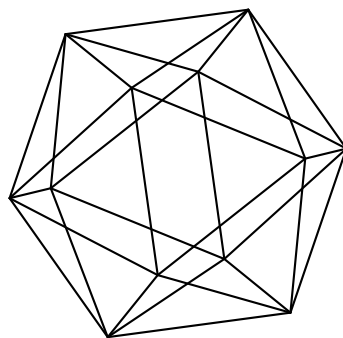


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The global topology of Pontrjagin duality

by

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Pontrjagin duality is implemented in the framework of fibre bundles. By means of Pontrjagin duality triples a Fourier transform is defined by a pull-push construction operating on sections of line bundles. This yields an isomorphism of Hilbert C^* -modules which generalises the classical isomorphism between the group C^* -algebra of a group and the continuous functions vanishing at infinity on the dual group.

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1 Introduction

The main tool of classical harmonic analysis on abelian groups is the Fourier transform. It maps a function $\alpha : G \rightarrow \mathbb{C}$ on a locally compact abelian group G to a function

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$\hat{\alpha} : \widehat{G} \rightarrow \mathbb{C}$ on the dual group $\widehat{G} := \text{Hom}(G, \text{U}(1))$ of G . Explicitly, it is given by the formula

$$\hat{\alpha}(\chi) := \int_G \alpha(g) \langle g, \chi \rangle dg, \quad \chi \in \widehat{G},$$

where $\langle \cdot, \cdot \rangle : G \times \widehat{G} \rightarrow \text{U}(1)$ is the canonical pairing and dg is the Haar measure on G . As it turns out, the Fourier transform of an integrable function is a continuous function vanishing at infinity, and in fact it extends to an isomorphism of C^* -algebras

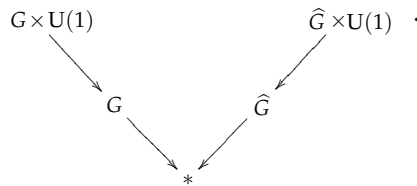
$$\widehat{\cdot} : C^*(G) \cong C_0(\widehat{G}) \tag{1}$$

between the group C^* -algebra of G (which contains the integrable functions as a dense subspace) and the continuous functions on \widehat{G} which vanish at infinity.

We reinterpret these data in a bundle theoretic set-up. By addition the space G has a free (right) action of the group G which means that the quotient map of this action $G \rightarrow *$ is a trivial G -principal fibre bundle over the one-point space $*$. (Recall that a G -principal fibre bundle $E \rightarrow B$ over a topological space B is a space E with a free (right) G -action and a homeomorphism $E/G \cong B$ such that the quotient map $E \rightarrow B$ has local sections.) A complex valued function $\alpha : G \rightarrow \mathbb{C}$ corresponds canonically to a section



of the line bundle $L := (G \times \text{U}(1)) \times_{\text{U}(1)} \mathbb{C}$ associated to the trivial $\text{U}(1)$ -principal fibre bundle $G \times \text{U}(1) \rightarrow G$. (Recall that the associated line bundle of a $\text{U}(1)$ -principal bundle $F \rightarrow E$ is just the quotient of $F \times \mathbb{C}$ by the diagonal $\text{U}(1)$ -action: $((x, z), t) \mapsto (x \cdot t, t^{-1}z)$.) Thus, on the domain side of the Fourier transform we have an underlying topological object $G \times \text{U}(1) \rightarrow G \rightarrow *$ which we call the trivial pair over the one-point space. On the range side of the Fourier transform we have by analogy a trivial dual pair which is the sequence $\widehat{G} \times \text{U}(1) \rightarrow \widehat{G} \rightarrow *$. Together these two objects form a diagram of principal fibre bundles over the one-point space



We extend this diagram step by step to a diagram of pullbacks

$$\begin{array}{ccccc}
 & G \times \widehat{G} \times U(1) & & G \times \widehat{G} \times U(1) & \\
 & \swarrow & & \searrow & \\
 G \times U(1) & & G \times \widehat{G} & & \widehat{G} \times U(1) \\
 & \searrow & & \swarrow & \\
 & G & & \widehat{G} & \\
 & \searrow & & \swarrow & \\
 & * & & &
 \end{array}$$

The canonical pairing $\langle \cdot, \cdot \rangle : G \times \widehat{G} \rightarrow U(1)$ gives rise to a $U(1)$ -principal fibre bundle morphism

$$\begin{array}{ccc}
 G \times \widehat{G} \times U(1) & \xrightarrow{\pi} & G \times \widehat{G} \times U(1) \\
 & \searrow & \swarrow \\
 & G \times \widehat{G} &
 \end{array}$$

just given by $\pi(g, \chi, t) := (g, \chi, \langle g, \chi \rangle t)$. Inserting this isomorphism into the previous diagram we obtain a diagram

$$\begin{array}{ccccc}
 & G \times \widehat{G} \times U(1) & \xrightarrow{\pi} & G \times \widehat{G} \times U(1) & \\
 & \swarrow & & \searrow & \\
 G \times U(1) & & G \times \widehat{G} & & \widehat{G} \times U(1) \\
 & \searrow & & \swarrow & \\
 & G & & \widehat{G} & \\
 & \searrow & & \swarrow & \\
 & * & & &
 \end{array} \tag{2}$$

which we call the trivial Pontrjagin duality triple over the one-point space. By this diagram we can define a Fourier transform in disguise which maps the sections of the associated line bundle $L \rightarrow G$ to the sections of the “dual” line bundle $\widehat{L} := (\widehat{G} \times U(1)) \times_{U(1)} \mathbb{C} \rightarrow \widehat{G}$. Explicitly, the Fourier transform of a section $\alpha : G \rightarrow L$ is the section $\widehat{\alpha} : \widehat{G} \rightarrow \widehat{L}$ given by

$$\widehat{\alpha}(\chi) := \int_G \text{pr}_{\widehat{L}} \left(\pi^{\mathbb{C}}(\alpha(h), \chi) \right) dh,$$

where $\pi^{\mathbb{C}} : L \times \widehat{G} \rightarrow G \times \widehat{L}$ is the isomorphism of line bundles which is induced by π , and $\text{pr}_{\widehat{L}} : G \times \widehat{L} \rightarrow \widehat{L}$ is the projection. By renaming the objects in (1) we have an isomorphism

$$\widehat{\cdot} : C^*(G, G \times U(1)) \cong \Gamma_0(\widehat{G}, \widehat{L}),$$

where $C^*(G, G \times U(1))$ is the C^* -algebra of the trivial pair $G \times U(1) \rightarrow G \rightarrow *$, and $\Gamma_0(\widehat{G}, \widehat{L})$ are the sections vanishing at infinity.

Now, the stage is set for topology. We want to consider family versions of diagram (2) glued by the topology of a space B which leads us to the notion of a general Pontrjagin duality triple. A Pontrjagin duality triple is a commutative diagram of principal fibre bundles

$$\begin{array}{ccccc}
 & F \times_B \widehat{E} & \xrightarrow{\kappa} & E \times_B \widehat{F} & \\
 & \swarrow & & \searrow & \\
 F \circlearrowleft U(1) & & & & \widehat{F} \circlearrowleft U(1) \\
 & \searrow & & \swarrow & \\
 & E \times_B \widehat{E} & & & \\
 & \swarrow & & \searrow & \\
 E \circlearrowleft G & & & & \widehat{E} \circlearrowleft \widehat{G} \\
 & \searrow & & \swarrow & \\
 & B & & &
 \end{array} \tag{3}$$

such that the restriction of (3) to each point $b \in B$ is isomorphic to (2). Let $F^{\mathbb{C}} := F \times_{U(1)} \mathbb{C}$ and $\widehat{F}^{\mathbb{C}} := \widehat{F} \times_{U(1)} \mathbb{C}$ denote the associated line bundles. For a horizontally integrable section $\gamma : E \rightarrow F^{\mathbb{C}}$ we can define the Fourier transform based on diagram (3) to be a section $\hat{\gamma} : \widehat{E} \rightarrow \widehat{F}^{\mathbb{C}}$. Namely, for $\hat{e} \in \widehat{E}$ over $b \in B$ take any $e \in E$ also over b and define

$$\hat{\gamma}(\hat{e}) := \int_G \text{pr}_{\widehat{F}^{\mathbb{C}}} \left(\kappa^{\mathbb{C}}(\gamma(e \cdot h), \hat{e}) \right) dh, \tag{4}$$

where $\kappa^{\mathbb{C}} : F^{\mathbb{C}} \times_B \widehat{E} \rightarrow E \times_B \widehat{F}^{\mathbb{C}}$ is the induced isomorphism of the top isomorphism κ in (3), and $\text{pr}_{\widehat{F}^{\mathbb{C}}} : E \times_B \widehat{F}^{\mathbb{C}} \rightarrow \widehat{F}^{\mathbb{C}}$ is the projection.

We wish to understand the Fourier transform (4) in the correct C^* -algebraic context. The answer will be that it defines an isomorphism of Hilbert C^* -modules (Theorem 8.1). In fact, these Hilbert C^* -modules are Hilbert C^* -modules of $U(1)$ -equivariant (self) Morita equivalences, namely F and \widehat{F} , between $U(1)$ -central extensions of groupoids. The task is to construct these groupoids.

Let us give diagrams (2) and (3) another purely topological interpretation. The dual of the group G , understood as a G -principal fibre bundle $G \rightarrow *$, is the dual group \widehat{G} , understood as a \widehat{G} -principal fibre bundle $\widehat{G} \rightarrow *$. Pontrjagin duality triples can give answer to the question of the dual of a general G -principal fibre bundle $E \rightarrow B$. Note that the a G -bundle $E \rightarrow B$ is the same amount of data as the bundle together with a trivial $U(1)$ -bundle $E \times U(1) \rightarrow E$ on its total space.

We introduce some terminology. A pair $F \rightarrow E \rightarrow B$ is a sequence of principal fibre bundles which is locally isomorphic to the trivial pair $B \times G \times U(1) \rightarrow B \times G \rightarrow B$. A dual pair $\widehat{F} \rightarrow \widehat{E} \rightarrow B$ is a sequence of principal fibre bundles which is locally isomorphic to the dual trivial pair $B \times \widehat{G} \times U(1) \rightarrow B \times \widehat{G} \rightarrow B$. We call $\widehat{F} \rightarrow \widehat{E} \rightarrow B$ a dual of $F \rightarrow E \rightarrow B$ if one can extend these two to a Pontrjagin duality triple (3).

So, if $E \rightarrow B$ is a G -principal fibre bundle, then $E \times U(1) \rightarrow E \rightarrow B$ is a pair, and we are concerned with the questions of existence and uniqueness of duals in the above sense.

These questions can be answered by analysing the topology of the classifying spaces of the automorphism groups of the corresponding local models. The automorphism group of the trivial pair over the point $G \times U(1) \rightarrow G \rightarrow *$ is the semi-direct product $A_{\text{Par}} := G \ltimes C(G, U(1))$, and the category of pairs is equivalent to the category of A_{Par} -principal fibre bundles (Proposition 2.1). The automorphism group of diagram (2) is the semi-direct product $A_{\text{Pon}} := G \ltimes (U(1) \times \widehat{G})$, and the category of Pontrjagin duality triples is equivalent to the category of A_{Pon} -principal fibre bundles (Proposition 7.2). A_{Pon} is the subgroup of A_{Par} consisting of those $(g, f) \in A_{\text{Par}}$ which admit a $t \in U(1)$ and a $\chi \in \widehat{G}$ such that $f(h) = t\langle h, \chi \rangle$. The inclusion $A_{\text{Pon}} \hookrightarrow A_{\text{Par}}$ induces a map between the classifying spaces $BA_{\text{Pon}} \rightarrow BA_{\text{Par}}$.

Expressed in homotopy theoretic terms, the question of the existence of a dual of a pair $F \rightarrow E \rightarrow B$ is the question of the existence of a (homotopy) lift of the classifying map $B \rightarrow BA_{\text{Par}}$ of $F \rightarrow E \rightarrow B$:

$$\begin{array}{ccc}
 & & BA_{\text{Pon}} \\
 & \nearrow \exists? & \downarrow \\
 B & \longrightarrow & BA_{\text{Par}}
 \end{array}$$

If such a lift exists, then the question of uniqueness is the question whether the (homotopy class of this) lift is unique. As the topology of the classifying spaces varies with the group G , the answers to these questions depend on the group G and are quite different for different G (s. section 7).

Remark 1.1 *The notion of Pontrjagin duality triples is similar to what has been introduced in [BRS] under the name T-duality triples (see also [Sch1, Sch2, BSST]). The analysis of Pontrjagin duality triples in this work consists of similar steps as the analysis of the C^* -algebraic content of T-duality triples [Sch1, Sch2], but the necessary tools for their investigation stay on a much more explicit level.*

Conventions. By a **space** B we will always mean a Hausdorff, paracompact topological space. Frequently, we use the word **bundle** as an abbreviation for principal fibre bundle. In the whole of this work we denote by G a Hausdorff, second countable, locally compact abelian group. Its dual group is $\widehat{G} := \text{Hom}(G, U(1))$. With the compact-open topology \widehat{G} is again a Hausdorff, second countable, locally compact abelian group. We denote by $\langle \cdot, \cdot \rangle : G \times \widehat{G} \rightarrow U(1)$ the canonical pairing. The integral of a function f on G with the Haar measure is simply denoted by $\int_G f(g) dg$.

We refer to [Ru] for the classical theory of Fourier analysis and to [La] for the language of Hilbert C^* -modules. An introduction to the language of stacks is found in [Hei] or [Moe].

Acknowledgement. The author is grateful to Ulrich Bunke and Ilya Shapiro for helpful discussions and suggestions.

2 Pairs

We start with the definition of a pair which is a collection of

$$\begin{array}{c} G \times U(1) \\ \downarrow \\ G \\ \downarrow \\ * \end{array}$$

glued together by the topology of a space B .

Definition 2.1 A **pair** over a space B is a sequence $F \rightarrow E \rightarrow B$ of a G -bundle $E \rightarrow B$ and a $U(1)$ -bundle $F \rightarrow E$ subject to the following local triviality axiom: There is an open cover $\{U_i\}$ of B together with bundle isomorphisms

$$\begin{array}{ccc} F|_{E|_{U_i}} & \longrightarrow & U_i \times G \times U(1) , \\ \downarrow & & \downarrow \\ E|_{U_i} & \longrightarrow & U_i \times G \\ \downarrow & & \downarrow \\ U_i & \xrightarrow{=} & U_i \end{array} \quad (5)$$

i.e. we require the $U(1)$ -bundle F to be trivialisable over the fibres of E .

The notion of a **dual pair** $\hat{F} \rightarrow \hat{E} \rightarrow B$ is defined by the same means, but with G replaced by its dual group \hat{G} .

The open cover $\{U_i\}$ of B together with the diagrams (5) is called an **atlas** for the pair, and each single diagram (5) is called a **chart**.

We define the category of pairs over a space B in the obvious way, i.e. a morphism of pairs is a diagramm

$$\begin{array}{ccc} F & \longrightarrow & F' \\ \downarrow & & \downarrow \\ E & \longrightarrow & E' \\ \downarrow & & \downarrow \\ B & \xrightarrow{=} & B \end{array}$$

with the horizontal arrows being bundle morphisms. This category clearly is a groupoid.

Let $F \rightarrow E \rightarrow B$ be a pair and let $\{U_i\}$ be an atlas. By the usual arguments, we obtain transition functions on twofold intersections $U_{ij} := U_i \cap U_j$

$$g_{ji} : U_{ji} \rightarrow G, \quad \zeta_{ji} : U_{ij} \times G \rightarrow U(1) \quad (6)$$

which satisfy

$$g_{kj}(u) g_{ji}(u) = g_{ki}(u), \quad \zeta_{kj}(u, g_{ji}(u)) \zeta_{ji}(u, g) = \zeta_{ji}(u, g) \quad (7)$$

on threefold intersections $U_{ijk} \ni u$. Conversely, given families $\{g_{ji}\}, \{\zeta_{ji}\}$ which satisfy (7) we (re-) obtain a pair by the usual quotients. The two families $\{g_{ji}\}, \{\zeta_{ji}\}$ of transition functions can also be considered as one single family

$$g_{ji} \times \zeta_{ji} : U_{ji} \rightarrow G \times C(G, U(1)),$$

where $G \times C(G, U(1))$ is the semi-direct product. Note that the (topological) group

$$A_{\text{Par}} := G \times C(G, U(1))$$

is the automorphism group of the trivial pair over the point $G \times U(1) \rightarrow G \rightarrow *$, where $(g, f) \in A_{\text{Par}}$ acts from the left on $(h, z) \in G \times U(1)$ by $(g, f) \cdot (h, z) := (g + h, f(h)z)$.

Let $P \rightarrow B$ be a A_{Par} -principal fibre bundle. We obtain a pair over B by associating the trivial pair over the point¹

$$F_P := P \times_{A_{\text{Par}}} (G \times U(1)), \quad E_P := P \times_{A_{\text{Par}}} G.$$

A morphism of A_{Par} -bundles over B

$$\begin{array}{ccc} P & \longrightarrow & P' \\ \downarrow & & \downarrow \\ B & \xrightarrow{=} & B \end{array}$$

induces a morphism of pairs over B

$$\begin{array}{ccc} F_P & \longrightarrow & F_{P'} \\ \downarrow & & \downarrow \\ E_P & \longrightarrow & E_{P'} \\ \downarrow & & \downarrow \\ B & \xrightarrow{=} & B \end{array}$$

i.e. we have constructed a functor from the category of A_{Par} -principal fibre bundles over B to the category of pairs over B .

¹ For a group H acting on a space X from the right and on a space Y from the left we denote by $X \times_H Y$ the quotient of $X \times Y$ by the induced right action $(x, y) \cdot h := (x \cdot h, h^{-1} \cdot y)$ as usual.

Proposition 2.1 *The functor*

$$(P \rightarrow B) \mapsto (F_P \rightarrow E_P \rightarrow B)$$

is an equivalence of categories.

Proof : We construct a functor in opposite direction. Let $F \xrightarrow{q} E \rightarrow B$ be a pair. For $b \in B$ we define P_b to be the set of tuples (e, s) of arbitrary elements $e \in E|_b$ and subsets $s \subset F|_{E|_b}$ with the property that $q|_s : s \rightarrow E|_b$ is a homeomorphism. Denote by $s(x)$ the unique element in s such that $q(s(x)) = x$, for $x \in E|_b$, so $s = \{s(x) | x \in E|_b\}$. The set P_b is a A_{Par} -torsor. In fact, for $(g, f) \in A_{\text{Par}}$ we have a right action on $(e, s) \in P_b$ by

$$(e, s) \diamond (g, f) := (e \cdot g, s \diamond_e (g, f)), \quad (8)$$

where $e \cdot g$ is the principal action, and $s \diamond_e (g, f)$ is the subset $\{s(e \cdot (h + g)) \cdot f(h) | h \in G\}$. As (the graph of) a function $f : G \rightarrow U(1)$ is a subset of $G \times U(1)$, an atlas U_i of the pair induces A_{Par} -equivariant bijections $\varphi_i : \coprod_{b \in U_i} P_b \cong U_i \times G \times C(G, U(1))$. We equip $\coprod_{b \in B} P_b$ with the coarsest topology such that all φ_i are homeomorphisms. This topology is independent of the chosen atlas. The canonical map $\coprod_{b \in B} P_b \rightarrow B$ is now an A_{Par} -principal fibre bundle. It is rather obvious that a morphism of pairs induces a morphism of A_{Par} principal bundles, and it is straight forward to verify that the compositions of the two functors in game are naturally isomorphic to the identity functors. ■

We give two simple but important examples of pairs which correspond to A_{Par} -bundles which have reductions of the structure group A_{Par} to two distinguished subgroups.

