HOMOTOPY MEANINGFUL CONSTRUCTIONS: HOMOTOPY COLIMITS

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Contents

Both authors would like to thank the Fields Institute, Toronto, the ETH, Zürich and the Max Planck Institute, Bonn.

This work has been partly supported by the Volkswagenstiftung Oberwolfach.

1. Introduction

The purpose of this paper is to give a homotopical framework for studying constructions like: push-outs, realization of simplicial objects, classifying spaces, Borel constructions of group actions, singular chains on spaces, twisted chains, étale spaces of presheaves etc.. All these constructions are very fundamental in homotopy theory, and they all are obtained as certain colimits in various categories. In order to consider homotopical properties of these constructions we are going to look at them in model categories. They were defined around thirty years ago by D. Quillen in his foundational book [34].

A model category is a category in which one can do homotopy theory. The key roles are played by three classes of morphisms called weak equivalences, fibrations and cofibrations, which are subject to five simple axioms (see Section 3). This way of thinking about homotopy theory has become very popular. For example, recent advances in localization theory (see in particular [2], [3], [4], [8], [10] and [13]) show that the category of spaces or spectra can be equipped with various model category structures, depending on what one wants to focus on. If the important maps are -say- homology equivalences for a chosen homology theory, the weak equivalences will not be anymore the plain homotopy equivalences, but rather these homology equivalences.

This general idea has been also used very fruitfully in studying classifying spaces of compact Lie groups. Started by D. Quillen [36] and continued in [29], [30] and [15] several homological decompositions of classifying spaces have been found. Such a homological decomposition is a weak equivalence (i.e. a certain homology equivalence!) between BG and the *homotopy colimit* of a diagram whose values are classifying spaces of proper subgroups of G.

This brings us to the core of the subject of this paper: the homotopy meaningful approximation of the colimit, which is called homotopy colimit and is denoted by *hocolim*. This paper gives a solution to the following problem:

How can one construct homotopy colimits in any closed model category?

In the case of classical homotopy theory the answer has been given by the work of A. Bousfield and D. Kan [6] (see also [42]). In this case the category $Fun(I, Spaces)$ can be given an appropriate model structure (see [19, Section 2]). A homotopy invariant approximation of the colimit can then be obtained as follows: for a given diagram F, take its cofibrant replacement and compute the usual colimit. This is indeed the way homotopy push-outs have been defined for decades: before taking the colimit replace the given push-out diagram by a weakly equivalent one where all the objects are cofibrant and both maps are cofibrations (see [28]). Similar methods were successfully applied in the category of Spectra in [41]. However, for a general model category $\mathcal{C}$, we still do not know how to put an appropriate model structure on $Fun(I, \mathcal{C})$. Only partial answers have been given. By [28] such a model structure can be found in the case when I is "very small", so for example when $\mathrm{Fun}(I, \mathcal{C})$ is the category of push-out diagrams (see Example 5.2). The work of C. Reedy [37] gives an appropriate model structure on $Fun(I, \mathcal{C})$, for example when $I = \Delta^{op}$, i.e. for simplicial objects in $\mathcal{C}$. But, quoting [28, page 121]) "..., it seems unlikely that $\mathrm{Fun}(I, \mathcal{C})$ has a natural model category structure".

A systematic study of homotopy properties of diagrams has been done by W. Dwyer and D. Kan in an extensive list of papers that includes [18]–[27]. Many of their ideas have found here an echo. The work of R. Thomason including [39] and [40] (see also [1]) should also be mentioned very strongly in this context. E. Dror Farjoun [11], [12] and A. Zabrodsky [14] have made important contributions in this subject as well.

A solution of the problem of constructing homotopy colimit in any model category has been finally given in the recent work of W. Dwyer, P. Hirschhorn, and D. Kan [16].

In this paper we propose an alternative solution, which has two advantages. First, our construction is very small (in some sense minimal). Second, the developed techniques are elementary and geometric. The general scheme is as follows: instead of imposing a model structure directly on $Fun(I, \mathcal{C})$, we approximate it by a model category. We find an "almost" Quillen equivalence between $Fun(I, \mathcal{C})$ and this model category. This is good enough to find an analogue of a cofibrant replacement in $Fun(I, \mathcal{C})$, which can be used in turn to construct the homotopy colimit by taking the ordinary colimit of this cofibrant replacement. Moreover, for various classes of indexing categories, a different model approximation of $Fun(I, \mathcal{C})$ can be used. This idea is applied to prove the so called Fubini Theorem (the homotopy colimit commutes with itself, see Corollary 22.7), as well as Thomason Theorem about homotopy colimits of diagrams indexed by Grothendieck constructions (see Corollary 24.6).

In order to construct the model category approximation for $Fun(I, \mathcal{C})$, we investigate the role of the *geometry* of an indexing category in the construction of the homotopy colimit. Since for an arbitrary small category it is difficult to make precise what its geometry is, we focus on the so called simplex categories, i.e., categories associated with simplicial sets (see Definition 6.1). In this way we can take advantage of the geometry of the underlying space. However, since simplex categories are big and complicated, in order to simplify the situation, we put restrictions on diagrams indexed by them. We define what we call bounded diagrams (see Definitions 10.1 and 12.1). The key result says then that there exists an appropriate model structure on the category of bounded diagrams $Fun^b(K, \mathcal{C})$ (see Theorem 17.3).

We investigate the local properties of this model structure. These local properties turn out to be essential in proving, among other things, Thomason Theorem (see Corollary 24.6).

The model structure on bounded diagrams is applied to define a functor denoted by *ocolim* (see Definition 18.1). It is not yet the desired homotopy colimit, but it has many of its good properties. The main feature of *hocolim* which is not shared by *ocolim* is additivity with respect to the indexing space (see Remark 18.3). The homotopy colimit $hocolim_I$ is then constructed using $ocolim$ and a forgetful functor $\epsilon : N(I) \to I$, where $N(I)$ is the simplex category of the nerve of I (see Section 6). We show that the composition:

$$Fun(I,\mathcal{C}) \xrightarrow{\epsilon^*} Fun^b\big(N(I),\mathcal{C}\big) \xrightarrow{ocolim_{N(I)}} Ho(\mathcal{C})$$

is the homotopy approximation of $colim_I$ (see Theorem 21.1).

2. Notations

The symbol Δ denotes the simplicial category (cf. [33, section 2]), in which the objects are the ordered sets $[n] = \{0 < \cdots < n\}$, and the morphisms are weakly monotone maps of sets. The morphisms of Δ are generated by coface maps $d_i : [n-1] \to [n]$ and codegeneracy maps $s_i : [n+1] \to [n]$ for $0 \le i \le n$, subject to well-known cosimplicial identities. A simplicial set is then a functor $K : \Delta^{op} \to Sets$ where $Sets$ denotes the category of sets. One usually denotes the set $K[n]$ by K_n. A morphism between two simplicial sets is by definition a natural transformation of functors. A simplicial set K can be interpreted as a collection of sets $(K_n)_{n\ge 0}$ together with face maps $d_i : K_n \to K_{n-1}$ and degeneracy maps $s_i : K_n \to K_{n+1}$ which satisfying the dual of the cosimplicial identities (simplicial identities). For a description of how to do homotopy theory in the category of simplicial sets see [6], [9], [33] and [34]. In this paper we use the symbol $Spaces$ to denote the category of simplicial sets, and by a space we always mean a simplicial set.

An element $\sigma \in K_n$ is called an n-dimensional simplex of K (or n-simplex for short). It is said to be degenerate if there exists $\sigma' \in K_{n-1}$ and $0 \le i \le n-1$ such that $s_i\sigma' = \sigma$.

The standard n-simplex $\Delta[n]$ is an important example of a space. By definition, its set of k-simplices is given by $(\Delta[n])_k := \mathrm{mor}_\Delta([k],[n])$. There is a distinguished n-dimensional simplex ι in $\Delta[n]$, namely the unique non-degenerate one which comes from the identity map $[n] \to [n]$. The assignment $f \mapsto f(\iota)$ yields a bijection of sets $mor_{Spaces}(\Delta[n], K) \to K_n$. Thus we do not distinguish between maps $\Delta[n] \to K$ and n-simplices in K. If $\sigma \in K_n$ is a simplex, we use the same symbol $\sigma : \Delta[n] \to K$ to denote the corresponding map.

The simplicial subset of $\Delta[n]$, which is generated by the simplices $\{d_i\iota \mid 0 \le i \le n\}$, is denoted by $\partial\Delta[n]$ and called the boundary of $\Delta[n]$. The simplicial subset of $\partial\Delta[n]$, which is generated by the simplices $\{d_i\iota \mid i \ne k\}$, is denoted by $\Delta[n,k]$ and called the horn. There are obvious inclusions $\Delta[n,k] \subset \partial\Delta[n] \subset \Delta[n]$.

Let $\mathcal{C}$ be a category and I be a small category. By $Fun(I,\mathcal{C})$ we denote the category whose objects are functors indexed by I with values in $\mathcal{C}$, and morphisms are natural transformations. For any object $X \in \mathcal{C}$, there is a constant diagram $X : I \to \mathcal{C}$ with value X. This assignment defines a functor $\mathcal{C} \to Fun(I,\mathcal{C})$. Its left adjoint is called the colimit and is denoted by $colim_I : Fun(I,\mathcal{C}) \to \mathcal{C}$. Its right adjoint is called the limit and is denoted by $lim_I : Fun(I,\mathcal{C}) \to \mathcal{C}$. If this left (respectively right) adjoint exists for any small category I, we say that $\mathcal{C}$ is closed under colimits (respectively limits).

Let $F : I \to \mathcal{C}$ be a functor. The object $colim_I F$ is equipped with a natural transformation $F \to colim_I F$, between F and the constant diagram with value $colim_I F$. This natural transformation has the following universal property. If X is an object in $\mathcal{C}$, then any natural transformation $F \to X$ factors uniquely as $F \to colim_I F \to X$.

For a detailed exposition on colimits and limits we refer the reader to [28] and [32]. For other categorical constructions used in this paper see Appendix A.

3. Model Categories

In this section we review classical homotopical properties of the coproduct, push-out and the sequential colimit constructions. In order to be able to consider their *homotopical* properties, we look at these constructions in *model categories*, i.e. in categories in which one can do homotopy theory. We refer the reader to [28], [34], and [35] for the necessary definitions and theorems concerning these categories. Here we just sketch some of their properties. However the reader should keep in mind that the notion of a model category is essential in this exposition, in fact this paper is about model categories.

A model category is a category, usually denoted by $\mathcal{C}$, together with three distinguished classes of morphisms: *weak equivalences*, *fibrations* and *cofibrations*. This structure is subject to five axioms **MC1**-**MC5** (see [28, Section 3]). A morphism which is both a weak equivalence and a fibration (respectively a cofibration) is called an *acyclic* fibration (respectively an *acyclic* cofibration). To denote a weak equivalence, a cofibration and a fibration we use respectively the symbols "$\xrightarrow{\sim}$","$\hookrightarrow$" and "$\twoheadrightarrow$".

Axiom **MC1** guarantees that model categories are equipped with colimits and limits. In particular there is a terminal object, denoted by $*$, as well as an initial one, denoted by $\emptyset$. An object A is said to be cofibrant if the morphism $\emptyset \to A$ is a cofibration. It is said to be fibrant if the morphism $A \to *$ is a fibration. This axiom also implies the existence of products and coproducts in $\mathcal{C}$, denoted respectively by the symbols "$\prod$" and "$\coprod$".

Axiom **MC2** says that for composable morphisms $f : A \to B$ and $g : B \to C$, if two out of $\{f, g, g \circ f\}$ are weak equivalence, then so is the third.

Axiom **MC3** guarantees that weak equivalences, fibrations and cofibrations are closed under retracts, i.e. in a commutative diagram:

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"'] \arrow[rr, bend left, "id"] & A' \arrow[r] \arrow[d, "h"] & A \arrow[d, "f"] \\
B \arrow[r] \arrow[rr, bend right, "id"'] & B' \arrow[r] & B
\end{tikzcd}$$

if h is a weak equivalence, a fibration or a cofibration, then so is f.

