CONVOLUTION EQUATIONS ON LATTICES: PERIODIC SOLUTIONS WITH VALUES IN A PRIME CHARACTERISTIC FIELD

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Dedicated to Yuri Lyubich
on occasion of his 75th birthday

Abstract. These notes are inspired by the theory of cellular automata which targets, in particular, to provide a model for inter-cellular or inter-molecular interactions. A cellular automaton on a lattice $\mathbb{Z}^s$ or on a toric grid can be regarded as a convolution operator $\Delta_a : f \longmapsto f * a$ with kernel a concentrated in the nearest neighborhood of 0. In [Za] we gave a survey on the most studied cellular automata $\Delta^{\pm}$ with kernels the star-functions $a^{\pm}$, where

$$a^+ = \delta_0 + a^- \quad \text{and} \quad a^- = \sum_{i=1}^{s} (\delta_{e_i} + \delta_{-e_i})$$

with $e_1, \ldots, e_s$ being a lattice basis. In the present paper we deal with general convolution operators. We propose an approach via harmonic analysis which works over a field of positive characteristic. It occurs that a standard spectral problem for a convolution operator is equivalent to counting points on an associate algebraic hypersurface in a torus according to their torsion multi-orders.

MP: - *Do you yourself perceive a fundamental difference between pure and applied mathematics?*
Stanislaw Ulam: - *I really don't. I think it's a question of language, and perhaps habits.*

Contents

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Introduction

0.1. We let $\bar{K}$ denote the algebraic closure of the Galois field $K = \mathrm{GF}(p)$ of characteristic $p > 0$ and $\bar{K}^\times$ the multiplicative group of $\bar{K}$. In the torus $(\bar{K}^\times)^s$ we consider an affine variety

$$\Sigma_{\bar{a}} = \{\sigma_{a_j} = 0 \,:\, j = 1, \ldots, t\}$$

defined by the Laurent polynomials in $x = (x_1, \ldots, x_s)$

$$\sigma_{a_j} = \sum_{u=(u_1,\ldots,u_s)\in\mathbb{Z}^s} a_j(u)x^u \in \bar{K}[x_1, x_1^{-1}, \ldots, x_s, x_s^{-1}],\;\; j = 1, \ldots, t,$$

where $\bar{a} = (a_1, \ldots, a_t)$ and $a_j : \mathbb{Z}^s \to \bar{K}$ are functions with finite supports.

The logarithm of the Weil zeta function counts the points on $\Sigma_{\bar{a}}$ over the Galois fields $\mathrm{GF}(q)$, where $q = p^r$, $r \geq 0$. Whereas our purpose is to count, for every multi-index $\bar{n} = (n_1, \ldots, n_s) \in \mathbb{N}^s$ with all n_i coprime to p, the number $d_{\bar{a}}(\bar{n})$ of $\bar{n}$-torsion points on $\Sigma_{\bar{a}}$. Namely

$$d_{\bar{a}}(\bar{n}) = \mathrm{card}\,(\Sigma_{\bar{a},\bar{n}})\,,$$

where

$$\Sigma_{\bar{a},\bar{n}} = \{\xi = (\xi_1, \ldots, \xi_s) \in \Sigma_{\bar{a}} \,:\, \xi_i^{n_i} = 1,\; i = 1, \ldots, s\}\,.$$

0.2. In this paper we provide different interpretations for the numbers $d_{\bar{a}}(\bar{n})$. In fact $d_{\bar{a}}(\bar{n})$ counts the number of linearly independent $\bar{n}$-periodic solutions $f : \mathbb{Z}^s \to \bar{K}$ of the system of convolution equations

$$\Delta_{a_j}(f) := f * a_j = 0,\;\; j = 1, \ldots, t\,,$$

whereas the Laurent polynomial σ_{a_j} appears as the symbol of the convolution operator Δ_{a_j}. We call these solutions $\bar{a}$-*harmonic*. Moreover $d_{\bar{a}}(\bar{n})$ is equal to the number of $\bar{a}$-harmonic characters $g^\vee : \mathbb{Z}^s \to \bar{K}^\times$ with multi-orders dividing $\bar{n}$. Indeed there is a one to one correspondence between the $\bar{a}$-harmonic characters and the points in $\Sigma_{\bar{a},\bar{n}}$, see Proposition 4.20 below. In the hypersurface case (i.e., for $t = 1$) the very existence of an $\bar{n}$-periodic $\bar{a}$-harmonic character is equivalent to non-periodicity of the orbit of δ_0 under the iterates $(\Delta_a^k)_{k\geq 0}$ acting on the quotient group $L/L_{\bar{n}}$, where $L = \mathbb{Z}^s$, $L_{\bar{n}} = \sum_{i=1}^s n_i\mathbb{Z}e_i \subseteq L$ and δ_0 stands for the delta-function concentrated at $0 \in L/L_{\bar{n}}$, see Theorem 4.21 and Remark 4.22.

0.3. These notes are inspired by the theory of cellular automata. This theory targets, in particular, to provide a model for inter-cellular or inter-molecular interactions. One can regard a linear cellular automaton on a group (or rather on the Caley graph of a group) as a discrete dynamical system generated by a convolution operator with kernel concentrated in a nearest neighborhood of the neutral element (cf. e.g., [MOW]).

In more detail, suppose we are given a collection (a colony) of 'cells' placed at the vertices of a locally finite graph Γ, which determines the relation 'neighbors' for cells.

Each cell can be in one of the n cyclically ordered states. The state of the whole collection at a moment t is codified via a function $f_t : \Gamma \to \mathbb{Z}/n\mathbb{Z}$. In the subsequent portions of time, the cells simultaneously change their states. The new state of a cell depends on the previous states of the given cell and of its neighbors, according to a certain local rule.

To define a cellular automaton, say, σ on Γ means to fix at each vertex v of Γ a local rule which does not depend on t. This collection of local rules determines a discrete dynamical system $\sigma : f_t \longmapsto f_{t+1}$. Usually the edges at v_0, say, $[v_0, v_1], \dots, [v_0, v_s]$, are ordered and the local rule at v_0 is a function $\phi_{v_0} : (\mathbb{Z}/n\mathbb{Z})^{s+1} \to \mathbb{Z}/n\mathbb{Z}$, so that

$$f_{t+1}(v_0) = \phi_{v_0}\left(f_t(v_0), f_t(v_1), \dots, f_t(v_s)\right) . \tag{1}$$

Suppose that Γ is homogeneous under a group action. In this case it is natural to assume that the family of local rules is as well homogeneous that is, globally stable under the group action. In particular every local rule ϕ_{v_0} must be stable under the stationary subgroup of v_0. Assume for instance that Γ is the Caley graph of a finitely generated group G with a generating set $\{g_1, \dots, g_s\}$. Given a local rule ϕ_e for the neutral element $e \in G$, for any vertex $g \in G$ we can define ϕ_g to be the shift of ϕ_e by g.

In the case of an additive cellular automaton, the local rule ϕ_e is a linear function. Such an automaton can be regarded as a convolution operator $\Delta_a : f \longmapsto f * a$ on G with kernel a, which is the coefficient function of ϕ_e supported on the nearest neighborhood of e. The evolution equation (1) in this case can be written as a heat equation

$$\partial_t(f_t) := f_{t+1} - f_t = \Delta_{a'}(f_t), \qquad \text{where} \qquad a' = a - \delta_e .$$

In the present paper we restrict to additive automata on lattices or on toric grids. These are the Caley graphs of finitely generated free abelian groups or of finite abelian groups, respectively.

Actually we deal with general convolution operators. So we allow distant interactions and not only the nearest ones as in the classical setting. We adopt the viewpoint of harmonic analysis in positive characteristic, which provides certain advantages. We extend some of the results in [Za] to our more general case.

0.4. In [Za] we gave a survey on the σ^+-automaton Δ_{a^+} on the integral lattice $\mathbb{Z}^s$ with kernel the star-function taking values in the field GF(2):

$$a^+ = \delta_0 + \sum_{i=1}^{s} (\delta_{e_i} + \delta_{-e_i}) , \tag{2}$$

where $e_1, \dots, e_s$ is a lattice basis. Already this particular case leads to numerous intriguing questions (see below). The σ^+-automaton on the plane lattice $\mathbb{Z}^2$ is related to the popular game 'Lights Out'. The latter one, commercialized by the 'Tiger Electronics', became a source of inspiration for the work of Sutner, Goldwasser-Klostermeyer-Ward, Barua-Sarkar, Hunziker-Machiavello-Park e.a., see [Za] for a survey.

Let us describe the game. Suppose that the offices in a department (which will be our table of game) correspond to the vertices of a grid $P_{m,n} = L_m \times L_n$, where L_m is the linear graph with m vertices. Suppose also that (by, say, a security reason) the interrupters are synchronized in such a way that turning off or on in a room changes automatically to the opposite the states in all neighborhooding (through a wall) rooms. How can the last person leave the department with all the lights off?

It is possible to reduce to an analogous question for the toric grid $\mathbb{T}_{m',n'} = C_{m'} \times C_{n'}$, where $m' = m+1$, $n' = n+1$ and C_m stands for the circular graph with m vertices. The initial state is then a binary function (a *pattern*) $f_0 \in \mathcal{F}(\mathbb{T}_{m',n'}, \mathrm{GF}(2))$. The move at a vertex v consists in the addition

$$f \longmapsto f + a_v^+ \mod 2\,,$$

where $a_v^+(u) = a^+(u+v)$ is the star function centered at $v \in \mathbb{T}_{m',n'}$. Thus the σ^+-game on the torus $\mathbb{T}_{m',n'}$ is winning starting with f_0 if and only if f_0 can be decomposed into a sum of shifts of the star function a^+ as in (2).

The linear invariants of the σ^+-game in the function space $\mathcal{F}(\mathbb{T}_{m,n}, \mathrm{GF}(2))$ endowed with the standard bilinear form, form a subspace $\mathcal{H}$ orthogonal to all shifts a_v^+ ($v \in \mathbb{T}_{m,n}$) of a^+. Indeed

$$h \in \mathcal{H} \quad \Longleftrightarrow \quad \langle h, f + a_v^+ \rangle \equiv \langle h, f\rangle \mod 2\,.$$

Moreover f_0 is winning if and only if $f_0 \in \mathcal{H}^\perp$. The functions $h \in \mathcal{H}$ are called *harmonic* [Za], justifying this by the following property: for any vertex v of the grid $\mathbb{T}_{m,n}$, the value $h(v)$ is the sum modulo 2 of the values of h over the neighbors of v in $\mathbb{T}_{m,n}$. Actually $\mathcal{H} = \ker(\Delta_{a^+})$. [1] The σ^+-game on a toric grid $\mathbb{T}_{m,n}$ is winning for any initial pattern if and only if

$$\ker(\Delta_{a^+}) = \{0\} \quad \Longleftrightarrow \quad 0 \notin \mathrm{spec}(\Delta_{a^+}) \quad \Longleftrightarrow \quad \gcd(T_m, T_n^+) = 1\,,$$

where T_m stands for the mth Chebyshev-Dickson polynomial and $T_n^+(x) = T_n(x+1)$, see [Za, 2.35] and the references therein.

0.5. Thus the game 'Lights Out' naturally leads to the following questions.

- *Determine the set of all winning toric grids $\mathbb{T}_{m,n}$.* [2] *Or, which is equivalent, determine the complementary set of all toric grids $\mathbb{T}_{m,n}$ admitting a nonzero binary harmonic function.*
- *Compute the dimension, say, $d(m,n)$ of the subspace $\mathcal{H}$ of all harmonic functions on $\mathbb{T}_{m,n}$ for all $(m,n) \in \mathbb{N}^2$ or, equivalently, the dimension of the subspace $\mathcal{H}^\perp$ of all winning patterns.*

Assuming that m,n are odd we will give several different interpretations for the numbers $d(m,n)$. In particular we will show that, over the algebraic closure $\bar{K}$ of the base field $K = \mathrm{GF}(2)$, there is an orthonormal basis of $\mathcal{H} \otimes \bar{K}$ consisting of characters $L \to \bar{K}^\times$ with values in the multiplicative group $\bar{K}^\times$, called *harmonic characters*. An initial pattern f_0 on $\mathbb{T}_{m,n}$ is winning if and only if f_0 is orthogonal to all harmonic characters on $\mathbb{T}_{m,n}$. The latter ones are in one to one correspondence with the (m,n)-bi-torsion points on the symbolic hypersurface, which is in our case the elliptic cubic in $\mathbb{A}^2_{\bar{K}}$, where $K = \mathrm{GF}(2)$, with equation

$$x^2y + xy^2 + xy + x + y = 0\,. \tag{3}$$

Thus to determine all toric grids $\mathbb{T}_{m,n}$ admitting a nonzero binary harmonic function is the same as to determine all bi-torsion orders of points on the cubic (3), see [Za].

[1] In the notation of [Za], $\Delta_{a^+} = \Delta^+$ and $\mathcal{H} = \ker(\Delta_{a^+}) = \mathrm{Harm}^+(\mathbb{T}_{m,n}, \mathrm{GF}(2))$.

[2] Or equivalently, the set of all pairs (m,n) such that the polynomials T_m and T_n^+ are coprime.

0.6. More generally, for a field K of characteristic $p > 0$ and for a group G we let $\mathcal{F}(G,K)$ and $\mathcal{F}^0(G,K)$, respectively, denote the vector space of all functions $f : G \to K$, of all those with finite support, respectively. We consider the convolution

$$* : \mathcal{F}(G,K) \times \mathcal{F}^0(G,K) \ni (f,a) \longmapsto f * a \in \mathcal{F}(G,K)\,,$$

where

$$f * a(g) = \sum_{h \in G} f(h)a(h^{-1}g) \qquad \forall g \in G\,.$$

Fixing a we get the convolution operator $\Delta_a : f \longmapsto f * a$ acting on the space $\mathcal{F}(G,K)$. These convolution operators form a K-algebra $\mathrm{Conv}_K(G)$ with $\Delta_{a_1} \circ \Delta_{a_2} = \Delta_{a_2 * a_1}$.

0.7. For a subgroup $H \subseteq G$ we let $\mathcal{F}_H(G,K)$ denote the subspace of all H-periodic functions in $\mathcal{F}(G,K)$ and $\Delta_a \mid H$ the restriction of Δ_a to $\mathcal{F}_H(G,K)$.

Later on we consider a lattice [3] L as G and a sublattice $L' \subseteq L$ as H. Letting $\mathcal{L}$ denote the set of all finite index sublattices in L, we aim to compute the following function d_a on $\mathcal{L}$:

$$d_a(L') = \dim\ker(\Delta_a \mid L'), \qquad L' \in \mathcal{L}\,. \tag{4}$$

Moreover for the restrictions $\Delta_a \mid L'$ we would like to compute the characteristic polynomials

$$\mathrm{CharPoly}_{a,L'} = \mathrm{CharPoly}(\Delta_a \mid L'), \qquad L' \in \mathcal{L}\,,$$

the spectra (and the spectral multiplicities) in the algebraic closure $\bar{K}$ of K. We will see that $d_a(L')$ coincides with the multiplicity of the zero root of the polynomial $\mathrm{CharPoly}_{a,L'}$, and as well with the number of a-harmonic characters of the quotient group $G = L/L'$. For a product sublattice L', it also coincides with the number of points on the symbolic hypersurface which have the corresponding torsion orders (see 0.9 below for more details).

In section 1 we show that it suffices to compute the functions d_a and $\mathrm{CharPoly}_a$ only on the subset $\mathcal{L}^0 \subseteq \mathcal{L}$ of all sublattices with index coprime to $p = \mathrm{char}\, K$.