First, bundles which have a reduction to the subgroup $G \subset A_{\text{Par}}$ lead to pairs isomorphic to

$$\begin{array}{c} E \times U(1) \\ \downarrow \\ E \\ \downarrow \\ B \end{array}$$

for a G -bundle $E \rightarrow B$.

Second, we have $\widehat{G} \subset C(G, U(1)) \subset A_{\text{Par}}$. Bundles which admit reductions to the subgroup $\widehat{G} \subset A_{\text{Par}}$ lead to pairs of the form

$$\begin{array}{c} F_{\widehat{E}} \\ \downarrow \\ B \times G \\ \downarrow \\ B \end{array} \quad (9)$$

where $F_{\widehat{E}} := \widehat{E} \times_{\widehat{G}} (G \times U(1))$, for a \widehat{G} -bundle $\widehat{E} \rightarrow B$.

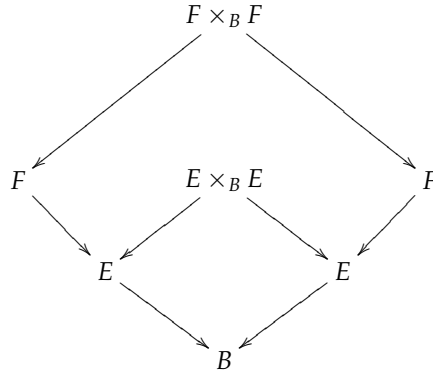
The latter example of a pair and its algebraic properties are the content of the next section.

3 Ring pairs

The trivial pair over the point admits a multiplication in the sense that

$$\begin{array}{ccc}
 (G \times U(1)) \times (G \times U(1)) & \xrightarrow{+ \times \cdot} & G \times U(1) \\
 \downarrow & & \downarrow \\
 G \times G & \xrightarrow{+} & G \\
 \downarrow & & \downarrow \\
 * & \xrightarrow{=} & *
 \end{array} \tag{10}$$

commutes, where $+ \times \cdot$ is the map $((g, t), (h, s)) \mapsto (g + h, ts)$. We take this as the local model for the notion of ring pairs which we introduce next. Consider an arbitrary pair $F \rightarrow E \rightarrow B$ and the following diagram of pullbacks.



The canonical map $F \times_B F \rightarrow E \times_B E$ is a $U(1) \times U(1)$ -bundle and $E \times_B E \rightarrow B$ is a $G \times G$ -bundle.

Definition 3.1 A **ring pair** is a pair $F \rightarrow E \rightarrow B$ together with a commutative diagram

$$\begin{array}{ccc}
 F \times_B F & \longrightarrow & F \\
 \downarrow & & \downarrow \\
 E \times_B E & \longrightarrow & E \\
 \downarrow & & \downarrow \\
 B & \xrightarrow{=} & B
 \end{array} \tag{11}$$

such that there exists an atlas with the property that for each chart U_i and $b \in U_i$ the restriction

$$\begin{array}{ccc} F|_{E|_b} & \longrightarrow & G \times U(1) \\ \downarrow & & \downarrow \\ E|_b & \longrightarrow & G \\ \downarrow & & \downarrow \\ b & \longrightarrow & b \end{array}$$

of the chart induces

$$\begin{array}{ccc} (G \times U(1)) \times (G \times U(1)) & \xrightarrow{+ \times \cdot} & G \times U(1) \\ \downarrow & & \downarrow \\ G \times G & \xrightarrow{+} & G \\ \downarrow & & \downarrow \\ b & \xrightarrow{=} & b \end{array}$$

from the restriction $(11)|_b$.

The horizontal maps in (11) are called the **multiplication maps** of the pair, and an atlas satisfying the conditions above is called a **ring atlas** for the ring pair.

An example of a ring pair can be made out of the pair $F_{\hat{E}} \rightarrow B \times G \rightarrow B$ from (9). Namely, this pair admits canonical maps

$$\begin{array}{ccc} F_{\hat{E}} \times_B F_{\hat{E}} & \rightarrow & F_{\hat{E}} & (B \times G) \times_B (B \times G) & \rightarrow & B \times G \\ ([\hat{e}, g, t], [\hat{e} \cdot \chi, h, s]) & \mapsto & [\hat{e}, g + h, \langle h, \chi \rangle ts] & ((b, g), (b, h)) & \mapsto & (b, g + h) \end{array} \quad (12)$$

which give this pair the structure of a ring pair. A ring atlas is obtained from the local trivialisations of the bundle $\hat{E} \rightarrow B$. We will see below, that all examples are of this form.

Remark 3.1 For a ring pair $F \rightarrow E \rightarrow B$ the map $F \rightarrow B$ is a bundle of groups with fibre isomorphic to $G \times U(1)$.

In particular, the multiplication map μ is associative and commutative, and there is a canonical section $\sigma : B \rightarrow F$. If $\mu : F \times_B F \rightarrow F$ denotes the multiplication map and $x \in F$ is an element over $b \in B$, then there is a unique $x^\dagger \in F$ over b with the property

$$\mu(x, x^\dagger) = \mu(x^\dagger, x) = \sigma(b).$$

$^\dagger : F \rightarrow F$ is anti-linear, i.e. $(x \cdot t)^\dagger = x^\dagger \cdot \bar{t}$, for the complex conjugate \bar{t} of $t \in U(1)$.

Proof : The associativity and commutativity of μ follows as the multiplication map is locally associative and commutative.

To define the section $\sigma : B \rightarrow F$ we define $\sigma(b) \in F|_{E|_b}$ to be the pre-image of $(0, 1)$ under any chart $F|_{E|_b} \rightarrow G \times U(1)$ of a ring atlas. We have to check that this is well-defined. So let $\{U_i\}$ be a ring atlas with transition functions g_{ij}, ζ_{ij} satisfying (7). Assume $b \in U_i$ and $b \in U_j$, then we have a commutative diagram

$$\begin{array}{ccc}
(U_i \times G \times U(1)) \times_{U_i} (U_i \times G \times U(1)) & & \\
\uparrow b & & \\
G \times G \times U(1) \times U(1) & \xrightarrow{+ \times \cdot} & G \times U(1) \\
\downarrow (g, h, t, s) & & \downarrow (g, t) \\
(G_{ji}(b) + g, g_{ji}(b) + h, \zeta_{ji}(b, g)t, \zeta_{ji}(b, h)s) & \xrightarrow{+ \times \cdot} & G \times U(1) \\
\downarrow b & & \downarrow (g_{ji}(b) + g, \zeta_{ji}(b, g)t) \\
(U_j \times G \times U(1)) \times_{U_j} (U_j \times G \times U(1)) & &
\end{array}$$

The commutativity of this diagram implies that

$$g_{ji}(b) = 0 \text{ and } \zeta_{ji}(b, g)\zeta_{ji}(b, h) = \zeta_{ji}(b, g + h), \quad (13)$$

so in particular $\zeta_{ji}(b, 0) = 1$. Thus, the element $(0, 1) \in G \times U(1)$ is fixed by any change of charts. It follows that $\sigma(b)$ is well-defined.

Fix an $x \in F$ over $b \in B$. The map $\mu(x, -) : F|_{E|_b} \rightarrow F|_{E|_b}$ is an isomorphism thus there exists a unique $x^\dagger \in F$ such that $\mu(x, x^\dagger) = \mu(x^\dagger, x) = \sigma(b)$. The anti-linearity of † is clear then. \blacksquare

The composition $B \xrightarrow{\sigma} F \rightarrow E$ is a section of $E \rightarrow B$, hence there is a canonical trivialisation

$$E \xrightarrow{\cong} B \times G.$$

By this trivialisation, $E \rightarrow B$ has a canonical structure of a bundle of groups. A bundle of groups consists of the same data as a groupoid where source and target maps are equal. Therefore the $U(1)$ -bundle $F \rightarrow E$ is a $U(1)$ -central extension of groupoids

$$\begin{array}{ccccc}
B \times U(1) & \twoheadrightarrow & F & \twoheadrightarrow & E \\
\Downarrow & & \Downarrow & & \Downarrow \\
B & \xrightarrow{=} & B & \xrightarrow{=} & B
\end{array} \quad (14)$$

Remark 3.2 In section 8 we will be concerned about the C^* -algebra $C^*(E, F)$ of a ring pair $F \rightarrow E \rightarrow B$. By thinking of ring pairs as central extensions it is possible to give a quick definition of $C^*(E, F)$. It is just the C^* -algebra of the central extension (14).

Equality (13) determines the possible transition functions in a ring atlas. There exist $\chi_{ji} : U_{ji} \rightarrow \widehat{G}$ such that $\zeta_{ji}(u, g) = \langle g, \chi_{ji}(u) \rangle$. Thus, in a ring atlas the transition functions are maps

$$U_{ji} \rightarrow \widehat{G} \subset A_{\text{Par}}.$$

Note that \widehat{G} is the automorphism group of the trivial ring pair over the point (10) in the category of ring pairs. Here a morphism in the category of ring pairs is a morphisms of pairs such that the induced diagram of multiplication maps commutes. The category of ring pairs and the category of \widehat{G} -bundles are proper subcategories (i.e. *not* full subcategories) of the categories of ring pairs and A_{Par} -bundles. Therefore the following proposition is not just a corollary of Proposition 2.1. However, the bundle theoretic argument used in the proof is quite the same.

Proposition 3.1 *The functor*

$$(\widehat{E} \rightarrow B) \mapsto (F_{\widehat{E}} \rightarrow B \times G \rightarrow B, (12)) \quad (15)$$

is an equivalence of categories from the category of \widehat{G} -bundles over B to the category of ring pairs over B .

Proof : We define a \widehat{G} -bundle from the data of a ring pair $F \rightarrow E \rightarrow B$. Let σ, μ be as in Remark 3.1, and let $r : F \rightarrow B \times G$ denote the composition $F \rightarrow E \cong B \times G$. For $b \in B$, let \widehat{E}_b be the set of all subsets $\hat{e} \subset F|_{E|_b}$ with the property that, first, $r|_{\hat{e}} : \hat{e} \rightarrow b \times G$ is a homoeomorphism, and second, $\mu(\hat{e}(g), \hat{e}(h)) = \hat{e}(g + h)$. Here we used the notation $\hat{e}(g) := (r|_{\hat{e}})^{-1}(b \times g) \in \hat{e}$. The set \widehat{E}_b is a \widehat{G} -torsor subject to the action

$$\hat{e} \diamond \chi := \{\hat{e}(g) \cdot \langle g, \chi \rangle \in F | g \in G\},$$

where $U(1)$ acts on F by the given principal action. Let $\widehat{E} := \bigsqcup_{b \in B} \widehat{E}_b$. We give \widehat{E} a topology such that the canonical map $\widehat{E} \rightarrow B$ becomes a \widehat{G} -principal bundle. Namely, if U_i is any ring atlas for the pair $F \rightarrow E \rightarrow B$, then $U_i \times G \times U(1) \cong F|_{E|_{U_i}}$ identifies the characters $\chi \in \widehat{G}$ understood as graphs $\chi \subset G \times U(1)$ with the elements of $\bigsqcup_{b \in U_i} \widehat{E}_b \subset \widehat{E}$. We take the coarsest topology on \widehat{E} such that all the bijections $\bigsqcup_{b \in U_i} \widehat{E}_b \cong U_i \times \widehat{G}$ are homoeomorphisms.

A morphism of ring pairs preserves the multiplication maps, it induces a morphism between the constructed \widehat{G} -bundles. So we have defined a functor from ring pairs to \widehat{G} -bundles.

To check that the compositions of the two functors are naturally isomorphic to the corresponding identity functors is a tedious manipulation. ■

We give an example of a (non-trivial) ring pair $F \rightarrow E \rightarrow B$ where both of the bundles $F \rightarrow E$ and $E \rightarrow B$ are trivial, but the corresponding \widehat{G} -bundle \widehat{E} is not, i.e. the non-triviality is hidden in the multiplication maps.

Example 3.1 Take $G := \mathcal{O}(1) \cong \widehat{G}$ the group with two elements, and let $B := \mathbb{RP}^1$ be the 1-dimensional projective space. The quotient map $S^1 \rightarrow \mathbb{RP}^1$ is a non-trivial \widehat{G} -bundle, but the $U(1)$ -bundle $F_{S^1} \rightarrow \mathbb{RP}^1 \times \mathcal{O}(1)$ is trivialisable in the category of $U(1)$ -bundles over $\mathbb{RP}^1 \times \mathcal{O}(1)$ since $H^2(\mathbb{RP}^1 \times \mathcal{O}(1)) = 0$. Thus, the choice of a trivialisaton $\mathbb{RP}^1 \times \mathcal{O}(1) \times U(1) \cong F_{S^1}$ and the multiplication map on F_{S^1} yield a non-trivial ring pair with an underlying trivial pair.

4 Module pairs

Having the notion of ring pairs at hand we can define the notion of a module pair over a ring pair.

Definition 4.1 A **module pair** over a ring pair $F_0 \rightarrow E_0 \rightarrow B$ is a pair $F \rightarrow E \rightarrow B$ together with a commutative diagram

$$\begin{array}{ccc} F \times_B F_0 & \longrightarrow & F \\ \downarrow & & \downarrow \\ E \times_B E_0 & \longrightarrow & E \\ \downarrow & & \downarrow \\ B & \xrightarrow{=} & B \end{array} \quad (16)$$

such that there exists an atlas for $F \rightarrow E \rightarrow B$ and a ring atlas for $F_0 \rightarrow E_0 \rightarrow B$ over the same open covering $\{U_i\}$ of B with the property that for each chart of the ring atlas over U_i there is a chart of $F \rightarrow E \rightarrow B$ over U_i such that for all $b \in U_i$ the restrictions

$$\begin{array}{ccc} F|_{E|_b} & \longrightarrow & G \times U(1) \\ \downarrow & & \downarrow \\ E|_b & \longrightarrow & G \\ \downarrow & & \downarrow \\ b & \longrightarrow & b \end{array} \quad \begin{array}{ccc} F_0|_{E_0|_b} & \longrightarrow & G \times U(1) \\ \downarrow & & \downarrow \\ E_0|_b & \longrightarrow & G \\ \downarrow & & \downarrow \\ b & \longrightarrow & b \end{array}$$

of these charts induce

$$\begin{array}{ccc} (G \times U(1)) \times (G \times U(1)) & \xrightarrow{+ \times \cdot} & G \times U(1) \\ \downarrow & & \downarrow \\ G \times G & \xrightarrow{+} & G \\ \downarrow & & \downarrow \\ b & \xrightarrow{=} & b \end{array}$$

from the restriction $(16)|_b$.

An atlas as above is called a **module atlas** for $F \rightarrow E \rightarrow B$. The horizontal maps in (16) are called **action maps**.

Remark 4.1 1. The multiplication maps of a ring pair and the action maps of a module pair are associative. I.e. if $\mu : F_0 \times_B F_0 \rightarrow F_0$ and $\varrho : F \times_B F_0 \rightarrow F$ denote the multiplication map and the action map of a ring pair and a module pair, then

$$\varrho(x, \mu(y_0, z_0)) = \varrho(\varrho(x, y_0), z_0)$$

holds.

2. Fix $x \in F$ in the fibre over $b \in B$. The map $\varrho(x, -) : F_0|_{E_0|_b} \rightarrow F|_{E|_b}$ is an isomorphism, thus there is a well-defined map $\sigma_\varrho : F \times_B F \rightarrow F_0$ unique with respect to the property

$$\varrho(x, \sigma_\varrho(x, y)) = y.$$

One should think of σ_ϱ as an F_0 -valued "scalar product" as it is linear and F_0 -linear in the second entry, i.e.

$$\sigma_\varrho(x, y \cdot t) = \sigma_\varrho(x, y) \cdot t, \quad \sigma_\varrho(x, \varrho(y, z_0)) = \mu(\sigma_\varrho(x, y), z_0),$$

it is anti-symmetric, i.e.

$$\sigma_\varrho(y, x) = \sigma_\varrho(x, y)^\dagger,$$

and it is positive definite, i.e.

$$\sigma_\varrho(x, x) = \sigma(b).$$

Here $\sigma : B \rightarrow F_0$ and ${}^\dagger : F_0 \rightarrow F_0$ are as in Remark 3.1.

Let us return to the groupoid point of view. As explained in diagram (14), we can think of a ring pair $F_0 \rightarrow E_0 \rightarrow B$ as of a central extension of groupoids

$$\begin{array}{ccccc} B \times \mathbf{U}(1) & \twoheadrightarrow & F_0 & \twoheadrightarrow & E_0 \\ \Downarrow & & \Downarrow & & \Downarrow \\ B & \xrightarrow{=} & B & \xrightarrow{=} & B \end{array}$$

Let $E \rightarrow B$ be a G -principal bundle. It can be thought of as a Morita self-equivalence

$$\begin{array}{ccccc} E_0 & \curvearrowright & E & \curvearrowleft & E_0 \\ \Downarrow & \swarrow & & \searrow & \Downarrow \\ B & & & & B \end{array}$$

where the left and the right action of $E_0 \rightrightarrows B$ coincide and commute by the abelianess of G . By the Remark 4.1, a module pair $F \rightarrow E \rightarrow B$ over $F_0 \rightarrow E_0 \rightarrow B$ gives rise to a $\mathbf{U}(1)$ -equivariant Morita self-equivalence

$$\left(\begin{array}{ccccc} F_0 & \curvearrowright & F & \curvearrowleft & F_0 \\ \Downarrow & \swarrow & & \searrow & \Downarrow \\ B & & & & B \end{array} \right) \circlearrowleft \mathbf{U}(1). \quad (17)$$

Remark 4.2 Viewing to module pairs as Morita equivalences of groupoids will be helpful in section 8. There we will define the Hilbert C^* -module $H(E, F)$ of a module pair $F \rightarrow E \rightarrow B$. The shortest way to do this is to define $H(E, F)$ as the Hilbert C^* -module of the $U(1)$ -equivariant Morita equivalence (17). It is a module over the C^* -algebra $C^*(E_0, F_0)$ of the ring pair $F_0 \rightarrow E_0 \rightarrow B$ (s. Remark 3.2).