Cofibrations and fibrations are linked together through the lifting axiom **MC4**. It says that in a commutative square:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] & Y \end{tikzcd}$$

if either $i : A \to B$ is an acyclic cofibration and $p : X \to Y$ is a fibration, or $i : A \to B$ is a cofibration and $p : X \to Y$ is an acyclic fibration, then there is a lift, i.e. a morphism $h : B \to X$ such that the resulting diagram with five arrows commutes. In such a situation we say that $i : A \to B$ has the lifting property with respect to $p : X \to Y$. This lifting property characterizes cofibrations and fibrations. A morphism $i : A \to B$ is a cofibration if it has the lifting property with respect to all acyclic fibrations. It is an acyclic cofibration if it has the lifting property with respect to all fibrations. Dually $p : X \to Y$ is a fibration if all acyclic cofibrations have the lifting property with respect to it. It is an acyclic fibration if all cofibrations have the lifting property with respect to it.

By checking the lifting criterion one can show (see [28, Proposition 3.13]):

3.1. **Proposition**. *Let the following be a push-out square in $\mathcal{C}$:*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & C \arrow[d] \\ B \arrow[r] & D \end{tikzcd}$$

If $A \to B$ is an (acyclic) cofibration, then so is $C \to D$.

Axiom **MC5** says that any morphism can be expressed as the composition of a cofibration followed by an acyclic fibration, and as the composition of an acyclic cofibration followed by a fibration. Sometimes we assume in addition that such factorizations can be chosen functorially.

By factoring the morphism $\emptyset \to A$ into a cofibration followed by an acyclic fibration $\emptyset \hookrightarrow QA \overset{\sim}{\twoheadrightarrow} A$ we get a *cofibrant* object QA weakly equivalent to A. Such an object is called a cofibrant replacement of A. For any morphism $f : A \to B$, by the lifting axiom **MC4**, there exists $Qf : QA \to QB$ which makes the compositions $QA \overset{\sim}{\twoheadrightarrow} A \xrightarrow{f} B$ and $QA \xrightarrow{Qf} QB \overset{\sim}{\twoheadrightarrow} B$ equal. Any such morphism Qf is called a cofibrant replacement of f. We say that we have chosen a cofibrant replacement Q in $\mathcal{C}$, if, for every object A, we have chosen a cofibrant replacement $QA \overset{\sim}{\twoheadrightarrow} A$ and, for every morphism $f : A \to B$, we have chosen $Qf : QA \to QB$.

Dually, by factoring the morphism $A \to *$ into an acyclic cofibration and a fibration $A \overset{\sim}{\hookrightarrow} RA \twoheadrightarrow *$ we get a *fibrant* object RA weakly equivalent to A. Such an object is called a fibrant replacement of A. For any morphism $f : A \to B$, by the lifting axiom

MC4, there exists $Rf : RA \to RB$ which makes the compositions $A \xrightarrow{f} B \overset{\sim}{\hookrightarrow} RB$ and $A \overset{\sim}{\hookrightarrow} RA \xrightarrow{Rf} RB$ equal. Any such morphism Rf is called a fibrant replacement of f. We say that we have chosen a fibrant replacement R in $\mathcal{C}$, if, for every object A, we have chosen a fibrant replacement $A \overset{\sim}{\hookrightarrow} RA$ and, for every morphism $f : A \to B$, we have chosen $Rf : RA \to RB$.

A model category is set up for defining the notion of homotopy between morphisms. For this purpose one uses so called cylinder and mapping objects. A cylinder object of X is an object $Cyl(X)$ which fits into a factorization of the the fold morphism $id \coprod id : X \coprod X \to X$ as a cofibration followed by a weak equivalence $X \coprod X \overset{i}{\hookrightarrow} Cyl(X) \overset{\sim}{\to} X$. Dually, a mapping object of Y is an object $Map(Y)$ which fits into a factorization of the diagonal $\Delta : Y \to Y \times Y$ as a weak equivalence followed by a fibration $Y \overset{\sim}{\to} Map(Y) \overset{p}{\twoheadrightarrow} Y \times Y$. A left, respectively right, homotopy between $f : X \to Y$ and $g : X \to Y$ is a morphism $H : Cyl(X) \to Y$, respectively $G : X \to Map(Y)$, for which the following triangles commute:

$$\begin{tikzcd} X \coprod X \arrow[r, "f \coprod g"] \arrow[d, hook, "i"'] & Y \\ Cyl(X) \arrow[ur, "H"] & \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "f \times g"] \arrow[dr, "G"] & Y \times Y \\ & Map(Y) \arrow[u, two heads, "p"'] \end{tikzcd}$$

For example, any two cofibrant replacements $Qf : QX \to QY$ and $Q'f : QX \to QY$ of a morphism $f : X \to Y$ are left homotopic. Dually, any two fibrant replacements $Rf : RX \to RY$ and $R'f : RX \to RY$ of f are right homotopic.

In the case X is cofibrant and Y is fibrant both notions of right and left homotopy between morphisms $X \to Y$ coincide (see [28, Lemma 4.21]) and define an equivalence relation. We use the symbol "$\simeq$" to denote this relation. The homotopy category $Ho(\mathcal{C})$ is then a category having same objects as $\mathcal{C}$, but where the morphisms from X to Y are the homotopy classes (right or left) from a *cofibrant* replacement QX of X to a *fibrant-cofibrant* replacement RQY of Y. We denote this set by $[X, Y]$. By sending an object $X \in \mathcal{C}$ to the same object $X \in Ho(\mathcal{C})$ and a morphism $f : X \to Y$ to the (right or left) homotopy class of the composition $QX \overset{Qf}{\to} QY \to RQY$, we get a functor $\mathcal{C} \to Ho(\mathcal{C})$. This functor satisfies the universal property of the localization of $\mathcal{C}$ with respect to all weak equivalences (see [28, Theorem 6.2]). If $f : X \to Y$ is a morphism in $\mathcal{C}$ we use the same symbol $f : X \to Y$ to denote the induced morphism in the homotopy category $Ho(\mathcal{C})$.

3.2. **Proposition**. *Let A and B be cofibrant objects. A morphism $f : A \to B$ is a weak equivalence if and only if, for any fibrant object Y and any morphism $\alpha : A \to Y$, there exists a unique (up to homotopy) $\beta : B \to Y$ such that α is homotopic to $\beta \circ f$.*

3.3. **Proposition**. *Let A and B be cofibrant, X fibrant, $i : A \hookrightarrow B$ a cofibration and the following be a diagram that commutes up to homotopy:*

$$\begin{tikzcd} A \arrow[r, hook, "i"] \arrow[d, "f"'] & B \arrow[dl, "g"] \\ X & \end{tikzcd}$$

Then there exists $B \to X$, homotopic to g, which makes this diagram strictly commutative.

Proof. The morphisms $g \circ i$ and f are homotopic, hence there exists a (right) homotopy $H : A \to Map(X)$ between them. Since X is a fibrant object, the composition $Map(X) \twoheadrightarrow X \times X \xrightarrow{p_1} X$ is a trivial fibration. Thus there exists a lift $G : B \to Map(X)$ in the following square:

$$\begin{tikzcd} A \arrow[r, "H"] \arrow[d, hook, "i"'] & Map(X) \arrow[d, two heads, "\wr"', "p_1"] \\ B \arrow[r, "g"] & X \end{tikzcd}$$

This lift is a homotopy between g and the composition $B \xrightarrow{G} Map(X) \xrightarrow{p_0} X$, which gives the desired morphism. □

Taking the push-out of an (acyclic) cofibration along any morphism is again an (acyclic) cofibration (see Proposition 3.1). Taking the push-out of a weak equivalence along a cofibration in general is no longer a weak equivalence. If it is so the category $\mathcal{C}$ is called proper (see [5, Definition 1.2]). Nevertheless in the case the involved objects are cofibrant, we have:

3.4. **Proposition**. *Let $A \hookrightarrow B$ be a cofibration, $f : A \xrightarrow{\sim} C$ a weak equivalence and the following be a push-out square:*

$$\begin{tikzcd} A \arrow[r, hook, "i"] \arrow[d, "f"', "\wr"] & B \arrow[d, "g"] \\ C \arrow[r, hook, "j"] & D \end{tikzcd}$$

If A, B and C are cofibrant, then $g : B \to D$ is a weak equivalence.

Proof. Since C is cofibrant and j is a cofibration, D is cofibrant. Hence, by Proposition 3.2, we have to show that for any fibrant object Y and a morphism $\alpha : B \to Y$, there exists a unique (up to homotopy) $\beta : D \to X$ such that $\beta \circ g \simeq \alpha$. Since f is a weak equivalence, there is a $\gamma : C \to X$ for which $i \circ \alpha \simeq \gamma \circ f$. By assumption i is a cofibration, thus according to Proposition 3.3 there exists $\alpha' : B \to X$, homotopic to α, such that $\gamma \circ f = \alpha' \circ i$. In this way we obtain a morphism $\beta : D \to X$, from the push-out D, with the desired property.

The uniqueness can be checked in a similar way by replacing X, in the previous argument, with $Map(X)$. □

Propositions 3.2, 3.3 and 3.4 can be used to study homotopy invariance of the coproduct, push-out and the sequential colimit constructions:

3.5. **Proposition**.

(1) *If for $i \in I$, $f_i : X_i \to Y_i$ is a weak equivalence between cofibrant objects, then so is the coproduct $\coprod_{i\in I} f_i : \coprod_{i\in I} A_i \to \coprod_{i\in I} B_i$.*

(2) *Consider the following natural transformation between push-out diagrams:*

$$\begin{tikzcd} A \arrow[d, "f"'] & = & colim & (& A_0 \arrow[d, "f_0"', "\wr"] & A_1 \arrow[l] \arrow[r, hook] \arrow[d, "\wr"', "f_1"] & A_2 \arrow[d, "\wr"', "f_2"] &) \\ B & = & colim & (& B_0 & B_1 \arrow[l] \arrow[r] & B_2 &) \end{tikzcd}$$

where, for $l = 0, 1, 2$, f_l is a weak equivalence and A_l, B_l are cofibrant. If $A_1 \hookrightarrow A_2$ and either $B_1 \to B_2$ or $B_1 \to B_0$ are cofibrations, then $f : A \to B$ is a weak equivalence.

(3) *Consider the following natural transformation between (possibly transfinite) telescope diagrams:*

$$\begin{tikzcd} A \arrow[d, "f"'] & = & colim & (& A_0 \arrow[r, hook, "i_0"] \arrow[d, "f_0"', "\wr"] & A_1 \arrow[r, hook, "i_1"] \arrow[d, "\wr"', "f_1"] & A_2 \arrow[r, hook, "i_2"] \arrow[d, "\wr"', "f_2"] & \cdots &) \\ B & = & colim & (& B_0 \arrow[r, hook, "j_0"] & B_1 \arrow[r, hook, "j_1"] & B_2 \arrow[r, hook, "j_0"] & \cdots &) \end{tikzcd}$$

where, for $l \geq 0$, f_l is a weak equivalence, A_l and B_l are cofibrant and i_l, j_l are cofibrations. Then $f : A \to B$ is a weak equivalence.

Since (1), (2) and (3) can be proved using similar methods, we show only (3).

Proof of (3). The colimit A is equipped with morphisms $\zeta_l : A_l \to A$ for all $l \geq 0$. By checking the lifting property, it is easy to see that $\zeta_0 : A_0 \to A$ is a cofibration and hence A is cofibrant. We can then apply Proposition 3.2 to prove that f is a weak equivalence. Let X be fibrant and $k : A \to X$ be a morphism. Since f_0 is a weak equivalence, there is $g_0 : B_0 \to X$ such that $g_0 \circ f_0 \simeq k \circ \zeta_0$. Using Proposition 3.3 we can modify k, up to homotopy, to get $k_0 : A \to X$ such that $g_0 \circ f_0 = k_0 \circ \zeta_0$.

In the next step, since f_1 is a weak equivalence, we can find $g_1' : B_1 \to X$ for which $g_1' \circ f_1 \simeq k_0 \circ \zeta_1$. By precomposing with i_0 we see that $g_0 \circ f_0 \simeq g_1' \circ j_0 \circ f_0$. Therefore $g_0 \simeq g_1' \circ j_0$. Hence we can replace g_1' by a homotopic morphism $g_1 : B_1 \to X$ which gives a strict equality $g_0 = g_1 \circ j_0$. We can again modify k_0, up to homotopy, to get $k_1 : A \to X$ such that $g_1 \circ f_1 = k_1 \circ \zeta_1$.

Continuing this process inductively we get a family of strictly compatible morphism $g_l : B_l \to X$ inducing $g : B \to X$ for which $g \circ f \simeq k$.

The homotopical uniqueness of g can be shown by a similar argument replacing X with $Map(X)$. $\square$

Under some circumstances the coproduct, push-put and the sequential colimit constructions preserve also cofibrations and acyclic cofibrations:

3.6. **Proposition**.