0.8. If $K = \mathrm{GF}(2)$, $a = a^+$, $L = \mathbb{Z}$ and $L' = n\mathbb{Z}$ then $\mathrm{CharPoly}_{a^+,L'}$ is just the classical Chebyshev-Dickson polynomial of first kind T_n [4] shifted by 1. So by analogy we call the characteristic polynomials $\mathrm{CharPoly}_{a,L'}$ *generalized Chebyshev-Dickson polynomials.*

The classical Chebyshev-Dickson polynomials (T_n) possess a number of interesting properties. They form a commutative composition semigroup i.e. $T_n \circ T_m = T_{mn}$. Furthermore T_m divides T_n if $m \mid n$ [5], etc. The composition property is not stable under shifts in the argument, and so we cannot expect it to hold in our general setting. It occurs however that the generalized Chebyshev-Dickson polynomials still satisfy the divisibility property. Namely if $L' \subset L''$ then $\mathrm{CharPoly}_{a,L''}$ divides $\mathrm{CharPoly}_{a,L'}$, see 3.8. Therefore $0 \in \mathrm{spec}(\Delta_a \mid L')$ as soon as $0 \in \mathrm{spec}(\Delta_a \mid L'')$. Hence one can restrict to maximal, for inclusion, a-harmonic sublattices. A sublattice $L' \subseteq L$ is called *a-harmonic* if $0 \in \mathrm{spec}(\Delta_a \mid L')$.

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[3] I.e. a free abelian group of finite rank.

[4] That is $T_n(x) = D_n(x,1)$ over GF(2), where the Dickson polynomial $D_n(x,a)$ over a finite field is the unique polynomial verifying the identity $D_n(x + a/x, a) = x^n + a^n/x^n$.

[5] Moreover $\gcd(T_m, T_n) = T_{\gcd(m,n)}$.

0.9. Given a basis $\mathcal{V} = (v_1, \ldots, v_s)$ of L, we restrict often to product sublattices

$$L' = L_{\bar n, \mathcal{V}} := \sum_{i=1}^{s} n_i \mathbb{Z} v_i \cong \bigoplus_{i=1}^{s} n_i \mathbb{Z} \subseteq \mathbb{Z}^s\,.$$

It occurs that to describe all maximal such $\bar a$-harmonic product sublattices, where $\bar a = (a_1, \ldots, a_t)$, is the same as to determine all multiplicative multi-torsion orders [6] of the points on the affine *symbolic variety* $\Sigma_{\bar a}$ of the torus $(\bar K^\times)^s$, see below.

0.10. Let us summarize some of the central results of the paper, see 3.8, 4.20-4.21. We let as before $\Sigma_{\bar a}$ be a closed affine subvariety (a *symbolic variety*) of the s-torus $(\bar K^\times)^s$ given by a system of equations

$$\sigma_{a_j} = 0, \quad j = 1, \ldots, t, \qquad \text{where} \quad \sigma_{a_j} \in \bar K[x_1, x_1^{-1}, \ldots, x_s, x_s^{-1}]$$

are Laurent polynomials with coefficients $a_j \in \mathcal{F}^0(\mathbb{Z}^s, \bar K)$. Given a lattice L of rang s and a basis $\mathcal{V} = (v_1, \ldots, v_s)$ of L, we associate with $\Sigma_{\bar a}$ a collection of convolution operators $\Delta_{\bar a} := \{\Delta_{a_j} : j = 1, \ldots, t\}$ with symbols $\{\sigma_{a_j}\}$. We say that a function $f \in \mathcal{F}(L, \bar K)$ is *$\bar a$-harmonic* if $\Delta_{a_j}(f) = 0\ \forall j = 1, \ldots, t$. We let

$$\mathrm{CharPoly}_{\bar a, L'} = \gcd\left(\mathrm{CharPoly}_{a_j, L'} \,:\, j = 1, \ldots, t\right)$$

and

$$d_{\bar a}(L') = \dim \ker(\Delta_{\bar a} \mid L'), \qquad \text{where} \qquad \ker(\Delta_{\bar a} \mid L') := \bigcap_{j=1}^{t} \ker\left(\Delta_{a_j} \mid L'\right)\,.$$

The set of zeros of $\mathrm{CharPoly}_{\bar a, L'}$ will be called the *spectrum* of $\Delta_{\bar a}$. Indeed $\mathrm{CharPoly}_{\bar a, L'}(\lambda) = 0$ if and only if there exists a nonzero L'-periodic eigenfunction $f \in \mathcal{F}_{L'}(L, \bar K)$ of $\Delta_{\bar a}$ with $\Delta_{a_j}(f) = \lambda \cdot f\ \forall j = 1, \ldots, t$.

Theorem 0.11. (a) *For any sublattice $L' \in \mathcal{L}^0$, the subspace $\ker(\Delta_{\bar a} \mid L')$ possesses an orthonormal basis of $\bar a$-harmonic characters. In particular there are $d_{\bar a}(L')$ such characters. Moreover $d_{\bar a}(L') = \mathrm{mult}_{\lambda=0}\left(\mathrm{CharPoly}_{\bar a, L'}\right)$.*

(b) *Given a base $\mathcal{V}$ of L there is a natural bijection $\sigma_{\mathcal{V}} : \mathrm{Char}(L, \bar K^\times) \xrightarrow{\cong} (\bar K^\times)^s$, where $\mathrm{Char}(L, \bar K^\times)$ denotes the set of all $\bar K^\times$-valued characters of L. This bijection restricts to*

$$\sigma_{\mathcal{V}} : \mathrm{Char}_{\bar a-\mathrm{harm}}(L, \bar K^\times) \xrightarrow{\cong} \Sigma_{\bar a} \subseteq (\bar K^\times)^s\,.$$

Moreover $\forall \bar n \in \mathbb{N}^s_{\mathrm{co}(p)}$ it further restricts to

$$\sigma_{\mathcal{V}} : \mathrm{Char}_{\bar a-\mathrm{harm}}(L/L_{\bar n, \mathcal{V}}, \bar K^\times) \xrightarrow{\cong} \Sigma_{\bar a, \bar n} := \Sigma_{\bar a} \cap \mu_{\bar n}\,,$$

where

$$\mu_{\bar n} := \{\xi = (\xi_1, \ldots, \xi_s) \in (\bar K^\times)^s \,:\, \xi_i^{n_i} = 1,\ i = 1, \ldots, s\}\,.$$

(c) *If $L' \subseteq L''$ then*

$$\mathrm{CharPoly}_{\bar a, L''} \mid \mathrm{CharPoly}_{\bar a, L'}\,.$$

[6] That is, the torsion orders of the coordinates in the multiplicative group $\bar K^\times$.

The author is grateful to Don Zagier for clarifying discussions. In particular the idea of processing in the present generality came from these discussions. Maxim Kontsevich suggested that the function $d_{\bar{a}} : \mathbb{N}^s \to \mathbb{N}$ should be studied via the technique of statistical sums within the framework of the Ising model. Hopefully this project will be realized one day. Our thanks also to Vladimir Berkovich for a kind assistance and to Dmitri Piontkovski for performing computer simulations.

1. Sylow p-subgroups and generalized Chebyshev-Dickson polynomials

We let K be a field of characteristic $p > 0$. Here we show that the maps $\mathrm{CharPoly}_a : \mathcal{L} \to K[x]$ and $d_a : \mathcal{L} \to \mathbb{Z}_{\geq 0}$ as in 0.7 can be recovered by their restrictions to $\mathcal{L}^0$. More precisely, we let $D_n(x,a)$ $(E_n(x,a))$ denote the classical Dickson polynomials of first (second) kind over a finite field of characteristic $p > 0$. They satisfy the relation $D_{p^\alpha m} = D_m^{p^\alpha}$ and $E_{p^\alpha m} = E_m^{p^\alpha}$, respectively, see e.g., [BhZi]. In Corollary 1.5(c) below we show that the same identity holds for the generalized Chebyshev-Dickson polynomials.

For a group G, δ_u stands for the delta-function on G concentrated on $u \in G$. That is $\delta_u(u) = 1$, $\delta_u(g) = 0\ \forall g \neq u$. For a subset $A \subseteq G$ we let $\delta_A = \sum_{u\in A} \delta_u$ denote the characteristic function of A. For a function $a = \sum_{u\in G} a(u)\delta_u$ on G we let

$$|(a \mid A)| = \sum_{u\in A} a(u), \quad \mathrm{CharPoly}_{a,G} = \det(\Delta_a - \lambda \cdot 1) \quad \text{and} \quad d_{a,G} = \dim\ker(\Delta_a)\,.$$

The following lemma is straightforward.

Lemma 1.1. *Let $\pi : G \twoheadrightarrow G/H$ be a surjection. Then for any $a \in \mathcal{F}^0(G,K)$ there is a unique $a_* = \pi_* a \in \mathcal{F}^0(G/H,K)$ such that $a_* \circ \pi = a * \delta_H$.*

Proof. Indeed the function

$$a_*(v+H) = \sum_{v'\in H} a(v+v') = |\,(a \mid v+H)\,| \tag{5}$$

satisfies the condition of the lemma. □

In the next proposition we let $A = S_A + N_A$ denote the Jordan decomposition of an endomorphism $A \in \mathrm{End}(\mathbb{A}^n_K)$. It is uniquely defined over the algebraic closure $\bar{K}$ of K.

Proposition 1.2. *For an abelian group $G = F \times H$, where $H = \bigoplus_{i=1}^n \mathbb{Z}/p^{r_i}\mathbb{Z}$, and for any $a \in \mathcal{F}^0(G,K)$, the following hold.*

(a) $S_{\Delta_a} = S_{\Delta_{a_*}} \otimes 1_H$, *where $a_* = \pi_* a \in \mathcal{F}(F,\bar{K})$ is as in (5) above.*

(b) $\mathrm{CharPoly}_{a,G} = (\mathrm{CharPoly}_{a_*,F})^{\mathrm{ord}\,(H)}$ *and* $d_{a,G} = \mathrm{ord}\,(H) \cdot d_{a_*,F}\,.$

Proposition 1.2 follows from Lemma 1.4 below by an easy induction on n. The idea of the proof is transperant in the following example.

Example 1.3. Letting in Proposition 1.2 $G = \mathbb{Z}/p^r\mathbb{Z}$ (so that $n = 1$, $F = 1$) we replace the convolution operator Δ_a by $\Delta_a - |a| \cdot 1 = \Delta_b$, where $|a| := \sum_{u\in G} a(u)$ and $b = a - |a|\delta_0 \in \mathcal{F}(G,K)$. Then $|b| = 0$ and so

$$\Delta_b^{p^r} = \left(\sum_{u\in G} b(u)\tau_u\right)^{p^r} = \sum_{u\in G} b(u)^{p^r} \cdot 1 = |b|^{p^r} \cdot 1 = 0\,.$$

Thus Δ_b is nilpotent, hence

$$\mathrm{CharPoly}_{b,G} = x^{p^r} \qquad \text{and so} \qquad \mathrm{CharPoly}_{a,G} = (x - |a|)^{p^r}\,.$$

For an arbitrary abelian group F and $H = \mathbb{Z}/p^r\mathbb{Z}$ we have the following result.

Lemma 1.4. *Proposition 1.2 is true for $n = 1$.*

Proof. Let us show that

$$\Delta_a^{p^r} = \Delta_{a_*}^{p^r} \otimes 1_H\,. \tag{6}$$

Indeed

$$\Delta_a^{p^r} = \sum_{u\in G} a(u)^{p^r}\tau_u^{p^r} = \sum_{u'\in F}\left(\sum_{u''\in H} a(u'+u'')^{p^r}\right)\tau_{u'}^{p^r}$$
$$= \bigoplus_{u''\in H}\left(\sum_{u'\in F} a_*(u')\tau_{u'}^{p^r}\right) = \Delta_{a_*}^{p^r}\otimes 1_H\,.$$

By (6) we have

$$S_{\Delta_a}^{p^r} + N_{\Delta_a}^{p^r} = S_{\Delta_{a*}}^{p^r}\otimes 1_H + N_{\Delta_{a*}}^{p^r}\otimes 1_H\,.$$

By the uniqueness of the Jordan decomposition, $S_{\Delta_a}^{p^r} = S_{\Delta_{a*}}^{p^r}\otimes 1_H$ and so (a) follows. Now (b) is immediate from (a). ☐

From Proposition 1.2 and Example 1.3 we deduce the following corollaries.

Corollary 1.5. (a) *Under the assumptions as in Proposition 1.2, the product $G = F\times H$ is a-harmonic if and only if F is a_*-harmonic.*

(b) *If $G = \bigoplus_{i=1}^k \mathbb{Z}/p^{r_i}\mathbb{Z}$, where $p = \mathrm{char}\,K$, then*

$$\mathrm{CharPoly}_{a,G} = (x - |a|)^{\mathrm{ord}\,(G)}\,.$$

(c) *Let L be a lattice and $L' \subseteq L$ a sublattice of index $p^\alpha q$, where $q \not\equiv 0 \mod p$. Then there exists a unique intermediate sublattice L'' of index q in L, where $L' \subseteq L'' \subseteq L$. Moreover*

$$\mathrm{CharPoly}_{\bar a,L'} = (\mathrm{CharPoly}_{\bar a,L''})^{p^\alpha} \qquad \forall \bar a \in (\mathcal{F}^0(L,K))^t\,. \tag{7}$$

Consequently for the function $d_{\bar a}$ as in (4) one has

$$d_{\bar a}(L') = p^\alpha\cdot d_{\bar a}(L'')\,. \tag{8}$$

Proof. The proofs of (a) and (b) are straightforward. To show (c) we assume first that $t = 1$ and $a_1 = a$. We decompose

$$G = L/L' = F \oplus G(p)\,,$$

where $G(p)$ is the Sylow p-subgroup of G and $\mathrm{ord}\,(F) = q$. We let $L'' = \pi^{-1}(G(p))$ (cf. 4.15 below) so that $L/L'' \cong F$. Due to 2.3 below,

$$\mathrm{CharPoly}_{a,L'} = \mathrm{CharPoly}_{\pi'_* a,G} \qquad \text{and} \qquad \mathrm{CharPoly}_{a,L''} = \mathrm{CharPoly}_{\pi''_* a,F}\,, \tag{9}$$

where $\pi' : L \twoheadrightarrow G$ and $\pi'' : L \twoheadrightarrow F$. Now (7) follows from (9) in view of Proposition 1.2(b). By virtue of Proposition 1.2(a), $d_a(L')$ and $d_a(L'')$ are equal to the multiplicities of the root $x = 0$ of the polynomials $\mathrm{CharPoly}_{a,L'}$ and $\mathrm{CharPoly}_{a,L''}$, respectively. (8) follows by virtue of (7).

Now for any $t \geq 1$ (7) and (8) follow easily. ☐

2. Convolution operators: Generalities

2.1. **Naive Fourier transform on convolution algebras.**

2.1. The convolution with a delta function $\delta_{u^{-1}}$ results in the right shift

$$\tau_u = \Delta_{\delta_{u^{-1}}} : f \longmapsto f * \delta_{u^{-1}} = f_u, \qquad \text{where} \quad f_u(g) = f(gu), \qquad g, u \in G\,.$$

The shifts $(\tau_u \,:\, u \in G)$ generate the K-algebra of convolution operators $\mathrm{Conv}_K\,(G)$ as a K-vector space. Indeed

$$\Delta_a = \sum_{g\in G} a(g)\tau_{g^{-1}} \qquad \forall a \in \mathcal{F}^0(G, K)\,.$$

Notice that $a = \Delta_a(\delta_e)$, where $e \in G$ is the unit. Any convolution operator commutes with shifts, and any linear operator on the space $\mathcal{F}^0(L, \bar{K})$ commuting with shifts is a convolution operator. For a finite group G there are natural isomorphisms

$$(\mathcal{F}(G, K), *) \xrightarrow{\varphi} \mathrm{Conv}_K\,(G) \xrightarrow{\psi} K[G]\,,$$

where $\varphi : a \longmapsto \Delta_a$ and $K[G]$ is the group algebra of G over K. The ideals of $\mathcal{F}(G, K)$ are called *convolution ideals.*

In particular the subspace

$$\mathcal{F}_H(G, K) = \{f \in \mathcal{F}(G, K) \,:\, \tau_h(f) = f \quad \forall h \in H\}\,,$$

where $H \subseteq G$ is a subgroup, is translation invariant, hence is a convolution ideal and a $\mathrm{Conv}_K(G)$-submodule.