Let $F \rightarrow E \rightarrow B$ be a pair and let $\widehat{E} \rightarrow B$ be a \widehat{G} -bundle. It is natural to ask whether or not one can turn $F \rightarrow E \rightarrow B$ into a module pair over $F_{\widehat{E}} \rightarrow B \times G \rightarrow B$. To answer this question consider first the cup product

$$\cup : \check{H}^1(B, \underline{G}) \times \check{H}^1(B, \widehat{G}) \rightarrow \check{H}^2(B, \underline{U}(1)),$$

i.e. if $g_{ji} : U_{ij} \rightarrow G$ and $\chi_{ji} : U_{ij} \rightarrow \widehat{G}$ are transition functions of the bundles E and \widehat{E} , then the class $[E] \cup [\widehat{E}]$ is represented by the cocycle $\alpha_{kji} : U_{ijk} \rightarrow U(1)$ with

$$\alpha_{kji}(u) := \langle g_{ij}(u), \chi_{jk}(u) \rangle \in U(1). \quad (18)$$

Assume that the class $[E] \cup [\widehat{E}]$ vanishes, so let $s_{ji} : U_{ji} \rightarrow U(1)$ be such that $\delta\{s_{ji}\} = \{\alpha_{kji}\}$. Then we can define

$$\zeta_{ji}(u, g) := \langle g, \chi_{ji} \rangle s_{ji}(u)^{-1}, \quad (19)$$

and it is immediate that g_{ji} and ζ_{ji} satisfy (7), i.e. ζ_{ij} is a family of transition functions for a $U(1)$ -bundle over E . As s_{ji} is unique only up to a 1-cocycle in $\check{Z}^1(\{U_i\}, \underline{U}(1))$ we see that the class of ζ_{ji} in $\check{H}^1(E, \underline{U}(1))$ is only determined up to the natural action of $\check{H}^1(B, \underline{U}(1))$ on $\check{H}^1(E, \underline{U}(1))$ given by pullback $p^* : \check{H}^1(B, \underline{U}(1)) \rightarrow \check{H}^1(E, \underline{U}(1))$ along the projection $p : E \rightarrow B$. In other words, if $[E] \cup [\widehat{E}] = 0$, then (19) defines a subset

$$[\widehat{E}]^\perp \subset \check{H}^1(E, \underline{U}(1)) \quad (20)$$

which is a $p^*(\check{H}^1(B, \underline{U}(1)))$ -torsor.

Proposition 4.1 $F \rightarrow E \rightarrow B$ is a module pair over $F_{\widehat{E}} \rightarrow B \times G \rightarrow B$ if and only if $[E] \cup [\widehat{E}] = 0$ and $[F] \in [\widehat{E}]^\perp$.

Proof : Let $F \rightarrow E \rightarrow B$ be a module pair. Let U_i be a module atlas with transition functions g_{ji}, ζ_{ji} . Let ζ_{ji}^0 denote the transition functions in the ring atlas, by equality (13) we have $\zeta_{ji}^0(b, h) = \langle h, \chi_{ji}(b) \rangle$. Similar to equality (13) we obtain that $\zeta_{ji}(b, g) \zeta_{ji}^0(b, h) = \zeta_{ji}(b, g + h)$. It follows that $\zeta_{ji}(b, h) = \langle h, \chi_{ji}(b) \rangle s_{ji}(b)^{-1}$, for $s_{ji}(b) := \zeta_{ji}(b, 0)^{-1}$. One computes $s_{kj}(b) s_{ki}(b)^{-1} s_{ji}(b) = \langle g_{ji}(b), \chi_{kj}(b) \rangle$. Hence $[E] \cup [\widehat{E}] = 0$ and $[F] \in [\widehat{E}]^\perp$.

Conversly, let $F \rightarrow E \rightarrow B$ be a pair such that $[E] \cup [\widehat{E}] = 0$ and $[F] \in [\widehat{E}]^\perp$. Take a ring atlas for the ring pair and an atlas of the pair such that the transition functions satisfy (19). Then we define the action maps by demanding that the chosen atlas is a module atlas. It is straight forward to check that this is well-defined. \blacksquare

Fix two bundles $E \rightarrow B$ and $\widehat{E} \rightarrow B$ with $[E] \cup [\widehat{E}] = 0$. The set $[\widehat{E}]^\perp$ is the set of isomorphism classes of $U(1)$ -bundles over E which admit a module structure, but the isomorphisms do not have to be compatible with the module structure.

Let $\text{Mod}(E, \widehat{E})$ be the groupoid of module pairs $F \rightarrow E \rightarrow B$ over $F_{\widehat{E}} \rightarrow B \times G \rightarrow B$, where a morphism from $F \rightarrow E \rightarrow B$ to $F' \rightarrow E \rightarrow B$ is a bundle morphism over E

$$\begin{array}{ccc} F & \xrightarrow{\cong} & F' \\ \downarrow & & \downarrow \\ E & \xrightarrow{=} & E \end{array} \quad (21)$$

such that the induced diagram

$$\begin{array}{ccc} F \times_B F_{\widehat{E}} & \longrightarrow & F \\ \downarrow & & \downarrow \\ F' \times_B F_{\widehat{E}} & \longrightarrow & F' \end{array} \quad (22)$$

commutes. Here the horizontal maps are the action maps of the module pairs.

Let $[\text{Mod}(E, \widehat{E})]$ be the set of isomorphism classes of $\text{Mod}(E, \widehat{E})$. By pullback, there is a well defined, natural action of $\check{H}^1(B, \underline{U(1)})$ on $[\text{Mod}(E, \widehat{E})]$. In fact, let η and η' be two $U(1)$ -bundles which represent the same class $[\eta] = [\eta'] \in \check{H}^1(B, \underline{U(1)})$, and let $F \rightarrow E \xrightarrow{p} B$ be a module pair. Then $F \otimes p^*\eta$ and $F \otimes p^*\eta'$ become module pairs in the obvious way, and they are isomorphic in $\text{Mod}(E, \widehat{E})$.

Proposition 4.2 *The set $[\text{Mod}(E, \widehat{E})]$ is a $\check{H}^1(B, \underline{U(1)})$ -torsor.*

Proof : The action is transitive: If $[F]$ and $[F']$ are classes in $[\text{Mod}(E, \widehat{E})]$, then it follows from (19) that there exists a $U(1)$ -bundle $\eta \rightarrow B$ such that F and $F' \otimes p^*\eta$ are isomorphic in $\text{Mod}(E, \widehat{E})$.

The action is free: Let $\{U_i\}$ be a sufficiently refined open cover of B such that the following algebraic arguments are meaningful. Let $\eta \rightarrow B$ be a bundle with transition functions $\eta_{ji} : U_i \rightarrow U(1)$ such that F and $F \otimes p^*\eta$ represent the same class in $\text{Mod}(E, \widehat{E})$. In particular, as they are isomorphic over E and as their transition functions vary by η_{ji} , there are local isomorphisms $\varphi_i : U_i \times G \rightarrow U(1)$ such that $\varphi_j(u, g_{ji}(u) + g)^{-1} \varphi_i(u, g) = \eta_{ji}(u)$. Using the commutativity of (22), it is easy to see that each φ_i is constant in the G -argument, so $\varphi_j(u, 0)^{-1} \varphi_i(u, 0) = \eta_{ji}(u)$, i.e. the class of η_{ji} is trivial in $\check{H}^1(B, \underline{U(1)})$. ■

Now, fix two bundles $E \rightarrow B$ and $\widehat{E} \rightarrow B$ without any further assumption about the class $[E] \cup [\widehat{E}] \in \check{H}^2(B, \underline{U(1)})$. For any open set $U \subset B$ we can consider the groupoid $\text{Mod}_{E, \widehat{E}}(U) := \text{Mod}(E|_U, \widehat{E}|_U)$, where $E|_U \rightarrow U$ and $\widehat{E}|_U \rightarrow U$ are the pulbacks of E and \widehat{E} along the inclusion $U \subset B$.

Proposition 4.3 *The assignment*

$$\text{Mod}_{E, \widehat{E}} : U \mapsto \text{Mod}(E|_U, \widehat{E}|_U) \quad (23)$$

is a $U(1)$ -banded gerbe over B , and its class given by $[E] \cup [\widehat{E}] \in \check{H}^2(B, \underline{U(1)})$.

Proof : It is immediate that (23) is a stack. It is also clear from the definition of module pairs that (23) is locally non empty, and that locally any two objects are isomorphic. Hence (23) is a gerbe.

Let $F \in \text{Mod}(E|_U, \widehat{E}|_U)$ be an object, and let $V \subset U$ be open. Pullback along $E|_U|_V \rightarrow V$ defines an injection $C(V, U(1)) \hookrightarrow C(E|_U|_V, U(1)) \hookrightarrow \text{Aut}(F|_{E|_U|_V})$. The commutativity of (22) implies that every automorphism of F is of this form. Thus for each open U the sheaf of automorphisms $\underline{\text{Aut}}(F)$ on U is isomorphic to $C(_, U(1))$. Hence the gerbe (23) is $U(1)$ -banded.

Let U_i be an open cover together with trivialisations $E|_{U_i} \rightarrow U_i \times G$ and $\widehat{E}|_{U_i} \rightarrow U_i \times \widehat{G}$ and transition functions $g_{ij} : U_{ij} \rightarrow G$ and $\chi_{ij} : U_{ij} \rightarrow \widehat{G}$. We choose for each i an $F_i \in \text{Mod}(E|_{U_i}, \widehat{E}|_{U_i})$, namely $F_i := E|_{U_i} \times U(1)$. The trivialisations of $E|_{U_i}$ and $\widehat{E}|_{U_i}$ define a unique module structure on $F_i \rightarrow E|_{U_i} \rightarrow U_i$ over $F_{\widehat{E}|_{U_i}} \rightarrow U_i \times G \rightarrow U_i$. Let $F_i|_j \in \text{Mod}(E|_{U_{ij}}, \widehat{E}|_{U_{ij}})$ denote the pullback of F_i along the composition $E|_{U_{ij}} \cong E|_{U_i}|_{U_{ij}} \hookrightarrow E|_{U_i}$. Interchanging i and j we also find $F_j|_i \in \text{Mod}(E|_{U_{ij}}, \widehat{E}|_{U_{ij}})$, and we define a morphism f_{ji} between these by

$$\begin{array}{ccc} F_i|_j & \xrightarrow{f_{ji}} & F_j|_i \\ \downarrow & & \uparrow \\ U_{ji} \times G \times U(1) & \xrightarrow{(u, g, z) \mapsto (u, g + g_{ji}(u), \langle g, \chi_{ji}(u) \rangle z)} & U_{ji} \times G \times U(1) \end{array} ,$$

where the vertical trivialisations are given by the corresponding trivialisations over U_i, U_j respectively, Then on threefold intersections we have the automorphism $f_{ki}^{-1} \circ f_{kj} \circ f_{ji} \in \underline{\text{Aut}}(F_i)(U_{ijk})$ which is mapped to a Čech cocycle $\beta_{kji} \in C(U_{ijk}, U(1))$ under the identification $\underline{\text{Aut}}(F_i)(U_{ijk}) \cong C(U_{ijk}, U(1))$. This cocycle represents the class of the gerbe. We have $\beta_{kji}(u) = \langle g_{ji}(u), \chi_{kj}(u) \rangle$, and if we compare with the cocycle (18) we find $\alpha_{kji} = \beta_{kji}$. Hence the class of the gerbe (23) is $[E] \cup [\widehat{E}]$. ■

5 Pontrjagin duality triples

To introduce the main objects of our interest we first define their local model.

Definition 5.1 The **trivial Pontrjagin duality triple** over the point is the diagram

$$\begin{array}{ccccc}
 & G \times \widehat{G} \times U(1) & \xrightarrow{\pi} & G \times \widehat{G} \times U(1) & \\
 & \swarrow & & \searrow & \\
 G \times U(1) & & & & \widehat{G} \times U(1) \\
 & \searrow & & \swarrow & \\
 & G & & \widehat{G} & \\
 & \searrow & & \swarrow & \\
 & & * & &
 \end{array} \tag{24}$$

of trivial bundles, where the top isomorphism π is given by the pairing $\langle \cdot, \cdot \rangle : G \times \widehat{G} \rightarrow U(1)$, i.e.

$$\pi(g, \chi, z) := (g, \chi, \langle g, \chi \rangle z). \tag{25}$$

A general Pontrjagin duality triple is a diagram which has the trivial Pontrjagin duality triple over the point as its local model.

Definition 5.2 A **Pontrjagin duality triple** over B is a commutative diagram

$$\begin{array}{ccccc}
 & F \times_B \widehat{E} & \xrightarrow{\kappa} & E \times_B \widehat{F} & \\
 & \swarrow & & \searrow & \\
 F & & & & \widehat{F} \\
 & \searrow & & \swarrow & \\
 & E \times_B \widehat{E} & & & \\
 & \swarrow & & \searrow & \\
 & E & & \widehat{E} & \\
 & \searrow & & \swarrow & \\
 & & B & &
 \end{array} \tag{26}$$

where $F \rightarrow E \rightarrow B$ is a pair, $\widehat{F} \rightarrow \widehat{E} \rightarrow B$ is a dual pair and κ is an isomorphism of $U(1)$ -bundles such that there exists atlases of the pair and the dual pair over the same open cover $\{U_i\}$ of B such that for each point $b \in B$ there are charts containing b such that the restrictions

$$\begin{array}{ccc}
 F|_{E|_b} \longrightarrow G \times U(1) & & \widehat{F}|_{\widehat{E}|_b} \longrightarrow \widehat{G} \times U(1) \\
 \downarrow & & \downarrow \\
 E|_b \longrightarrow G & & \widehat{E}|_b \longrightarrow \widehat{G} \\
 \downarrow & & \downarrow \\
 b \longrightarrow * & & b \longrightarrow *
 \end{array}$$

of these charts induce the trivial Pontrjagin duality diagram (24) from the restriction (26)|_b.

A **chart** of a Pontrjagin duality triple is the datum of two charts with the property spelled out above. An **atlas** of a Pontrjagin duality triple is a collection of charts covering B .

Let $\hat{p} : \hat{E} \rightarrow B$ be a \widehat{G} -bundle, and let \hat{E}^{op} be the opposite \widehat{G} -bundle which is \hat{E} as a space but with \widehat{G} -action: $(\hat{e}, \chi) \mapsto \hat{e} \cdot (-\chi)$. The simplest possible non-trivial example of a Pontrjagin duality triple is

$$\begin{array}{ccccc}
 & F_{\hat{E}^{\text{op}}} \times_B \hat{E} & \xrightarrow{\kappa_{\hat{E}}} & (B \times G) \times_B \hat{E} \times \text{U}(1) & , & (27) \\
 & \swarrow & & \swarrow & & \\
 F_{\hat{E}^{\text{op}}} & & & (B \times G) \times_B \hat{E} & & \hat{E} \times \text{U}(1) \\
 & \searrow & & \searrow & & \swarrow \\
 & B \times G & & & & \hat{E} \\
 & & & & & \searrow \\
 & & & & & B
 \end{array}$$

where

$$\kappa_{\hat{E}}([\hat{e}, g, t], \hat{e} \cdot \chi) := (\hat{p}(\hat{e}), g, \hat{e} \cdot \chi, \langle g, \chi \rangle t)$$

and the obvious maps elsewhere. Note that it is necessary to deal with $F_{\hat{E}^{\text{op}}}$ instead of $F_{\hat{E}}$ to make the top isomorphism $\kappa_{\hat{E}}$ well-defined with the correct local structure (25).

Let $\{U_i\}$ be an atlas of a general Pontrjagin duality triple (26). We have transition functions $g_{ji} : U_{ij} \rightarrow G, \zeta_{ji} : U_{ij} \times G \rightarrow \text{U}(1)$ for the pair and $\hat{g}_{ji} : U_{ij} \rightarrow \widehat{G}, \hat{\zeta}_{ji} : U_{ij} \times \widehat{G} \rightarrow \text{U}(1)$ for the dual pair. As over each point the isomorphism κ reduces to π from (24), the relation

$$\hat{\zeta}_{ji}(u, \chi) \langle g, \chi \rangle = \langle g + g_{ji}(u), \chi + \hat{\chi}_{ji}(u) \rangle \zeta_{ji}(u, g) \quad (28)$$

holds. By putting $\chi = 0$ or $g = 0$ we find that

$$\begin{aligned}
 \zeta_{ji}(u, g) &= \langle g, \hat{\chi}_{ji}(u) \rangle^{-1} s_{ji}(u), \\
 \hat{\zeta}_{ji}(u, \chi) &= \langle g_{ji}(u), \hat{\chi} \rangle \langle g_{ji}(u), \hat{\chi}_{ji}(u) \rangle s_{ji}(u),
 \end{aligned} \quad (29)$$

where $s_{ji}(u) := \zeta_{ji}(u, 0)$. Thus, if we compare with (19), we see that Pontrjagin duality triples contain the same Čech theoretic amount of data as module pairs. In the following we make this correspondence more precise by constructing an explicit equivalence of gerbes between the gerbe $\text{Mod}_{E, \hat{E}}$ of module pairs of Proposition 4.3 and the gerbe of Pontrjagin duality triples we introduce next.

Let B be a space and let $E \rightarrow B$ and $\hat{E} \rightarrow B$ be a G -bundle and a \widehat{G} -bundle, respectively. Let us denote by $\text{Pon}(E, \hat{E})$ the groupoid of Pontrjagin duality triples with $E \rightarrow B$ and $\hat{E} \rightarrow B$ fixed, i.e. a morphism

$$\left(\begin{array}{ccccc}
 & F \times_B \hat{E} & \xrightarrow{\kappa} & E \times_B \hat{F} & \\
 & \swarrow & & \swarrow & \\
 F & & & E \times_B \hat{E} & \\
 & \searrow & & \searrow & \\
 & E & & & \hat{E} \\
 & & & & \searrow \\
 & & & & B
 \end{array} \right) \rightarrow \left(\begin{array}{ccccc}
 & F' \times_B \hat{E} & \xrightarrow{\kappa'} & E \times_B \hat{F}' & \\
 & \swarrow & & \swarrow & \\
 F' & & & E \times_B \hat{E} & \\
 & \searrow & & \searrow & \\
 & E & & & \hat{E} \\
 & & & & \searrow \\
 & & & & B
 \end{array} \right)$$

between two Pontrjagin duality triples with fixed E and \widehat{E} consists of two bundle morphisms

$$\begin{array}{ccc} F & \xrightarrow{\cong} & F' \\ \downarrow & & \downarrow \\ E & \xrightarrow{=} & E \end{array} \quad \begin{array}{ccc} \widehat{F} & \xrightarrow{\cong} & \widehat{F}' \\ \downarrow & & \downarrow \\ \widehat{E} & \xrightarrow{=} & \widehat{E} \end{array}$$

such that the induced diagram

$$\begin{array}{ccc} F \times_B \widehat{E} & \xrightarrow{\kappa} & E \times_B F \\ \downarrow & & \downarrow \\ F' \times_B \widehat{E} & \xrightarrow{\kappa'} & E \times_B F' \end{array}$$

commutes. For each open $U \subset B$ let $\text{Pon}_{E, \widehat{E}}(U) := \text{Pon}(E|_U, \widehat{E}|_U)$ be the groupoid of Pontrjagin duality triples over $E|_U \rightarrow U$ and $\widehat{E}|_U \rightarrow U$.

