\setminus\bigcup_{W\neq W^{\xi_i}}W$ is called a tile.

Let $\varphi:\{1,\ldots,l\}\longrightarrow\{1,-1\}$ be a map. We have the chamber $C(\varphi)$ satisfying the following conditions:

- The closure of $C(\varphi)$ contains T.
- Let H_φ be any ample line bundle contained in $C(\varphi)$. Then $\mathrm{sign}\bigl(\xi_i,H\bigr)=\varphi(i)$.

We take line bundles $H_\varphi\in C(\varphi)$. We put $k(\varphi):=\#\{i\,|\,\varphi(i)=-1\}$. For $\Phi=P(\widehat{E}^u)\in\mathcal{R}(\widehat{E}^u)$, we put as follows:

$$\mathcal{D}^T_{\boldsymbol{\xi}}\Phi(\widehat{y}):=\sum_{\varphi}(-1)^{k(\varphi)}\cdot\Phi_{H_\varphi}(\widehat{y})$$

Here, $\Phi_{H_\varphi}(\widehat{y})$ is given as in (263).

Theorem 7.52 *Assume $p_g=0$. Let y be an element of $\mathcal{T}ype$, and let $\boldsymbol{\xi}=(\xi_1,\ldots,\xi_l)$ and T be as above.*

- *In the case $l\geq\mathrm{rank}(y)$, we have $\mathcal{D}^T_{\boldsymbol{\xi}}\Phi(y)=0$.*
- *In the case $l=\mathrm{rank}(y)-1$, the number $\mathcal{D}^T_{\boldsymbol{\xi}}\Phi(y)$ is independent of the choice of T, and the following equality holds:*

$$\mathcal{D}_{\boldsymbol{\xi}}\Phi(y)=\sum_{R\in\mathfrak{T}(r)}\sum_{\boldsymbol{a}\in S(r,a,\boldsymbol{\xi},R)}\int_{\mathcal{M}(\boldsymbol{a},y)}\Psi(\boldsymbol{a},R)\tag{285}$$

Here, $\Psi(\boldsymbol{a},R)$ are given in (284).

The theorem will be proved in the subsubsection 7.8.4 after the combinatorial preparation. We have already written down the formula (285) for the rank 3 case in the subsubsection 7.7.4.

7.8.3 Preparation from combinatorics

Let S be a finite set. Let $\mathrm{Map}(S,\{\pm\})$ denote the set of maps of S to $\{\pm\}$. For any $\varphi\in\mathrm{Map}(S,\{\pm\})$, let $k(\varphi)$ denote the number of $\bigl\{i\bigm|\varphi(i)=-\bigr\}$. Let I be a subset of S. For any $\varphi_I\in\mathrm{Map}(I,\{\pm\})$, let $\mathrm{Ind}_S^-(\varphi_I)$ and $\mathrm{Ind}_S^+\varphi_I$ be elements of $\mathrm{Map}(S,\{\pm\})$ determined as follows:

$$\mathrm{Ind}_S^-\bigl(\varphi_I\bigr)(i):=\begin{cases}\varphi_I(i) & (i\in I)\\ - & (i\notin I)\end{cases}\qquad \mathrm{Ind}_S^+\bigl(\varphi_I\bigr)(i):=\begin{cases}\varphi_I(i) & (i\in I)\\ + & (i\notin I)\end{cases}$$

Lemma 7.53 *For any functions $F_i:\mathrm{Map}(S,\{\pm\})\longrightarrow\mathbb{C}$ $(i=1,2)$, we have the following equality:*

$$\sum_{\varphi\in\mathrm{Map}(S,\{\pm\})}(-1)^{k(\varphi)}F_1(\varphi)\cdot F_2(\varphi)=$$
$$\sum_{I\sqcup J=S}\left(\sum_{\varphi_I\in\mathrm{Map}(I,\{\pm\})}(-1)^{k(\varphi_I)}F_1\bigl(\mathrm{Ind}_S^-(\varphi_I)\bigr)\right)\cdot\left(\sum_{\varphi_J\in\mathrm{Map}(J,\{\pm\})}(-1)^{k(\varphi_J)}F_2\bigl(\mathrm{Ind}_S^+(\varphi_J)\bigr)\right)\qquad(286)$$

Proof We use an induction on $|S|$. In the case $|S|=1$, we can identify $\mathrm{Map}(S,\{\pm\})$ and $\{\pm\}$. We have the following:

$$F_1(+)\cdot F_2(+)-F_1(-)\cdot F_2(-)=\bigl(F(+)-F(-)\bigr)\cdot G(+)+F(-)\cdot\bigl(G(+)-G(-)\bigr)$$

Thus, the claim holds in this case.