2.2. For any $a \in \mathcal{F}^0(G, K)$ there is a commutative diagram

$$\begin{array}{ccc}
\mathcal{F}_H(G, K) & \xrightarrow{\Delta_a} & \mathcal{F}_H(G, K) \\
\Big\downarrow \cong & & \Big\downarrow \cong \\
\mathcal{F}(G/H, K) & \xrightarrow{\Delta_{a_*}} & \mathcal{F}(G/H, K)
\end{array}$$

where $\pi : G \twoheadrightarrow G/H$ stands for the canonical surjection.

2.3. The composition $F_{\mathrm{naive}} = \psi \circ \varphi : \mathcal{F}(G, K) \to K[G], \quad a \longmapsto \tilde{a}$, is called a *naive Fourier transform.* The convolution operator Δ_a on $\mathcal{F}(G, K)$ corresponds to the operator of multiplication by $\tilde{a}$ in $K[G]$. So $\ker(\Delta_a)$ is sent by F_{naive} to the annihilator ideal $\mathrm{Ann}(\tilde{a})$ of the principal ideal $(\tilde{a}) \subseteq K[G]$.

2.2. **Harmonic functions, harmonic groups and lattices.** We use below the following terminology.

Definition 2.4. We let as before $\ker(\Delta_{\bar{a}}) = \bigcap_{j=1}^t \ker(\Delta_{a_j}) \subseteq \mathcal{F}(G, \bar{K})$, where $\bar{a} = (a_1, \ldots, a_j) \in (\mathcal{F}(G, \bar{K}))^t$. By analogy with the case where $t = 1$ and $a_1 = a^+$, the functions in $\ker(\Delta_{\bar{a}})$ will be called *$\bar{a}$-harmonic.* We say that the group G is *$\bar{a}$-harmonic* if $\ker(\Delta_{\bar{a}}) \neq \{0\}$ or, equivalently, if the annihilator ideal $\mathrm{Ann}(\tilde{a}_1, \ldots, \tilde{a}_t) \subseteq \bar{K}[G]$ is not the zero ideal. In the case where $t = 1$ and $\bar{a} = a$ this is the same as to say that $\tilde{a} \in \bar{K}[G]$ is a zero divisor.

We let L be a lattice and $L' \subset L$ a sublattice. Hereafter $\pi = \pi_{L'} : L \twoheadrightarrow L/L'$ stands for the canonical surjection. We say that L' is *$\bar{a}$-harmonic* if there exists a non-zero L'-periodic $\bar{a}$-harmonic function $f \in \ker(\Delta_{\bar{a}}) \subseteq \mathcal{F}(L, \bar{K})$. That is $L' \subseteq L(f)$, where

$$L(f) := \{u \in L \,:\, \tau_u(f) = f\}$$

denotes the *sublattice of periods* of f. This is equivalent to the $\bar{a}_*$-harmonicity of the quotient group $G = L/L'$, where $\bar{a}_* = \pi_* \bar{a} \in (\mathcal{F}^0(G, K))^t$ is as in (5).

Example 2.5. Suppose that $K = \mathrm{GF}(2)$, $L = \mathbb{Z}^2$, $t = 1$ and $a_1 = a^+$, see (2). For a primitive vector $u_0 = (k, l) \in \mathbb{Z}^2 = L$ we let $L_0 = \mathbb{Z}v_0$, where $v_0 = (-l, k) \perp u_0$. Since $\gcd(k, l) = 1$ we have $L/L_0 \cong \mathbb{Z}$. We let $\pi_0 : L \twoheadrightarrow L/L_0 \cong \mathbb{Z}$, $u \longmapsto \langle u, u_0 \rangle$. The induced convolution operator $\Delta_{a_*^+} \in \mathrm{End}(\mathcal{F}(L/L_0, K))$ corresponds to the following function on $\mathbb{Z}$:

$$a_*^+ = \delta_0 + \delta_k + \delta_{-k} + \delta_l + \delta_{-l}\,.$$

For a function $f \in \mathcal{F}(L/L_0, \bar{K})$, one has $\tilde{f} := f(-lx + ky) \in \ker(\Delta_{a^+})$ if and only if f satisfies the equation

$$f(z) + f(z-k) + f(z+k) + f(z-l) + f(z+l) = 0, \qquad \forall z \in \mathbb{Z}\,. \tag{10}$$

In particular if $u_0 = (0, 1)$ then $\tilde{f} \in \ker(\Delta_{a^+} \mid \mathbb{Z}^2) \iff f \in \ker(\Delta_{a^+} \mid \mathbb{Z})$. The only a^+-harmonic sublattice $L' \in \mathcal{L}^0$ that contains the vector e_1 is $L' = \mathbb{Z}e_1 + 3\mathbb{Z}e_2$ (cf. 4.5 below).

Further, if $u_0 = \pm e_1 \pm e_2$ then $\tilde{f} \in \ker(\Delta_{a^+}) \iff f = 0$. Consequently none of the a^+-harmonic sublattices $L' \in \mathcal{L}$ contains a vector of the form $\pm e_1 \pm e_2$.

Next we let $L' = \mathbb{Z}u_0 + \mathbb{Z}v_0 \subseteq \mathbb{Z}^2 = L$, where $u_0 = (k, l)$ and $v_0 = (k', l')$ are primitive lattice vectors different from $\pm e_1, \pm e_2, \pm e_1 \pm e_2$. Suppose that $m = \mathrm{ind}_L(L') = |\det(u_0, v_0)|$ is odd. Then L' is a^+-harmonic if and only if there exists a nonzero solution $f \in \mathcal{F}(\mathbb{Z}, \bar{K})$ of (10), if and only if there is a primitive mth root of unity $\zeta \in \bar{K}^\times$ satisfying

$$1 + \zeta^k + \zeta^{-k} + \zeta^l + \zeta^{-l} = 0\,.$$

Such a root ζ determines an a_*^+-harmonic character $\theta \in \mathrm{Char}_{a_*^+-\mathrm{harm}}(L/L_0, \bar{K}^\times)$ of order m, where $\theta : x \longmapsto \zeta^x$. It also defines an a^+-harmonic character $\theta \circ \pi_0 \in \mathrm{Char}_{a^+-\mathrm{harm}}(L, \bar{K}^\times)$ and the corresponding a^+-harmonic sublattice $L' = \ker(\theta \circ \pi_0)$.

2.3. **Symbol of a convolution operator. Examples.**

Definition 2.6. Fixing a lattice L, a basis $\mathcal{V} = (v_1, \ldots, v_s)$ of L and an n-tuple $\bar{n} = (n_1, \ldots, n_s)$, where $n_i \in \mathbb{N}$, we let

$$L_{\bar{n},\mathcal{V}} = \sum_{i=1}^{s} n_i \mathbb{Z} v_i \cong \bigoplus_{i=1}^{n} n_i \mathbb{Z}$$

be the product sublattice of L generated by $n_1 v_1, \ldots, n_s v_s$. There is an isomorphism of K-algebras

$$\sigma_{\mathcal{V}} : \mathrm{Conv}_K(L) \xrightarrow{\cong} K[x_1, x_1^{-1}, \ldots, x_s, x_s^{-1}], \qquad \Delta_a \longmapsto \sigma_{a,\mathcal{V}}\,,$$

which associates to a convolution operator Δ_a on L its *$\mathcal{V}$-symbol*, that is the Laurent polynomial in s variables

$$\sigma_{a,\mathcal{V}} = \sum_{v \in L} a(v) x^{-\alpha(v)} = \sum_{v = \sum_{i=1}^{s} \alpha_i e_i} a(v) x_1^{-\alpha_1} \cdot \ldots \cdot x_s^{-\alpha_s}\,.$$

The inverse $\sigma_{\mathcal{V}}^{-1}$ is given via

$$x_i^{-1} \longmapsto \tau_{v_i} \qquad \text{and} \qquad x_i \longmapsto \tau_{-v_i}, \qquad i = 1, \ldots, s\,.$$

The algebraic hypersurface in the s-torus

$$\Sigma_{a,\mathcal{V}} = \sigma_{a,\mathcal{V}}^{-1}(0) \subseteq (K^\times)^s$$

associate with Δ_a will be called the *symbolic hypersurface*. Given a sequence of convolution operators $\Delta_{\bar{a}} = (\Delta_{a_j} \,:\, j = 1, \ldots, t)$ the affine subvariety

$$\Sigma_{\bar{a},\mathcal{V}} = \bigcap_{j=1}^{t} \sigma_{a_j,\mathcal{V}}^{-1}(0) \subseteq (K^\times)^s$$

is called the *symbolic variety*.

Example 2.7. (cf. e.g., [MOW]) The group ring of a finite abelian group

$$G = L/L_{\bar{n},\mathcal{V}} \cong \mathbb{Z}_{\bar{n}} := \bigoplus_{i=1}^{s} \mathbb{Z}/n_i\mathbb{Z}$$

is the truncated polynomial ring

$$K[G] = \bigotimes_{i=1}^{s} K[x_i]/(x_i^{n_i} - 1)\,.$$

The group G and the sublattice $L_{\bar{n},\mathcal{V}} \subseteq L$ are a-harmonic if and only if the image in $K[G]$ of the symbol $\sigma_{a,\mathcal{V}}$ is a zero divisor. Indeed the shift by $\pm v_i$ on L corresponds to the multiplication by $x_i^{\mp 1}$ in $K[G]$. Hence the image $\tilde{a} = F_{\text{naive}}(a) \in K[G]$ coincides with the symbol $\sigma_{a,\mathcal{V}}$ modulo the corresponding ideal.

Example 2.8. (see [Za]) We let $K = \mathrm{GF}(2)$, $L = \mathbb{Z}^s$ and $a = a^+$. With $\mathcal{V} = (e_1, \ldots, e_s)$ being the standard basis in L, the following hold.

• A product sublattice $L_{\bar{n},\mathcal{V}}$ is a^+-harmonic if and only if the image $\tilde{a}^+$ of the symbol

$$\sigma_{a^+,\mathcal{V}} = 1 + \sum_{i=1}^{s} (x_i + x_i^{-1}) \tag{11}$$

is a zero divisor in $K[G]$.

• If $s = 1$ then a sublattice $n\mathbb{Z} \subseteq \mathbb{Z}$ is a^+-harmonic (equivalently, $1 + x + x^{-1}$ is a zero divisor in $K[x]/(1 + x^n)$) if and only if $n \equiv 0 \mod 3$.

• Similarly for every $\bar{n} = (n_1, \ldots, n_s) \in \mathbb{N}^s$ with $n_1 \equiv 0 \mod 3$, the group $\mathbb{Z}_{\bar{n}}$ is a^+-harmonic.

• The group $\mathbb{Z}_{5,5} = (\mathbb{Z}/5\mathbb{Z})^2$ is a^+-harmonic, while $\mathbb{Z}_{7,7} = (\mathbb{Z}/7\mathbb{Z})^2$ is not.[7]

• For $s = 2$ the symbolic hypersurface $\Sigma_{a^+,\mathcal{V}}$ is the elliptic cubic with equation

$$x^2y + xy^2 + xy + x + y = 0\,.$$

[7]Many more examples of this kind were computed by Makar-Limanov (a letter to the author, 2004, 5p.) along the same lines, and in [Za, Appendix 1] by different methods. Cf. also 4.24 below.

3. Generalized Chebyshev-Dickson polynomials: the p-free case

3.1. **Dual groups and the Fourier transform.** Throughout this section we let $K = \mathrm{GF}(p^r)$. Thus the multiplicative group $\bar{K}^\times$ of the algebraic closure $\bar{K}$ is a torsion group, with torsion orders coprime with p. We let G be a finite abelian group of order $\mathrm{ord}\,(G)$ not divisible by p, so invertible in K, and we let $\mathbb{N}_{\mathrm{co}(p)}$ denote the set of all positive integers coprime to p.

3.1. Since $\bar{K}$ contains all roots of unity with orders dividing $\mathrm{ord}\,(G)$, the characters of G can be realized via homomorphisms $G \to \bar{K}^\times$. Hence they can be viewed as functions on G with values in $\bar{K}$. So the dual group $G^\vee$ of G admits a natural embedding into $\mathcal{F}(G, \bar{K})$.

3.2. The Fourier transform $F : \mathcal{F}(G, \bar{K}) \to \mathcal{F}(G^\vee, \bar{K})$ is defined via[8]

$$F : f \longmapsto \widehat{f}, \qquad \text{where} \qquad \widehat{f}(g^\vee) = \sum_{g\in G} f(g) g^\vee(g), \qquad g^\vee \in G^\vee\,,$$

and its inverse $F^{-1} : \mathcal{F}(G^\vee, \bar{K}) \to \mathcal{F}(G, \bar{K})$ via

$$F^{-1} : \varphi \longmapsto \widehat{\varphi}, \qquad \text{where} \qquad \widehat{\varphi}(g) = \frac{1}{\mathrm{ord}\,(G)} \sum_{g^\vee\in G^\vee} \varphi(g^\vee) g^\vee(g^{-1}), \qquad g \in G\,.$$

Thus $\hat{\hat{f}} = f$ and $\hat{\hat{\varphi}} = \varphi$. This notation will not lead to a confusion as we never exploit the Fourier transform on the dual group $G^\vee$.

3.3. Up to constant factors, both F and F^{-1} send δ-functions to characters and vice versa. Namely $\forall g \in G$, $\forall g^\vee \in G^\vee$,

$$\widehat{\delta_g} = g, \quad \widehat{g} = \delta_g \quad \text{and} \quad \widehat{\delta_G} = \sum_{g\in G} \widehat{\delta_g} = \mathrm{ord}\,(G)\delta_{e^\vee}\,,$$

respectively,

$$\mathrm{ord}\,(G)\widehat{\delta_{g^\vee}} = (g^\vee)^{-1}, \quad \widehat{g^\vee} = \mathrm{ord}\,(G)\delta_{(g^\vee)^{-1}} \quad \text{and} \quad \widehat{\delta_{G^\vee}} = \delta_e\,.$$

Furthermore F sends the convolution in the ring $\mathcal{F}(G, \bar{K})$ into the pointwise multiplication on $\mathcal{F}(G^\vee, \bar{K})$ giving an isomorphism of $\bar{K}$-algebras. The convolution operator Δ_a is sent to the operator of multiplication by $\hat{a}$. The Fourier transform of a character being a delta-function, any character $g^\vee \in G^\vee$ is a convolution idempotent. Moreover any idempotent of the ring $(\mathcal{F}(G, \bar{K}), *)$ is a sum of pairwise distinct characters.

3.4. In the $\bar{K}$-vector space $\mathcal{F}(G, \bar{K})$ we consider the non-degenerate symmetric bilinear form

$$\langle f_1, f_2\rangle_2 = \frac{1}{\mathrm{ord}\,(G)} \sum_{g\in G} f_1(g) f_2(g^{-1}) = \frac{1}{\mathrm{ord}\,(G)} f_1 * f_2(e)\,,$$

and in $\mathcal{F}(G^\vee, \bar{K})$ the bilinear form

$$\langle \varphi_1, \varphi_2\rangle_1 = \frac{1}{(\mathrm{ord}\,(G))^2} \sum_{g^\vee\in G^\vee} \varphi_1(g^\vee)\varphi_2(g^\vee)\,.$$

We have $\langle \widehat{f_1}, \widehat{f_2}\rangle_1 = \langle f_1, f_2\rangle_2$. The characters $(g^\vee : g^\vee \in G^\vee)$ form an orthonormal basis in $\mathcal{F}(G, \bar{K})$ w.r.t. the form $\langle \cdot, \cdot\rangle_2$.

[8]See e.g. [Nic].

3.2. **Convolution ideals.** We assume in the sequel that G is a finite abelian group and so the group ring $\bar{K}[G]$ is commutative. We note that a subspace $I \subseteq \mathcal{F}(G, K)$ is a convolution ideal if and only if it is a $\mathrm{Conv}_K(G)$-submodule, if and only if it is translation invariant that is, stable under shifts.

The next result follows immediately from the Burnside Theorem. Alternatively it can be easily deduced using the Fourier transform.