(1) *If for all $i \in I$, $f_i : A_i \to B_i$ is an (acyclic) cofibration, then so is the coproduct $\coprod_{i\in I} f_i : \coprod_{i\in I} A_i \to \coprod_{i\in I} B_i$.*
(2) *Consider the following natural transformation between push-out diagrams:*

$$\begin{tikzcd} A \arrow[d, "f"'] & = & colim \Big(& A_0 \arrow[d, "f_0"'] & A_1 \arrow[l] \arrow[r] \arrow[d, "f_1"] & A_2 \arrow[d, "f_2"] & \Big) \\ B & = & colim \Big(& B_0 & B_1 \arrow[l] \arrow[r] & B_2 & \Big) \end{tikzcd}$$

Let $M = colim(B_1 \xleftarrow{f_1} A_1 \to A_2)$ and $g : M \to B_2$ be induced by the commutativity of the above diagram. Then:
- *If f_0 and g are cofibrations, then so is f.*
- *If f_0, f_1 are acyclic cofibrations, f_2 is a weak equivalence and g is a cofibration, then g and f are acyclic cofibrations.*

(3) *Consider the following natural transformation between (possibly transfinite) telescope diagrams:*

$$\begin{tikzcd} A \arrow[d, "f"'] & = & colim \Big(& A_0 \arrow[r] \arrow[d, "f_0"'] & A_1 \arrow[r] \arrow[d, "f_1"] & A_2 \arrow[r] \arrow[d, "f_2"] & \cdots & \Big) \\ B & = & colim \Big(& B_0 \arrow[r] & B_1 \arrow[r] & B_2 \arrow[r] & \cdots & \Big) \end{tikzcd}$$

If f_0 and, for all i, $colim(B_i \leftarrow A_i \to A_{i+1}) \to B_{i+1}$ are (acyclic) cofibrations, then so is f.

Since the proofs are analogous we show only the second part of (2).

Proof of the second part of (2). The morphism $f_2 : A_2 \to B_2$ factors as a composition $A_2 \to M \xrightarrow{g} B_2$. By the assumption f_1 is an acyclic cofibration and hence so is $A_2 \to M$. Therefore, since f_2 is a weak equivalence, g is an *acyclic* cofibration.

To prove that f is an acyclic cofibration we have to show that it has the lifting property with respect to all fibrations, i.e. for any fibration $E \twoheadrightarrow X$ and commutative square:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & E \arrow[d] \\ B \arrow[r] & X \end{tikzcd}$$

we need to construct a lift $h : B \to E$.

For this purpose consider the following commutative diagram:

$$
\begin{tikzcd}
A_0 \arrow[dr] \arrow[dd, hook, "f_0"', "\wr"] & & A_1 \arrow[ll] \arrow[rr] \arrow[dd, hook, "\wr"', "f_1"] & & A_2 \arrow[dlll] \arrow[dl, hook, "\sim"'] \arrow[dd, "\wr"', "f_2"] \\
& E \arrow[dd, two heads] & & M \arrow[ll, "s"'] \arrow[dr, hook, "g"] & \\
B_0 \arrow[ur, "h_0"] \arrow[dr] & & B_1 \arrow[ll] \arrow[rr] \arrow[ur] & & B_2 \arrow[ulll, "h_2"'] \arrow[dlll] \\
& X & & &
\end{tikzcd}
$$

where:

- the morphism $h_0 : B_0 \to E$ is constructed by the lifting property of the acyclic cofibration $f_0 : A_0 \overset{\sim}{\hookrightarrow} B_0$ with respect to the fibration $E \twoheadrightarrow X$,
- the morphism $s : M \to E$ is then induced by the universal property of the colimit $M = colim(B_1 \leftarrow A_1 \to A_2)$,
- finally, $h_2 : B_2 \to E$ is constructed by the lifting property of the acyclic cofibration $g : M \overset{\sim}{\hookrightarrow} B_2$ with respect to the fibration $E \twoheadrightarrow X$.

The desired lift $h : B \to E$ can be now constructed using the morphisms h_0 and h_2. □

4. Left derived functors

In this section we recall the notion of a left derived functor (see [28, Definition 9.1] and [34, Definition 4.1]). Let $\mathcal{D}$ be a category with a distinguished class of morphisms called weak equivalences. We use the symbol "$\overset{\sim}{\to}$" to denote the morphisms in this class. In the case $\mathcal{D}$ is a model category we focus our attention only on the class of weak equivalences.

4.1. **Definition**. *Let $\mathcal{E}$ be a category. We say that a functor $\mathcal{H} : \mathcal{D} \to \mathcal{E}$ is homotopy invariant if, for any weak equivalence f in $\mathcal{D}$, $\mathcal{H}(f)$ is an isomorphism in $\mathcal{E}$.*

If $\mathcal{D}$ admits a localization functor with respect to weak equivalences $\mathcal{D} \to Ho(\mathcal{D})$, then, by the universal property, $\mathcal{H} : \mathcal{D} \to \mathcal{E}$ is homotopy invariant if and only if it can be expressed as a composition $\mathcal{D} \to Ho(\mathcal{D}) \to \mathcal{E}$.

Of course an arbitrary functor usually is not homotopy invariant. We can ask then if its homotopy invariant approximation exists, the so called derived functor:

4.2. **Definition**. *A functor $L(\mathcal{H}) : \mathcal{D} \to \mathcal{E}$ together with a natural transformation $L(\mathcal{H}) \to \mathcal{H}$ is called the left derived functor of $\mathcal{H} : \mathcal{D} \to \mathcal{E}$ if:*

- *$L(\mathcal{H})$ is homotopy invariant.*
- *If $\mathcal{G} : \mathcal{D} \to \mathcal{E}$ is a homotopy invariant functor, then any natural transformation $\mathcal{G} \to \mathcal{H}$ factors uniquely as a composition $\mathcal{G} \to L(\mathcal{H}) \to \mathcal{H}$.*

Let $\mathcal{C}$ be a model category and $\mathcal{H} : \mathcal{D} \to \mathcal{C}$ be a functor. The left derived functor of the composition $\mathcal{D} \xrightarrow{\mathcal{H}} \mathcal{C} \to Ho(\mathcal{C})$ is called the *total* left derived functor of $\mathcal{H}$ (see [28, Definition 9.5]). The essential data to define the total left derived functor of $\mathcal{H} : \mathcal{D} \to \mathcal{C}$ is the choice of weak equivalences in $\mathcal{D}$. However for its construction it is very helpful to have an appropriate model structure on $\mathcal{D}$, i.e. in addition to weak equivalences, an appropriate choice of cofibrations and fibrations. In such a situation, for the construction of $L(\mathcal{H})$, cofibrant objects and cofibrant replacement play the crucial role.

4.3. **Definition**. *Let $\mathcal{D}$ be model category and $\mathcal{C}$ be a category with a distinguished class of weak equivalences. A functor $\mathcal{H} : \mathcal{D} \to \mathcal{C}$ is called homotopy meaningful on cofibrant objects if, for any weak equivalence $f : X \to Y$ in $\mathcal{D}$ between cofibrant objects X and Y, $\mathcal{H}(f)$ is a weak equivalence in $\mathcal{C}$.*

4.4. **Proposition**. *Let $\mathcal{C}$ and $\mathcal{D}$ be model categories. If $\mathcal{H} : \mathcal{D} \to \mathcal{C}$ is homotopy meaningful on cofibrant objects, then its total left derived functor $L(\mathcal{H})$ exists. It can be constructed by choosing a cofibrant replacement Q in $\mathcal{D}$ and assigning to $X \in \mathcal{D}$ the object $L(\mathcal{H})(X) := \mathcal{H}(QX) \in Ho(\mathcal{C})$. The natural transformation $L(\mathcal{H}) \to \mathcal{H}$ is induced by the morphisms $\mathcal{H}(QX \overset{\sim}{\twoheadrightarrow} X)$.*

4.5. **Lemma**. *Let $\mathcal{C}$, $\mathcal{D}$ be model categories and $\mathcal{H} : \mathcal{D} \to \mathcal{C}$ be homotopy meaningful on cofibrant objects. If $f : X \to Y$ and $g : X \to Y$ are left homotopic in $\mathcal{D}$, and X is cofibrant, then $\mathcal{H}(f)$ and $\mathcal{H}(g)$ are left homotopic in $\mathcal{C}$.*

Proof. Since f and g are left homotopic, we can form a commutative diagram in $\mathcal{D}$:

$$\begin{tikzcd}
X \coprod X \arrow[r, "f \coprod g"] \arrow[d, "id \coprod id"'] \arrow[dr, hook] & Y \\
X & Cyl(X) \arrow[l, "\sim"'] \arrow[u]
\end{tikzcd}$$

By applying $\mathcal{H}$ we get:

$$\begin{tikzcd}
\mathcal{H}(X) \coprod \mathcal{H}(X) \arrow[r] \arrow[rr, bend left, "\mathcal{H}(f) \coprod \mathcal{H}(g)"] \arrow[dr, "\mathcal{H}(id) \coprod \mathcal{H}(id)"'] & \mathcal{H}(X \coprod X) \arrow[r] \arrow[d] \arrow[dr] & \mathcal{H}(Y) \\
 & \mathcal{H}(X) & \mathcal{H}\big(Cyl(X)\big) \arrow[l] \arrow[u]
\end{tikzcd}$$

Since X is cofibrant, then so is $Cyl(X)$. As $\mathcal{H}$ is homotopy meaningful on cofibrant objects, $\mathcal{H}\big(Cyl(X)\big) \to \mathcal{H}(X)$ is a weak equivalence. Let $Cyl\big(\mathcal{H}(X)\big) \xrightarrow{\sim} \mathcal{H}\big(Cyl(X)\big)$ be a morphism which fits into a factorization of $\mathcal{H}(X) \coprod \mathcal{H}(X) \to \mathcal{H}\big(Cyl(X)\big)$ by a cofibration followed by a weak equivalence:

$$\mathcal{H}(X) \coprod \mathcal{H}(X) \hookrightarrow Cyl\big(\mathcal{H}(X)\big) \xrightarrow{\sim} \mathcal{H}\big(Cyl(X)\big)$$

This is then indeed a cylinder object for $\mathcal{H}(X)$ and we get a left homotopy between $\mathcal{H}(f)$ and $\mathcal{H}(g)$ by taking the composition $Cyl\big(\mathcal{H}(X)\big) \xrightarrow{\sim} \mathcal{H}\big(Cyl(X)\big) \to \mathcal{H}(Y)$. $\square$

Proof of Proposition 4.4. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms in $\mathcal{D}$. Since:

$$QX \xrightarrow{Q(g\circ f)} Q(Z) \quad , \quad QX \xrightarrow{Qf} QY \xrightarrow{Qg} QZ$$

are cofibrant replacements of the same morphism $g \circ f$, they are left homotopic. Lemma 4.5 implies therefore that the assignment $\mathcal{D} \ni X \mapsto \mathcal{H}(QX) \in Ho(\mathcal{C})$ is a well defined functor.

We have to show that the natural transformation $\mathcal{H}(QX \overset{\sim}{\twoheadrightarrow} X)$ satisfies the appropriate universal property. Let $\mathcal{G} : \mathcal{D} \to Ho(\mathcal{C})$ be a homotopy invariant functor and $\mathcal{G} \to \mathcal{H}$ be a natural transformation. Since $\mathcal{G}$ is homotopy invariant, $\mathcal{G}(QX) \to \mathcal{G}(X)$ is an isomorphism and hence, for any object X in $\mathcal{D}$, there exists a unique morphism $\mathcal{G}(X) \to \mathcal{H}(QX)$ for which the following square commutes in $Ho(\mathcal{C})$:

$$\begin{tikzcd}
\mathcal{G}(QX) \arrow[r, "\cong"] \arrow[d] & \mathcal{H}(QX) \arrow[d] \\
\mathcal{G}(X) \arrow[r] \arrow[ur] & \mathcal{H}(X)
\end{tikzcd}$$

These morphisms form the desired unique natural transformation $\mathcal{G} \to \mathcal{H}(Q-)$. $\square$

5. Homotopy invariant approximation of colimit

Let I be a small category. Let us consider the category of functors $Fun(I, \mathcal{C})$ with values in a model category $\mathcal{C}$. Let $F : I \to \mathcal{C}$ and $G : I \to \mathcal{C}$ be functors. A natural transformation $\Psi : F \to G$ is called a *weak equivalence* if it is an objectwise weak equivalence, i.e. if for all $i \in I$, $\Psi_i : F(i) \to G(i)$ is a weak equivalence in $\mathcal{C}$.