Proposition 5.1 *The assignment*

$$\text{Pon}_{E, \widehat{E}} : U \mapsto \text{Pon}(E|_U, \widehat{E}|_U)$$

is a $U(1)$ -banded gerbe.

Proof : The proof is straight forward and analogue to the proof of Proposition 4.3. \blacksquare

As in the proof of Proposition 4.3, we can also determine the class of the gerbe $\text{Pon}_{E, \widehat{E}}$ but the result of next section will do this job equally well. Namely, we construct an explicit equivalence of gerbes $\text{Mod}_{E, \widehat{E}} \cong \text{Pon}_{E, \widehat{E}}$.

6 Module pairs vs. Pontrjagin duality triples

Let $\widehat{E} \rightarrow B$ be a \widehat{G} -bundle, and let $F_{\widehat{E}} \rightarrow B \times G \rightarrow B$ be the ring pair as defined in (12), i.e. $F_{\widehat{E}} := \widehat{E} \times_{\widehat{G}} (G \times U(1))$. There is a canonical map $\iota : \widehat{E} \times G \rightarrow F_{\widehat{E}}$ given by $\iota(\hat{e}, g) := [\hat{e}, g, 1]$. Let $F \xrightarrow{q} E \rightarrow B$ be a module pair over $F_{\widehat{E}} \rightarrow B \times G \rightarrow B$ with action map $q : F \times_B F_{\widehat{E}} \rightarrow F$. By use of the maps ρ and ι we define a G -action on the space $F \times_B \widehat{E}$ by

$$\begin{aligned} (F \times_B \widehat{E}) \times G &\rightarrow F \times_B \widehat{E}. \\ (x, \hat{e}, g) &\mapsto (q(x, \iota(\hat{e}, g)), \hat{e}) \end{aligned} \quad (30)$$

The corresponding quotient $\widehat{F}_q := (F \times_B \widehat{E})/G$ comes along with two natural maps:

$$\begin{aligned} \hat{q} : \widehat{F}_q &\rightarrow \widehat{E} & \text{and} & & \kappa_q : F \times_B \widehat{E} &\rightarrow E \times_B \widehat{F}_q. \\ [x, \hat{e}] &\mapsto \hat{e} & & & (x, \hat{e}) &\mapsto (q(x), [x, \hat{e}]) \end{aligned}$$

The map \hat{q} is a principal $U(1)$ -bundle with action induced by the principal action of $U(1)$ on F .

Proposition 6.1 *The diagram*

$$\begin{array}{ccccc}
 & F \times_B \widehat{E}^{\text{op}} & \xrightarrow{\kappa_\varrho} & E \times_B \widehat{F}_\varrho & \\
 & \swarrow & & \swarrow & \\
 F & & & & \widehat{F}_\varrho \\
 & \searrow & & \searrow & \\
 & E \times_B \widehat{E}^{\text{op}} & & & \\
 & \swarrow & & \swarrow & \\
 E & & & & \widehat{E}^{\text{op}} \\
 & \searrow & & \searrow & \\
 & B & & &
 \end{array} \tag{31}$$

is a Pontrjagin duality triple. Here \widehat{E}^{op} the space \widehat{E} equipped with the opposite \widehat{G} -action $(\hat{e}, \chi) \mapsto \hat{e} \cdot (-\chi)$.

Proof : Let U_i be a module atlas, so there are trivialisations $\widehat{E}|_{U_i} \cong U_i \times \widehat{G}$ and $F|_{E|_{U_i}} \cong U_i \times G \times \text{U}(1)$ such that over each point of U_i the action map induced from the action map ϱ takes the form $+ \times \cdot : \text{U}(1) \times G \times \text{U}(1) \times G \rightarrow \text{U}(1) \times G$. These induce trivialisations

$$\widehat{F}_\varrho|_{\widehat{E}|_{U_i}} \cong ((U_i \times G \times \text{U}(1)) \times_{U_i} (U_i \times \widehat{G}))/G,$$

where the quotient of the right hand side is by the induced action:

$$((u, g, t), (u, \chi)) \cdot h := ((u, g + h, \langle h, \chi \rangle t), (u, \chi)).$$

We can identify further

$$\begin{aligned}
 ((U_i \times G \times \text{U}(1)) \times_{U_i} (U_i \times \widehat{G}))/G &\cong U_i \times \widehat{G} \times \text{U}(1) \\
 [(u, g, t), (u, \chi)] &\mapsto (u, \chi, \langle g, \chi \rangle^{-1} t)
 \end{aligned}$$

From this one can deduce that, firstly, $\widehat{F}_\varrho \rightarrow \widehat{E} \rightarrow B$ is a dual pair, and that, secondly, the isomorphism κ_ϱ induces

$$\begin{array}{ccc}
 (F \times_B \widehat{E})|_{U_i} & \xrightarrow{\kappa_\varrho|_{U_i}} & (E \times_B \widehat{F}_\varrho)|_{U_i} \\
 \downarrow & & \downarrow \\
 (U_i \times G \times \text{U}(1)) \times_{U_i} (U_i \times \widehat{G}) & \longrightarrow & (U_i \times G) \times_{U_i} (U_i \times \widehat{G} \times \text{U}(1)) \\
 ((u, g, t), (u, \chi)) & \longmapsto & ((u, g), (u, \chi, \langle g, \chi \rangle^{-1} t))
 \end{array}$$

where the vertical arrows are the chosen trivialisations. It follows that $\kappa_\varrho : F \times_B \widehat{E} \rightarrow E \times_B \widehat{F}_\varrho$ does not have the correct local structure of (25), due to the power of -1 in the above diagram. However, the minus sign can be absorbed by a modification of the trivialisations by the map $\chi \mapsto -\chi$ (which is not a \widehat{G} -bundle morphism). This means nothing but dealing with \widehat{E}^{op} instead of \widehat{E} . This proves the proposition. \blacksquare

The above construction is completely natural, therefore it defines a morphism of gerbes

$$\text{Mod}_{E, \widehat{E}} \rightarrow \text{Pon}_{E, \widehat{E}^{\text{op}}} \text{ or likewise } \text{Mod}_{E, \widehat{E}^{\text{op}}} \rightarrow \text{Pon}_{E, \widehat{E}}. \quad (32)$$

An example of the above construction is the following. Let $\widehat{E} \rightarrow B$ be a \widehat{G} -bundle. Consider the opposite \widehat{E}^{op} and note that $(\widehat{E}^{\text{op}})^{\text{op}} = \widehat{E}$. Take the ring pair $F_{\widehat{E}^{\text{op}}} \rightarrow B \times G \rightarrow B$ as a module pair over itself. Then the above construction yields a Pontrjagin duality triple

$$\begin{array}{ccccc}
 & F_{\widehat{E}^{\text{op}}} \times_B \widehat{E} & \xrightarrow{\kappa} & (B \times G) \times_B (F_{\widehat{E}^{\text{op}}} \times_B \widehat{E}) / G & \\
 & \swarrow & & \swarrow & \searrow \\
 F_{\widehat{E}^{\text{op}}} & & (B \times G) \times_B \widehat{E} & & (F_{\widehat{E}^{\text{op}}} \times_B \widehat{E}) / G \\
 & \searrow & \swarrow & \searrow & \swarrow \\
 & B \times G & & \widehat{E} & \\
 & \searrow & & \swarrow & \\
 & & B & &
 \end{array}$$

which is naturally isomorphic to (27) by the $U(1)$ -bundle isomorphism

$$\begin{aligned}
 (F_{\widehat{E}^{\text{op}}} \times_B \widehat{E}) / G &\cong \widehat{E} \times U(1). \\
 [[\hat{e}, g, t], \hat{e} \cdot \chi] &\mapsto (\hat{e} \cdot \chi, \langle g, \chi \rangle t)
 \end{aligned}$$

Let us start with the construction of a morphism in opposite direction of (32). Consider a Pontrjagin duality triple

$$\begin{array}{ccccc}
 & F \times_B \widehat{E} & \xrightarrow{\kappa} & E \times_B \widehat{F} & \\
 & \swarrow \text{pr}_F & & \swarrow & \searrow \\
 F & & E \times_B \widehat{E} & & \widehat{F} \\
 & \searrow & \swarrow & \searrow & \swarrow \\
 & E & & \widehat{E} & \\
 & \searrow & & \swarrow & \\
 & & B & &
 \end{array}$$

The \widehat{G} -bunde \widehat{E} defines the ring pair $F_{\widehat{E}^{\text{op}}} \rightarrow B \times G \rightarrow B$, i.e. $F_{\widehat{E}^{\text{op}}} := \widehat{E}^{\text{op}} \times_{\widehat{G}} (G \times U(1))$, and we define

$$\begin{aligned}
 \varrho_\kappa : F \times_B F_{\widehat{E}^{\text{op}}} &\rightarrow F. \\
 (x, [\hat{e}, g, t]) &\mapsto \text{pr}_F(\kappa^{-1}(\kappa(x, \hat{e}) \cdot g)) \cdot t
 \end{aligned} \quad (33)$$

This is well-defined, because κ is locally of the form (25) and therefore satisfies

$$\kappa^{-1}(\kappa(x, \hat{\ell} \cdot \chi) \cdot g) = (\kappa^{-1}(\kappa(x, \hat{\ell}) \cdot g) \cdot \chi) \cdot \langle g, \chi \rangle^{-1}.$$

Here the action of G on $E \times_B \hat{F}$ and the actions of \hat{G} and $U(1)$ on $F \times_B \hat{E}$ are the obvious ones.

Proposition 6.2 *Via the map q_κ the pair $F \rightarrow E \rightarrow B$ is a module pair over the ring pair $F_{\hat{E}^{\text{op}}} \rightarrow B \times G \rightarrow B$.*

Proof : It is immediate that q_κ satisfies the necessary local condition. ■

As the construction of q_κ is natural, we obtain a morphism of gerbes

$$\text{Pon}_{E, \hat{E}} \rightarrow \text{Mod}_{E, \hat{E}^{\text{op}}}. \quad (34)$$

Proposition 6.3 (32) : $\text{Mod}_{E, \hat{E}^{\text{op}}} \rightleftarrows \text{Pon}_{E, \hat{E}} : (34)$ is an equivalence of gerbes.

Proof : Consider the composition (34) \circ (32) applied to a module pair $F \rightarrow E \rightarrow B$ with action map $q : F \times_B F_{\hat{E}} \rightarrow F$. This gives a new module pair, where the underlying pairs are unchanged, but with a new action map $q_{\kappa_q} : F \times F_{\hat{E}^{\text{op}}} \rightarrow F$ as obtained from the top isomorphism κ_q . Note that κ_q is G -equivariant with respect to the G -action (30) on $F \times_B \hat{E}$ and the obvious G -action on $E \times_B \hat{F}$. Therefore we just can compute q_{κ_q} :

$$\begin{aligned} q_{\kappa_q}(x, [\hat{\ell}, g, t]) &= \text{pr}_F(\kappa_q^{-1}(\kappa_q(x, \hat{\ell}) \cdot g)) \cdot t \\ &= \text{pr}_F(q(x, \iota(\hat{\ell}, g)), \hat{\ell}) \cdot t \\ &= q(x, \iota(\hat{\ell}, g)) \cdot t \\ &= q(x, [\hat{\ell}, g, t]). \end{aligned}$$

Thus, the composition (34) \circ (32) is the identity on $\text{Mod}_{E, \hat{E}^{\text{op}}}$.

Consider the composition (32) \circ (34) applied to a Pontrjagin duality triple with top isomorphism $\kappa : F \times_B \hat{E} \rightarrow E \times_B \hat{F}$. Define q_κ , \hat{F}_{q_κ} and $\kappa_{q_\kappa} : F \times_B \hat{E} \rightarrow E \times_B \hat{F}_{q_\kappa}$ as above. Note that $\kappa : F \times_B \hat{E} \rightarrow E \times_B \hat{F}$ is G -equivariant with respect to the G -action (30) induced by q_κ and the obvious G -action on $E \times_B \hat{F}$. Thus, there are well-defined bundle isomorphisms

$$\begin{array}{ccc} F & \xrightarrow{=} & F \\ \downarrow & & \downarrow \\ E & \xrightarrow{=} & E \end{array} \quad \begin{array}{ccc} \hat{F}_{q_\kappa} & \xrightarrow{[x, \hat{\ell}] \mapsto \text{pr}_{\hat{F}}(\kappa(x, \hat{\ell}))} & \hat{F} \\ \downarrow & & \downarrow \\ \hat{E} & \xrightarrow{=} & \hat{E} \end{array},$$

and the induced diagram

$$\begin{array}{ccc} F \times_B \hat{E} & \xrightarrow{\kappa_{q_\kappa}} & E \times_B F_{q_\kappa} \\ \downarrow = & & \downarrow \cong \\ F \times_B \hat{E} & \xrightarrow{\kappa} & E \times_B F \end{array}$$

commutes. This construction is a natural isomorphism of Pontrjagin duality triples, therefore the composition (32) \circ (34) is naturally isomorphic to the identity transformation on $\text{Pon}_{E, \widehat{E}}$. \blacksquare

Corollary 6.1 *Let $E \rightarrow B$ and $\widehat{E} \rightarrow B$ be a G -bundle and a \widehat{G} -bundle, respectively.*

- (i) *The class of the gerbe $\text{Pon}_{E, \widehat{E}}$ is given by $-[E] \cup [\widehat{E}] \in \check{H}^2(B, \underline{\mathbb{U}(1)})$.*
- (ii) *If $[E] \cup [\widehat{E}] = 0$, then the set of isomorphism classes of the groupoid $\text{Pon}(E, \widehat{E})$ is a $\check{H}^1(B, \underline{\mathbb{U}(1)})$ -torsor.*

Proof : Proposition 6.3 and Proposition 4.3 together with the fact $[\widehat{E}^{\text{op}}] = -[\widehat{E}]$ imply (i). Proposition 6.3 and Proposition 4.2 imply (ii). \blacksquare

Example 6.1 *Let $B := \Sigma$ be a two-dimensional, connected, closed manifold, then $\check{H}^2(\Sigma, \underline{\mathbb{U}(1)}) \cong H^3(\Sigma, \mathbb{Z}) = 0$, so all choices of bundles $E \rightarrow \Sigma \leftarrow \widehat{E}$ can be extended to a Pontrjagin duality triple.*

If Σ is non-orientable, then also $\check{H}^1(\Sigma, \underline{\mathbb{U}(1)}) \cong H^2(\Sigma, \mathbb{Z}) = 0$, so up to isomorphism in $\text{Pon}(E, \widehat{E})$ this Pontrjagin duality triple is unique. Otherwise, if Σ is orientable, then $\check{H}^1(\Sigma, \underline{\mathbb{U}(1)}) \cong H^2(\Sigma, \mathbb{Z}) \cong \mathbb{Z}$, so there are infinitely many different isomorphism classes in $\text{Pon}(E, \widehat{E})$.

7 Extensions to Pontrjagin duality triples

Definition 7.1 *Let X be a commutative diagram of topological spaces. An **extension** of X to a Pontrjagin duality triple is a Pontrjagin duality triple which contains X as a sub-diagram.*

Let $E \rightarrow B$ be a G -bundle, and $\widehat{E} \rightarrow B$ be a \widehat{G} -bundle. The first part of Corollary 6.1 answers the question whether or not the diagram

$$\begin{array}{ccc}
 & E \times_B \widehat{E} & \\
 \swarrow & & \searrow \\
 E & & \widehat{E} \\
 \searrow & & \swarrow \\
 & B &
 \end{array} \tag{35}$$

admits an extension. Namely, a global object of the gerbe $\text{Pon}_{E, \widehat{E}}$ exists, i.e. it exists an object in $\text{Pon}(E, \widehat{E}) = \text{Pon}_{E, \widehat{E}}(B)$, if and only if the class of the gerbe is trivial. Thus, diagram (35) can be extended to a Pontrjagin duality triple if and only if $[E] \cup [\widehat{E}] = 0$. The second part of Corollary 6.1 states that such extensions are not unique if they exist, but (up to isomorphism) we know exactly about this ambiguity which only depends on the topology of B (and not on G or \widehat{G}).

We wish to understand another extension problem. Namely, if given a pair $F \rightarrow E \rightarrow B$, we want to understand the existence and uniqueness problem of extensions of the diagram

$$\begin{array}{ccc}
 F & & \\
 & \searrow & \\
 & E & \\
 & & \searrow \\
 & & B
 \end{array}
 \quad . \quad (36)$$

To manage this we take a closer look at the category $\text{Pon}(B)$ of (all) Pontrjagin duality triples over a space B . A morphisms in this category

$$(a, \hat{a}) : \left(\begin{array}{ccccc}
 & F \times_B \hat{E} & \xrightarrow{\kappa} & E \times_B \hat{F} & \\
 & \swarrow & & \swarrow & \\
 F & & E \times_B \hat{E} & & \hat{F} \\
 & \searrow & & \searrow & \\
 & E & & \hat{E} & \\
 & & \searrow & & \\
 & & B & &
 \end{array} \right) \rightarrow \left(\begin{array}{ccccc}
 & F' \times_B \hat{E}' & \xrightarrow{\kappa'} & E' \times_B \hat{F}' & \\
 & \swarrow & & \swarrow & \\
 F' & & E' \times_B \hat{E}' & & \hat{F}' \\
 & \searrow & & \searrow & \\
 & E' & & \hat{E}' & \\
 & & \searrow & & \\
 & & B & &
 \end{array} \right) \quad (37)$$

consists of a morphism of pairs $a : (F \rightarrow E \rightarrow B) \rightarrow (F' \rightarrow E' \rightarrow B)$ over B and of a morphism of dual pairs $\hat{a} : (\hat{F} \rightarrow \hat{E} \rightarrow B) \rightarrow (\hat{F}' \rightarrow \hat{E}' \rightarrow B)$ over B such that the induced diagram

$$\begin{array}{ccc}
 F \times_B \hat{E} & \xrightarrow{\kappa} & E \times_B \hat{F} \\
 \downarrow & & \downarrow \\
 F' \times_B \hat{E}' & \xrightarrow{\kappa'} & E' \times_B \hat{F}'
 \end{array}$$

commutes. Recall from section 2 that the automorphism group of the trivial pair over the point is the semi-direct product $A_{\text{Par}} = G \ltimes C(G, U(1))$. The automorphism group of the trivial dual pair over the point is $\hat{A}_{\text{Par}} := \hat{G} \ltimes C(\hat{G}, U(1))$. These two groups contain two isomorphic subgroups

$$A_{\text{Par}} \supset G \ltimes (U(1) \times \hat{G}) \xrightarrow{\phi} \hat{G} \ltimes (U(1) \times G) \subset \hat{A}_{\text{Par}}.$$

Here we take the isomorphism $\phi : (g, t, \chi) \mapsto (-\chi, t\langle g, -\chi \rangle, g)$, and $U(1) \times \hat{G}$ is the subgroup of $C(G, U(1))$ consisting of those f such that $f(h) = t\langle h, \chi \rangle$ for some $t \in U(1)$ and $\chi \in \hat{G}$. Similarly, the inclusion $U(1) \times G \subset C(\hat{G}, U(1))$ is understood by the identification $G \cong \hat{\hat{G}}$.