Proposition 3.5. (a) *$\mathcal{F}(G, \bar{K})$ admits a decomposition into a direct sum of one-dimensional $\mathrm{Conv}_{\bar{K}}(G)$-submodules generated by characters:*

$$\mathcal{F}(G, \bar{K}) = \bigoplus_{g^\vee \in G^\vee} (g^\vee)\,.$$

(b) *Any convolution ideal $I \subseteq \mathcal{F}(G, \bar{K})$ is principal, generated by the sum of characters contained in I. Furthermore there is a decomposition*

$$\mathcal{F}(G, \bar{K}) = I \oplus \mathrm{Ann}(I)\,.$$

3.3. **Divisibility of generalized Chebyshev-Dickson polynomials.** Let us recall the notation. Given $\bar{a} = (a_1, \dots a_t) \in (\mathcal{F}(G, \bar{K}))^t$ we let $\ker(\Delta_{\bar{a}}) = \bigcap_{j=1}^t \ker(\Delta_{a_j})$ and

$$\mathrm{CharPoly}_{\bar{a},G} = \mathrm{CharPoly}(\Delta_{\bar{a}}) = \gcd\big(\mathrm{CharPoly}(\Delta_{a_j}) \,:\, j = 1, \dots, t\big)\,.$$

For a subgroup $H \subseteq G$ we let

$$\mathrm{CharPoly}_{\bar{a}*,G/H} = \mathrm{CharPoly}(\Delta_{\bar{a}} \mid \mathcal{F}_H(G, \bar{K}))\,.$$

From Proposition 3.5 one can readily deduce the following corollary.

Corollary 3.6. (a) *For any $a \in \mathcal{F}(G, \bar{K})$, the matrix of Δ_a in the basis of characters in $\mathcal{F}(G, \bar{K})$ is diagonal and*

$$\mathrm{CharPoly}(\Delta_a) = \prod_{g^\vee \in G^\vee} (x - \hat{a}(g^\vee))\,.$$

Hence for any convolution ideal $I \subseteq \mathcal{F}(G, \bar{K})$,

$$\mathrm{CharPoly}(\Delta_a \mid I) \mid \mathrm{CharPoly}(\Delta_a)\,.$$

(b) *Furthermore*

$$\mathrm{spec}(\Delta_a) = \hat{a}(G^\vee) \subseteq \bar{K} \qquad \textit{and} \qquad \ker(\Delta_a) = \left(\widehat{\delta_{V(\hat{a})}}\right),$$

where

$$V(\hat{a}) = \{g^\vee : \hat{a}(g^\vee) = 0\} \subseteq G^\vee\,.$$

Consequently

$$d_a = \dim\ker(\Delta_a) = \mathrm{card}\,(V(\hat{a}))\,.$$

(c) *Similarly given $\bar{a} = (a_1, \dots, a_t) \in (\mathcal{F}(G, \bar{K}))^t$ we have*

$$\ker(\Delta_{\bar{a}}) = \left(\widehat{\delta_{V(\widehat{\bar{a}})}}\right), \qquad \textit{where} \qquad V(\widehat{\bar{a}}) := \bigcap_{j=1}^t V(\widehat{a_j}) \subseteq G^\vee\,.$$

So

$$d_{\bar{a},G} := \dim\ker(\Delta_{\bar{a}}) = \mathrm{card}\,\big(V(\widehat{\bar{a}})\big)\,.$$

Examples 3.7. 1. For any subgroup $H \subseteq G$,

$$\mathcal{F}(G, \bar{K}) = \mathcal{F}_H(G, \bar{K}) \oplus \mathrm{Ann}(\mathcal{F}_H(G, \bar{K}))\,,$$

where

$$\mathcal{F}_H(G, \bar{K}) = \left(\sum_{H \subseteq \ker(g^\vee)} g^\vee \right) = \sum_{H \subseteq \ker(g^\vee)} (g^\vee)\ .$$

Consequently for any $\bar{a} \in (\mathcal{F}(G, \bar{K}))^t$,

$$\mathrm{CharPoly}_{\bar{a}, G/H} \mid \mathrm{CharPoly}_{\bar{a}, G}\,. \tag{12}$$

2. We let $L' \subseteq L'' \subseteq L$ be a chain of sublattices, where L' is of finite index in L coprime with p. We let also $H = L''/L'$, $G = L/L'$ (so that $G/H = L/L''$) and

$$\bar{a}' = \pi'_* \bar{a} \in \mathcal{F}(G, \bar{K}), \qquad \bar{a}'' = \pi''_* \bar{a} \in \mathcal{F}(G/H, \bar{K})\,,$$

where $\pi : G \twoheadrightarrow G/H$, $\pi' : L \twoheadrightarrow G$ and $\pi'' = \pi \circ \pi' : L \twoheadrightarrow G/H$. By virtue of 2.3 we obtain

$$\mathrm{CharPoly}_{\bar{a}, L'} = \mathrm{CharPoly}_{\bar{a}', G} \quad \text{and} \quad \mathrm{CharPoly}_{\bar{a}, L''} = \mathrm{CharPoly}_{\bar{a}'', G/H}\,.$$

Hence by (12)

$$\mathrm{CharPoly}_{\bar{a}, L''} \mid \mathrm{CharPoly}_{\bar{a}, L'}\,. \tag{13}$$

Letting for instance $L'' = L$ we deduce that $(x - |a|) \mid \mathrm{CharPoly}_{a, L'}$ $\forall L' \in \mathcal{L}^0$, $\forall a \in \mathcal{F}^0(L, \bar{K})$. Indeed for the eigenvalue $|a|$ of Δ_a, the corresponding eigenspace contains the one dimensional subspace $\bar{K} \cdot 1 \subseteq \mathcal{F}(L, \bar{K})$ of constant functions.

Let us show finally that (13) holds disregarding the assumption of p-freeness.

Proposition 3.8. *Suppose that $L_1, L_2 \in \mathcal{L}$.*

(a) *If $L_1 \subseteq L_2$ then* $\mathrm{CharPoly}_{\bar{a}, L_2} \mid \mathrm{CharPoly}_{\bar{a}, L_1}$.

(b) *In general*

$$\mathrm{CharPoly}_{\bar{a}, L_1 + L_2} \mid \gcd(\mathrm{CharPoly}_{\bar{a}, L_1}, \mathrm{CharPoly}_{\bar{a}, L_2})$$

and

$$\mathrm{lcm}(\mathrm{CharPoly}_{\bar{a}, L_1}, \mathrm{CharPoly}_{\bar{a}, L_2}) \mid \mathrm{CharPoly}_{\bar{a}, L_1 \cap L_2}\,.$$

(c) *Consequently*

$$d_{\bar{a}}(L_1 + L_2) \le \min\{d_a(L_1), d_{\bar{a}}(L_2)\} \le \max\{d_{\bar{a}}(L_1), d_{\bar{a}}(L_2)\} \le d_{\bar{a}}(L_1 \cap L_2)\,.$$

Proof. It suffices to show (a), then (b) and (c) follow immediately. Assuming that $L_1 \subseteq L_2$ we consider the decomposition $G_1 = L/L_1 = F \oplus G_1(p)$, where $G_1(p) \subseteq G_1$ is the Sylow p-subgroup. Letting $G_2 = L_2/L_1 \subseteq G_1$ we obtain $G_2(p) = G_1(p) \cap G_2$. For $L''_i := \pi^{-1}(G_i(p)) \subseteq L$, where $\pi : L \twoheadrightarrow G_1$, we have $L''_i \supseteq L_i$, $i = 1, 2$, and $L''_1, L''_2 \in \mathcal{L}^0$, $L''_1 \subseteq L''_2$. By Corollary 1.5, $\mathrm{CharPoly}_{a, L''_2} \mid \mathrm{CharPoly}_{a, L''_1}$. By (13), $\mathrm{CharPoly}_{a, L_i} = (\mathrm{CharPoly}_{a, L''_i})^{p^{\alpha_i}}$, $i = 1, 2$, where $\alpha_2 \le \alpha_1$. Now (a) follows. □

Remarks 3.9. 1. By virtue of Corollary 3.6.b, for any finite abelian group G and any $\bar a=(a_1,\dots,a_t)\in(\mathcal{F}(L,\bar K))^t$ the $\bar a$-harmonic characters form an orthonormal basis in $\ker(\Delta_{\bar a})=\bigcap_{j=1}^t\ker(\Delta_{a_j})$. Consequently the group G is $\bar a$-harmonic if and only if it admits an $\bar a$-harmonic character. Moreover there is a bijection

$$G^\vee_{\bar a-\mathrm{harm}}\cong V(\widehat{\bar a})=\bigcap_{j=1}^t V(\widehat{a_j})\,.$$

2. For an abelian group G and a subgroup $H\subseteq G$ there is a natural injection $(G/H)^\vee\hookrightarrow G^\vee$, which is dual to $G\twoheadrightarrow G/H$. For any $a\in\mathcal{F}^0(G,\bar K)$ this map fits in the commutative diagram

$$\begin{array}{ccc}(G/H)^\vee & \longrightarrow & G^\vee\\ \Big\downarrow{\scriptstyle\widehat{a_*}} & & \Big\downarrow{\scriptstyle\hat a}\\ \bar K & \xrightarrow{\ \mathrm{id}\ } & \bar K\end{array}$$

Hence $V(\widehat{a_*})\hookrightarrow V(\hat a)$ and so $(G/H)^\vee_{a_*-\mathrm{harm}}\hookrightarrow G^\vee_{a-\mathrm{harm}}$. By virtue of 3.6(c) this implies the inequality $d_{a_*,G/H}\le d_{a,G}$. Replacing a by $a-\lambda\delta_e$ with $\lambda\in\bar K$ we see that a similar inequality holds for the multiplicities of all roots of the polynomials $\mathrm{CharPoly}_{a_*,G/H}$ and $\mathrm{CharPoly}_{a,G}$, hence

$$\mathrm{CharPoly}_{a_*,G/H}\mid\mathrm{CharPoly}_{a,G}\,.$$

This provides an alternative proof of (12).

Examples 3.10. 1. Letting $G=G_1\times G_2$ and $a=a_1\otimes a_2\in\mathcal{F}(G,\bar K)$, where $a_i\in\mathcal{F}(G_i,\bar K)$, $i=1,2$, we obtain

$$\mathrm{CharPoly}(\Delta_a)=\prod_{i,j}(x-\lambda_i\mu_j)\,,$$

where $\lambda_1,\dots,\lambda_{\mathrm{ord}\,(G_1)}$ and $\mu_1,\dots,\mu_{\mathrm{ord}\,(G_2)}$ denote the eigenvalues of Δ_{a_1} and Δ_{a_2}, respectively.

2. Similarly, letting $a=a_1\otimes 1\oplus 1\otimes a_2\in\mathcal{F}(G,\bar K)$ we obtain

$$\mathrm{CharPoly}(\Delta_a)=\prod_{i,j}(x-(\lambda_i+\mu_j))\,.$$

In particular this applies to $a_i^-=a^+_{G_i}-\delta_0$, $i=1,2$, and $a^-=a^+_G-\delta_0$ (Bacher's Lemma; cf. Corollary 2.10(a) in [Za] and its proof).

3. Letting $K=\mathrm{GF}(2)$ and $a=a^+$, for a finite abelian group $G=\bigoplus_{i=1}^s\mathbb{Z}/n_i\mathbb{Z}$ of odd order we have by 3.6(b)

$$\mathrm{spec}\,(\Delta_{a^+,G})=\widehat{a^+}(G^\vee)\subset\bar K\,.$$

If $g^\vee\in G^\vee$ is a character with $g^\vee(e_j)=\xi_j=\zeta_j^{k_j}$, where $\zeta_j\in\mu_{n_j}$ is a primitive n_jth root of unity and $0\le k_j\le n_j-1$, then (cf. (11))

$$\widehat{a^+}(g^\vee)=1+\sum_{j=1}^s\left(g^\vee(e_j)+g^\vee(e_j)^{-1}\right)=1+\sum_{j=1}^s\left(\zeta_j^{k_j}+\zeta_j^{-k_j}\right)\,.$$

Therefore

$$\mathrm{CharPoly}(\Delta_{a^+,G}) = \prod_{(k_1,\ldots,k_s)\in\mathbb{Z}_{\bar n}} \left(x - \left(1 + \sum_{j=1}^{s} \left(\zeta_j^{k_j} + \zeta_j^{-k_j}\right)\right)\right).$$

3.4. **Generalized Chebyshev-Dickson polynomials as iterated resultants.** For a finite abelian group $G = \mathbb{Z}_{\bar n} = \bigoplus \mathbb{Z}/n_i\mathbb{Z}$, where $\bar n = (n_1, \ldots, n_s) \in \mathbb{N}^s_{\mathrm{co}(p)}$, we let $\mathcal{V} = (e_1, \ldots, e_s)$ denote the standard base. We let also

$$\mu_{\bar n} = \bigoplus_{i=1}^{s} \mu_{n_i} \subseteq (\bar K^\times)^s,$$

where $\mu_n \subseteq \bar K^\times$ stands for the cyclic group of nth roots of unity. The correspondence

$$g^\vee \longmapsto (g^\vee(e_1), \ldots, g^\vee(e_1))$$

yields an isomorphism

$$\varphi : G^\vee \xrightarrow{\cong} \mu_{\bar n}.$$

Lemma 3.11. *For any $a \in \mathcal{F}(G, \bar K)$ we have*

$$\hat a = (\sigma_{a,\mathcal{V}} \mid \mu_{\bar n}) \circ \varphi.$$

Consequently

$$\mathrm{CharPoly}_{a,\bar n,\mathcal{V}} := \mathrm{CharPoly}_{a,L_{\bar n,\mathcal{V}}} = \prod_{\xi\in\mu_{\bar n}} (x - \sigma_{a,\mathcal{V}}(\xi)). \tag{14}$$

Proof. Indeed $\forall g \in G$, $\forall g^\vee \in G^\vee$, letting $\xi_i = g^\vee(e_i)$, $i = 1, \ldots, s$, we obtain:

$$\hat a(g) \cdot g^\vee(g) = \Delta_a(g^\vee)(g) = \left(\sum_{v\in G} a(v)\tau_{-v}\right)(g^\vee)(g) = \sum_{v\in G} a(v)\tau_{-v}(g^\vee)(g)$$

$$= \left(\sum_{v=\sum_{j=1}^s \alpha_j e_j \in G} a(v) g^\vee(-v)\right)(g^\vee)(g) = \sigma_{a,\mathcal{V}}(\xi_1, \ldots, \xi_s) \cdot g^\vee(g) = \sigma_{a,\mathcal{V}}(\xi),$$

where $\xi = (\xi_1, \ldots, \xi_s) \in \mu_{\bar n}$. The last equality follows now from Proposition 3.6. $\square$

Definition 3.12. Given a multi-index $\bar n = (n_1, \ldots, n_s) \in \mathbb{N}^s$ and a Laurent polynomial $\sigma = p/y^\alpha$, where $y^\alpha = y_1^{\alpha_1} \cdot \ldots \cdot y_s^{\alpha_s}$ and $p \in \bar K[y_1, \ldots, y_s]$ are coprime, we consider the *iterated resultant* $\mathrm{res}_{\bar n}(\sigma) = q_s \in \bar K[x]$, where

$$q_0(x, y_1, \ldots, y_s) = y^\alpha x - p(y_1, \ldots, y_s)$$

and $q_i \in \bar K[x, y_{i+1}, \ldots, y_s]$ are defined recursively via

$$q_i(x, y_{i+1}, \ldots, y_s) = \mathrm{res}_{y_i}\left(q_{i-1}(x, y_i, \ldots, y_s), y_i^{n_i} - 1\right), \qquad i = 1, \ldots, s.$$

In detail

$$\mathrm{res}_{\bar n}(\sigma) = \mathrm{res}_{y_s}\left(\ldots \mathrm{res}_{y_1}\left(y_1^{\alpha_1} \ldots y_s^{\alpha_s} x - p(y_1, \ldots, y_s), y_1^{n_1} - 1\right), \ldots, y_s^{n_s} - 1\right).$$

Clearly $\lambda = \sigma(\xi) = p(\xi)/\xi^\alpha$ for some $\xi \in \mu_{\bar n}$ if and only if $\mathrm{res}_{\bar n}(\sigma)(\lambda) = 0$.