In general, the colimit construction $colim_I : Fun(I, \mathcal{C}) \to \mathcal{C}$ is not homotopy meaningful: a weak equivalence between diagrams usually does not induce a weak equivalence on colimits. We can ask then:

Does a homotopy meaningful approximation of the colimit exist?

5.1. **Definition**. *The total left derived functor of* $colim_I : Fun(I, \mathcal{C}) \to \mathcal{C}$ *is called homotopy colimit and is denoted by* $hocolim_I : Fun(I, \mathcal{C}) \to Ho(\mathcal{C})$.

What is a general strategy for constructing a homotopy meaningful approximation of $colim_I$? According to Proposition 4.4 it suffices to put a model category structure on $Fun(I, \mathcal{C})$ for which the colimit functor is homotopy meaningful on cofibrant objects.

5.2. **Example**. Let I be the push-out category, i.e. a category associated with a graph $*_0 \leftarrow *_1 \rightarrow *_2$. Let $\Psi : F \to G$ be a morphism in $Fun(I, \mathcal{C})$ given by a commutative diagram:

$$\begin{tikzcd} F \arrow[d, "\Psi"'] & = & \big(& F_0 \arrow[d, "\Psi_0"'] & F_1 \arrow[l] \arrow[r] \arrow[d, "\Psi_1"] & F_2 \arrow[d, "\Psi_2"] & \big) \\ G & = & \big(& G_0 & G_1 \arrow[l] \arrow[r] & G_2 & \big) \end{tikzcd}$$

- We call Ψ a weak equivalence if it is an objectwise weak equivalence, i.e. if Ψ_i is a weak equivalence in $\mathcal{C}$ for $0 \le i \le 2$.
- We call Ψ a fibration if it is an objectwise fibration, i.e. if Ψ_i is a fibration in $\mathcal{C}$ for $0 \le i \le 2$.
- We call Ψ a cofibration if the morphisms $colim\big(F(i) \leftarrow F(1) \xrightarrow{\Psi_1} G(1)\big) \to G(i)$ (for $i \in \{0, 2\}$) and $\Psi_1 : F(1) \to G(1)$ are cofibrations in $\mathcal{C}$.

It is not difficult to check that $Fun(I, \mathcal{C})$, with the above choice of weak equivalences, fibrations and cofibrations, satisfies the axioms of a model category (see [28, Section 10] for a more detailed discussion of push-out diagrams). With this model structure a diagram $F(0) \leftarrow F(1) \rightarrow F(2)$ is cofibrant if the object $F(1)$ is cofibrant in $\mathcal{C}$ and the morphisms $F(1) \to F(0)$, $F(1) \to F(2)$ are cofibrations in $\mathcal{C}$.

Proposition 3.5.(2) implies that the functor $colim_I : Fun(I, \mathcal{C}) \to \mathcal{C}$ is homotopy meaningful on cofibrant objects. Therefore, according to Proposition 4.4, its total left derived functor $hocolim_I$ exists. It can be constructed by taking the colimit of a cofibrant replacement. Moreover, Proposition 3.5.(2) implies that if $F = \big(F(0) \leftarrow F(1) \rightarrow F(2)\big)$ is a push-out diagram of cofibrant objects where either $F(1) \to F(0)$ or $F(1) \to F(2)$ is a cofibration, then the morphism $hocolim(F) \to colim(F)$ is a weak equivalence.

5.3. **Example**. Let I be a telescope category, i.e. a category associated with a graph $(*_0 \to *_1 \to *_2 \to \cdots)$. Let $\Psi : F \to G$ be a morphism in $Fun(I, \mathcal{C})$ given by a commutative diagram:

$$\begin{tikzcd} F \arrow[d, "\Psi"'] & = & \big(& F_0 \arrow[r] \arrow[d, "\Psi_0"'] & F_1 \arrow[r] \arrow[d, "\Psi_1"] & F_2 \arrow[r] \arrow[d, "\Psi_2"] & \cdots & \big) \\ G & = & \big(& G_0 \arrow[r] & G_1 \arrow[r] & G_2 \arrow[r] & \cdots & \big) \end{tikzcd}$$

- We call Ψ a weak equivalence if it is an objectwise weak equivalence, i.e. if Ψ_i is a weak equivalence in $\mathcal{C}$ for $i \geq 0$.
- We call Ψ a fibration if it is an objectwise fibration, i.e. if Ψ_i is a fibration in $\mathcal{C}$ for $i \geq 0$.
- We call Ψ a cofibration if the morphisms $\Psi_0 : F(0) \to G(0)$ and, for $i \geq 0$, $colim\big(G(i) \xleftarrow{\Psi_i} F(i) \to F(i+1)\big) \to G(i+1)$ are cofibrations in $\mathcal{C}$.

It is not difficult to check that $Fun(I, \mathcal{C})$, with the above choice of weak equivalences, fibrations and cofibrations, satisfies the axioms of a model category. With this model structure a diagram $\big(F(0) \to F(1) \to F(2) \to \cdots\big)$ is cofibrant if the object $F(0)$ is cofibrant and, for $i \geq 0$, the morphisms $F(i) \to F(i+1)$ are cofibrations.

Proposition 3.5.(3) implies that the functor $colim_I : Fun(I, \mathcal{C}) \to \mathcal{C}$ is homotopy meaningful on cofibrant objects. Therefore, according to Proposition 4.4, its total left derived functor $hocolim_I$ exists. It can be constructed by taking the colimit of a cofibrant replacement.

For a general category I however, we do not know how to put a model structure on $Fun(I, \mathcal{C})$ for which the colimit is homotopy meaningful on cofibrant objects. Therefore we can not use directly Proposition 4.4 to construct $hocolim_I$. Instead we are going to approximate $Fun(I, \mathcal{C})$ by a model category. Using this approximation we can find a candidate for a "cofibrant replacement" in $Fun(I, \mathcal{C})$. We can then construct the total left derived functor of $colim_I$ in an analogous way to the construction given in Proposition 4.4. More precisely:

5.4. **Proposition**. *Let $\mathcal{M}$ be a model category. Assume that we have a commutative diagram of functors (both triangles commute):*

$$\begin{tikzcd} Fun(I,\mathcal{C}) \arrow[rr, shift left, "r"] \arrow[dr, "colim_I"'] & & \mathcal{M} \arrow[ll, shift left, "l"] \arrow[dl, "\mathcal{H}"] \\ & C & \end{tikzcd}$$

where:

- *l is left adjoint to r.*
- *r is homotopy meaningful, i.e. if Ψ is a weak equivalence in $Fun(I, \mathcal{C})$, then $r\Psi$ is a weak equivalence in $\mathcal{M}$.*
- *For any diagram $F : I \to \mathcal{C}$ and any cofibrant object X in $\mathcal{M}$, if a morphism $X \to rF$ is a weak equivalence in $\mathcal{M}$, then so is its adjoint $lX \to F$ in $Fun(I, \mathcal{C})$.*
- *$\mathcal{H}$ is homotopy meaningful on cofibrant objects.*

Then:

(1) *The total left derived functor of $colim_I$ exists. It can be constructed by taking the composition $Fun(I, \mathcal{C}) \xrightarrow{r} \mathcal{M} \xrightarrow{L(\mathcal{H})} Ho(\mathcal{C})$.*

Moreover, if l is homotopy meaningful on cofibrant objects, then:

(2) *The localization of $Fun(I,\mathcal{C})$ with respect to weak equivalences exists. The homotopy category of $Fun(I,\mathcal{C})$ can be constructed by taking the full subcategory in $Ho(\mathcal{M})$ generated by the image of $Fun(I,\mathcal{C}) \xrightarrow{r} \mathcal{M} \to Ho(\mathcal{M})$.*

Proof of Proposition 5.4.(1). Since $\mathcal{H}$ is homotopy meaningful on cofibrant objects, according to Proposition 4.4, its total left derived functor exists. Moreover, it can be constructed by choosing a cofibrant replacement Q in $\mathcal{M}$ and taking $L(\mathcal{H})(X) = \mathcal{H}(QX)$.

For any $F : I \to \mathcal{C}$, define:

- $QF : I \to \mathcal{C}$ to be $lQrF$,
- $QF = lQrF \to F$ to be the adjoint of $QrF \overset{\sim}{\twoheadrightarrow} rF$.

Observe that the assignment $Fun(I,\mathcal{C}) \ni F \mapsto colim_I QF \in Ho(\mathcal{C})$ is a well defined functor as it coincides with $L(\mathcal{H}) \circ r$.

We are going to show that $colim_I Q(-) : Fun(I,\mathcal{C}) \to Ho(\mathcal{C})$, together with the natural transformation induced by $colim_I(QF \to F)$, is the total left derived functor of $colim_I$. Let $\mathcal{G} : Fun(I,\mathcal{C}) \to Ho(\mathcal{C})$ be homotopy invariant and $\mathcal{G} \to colim_I$ be a natural transformation. For any $F : I \to \mathcal{C}$, define $\mathcal{G}(F) \to colim_I QF$ to be the unique morphism that fits into the following commutative diagram:

$$\begin{tikzcd} \mathcal{G}(QF) \arrow[r] \arrow[d] & colim_I QF \arrow[d] \\ \mathcal{G}(F) \arrow[r] \arrow[ur] & colim_I F \end{tikzcd}$$

Since QrF is cofibrant, $QF \to F$ is a weak equivalence in $Fun(I,\mathcal{C})$ as its adjoint is one. It follows that $\mathcal{G}(QF) \to \mathcal{G}(F)$ is an isomorphism and hence such a $\mathcal{G}(F) \to colim_I QF$ does exist. These morphisms form the appropriate unique natural transformation $\mathcal{G} \to colim_I Q(-)$. $\square$

5.5. **Lemma**. *Under the assumption of Proposition 5.4, a natural transformation $\Psi : F \to G$ in $Fun(I,\mathcal{C})$ is a weak equivalence if and only if $r\Psi$ is a weak equivalence in $\mathcal{M}$.*

Proof. Let us assume that $r\Psi$ is a weak equivalence in $\mathcal{M}$. By taking cofibrant replacements we get a commutative square where all morphisms are weak equivalences:

$$\begin{tikzcd} QrF \arrow[r, "Qr\Psi"] \arrow[d] & QrG \arrow[d] \\ rF \arrow[r, "r\Psi"] & rG \end{tikzcd}$$

Since QrF is cofibrant, the morphisms $lQrF \to F$ and $lQrF \to G$ are weak equivalences as their adjoints are so. Commutativity of the triangle

$$\begin{tikzcd} lQrF \arrow[d, "\wr"'] \arrow[dr, "\sim"] & \\ F \arrow[r, "\Psi"] & G \end{tikzcd}$$

and axiom **MC2** prove that Ψ is a weak equivalence as well. □

Proof of Proposition 5.4.(2). Let us denote by $Ho\big(Fun(I,\mathcal{C})\big)$ the full subcategory in $Ho(\mathcal{M})$ generated by the image of $Fun(I,\mathcal{C}) \xrightarrow{r} \mathcal{M} \to Ho(\mathcal{M})$. By Lemma 5.5, to prove that $Fun(I,\mathcal{C}) \to Ho\big(Fun(I,\mathcal{C})\big)$ satisfies the universal property of the localization functor with respect to weak equivalence, we only have to show that the category $Ho\big(Fun(I,\mathcal{C})\big)$ does not have to many morphisms.

Let $\alpha : Fun(I,\mathcal{C}) \to \mathcal{D}$ be a homotopy invariant functor. We prove that there exists a unique functor $\beta : Ho\big(Fun(I,\mathcal{C})\big) \to \mathcal{D}$ for which the composition $Fun(I,\mathcal{C}) \to Ho\big(Fun(I,\mathcal{C})\big) \xrightarrow{\beta} \mathcal{D}$ equals α. On objects we have no choice, we define $\beta([F]) := \alpha(F)$.