Assume that the claim holds in the case $|S|=k-1$, and we will show that the claim in the case $|S|=k$. We take any element $s\in S$, and we put $S':=S-\{s\}$. The left hand side of (286) can be rewritten as follows:

$$\sum_{\substack{\varphi\in\mathrm{Map}(S,\{\pm\})\\ \varphi(s)=+}}(-1)^{k(\varphi)}\cdot F_1(\varphi)\cdot F_2(\varphi)+\sum_{\substack{\varphi\in\mathrm{Map}(S,\{\pm\})\\ \varphi(s)=-}}(-1)^{k(\varphi)}\cdot F_1(\varphi)\cdot F_2(\varphi)$$
$$=\sum_{\varphi'\in\mathrm{Map}(S',\{\pm\})}(-1)^{k(\varphi')}F_1\bigl(\mathrm{Ind}_S^+(\varphi')\bigr)\cdot F_2\bigl(\mathrm{Ind}_S^+(\varphi')\bigr)-\sum_{\varphi'\in\mathrm{Map}(S',\{\pm\})}(-1)^{k(\varphi')}F_1\bigl(\mathrm{Ind}_S^-(\varphi')\bigr)\cdot F_2\bigl(\mathrm{Ind}_S^+(\varphi')\bigr)\qquad(287)$$

By using the hypothesis of the induction, the right hand side of (287) can be rewritten as follows:

$$\sum_{I\sqcup J=S'}\sum_{\substack{\varphi_I\in\mathrm{Map}(I,\{\pm\})\\ \varphi_J\in\mathrm{Map}(J,\{\pm\})}}(-1)^{k(\varphi_I)+k(\varphi_J)}F_1\bigl(\mathrm{Ind}_S^+(\mathrm{Ind}_{S'}^-(\varphi_I))\bigr)\cdot F_2\bigl(\mathrm{Ind}_S^+\,\mathrm{Ind}_{S'}^+(\varphi_J)\bigr)$$
$$-\sum_{I\sqcup J=S'}\sum_{\substack{\varphi_I\in\mathrm{Map}(I,\{\pm\})\\ \varphi_J\in\mathrm{Map}(J,\{\pm\})}}(-1)^{k(\varphi_I)+k(\varphi_J)}F_1\bigl(\mathrm{Ind}_S^-(\mathrm{Ind}_{S'}^-(\varphi_I))\bigr)\cdot F_2\bigl(\mathrm{Ind}_S^-(\mathrm{Ind}_{S'}^+(\varphi_J))\bigr)$$
$$=\sum_{I\sqcup J=S'}\sum_{\substack{\varphi_I\in\mathrm{Map}(I,\{\pm\})\\ \varphi_J\in\mathrm{Map}(J,\{\pm\})}}(-1)^{k(\varphi_I)+k(\varphi_J)}\Bigl(F_1\bigl(\mathrm{Ind}_S^+(\mathrm{Ind}_{S'}^-(\varphi_I))\bigr)-F_1\bigl(\mathrm{Ind}_S^-(\mathrm{Ind}_S'^-(\varphi_I))\bigr)\Bigr)\cdot F_2\bigl(\mathrm{Ind}_S^+\,\mathrm{Ind}_{S'}^+(\varphi_J)\bigr)$$
$$+\sum_{I\sqcup J=S'}\sum_{\substack{\varphi_I\in\mathrm{Map}(I,\{\pm\})\\ \varphi_J\in\mathrm{Map}(J,\{\pm\})}}(-1)^{k(\varphi_I)+k(\varphi_J)}F_1\bigl(\mathrm{Ind}_S^-(\mathrm{Ind}_{S'}^-(\varphi_I))\bigr)\cdot\Bigl(F_2\bigl(\mathrm{Ind}_S^+(\mathrm{Ind}_{S'}^+(\varphi_J))\bigr)-F_2\bigl(\mathrm{Ind}_S^-(\mathrm{Ind}_{S'}^+(\varphi_J))\bigr)\Bigr)\qquad(288)$$