Given a lattice L of rank s with a basis $\mathcal{V}$ and a multi-index $\bar{n} \in \mathbb{N}^s_{\mathrm{co}(p)}$, we let as before $L_{\bar{n},\mathcal{V}}$ denote the product sublattice $\bigoplus n_i\mathbb{Z}v_i$ of L. From Lemma 3.11 we deduce (in the p-free case) the following expressions for the multivariate generalized Chebyshev-Dickson polynomials $\mathrm{CharPoly}_{a,\bar{n},\mathcal{V}}$ as in (14).

Proposition 3.13. *In the notation as above, the characteristic polynomial of the restriction $\Delta_a \mid L_{\bar{n},\mathcal{V}}$, where $a \in \mathcal{F}(L, \bar{K})$ and $\bar{n} \in \mathbb{N}^s_{\mathrm{co}(p)}$, can be expressed as follows:*

$$\mathrm{CharPoly}_{a,\bar{n},\mathcal{V}} = \mathrm{res}_{\bar{n}}(\sigma_{a,\mathcal{V}})\,.$$

Example 3.14. For the classical Chebyshev-Dickson polynomials T_n of first kind the proposition gives

$$T_n(x) = \mathrm{res}_y(xy + y^2 + 1, y^n + 1)\,.$$

An alternative expression for the characteristic polynomials of σ^+-automata on multidimensional grids in terms of iterative resultants can be found e.g. in [HMP, 3.3].

Remark 3.15. To a function $a \in \mathcal{F}(L, \bar{K})$ we have associated a family of generalized Chebyshev-Dickson polynomials $\mathrm{CharPoly}_a : \mathcal{L}^0 \to \bar{K}[x]$. Despite the fact that this family satisfies the division property, its individual members can be arbitrary polynomials. Let us show for instance that given a degree $d > 0$ polynomial $p \in \bar{K}[x]$, where $K = \mathrm{GF}(q)$, there exists a function $a \in \mathcal{F}^0(\mathbb{Z}, \bar{K})$ such that $p = \mathrm{CharPoly}_{a,d,e_1}$. Indeed enumerating arbitrarily the roots $z_1, \ldots, z_d \in \bar{K}$ of p we consider a function $\hat{a}_*$ on $G^\vee$ with $\hat{a}_*(i) = z_{i+1}$, $i = 0, \ldots, d-1$, where $G = \mathbb{Z}/d\mathbb{Z}$. Letting $a_* = F^{-1}(\hat{a}_*) \in \mathcal{F}(G, \bar{K})$ we push a_* backward via $\mathbb{Z} \twoheadrightarrow G$ to a function $a \in \mathcal{F}^0(\mathbb{Z}, \bar{K})$ supported on the interval $[0, \ldots, d-1]$. Then a is as required.

4. Characteristic sublattices and translation invariant subspaces

4.1. **Lattice characters with values in $\bar{K}^\times$.** Given a lattice L, we call a homomorphism $g^\vee : L \to \bar{K}^\times$ a *character of L with values in $\bar{K}^\times$*. It is called *$\bar{a}$-harmonic* for $\bar{a} = (a_1, \ldots, a_t) \in (\mathcal{F}^0(L, \bar{K}))^t$ if $\Delta_{a_j}(g^\vee) = 0$ $\forall j = 1, \ldots, t$. We let $\mathrm{Char}(L, \bar{K}^\times)$ be the set of all characters on L with values in $\bar{K}^\times$ and $\mathrm{Char}_{\bar{a}-\mathrm{harm}}(L, \bar{K}^\times)$ the subset of all $\bar{a}$-harmonic characters.

Example 4.1. For $K = \mathrm{GF}(2)$, $L = \mathbb{Z}^2$, $t = 1$, $a_1 = a^+$ and for a primitive cubic root of unity $\omega \in \mu_3$,

$$\theta = \begin{pmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & 1 & \omega & \omega^2 & 1 & \omega & \cdots \\ \cdots & 1 & \omega & \omega^2 & 1 & \omega & \cdots \\ \cdots & 1 & \omega & \omega^2 & 1 & \omega & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

is an a^+-harmonic character with values in $\mu_3 \subseteq \bar{K}^\times$.

Remarks 4.2. 1. Clearly $L/\ker(\theta) \cong \mathbb{Z}/m\mathbb{Z}$ for a character $\theta \in \mathrm{Char}(L, \bar{K}^\times)$ if and only if $\mathrm{ord}(\theta) = m$. Vice versa given a cocyclic sublattice $L' \subseteq L$ with $L/L' \cong \mathbb{Z}/m\mathbb{Z}$, where $m \in \mathbb{N}_{\mathrm{co}(p)}$, we have $L' = \ker(\theta)$ for the character θ on L defined via

$$\theta : L \twoheadrightarrow L/L' \xrightarrow{\cong} \mathbb{Z}/m\mathbb{Z} \xrightarrow{\cong} \mu_m \hookrightarrow \bar{K}^\times\,.$$

2. We note that $L(\theta') = \varepsilon(L(\theta))$ for two characters $\theta, \theta' \in \mathrm{Char}(\mathbb{Z}^s, \bar{K}^\times)$ and for some $\varepsilon \in \mathbf{SL}(s, \mathbb{Z})$ if and only if $\mathrm{ord}(\theta') = \mathrm{ord}(\theta)$. In the latter case $\theta' = \delta \circ \theta \circ \varepsilon$, where $\delta \in \mathrm{Aut}(\mathbb{Z}/m\mathbb{Z}) \cong (\mathbb{Z}/m\mathbb{Z})^\times$ with $m = \mathrm{ord}(\theta)$.

Fixing a basis $\mathcal{V}=(v_1,\dots,v_s)$ of L, for a lattice vector $v=\sum_{i=1}^s \alpha_i(v)v_i\in L$ we let $\vec{v}=(\alpha_1,\dots,\alpha_s)\in\mathbb{Z}^s$, where $\alpha_i=\alpha_i(v)$. Below $\langle\cdot,\cdot\rangle$ stands for the standard bilinear form on $\mathbb{Z}^s$. We observe the following.

Proposition 4.3. (a) *Given a basis $\mathcal{V}=(v_1,\dots,v_s)$ of L, a primitive mth root of unity $\zeta\in\mu_m$ $(m\in\mathbb{N}_{\mathrm{co}(p)})$ and a primitive lattice vector $v_0\in L$, the formula*

$$\theta(v)=\zeta^{\langle\vec{v},\vec{v_0}\rangle} \tag{15}$$

defines a character $\theta\in\mathrm{Char}(L,\bar{K}^\times)$ of order m.

(b) *Vice versa any character $\theta\in\mathrm{Char}(L,\bar{K}^\times)$ can be written as in (15) for a suitable primitive mth root of unity $\zeta\in\mu_m$ of order $m=\mathrm{ord}\,(\theta)\in\mathbb{N}_{\mathrm{co}(p)}$ and a suitable primitive lattice vector $v_0=v_\theta=\sum_{i=1}^s\alpha_i(v_\theta)v_i\in L$ with $\alpha_i(v_\theta)\in\{0,\dots,m-1\}$.*

(c) *Consequently the period lattice of θ:*

$$L(\theta)=\ker(\theta)=\{v\in L\,:\,\langle\vec{v},\vec{v_\theta}\rangle\equiv 0\mod m\}$$

is an index m sublattice of L.

(d) *The character θ as in (15) and the corresponding sublattice $L'=\ker(\theta)$ are a-harmonic, where $a\in\mathcal{F}^0(L,\bar{K})$, if and only if*

$$(\theta*a)(0)=\sigma_{a,\mathcal{V}}(\xi)=0\quad\text{that is}\quad \xi\in\Sigma_{a,\mathcal{V}}\,, \tag{16}$$

where $\xi=(\zeta^{\alpha_1(v_\theta)},\dots,\zeta^{\alpha_s(v_\theta)})$ and $\sigma_{a,\mathcal{V}}$ is the symbol of Δ_a w.r.t. the basis $\mathcal{V}$.

Proof. The proof of (a) is straightforward. To show the converse we let $\xi_i=\theta(e_i)\in\bar{K}^\times$ and $n_i=\mathrm{ord}\,(\xi_i)$, $i=1,\dots,s$. We let also $m=m(\theta)=\mathrm{lcm}(n_1,\dots,n_s)$ be the exponent of the group $\mu_{\bar{n}}$. For a primitive mth root of unity $\tilde{\zeta}\in\mu_m$ we write $\xi_i=\tilde{\zeta}^{b_i}$, where $n_i=m/\gcd(b_i,m)$. Letting further $d=\gcd(b_1,\dots,b_s)$, $b_i=d\alpha_i$ and $d_i=\gcd(b_i,m)$ we obtain $\gcd(d_1,\dots,d_s)=1$ and so $\gcd(d,m)=1$. Hence $\zeta=\tilde{\zeta}^d$ is again a primitive mth root of unity and $\xi_i=\zeta^{\alpha_i}$, $i=1,\dots,s$. Therefore $\theta: v\longmapsto\zeta^{\langle\vec{v},\vec{v_\theta}\rangle}$ for a primitive vector $v_\theta=\sum_{i=1}^s\alpha_i(v_\theta)v_i\in L$, where $\alpha_i\in\{0,\dots,m-1\}$, $i=1,\dots,s$.

Now (16) follows from the equalities

$$\theta*a(0)=\sum_{u\in L}\theta(u)a(-u)=\sum_{u\in L}a(u)\zeta^{-\langle\vec{u},\vec{v_\theta}\rangle}$$
$$=\sum_{u\in L}a(u)\zeta^{-\alpha_1(u)\alpha_1(v_\theta)}\cdot\ldots\cdot\zeta^{-\alpha_s(u)\alpha_s(v_\theta)}=\sigma_{a,\mathcal{V}}(\xi)\,.$$

□

Example 4.4. A character $\theta\in\mathrm{Char}(\mathbb{Z}^s,\bar{K}^\times)$, $v\longmapsto\zeta^{\langle v,v_0\rangle}$ as in (15), where $K=\mathrm{GF}(2)$, and the corresponding period sublattice $L'=\ker(\theta)$ of full rank s are a^+-harmonic if and only if

$$(\theta*a^+)(0)=1+\sum_{i=1}^s\left(\zeta^{\alpha_i(v_0)}+\zeta^{-\alpha_i(v_0)}\right)=0\,. \tag{17}$$

Remark 4.5. A primitive lattice vector $v_0\in L$ will be called *a-exceptional* if the symbol $\sigma_{a_*}\in\bar{K}[x,x^{-1}]$ is a Laurent monomial, where $a_*=\pi_*a\in\mathcal{F}(\mathbb{Z},\bar{K})$ for the surjection $\pi: L\twoheadrightarrow\mathbb{Z}$, $v\longmapsto\langle\vec{v},\vec{v}_0\rangle$. Clearly there exist many non a-exceptional vectors $v_0\in L$ as soon as $\mathrm{card}\,(\mathrm{supp}(a))\geq 2$. For such a vector v_0 any nonzero root ζ of the Laurent polynomial σ_{a_*} gives rise to an a-harmonic character $\theta \in \mathrm{Char}(L, \bar{K}^\times)$, $v \longmapsto \zeta^{\langle \vec{v}, \vec{v}_0 \rangle}$.

4.2. **Periodicity of solutions of convolution equations.** A general convolution equation on a lattice L of rank ≥ 2 does admit aperiodic solutions. For instance on $L = \mathbb{Z}^2$ there are aperiodic a^+-harmonic functions with values in GF(2). Indeed consider the strip $S := \mathbb{Z} \times \{0, -1\} \subseteq L$ of width 2. Any function $f_0 : S \to \mathrm{GF}(2)$ on S admits a unique a^+-harmonic extension $f_0 \rightsquigarrow f$ to L given via

$$f(m,n) = f_0(m,-1) + f_0(m,0) + f_0(m+n,0) + f_0(m-n,0), \quad m > 0,$$

on the upper halfplane and symmetrically on the lower one. Clearly for a generic f_0 this extension f is aperiodic that is $L(f) = \{0\}$.

Furthermore there are bi-periodic a^+-harmonic functions on L with arbitrarily large pairs of periods, see [Za].

However all solutions of convolution equations on rank 1 lattices are periodic with a period depending only on the equation. Indeed the following holds.

Lemma 4.6. *For any $a \in \mathcal{F}^0(\mathbb{Z}, \bar{K}) \setminus \{0\}$, every a-harmonic function $f \in \ker(\Delta_a)$ is m_a-periodic for a certain $m_a > 0$ depending only on a. Consequently the subspace*

$$\ker(\Delta_a) \subseteq \mathcal{F}_{L'}(L, \bar{K}), \qquad \textit{where} \quad L' = m_a\mathbb{Z} \subseteq L := \mathbb{Z}\,,$$

is of finite dimension.

Proof. Replacing a by $a * \delta_n$ with a suitable n we may suppose that $a = \sum_{i=0}^N a(i)\delta_{-i}$ with $N \geq 0$ and $a(0)$, $a(N) \neq 0$. For any $f \in \ker(\Delta_a)$ we have

$$0 = f * a(0) = a(0)f(0) + a(1)f(1) + \ldots + a(N)f(N)\,.$$

Hence

$$f(N) = b_0 f(0) + \ldots + b_{N-1} f(N-1)\,,$$

where $b_i = -\frac{a(i)}{a(N)}$ and so $b_0 \neq 0$. Therefore the linear transformation

$$\varphi : \mathbb{A}^N_{\bar{K}} \to \mathbb{A}^N_{\bar{K}}, \qquad (f_0, \ldots, f_{N-1}) \longmapsto (f_1, \ldots, f_N)$$

with $\det(\varphi) = \pm b_0$ is invertible, whence of finite order, say, m_a. This shows that f is periodic of period m_a. □

For instance any a^+-harmonic function on $\mathbb{Z}$ is periodic of period 3. Moreover any a^+-harmonic function on $L = \mathbb{Z}^2$ which is periodic on the strip $S = \mathbb{Z} \times \{0, -1\}$ is also bi-periodic. In the same fachion, one can easily prove the following fact.

Corollary 4.7. *Let $\bar{a} = (a_1, \ldots, a_l) \in (\mathcal{F}^0(L, \bar{K}))^l$ and suppose that the convolution ideal $(a_1, \ldots, a_l) \subseteq \mathcal{F}^0(L, \bar{K})$ contains $s = \mathrm{rk}(L)$ nonzero functions $b_1, \ldots, b_s \in \mathcal{F}^0(L, \bar{K})$ such that $\mathrm{supp}(b_j) \subseteq \mathbb{Z}v_j$ ($j = 1, \ldots, s$), where $v_1, \ldots, v_s \in L$ are linearly independent. Then any function $f \in \ker(\Delta_{\bar{a}}) := \bigcap_{j=1}^l \ker(\Delta_{a_j})$ is pluri-periodic and its period lattice $L(f)$ contains a rank s sublattice $L' = \sum_{j=1}^s m_j\mathbb{Z}v_j$, where $m_j > 0$, $j = 1, \ldots, s$.*

4.3. **Translation invariant subspaces.**

4.8. The following observation will be used in the proofs. For any finitely generated abelian group G with a Sylow p-subgroup G_p one has

$$G_p = \bigcap_{\theta\in\mathrm{Char}(G,\bar K^\times)} \ker(\theta) \quad \text{and} \quad \mathrm{Char}(G,\bar K^\times) = \pi^* \,\mathrm{Char}(G/G_p,\bar K^\times)\,,$$

where $\pi : G \twoheadrightarrow G/G_p$.

Let us introduce the following notions.

Definition 4.9. A subspace $E \subset \mathcal{F}(L,\bar K)$ will be called a *characteristic subspace* if it is spanned by a finite set of characters $\theta_1,\ldots,\theta_m \in \mathrm{Char}(L,\bar K^\times)$. We note that any characteristic subspace is translation invariant.