Let $F : I \to \mathcal{C}$ and $G : I \to \mathcal{C}$ be functors. Consider a morphism $[f] : [F] \to [G]$ in $Ho\big(Fun(I,\mathcal{C})\big) \subset Ho(\mathcal{M})$ induced by a morphism $f : QrF \to RQrG$ in $\mathcal{M}$, where Q and R are appropriate cofibrant and fibrant replacements (see Section 3 for the definition of a morphism in the homotopy category). Consider the following natural transformations:

$$F \leftarrow lQrF \xrightarrow{lf} lRQrG \leftarrow lQrG \to G$$

Observe that $lQrF \to F$ and $lQrG \to G$ are weak equivalences as their adjoints are so. Since l is homotopy meaningful on cofibrant objects, $lRQrG \leftarrow lQrG$ is also a weak equivalence. We define $\beta([f])$ to be the unique morphism which makes the following diagram to commute in $\mathcal{D}$:

$$\begin{tikzcd} \alpha(lQrF) \arrow[r] \arrow[d, "\cong"'] & \alpha(lRQrG) \arrow[dr] & \alpha(lQrG) \arrow[l, "\cong"'] \arrow[d, "\cong"] \\ \alpha(F) \arrow[ur] \arrow[rr, "\beta([f])"] & & \alpha(G) \end{tikzcd}$$

Since α converts weak equivalences into isomorphisms, such $\beta([f])$ does exists. One can check that this process defines the desired functor $\beta : Ho\big(Fun(I,\mathcal{C})\big) \to \mathcal{D}$. □

6. Spaces and small categories

In this section we recall the definitions and some basic functorial properties of two constructions which intertwine categories with spaces. To a space one can associate its so called simplex category (see [31] and [38]). To a small category one can associate a space called its nerve.

6.1. **Definition**. *Let K be a simplicial set. The simplex category of K is a category whose objects are maps of the form $\sigma : \Delta[n] \to K$ and a morphism between $\sigma : \Delta[n] \to K$ and $\tau : \Delta[m] \to K$ is a commutative triangle:*

$$\begin{tikzcd} \Delta[n] \arrow[rr] \arrow[dr, "\sigma"'] & & \Delta[m] \arrow[dl, "\tau"] \\ & K & \end{tikzcd}$$

To avoid a proliferation of notations the simplex category of K is denoted by the same symbol K. The terminology we use comes from the fact that the objects of K can be identified with the simplices of K. For each n-dimensional simplex $\sigma \in K_n$, there are $n+1$ *different* morphisms called degeneracies $\{s_i : s_i\sigma \to \sigma\}_{0\le i\le n}$ and, if $n > 0$, there are $n+1$ *different* morphisms called faces $\{d_i : d_i\sigma \to \sigma\}_{0\le i\le n}$. Subject to the usual cosimplicial relations they generate all the morphisms in K.

Taking the simplex category of K is natural in K and hence this process defines a functor $Spaces \to Cat$.

6.2. **Example**. The simplex category of $\Delta[0]$ is the category Δ. Thus diagrams indexed by $\Delta[0]$ are cosimplicial objects. Simplex categories are therefore big and complicated. In order to simplify the situation we will put restrictions on diagrams indexed by them (see Sections 10 and 12).

Although the category $\Delta[n]$ is rather complicated, it is easy to calculate colimits of diagrams indexed by it. The only non-degenerate n-dimensional simplex $\iota \in \Delta[n]$, i.e. the simplex corresponding to the identity map $id : \Delta[n] \to \Delta[n]$, is a terminal object in the category $\Delta[n]$. Thus, if $F : \Delta[n] \to \mathcal{C}$ is a functor, then the morphism $F(\iota) \to colim_{\Delta[n]}F$ is an isomorphism.

6.3. **Definition**. *Let I be a small category. The nerve of I is a simplicial set $N(I)$ whose set of n-dimensional simplices is given by:*

$$N(I)_n := \{i_n \xrightarrow{\alpha_n} \cdots \xrightarrow{\alpha_1} i_0 \mid \alpha_k \text{ is a morphism in } \mathcal{C}\}$$

The simplicial operators d_l and s_l are defined as follows:

$$s_0 : N(I)_0 \to N(I)_1\ ,\ s_0(i) := i \xrightarrow{id} i$$

Let $\sigma = (i_n \xrightarrow{\alpha_n} i_{n-1} \xrightarrow{\alpha_{n-1}} \cdots \xrightarrow{\alpha_1} i_0)$ be an element of $N(I)_n$. For $n > 0$ and $0 \le l \le n$, the maps $N(I)_{n+1} \xleftarrow{s_l} N(I)_n \xrightarrow{d_l} N(I)_{n-1}$ are defined as:

$$s_l(\sigma) := (i_n \xrightarrow{\alpha_n} \cdots \xrightarrow{\alpha_{l+1}} i_l \xrightarrow{id} i_l \xrightarrow{\alpha_l} \cdots \xrightarrow{\alpha_1} i_0)$$

$$d_l(\sigma) := \begin{cases} (i_n \xrightarrow{\alpha_n} \cdots \xrightarrow{\alpha_2} i_1) & \text{if } l = 0 \\ (i_n \xrightarrow{\alpha_n} \cdots \xrightarrow{\alpha_{l+2}} i_{l+1} \xrightarrow{\alpha_l \circ \alpha_{l+1}} i_{l-1} \xrightarrow{\alpha_{l-1}} \cdots \xrightarrow{\alpha_1} i_0) & \text{if } 0 < l < n \\ (i_{n-1} \xrightarrow{\alpha_{n-1}} \cdots \xrightarrow{\alpha_1} i_0) & \text{if } l = n \end{cases}$$

Taking the nerve $N(I)$ of a category I is natural in I, i.e. this process defines a functor $N : Cat \to Spaces$.

6.4. **Remark**. For any $n \geq 0$, let $[n]$ to be the category associated with a graph $(*_n \to *_{n-1} \to \cdots \to *_0)$. These categories can be assembled together to form a cosimplicial object $[-] : \Delta \to Cat$. The nerve of I can be identified with a simplicial set whose set of n-dimensional simplices is $Fun([n], I)$ and the simplicial operators are induced by the cosimplicial operators in $[-]$, i.e. $N(I) = Fun([-], I)$.

6.5. **Example**. The nerve of the category $[n]$ (see Remark 6.4) coincide with the n-dimensional simplex $\Delta[n]$, i.e. $N([n]) = \Delta[n]$.

There are two forgetful functors $\epsilon : N(I) \to I$ and $\epsilon : N(I)^{op} \to I$ associated with the nerve construction:

- $\epsilon : N(I) \to I$ is defined as:

$$(i_n \xrightarrow{\alpha_n} \cdots \xrightarrow{\alpha_1} i_0) = \sigma \mapsto i_0\ ,\ (s_l\sigma \xrightarrow{s_l} \sigma) \mapsto id_{i_0}\ ,\ (d_l\sigma \xrightarrow{d_l} \sigma) \mapsto \begin{cases} id_{i_0} & \text{if } \ l > 0 \\ \alpha_1 & \text{if } \ l = 0 \end{cases}$$

- $\epsilon : N(I)^{op} \to I$ is defined as:

$$(i_n \xrightarrow{\alpha_n} \cdots \xrightarrow{\alpha_1} i_0) = \sigma \mapsto i_n\ ,\ (s_l\sigma \xrightarrow{s_l} \sigma) \mapsto id_{i_n}\ ,\ (d_l\sigma \xrightarrow{d_l} \sigma) \mapsto \begin{cases} id_{i_n} & \text{if } \ l < n \\ \alpha_n & \text{if } \ l = n \end{cases}$$

6.6. **Example**. Let I be a small category. Consider the functor $N(I\!\downarrow\!-) : I \to Spaces$ which assigns to an object i the nerve of the over category $I\!\downarrow\! i$ (see A.1). By sending an n-simplex $(i_n \to \cdots \to i_0 \to i)$ in $N(I\!\downarrow\! i)$ to $(i_n \to \cdots \to i_0) \in N(I)$ we get a map $N(I\!\downarrow\! i) \to N(I)$. This map fits into the following pull-back square in Cat:

$$\begin{tikzcd} N(I\downarrow i) \arrow[r] \arrow[d, "\epsilon"'] & N(I) \arrow[d, "\epsilon"] \\ I\downarrow i \arrow[r] & I \end{tikzcd}$$

where $\epsilon : N(I) \to I$ and $\epsilon : N(I\!\downarrow\! i) \to I\!\downarrow\! i$ are the forgetful functors. In particular this implies that $N(I\!\downarrow\! i)$ can be identified with $\epsilon\!\downarrow\! i$ (see A.2).

The maps $N(I\!\downarrow\! i) \to N(I)$ form a natural transformation between the diagram $N(I\!\downarrow\!-) : I \to Spaces$ and the space $N(I)$. This natural transformation satisfies the universal property of the colimit of $N(I\!\downarrow\!-)$ and hence the induced map $colim_I N(I\!\downarrow\!-) \to N(I)$ is an isomorphism.

Here is a list of some basic functorial properties of the nerve functor and the simplex category functor:

6.7. The nerve $N : Cat \to Spaces$ has a left adjoint $c : Spaces \to Cat$ (see [31]), i.e.:

$$mor_{Spaces}\big(K, N(I)\big) = Fun\big(c(K), I\big)$$

Hence the nerve converts pull-backs in Cat into pull-backs in $Spaces$. In particular $N(I \times J) = N(I) \times N(J)$.

6.8. The functor $Spaces \to Cat$ has a right adjoint $S : Cat \to Spaces$ (see [31]), i.e.:

$$Fun(K, I) = mor_{Spaces}\big(K, S(I)\big)$$

This right adjoint can be defined as $S(I) := Fun(\Delta[-], I)$. It follows that the functor $Spaces \to Cat$ converts colimits in $Spaces$ into colimits in Cat.

6.9. By a direct verification one can show that the functor $Spaces \to Cat$ converts pull-backs in $Spaces$ into pull-backs in Cat. Thus in particular the simplex category of the product of two spaces $K \times N$ can be identified with the pull-back $K \times_{\Delta[0]} N = lim(K \to \Delta[0] \leftarrow N)$ in Cat. Nevertheless the functor $Spaces \to Cat$ does not have a left adjoint. If it had one, it would convert products in $Spaces$ into products in Cat. However, since the simplex category of $\Delta[0]$ is not a terminal object in Cat, the product of simplex categories is much bigger than the simplex category of the product. We use therefore the following notation:

Notation. The product of simplex categories of K and N is denoted by $K \tilde{\times} N$.

6.10. The composition $Spaces \to Cat \xrightarrow{N} Spaces$ has a right adjoint $Ex : Spaces \to Spaces$, i.e.

$$mor_{Spaces}\big(N(K), L\big) = mor_{Spaces}\big(K, Ex(L)\big)$$

The functor Ex can be defined as $Ex(L) := mor_{Spaces}\big(N(\Delta[-]), L\big)$. It follows that the functor $Spaces \to Cat \xrightarrow{N} Spaces$ commutes with colimits. Explicitly, for any diagram of spaces $H : I \to Spaces$, the natural morphisms $colim_I N(H) \to N(colim_I H)$ and $colim_I N(H^{op}) \to N\big((colim_I H)^{op}\big)$ are isomorphisms.

7. Pull-back process and local properties

Let $f : L \to K$ be a map of spaces. We can think about f as a functor between simplex categories and hence consider the pull-back process along f (see A.3). By definition it is a functor $f^* : Fun(K, \mathcal{C}) \to Fun(L, \mathcal{C})$ which assigns to a diagram $F : K \to \mathcal{C}$ the composition $L \xrightarrow{f} K \xrightarrow{F} \mathcal{C}$. The pull-back process commutes with compositions of maps, i.e. for any $N \xrightarrow{h} L \xrightarrow{f} K$, $(f \circ h)^*$ coincide with $h^* \circ f^*$.

In this paper we are particularly interested in those properties of diagrams indexed by simplex categories which are preserved by the process of pulling-back along maps of spaces.

7.1. **Definition**. *We say that a property of diagrams indexed by simplex categories is local if the following statements are equivalent:*

- *A diagram $F : K \to \mathcal{C}$ has this property.*
- *For any simplex $\Delta[n] \to K$, the composition $\Delta[n] \to K \xrightarrow{F} \mathcal{C}$ has this property.*

Local properties are preserved by the pull-back process, i.e. if $F : K \to \mathcal{C}$ satisfies some local property, then, for any map $L \to K$, so does the composition $L \to K \xrightarrow{F} \mathcal{C}$.

8. Colimits of diagrams indexed by spaces

Simplex categories are associated with geometric objects. We would like to take advantage of the intuition coming from this geometry to understand diagrams indexed by such categories and constructions on them.