It is easy to observe the the right hand side of (288) can be rewritten as follows:

$$\sum_{\substack{I\sqcup J=S\\ s\in I}}(-1)^{k(\varphi_I)+k(\varphi_J)}F_1\bigl(\mathrm{Ind}_S^-(\varphi_I)\bigr)\cdot F_2\bigl(\mathrm{Ind}_S^+(\varphi_J)\bigr)+\sum_{\substack{I\sqcup J=S\\ s\in J}}(-1)^{k(\varphi_I)+k(\varphi_J)}F_1\bigl(\mathrm{Ind}_S^-(\varphi_I)\bigr)\cdot F_2\bigl(\mathrm{Ind}_S^+(\varphi_J)\bigr)$$
$$=\sum_{I\sqcup J=S}(-1)^{k(\varphi_I)+k(\varphi_J)}F_1\bigl(\mathrm{Ind}_S^-(\varphi_I)\bigr)\cdot F_2\bigl(\mathrm{Ind}_S^+(\varphi_J)\bigr)\qquad(289)$$

Thus we are done. ∎

7.8.4 Proof of Theorem 7.52

We consider the following statements:

$P(r)$**:** Assume $\mathrm{rank}(\boldsymbol{y}) \leq r$. For any $\boldsymbol{\xi} = (\xi_1, \dots, \xi_l)$ such that $l \geq \mathrm{rank}(\boldsymbol{y})$, we have $\mathcal{D}^T_{\boldsymbol{\xi}} \Phi(\widehat{y}) = 0$.

$Q(r)$**:** Assume $\mathrm{rank}(\boldsymbol{y}) \leq r$. For any $\boldsymbol{\xi} = (\xi_1, \dots, \xi_l)$ such that $l = \mathrm{rank}(\boldsymbol{y}) - 1$, the number $\mathcal{D}^T_{\boldsymbol{\xi}} \Phi(\widehat{y})$ is independent of the choice of the tile T. Moreover, the equality (285) holds.

It is easy to see $Q(r)$ implies $P(r)$. We have already known that $Q(1)$ and $Q(2)$ hold. We will prove $P(r)$, and hence $Q(r)$, by an induction on r.

Let y_i $(i = 1, \dots, s)$ be elements of $\mathcal{T}ype$, and let Φ_i be elements of $\mathcal{R}(\widehat{E}_i)$. We put as follows:

$$\mathcal{D}^T_{\boldsymbol{\xi}} \left(\prod_{i=1}^{s} \Phi_i(y_i) \right) := \sum_{\varphi} (-1)^{k(\varphi)} \prod_{i=1}^{s} \Phi_{i,H_\varphi}(y_i)$$

For a given $\boldsymbol{\xi} = (\xi_1, \dots, \xi_l)$ and a subset $I \subset \{1, \dots, l\}$, we put $\boldsymbol{\xi}_I := \bigl(\xi_i \,|\, i \in I\bigr)$. For a tile T in $W^{\boldsymbol{\xi}}$, let T^I_- be the tile of $W^{\boldsymbol{\xi}_I}$ such that $(H, \xi_j) < 0$ for any $j \in \{1, \dots, l\} \setminus I$ and for any element $H \in T^I_-$. Similarly, let T^I_+ be the tile of $W^{\boldsymbol{\xi}_I}$ such that $(H, \xi_j) > 0$ for any $j \in \{1, \dots, l\} \setminus I$ and for any element $H \in T^I_+$. We put $S = \{1, \dots, l\}$. Then, we obtain the following equality from Lemma 7.53:

$$\mathcal{D}^T_{\boldsymbol{\xi}} \left(\prod_{i=1}^{s} \Phi_i(\widehat{y}_i) \right) = \sum_{I \sqcup J = S} \mathcal{D}^{T^I_-}_{\boldsymbol{\xi}_I} \bigl(\Phi_1(\widehat{y}_1)\bigr) \cdot \mathcal{D}^{T^J_+}_{\boldsymbol{\xi}_J} \left(\prod_{i=2}^{s} \Phi_i(\widehat{y}_i) \right) \tag{290}$$

Lemma 7.54 *Assume that $P(r-1)$ holds. Let $y_1, \dots, y_s$ $(s \geq 2)$ be elements of $\mathcal{T}ype$ such that $\mathrm{rank}\, y_i \leq r-1$. For $\boldsymbol{\xi} = (\xi_1, \dots, \xi_l)$ such that $l \geq \sum_{i=1}^{s} \mathrm{rank}(y_i) - s$, we have $\mathcal{D}^T_{\boldsymbol{\xi}} \prod_{i=1}^{s} \Phi_i(y_i) = 0$.*