A sublattice of the form

$$L' = \bigcap_{i=1}^{m} \ker(\theta_i)\,,$$

where $\theta_1,\ldots,\theta_m \in \mathrm{Char}(L,\bar K^\times)$, will be called a *characteristic sublattice.*

Lemma 4.10. *$L' \in \mathcal{L}$ is a characteristic sublattice if and only if $L' \in \mathcal{L}^0$.*

Proof. If L' is a characteristic sublattice then $L' = \ker(\varphi)$, where $\varphi = (\theta_1,\ldots,\theta_m) : L \to \bigoplus_{i=1}^m \mu_{n_i}$ with $n_i = \mathrm{ord}\,(\theta_i) \in \mathbb{N}_{\mathrm{co}(p)}$, $i = 1,\ldots,m$. Hence the index $\mathrm{ind}_L(L')$ is coprime with p.

Conversely assuming that $L' \in \mathcal{L}^0$, the order of the group $G = L/L'$ is coprime with p. Hence G is a product of cyclic groups μ_{m_i} of orders $m_i \in \mathbb{N}_{\mathrm{co}(p)}$, $i = 1,\ldots,m$. The compositions

$$L \twoheadrightarrow G \twoheadrightarrow \mu_{m_i}, \quad i = 1,\ldots,n\,,$$

yield characters $\theta_i \in \mathrm{Char}(L,\bar K^\times)$ with $L' = \bigcap_{i=1}^n \ker(\theta_i)$, and so the sublattice L' is characteristic. ☐

Remark 4.11. Given a finite group G of order coprime with p and a function $f \in \mathcal{F}(G,\bar K)$, the subgroup of periods $L(f) \subseteq G$ is a characteristic subgroup. It can be recovered by $\mathrm{supp}(\hat f)$ as follows:

$$L(f) = \bigcap_{g^\vee\in\mathrm{supp}(\widehat{f})} \ker\,(g^\vee)\,.$$

Indeed for $g \in G$, $f_g = f \iff \widehat{f}\cdot g = \widehat{f} \iff g \mid \mathrm{supp}(\widehat{f}) = 1 \iff g \in \bigcap_{g^\vee\in\mathrm{supp}(\widehat{f})} \ker(g^\vee)\,.$

4.12. For a translation invariant subspace $E \subseteq \mathcal{F}(L,\bar K)$ of finite dimension[9] we let $G(E)$ and $L(E)$ denote the image and the kernel, respectively, of the homomorphism

$$\rho : L \to \mathrm{Aut}\,(E) \cong \mathbf{GL}(n,\bar K), \qquad v \longmapsto \tau_v\,.$$

We let $L(E)$ denote the period lattice of E that is $L(E) = \bigcap_{f\in E} L(f)$. Since E is translation invariant we have

$$E(f) := \mathrm{span}\,(\tau_v(f))_{v\in L} \subseteq E \quad \forall f \in E\,.$$

[9]We notice that such a subspace E consists of pluri-periodic functions. Indeed the restrictions to E of any shift has finite order.

The group $G(E) = \rho(L) \subseteq \mathrm{Aut}(E)$ is finite being a finitely generated abelian torsion group. We let $G_p(E)$ denote the Sylow p-subgroup of the group $G(E)$.

Proposition 4.13. *A translation invariant subspace $E \subseteq \mathcal{F}(\mathbb{Z}^s, \bar{K})$ of finite dimension is a characteristic subspace if and only if the sublattice $L(E) \subseteq L$ is characteristic*[10].

Proof. If E is spanned by characters, say, $\theta_1, \ldots, \theta_n$ then

$$\bigcap_{i=1}^{n} \ker(\theta_i) \subseteq L(f) \quad \forall f \in E\,,$$

hence $L(E) = \bigcap_{i=1}^{n} \ker(\theta_i)$ is a characteristic sublattice.

Conversely suppose that $\mathrm{ind}_L(L(E)) = \mathrm{ord}(G(E))$ is coprime with p. We have $E = \pi^*(E')$ for a suitable translation invariant subspace $E' \subseteq \mathcal{F}(G(E), \bar{K})$. By 4.9, E' is a convolution ideal spanned by characters. Hence E is spanned by characters too. □

The period lattice of an $\bar{a}$-harmonic function on L is not necessarily a characteristic one, as the following example shows.

Example 4.14. (cf. [Za, Example 2.33].) Letting $K = \mathrm{GF}(2)$, $t = 1$, $a_1 = a^+$ and $G = \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}$, the a^+-harmonic function on G

$$h = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

lifts to $f = h \circ \pi \in \ker_{a^+}(L)$ via $\pi : L = \mathbb{Z}^2 \twoheadrightarrow G$. Thus f is a^+-harmonic and has the period lattice $L(f) = \pi^{-1}(L(h)) = 3\mathbb{Z}e_1 + 6\mathbb{Z}e_2 \subseteq L$ of even index. By virtue of Proposition 4.13, f cannot be represented as a linear combination of characters with values in $\bar{K}^\times$.

4.15. Any sublattice $L' \in \mathcal{L}$ is contained in a unique minimal characteristic sublattice $L'' \in \mathcal{L}^0$, where

$$L'' = \bigcap_{\tilde{L} \in \mathcal{L}^0,\, \tilde{L} \supseteq L'} \tilde{L} = \bigcap_{\theta \in \mathrm{Char}(L, \bar{K}^\times),\, \ker(\theta) \supseteq L'} \ker(\theta)\,.$$

These data fit in the following commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & & & \\
0 \arrow[r] & L(E) \arrow[r] \arrow[d] & L \arrow[r, "\rho"] \arrow[d, "\mathrm{id}"] & G(E) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & L''(E) \arrow[r] & L \arrow[r] & G(E)/G_p(E) \arrow[r] \arrow[d] & 0 \\
 & & & 0 &
\end{tikzcd}$$

[10] That is $L(E) \in \mathcal{L}^0$, see Lemma 4.10 above.

Any translation invariant subspace E as in 4.12 contains a unique maximal characteristic subspace $E^0 \subseteq E$, where

$$E^0 = \text{span}\,\left(\theta \,:\, \theta \in E \cap \text{Char}(L, \bar{K}^\times)\right)\,.$$

Actually E^0 is the fixed point subspace for the action of $G_p(E)$ on E by shifts. The next lemma says that in characteristic $p = 2$, $E^0 \neq 0$ as soon as $E \neq 0$.

Lemma 4.16. *Suppose that $p = \text{char}\, K = 2$ and let $E \subseteq \mathcal{F}(L, \bar{K})$ be a non-trivial translation invariant subspace of finite dimension. Then there exists a nonzero function $h \in E^0$ such that the sublattice of periods $L(h)$ is a characteristic one i.e., $L(h)$ is of odd index in L.*

Proof. If $2v \in L$ is a period of a nonzero function $f \in E$ then so is v, maybe, for another such function $h \in E$. Indeed since $f = f_{2v} \neq 0$ then either $f = f_v = h$ or $f \neq f_v$, and so $h = f + f_v \neq 0 \in E$ is v-periodic, as required. In this way we arrive finally at a sublattice $L'' = L(h'') \subseteq L$ of odd index which contains $L(E)$. This corresponds to passing from the group $G(E) = L/L(E)$ to its quotient group $G(E)/G_2(E)$ of odd order, where $G_2(E)$ stands for the Sylow 2-subgroup of $G(E)$. By Proposition 4.13, $h'' \in E^0$ as required. $\square$

4.17. For any sublattice $L' \in \mathcal{L}$ the translation invariant subspace

$$E_{L'} := \mathcal{F}_{L'}(L, \bar{K}) = \{f \in \mathcal{F}(L, \bar{K}) \,:\, L(f) \supseteq L'\} \tag{18}$$

is a maximal one with period lattice L'. It is easily seen that $L'' = L(E^0_{L'})$. The regular representation $\rho : L \to \text{Aut}(E_{L'}),\ v \longmapsto \tau_v$, factorizes through a representation $L \to S_n$ into the symmetric group, where $n = \text{ind}_L(L') = \dim_{\bar{K}}(E_{L'})$ and S_n acts by permutations of the orthonormal basis $((\delta_{v+L'})_{v\in L}$ of $E_{L'}$. The latter can be identified with the basis of δ-functions in $\mathcal{F}(G, \bar{K}) \cong E_{L'}$, where $G = L/L'$. The representation ρ is induced via $L \twoheadrightarrow G$ by the regular representation of G on $\mathcal{F}(G, \bar{K})$. The matrix elements of ρ are the δ-functions δ_g $(g \in G)$. Every function $f \in \mathcal{F}(G, \bar{K})$ is a *state* i.e., a linear combination of matrix elements.

4.4. **Harmonic characters as points on the symbolic variety.** We let as before G denote a finite abelian group of order coprime with $p = \text{char}\, \bar{K}$, where $K = \text{GF}(p^r)$. Given $\bar{a} = (a_1, \ldots, a_t) \in (\mathcal{F}(L, \bar{K}))^t$, there is a natural bijection between the set of points on the symbolic variety $\Sigma_{\bar{a},\mathcal{V}}$ and the set of $\bar{a}$-harmonic characters, see Proposition 4.20 below.

4.18. Given a basis $\mathcal{V} = (v_1, \ldots, v_s)$ of L, any character $g^\vee \in \text{Char}(L, \bar{K}^\times)$ is $L_{\bar{n},\mathcal{V}}$-periodic for $\bar{n} = (n_1, \ldots, n_s)$, where $n_i = \text{ord}\,(g^\vee(v_i)) \in \mathbb{N}_{\text{co}(p)}$, $i = 1, \ldots, s$, because $\bar{K}^\times$ is a torsion group. Letting $G = G_{\bar{n},\mathcal{V}} = L/L_{\bar{n},\mathcal{V}}$ we have $g^\vee = h^\vee \circ \pi$ for a character $h^\vee \in G^\vee$, where $\pi : L \twoheadrightarrow G$. By virtue of 2.3, $g^\vee$ is $\bar{a}$-harmonic if and only if $h^\vee$ is $\bar{a}_*$-harmonic. Consequently

$$\text{Char}_{\bar{a}-\text{harm}}(L, \bar{K}^\times) = \bigcup_{\bar{n}\in\mathbb{N}^s_{\text{co}(p)}} (G_{\bar{n},\mathcal{V}})^\vee_{\bar{a}_*-\text{harm}}\,.$$

4.19. For any t-tuple $\bar{a} = (a_1, \ldots, a_t) \in (\mathcal{F}^0(L, \bar{K}))^t$ and s-tuple $\bar{n} = (n_1, \ldots, n_s) \in \mathbb{N}^s_{\text{co}(p)}$, we consider the following overdetermined system of algebraic equations (cf. 2.6):

$$\sigma_{a_j,\mathcal{V}}(x_1, \ldots, x_s) = 0, \quad x_i^{n_i} = 1, \quad i = 1, \ldots, s,\ \ j = 1, \ldots, t\,. \tag{19}$$

We let $\Sigma_{\bar{a},\bar{n},\mathcal{V}}$ denote the set of all solutions of (19), or in other words the set of all points on the symbolic variety $\Sigma_{\bar{a},\mathcal{V}}$ with multi-torsion orders[11] which divide $\bar{n} = (n_1, \ldots, n_s)$.

Proposition 4.20. *There are natural bijections*

$$\sigma_{\mathcal{V}} : \mathrm{Char}_{\bar{a}-\mathrm{harm}}(L, \bar{K}^\times) \xrightarrow{\cong} \Sigma_{\bar{a},\mathcal{V}}$$

and

$$\Sigma_{\bar{a},\bar{n},\mathcal{V}} \cong (G_{\bar{n},\mathcal{V}})^\vee_{\bar{a}_*-\mathrm{harm}} \cong V(\widehat{\bar{a}_*})\,, \tag{20}$$

where $\pi : L \twoheadrightarrow G_{\bar{n},\mathcal{V}} = L/L_{\bar{n},\mathcal{V}}$. *Consequently*

$$d_{\bar{a}}(L_{\bar{n},\mathcal{V}}) = \mathrm{card}\,\left(V(\widehat{\bar{a}_*})\right) = \mathrm{card}\,(\Sigma_{\bar{a},\bar{n},\mathcal{V}})\ .$$

Proof. For $h^\vee \in G^\vee_{\bar{n},\mathcal{V}}$, letting $g^\vee = h^\vee \circ \pi \in \mathrm{Char}(L, \bar{K}^\times)$ and $\xi_i = g^\vee(v_i) \in \bar{K}^\times$, we have $\xi_i^{n_i} = 1\ \forall i = 1, \ldots, s$ (indeed $n_i v_i \in L_{\bar{n},\mathcal{V}}\ \forall i$). Moreover $h^\vee \in (G_{\bar{n},\mathcal{V}})^\vee_{\bar{a}_*-\mathrm{harm}}$ if and only if $\forall j = 1, \ldots, t$,

$$h^\vee * a_{j*} = 0 \iff g^\vee * a_j = 0 \iff g^\vee * \left(\sum_{g\in L} a_j(g)\delta_g\right) = 0$$

$$\iff \sum_{g=\sum_{i=1}^s \alpha_i v_i \in L} a_j(g)(g^\vee)^{-1}(g) = 0 \iff \sigma_{a_j,\mathcal{V}}(\xi_1, \ldots, \xi_s) = 0\,,$$

and so $\xi = (\xi_1, \ldots, \xi_s) \in \bigcap_{j=1}^t \Sigma_{a_j,\mathcal{V}} = \Sigma_{\bar{a},\mathcal{V}}$. Vice versa, given a solution $\xi = (\xi_1, \ldots, \xi_s) \in (\bar{K}^\times)^s$ of (19), letting $g^\vee(v_i) = \xi_i$ defines an $L_{\bar{n},\mathcal{V}}$-periodic character $g^\vee \in \mathrm{Char}(L, \bar{K}^\times)$ and also a character $h^\vee = \pi_*(g^\vee) \in (G_{\bar{n},\mathcal{V}})^\vee$. By the same argument as above, these characters are $\bar{a}$- and $\bar{a}_*$-harmonic, respectively. The correspondence $h^\vee \longleftrightarrow \xi = (\xi_1, \ldots, \xi_s)$ yields the second bijection in (20). ☐

4.5. **Criteria of harmonicity.**

The preceding results lead to the following harmonicity criteria.

Theorem 4.21. *We let* $K = \mathrm{GF}(p)$. *Given a basis* $\mathcal{V}$ *of* L, *a sequence* $\bar{a} \in (\mathcal{F}^0(L, \bar{K}))^t$ *and a multi-index* $\bar{n} \in \mathbb{N}^s_{\mathrm{co}(p)}$, *we let* $G = L/L_{\bar{n},\mathcal{V}}$ *and* $\bar{a}_* = \pi_*\bar{a}$, *where* $\pi : L \twoheadrightarrow G$. *We fix a minimal* $q_0 = q(\bar{a}_*) = p^{r_0}$ $(r_0 > 0)$ *such that* $\widehat{a}_j(G^\vee) \subseteq \mathrm{GF}(q_0)\ \forall j = 1, \ldots, t$. *With this notation the following conditions are equivalent.*

- (i) *The sublattice* $L_{\bar{n},\mathcal{V}}$ *is* $\bar{a}$*-harmonic (equivalently, the group* G *is* $\bar{a}_*$*-harmonic).*
- (ii) $V(\widehat{\bar{a}_*}) := \bigcap_{j=1}^t V(\widehat{a_{j*}}) \neq \emptyset$.
- (iii) *The system (19) has a solution* $\xi = (\xi_1, \ldots, \xi_s) \in \Sigma_{\bar{a},\bar{n},\mathcal{V}} \subseteq (\bar{K}^\times)^s$.