Let $H : I \to Spaces$ be a diagram of spaces and $\{H(i) \to colim_I H\}_{i \in I}$ be a natural transformation which satisfies the universal property of the colimit of H. To describe a functor $F : colim_I H \to \mathcal{C}$ it is sufficient and necessary to have the following data (compare with A.9):

(1) for every object $i \in I$, a functor $F_i : H(i) \to \mathcal{C}$;
(2) for every morphism $\alpha : i \to j$ in I, $F_i : H(i) \to \mathcal{C}$ should coincide with the pull-back $H(\alpha)^* F_j$, i.e. F_i should be equal to the composition $H(i) \xrightarrow{H(\alpha)} H(j) \xrightarrow{F_j} \mathcal{C}$.

If $F : colim_I H \to \mathcal{C}$ is a diagram, then $F_i : H(i) \to \mathcal{C}$ is given by the composition $H(i) \to colim_I H \xrightarrow{F} \mathcal{C}$.

In the case H is a push-out diagram we get:

8.1. **Proposition**. *Let the following be a push-out square of spaces:*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & L \arrow[d] \\ B \arrow[r] & M \end{tikzcd}$$

A diagram $F : L \to \mathcal{C}$ is isomorphic to one of the form $L \to M \to \mathcal{C}$ if and only if the pull-back $A \to L \xrightarrow{F} \mathcal{C}$ is isomorphic to one of the form $A \to B \to \mathcal{C}$.

Analogously, to describe a functor $G : (colim_I H)^{op} \to \mathcal{C}$ it is sufficient and necessary to have the following data:

(1) for every object $i \in I$, a functor $G_i : H(i)^{op} \to \mathcal{C}$;
(2) for every morphism $\alpha : i \to j$ in I, $G_i : H(i)^{op} \to \mathcal{C}$ should coincide with the composition $H(i)^{op} \xrightarrow{H(\alpha)} H(j)^{op} \xrightarrow{G_j} \mathcal{C}$.

If $G : (colim_I H)^{op} \to \mathcal{C}$ is a diagram, then $G_i : H(i) \to \mathcal{C}$ is given by the composition $H(i)^{op} \to (colim_I H)^{op} \xrightarrow{G} \mathcal{C}$.

The geometry of an indexing space can be used to calculate colimits. The following proposition can be used to construct colimits by induction on the cell decomposition of the indexing space.

8.2. **Proposition**. *Let $H : I \to Spaces$ be a functor.*

(1) *For any $F : colim_I H \to \mathcal{C}$, $colim_{colim_I H} F = colim_I colim_{H(i)} F_i$.*
(2) *For any $G : (colim_I H)^{op} \to \mathcal{C}$, $colim_{(colim_I H)^{op}} G = colim_I colim_{H(i)^{op}} G_i$*

Since the proofs are analogous we show only (1).

Proof of (1). Let us denote the map $H(i) \to colim_I H$ by ξ_i. Let $colim_{H(i)} F_i \to colim_{colim_I H} F$ be induced by ξ_i. We show that $colim_{colim_I H} F$, together with the natural transformation induced by these morphisms, satisfies the universal property of the colimit of the diagram $i \mapsto colim_{H(i)} F_i$. Let us choose a compatible family of morphism $\{h_i : colim_{H(i)} F_i \to X\}_{i \in I}$. For every simplex σ in $colim_I H$, there exists $j \in I$ and $\tau \in H(j)$ such that $\xi_j(\tau) = \sigma$. Define $F(\sigma) \to X$ to be the composition $F(\sigma) = F\big(\xi_j(\tau)\big) \to colim_{H(j)} F_j \xrightarrow{h_j} X$. It is easy to check that, for all $\sigma \in colim_I H$, these morphisms are well defined (they depend only on $\sigma \in colim_I H$) and they are compatible over $colim_I H$. Hence they induce a morphism $colim_{colim_I H} F \to X$. It is clear from the above description that this morphism is unique. $\square$

8.3. **Remark**. Let $H : I \to Spaces$ be a functor. We can perform two operations on H. We can take its colimit $colim_I H$, or we can think about the values of H as simplex categories and take its Grothendieck construction $Gr_I H$ (see Definition A.5). There is a functor connecting these two categories $Gr_I H \to colim_I H$. It sends an object (i, τ) to the image of τ under the map $H(i) \to colim_I H$. Propositions 8.2 and A.10 say that this functor is cofinal with respect to taking colimits.

The following particular cases of Proposition 8.2 are of special interest:

8.4. **Corollary**.

(1) *Let $M = colim(B \leftarrow A \to L)$ and $F : M \to \mathcal{C}$ be a functor. Then the following is a push-out square:*

$$\begin{tikzcd} colim_A F \arrow[r] \arrow[d] & colim_L F \arrow[d] \\ colim_B F \arrow[r] & colim_M F \end{tikzcd}$$

(2) *Let $K = colim(K_0 \to K_1 \to K_2 \to \cdots)$ where the telescope is possibly transfinite. Let $F : K \to \mathcal{C}$ be a functor. Then:*

$$colim_K F = colim(colim_{K_0} F \to colim_{K_1} F \to colim_{K_2} F \to \cdots)$$

Calculating colimits of diagrams indexed by arbitrary small categories can always be reduced to calculating colimits of diagrams indexed by simplex categories. The following proposition can be shown by checking that, for all $i \in I$, the categories $I\!\downarrow\! N(I)$ and $i\!\downarrow\!(NI)^{op}$ (see A.2) are non-empty and connected (see A.4 and [32, Theorem IX.3.1]).

8.5. **Proposition**. *The forgetful functors $\epsilon : N(I) \to I$ and $\epsilon : N(I)^{op} \to I$ (see Section 6) are cofinal with respect to taking colimits, i.e. for any $F : I \to \mathcal{C}$, the induced morphisms $colim_{N(I)}\epsilon^* F \to colim_I F$ and $colim_{N(I)^{op}}\epsilon^* F \to colim_I F$ are isomorphisms.*

9. Kan extensions

Let $\mathcal{C}$ be a category closed under colimits. Consider a map of spaces $f : L \to K$. In addition to the pull-back process $f^* : Fun(K, \mathcal{C}) \to Fun(L, \mathcal{C})$, one can associate with f a functor going the "other direction" $f^k : Fun(L, \mathcal{C}) \to Fun(K, \mathcal{C})$. This functor is called the Kan extension of f (see A.3 and [32, Section X.3]). It can be characterized as the left adjoint of f^*. In order to give an explicit construction of f^k we first have to decompose f into a diagram $df : K \to Spaces$. For a simplex $\sigma : \Delta[n] \to K$ define $df(\sigma)$ to be the space that fits into the following pull-back square:

$$\begin{tikzcd} df(\sigma) \arrow[r] \arrow[d] & L \arrow[d, "f"] \\ \Delta[n] \arrow[r, "\sigma"] & K \end{tikzcd}$$

The maps $df(\sigma) \to L$ form a natural transformation between the functor df and the space L. This natural transformation satisfies the universal property of the colimit of df, and hence the induced map $colim_K df \to L$ is an isomorphism. Moreover f can be expressed as $colim_{\sigma\in K}\big(df(\sigma) \to \Delta[dim(\sigma)]\big)$. This suggests that one should think about the diagram df as a decomposition of f into pieces lying over small parts of K.

On the level of simplex categories, for any $\sigma : \Delta[n] \to K$, $\Delta[n]$ is isomorphic to $K\!\downarrow\!\sigma$, $df(\sigma)$ is isomorphic to $f\!\downarrow\!\sigma$ and the above diagram corresponds to the following pull-back square in Cat (see A.2):

$$\begin{tikzcd} f\downarrow\sigma \arrow[r] \arrow[d] & L \arrow[d, "f"] \\ K\downarrow\sigma \arrow[r] & K \end{tikzcd}$$

9.1. **Definition**. *Let $f : L \to K$ be a map of spaces. The Kan extension of f is a functor $f^k : Fun(L, \mathcal{C}) \to Fun(K, \mathcal{C})$, which assigns to $F : L \to \mathcal{C}$ a diagram $f^k F : K \to \mathcal{C}$ defined as $K \ni \sigma \mapsto colim_{df(\sigma)}F$, i.e. $f^k F = colim_{df(-)}F$.*

The Kan extension process does not modify colimits of diagrams.

9.2. **Proposition**. *For any map $f : L \to K$, the following triangle commutes:*

$$\begin{tikzcd} Fun(L,\mathcal{C}) \arrow[rr, "f^k"] \arrow[dr, "colim_L"'] & & Fun(K,\mathcal{C}) \arrow[dl, "colim_K"] \\ & \mathcal{C} & \end{tikzcd}$$

Proof. Let $F : L \to \mathcal{C}$ be a diagram. Since $L = colim_K df$, according to Proposition 8.2.(1), $colim_L F = colim_{colim_K df} F = colim_K colim_{df} F = colim_K f^k F$. $\square$

The Kan extension and the pull-back process are closely related (see A.3):

9.3. **Proposition**. *Let $f : L \to K$ be a map of spaces. The Kan extension functor $f^k : Fun(L, \mathcal{C}) \to Fun(K, \mathcal{C})$ is left adjoint to $f^* : Fun(K, \mathcal{C}) \to Fun(L, \mathcal{C})$.*

9.4. **Corollary**. *The Kan extension $f^k : Fun(L, \mathcal{C}) \to Fun(K, \mathcal{C})$ commutes with colimits. Explicitly, for any diagram $F : I \to Fun(L, \mathcal{C})$, the natural transformation $colim_I(f^k F) \to f^k(colim_I F)$ is an isomorphism.*

Proof. The corollary is a consequence of f^k being a left adjoint. We can also argue directly. Let $\sigma \in K$ be a simplex. By definition $(colim_I f^k F)(\sigma) = colim_I(f^k F(\sigma)) = colim_{i \in I} colim_{df(\sigma)} F_i$. Since the colimit functor commutes with itself, we can conclude that $(colim_I f^k F)(\sigma) = colim_{df(\sigma)}(colim_I F)(\sigma) = f^k(colim_I F)(\sigma)$. $\square$

9.5. **Corollary**. *The Kan extension process commutes with composition of maps, i.e, for any $N \xrightarrow{h} L \xrightarrow{f} K$, $(f \circ h)^k = f^k \circ h^k$.*

Proof. Since f^k is left adjoint to f^* and h^k is left adjoint to h^*, the composition $f^k \circ h^k$ is left adjoint to $h^* \circ f^*$. Hence $f^k \circ h^k$ coincide with $(f \circ h)^k$, as $h^* \circ f^* = (f \circ h)^*$. $\square$

In Sections 10–13 we introduce and study the notion of boundness for diagrams indexed by simplex categories. It is a fundamental concept in this paper and plays a vital role in subsequent sections.

10. Bounded diagrams

Since simplex categories are very complicated, in order to simplify the situation, we put restrictions on diagrams indexed by them:

10.1. **Definition**. *A functor $F : K \to \mathcal{C}$ is called bounded if, for any degeneracy map $s_i : s_i\sigma \to \sigma$ in K, the morphism $F(s_i) : F(s_i\sigma) \to F(\sigma)$ is an isomorphism. It is called strongly bounded if, for any s_i, $F(s_i) : F(s_i\sigma) \to F(\sigma)$ is the identity.*

Using the simplicial identities $d_i \circ s_i = d_{i+1} \circ s_i = id$ the boundness condition can also be expressed in terms of the boundary morphisms in K.

10.2. **Proposition**. *A diagram $F : K \to \mathcal{C}$ is (strongly) bounded if and only if, for any simplex $\sigma \in K$ of the form $\sigma = s_i\xi$, the morphisms $F(d_i) : F(d_i\sigma) \to F(\sigma)$ and $F(d_{i+1}) : F(d_{i+1}\sigma) \to F(\sigma)$ are (identities) isomorphisms.*

Bounded diagrams indexed by K, with natural transformations as morphisms, form a category. This category is be denoted by $Fun^b(K, \mathcal{C})$. It is a full subcategory of $Fun(K, \mathcal{C})$. If S denotes the set of all degeneracy morphisms in K, then $Fun^b(K, \mathcal{C})$ can be identified with $Fun(K[S^{-1}], \mathcal{C})$.

The full subcategory of $Fun^b(K, \mathcal{C})$ consisting of strongly bounded diagrams is be denoted by $Fun^{sb}(K, \mathcal{C})$. The inclusion $Fun^{sb}(K, \mathcal{C}) \subset Fun^b(K, \mathcal{C})$ is "almost" an isomorphism.