Proposition 7.1 *The automorphism group of the trivial Pontrjagin duality triple over the point is*

$$A_{\text{Pon}} := \{(a, \phi(a)) \mid a \in G \ltimes (U(1) \times \hat{G})\}.$$

Proof : Given $(g, f) \in A_{\text{Par}}$ and $(\chi, F) \in \widehat{A}_{\text{Par}}$, then it is easy to see that

$$\begin{array}{ccc} G \times \mathbf{U}(1) \times \widehat{G} & \xrightarrow{\pi} & G \times \widehat{G} \times \mathbf{U}(1) \\ \downarrow (g,f) \times \chi & & \downarrow g \times (\chi, F) \\ G \times \mathbf{U}(1) \times \widehat{G} & \xrightarrow{\pi} & G \times \widehat{G} \times \mathbf{U}(1) \end{array}$$

commutes if and only if $f(h) = f(0)\langle h, \chi \rangle^{-1}$ and $F(\psi) = f(0)\langle g, \chi \rangle \langle g, \psi \rangle$. This encodes precisely the isomorphism ϕ . \blacksquare

To an A_{Pon} -principal bundle $P \rightarrow B$ we can associate a Pontrjagin duality triple

$$P \times_{A_{\text{Pon}}} \left(\begin{array}{ccccc} & & G \times \widehat{G} \times \mathbf{U}(1) & \xrightarrow{\pi} & G \times \widehat{G} \times \mathbf{U}(1) & \\ & \swarrow & \downarrow & & \downarrow & \searrow \\ G \times \mathbf{U}(1) & & G \times \widehat{G} & & \widehat{G} \times \mathbf{U}(1) & \\ & \swarrow & \downarrow & & \downarrow & \searrow \\ & & G & & \widehat{G} & \\ & & \downarrow & & \downarrow & \\ & & * & & * & \end{array} \right)$$

over B which defines a functor from the category of A_{Pon} -principal fibre bundles over B to the category $\text{Pon}(B)$ of Pontrjagin duality triples over B .

Proposition 7.2 *The functor*

$$(P \rightarrow B) \mapsto P \times_{A_{\text{Pon}}} \left(\begin{array}{ccccc} & & G \times \widehat{G} \times \mathbf{U}(1) & \xrightarrow{\pi} & G \times \widehat{G} \times \mathbf{U}(1) & \\ & \swarrow & \downarrow & & \downarrow & \searrow \\ G \times \mathbf{U}(1) & & G \times \widehat{G} & & \widehat{G} \times \mathbf{U}(1) & \\ & \swarrow & \downarrow & & \downarrow & \searrow \\ & & G & & \widehat{G} & \\ & & \downarrow & & \downarrow & \\ & & * & & * & \end{array} \right)$$

is an equivalence of categories.

Proof : The proof is analogous to the proofs of Proposition 2.1 and Proposition 3.1. We sketch how to define an inverse functor up to equivalence. Let

$$\begin{array}{ccccc} F \times_B \widehat{E} & \xrightarrow{\kappa} & E \times_B \widehat{F} & & \\ \swarrow & & \downarrow & & \swarrow \\ F & & E \times_B \widehat{E} & & \widehat{F} \\ \downarrow q & & \downarrow & & \downarrow \hat{q} \\ E & & \widehat{E} & & \\ \downarrow & & \downarrow & & \\ B & & B & & \end{array}$$

be a Pontrjagin duality triple. For a $b \in B$ we define P_b to be the set of all tuples (e, s, \hat{e}, \hat{s}) with $e \in E|_b, \hat{e} \in \widehat{E}|_b$ and $s \in F|_{E|_b}, \hat{s} \in \widehat{F}|_{\widehat{E}|_b}$ such that $q|_s : s \rightarrow E|_b$ and $\hat{q}|_{\hat{s}} : \hat{s} \rightarrow \widehat{E}|_b$ are homoeomorphisms, and such that κ restricts to homoeomorphisms

$$\{s(e)\} \times \widehat{E}|_b \cong \{e\} \times \hat{s}, \quad s \times \{\hat{e}\} \cong E \times \{\hat{s}(\hat{e})\},$$

where $s(e) := (q|_s)^{-1}(e)$ and $\hat{s}(\hat{e}) := (\hat{q}|_{\hat{s}})^{-1}(\hat{e})$. P_b is a A_{Pon} -torsor subject to the action

$$\begin{aligned} P_b \times \mathbf{BA}_{\text{Pon}} &\rightarrow P_b, \\ ((e, s, \hat{e}, \hat{s}), (a, \phi(a))) &\mapsto ((e, s) \diamond a, (\hat{e}, \hat{s}) \hat{\diamond} \phi(a)), \end{aligned}$$

where \diamond is as in (8), and $\hat{\diamond}$ is defined by the same formula but with G and \hat{G} exchanged. In fact, this action is well-defined, free and transitive. The local trivialisations of the triple induce on $\coprod_{b \in B} P_b$ a topology such that the projection $\coprod_{b \in B} P_b \rightarrow B$ becomes a principal A_{Pon} -fibre bundle. This way we obtain a functor from Pontrjagin duality triples to A_{Pon} -bundles which is up to equivalence inverse to the functor in the proposition. \blacksquare

Let us denote by $\text{Pon}^{\text{full}}(F, E)$ the full subcategory of $\text{Pon}(B)$ whose objects are the extensions of the pair $F \rightarrow E \rightarrow B$. By $\text{Pon}(F, E)$ we denote the proper subcategory of $\text{Pon}(B)$ whose objects are the extensions of $F \rightarrow E \rightarrow B$ and whose morphisms (37) are only those of the form $(a, \hat{a}) = (\text{id}, \hat{a})$, i.e. only those which are the identity on the underlying pair. The task of understanding the extension problem of a pair $F \rightarrow E \rightarrow B$ is to understand the categories $\text{Pon}^{\text{full}}(F, E)$ and $\text{Pon}(F, E)$ or, at least, to understand the structure of their isomorphism classes $[\text{Pon}^{\text{full}}(F, E)]$ and $[\text{Pon}(F, E)]$. We will see next that this is equivalent to understand corresponding categories of A_{Pon} -reductions of the A_{Par} -principal bundle given by the pair (Proposition 2.1).

Let $P \rightarrow B$ be a principal A_{Par} -bundle. The category of reductions $\text{Red}(P)$ of P to A_{Pon} -bundles has as objects commutative diagrams

$$\begin{array}{ccc} P_r & \longrightarrow & P, \\ \downarrow & & \downarrow \\ B & \xrightarrow{=} & B \end{array} \quad (38)$$

where $P_r \rightarrow B$ is a A_{Pon} -principal bundle and $P_r \rightarrow P$ is an A_{Pon} -equivariant map with closed image. A morphism in this category is a commutative diagram

$$\begin{array}{ccccc} P_r & & & & P \\ & \searrow & & & \downarrow \\ & & P_s & \longrightarrow & P \\ & & \downarrow & & \downarrow \\ B & & B & \xrightarrow{=} & B \\ & \searrow & & & \downarrow \\ & & B & \xrightarrow{=} & B \end{array}$$

where $P_r \rightarrow P_s$ is an A_{Pon} -bundle isomorphism over B .

Proposition 7.3 *Let $P \rightarrow B$ be an A_{Par} -bundle, and let $F \rightarrow E \rightarrow B$ be the pair associated to P by the functor of Proposition 2.1. Then there is an equivalence of categories*

$$\text{Red}(P) \simeq \text{Pon}(F, E).$$

Proof : The proof is straight forward. From a reduction P_r of P one obtains an extension X of $F \rightarrow E \rightarrow B$ by associating the trivial Pontrjagin duality triple to P_r and by observing that $F \rightarrow E \rightarrow B$ and the pair of X are isomorphic by the A_{Pon} -equivariant map $P_r \rightarrow P$.

From an extension X of $F \rightarrow E \rightarrow B$ one obtains a reduction of P by the functor from $\text{Pon}(B)$ to A_{Pon} -bundles as constructed in the proof of Proposition 7.2.

Keeping track of the particular morphisms in the two categories one obtains two functors by this procedure which are inverses of each other (up to natural equivalence). ■

The isomorphism classes of the category of reductions are a well-known object. If the quotient map $A_{\text{Par}} \rightarrow A_{\text{Par}}/A_{\text{Pon}}$ has local sections (e.g. if G is compact), then by [Br, Theorem V.3.1] we have that the isomorphism classes $[\text{Red}(P)]$ of reductions of P are naturally bijective to the set of sections $\Gamma(B, P/A_{\text{Pon}})$ of $P/A_{\text{Pon}} \cong P \times_{A_{\text{Par}}} (A_{\text{Par}}/A_{\text{Pon}}) \rightarrow B$ which is a fibre bundle with fibre $A_{\text{Par}}/A_{\text{Pon}}$, and we can identify the space of sections with the A_{Par} -equivariant maps $C(P, A_{\text{Par}}/A_{\text{Pon}})^{A_{\text{Par}}}$. However, if the quotient $A_{\text{Par}}/A_{\text{Pon}}$ behaves badly, i.e. the quotient map does not have local sections, then the homotopy quotients $A_{\text{Par}} // A_{\text{Pon}} := (A_{\text{Par}} \times EA_{\text{Pon}})/A_{\text{Pon}}$ and $P // A_{\text{Pon}} := (P \times EA_{\text{Pon}})/A_{\text{Pon}} \cong P \times_{A_{\text{Par}}} (A_{\text{Par}} // A_{\text{Pon}})$ must be concerned. The technical issue we have to take care of here is that $P \rightarrow P/A_{\text{Pon}}$ is only a principal bundle, i.e. it has local sections, if $A_{\text{Par}} \rightarrow A_{\text{Par}}/A_{\text{Pon}}$ has.

Generalising the proof of [Br, Theorem V.3.1] we obtain the following classification result.

Proposition 7.4 *Let P and $F \rightarrow E \rightarrow B$ be as above.*

i) There are natural bijections

$$[\text{Pon}(F, E)] \cong [\text{Red}(P)] \cong \text{im}(\gamma_*) \cong \text{im}(\varepsilon_*),$$

where $\gamma_ : \Gamma(B, P // A_{\text{Pon}}) \rightarrow \Gamma(B, P/A_{\text{Pon}})$ is the induced map of the canonical map $\gamma : P // A_{\text{Pon}} \rightarrow P/A_{\text{Pon}}$, and $\varepsilon_* : C(P, A_{\text{Par}} // A_{\text{Pon}})^{A_{\text{Par}}} \rightarrow C(P, A_{\text{Par}}/A_{\text{Pon}})^{A_{\text{Par}}}$ is induced by $\varepsilon : A_{\text{Par}} // A_{\text{Pon}} \rightarrow A_{\text{Par}}/A_{\text{Pon}}$.*

ii) If $A_{\text{Par}} \rightarrow A_{\text{Par}}/A_{\text{Pon}}$ has local sections, then γ_ and ε_* are surjective, so*

$$[\text{Pon}(F, E)] \cong [\text{Red}(P)] \cong \Gamma(B, P/A_{\text{Pon}}) \cong C(P, A_{\text{Par}}/A_{\text{Pon}})^{A_{\text{Par}}},$$

Proof : *i)* The first and the last bijection are immediate.

To construct a map from $[\text{Red}(P)]$ to $\text{im}(\gamma_*)$ let $P_r \rightarrow P$ be a reduction of P , it induces a map $\sigma : B = P_r/A_{\text{Pon}} \rightarrow P/A_{\text{Pon}}$. σ is a section of P/A_{Pon} and does not vary inside the isomorphism class of the reduction. We show that σ is in the image of γ_* . Choose any classifying bundle map $P_r \rightarrow EA_{\text{Pon}}$, then we have an equivariant factorisation $P_r \rightarrow P \times EA_{\text{Pon}} \rightarrow P$ of the reduction $P_r \rightarrow P$. It induces a factorisation $B \rightarrow P // A_{\text{Pon}} \rightarrow P/A_{\text{Pon}}$ of σ .

Given a section $\sigma \in \text{im}(\gamma_*)$ we define a reduction P_σ to be the pullback in

$$\begin{array}{ccc} P_\sigma & \longrightarrow & P \\ \downarrow & & \downarrow \\ B & \xrightarrow{\sigma} & P/A_{\text{Pon}} \end{array} .$$

Despite the fact that $P \rightarrow P/A_{\text{Pon}}$ may not have local sections the pullback $P \times EA_{\text{Pon}}$ along γ has (as indicated by the dashed arrow)

$$\begin{array}{ccc} P \times EA_{\text{Pon}} & \longrightarrow & P \\ \downarrow \dashrightarrow & & \downarrow \\ P // A_{\text{Pon}} & \xrightarrow{\gamma} & P/A_{\text{Pon}} \end{array} ,$$

and as σ has a factorisation over γ we find that P_σ is also the pullback in

$$\begin{array}{ccccc} P_\sigma & \longrightarrow & P \times EA_{\text{Pon}} & \longrightarrow & P \\ \downarrow \dashrightarrow & & \downarrow \dashrightarrow & & \downarrow \\ B & \longrightarrow & P // A_{\text{Pon}} & \xrightarrow{\gamma} & P/A_{\text{Pon}} \\ & \searrow \sigma & & & \downarrow \end{array} .$$

Therefore $P_\sigma \rightarrow B$ has local sections and is a principal A_{Pon} -bundle.

The two constructions are easily seen to be inverses of each other.

ii) In case $P \rightarrow P/A_{\text{Pon}}$ is a principal A_{Pon} -bundle it has local sections, and each section in $\Gamma(B, P/A_{\text{Pon}})$ factors over γ . ■

To classify the extensions of a pair $F \rightarrow E \rightarrow B$ up to isomorphism in the category $\text{Pon}(F, E)$ is kind of a too fine way of classifying objects. For instance, even up to isomorphism the trivial pair over the one-point space $*$ does not have a unique extension, as $\Gamma(*, A_{\text{Par}}/A_{\text{Pon}}) = A_{\text{Par}}/A_{\text{Pon}} \neq \{*\}$. A more appropriate classification is the classification up to isomorphism in the full subcategory $\text{Pon}^{\text{full}}(F, E)$ of $\text{Pon}(B)$. By definition, over the one-point space there exists only one Pontrjagin duality triple up to isomorphism.

For an A_{Par} -bundle $P \rightarrow B$, let $\text{Red}^{\text{full}}(P)$ be the “full” category of reductions, i.e. its objects are reductions to A_{Par} -bundles (38) and its morphisms are commutative diagrams

$$\begin{array}{ccccc} P_r & \longrightarrow & P & & \\ \downarrow & \searrow & \downarrow & \searrow & \\ & & P_s & \longrightarrow & P \\ & & \downarrow & & \downarrow \\ B & \longrightarrow & B & & B \\ \downarrow & \searrow & \downarrow & \searrow & \\ & & B & \longrightarrow & B \end{array}$$

(Note: The diagram above is a simplified representation of the commutative diagram in the image, showing the relationships between the various bundles and their reductions.)

where $P_r \rightarrow P_s$ is an A_{Pon} -bundle isomorphism over B and $P \rightarrow P$ is an A_{Par} -bundle automorphism over B . There is a direct analogue of Proposition 7.3 for the two “full” categories:

Proposition 7.5 *Let $P \rightarrow B$ be an A_{Par} -bundle, and let $F \rightarrow E \rightarrow B$ be its associated pair. Then there is an equivalence of categories*

$$\text{Pon}^{\text{full}}(F, E) \simeq \text{Red}^{\text{full}}(P).$$

We denote by $\text{Aut}(P)$ the automorphism group of A_{Par} -bundle automorphisms of P over B . The maps γ_*, ε_* of Proposition 7.4 are $\text{Aut}(P)$ -equivariant with respect to the obvious actions of $\text{Aut}(P)$. Concerning the isomorphism classes of the two “full” categories in the proposition above, we obtain the following result which can be proven by the same means as Proposition 7.4.

Proposition 7.6 *i) There are natural bijections*

$$[\text{Pon}^{\text{full}}(F, E)] \cong [\text{Red}^{\text{full}}(P)] \cong \frac{\text{im}(\gamma_*)}{\text{Aut}(P)} \cong \frac{\text{im}(\varepsilon_*)}{\text{Aut}(P)}.$$

ii) If $A_{\text{Par}} \rightarrow A_{\text{Par}}/A_{\text{Pon}}$ has local sections, then

$$[\text{Pon}^{\text{full}}(F, E)] \cong [\text{Red}^{\text{full}}(P)] \cong \frac{\Gamma(B, P/A_{\text{Pon}})}{\text{Aut}(P)} \cong \frac{C(P, A_{\text{Par}}/A_{\text{Pon}})^{A_{\text{Par}}}}{\text{Aut}(P)}.$$

If $F \rightarrow E \rightarrow B$ is a pair, it has a classifying map $B \rightarrow \text{BA}_{\text{Par}}$ into to the classifying space of A_{Par} -principal bundles which is unique up to homotopy. The inclusion (projection) $A_{\text{Pon}} \hookrightarrow A_{\text{Par}}, (a, \phi(a)) \mapsto a$ induces a map of the classifying spaces $\text{BA}_{\text{Pon}} \rightarrow \text{BA}_{\text{Par}}$, and an extension of the pair exists if and only if there exists a map $B \rightarrow \text{BA}_{\text{Pon}}$ such that the diagram

$$\begin{array}{ccc} & & \text{BA}_{\text{Pon}} \\ & \nearrow & \downarrow \\ B & \longrightarrow & \text{BA}_{\text{Par}} \end{array} \quad (39)$$

commutes up to homotopy. We have a commutative diagram for the set of isomorphism classes of triples $[\text{Pon}(B)]$ over B and the set of isomorphism classes of pairs $[\text{Par}(B)]$ over B

$$\begin{array}{ccc} [\text{Pon}(B)] & \xrightarrow{\cong} & [B, \text{BA}_{\text{Pon}}] \\ \downarrow \text{forget} & & \downarrow \\ [\text{Par}(B)] & \xrightarrow{\cong} & [B, \text{BA}_{\text{Par}}] \end{array}$$

where we put the homotopy classes of maps from B to BA_{Par} or BA_{Pon} in the right column and the isomorphisms are given by sending an isomorphism class to the homotopy class of a classifying map. Let $[F, E]$ denote any element of $[\text{Par}(B)]$, then there

is a canonical correspondence: $\text{forget}^{-1}([F, E]) \cong [\text{Pon}^{\text{full}}(F, E)]$, and in case of *ii*) of Proposition 7.6 we end up with a commutative diagram

$$\begin{array}{ccccc}
& & [\text{Pon}^{\text{full}}(F, E)] & \xrightarrow{\cong} & \frac{C(P, A_{\text{Par}}/A_{\text{Pon}})^{A_{\text{Par}}}}{\text{Aut}(P)} \\
& \nearrow & \downarrow & & \downarrow u_* \\
\text{forget}^{-1}([F, E]) & \hookrightarrow & [\text{Pon}(B)] & \xrightarrow{\cong} & [B, \text{BA}_{\text{Pon}}] \\
\downarrow & & \downarrow \text{forget} & & \downarrow \\
\{[F, E]\} & \hookrightarrow & [\text{Par}(B)] & \xrightarrow{\cong} & [B, \text{BA}_{\text{Par}}]
\end{array}$$

where $u : P/A_{\text{Pon}} \rightarrow \text{BA}_{\text{Pon}}$ is the classifying map of the bundle $P \rightarrow P/A_{\text{Pon}}$ and $u_*([\sigma]) := [u \circ \sigma]$.