In the case where $t = 1$ *and* $a_1 = a$, *(i)-(iii) are also equivalent to*

- (iv) $(\Delta_{a_*})^{q_0-1}(\delta_e) \neq \delta_e$ *or, equivalently,* $(\Delta_{a_*})^{q_0-1} \neq 1$.
- (v) *The sequence* $\left(\Delta^k_{a_*}(\delta_e)\right)_{k\geq 0} \subseteq \mathcal{F}(G, \bar{K})$ *is not periodic.*

Furthermore

$$\prod_{j=1}^t \left(1 - \Delta^{q_0-1}_{a_{j*}}\right) : \mathcal{F}(G, \bar{K}) \twoheadrightarrow \ker(\Delta_{\bar{a}_*})$$

[11] See (22) below.

is the orthogonal projection. [12] *Moreover the convolution ideal* $\ker(\Delta_{\bar{a}_*}) \subseteq \big(\mathcal{F}(G,\bar{K}),*\big)$ *is the principal ideal generated by the function* $\prod_{j=1}^{t}\Big(\delta_e - \Delta_{a_{j*}}^{q_0-1}(\delta_e)\Big)$.

Proof. The equivalences (i)$\Longleftrightarrow$(ii)$\Longleftrightarrow$(iii) follow immediately from 3.6 and 4.20. We have $\delta_{V(\widehat{a_{j*}})} = 1 - \widehat{a_{j*}}^{\,q_0-1}$. Indeed the function $\widehat{a_{j*}} \in \mathcal{F}(G^\vee, \bar{K})$ takes values in the field $\mathrm{GF}(q_0)$. The Fourier transform sends $1 - \Delta_{a_{j*}}^{q_0-1}$ into the operator of multiplication by $\delta_{V(a_{j*})}$, $j = 1, \dots, t$. This yields the equivalences (ii)$\Longleftrightarrow$(iv)$\Longleftrightarrow$(v) (for $t = 1$) and the last two assertions. ☐

Remarks 4.22. 1. We let $t = 1$ and $a = a_1$. Since $\widehat{a_*}^{\,q_0} = \widehat{a_*}$ we have $\Delta_{a_*}^{q_0+1}(\delta_e) = \Delta_{a_*}(\delta_e) = a_*$. Consequently the truncated sequence $\big(\Delta_{a_*}^k(\delta_e)\big)_{k\ge 1}$ starting with a_* is periodic with period, say, l which divides $q_0 - 1$. Whereas the sequence in (iv) starting with δ_e is periodic if and only if $L_{\bar{n},\mathcal{V}}$ is not a-harmonic. In the latter case Δ_{a_*} is invertible and so has finite order equal l in the group $\mathrm{Aut}(L/L_{\bar{n},\mathcal{V}}, \bar{K})$.

2. For $K = \mathrm{GF}(2)$ and $G = \mathbb{Z}/n\mathbb{Z}$ we have

$$K[G]^\times = (K[x]/(x^n - 1))^\times \cong \mathbb{Z}/\nu\mathbb{Z}\,.$$

Here according to [J, 1.1.7] or [MOW]

$$\nu = \nu(n) = 2^n \prod_{d|n}\left(1 - \frac{1}{2^{f(d)}}\right)^{g(d)}$$

with $f(n) = \mathrm{ord}_n(2) = \min\{j \,:\, 2^j \equiv 1 \mod n\}$ and $g(n) = \varphi(n)/f(n)$, where φ stands for the Euler totient function. We recall that G is a^+-harmonic if and only if $n \equiv 0 \mod 3$. In the opposite case the minimal period l as in (1) above coincides with the order of $\tilde{a}^+$ in the cyclic group $\mathbb{Z}/\nu\mathbb{Z}$, so $l \mid \nu$.

Next we provide examples of explicit calculations of harmonic characters.

4.6. **Examples.** We let below $K = \mathrm{GF}(2)$, $L = \mathbb{Z}^s$, $t = 1$, $a = a^+$, and we denote a_* again by a^+.

4.23. ($p = 2$, $s = 1$) 1. For $L = 2\mathbb{Z} \subseteq \mathbb{Z}$ the representation

$$\rho : \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to S_2 \hookrightarrow \mathbf{GL}(2, K), \qquad 1 \longmapsto \beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{with} \quad \beta^2 = 1$$

is equivalent to

$$\rho' : 1 \longmapsto \alpha = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{with} \quad \alpha^2 = 1\,.$$

The matrix elements of ρ provide the 2-periodic function $\delta_L = (\dots 0, 1, 0, 1, 0, 1 \dots)$ on $\mathbb{Z}$ and its shifts, whereas the other states are constant functions.

2. Similarly for $L = 3\mathbb{Z} \subseteq \mathbb{Z}$, ρ factorizes through a faithful representation $\mathbb{Z}/3Z \to S_3$. The matrix elements give rise to the shifts of the periodic function

$$\delta_L = (\dots 0, 0, 1, 0, 0, 1, 0, 0, 1 \dots)$$

[12] Let us note that for $K = \mathrm{GF}(2)$, $t = 1$ and $a = a^+$, $\Delta_{a_*}^{q_0-1} : \mathcal{F}(G, \bar{K}) \twoheadrightarrow (\ker(\Delta_{a_*}))^\perp$ is the orthogonal projection onto the space $(\ker(\Delta_{a_*}))^\perp$ of all winning patterns for the game 'Lights Out' on the toric grid $G = \mathbb{Z}_{\bar{n}}$ (see [Za, §2.8] or 0.4 in the Introduction).

on $\mathbb{Z}$. The a^+-harmonic function $(\dots 0,1,1,0,1,1,0,1,1\dots)$ and its shifts are states too.

3. A cyclic group $G=\mathbb{Z}/m\mathbb{Z}$ is a^+-harmonic if and only if $m\equiv 0 \mod 3$, see 2.7.2. Letting $m=3l$, $l\in\mathbb{N}$, we fix a primitive mth root of unity $\zeta\in\mu_{3l}$, a primitive cubic root of unity $\omega\in\mu_3$, and we let $g^\vee: ne_1\longmapsto\zeta^n$ be the corresponding character of $G=\mathbb{Z}/3l\mathbb{Z}$. For $\theta=(g^\vee)^t$ we have

$$\theta\in G^\vee_{a^+-\mathrm{harm}}\quad\Longleftrightarrow\quad\zeta^t+\zeta^{-t}=1\quad\Longleftrightarrow\quad\zeta^t=\omega^{\pm1}\quad\Longleftrightarrow\quad t\equiv\pm l\mod 3l.$$

So $\theta=(g^\vee)^l:\mathbb{Z}/3l\mathbb{Z}\to\mathbb{F}_4^\times$ is an a^+-harmonic character with trace

$$h=\mathrm{Tr}_{\mathbb{F}_4}(\theta):ne_1\longmapsto\omega^n+\omega^{2n}=\begin{cases}0 & \text{if } n\equiv 0 \mod 3,\\ 1 & \text{otherwise}.\end{cases}$$

Furthermore

$$d_{a^+,G}=2,\quad V(a^+)=G^\vee_{a^+-\mathrm{harm}}=\{\theta,\theta^{-1}\}\quad\text{and}\quad\ker(a^+)=\mathrm{span}(h,h^+)\,,$$

where $h^+(x)=h(1+x)$.

Similarly for $\theta(x)=\omega^x\in\mathrm{Char}_{a^+-\mathrm{harm}}(\mathbb{Z},\bar K^\times)$, $L(\theta)=3\mathbb{Z}$ is a maximal proper a^+-harmonic sublattice of $\mathbb{Z}$. Moreover h,h^+ lifted to $\mathbb{Z}$ give a basis of $\ker(\Delta_{a^+}\mid 3\mathbb{Z})$.

4.24. ($p=2$, $s=2$) As another example, we consider the group $G=\mathbb{Z}/5\mathbb{Z}\oplus\mathbb{Z}/5\mathbb{Z}$. We fix a primitive 5th root of unity $\zeta\in\mu_5$. We have $d_{a^+,G}=8$, see e.g., [Za, Appendix 1]. The relation $\zeta+\zeta^2+\zeta^3+\zeta^4=1$ yields the 8 solutions of (19) with $s=2$, $t=1$, $a_1=a^+$, $\sigma_{a^+}=x_1+x_1^{-1}+x_2+x_2^{-1}+1$ and $n_1=n_2=5$. These solutions can be obtained from $(x_1,x_2)=(\zeta,\zeta^3)$ by suitable transformations. The locus of a^+-harmonic characters

$$V(a^+)=G^\vee_{a^+-\mathrm{harm}}=\begin{pmatrix}0&0&0&0&0\\0&0&1&1&0\\0&1&0&0&1\\0&1&0&0&1\\0&0&1&1&0\end{pmatrix}$$

consists of two orbits of the cyclic group $\langle D_2\rangle\cong\mathbb{Z}/4\mathbb{Z}$ acting on $G^\vee$ via $D_2:g^\vee\longmapsto(g^\vee)^2$.

The solution (ζ,ζ^3) ((ζ^3,ζ), respectively) of (19) gives rise to the a^+-harmonic character $g^\vee:me_1+ne_2\longmapsto\zeta^{m+3n}$ (${}^tg^\vee:me_1+ne_2\longmapsto\zeta^{3m+n}$, respectively), where

$$g^\vee=\begin{pmatrix}1&\zeta^3&\zeta&\zeta^4&\zeta^2\\\zeta&\zeta^4&\zeta^2&1&\zeta^3\\\zeta^2&1&\zeta^3&\zeta&\zeta^4\\\zeta^3&\zeta&\zeta^4&\zeta^2&1\\\zeta^4&\zeta^2&1&\zeta^3&\zeta\end{pmatrix}\quad\text{resp.}\quad{}^tg^\vee=\begin{pmatrix}1&\zeta&\zeta^2&\zeta^3&\zeta^4\\\zeta^3&\zeta^4&1&\zeta&\zeta^2\\\zeta&\zeta^2&\zeta^3&\zeta^4&1\\\zeta^4&1&\zeta&\zeta^2&\zeta^3\\\zeta^2&\zeta^3&\zeta^4&1&\zeta\end{pmatrix}$$

has trace[13]

$$h=\mathrm{Tr}_{\mathbb{F}_{16}}(g^\vee)=\begin{pmatrix}0&1&1&1&1\\1&1&1&0&1\\1&0&1&1&1\\1&1&1&1&0\\1&1&0&1&1\end{pmatrix}\quad\text{resp.}\quad{}^th=\begin{pmatrix}0&1&1&1&1\\1&1&0&1&1\\1&1&1&1&0\\1&0&1&1&1\\1&1&1&0&1\end{pmatrix}.$$

[13] See 5.4 below.

The doubling of periods as in [Za, 2.35] produces the following a^+-harmonic function on $G = (\mathbb{Z}/10\mathbb{Z})^2$:

$$\delta(h) = \begin{pmatrix} 0 & \mathbf{1} & \mathbf{1} & 0 & \mathbf{1} & 0 & \mathbf{1} & 0 & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 \\ \mathbf{1} & 0 & \mathbf{1} & 0 & \mathbf{1} & \mathbf{1} & 0 & \mathbf{1} & \mathbf{1} & 0 \\ 0 & 0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 \\ \mathbf{1} & \mathbf{1} & 0 & \mathbf{1} & \mathbf{1} & 0 & \mathbf{1} & 0 & \mathbf{1} & 0 \\ 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 \\ \mathbf{1} & 0 & \mathbf{1} & 0 & \mathbf{1} & 0 & \mathbf{1} & \mathbf{1} & 0 & \mathbf{1} \\ 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & 0 \\ \mathbf{1} & 0 & \mathbf{1} & \mathbf{1} & 0 & \mathbf{1} & \mathbf{1} & 0 & \mathbf{1} & 0 \\ \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

composed of five crosses

$$\begin{pmatrix} & & \mathbf{1} & & \\ & & \mathbf{1} & & \\ \mathbf{1} & \mathbf{1} & 0 & \mathbf{1} & \mathbf{1} \\ & & \mathbf{1} & & \\ & & \mathbf{1} & & \end{pmatrix}.$$

Letting $\pi_1 : L = \mathbb{Z}^2 \twoheadrightarrow (\mathbb{Z}/5\mathbb{Z})^2$ and $\pi_2 : \mathbb{Z}^2 \twoheadrightarrow (\mathbb{Z}/10\mathbb{Z})^2$ it is easily seen that $L_1 = L(h \circ \pi_1) = \mathbb{Z}u + \mathbb{Z}v$ and $L_2 = L(\delta(h) \circ \pi_2) = 2\mathbb{Z} + 2\mathbb{Z}v$, where $u = (1, 2)$ and $v = (-2, 1)$. Hence $\mathrm{ind}_L(L_1) = 5$ and $\mathrm{ind}_L(L_2) = 20$. Actually $L_1 = L(\theta)$, where $\theta = g \circ \pi_1 \in \mathrm{Char}_{a^+-\mathrm{harm}}(L)$. In contrast the harmonic lattice $L_2 \subseteq \mathbb{Z}^2$ of even index is not a characteristic one (see Definition 4.9.a) and so by virtue of Proposition 4.13, $\delta(h) \circ \pi_2$ is not a linear combination of characters with values in $\bar{K}^\times$.

5. Convolution operators over finite fields

5.1. We fix a Galois field $K = \mathrm{GF}(q)$ with $q = p^r$ and a finite abelian group G of order coprime with p. The Fourier transform $F : \mathcal{F}(G, \bar{K}) \to \mathcal{F}(G^\vee, \bar{K})$ and its inverse F^{-1} do not respect the subrings $\mathcal{F}(G, K) \subseteq \mathcal{F}(G, \bar{K})$ and $\mathcal{F}(G^\vee, K) \subseteq \mathcal{F}(G^\vee, \bar{K})$. The latter are the rings of invariants for the induced action $\phi^* : f \longmapsto f^\phi = f^q$ on $\mathcal{F}(G, \bar{K})$ and $\mathcal{F}(G^\vee, \bar{K})$, respectively, of the Frobenius automorphism $\phi : \xi \longmapsto \xi^q$ of $\bar{K} = \overline{\mathrm{GF}(q)}$.

The next well known lemma describes the Fourier dual of the Frobenius automorphism. We let α denote the automorphism $\phi^* \circ (D_q^*)^{-1}$ of the algebra $\mathcal{F}(G^\vee, \bar{K})$, where $D_q \in \mathrm{Aut}(G^\vee)$, $D_q(g^\vee) = (g^\vee)^q$, and D_q^* stands for the adjoint action of D_q on $\mathcal{F}(G^\vee, \bar{K})$.

Lemma 5.2. *The automorphism α is the Fourier dual of ϕ^*. Hence the Fourier image*

$$F\left(\mathcal{F}(G^\vee, K)\right) = (\mathcal{F}(G^\vee, \bar{K}))^\alpha$$

is the subalgebra of α-invariants.

Proof. Indeed $\forall f \in \mathcal{F}(G, \bar{K})$,

$$\left(\hat{f}(g^\vee)\right)^q = \left(\sum_{h \in G} f(h) g^\vee(h)\right)^q = \sum_{h \in G} f^\phi(h)(g^\vee)^\phi(h) = \widehat{f^\phi}((g^\vee)^\phi).$$

Hence $(\widehat{f})^\phi = \widehat{f^\phi} \circ D_q^*$. Therefore

$$f \in \mathcal{F}(G,K) \iff f = f^\phi \iff \widehat{f} = \widehat{f^\phi} \quad \iff \quad \widehat{f} \circ D_q^* = (\widehat{f})^\phi \iff \widehat{f} = \alpha(\widehat{f}),$$

as stated. $\square$

We can easily deduce the following fact.