10.3. **Proposition**. *If $F : K \to \mathcal{C}$ is a bounded diagram, then there exists an isomorphism $F \to F'$, depending functorially on F, such that F' is strongly bounded.*

Proof. Any simplex $\sigma : \Delta[m] \to K$ can be expressed uniquely as a composition:

$$\Delta[m] \xrightarrow{s_{i_1}} \Delta[m-1] \xrightarrow{s_{i_2}} \cdots \xrightarrow{s_{i_k}} \Delta[m-k] \xrightarrow{\sigma'} K$$

where $i_1 < i_2 < \cdots < i_k$ and σ' is non-degenerate (see [6, Section VIII 2.3]).

Let $F : K \to \mathcal{C}$ be a bounded diagram. For any $\sigma \in K$, define $F'(\sigma) := F(\sigma')$ and, for any $\alpha : \sigma \to \tau$ in K, define $F'(\alpha)$ to be the composition:

$$F(\sigma') \xrightarrow{F(\sigma \to \sigma')^{-1}} F(\sigma) \xrightarrow{F(\alpha)} F(\tau) \xrightarrow{F(\tau \to \tau')} F(\tau')$$

This assignment clearly defines a functor $F' : K \to \mathcal{C}$. It is also clear that, for any degeneracy morphism s_i in K, $F'(s_i)$ is the identity, i.e. F' is strongly bounded.

Let $F \to F'$ be a natural transformation induced by $F(\sigma) \to F(\sigma') = F'(\sigma)$. Since these morphisms are isomorphisms, the proposition has been proven. $\square$

Boundness is a local property (see Definition 7.1).

10.4. **Proposition**. *A diagram $F : K \to \mathcal{C}$ is bounded if and only if, for any simplex $\Delta[n] \to F$, the composition $\Delta[n] \to K \xrightarrow{F} \mathcal{C}$ is a bounded diagram.*

10.5. **Corollary**. *Let $f : L \to K$ be a map of spaces. If $F : K \to \mathcal{C}$ is a bounded diagram, then so is the composition $L \xrightarrow{f} K \xrightarrow{F} \mathcal{C}$, i.e. the pull-back process along $f : L \to K$ induces a functor $f^* : Fun^b(K, \mathcal{C}) \to Fun^b(L, \mathcal{C})$.*

Values of a strongly bounded diagram are entirely determined by its values on the non-degenerate simplices. However, in general the category $Fun^{sb}(K, \mathcal{C})$ can not be identified with $Fun(dK, \mathcal{C})$, where dK is the full subcategory of K having only non-degenerate simplices as objects (see Example 10.8).

10.6. **Example**. The simplex category of $\Delta[0]$ is isomorphic to Δ. Thus $Fun(\Delta[0], \mathcal{C})$ can be identified with the category of cosimplicial objects in $\mathcal{C}$. A strongly bounded diagram indexed by $\Delta[0]$ is entirely determined by the value it takes on the unique 0-dimensional simplex (the only non-degenerate one). Thus $Fun^{sb}(\Delta[0], \mathcal{C})$ can be identified with $\mathcal{C}$. In this case $Fun^{sb}(\Delta[0], \mathcal{C})$ coincide with the category $Fun(d\Delta[0], \mathcal{C})$.

A strongly bounded diagram $F : \Delta[1] \to \mathcal{C}$ is entirely determined by the pull-back diagram $\big(F(0) \xrightarrow{F(d_1)} F(0,1) \xleftarrow{F(d_0)} F(1)\big)$. Therefore $Fun^{sb}(\Delta[1], \mathcal{C})$ is isomorphic to the category of diagrams in $\mathcal{C}$ of the shape:

$$\begin{tikzcd} & * \arrow[d] \\ * \arrow[r] & * \end{tikzcd}$$

i.e. $Fun^{sb}(\Delta[1], \mathcal{C})$ coincide with the category of almost square diagrams in $\mathcal{C}$ with the initial object missing. Again $Fun^{sb}(\Delta[1], \mathcal{C})$ can be identified with $Fun(d\Delta[1], \mathcal{C})$.

The category $Fun^{sb}(\Delta[2], \mathcal{C})$ can be identified with the category of diagrams in $\mathcal{C}$ of the shape:

$$\begin{tikzcd} & & & * \arrow[dl] \arrow[dd] \\ * \arrow[rr] \arrow[dd] & & * \arrow[dd] & \\ & * \arrow[rr] \arrow[dl] & & * \arrow[dl] \\ * \arrow[rr] & & * & \end{tikzcd}$$

i.e. $Fun^{sb}(\Delta[2], \mathcal{C})$ coincide with the category of almost 3-dimensional cubical diagrams in $\mathcal{C}$ with the initial object missing. In general, for all n, $Fun^{sb}(\Delta[n], \mathcal{C})$ can be identified with the category of almost $(n+1)$-dimensional cubical diagrams in $\mathcal{C}$ with the initial object missing. For all n, $Fun^{sb}(\Delta[n], \mathcal{C})$ is isomorphic to $Fun(d\Delta[n], \mathcal{C})$.

10.7. **Example**. Let us depict the space $\Delta[2, 1]$ and its non-degenerate simplices as:

$$\begin{tikzcd} 0 \arrow[r, dash, "(0{,}1)"] & 1 \arrow[r, dash, "(0{,}2)"] & 2 \end{tikzcd}$$

A strongly bounded diagram $F : \Delta[2,1] \to \mathcal{C}$ is determined by the following data:

$$F(0) \xrightarrow{F(d_1)} F(0,1) \xleftarrow{F(d_0)} F(1) \xrightarrow{F(d_1)} F(1,2) \xleftarrow{F(d_0)} F(2)$$

Therefore $Fun^{sb}(\Delta[2,1],\mathcal{C})$ can be identified with the category of diagrams in $\mathcal{C}$ of the shape $* \to * \leftarrow * \to * \leftarrow *$. In this case $Fun^{sb}(\Delta[2,1],\mathcal{C})$ coincides with $Fun(d\Delta[2,1],\mathcal{C})$.

10.8. **Example**. For $n > 0$, let $S^n := \Delta[n]/\partial(\Delta[n])$. A strongly bounded diagram $F : S^1 \to \mathcal{C}$ is determined by the data:

$$F(0) \underset{F(d_1)}{\overset{F(d_0)}{\rightrightarrows}} F(0,1)$$

Thus $Fun^{sb}(S^1,\mathcal{C})$ can be identified with the category of diagrams in $\mathcal{C}$ of the shape $* \rightrightarrows *$. In this case $Fun^{sb}(S^1,\mathcal{C})$ is again the same as $Fun(dS^1,\mathcal{C})$.

Let 0 be the only zero-dimensional simplex in S^n, τ be the only non-degenerate n-dimensional simplex in S^n and $\alpha : 0 \to \tau$ be any morphism in S^n. Let $F : S^n \to \mathcal{C}$ be strongly bounded. If $n > 1$, using the simplicial identities, one can show that the morphism $F(\alpha) : F(0) \to F(\tau)$ does not depend on the choice of α. This morphism determines the entire diagram F. Thus, for $n > 1$, $Fun^{sb}(S^n,\mathcal{C})$ can be identified with the category of morphisms in $\mathcal{C}$, i.e. with the category of diagrams in $\mathcal{C}$ of the shape $* \to *$. It follows that if $F : S^n \to \mathcal{C}$ is bounded, then $colim_{S^n} F$ is isomorphic to $F(\tau)$. In this case $Fun^{sb}(S^n,\mathcal{C})$ and $Fun(dS^n,\mathcal{C})$ do *NOT* coincide.

10.9. **Example**. Let I be a small category and $N(I)$ be its nerve. Let us consider the forgetful functor $\epsilon : N(I) \to I$, $(i_n \to \cdots \to i_0) \mapsto i_0$ (see Section 6). By precomposing with this functor diagrams indexed by I pull-back to *bounded* diagrams indexed by the nerve $N(I)$. This gives a monomorphism $\epsilon^* : Fun(I,\mathcal{C}) \to Fun^b\big(N(I),\mathcal{C}\big)$.

Observe that the pull-back $\epsilon^* F$ of a functor $F : I \to \mathcal{C}$ sends not only degeneracy morphisms $s_i : s_i\sigma \to \sigma$ in $N(I)$ to isomorphisms (it is a bounded diagram), but also boundary morphisms $d_i : d_i\sigma \to \sigma$ for $i > 0$.

10.10. **Example**. Let $f : L \to K$ be a map of spaces. The diagram $df : K \to Spaces$ (see Section 9) is never bounded.

11. Kan extensions and bounded diagrams

In this section we show that the Kan extension process (see Definition 9.1) preserves boundness.

11.1. **Theorem**. *Let $f : L \to K$ be a map of spaces. If $F : L \to \mathcal{C}$ is a bounded diagram, then so is $f^k F : K \to \mathcal{C}$, i.e. the Kan extension of f induces a functor $f^k : Fun^b(L,\mathcal{C}) \to Fun^b(K,\mathcal{C})$.*

11.2. **Lemma**. *Let the following be a pull-back square:*

$$\begin{tikzcd} P \arrow[r] \arrow[d] & \Delta[n+1] \arrow[d, "s_i"] \\ \Delta[m] \arrow[r, "\sigma"] & \Delta[n] \end{tikzcd}$$

If $F : \Delta[m] \to \mathcal{C}$ is bounded, then the induced morphism $colim_P F \to colim_{\Delta[m]} F$ is an isomorphism.

Proof. If σ does not contain the vertex i, then, according to Proposition B.4.(1), the map $P \to \Delta[m]$ coincide with $id : \Delta[m] \to \Delta[m]$. Hence in this case the lemma is obvious. We can assume therefore that σ does contain the vertex i. We can go further and assume that the map $\sigma : \Delta[m] \to \Delta[n]$ is onto all together. If not let $\Delta[l] \hookrightarrow \Delta[n]$ be the simplex corresponding to the image of σ. Since this simplex contains i, according to Proposition B.4.(2), for some j, the following are pull-back squares:

$$\begin{tikzcd} P \arrow[r] \arrow[d] & \Delta[l+1] \arrow[r] \arrow[d, "s_j"] & \Delta[n+1] \arrow[d, "s_i"] \\ \Delta[m] \arrow[r] \arrow[rr, bend right, "\sigma"] & \Delta[l] \arrow[r, hook] & \Delta[n] \end{tikzcd}$$

Thus by considering $\Delta[m] \to \Delta[l]$, instead of $\sigma : \Delta[m] \to \Delta[n]$, we can reduce the problem to the case when the map is onto.

Let $\sigma : \Delta[m] \to \Delta[n]$ be an epimorphism. We are going to show that, for any bounded diagram $F : \Delta[m] \to \mathcal{C}$, the morphism $colim_P F \to colim_{\Delta[m]} F$ is an isomorphism. We prove it by induction on the difference $m - n$.

If $m - n = 0$, then σ corresponds to the identity $\Delta[n] \to \Delta[n]$. Hence we can identify $P \to \Delta[n]$ with $s_i : \Delta[n+1] \to \Delta[n]$. In this case the lemma follows easily from the boundness assumption on F.

Let us assume that the statement is true for all maps $\sigma : \Delta[m] \to \Delta[n]$ where $m - n < k$. Choose a simplex $\sigma : \Delta[m] \to \Delta[n]$ for which $m - n = k$. If the pre-image $\sigma^{-1}(i)$ consists of only one element, then the map $P \to \Delta[m]$ corresponds to $s_j : \Delta[m+1] \to \Delta[m]$ (see Proposition B.4.(2)) and thus, by the boundness assumption on F, this case is clear.

Let us assume that the pre-image $\sigma^{-1}(i)$ consists of more than one element. In this case σ can be expressed as a composition $\Delta[m] \xrightarrow{\sigma'} \Delta[n+1] \xrightarrow{s_i} \Delta[n]$, where σ' is an epimorphism. Let P' be a space that fits into the following pull-back square:

$$\begin{tikzcd} P' \arrow[r] \arrow[d] & \Delta[n+2] \arrow[d, "s_{i+1}"] \\ \Delta[m] \arrow[r] & \Delta[n+1] \end{tikzcd}$$

According to Proposition B.4.(3) the map $P \to \Delta[m]$ can be expressed as a push-out:

$$\begin{tikzcd} P \arrow[d] & = & colim \Big(& \Delta[m+1] \arrow[dr, "s_i"] & \Delta[m] \arrow[l, "d_{i+1}"'] \arrow[r] \arrow[d, "id"] & P' \arrow[dl] & \Big) \\ \Delta[m] & & = & & \Delta[m] & & \end{tikzcd}$$

Hence:

$$colim_P F = colim\big(colim_{\Delta[m+1]} F \leftarrow colim_{\Delta[m]} F \to colim_{P'} F\big)$$

The boundness condition on F implies that $colim_{\Delta[m]} F \to colim_{\Delta[m+1]} F$ is an isomorphism. It follows that so is $colim_{P'} F \to colim_P F$. As by inductive assumption $colim'_P F \to colim_{\Delta[m]} F$ is an isomorphism, we can conclude that $colim_P F \to colim_{\Delta[m]} F$ is an isomorphism as well. □

11.3. **Lemma**. *Let $K \to \Delta[n]$ be a map. Consider the following pull-back square:*

$$\begin{tikzcd} P \arrow[r] \arrow[d] & \Delta[n+1] \arrow[d, "s_i"] \\ K \arrow[r] & \Delta[n] \end{tikzcd}$$

If $F : K \to \mathcal{C}$ is a bounded diagram, then the induced morphism $colim_P F \to colim_K F$ is an isomorphism.