Example 7.1 Let $G := S^1$ be the circle, so $\widehat{G} \cong \mathbb{Z}$. Then the inclusion $A_{\text{Pon}} \hookrightarrow A_{\text{Par}}$ is a homotopy equivalence. It follows that the map $\text{BA}_{\text{Pon}} \rightarrow \text{BA}_{\text{Par}}$ between the classifying spaces is a weak homotopy equivalence.

So if one considers pairs $F \rightarrow E \rightarrow B$ over a CW-complex B , then up to isomorphism in $\text{Pon}^{\text{full}}(F, E)$ there always exists a unique extension to a Pontrjagin duality triple. (Cp. also Example 7.2)

Next we try to understand, i.e. simplify, the quotient $\frac{C(P, A_{\text{Par}}/A_{\text{Pon}})^{A_{\text{Par}}}}{\text{Aut}(P)}$. Let us assume the pair $F \rightarrow E \rightarrow B$ has an extension, so the A_{Par} -bundle $P \rightarrow B$ has a reduction to a A_{Pon} -bundle $Q \rightarrow B$. There are canonical identifications

$$C(P, A_{\text{Par}}/A_{\text{Pon}})^{A_{\text{Par}}} \cong C(Q, A_{\text{Par}}/A_{\text{Pon}})^{A_{\text{Pon}}}$$

and

$$\text{Aut}(P) \cong C(Q, A_{\text{Par}}^{\text{ad}})^{A_{\text{Pon}}},$$

where $A_{\text{Pon}} \ni b$ acts from the right on $A_{\text{Par}}^{\text{ad}} := A_{\text{Par}} \ni a$ by conjugation: $a \cdot b := b^{-1}ab$.

Because the normal subgroup $(\text{U}(1) \times \widehat{G}) \subset G \times (\text{U}(1) \times \widehat{G}) = A_{\text{Pon}}$ acts trivially on $A_{\text{Par}}/A_{\text{Pon}}$, we have a further identification

$$C(Q, A_{\text{Par}}/A_{\text{Pon}})^{A_{\text{Pon}}} \cong C(E, A_{\text{Par}}/A_{\text{Par}})^G,$$

where we used that $Q/(\text{U}(1) \times \widehat{G}) \cong E$. Topologically, the quotient $A_{\text{Par}}/A_{\text{Pon}}$ is isomorphic to $C_*(G, \text{U}(1))/\widehat{G}$, where we denoted by $C_*(G, \text{U}(1))$ the base-point preserving continuous functions. In fact, we have a diagram of inclusions and quotients of

topological spaces

$$\begin{array}{ccc}
G \times U(1) \hookrightarrow G \times (U(1) \times \widehat{G}) & \xleftrightarrow{\quad} & \widehat{G} \\
\downarrow & & \downarrow \\
G \times U(1) \hookrightarrow G \times C(C, U(1)) & \xleftrightarrow{\quad} & C_*(G, U(1)) \\
\downarrow & & \downarrow \\
\frac{G \times C(C, U(1))}{G \times (U(1) \times \widehat{G})} & \xrightarrow{\cong} & \frac{C_*(G, U(1))}{\widehat{G}}
\end{array}$$

The action of G on $A_{\text{Par}}/A_{\text{Pon}}$ induces an action of G on $C_*(G, U(1))/\widehat{G}$ which turns out to be the action induced by the shift action of G on $C_*(G, U(1))$:

$$f\widehat{G} \cdot g := (f(g)^{-1}f(g + \cdot))\widehat{G} \in C_*(G, U(1))/\widehat{G}.$$

Let $Q \ni q \mapsto (g_q, f_q) \in A_{\text{Par}}^{\text{ad}}$ be an A_{Pon} -equivariant map, i.e.

$$\begin{aligned}
(g_{q(h,z,\chi)}, f_{q(h,z,\chi)}) &= (h, z, \chi)^{-1}(g_q, f_q)(h, z, \chi) \\
&= (-h, z^{-1}\langle h, \chi \rangle, -\chi)(g_q, f_q)(h, z, \chi) \\
&= (g_q, \langle g_q, \chi \rangle^{-1}f_q(h + \cdot)).
\end{aligned}$$

In particular, $q \mapsto g_q$ is A_{Pon} -invariant and $q \mapsto (g_q, f_q)$ is $U(1)$ -invariant. Furthermore the \widehat{G} -equivariance only contributes by the scalar $q \mapsto \langle g_q, \chi \rangle^{-1} \in U(1)$ which itself acts trivially on $C(E, A_{\text{Par}}/A_{\text{Pon}})^G$, and the mapping $q \mapsto f_q(0)^{-1}f_q \in C_*(G, U(1))$ is G -equivariant. So by use of the decomposition $G \times C(G, U(1)) = G \times (U(1) \times C_*(G, U(1)))$ we can identify

$$\frac{C(E, C_*(G, U(1))/\widehat{G})^G}{C(Q, (G \times C(G, U(1)))^{\text{ad}} A_{\text{Pon}})^G} \cong \frac{C(E, C_*(G, U(1))/\widehat{G})^G}{C(B, G) \times C(E, C_*(G, U(1)))^G}.$$

Lemma 7.1 *Inside $C(E, C_*(G, U(1))/\widehat{G})^G$ the orbits of $C(B, G) \times C(E, C_*(G, U(1)))^G$ coincide with the orbits of $C(E, C_*(G, U(1)))^G$, so*

$$\frac{C(E, C_*(G, U(1))/\widehat{G})^G}{C(B, G) \times C(E, C_*(G, U(1)))^G} \cong \frac{C(E, C_*(G, U(1))/\widehat{G})^G}{C(E, C_*(G, U(1)))^G}.$$

Proof : Let $e \mapsto F_e \in C_*(G, U(1))/\widehat{G}$ be G -equivariant, and let $(b \mapsto f_b) \in C(B, G)$. For each F_e choose (non-continuously) a $\bar{F}_e \in C_*(G, U(1))$ such that $\bar{F}_e \widehat{G} = F_e$. Denote by $p : E \rightarrow B$ the projection, then

$$\begin{aligned}
(f \cdot F)_e &= (\bar{F}_e(f_{p(e)})^{-1}\bar{F}_e(f_{p(e)} + \cdot))\widehat{G} \\
&= \underbrace{(\bar{F}_e(f_{p(e)})^{-1}\bar{F}_e(f_{p(e)} + \cdot)\bar{F}_e(-)^{-1})}_{=: d(\bar{F}_e)(f_{p(e)}, -)}\widehat{G} F_e
\end{aligned}$$

here $d : C(G, U(1)) \rightarrow C(G \times G, U(1))$ is the boundary operator of group cohomology which has kernel \widehat{G} , so it factors (continuously)

$$\begin{array}{ccc} C(G, U(1)) & \xrightarrow{d} & C(G \times G, U(1)) , \\ \downarrow & \nearrow d & \\ \frac{C(G, U(1))}{\widehat{G}} & & \end{array}$$

and we have $d(\overline{F}_e) = \underline{d}(F_e)$. Therefore $f \cdot F$ differs from F by action of

$$e \mapsto \underline{d}(F_e)(f_{p(e)} + \cdot) \in C_*(G, U(1))$$

which is easily checked to be G -equivariant. This proves the lemma. \blacksquare

Corollary 7.1 *If a pair $F \rightarrow E \rightarrow B$ has an extension, then there is a bijection*

$$[\text{Pon}^{\text{full}}(F, E)] \cong \frac{C(E, C_*(G, U(1))/\widehat{G})^G}{C(E, C_*(G, U(1)))^G}.$$

Example 7.2 *Let $G := S^1$ be the circle group. Then $\widehat{G} \cong \mathbb{Z}$ and the inclusion $\mathbb{Z} \hookrightarrow C_*(S^1, U(1))$ is a homotopy equivalence. $C_*(S^1, U(1))/\mathbb{Z}$ is G -equivariantly isomorphic to the null-homotopy functions which leads to a G -equivariant section*

$$C_*(S^1, U(1))/\mathbb{Z} \xrightarrow{\cong} C_*(S^1, U(1))_{\text{null}} \xrightarrow{\subset} C_*(S^1, U(1))$$

of the quotient map $C_*(S^1, U(1)) \rightarrow C_*(S^1, U(1))/\mathbb{Z}$. Therefore the quotient $\frac{C(E, C_*(S^1, U(1))/\mathbb{Z})^{S^1}}{C(E, C_*(S^1, U(1)))^{S^1}}$ consists of a single element only. This means that an extension of a pair is unique up to isomorphism.

Example 7.3 *Let $G := \mathbb{Z}$ be the integers, so $\widehat{G} \cong S^1$ is the circle group. The quotient map $C_*(\mathbb{Z}, U(1)) \rightarrow C_*(\mathbb{Z}, U(1))/S^1$ is a canonically trivialisable S^1 -bundle*

$$\begin{array}{ccc} C_*(\mathbb{Z}, U(1)) & \xrightarrow{\cong} & \prod_{\mathbb{Z} \setminus \{0\}} U(1) \xrightarrow{\cong_{\theta}} \prod_{\mathbb{Z} \setminus \{0,1\}} U(1) \times S^1 , \\ \downarrow & & \downarrow \qquad \qquad \qquad \downarrow \\ \frac{C_*(\mathbb{Z}, U(1))}{S^1} & \xrightarrow{\cong} & \prod_{\mathbb{Z} \setminus \{0\}} U(1) \xrightarrow{\cong} \prod_{\mathbb{Z} \setminus \{0,1\}} U(1) \end{array} \quad (40)$$

where $\theta : (z_k)_k \mapsto \left(\left(\frac{z_k}{z_1^k} \right)_k, z_1 \right)$, and the action of $S^1 \ni t$ on the product $\prod_{\mathbb{Z} \setminus \{0\}} U(1) \ni (z_k)_k$ is $(z_k)_k \cdot t = (z_k t^k)_k$. The action of $\mathbb{Z} \ni n$ on $\prod_{\mathbb{Z} \setminus \{0\}} U(1) \ni (z_k)_k$ is given by $(z_k)_k \cdot n = \left(\frac{z_{k+n}}{z_n^k} \right)_k$. The induced action of $\mathbb{Z} \ni n$ on $\prod_{\mathbb{Z} \setminus \{0,1\}} U(1) \times S^1 \ni ((z_k)_k, s)$ is

$$((z_k)_k, s) \cdot n = \left(\left(\frac{z_{k+n} z_n^{k-1}}{z_{1+n}^k} \right)_k, \frac{z_{1+n}}{z_n} s \right).$$

Let $E \rightarrow B$ be a \mathbb{Z} -bundle, then we are faced with the \mathbb{Z} -equivariant lifting problem: Does there exist a $\psi : E \rightarrow S^1$ such that

$$\begin{array}{ccc} & & \prod_{\mathbb{Z} \setminus \{0,1\}} \mathbb{U}(1) \times S^1 \\ & \nearrow^{\varphi \times \psi} & \downarrow \\ E & \xrightarrow{\varphi} & \prod_{\mathbb{Z} \setminus \{0,1\}} \mathbb{U}(1) \end{array} \quad (41)$$

commutes \mathbb{Z} -equivariantly for a given \mathbb{Z} -equivariant φ ? The equivariance of $\varphi = (\varphi_k)_k$ gives

$$\varphi_k(e \cdot n) = \frac{\varphi_{k+n}(e) \varphi_n(e)^{k-1}}{\varphi_{1+n}(e)^k}, \quad e \in E, n \in \mathbb{Z},$$

which for $k = 2$ implies

$$\varphi_{2+n}(e) = \varphi_2(e \cdot n) \varphi_{1+n}(e)^2 \varphi_n(e)^{-1}. \quad (42)$$

As $\varphi_0 = \varphi_1 = 1$, this shows that all φ_k can be expressed in terms of φ_2 , and, conversely, any $\varphi_2 : E \rightarrow \mathbb{U}(1)$ determines a φ via (42). If ψ exists, then by equivariance $\psi(e \cdot n) = \varphi_{1+n}(e) / \varphi_n(e) \psi(e)$. In particular

$$\psi(e \cdot 1) = \varphi_2(e) \psi(e) \quad (43)$$

(note again $\varphi_0 = \varphi_1 = 1$). The solvability of the lifting problem (41) is equivalent to to solvability of (43), and it depends on E whether or not this equality can be always solved:

- i) Let $E := \mathbb{R} \rightarrow T := \mathbb{R}/\mathbb{Z}$ be the universal covering of the 1-torus $T := \mathbb{R}/\mathbb{Z}$. Then (43) can always be solved by the formula

$$\psi(x) := \begin{cases} \prod_{k=0}^{n-1} \varphi_2(\varepsilon + k) \eta(\varepsilon), & \text{if } x = \varepsilon + n, \varepsilon \in [0, 1), n \in \{0, 1, 2, \dots\} \\ \prod_{k=1}^n \varphi_2(\varepsilon - k)^{-1} \eta(\varepsilon), & \text{if } x = \varepsilon - n, \varepsilon \in [0, 1), n \in \{1, 2, \dots\} \end{cases} ,$$

where $\eta : [0, 1] \rightarrow \mathbb{U}(1)$ is any continuous map satisfying $\eta(1) = \varphi_2(0) \eta(0)$. Therefore the pair

$$\mathbb{R} \times \mathbb{U}(1) \rightarrow \mathbb{R} \rightarrow T$$

has a unique extension.

- ii) Let $E := T \times \mathbb{R} \rightarrow T^2 := B$. Let $\varphi_2 : T \times \mathbb{R} \rightarrow T \xrightarrow{k} \mathbb{U}(1)$ be a map with winding number $k \in \mathbb{Z}$. The equality $\psi(t, x+1) = \varphi_2(t, x) \psi(t, x)$ can be solved iff $k = 0$, and each $k \in \mathbb{Z}$ defines a class in the quotient $\frac{C(E, C_*(G, \mathbb{U}(1)) / \widehat{G})^G}{C(E, C_*(G, \mathbb{U}(1)))^G}$. Therefore the pair

$$T \times \mathbb{R} \times \mathbb{U}(1) \rightarrow T \times \mathbb{R} \rightarrow T^2$$

has \mathbb{Z} -many extensions.

Example 7.4 Let $G := O(1) \cong \widehat{G}$ be the group with two elements. We have a commutative diagram

$$\begin{array}{ccc} C_*(O(1), U(1)) & \xrightarrow{\cong} & U(1) , \\ \downarrow & & \downarrow 2 \\ \frac{C_*(O(1), U(1))}{O(1)} & \xrightarrow{\cong} & U(1) \end{array}$$

where $2 : U(1) \rightarrow U(1)$ is the two-fold covering. The induced action by the non-trivial element of G on the two occurring $U(1)$ is just complex conjugation (inversion).

Let $E := S^n \rightarrow \mathbb{R}P^n =: B$ be the two-fold covering, so the non-trivial element of $O(1)$ acts by mapping a point of S^n to its antipodal point.

Then the two constant maps $S^n \rightarrow U(1) : x \mapsto \pm 1$ to the fixpoints of the conjugation action are G -equivariant. Clearly, $x \mapsto +1$ can be lifted G -equivariantly:

$$\begin{array}{ccc} & & U(1) \\ & \nearrow^{x \mapsto +1} & \downarrow 2 \\ S^n & \xrightarrow{x \mapsto +1} & U(1) \end{array}$$

which is not the case for $x \mapsto -1$. In fact, these two maps represent two classes such that

$$\frac{C(S^n, C_*(O(1), U(1)) / \widehat{O}(1))^{O(1)}}{C(S^n, C_*(O(1), U(1)))^{O(1)}} \cong \{[x \mapsto +1], [x \mapsto -1]\}$$

Therefore the pair

$$S^n \times U(1) \rightarrow S^n \rightarrow \mathbb{R}P^n$$

has two non-isomorphic extensions (cp. Example 3.1).

Let us consider the case of ring pairs $F_0 \rightarrow E_0 \rightarrow B$, so $E_0 = B \times G$ is the trivial bundle. In this case G -equivariant maps on E_0 can be identified with (non-equivariant) maps on the base, so

$$[\text{Pon}^{\text{full}}(F_0, E_0)] \cong \frac{C(B, C_*(G, U(1)) / \widehat{G})}{C(B, C_*(G, U(1)))}. \quad (44)$$

Example 7.5 Let $G := \mathbb{Z}$ be the integers, so $\widehat{G} \cong \mathbb{Z}$. As (40) is a trivial bundle the quotient (44) is trivial. Therefore all ring pairs $F_0 \rightarrow B \times \mathbb{Z} \rightarrow B$ admit a unique extensions.

Let us introduce some self-explaining terminology.

Definition 7.2 A **dual** of a pair $F \rightarrow E \rightarrow B$ is a dual pair $\widehat{F} \rightarrow \widehat{E} \rightarrow B$ such that the diagram

$$\begin{array}{ccc}
 F & & \widehat{F} \\
 \searrow & & \swarrow \\
 & E & \widehat{E} \\
 & \searrow & \swarrow \\
 & & B
 \end{array} \tag{45}$$

admits an extension. In that case we also call the pair a **dual** of the dual pair.

For a pair and a dual of it the discussion of the extension problem (45) is to answer the question of how many different ways there are for being dual to each other.

Let $\text{Pon}^{\text{top}}(F, E, \widehat{F}, \widehat{E})$ be the full subcategory of $\text{Pon}(E, \widehat{E})$ (p. 19) whose objects are extensions of (45). By Corollary 6.1, we know that the isomorphism classes of $\text{Pon}(E, \widehat{E})$ are a $\check{H}^1(B, \underline{\mathbb{U}(1)})$ -torsor. The projections $p : E \rightarrow B$ and $\hat{p} : \widehat{E} \rightarrow B$ induce homomorphisms $p^* : \check{H}^1(B, \underline{\mathbb{U}(1)}) \rightarrow \check{H}^1(E, \underline{\mathbb{U}(1)})$ and $\hat{p}^* : \check{H}^1(B, \underline{\mathbb{U}(1)}) \rightarrow \check{H}^1(\widehat{E}, \underline{\mathbb{U}(1)})$. The subgroup $N(E, \widehat{E}) := \ker(p^*) \cap \ker(\hat{p}^*) \subset \check{H}^1(B, \underline{\mathbb{U}(1)})$ still acts on the set

$$[\text{Pon}^{\text{top}}(F, E, \widehat{F}, \widehat{E})] \subset [\text{Pon}(E, \widehat{E})]$$

of isomorphism classes of $\text{Pon}^{\text{top}}(F, E, \widehat{F}, \widehat{E})$, and this action is still free and transitive:

Corollary 7.2 The set of isomorphism classes $[\text{Pon}^{\text{top}}(F, E, \widehat{F}, \widehat{E})]$ is a $N(E, \widehat{E})$ -torsor.