Corollary 5.3. *The locus $V(\widehat{\bar{a}}) \subset G^\vee$ of $\bar{a}$-harmonic characters is D_q-stable for any $\bar{a} \in (\mathcal{F}(G,K))^t$.*

5.4. For a function $f \in \mathcal{F}(G,\bar{K})$ we let $q(f) = q^{r(f)}$ be the minimal power of q such that the subfield $GF(q(f)) = K(f(G))$ is generated by the image $f(G)$ in $\bar{K}$. We define the *trace* of f as

$$\mathrm{Tr}(f) = \mathrm{Tr}_{GF(q(f)):GF(q)}(f) := f + f^q + \ldots + f^{q^{r(f)-1}}.$$

Proposition 5.5. *For any $\bar{a} \in (\mathcal{F}(G,K))^t$ the following hold.*

(a) *There is a bijection between the set of all $\bar{a}$-harmonic trace functions $\mathrm{Tr}(g^\vee)$ on G, where $g^\vee \in G^\vee_{\bar{a}-\mathrm{harm}} = V(\widehat{\bar{a}})$, and the orbit space of the cyclic group $\langle D_q\rangle$ acting on $V(\widehat{\bar{a}})$.*

(b) *Chosen representatives $g_1^\vee, \ldots, g_N^\vee$ of the $\langle D_q\rangle$-orbits on $V(\widehat{\bar{a}})$, there is a decomposition into orthogonal direct sum of $\mathrm{Conv}_K(G)$-submodules*

$$\ker(\Delta_{\bar{a}}) = \bigoplus_{i=1}^N (\mathrm{Tr}(g_i^\vee)) \subseteq \mathcal{F}(G,K).$$

(c) $\forall g^\vee \in G^\vee$,

$$g^\vee = \frac{1}{\mathrm{ord}(G)} \sum_{g\in G} g^\vee(g^{-1}) h_g, \qquad \textit{where} \qquad h_g(x) = \mathrm{Tr}(g^\vee(gx)) = \tau_g\left(\mathrm{Tr}(g^\vee(x))\right).$$

Proof. By virtue of 5.3 $\forall g^\vee \in G^\vee$,

$$h := \mathrm{Tr}(g^\vee) = g^\vee + (g^\vee)^q + \ldots + (g^\vee)^{q^{r(g^\vee)-1}} \in \ker(\Delta_{\bar{a}}) \cap \mathcal{F}(G,K).$$

Letting

$$O(g^\vee) = \{g^\vee, (g^\vee)^q, \ldots, (g^\vee)^{q^{r(g^\vee)-1}}\}$$

be the orbit of $g^\vee$ under the action of the cyclic group $\langle D_q\rangle$ on $V(\widehat{\bar{a}})$, using 3.3 we obtain $\mathrm{card}\,(O(g^\vee)) = r(g^\vee)$ and

$$\widehat{h} = \mathrm{ord}(G) \sum_{i=0}^{r(g^\vee)-1} \delta_{(g^\vee)^{-q^i}} = \mathrm{ord}(G)\delta_{O((g^\vee)^{-1})}.$$

This shows (a) and in turn implies (b). Whereas (c) follows from the orthogonality relations for characters. Indeed using 3.2-3.3, $\forall x \in G$ one has

$$\sum_{g\in G} g^\vee(g^{-1})h_g(x) = \sum_{g\in G} g^\vee(g^{-1}) \sum_{i=0}^{r(g^\vee)-1} (g^\vee)^{q^i}(gx) = \sum_{i=0}^{r(g^\vee)-1} \left(\sum_{g\in G} (g^\vee)^{-q^i}(g^{-1}) g^\vee(g^{-1})\right) (g^\vee)^{q^i}(x)$$

$$= \sum_{i=0}^{r(g^\vee)-1} \widehat{(g^\vee)^{-q^i}}(g^\vee)(g^\vee)^{q^i}(x) = \mathrm{ord}(G) \sum_{i=0}^{r(g^\vee)-1} \delta_{(g^\vee)^{q^i}}(g^\vee)(g^\vee)^{q^i}(x) = \mathrm{ord}(G) g^\vee(x).$$

$\square$

The following corollary is straightforward from 4.20 and 5.5(c).

Corollary 5.6. *For a finite abelian group G of order coprime to p and for a sequence $\bar a \in (\mathcal{F}(G,K))^t$, where $K = \mathrm{GF}(p^r)$, the kernel* $\ker(\Delta_{\bar a} \mid \mathcal{F}(G,K))$ *is spanned over K by the shifts of traces of $\bar a$-harmonic characters $g^\vee \in G^\vee_{\bar a-\mathrm{harm}}$.*

6. Multi-orders table and Partnership graph

In this section we let again $K = \mathrm{GF}(q)$, where $q = p^r$ with $p, r > 0$, so that $\bar K^\times$ is a torsion group.

6.1. Given a lattice L, a base $\mathcal{V}$ of L and a system $\Delta_{\bar a}$ of convolution operators on L, where $\bar a = (a_1, \ldots, a_t) \in \left(\mathcal{F}(L, \bar K)\right)^t$, we compose a table $\mathcal{D}_{\bar a,\mathcal{V}} = \{d_{\bar a,\bar n,\mathcal{V}}\}_{\bar n \in \mathbb{N}^s}$, where

$$d_{\bar a,\bar n,\mathcal{V}} = \dim\ker(\Delta_{\bar a} \mid L_{\bar n,\mathcal{V}}) = \dim\left(\bigcap_{j=1}^{t} \ker\left(\Delta_{a_j} \mid L_{\bar n,\mathcal{V}}\right)\right),$$

cf. (4). Thus the entries $d_{\bar a,\bar n,\mathcal{V}}$ are nonzero for all $\bar n \in \mathbb{N}^s$ such that the sublattice $L_{\bar n,\mathcal{V}} \subseteq L$ is $\bar a$-harmonic or, equivalently, such that $\Sigma_{\bar a,\bar n,\mathcal{V}} \neq \emptyset$ (see 4.21). By virtue of (8) for $\bar n = (n_1, \ldots, n_s)$ with $n_i = p^{\alpha_i} n_i'$, where $n_i' \not\equiv 0 \mod p$, letting $\bar n' = (n_1', \ldots, n_s')$ we have

$$d_{\bar a,\bar n,\mathcal{V}} = p^{|\alpha|} \cdot d_{\bar a,\bar n',\mathcal{V}}, \quad \text{where} \quad |\alpha| = \sum_{i=1}^{s} \alpha_i\,.$$

Furthermore by virtue of 4.20, $\forall \bar m = (m_1, \ldots, m_s) \in \mathbb{N}^s$, $\forall n = (n_1, \ldots, n_s) \in \mathbb{N}^s$

$$d_{\bar a,\gcd(\bar m,\bar n),\mathcal{V}} \le \min\{d_{\bar a,\bar m,\mathcal{V}},\, d_{\bar a,\bar n,\mathcal{V}}\}\,, \tag{21}$$

where $\gcd(\bar m, \bar n) := (\gcd(m_1, n_1), \ldots, \gcd(m_s, n_s))$. Indeed by 4.19,

$$\Sigma_{\bar a,\gcd(\bar m,\bar n),\mathcal{V}} = \Sigma_{\bar a,\bar m,\mathcal{V}} \cap \Sigma_{\bar a,\bar n,\mathcal{V}}\,.$$

Remark 6.2. We note that for $s = 1$, the classical Chebyshev-Dickson polynomials T_n of first kind satisfy the identity $\gcd(T_m, T_n) = T_{\gcd(m,n)}$. Whereas for $s \ge 2$ the inequality (21) is strict in general. For instance for $K = \mathrm{GF}(2)$, $s = 2$, $t = 1$ and $a_1 = a^+$,

$$d_{a^+,(3,3)} = 4 < \min\{d_{a^+,(9,21)},\, d_{a^+,(21,9)}\} = 16\,.$$

Thus the above identity does not hold any more for the multivariate Chebyshev-Dickson polynomials.

6.3. We consider also the *table of multi-orders* $\mathcal{S}_{\bar a,\mathcal{V}} = \{s_{\bar a,\bar n,\mathcal{V}}\}_{\bar n \in \mathbb{N}^s_{\mathrm{co}(p)}}$, where $s_{\bar a,\bar n,\mathcal{V}}$ stands for the number of points $\xi = (\xi_1, \ldots, \xi_s)$ on the symbolic hypersurface $\Sigma_{\bar a,\mathcal{V}}$ which have the given *multi-order*

$$\bar n = \text{multi-ord}(\xi) := (\mathrm{ord}(\xi_1), \ldots, \mathrm{ord}(\xi_s)) \in \mathbb{N}^s_{\mathrm{co}(p)}\,. \tag{22}$$

These two tables $\mathcal{D}_{\bar a,\mathcal{V}}$ and $\mathcal{S}_{\bar a,\mathcal{V}}$ are related via

$$d_{\bar a,\bar n,\mathcal{V}} = \sum_{\bar d \mid \bar n} s_{\bar a,\bar d,\mathcal{V}}\,,$$

where $\bar n \in \mathbb{N}_{\mathrm{co}(p)}$ and $\bar d$ runs over all s-tuples $\bar d = (d_1, \ldots, d_s) \in \mathbb{N}^s_{\mathrm{co}(p)}$ with $d_i \mid n_i$ $\forall i = 1, \ldots, s$.

Letting $\bar{n} = (n_1, \ldots, n_s) = (\bar{n}', n_s)$, for any fixed $\bar{\xi}' = (\xi_1, \ldots, \xi_{s-1}) \in \mu_{\bar{n}'} = \bigoplus_{i=1}^{s-1} \mu_{n_i}$ the system $\sigma_{a_j}(\bar{\xi}', x_s) = 0$, $j = 1, \ldots, t$, has a finite set of solutions $x_s = \eta \in \bar{K}^\times$. Hence every line $(s_{a,(\bar{n}',n_s),\mathcal{V}})$ of the table $\mathcal{S}_{\bar{a},\mathcal{V}}$ with $\bar{n}'$ fixed represents a function on $\mathbb{Z}$ with finite support.

We let

$$l(\bar{n}') = \operatorname{lcm}\left(\operatorname{ord}(\eta) \,:\, \exists \bar{\xi}' \in \mu_{\bar{n}'},\, \sigma_{a_j}(\bar{\xi}', \eta) = 0 \quad \forall j = 1, \ldots, t\right).$$

Then clearly the line $(d_{a,(\bar{n}',n_s),\mathcal{V}})_{n_s}$ of the table $\mathcal{D}_{\bar{a},\mathcal{V}}$ is periodic with minimal period $l(\bar{n}')$ so that

$$d_{a,(\bar{n}',n_s+l(\bar{n}')),\mathcal{V}} = d_{a,(\bar{n}',n_s),\mathcal{V}} \qquad \forall \bar{n}' \in \mathbb{N}^{s-1}_{\mathrm{co}(p)}.$$

Whereas the set $(l(\bar{n}'))_{\bar{n}' \in \mathbb{N}^{s-1}_{\mathrm{co}(p)}}$ of all such periods is unbounded in general.

For instance for $K = \mathrm{GF}(2)$, $s = 2$, $t = 1$ and $a_1 = a^+$ we have

$$\max_{m \in \mathbb{N}_{\mathrm{odd}}} \{d_{a^+,(n,m)}\} = d_{a^+,(n,l(n))} = \begin{cases} 2n, & n \not\equiv 0 \mod 3 \\ 2n-2, & \text{otherwise}. \end{cases}$$

6.4. Following [Za] we let

$$\mathcal{E}_{\bar{a},\mathcal{V}} = \{\bar{n} \in \mathbb{N}^s_{\mathrm{co}(p)} \mid d_{\bar{a},\bar{n},\mathcal{V}} \neq 0\} \qquad \text{and} \qquad \mathcal{E}^0_{\bar{a},\mathcal{V}} = \{\bar{n} \in \mathbb{N}^s_{\mathrm{co}(p)} \mid s_{\bar{a},\bar{n},\mathcal{V}} \neq 0\}.$$

By 4.21, $\mathcal{E}^0_{\bar{a},\mathcal{V}} \subseteq \mathcal{E}_{\bar{a},\mathcal{V}}$. Letting $\bar{k}\bar{n} = (k_1 n_1, \ldots, k_s n_s)$ we obtain a natural covering $\pi : L/L_{\bar{k}\bar{n},\mathcal{V}} \to L/L_{\bar{n},\mathcal{V}}$. Any $\bar{a}_*$-harmonic function on the second group lifts to such a function on the first one. Therefore $\mathcal{E}_{\bar{a},\mathcal{V}}$ is generated by $\mathcal{E}^0_{\bar{a},\mathcal{V}}$ as an $\mathbb{N}^s_{\mathrm{co}(p)}$-module. So in order to determine $\mathcal{E}_{\bar{a},\mathcal{V}}$ it is enough to determine $\mathcal{E}^0_{\bar{a},\mathcal{V}}$.

6.5. If the symbols σ_{a_j}, $j = 1, \ldots, t$, are symmetric (i.e. stable under the natural action of the symmetric group S_s on the Laurent polynomial ring $\bar{K}[x_1, x_1^{-1}, \ldots, x_s, x_s^{-1}]$) then it is convinient to replace the multi-orders table $\mathcal{S}_{\bar{a},\mathcal{V}}$ by the *labeled partnership hypergraph* $\mathcal{P}_{\bar{a}}$. The latter one has the set of naturals $\mathbb{N}_{\mathrm{co}(p)}$ as the set of vertices, and consists of all the $(s-1)$-simplices $\bar{n} \in \mathcal{E}^0_{\bar{a},\mathcal{V}}$ labeled with $s_{\bar{a},\bar{n},\mathcal{V}} \neq 0$. For $s = 2$, $\mathcal{P}_{\bar{a}}$ is just the infinite labeled graph with the set of vertices $\mathbb{N}_{\mathrm{co}(p)}$ and with the edges $[m,n] \in \mathcal{E}^0_{a^+,\mathcal{V}}$ labeled with $s_{a^+,(m,n),\mathcal{V}} \neq 0$.

6.6. In the case where $s = 2$, $K = \mathrm{GF}(2)$, $t = 1$ and $a_1 = a^+$, all connected components of the partnership graph $\mathcal{P}_a$ are finite, see [Za, Theorem 3.11]. Actually every such component is contained in a level set $f_0^{-1}(r)$ of the *suborder function* $f_0 : \mathbb{N}_{\mathrm{odd}} \to \mathbb{N}$ given via

$$f_0(n) = \mathrm{sord}_n(2) := \min\{j \,:\, 2^j \equiv \pm 1 \mod n\}$$

(see e.g. [MOW]). Indeed for a primitive nth root of unity $\zeta \in \mu_n$ we have $f_0(n) = \deg(\zeta+\zeta^{-1})$, see [Za, 5.10(a)]. Let $(\zeta, \eta) \in \sigma_{a^+}$ be a point with bi-order $(\operatorname{ord}(\zeta), \operatorname{ord}(\eta)) = (m,n)$. Since (ζ, η) satisfies the symbolic equation

$$\zeta + \zeta^{-1} + \eta + \eta^{-1} = 1$$

we have $K(\zeta + \zeta^{-1}) = K(\eta + \eta^{-1})$ and so $f_0(m) = f_0(n)$.

It is plausible [Za, 4.1] that the connected components of the partnership graph $\mathcal{P}_{a^+}$ coincide with the corresponding level sets of the suborder function f_0 except for $r = 5$, where $f_0^{-1}(5)$ consists of two components. This conjecture is based on computations of the first 13 of these components done by Zagier with PARI, see [Za, Appendix 1].

6.7. Some natural questions arise. For instance,

- Given a lattice L of rank $s = 2$, it would be interesting to determine for which Galois fields $K = \mathrm{GF}(q)$ and for which functions $a \in \mathcal{F}^0(L, K)$, all the components of the graph $\mathcal{P}_a$ are finite.
- Is it possible to reconstruct the function a from the graph $\mathcal{P}_a$?
- Given $a \in \mathcal{F}^0(L, K)$, formulate a criterion as to when the irreducible factors of the generalized Chebyshev-Dickson polynomials $\mathrm{CharPoly}_{a,\bar{n},\mathcal{V}}$ exhaust all the irreducible polynomials over a finite field K. This is indeed the case for the classical Chebyshev-Dickson polynomials T_n, see e.g. [HMP, 2.8-2.10] and the references therein.
- [14] Does there exist any reasonable (multivariate) generating function for at least one of the tables $\mathcal{D}_{\bar{a},\mathcal{V}}$ or $\mathcal{S}_{\bar{a},\mathcal{V}}$? The direct analog of the logarithm of the Weil zeta function is unlikely to play this role.

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[14]This question is due to Roland Bacher in the case of σ^+-automata.