Proof We use an induction on s. In the case $s = 1$, the claim follows from the assumption $P(r-1)$. Assume that the claim holds in the case $s-1$, and we will show the claim in the case s. Due to the assumption $P(r-1)$, the contribution from (I, J) is 0 in (290), if the inequality $|I| \geq \mathrm{rank}(y_1)$ holds. If $|I| < \mathrm{rank}(y_1)$ holds, the inequality $|J| \geq \sum_{i=2}^{s} \mathrm{rank}\, y_i - s + 1$ holds. Therefore, we obtain the vanishing of the contribution due to the hypothesis of the induction. Thus we are done. ∎

Lemma 7.55 *Assume that $Q(r-1)$ holds. Let y_i $(i = 1, 2)$ be elements of $\mathcal{T}ype$ such that $\mathrm{rank}\, y_i \leq r-1$. For $\boldsymbol{\xi} = (\xi_1, \dots, \xi_{r-2})$, the number $\mathcal{D}^T_{\boldsymbol{\xi}}(\Phi_1(\widehat{y}_1) \cdot \Phi_2(\widehat{y}_2))$ is independent of the choice of a tile T, and the following equality holds:*

$$\mathcal{D}_{\boldsymbol{\xi}}\bigl(\Phi_1(\widehat{y}_1) \cdot \Phi_2(\widehat{y}_2)\bigr) = \sum_{\substack{I_1 \sqcup I_2 = S \\ |I_i| = \mathrm{rank}\, y_i}} \mathcal{D}_{\boldsymbol{\xi}_{I_1}}\bigl(\Phi_1(\widehat{y}_1)\bigr) \cdot \mathcal{D}_{\boldsymbol{\xi}_{I_2}}\bigl(\Phi_2(\widehat{y}_2)\bigr)$$

Proof It can be shown by an argument similar to the proof of Lemma 7.54. ∎

Let us show the claim $Q(r)$ assuming that $Q(r-1)$ holds. We put $S' := \{2, \dots, l\}$. For a map $\lambda : S' \longrightarrow \{\pm\}$, let $T(\lambda)$ denote the tile of W^{ξ_1} determined by the following conditions:

- The closure of $T(\lambda)$ contains T.
- Let H be any ample line bundle contained in $T(\lambda)$. Then we have $\mathrm{sign}(H, \xi_i) = \lambda(i)$.

From the definition of $\mathcal{D}^T_{\boldsymbol{\xi}} \Phi(\widehat{y})$, we obtain the following equality:

$$\mathcal{D}^T_{\boldsymbol{\xi}}\bigl(\Phi(\widehat{y})\bigr) = \sum_{\lambda} (-1)^{k(\lambda)} \cdot \mathcal{D}^{T(\lambda)}_{\xi_1}\bigl(\Phi(\widehat{y})\bigr) \tag{291}$$

Let $H_{\kappa,\lambda}$ denote an ample line bundle contained in the chamber corresponding to $\mathrm{Ind}^{\kappa}_{S}(\lambda)$. We can rewrite (291) as follows, by using Theorem 7.47:

$$\mathcal{D}^T_{\boldsymbol{\xi}} \Phi(\widehat{y}) + \sum_{\lambda} (-1)^{k(\lambda)} \sum_{\mathfrak{y} \in S_+} \overline{B}(\mathfrak{y}) \cdot \int_{\mathcal{M}_{H_{+\lambda}}(\widehat{\mathfrak{y}})} \Psi(\mathfrak{y}) - \sum_{\lambda} (-1)^{k(\lambda)} \sum_{\mathfrak{y} \in S_-} \overline{B}(\mathfrak{y}) \cdot \int_{\mathcal{M}_{H_{-\lambda}}(\widehat{\mathfrak{y}})} \Psi(\mathfrak{y}) = 0 \tag{292}$$

We put $\boldsymbol{\xi}' := (\xi_2, \dots, \xi_l)$. Let $T^{\boldsymbol{\xi}'}_{\kappa}$ $(\kappa = \pm)$ be the tiles of $W^{\boldsymbol{\xi}'}$ determined by the following conditions:

- The closure of $T_\kappa^{\boldsymbol{\xi}'}$ contain T.
- Let H_κ be any element of $T_\kappa^{\boldsymbol{\xi}'}$. Then, we have $(H_-,\xi_1) < 0 < (H_+,\xi_1)$.