Let us denote by $\text{Pon}^{\text{full}}(F, E, \widehat{F}, \widehat{E})$ the full subcategory of $\text{Pon}(B)$ whose objects are the extensions of (45). The inclusion functor $\text{Pon}^{\text{top}}(F, E, \widehat{F}, \widehat{E}) \hookrightarrow \text{Pon}^{\text{full}}(F, E, \widehat{F}, \widehat{E})$ induces a surjection on isomorphism classes

$$[\text{Pon}^{\text{top}}(F, E, \widehat{F}, \widehat{E})] \twoheadrightarrow [\text{Pon}^{\text{full}}(F, E, \widehat{F}, \widehat{E})]$$

which is $N(E, \widehat{E})$ -equivariant for the descended action of $N(E, \widehat{E})$ on $[\text{Pon}^{\text{full}}(F, E, \widehat{F}, \widehat{E})]$. In fact, this action is well-defined and therefore still transitive, and we shall describe its stabiliser group next.

The cup product $\cup : \check{H}^0(B, \underline{\mathbb{G}}) \times \check{H}^1(B, \widehat{\mathbb{G}}) \rightarrow \check{H}^1(B, \underline{\mathbb{U}(1)})$ and the class $[\widehat{E}] \in \check{H}^1(B, \widehat{\mathbb{G}})$ of $\widehat{E} \rightarrow B$ define a map $_ \cup [\widehat{E}] : \check{H}^0(B, \underline{\mathbb{G}}) \rightarrow \check{H}^1(B, \underline{\mathbb{U}(1)})$. Its image $\text{im}(_ \cup [\widehat{E}])$ is contained in $\ker(\hat{p}^*)$, as the pullback $\hat{p}^* \widehat{E} \rightarrow \widehat{E}$ is trivialisable. Dually, $\text{im}([E] \cup _) \subset \ker(p^*)$ for the cup product $\cup : \check{H}^1(B, \underline{\mathbb{G}}) \times \check{H}^0(B, \widehat{\mathbb{G}}) \rightarrow \check{H}^1(B, \underline{\mathbb{U}(1)})$. Let us denote by $M(E, \widehat{E})$ the following subgroup

$$\begin{aligned}
 M(E, \widehat{E}) &:= (\text{im}(_ \cup [\widehat{E}]) \cap \ker(p^*)) + (\text{im}([E] \cup _) \cap \ker(\hat{p}^*)) \\
 &= (\text{im}(_ \cup [\widehat{E}]) + \text{im}([E] \cup _)) \cap N(E, \widehat{E}) \\
 &\subset N(E, \widehat{E}).
 \end{aligned}$$

Proposition 7.7 The stabiliser of the $N(E, \widehat{E})$ -action on $[\text{Pon}^{\text{full}}(F, E, \widehat{F}, \widehat{E})]$ is $M(E, \widehat{E})$, so $[\text{Pon}^{\text{full}}(F, E, \widehat{F}, \widehat{E})]$ is a $N(E, \widehat{E})/M(E, \widehat{E})$ -torsor.

Proof : Note first that we have canonical identifications

$$\check{H}^0(B, \underline{G}) = C(B, G) \cong \text{Aut}(E), \text{ and } \check{H}^0(B, \widehat{G}) = C(B, \widehat{G}) \cong \text{Aut}(\widehat{E}),$$

where $\text{Aut}(E), \text{Aut}(\widehat{E})$ are the bundle automorphisms over B . Pick an arbitrary $\varphi \in C(B, G)$, we denote by φ_E the corresponding bundle automorphism of E . Dually, $\hat{\varphi}_{\widehat{E}}$ is the bundle automorphism of \widehat{E} corresponding to some $\hat{\varphi} \in C(B, \widehat{G})$. An explicit bundle $\varphi \cup \widehat{E}$ representing the class $\varphi \cup [\widehat{E}]$ is given by the pullback

$$\begin{array}{ccc} \varphi \cup \widehat{E} & \longrightarrow & F_{\widehat{E}} \\ \downarrow & & \downarrow \\ B & \xrightarrow{\text{id} \times \varphi} & B \times G \end{array} \quad , \quad (46)$$

where $F_{\widehat{E}} := \widehat{E} \times_{\widehat{G}} (G \times \text{U}(1))$ as before. We have a canonical isomorphism $(-\varphi) \cup \widehat{E}^{\text{op}} \cong \varphi \cup \widehat{E}$. Dually, we define the pullback $E \cup \hat{\varphi}$ which represents the class $[E] \cup \hat{\varphi}$. Now, let

$$\begin{array}{ccccc} & F \times_B \widehat{E} & \xrightarrow{\kappa} & E \times_B \widehat{F} & \\ & \swarrow & & \searrow & \\ F & & E \times_B \widehat{E} & & \widehat{F} \\ & \swarrow & & \searrow & \\ & E & & \widehat{E} & \\ & \searrow p & & \swarrow \hat{p} & \\ & & B & & \end{array} \quad (47)$$

be a Pontrjagin duality triple, and consider the induced triple given by tensoring the $\text{U}(1)$ -bundles with the $\text{U}(1)$ -bundle $(\varphi \cup \widehat{E}) \otimes (E \cup \hat{\varphi}) \rightarrow B$:

$$\begin{array}{ccccc} & F \otimes p^*(\varphi \cup \widehat{E} \otimes E \cup \hat{\varphi}) \times_B \widehat{E} & \xrightarrow{\kappa \otimes \text{flip}} & E \times_B \widehat{F} \otimes \hat{p}^*(\varphi \cup \widehat{E} \otimes E \cup \hat{\varphi}) & \\ & \swarrow & & \searrow & \\ F \otimes p^*(\varphi \cup \widehat{E} \otimes E \cup \hat{\varphi}) & & E \times_B \widehat{E} & & \widehat{F} \otimes \hat{p}^*(\varphi \cup \widehat{E} \otimes E \cup \hat{\varphi}) \\ & \swarrow & & \searrow & \\ & E & & \widehat{E} & \\ & \searrow p & & \swarrow \hat{p} & \\ & & B & & \end{array} \quad . (48)$$

To prove the proposition it is sufficient to prove that $(-\varphi)_E : E \rightarrow E$ and $(-\hat{\varphi})_{\widehat{E}} : \widehat{E} \rightarrow \widehat{E}$ can be extended to an isomorphism from (47) to (48) in $\text{Pon}(B)$.

First note that we have a canonical trivialisation $\hat{p}^*(\varphi \cup \widehat{E}) \cong \widehat{E} \times \text{U}(1)$ which is induced by the canonical map

$$\begin{array}{ccc} & & F_{\widehat{E}} \\ & \searrow [\text{id} \times (\varphi \circ p) \times 1] & \downarrow \\ \widehat{E} & \xrightarrow{\hat{p}} B & \xrightarrow{\text{id} \times \varphi} B \times G \end{array}$$

Dually, we have a canonical identification $p^*(E \cup \hat{\varphi}) \cong E \times \mathbf{U}(1)$. We define a bundle isomorphism over $(-\varphi)_E$ by

$$\begin{array}{ccc}
F \otimes p^*(\varphi \cup \hat{E}) & \xrightarrow{\cong} & F \otimes p^*((-\varphi) \cup \hat{E}^{\text{op}}) & \longrightarrow & F \\
\downarrow & & \downarrow & \nearrow \scriptstyle \varrho_\kappa & \downarrow \\
E & \xrightarrow{(-\varphi)_E} & F \times_B F_{\hat{E}^{\text{op}}} & & E
\end{array}$$

where ϱ_κ is from (33), and the dotted arrow sends an element $[x, (e, [\hat{e}^{\text{op}}, -\varphi(p(e)), t])]$ to $(x, [\hat{e}^{\text{op}}, -\varphi(p(e)), t])$ which is not well-defined, but its composition with ϱ_κ is. Dually, we obtain a bundle isomorphism $\hat{F} \otimes \hat{p}^*(E \cup \hat{\varphi}) \rightarrow \hat{F}$ over $(-\hat{\varphi})_{\hat{E}}$. It is then a straight forward calculation to show that the diagram

$$\begin{array}{ccc}
F \otimes p^*(\varphi \cup E \otimes E \cup \hat{\varphi}) \times_B \hat{E} & \xrightarrow{\kappa \otimes \text{flip}} & E \times_B \hat{F} \otimes \hat{p}^*(E \cup \hat{\varphi} \otimes E \cup \hat{\varphi}) \\
\downarrow \cong & & \downarrow \cong \\
F \times_B \hat{E} & \xrightarrow{\kappa} & E \times_B \hat{F}
\end{array}$$

commutes. This proves the proposition. \blacksquare

Remark 7.1 Let $\text{Pon}^{\text{full}}(E, \hat{E})$ be the full subcategory of $\text{Pon}(B)$ consisting of extensions of $E \rightarrow B \leftarrow \hat{E}$. In fact, the proof of Proposition 7.7 shows that $[\text{Pon}^{\text{full}}(E, \hat{E})]$ is a torsor for the group $\check{H}^1(B, \mathbf{U}(1)) / (\text{im}(- \cup [\hat{E}]) + \text{im}([E] \cup -))$. The set $[\text{Pon}^{\text{full}}(E, \hat{E})]$ decomposes into a disjoint union

$$[\text{Pon}^{\text{full}}(E, \hat{E})] = \coprod [\text{Pon}^{\text{full}}(F, E, \hat{F}, \hat{E})]$$

such that each component is a torsor for the group $N(E, \hat{E}) / M(E, \hat{E})$.

Example 7.6 Let $B := \Sigma$ be a connected, two-dimensional, orientable, closed manifold, then $\check{H}^2(\Sigma, \mathbf{U}(1)) \cong H^3(\Sigma, \mathbf{Z}) = 0$, so all choices of bundles $E \rightarrow \Sigma \leftarrow \hat{E}$ can be extended to a Pontrjagin duality triple. Let $G := S^1$ be the circle group, so $\hat{G} \cong \mathbf{Z}$, and let $E_1 \rightarrow \Sigma$ be an S^1 -bundle representing a generator of $\check{H}^1(\Sigma, \mathbf{U}(1)) \cong H^2(\Sigma, \mathbf{Z}) \cong \mathbf{Z}$. Then $[E_1] \cup - : \check{H}^0(\Sigma, \mathbf{Z}) \rightarrow \check{H}^1(\Sigma, \mathbf{U}(1))$ is surjective. In fact, it is the identity map after identifying $\check{H}^0(\Sigma, \mathbf{Z}) \cong \mathbf{Z}$ and $\check{H}^1(\Sigma, \mathbf{U}(1)) \cong \mathbf{Z}$. Therefore

$$\check{H}^1(\Sigma, \mathbf{U}(1)) / (\text{im}(- \cup [\hat{E}]) + \text{im}([E_1] \cup -)) = 0$$

for any \mathbf{Z} -bundle $\hat{E} \rightarrow \Sigma$, and all diagrams $E_1 \rightarrow \Sigma \leftarrow \hat{E}$ have (up to isomorphism) a unique extension in $\text{Pon}^{\text{full}}(E_1, \hat{E})$.

Example 7.7 Let $T := \mathbb{R}/\mathbb{Z}$, and let $B := T^2$ be the two-dimensional torus. Let $G := S^1$ be the circle group, so $\widehat{G} \cong \mathbb{Z}$. A $U(1)$ -bundle representing the generator of $H^2(T^2, \mathbb{Z})$ is

$$E_1 := F_{\mathbb{R}} := \mathbb{R} \times_{\mathbb{Z}} (S^1 \times U(1)) .$$

$$\downarrow$$

$$T^2 = T \times S^1$$

Let $\widehat{E}_{m,n} \rightarrow T^2$ be the \mathbb{Z} -bundle given by pullback

$$\begin{array}{ccc} \widehat{E}_{m,n} & \longrightarrow & \mathbb{R} \\ \downarrow & & \downarrow \\ T^2 & \xrightarrow{f_{m,n}} & T \end{array}$$

along the map $f_{m,n} : (x, y) \mapsto mx + ny$ which represents $(m, n) \in \mathbb{Z} \oplus \mathbb{Z} \cong H^1(T^2, \mathbb{Z})$. For an integer k consider the extension problem

$$\begin{array}{ccc} E_1^{\otimes k} & & \widehat{E}_{m,n} \\ & \searrow & \swarrow \\ & T^2 & \end{array} \quad (49)$$

The image of $[E_1^{\otimes k}] \cup _$ in $\check{H}^1(T^2, \underline{U}(1))$ is the subgroup corresponding to $k\mathbb{Z} \subset \mathbb{Z} \cong H^2(T^2, \mathbb{Z})$. To understand the image of $_ \cup [\widehat{E}_{m,n}]$ in $\check{H}^1(T^2, \underline{U}(1))$ we have to consider the diagram of pull-backs according to (46)

$$\begin{array}{ccccc} \varphi \cup \widehat{E}_{m,n} & \longrightarrow & \widehat{E}_{m,n} \times_{\mathbb{Z}} (S^1 \times U(1)) & \longrightarrow & \mathbb{R} \times_{\mathbb{Z}} (S^1 \times U(1)) \\ \downarrow & & \downarrow & & \downarrow \\ T^2 & \xrightarrow{\text{id} \times \varphi} & T^2 \times S^1 & \xrightarrow{f_{m,n} \times \text{id}} & T \times S^1 \\ & \searrow & & \nearrow & \\ & & & & f_{m,n} \times \varphi \end{array}$$

For $\varphi = \text{pr}_2 : (x, y) \mapsto y$ it follows that $\text{pr}_2 \cup \widehat{E}_{m,n}$ is isomorphic to $E_1^{\otimes m}$, and for $\varphi = \text{pr}_1 : (x, y) \mapsto x$ it follows that $\text{pr}_1 \cup \widehat{E}_{m,n}$ is isomorphic to $E_1^{\otimes (-n)}$. Therefore, if any two of the three numbers k, m and n are coprime, then

$$\text{im}([E_1^{\otimes k}] \cup _) + \text{im}(_ \cup [\widehat{E}_{m,n}]) = \check{H}^1(T^2, \underline{U}(1))$$

which means that in this case the extension problem (49) has a solution which is unique (up to isomorphism in $\text{Pon}^{\text{full}}(E_1^{\otimes k}, \widehat{E}_{m,n})$).

8 The Fourier transform

In this section we show that the Fourier transform implements a functor from Pontrjagin duality triples to the category of tuples consisting of two isomorphic Hilbert C^* -modules. Let us explain what is meant by this.

Let $\text{Pon}(B)$ denote the category of Pontrjagin duality triples over a space B , and let Pon denote the associated total category of all Pontrjagin duality triples over all spaces. I.e. a morphism from a Pontrjagin duality triple X over B to a Pontrjagin duality triple X' over B' consists of a continuous map $f : B \rightarrow B'$ and an isomorphism $X \rightarrow f^*X'$ in $\text{Pon}(B)$. Let $(0 \cong 1)$ denote the category with two objects $0, 1$ and two non-trivial morphisms $0 \rightarrow 1$ and $1 \rightarrow 0$ which are inverse to each other. Let $\text{Hilb-}C^*\text{-mod}$ denote the category of Hilbert C^* -modules. I.e. a morphism from (H, C) to (H', C') consists of a continuous linear map $H \rightarrow H'$ of Banach spaces and a morphism of C^* -algebras $C \rightarrow C'$ such that the obvious conditions are satisfied. In this section we define a bifunctor

$$(\text{Pon})^{\text{op}} \times (0 \cong 1) \rightarrow \text{Hilb-}C^*\text{-mod},$$

equivalently, a functor

$$(\text{Pon})^{\text{op}} \rightarrow (\text{Hilb-}C^*\text{-mod})^{(0 \cong 1)},$$

where the target category is a functor category. This functor will be constructed such that the trivial Pontrjagin duality triple over the one-point space is mapped to the classical isomorphism of Pontrjagin duality:

$$\left(\begin{array}{ccccc} & G \times \widehat{G} \times U(1) & \xrightarrow{\pi} & G \times \widehat{G} \times U(1) & \\ & \swarrow & & \searrow & \\ G \times U(1) & & G \times \widehat{G} & & \widehat{G} \times U(1) \\ & \swarrow & & \searrow & \\ & G & & \widehat{G} & \\ & \swarrow & & \searrow & \\ & * & & * & \end{array} \right) \mapsto \left((C^*(G), C^*(G)) \cong (C_0(\widehat{G}), C_0(\widehat{G})) \right).$$

Here the group C^* -algebra of G and the at infinity vanishing functions on \widehat{G} are understood as Hilbert C^* -modules over themselves.

Let us now introduce the Fourier transform based on a Pontrjagin duality triple. Let

$$\begin{array}{ccccc} & F \times_B \widehat{E} & \xrightarrow{\kappa} & E \times_B \widehat{F} & \\ & \swarrow & & \searrow & \\ F & & E \times_B \widehat{E} & & \widehat{F} \\ & \swarrow & & \searrow & \\ & E & & \widehat{E} & \\ & \swarrow & & \searrow & \\ & B & & B & \end{array} \quad (50)$$

be any Pontrjagin duality triple. We denote by $F^{\mathbb{C}}$ and $\widehat{F}^{\mathbb{C}}$ the associated line bundles of F and \widehat{F} , respectively. We define the **Fourier transform based on diagram (50)** to be a map

$$\widehat{\cdot} : \Gamma_c(E, \widehat{F}^{\mathbb{C}}) \rightarrow \Gamma_0(\widehat{E}, \widehat{F}^{\mathbb{C}}).$$

Here $\Gamma_c(E, \widehat{F}^{\mathbb{C}})$ and $\Gamma_0(\widehat{E}, \widehat{F}^{\mathbb{C}})$ are the bounded continuous section which have fibre-wise compact support or which vanish at fibre-wise infinity as indicated by Γ_c and Γ_0 , respectively. The top isomorphism κ induces an isomorphism of line bundles $\kappa^{\mathbb{C}} : F^{\mathbb{C}} \times_B \widehat{E} \rightarrow E \times_B \widehat{F}^{\mathbb{C}}$. For a fibre-wise compactly supported section $\gamma : E \rightarrow F^{\mathbb{C}}$ we define its Fourier transform $\widehat{\gamma} : \widehat{E} \rightarrow \widehat{F}^{\mathbb{C}}$ by

$$\widehat{\gamma}(\widehat{e}) := \int_G \text{pr}_{\widehat{F}^{\mathbb{C}}}(\kappa^{\mathbb{C}}(\gamma(e \cdot h), \widehat{e})) dh, \quad (51)$$

where $e \in E$ is any point over the image of \widehat{e} in B . Note that $\widehat{\gamma}$ is a well-defined section as $\text{pr}_{\widehat{F}^{\mathbb{C}}}(\kappa^{\mathbb{C}}(\gamma(e \cdot h), \widehat{e}))$ stays in the same fibre $\widehat{F}^{\mathbb{C}}|_{\widehat{e}}$ independent of $h \in G$.