For $\mathfrak{y} = (y_0,\ldots,y_k)$, we put as follows:

$$\mathcal{D}_{\boldsymbol{\xi}'}^{T_\kappa^{\boldsymbol{\xi}'}}\big(\Psi(\widehat{\mathfrak{y}})\big) := \mathcal{D}_{\boldsymbol{\xi}'}^{T_\kappa^{\boldsymbol{\xi}'}}\Psi(\mathfrak{y})(\widehat{y}_0,\ldots,\widehat{y}_k).$$

Then, we can rewrite (292) as follows:

$$\mathcal{D}_{\boldsymbol{\xi}}^{T}\big(\Phi(\widehat{y})\big) + \sum_{\mathfrak{y}\in S_+}\overline{B}(\mathfrak{y})\cdot\mathcal{D}_{\boldsymbol{\xi}'}^{T_+^{\boldsymbol{\xi}'}}\big(\Psi(\widehat{\mathfrak{y}})\big) - \sum_{\mathfrak{y}\in S_-}\overline{B}(\mathfrak{y})\cdot\mathcal{D}_{\boldsymbol{\xi}'}^{T_-^{\boldsymbol{\xi}'}}\big(\Psi(\widehat{\mathfrak{y}})\big) = 0 \tag{293}$$

Due to Lemma 7.54, the contribution from $\mathfrak{y} = (y_0,\ldots,y_k)$ is 0, if $k \geq 2$ holds. In the case $k=1$, the contributions do not depend on $T_\kappa^{\boldsymbol{\xi}'}$. So, we can omit to denote them in the following argument.

We put as follows:

$$S_1 := \big\{(y_0,y_1)\in\mathcal{T}ype^2 \,\big|\, y_0+y_1=y,\ a_0/r_0 - a_1/r_1 = A\cdot\xi_1\ (A<0)\big\},$$

$$S_2 := \big\{(y_0,y_1)\in\mathcal{T}ype^2 \,\big|\, y_0+y_1=y,\ a_0/r_0 - a_1/r_1 = A\cdot\xi_1\ (A>0)\big\}$$

$$S_3 := \big\{(y_0,y_1)\in\mathcal{T}ype^2 \,\big|\, y_0+y_1=y,\ a_0/r_0 = a_1/r_1,\ b_0<b_1\big\}$$

Then, we obtain the following equality from (293) and Lemma 7.55:

$$\begin{aligned} 0 = \mathcal{D}_{\boldsymbol{\xi}}^{T}\big(\Phi(\widehat{y})\big) &+ \sum_{(y_0,y_1)\in S_1}\frac{r_0}{r}\mathcal{D}_{\boldsymbol{\xi}'}\big(\Psi(\widehat{y}_0,\widehat{y}_1)\big) + \sum_{(y_0,y_1)\in S_3}\frac{r_0}{r}\mathcal{D}_{\boldsymbol{\xi}'}\big(\Psi(\widehat{y}_0,\widehat{y}_1)\big) \\ &- \sum_{(y_0,y_1)\in S_2}\frac{r_0}{r}\mathcal{D}_{\boldsymbol{\xi}'}\big(\Psi(\widehat{y}_0,\widehat{y}_1)\big) - \sum_{(y_0,y_1)\in S_3}\frac{r_0}{r}\mathcal{D}_{\boldsymbol{\xi}'}\big(\Psi(\widehat{y}_0,\widehat{y}_1)\big) \end{aligned} \tag{294}$$

The contributions from S_3 are cancelled out. We have the bijection $S_1 \longrightarrow S_2$ given by $(y_0,y_1)\longmapsto(y_1,y_0)$. We also have $\mathcal{D}_{\boldsymbol{\xi}'}\Psi(\widehat{y}_0,\widehat{y}_1) = -\mathcal{D}_{\boldsymbol{\xi}'}\Psi(\widehat{y}_1,\widehat{y}_0)$ for $(y_0,y_1)\in S_1$ and $(y_1,y_0)\in S_2$. Therefore, we obtain the following equality:

$$\mathcal{D}_{\boldsymbol{\xi}}^{T}\Phi(\widehat{y}) = \sum_{(y_0,y_1)\in S_2}\mathcal{D}_{\boldsymbol{\xi}'}\Psi(\widehat{y}_0,\widehat{y}_1) \tag{295}$$

By using (295) inductively, we can obtain the formula (285). Then, it is clear that $\mathcal{D}_{\boldsymbol{\xi}}^{T}\Phi(y)$ is independent of the choice of T. Applying the hypothesis of the induction to (295) we can easily derive the equality (285). Thus, the claim $Q(r)$ is proved, and the proof of Theorem 7.52 is finished. ∎

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