As an example take the Pontrjagin duality triple (27) given by a \widehat{G} -bundle $\widehat{E} \rightarrow B$:

$$\begin{array}{ccccc} & F_{\widehat{E}^{\text{op}}} \times_B \widehat{E} & \xrightarrow{\kappa_{\widehat{E}}} & (B \times G) \times_B \widehat{E} \times \text{U}(1) & . \\ & \swarrow & & \swarrow & \\ F_{\widehat{E}^{\text{op}}} & & (B \times G) \times_B \widehat{E} & & \widehat{E} \times \text{U}(1) \\ & \searrow & \swarrow & \searrow & \swarrow \\ & B \times G & & \widehat{E} & \\ & \searrow & & \swarrow & \\ & B & & & \end{array} \quad (52)$$

The Fourier transform based on (52) maps a section $\alpha : B \times G \rightarrow F_{\widehat{E}^{\text{op}}}^{\mathbb{C}}$ to a section $\widehat{\alpha} : \widehat{E} \rightarrow \widehat{E} \times \mathbb{C}$ of the trivial line bundle, so it is a map

$$\Gamma_c(B \times G, F_{\widehat{E}^{\text{op}}}^{\mathbb{C}}) \rightarrow C_0(\widehat{E}),$$

where $C_0(\widehat{E})$ are the bounded continuous functions vanishing at fibre-wise infinity.

The intention of this section is to embed the Fourier transform based on a Pontrjagin duality triple into a C^* -algebraic context.

We start with the definition of the C^* -algebra $C^*(E_0, F_0)$ of a ring pair $F_0 \rightarrow E_0 \rightarrow B$, $F_0 = F_{\widehat{E}}$, $E_0 = B \times G$. As we observed in section 3, a ring pair can be thought of as a $\text{U}(1)$ -central extension of groupoids

$$\begin{array}{ccccc} B \times \text{U}(1) & \twoheadrightarrow & F_0 & \twoheadrightarrow & E_0 . \\ \Downarrow & & \Downarrow & & \Downarrow \\ B & \twoheadrightarrow & B & \twoheadrightarrow & B \end{array}$$

In short, the C^* -algebra $C^*(E_0, F_0)$ of the ring pair is the C^* algebra of this central extension [TXL, Section 3].

Let us spell out what this means. Denote by $F_0^{\mathbb{C}} := F_0 \times_{U(1)} \mathbb{C}$ the associated line bundle. The multiplication map μ of the ring pair induces an associated map $\mu^{\mathbb{C}} : F_0^{\mathbb{C}} \times_B F_0^{\mathbb{C}} \rightarrow F_0^{\mathbb{C}}$ by $\mu^{\mathbb{C}}([x, z], [x', z']) := [\mu(x, x'), zz']$, and we let $[x, z]^{\dagger} := [x^{\dagger}, \bar{z}]$, where $x^{\dagger} \in F_0$ is as in Remark 3.1 and \bar{z} is the complex conjugate of $z \in \mathbb{C}$. Let $\Gamma_c(E_0, F_0^{\mathbb{C}})$ denote the vector space of bounded continuous sections with compact support along the fibres. We define a convolution for two such sections α, β by

$$(\alpha *_\mu \beta)(b, g) := \int_G \mu^{\mathbb{C}}(\alpha(b, g-h), \beta(b, h)) dh,$$

a star operation by

$$\alpha^*(b, g) := (\alpha(b, -g))^{\dagger},$$

and a norm by taking the supremum over all fiber-wise L^1 -norms, i.e.

$$\|\alpha\|_{(\infty, 1)} := \sup_{b \in B} \int_G |\alpha(b, g)| dg,$$

where $\|[x, z]\| := |z|$. The completion of $\Gamma_c(E_0, F_0^{\mathbb{C}})$ with respect to this norm yields a Banach $*$ -algebra. We call its enveloping C^* -algebra $C^*(E_0, F_0)$ the C^* -algebra of the ring pair $F_0 \rightarrow E_0 \rightarrow B$.

Next we give the definition of the Hilbert C^* -module $H(E, F)$ of a module pair $F \rightarrow E \rightarrow B$ over $F_0 \rightarrow E_0 \rightarrow B$. We observed in section 4 that we can interpret a module pair as a $U(1)$ -equivariant Morita self-equivalence

$$\left(\begin{array}{ccc} F_0 & \begin{array}{c} \circlearrowleft \\ \circlearrowright \end{array} & F & \begin{array}{c} \circlearrowleft \\ \circlearrowright \end{array} & F_0 \\ \Downarrow & \swarrow & \searrow & \Downarrow \\ & B & & B \end{array} \right) \circlearrowleft U(1).$$

of the groupoid central extension given by the ring pair. The Hilbert C^* -module $H(E, F)$ is the Hilbert C^* -module of this Morita equivalence. It is a module over $C^*(E_0, F_0)$.

Again we spell out some of this construction. Denote by $\varrho : F \times_B F_0 \rightarrow F$ the action map, and denote by $F^{\mathbb{C}}$ the associated line bundle to F . We use the same formula we have used to define $\mu^{\mathbb{C}}$ to define $\varrho^{\mathbb{C}} : F^{\mathbb{C}} \times_B F_0^{\mathbb{C}} \rightarrow F^{\mathbb{C}}$. The vector space of bounded, fibre-wise compactly supported sections $\Gamma_c(E, F^{\mathbb{C}})$ has the structure of a $\Gamma_c(E_0, F_0^{\mathbb{C}})$ -right module by

$$(\gamma *_\varrho \alpha)(e) := \int_G \varrho^{\mathbb{C}}(\gamma(e \cdot (-h)), \alpha(b, h)) dh.$$

Here $b \in B$ is the image of $e \in E$ by $E \rightarrow B$. We define $\sigma_\varrho^{\mathbb{C}} : F^{\mathbb{C}} \times_B F^{\mathbb{C}} \rightarrow F_0^{\mathbb{C}}$ by $\sigma_\varrho^{\mathbb{C}}([y, z], [y', z']) := [\sigma_\varrho(y, y'), \bar{z}z']$, then $\sigma_\varrho^{\mathbb{C}}([y', z'], [y, z]) = \sigma_\varrho^{\mathbb{C}}([y, z], [y', z'])^{\dagger}$ (cp. Remark 4.1). For $\gamma, \delta \in \Gamma_c(E, F^{\mathbb{C}})$ we have a $\Gamma_c(E_0, F_0^{\mathbb{C}})$ -valued inner product by

$$\langle \gamma, \delta \rangle_c(b, g) := \int_G \sigma_\varrho^{\mathbb{C}}(\gamma(e \cdot h), \delta(e \cdot (h+g))) dh \in F_0^{\mathbb{C}}|_{b \times g}.$$

One can check that this is a sesquilinear form which is $\Gamma_c(E_0, F_0^{\mathbb{C}})$ -linear, i.e. $\langle \gamma, \delta \rangle_c * \mu \alpha = \langle \gamma, \delta * \mu \alpha \rangle_c$, which is anti-symmetric, i.e. $\langle \delta, \gamma \rangle_c = \langle \gamma, \delta \rangle_c^*$, which is positive, i.e. $\langle \gamma, \gamma \rangle_c \geq 0 \in C^*(E_0, F_0^{\mathbb{C}})$, and which is definite, i.e. $\langle \gamma, \gamma \rangle_c = 0$ implies $\gamma = 0$. We define a norm on $\Gamma_c(E, F^{\mathbb{C}})$ by

$$\|\gamma\|_{H(E,F)} := \left(\|\langle \gamma, \gamma \rangle_c\|_{C^*(E_0, F_0)} \right)^{\frac{1}{2}},$$

and $H(E, F)$ is the completion of $\Gamma_c(E, F^{\mathbb{C}})$ with respect to this norm.

Lemma 8.1 *The maps $*_q$ and $\langle \cdot, \cdot \rangle_c$ as defined above extend together with its properties to $H(E, F)$ and $C^*(E_0, F_0)$ such that $H(E, F)$ becomes a full Hilbert C^* -module over $C^*(E_0, F_0)$.*

Proof : The proof is standard. ■

Now consider a Pontrjagin duality triple

$$\begin{array}{ccccc}
 & F \times_B \widehat{E} & \xrightarrow{\kappa} & E \times_B \widehat{F} & \\
 & \swarrow & & \searrow & \\
 F & & & & \widehat{F} \\
 & \searrow & & \swarrow & \\
 & E & & \widehat{E} & \\
 & \swarrow & & \searrow & \\
 & & B & &
 \end{array}$$

By Proposition 6.3 the pair of the Pontrjagin duality triple has the structure of a module pair over the ring pair $F_{\widehat{E}^{\text{op}}} \rightarrow B \times G \rightarrow B$, and we can consider the Hilbert C^* -module $H(E, F)$ of this module pair over the C^* -algebra $C^*(B \times G, F_{\widehat{E}^{\text{op}}})$.

There is a second Hilbert C^* -module one is ought to consider. Let $\Gamma_0(\widehat{E}, \widehat{F}^{\mathbb{C}})$ be the bounded continuous sections of the associated line bundle $\widehat{F}^{\mathbb{C}}$ of \widehat{F} which vanish at fibre-wise infinity. $\Gamma_0(\widehat{E}, \widehat{F}^{\mathbb{C}})$ is a full Hilbert C^* -module over the C^* -algebra $C_0(\widehat{E})$ of continuous functions on \widehat{E} which vanish at fibre-wise infinity. The module structure is given by point-wise multiplication, and the $C_0(\widehat{E})$ -valued inner product $\langle \cdot, \cdot \rangle_0$ on $\Gamma_0(\widehat{E}, \widehat{F}^{\mathbb{C}})$ is given for two sections $\hat{\gamma}, \hat{\delta}$ by

$$\langle \hat{\gamma}, \hat{\delta} \rangle_0 : \widehat{E} \xrightarrow{\hat{\gamma} \times \hat{\delta}} \widehat{F}^{\mathbb{C}} \times_{\widehat{E}} \widehat{F}^{\mathbb{C}} \xrightarrow{\Theta} \mathbb{C}. \quad (53)$$

Here $\Theta : \widehat{F}^{\mathbb{C}} \times_{\widehat{E}} \widehat{F}^{\mathbb{C}} \rightarrow \mathbb{C}$ is the canonical map $([\hat{x}, z], [\hat{x} \cdot t, z']) \mapsto \bar{z}tz'$. From a groupoid point of view $C_0(\widehat{E})$ is the C^* -algebra of the $U(1)$ -central extension

$$\begin{array}{ccccc}
 \widehat{E} \times U(1) & \twoheadrightarrow & \widehat{E} \times U(1) & \twoheadrightarrow & \widehat{E} \\
 \Downarrow & & \Downarrow & & \Downarrow \\
 \widehat{E} & \xrightarrow{=} & \widehat{E} & \xrightarrow{=} & \widehat{E}
 \end{array}$$

and $\Gamma_0(\widehat{E}, \widehat{F}^{\mathbb{C}})$ is the Hilbert C^* -module of a $U(1)$ -equivariant Morita self-equivalence

$$\left(\widehat{E} \times U(1) \begin{array}{c} \circlearrowleft \\ \circlearrowright \end{array} F \begin{array}{c} \circlearrowright \\ \circlearrowleft \end{array} \widehat{E} \times U(1) \right) \circlearrowright U(1).$$

$$\begin{array}{ccc} \Downarrow & & \Downarrow \\ \widehat{E} & & \widehat{E} \end{array}$$

The following main theorem states that the Fourier transform identifies the two Hilbert C^* -modules we are concerned with.

Theorem 8.1 *Let*

$$\begin{array}{ccccc} & F \times_B \widehat{E} & \xrightarrow{\kappa} & E \times_B \widehat{F} & \\ & \swarrow & & \searrow & \\ F & & E \times_B \widehat{E} & & \widehat{F} \\ & \searrow q & & \swarrow \hat{q} & \\ & E & & \widehat{E} & \\ & \swarrow p & & \searrow \hat{p} & \\ & B & & & \end{array} \quad (54)$$

be a Pontrjagin duality triple, and let

$$\begin{array}{ccccc} & F_{\widehat{E}^{\text{op}}} \times_B \widehat{E} & \xrightarrow{\kappa_{\widehat{E}}} & (B \times G) \times_B \widehat{E} \times U(1) & \\ & \swarrow & & \searrow & \\ F_{\widehat{E}^{\text{op}}} & & (B \times G) \times_B \widehat{E} & & \widehat{E} \times U(1) \\ & \searrow & & \swarrow & \\ & B \times G & & \widehat{E} & \\ & \swarrow & & \searrow & \\ & B & & & \end{array} \quad (55)$$

be the Pontrjagin duality triple given by the bundle $\widehat{E} \rightarrow B$ of (54) as defined in (27).

Then the Fourier transforms based on diagrams (54) and (55) extend to an isomorphism of Hilbert C^* -modules

$$\left(H(E, F), C^*(B \times G, F_{\widehat{E}^{\text{op}}}) \right) \cong \left(\Gamma_0(\widehat{E}, \widehat{F}^{\mathbb{C}}), C_0(\widehat{E}) \right).$$

$$(\gamma, \alpha) \mapsto (\hat{\gamma}, \hat{\alpha})$$

Moreover this isomorphism is natural in the base, i.e. it defines a functor

$$\text{Pon}^{\text{op}} \rightarrow (\text{Hilb-}C^*\text{-mod})^{(0 \cong 1)},$$

such that the value on the trivial Pontrjagin duality triple over the one-point space is just the classical isomorphism of Pontrjagin duality

$$(C^*(G), C^*(G)) \cong (C_0(\widehat{G}), C_0(\widehat{G})),$$

where we regard the C^* -algebras as modules over themselves.

Proof : For two sections $\gamma, \delta : E \rightarrow F^{\mathbb{C}}$ let us compute the Fourier transform of the inner product $\langle \gamma, \delta \rangle_c : B \times G \rightarrow F_{\widehat{E}^{\text{op}}}^{\mathbb{C}}$. This is a function $\widehat{\langle \gamma, \delta \rangle}_c : \widehat{E} \rightarrow \mathbb{C}$, namely

$$\begin{aligned} \widehat{\langle \gamma, \delta \rangle}_c(\hat{e}) &= \int_G \text{pr}_{\mathbb{C}}(\kappa_{\widehat{E}}^{\mathbb{C}}(\langle \gamma, \delta \rangle_c(b, g), \hat{e})) dg \\ &= \int_G \text{pr}_{\mathbb{C}}\left(\kappa_{\widehat{E}}^{\mathbb{C}}\left(\int_G \sigma_q^{\mathbb{C}}(\gamma(e \cdot h), \delta(e \cdot (h + g))) dh, \hat{e}\right)\right) dg \\ &= \int_{G \times G} \text{pr}_{\mathbb{C}}\left(\kappa_{\widehat{E}}^{\mathbb{C}}(\sigma_q^{\mathbb{C}}(\gamma(e \cdot h), \delta(e \cdot (h + g))), \hat{e})\right) d(g, h). \end{aligned}$$

To understand the integrated function we need to know the map $\sigma_q : F \times_E F \rightarrow F_{\widehat{E}^{\text{op}}}$ explicitly. For $(x, x') \in F \times_E F$ let $g \in G$ be such that $q(x') = q(x) \cdot g$, and for any $\hat{e} \in E|_{p(q(x))}$ let $t \in \text{U}(1)$ be such that $\text{pr}_{\widehat{F}}(\kappa(x', \hat{e})) = \text{pr}_{\widehat{F}}(\kappa(x, \hat{e})) \cdot t$. Then, by use of (33), we see that $\varrho(x, [\hat{e}, g, t]) = x'$, i.e. $\sigma_q(x, x') = [\hat{e}, g, t]$. So for $[x, z] := \gamma(e \cdot h)$ and $[x', z'] := \delta(e \cdot (h + g))$ we have $\sigma_q^{\mathbb{C}}(\gamma(e \cdot h), \delta(e \cdot (h + g))) = [[\hat{e}, g, t], \bar{z}z']$. Thus the function we want to integrate is

$$\begin{aligned} &\text{pr}_{\mathbb{C}}\left(\kappa_{\widehat{E}}^{\mathbb{C}}(\sigma_q^{\mathbb{C}}(\gamma(e \cdot h), \delta(e \cdot (h + g))), \hat{e})\right) \\ &= \text{pr}_{\mathbb{C}}\left(\kappa_{\widehat{E}}^{\mathbb{C}}([\hat{e}, g, t], \bar{z}z'), \hat{e})\right) \\ &= t\bar{z}z' \\ &= \Theta([\text{pr}_{\widehat{F}}(\kappa(x, \hat{e})), z], [\text{pr}_{\widehat{F}}(\kappa(x', \hat{e})), z']) \\ &= \Theta\left(\text{pr}_{\widehat{F}^{\mathbb{C}}}(\kappa^{\mathbb{C}}(\gamma(e \cdot h), \hat{e})), \text{pr}_{\widehat{F}^{\mathbb{C}}}(\kappa^{\mathbb{C}}(\delta(e \cdot (h + g)'), \hat{e}))\right) \end{aligned}$$

with Θ as in (53). So if we do the dg -integration before the dh -integration in the above integral, we see that

$$\widehat{\langle \gamma, \delta \rangle}_c(\hat{e}) = \langle \hat{\gamma}, \hat{\delta} \rangle_0(\hat{e}).$$

By similar arguments, we find that the Fourier transform $\widehat{\cdot} : \Gamma_c(B \times G, F_{\widehat{E}^{\text{op}}}^{\mathbb{C}}) \rightarrow C_0(\widehat{E})$ is a morphism of algebras, i.e. $\widehat{\alpha *_{\mu} \beta} = \widehat{\alpha} \widehat{\beta}$, and that the Fourier transform $\widehat{\cdot} : \Gamma_c(E, F^{\mathbb{C}}) \rightarrow \Gamma_0(\widehat{E}, \widehat{F}^{\mathbb{C}})$ is a morphism that preserves the actions, i.e. $\widehat{\gamma *_{\varrho} \alpha} = \widehat{\gamma} \widehat{\alpha}$. By continuity, we get a map of Hilbert C^* -modules.

The naturality of this map is obvious, and it is also clear that over the one-point space we just reobtain the classical Fourier transform which is an isomorphism. Using a partition of unity on the base space B we find that we have an isomorphism of Hilbert C^* -modules, as it is an isomorphism locally. \blacksquare

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