Proof. Assume first that K is finite dimensional. In this case we prove the lemma by induction on the dimension of K.

If $dim(K) = 0$, then $K = \coprod \Delta[0]$. Thus the lemma follows from Lemma 11.2 and the fact that the colimit commutes with coproducts.

Assume that the lemma is true for those spaces whose dimension is less than m. Let $dim(K) = m$. We assume for simplicity that K has only one non-degenerate simplex of dimension m, i.e. K fits into the following push-out square:

$$\begin{tikzcd} \partial(\Delta[m]) \arrow[r, hook] \arrow[d] & \Delta[m] \arrow[d] & \\ L \arrow[r] & K \arrow[r] & \Delta[n] \end{tikzcd} \qquad \text{where } dim(L) < m.$$

The general case, when K contains more than one non-degenerate simplex of dimension m, can be proven analogously.

By pulling back $s_i : \Delta[n+1] \to \Delta[n]$ along the maps of the above diagram we get a commutative cube, where all the side squares are pull-backs and the bottom and top squares are push-outs:

$$\begin{tikzcd}
 & P_2 \arrow[rr, hook] \arrow[dl] \arrow[dd] & & P_3 \arrow[dl] \arrow[dd] & \\
P_1 \arrow[rr] \arrow[dd] & & P \arrow[rr] \arrow[dd] & & \Delta[n+1] \arrow[dd, "s_i"] \\
 & \partial(\Delta[m]) \arrow[rr, hook] \arrow[dl] & & \Delta[m] \arrow[dl] & \\
L \arrow[rr] & & K \arrow[rr] & & \Delta[n]
\end{tikzcd}$$

We can now apply Corollary 8.4.(1) to get the following diagram:

$$\begin{array}{ccccc}
colim_P F & = & colim & \Big(\; colim_{P_1} F \longleftarrow colim_{P_2} F \longrightarrow colim_{P_3} F \; \Big) \\
\downarrow & & & \quad\downarrow \qquad\qquad\quad \downarrow \qquad\qquad\quad \downarrow \\
colim_K F & = & colim & \Big(\; colim_L F \longleftarrow colim_{\partial(\Delta[m])} F \longrightarrow colim_{\Delta[m]} F \; \Big)
\end{array}$$

By inductive assumption the morphisms:

$$colim_{P_1} F \to colim_L F \quad , \quad colim_{P_2} F \to colim_{\partial(\Delta[m])} F$$

are isomorphisms. Moreover Lemma 11.2 implies that so is $colim_{P_3} F \to colim_{\Delta[m]} F$. This shows that $colim_P F \to colim_K F$ is an isomorphism as well.

So far we have proven the lemma in the case when K is finite dimensional. If K is infinite dimensional, by considering its skeleton filtration and applying Corollary 8.4.(2), we can conclude that the lemma is also true in this case. □

Proof of Theorem 11.1. Let $\sigma : \Delta[n] \to K$ be a simplex. Consider a degeneracy morphism $\Delta[n+1] \xrightarrow{s_i} \Delta[n] \xrightarrow{\sigma} K$. By definition the spaces $df(\sigma)$ and $df(s_i\sigma)$ fit into pull-back squares:

$$\begin{tikzcd}
df(s_i\sigma) \arrow[r, "df(s_i)"] \arrow[d] & df(\sigma) \arrow[r] \arrow[d] & L \arrow[d] \\
\Delta[n+1] \arrow[r, "s_i"] & \Delta[n] \arrow[r, "\sigma"] & K
\end{tikzcd}$$

If $F : L \to \mathcal{C}$ is bounded, then so is the composition $df(\sigma) \to L \xrightarrow{F} \mathcal{C}$. Thus Lemma 11.3 implies that the induced morphism $colim_{df(s_i\sigma)} F \to colim_{df(\sigma)} F$ is an isomorphism. This shows that $f^k F : K \to \mathcal{C}$ is a bounded diagram. □

11.4. **Corollary**. *Let $f : L \to K$ be a map of spaces. The Kan extension functor $f^k : Fun^b(L,\mathcal{C}) \to Fun^b(K,\mathcal{C})$ is left adjoint to the pull-back $f^* : Fun^b(K,\mathcal{C}) \to Fun^b(L,\mathcal{C})$.*

12. Relative boundness

In this section we introduce the notion of relative boundness extending Definition 10.1.

12.1. **Definition**. *Let $f : L \to K$ be a map of spaces and $F : L \to \mathcal{C}$ be a functor. We say that F is f-bounded, or that it is bounded relative to f if, for any simplex $\sigma \in L$ such that $f(\sigma) = s_i\xi$ in K, the morphisms $F(d_i) : F(d_i\sigma) \to F(\sigma)$ and $F(d_{i+1}) : F(d_{i+1}\sigma) \to F(\sigma)$ are isomorphisms (compare with Proposition 10.2).*

A simplex $\sigma \in L$ is called f-non-degenerate if $f(\sigma)$ is non-degenerate in K. A bounded diagram relative to f is determined, up to isomorphism, by the values it takes on the f-non-degenerate simplices in L.

The full subcategory of $Fun(L,\mathcal{C})$ consisting of diagrams which are bounded relative to $f : L \to K$ is denoted by $Fun^b_f(L,\mathcal{C})$. If $F : L \to \mathcal{C}$ is f-bounded, then it is a bounded diagram, i.e. there is an inclusion $Fun^b_f(L,\mathcal{C}) \subseteq Fun^b(L,\mathcal{C})$.

12.2. **Example**. Let $f : L \to K$ be a map which sends non degenerate simplices in L to non-degenerate simplices in K (such a map is called reduced, see Definition 13.4). A diagram $F : L \to \mathcal{C}$ is f-bounded if and only if it is a bounded diagram, i.e. the inclusion $Fun^b_f(L,\mathcal{C}) \subseteq Fun^b(L,\mathcal{C})$ is an isomorphism. In particular $F : K \to \mathcal{C}$ is bounded if and only if it is id_K-bounded, i.e. $Fun^b_{id}(K,\mathcal{C}) = Fun^b(K,\mathcal{C})$. Diagrams bounded relative to id are also called *absolutely* bounded.

12.3. **Example**. A diagram $F : L \to \mathcal{C}$ is bounded relative to $L \to \Delta[0]$ if and only if it is isomorphic to a constant diagram.

Relative boundness is a local property (see Definition 7.1).

12.4. **Proposition**. *A diagram $F : L \to \mathcal{C}$ is bounded relative to $f : L \to K$ if, for any simplex $\Delta[n] \to L$, the pull-back $\Delta[n] \to L \xrightarrow{F} \mathcal{C}$ is bounded relative to the composition $\Delta[n] \to L \xrightarrow{f} K$.*

As a corollary we get that the pull-back process preserves relative boundness.

12.5. **Corollary**. *Let $L \xrightarrow{h} M \xrightarrow{g} K$ be maps of spaces and $F : M \to \mathcal{C}$ be a diagram.*

(1) *If F is g-bounded, then the pull-back $L \xrightarrow{h} M \xrightarrow{F} \mathcal{C}$ is $(g \circ h)$-bounded, i.e. h induces a functor $h^* : Fun^b_g(M,\mathcal{C}) \to Fun^b_{g\circ h}(L,\mathcal{C})$.*
(2) *If $h : L \to M$ is an epimorphism, then $L \xrightarrow{h} M \xrightarrow{F} \mathcal{C}$ is $(g \circ h)$-bounded if and only if $F : M \to \mathcal{C}$ is g-bounded.*

Even though Kan extensions preserve absolute boundness (see Theorem 11.1), in general they do not preserve relative boundness. Consider maps $\Delta[0] \overset{d_1}{\hookrightarrow} \Delta[1] \to \Delta[0]$. Let X be an object in $\mathcal{C}$ which is not an initial one. It is clear that a constant diagram $X : \Delta[0] \to \mathcal{C}$ is bounded relative to the composition $\Delta[0] \overset{d_1}{\hookrightarrow} \Delta[1] \to \Delta[0]$. However $(d_1)^k X$ corresponds to a diagram in $Fun^b(\Delta[1], \mathcal{C})$ given by a graph $X \overset{id}{\to} X \leftarrow \emptyset$ (see Example 10.6). Thus this diagram is not bounded relative to $\Delta[1] \to \Delta[0]$ (see Example 12.3).

13. Reduction process

In this section we introduce a reduction process. It is used to reduce the study of diagram bounded relative to a map to the study of absolutely bounded diagrams (bounded relative to id).

The reduction process is motivated by the following property of relative boundness:

13.1. **Lemma**. *A diagram $F : \Delta[n{+}1] \to \mathcal{C}$ is bounded relative to $s_i : \Delta[n{+}1] \to \Delta[n]$ if and only if there exists a bounded diagram $F' : \Delta[n] \to \mathcal{C}$ for which the composition $\Delta[n+1] \overset{s_i}{\to} \Delta[n] \overset{F'}{\to} \mathcal{C}$ is isomorphic to F. Such a diagram F' is necessarily unique up to isomorphism.*

Proof. We show that F' is given by the composition $\Delta[n] \overset{d_i}{\hookrightarrow} \Delta[n+1] \overset{F}{\to} \mathcal{C}$. We have to construct a natural transformation $s_i^* F' \to F$ which is an isomorphism. Since both diagrams are bounded it is enough to construct isomorphisms $s_i^* F'(\sigma) \to F(\sigma)$ for non-degenerate simplices $\sigma \in \Delta[n+1]$ (these isomorphism should be natural with respect to σ).

Let $\sigma = (l_0 < \cdots < l_m)$ be a non-degenerate simplex in $\Delta[n+1]$. If σ does not contain the vertex i, then $(d_i \circ s_i)(\sigma) = \sigma$. In this case we define the morphism $s_i^* F'(\sigma) = F\big((d_i \circ s_i)(\sigma)\big) \to F(\sigma)$ to be the identity.

Assume that σ contains the vertex i, i.e. there is k such that $l_k = i$. We consider two cases. First, assume in addition that σ contains also the vertex $i+1$, i.e. $l_{k+1} = i+1$. In this case $d_i \circ s_i(\sigma) = (l_0 < \cdots < i = i < l_{k+2} < \cdots < l_m) \in \Delta[n+1]$, i.e. $(d_i \circ s_i)(\sigma) = s_k d_{k+1} \sigma$. Since the simplex $s_i(\sigma) \in \Delta[n]$ is of the form $s_k \tau$, the morphisms $F(d_k) : F(d_k \sigma) \to F(\sigma)$ and $F(d_{k+1}) : F(d_{k+1}\sigma) \to F(\sigma)$ are isomorphisms (F is bounded relative s_i). We define $s_i^* F'(\sigma) \to F(\sigma)$ to be the composition:

$$s_i^* F'(\sigma) = F\big((d_i \circ s_i)(\sigma)\big) = F(s_k d_{k+1}\sigma) \xrightarrow{F(s_k)} F(d_{k+1}\sigma) \xrightarrow{F(d_{k+1})} F(\sigma)$$

It is clear that $F(s_k)$ is an isomorphism (F is a bounded diagram).

Assume that σ does not contain the vertex $i+1$. Consider the simplex $\tau = (i_0 < \cdots < i_k < i+1 < i_{k+1} < \cdots < l_m) \in \Delta[n+1]$. The above discussion shows that $F(d_k) : F(d_k \tau) \to F(\tau)$ and $F(d_{k+1}) : F(d_{k+1}\tau) \to F(\tau)$ are isomorphisms.

Observe that $(d_i \circ s_i)(\sigma) = d_k\tau$ and $\sigma = d_{k+1}\tau$. We define $s_i^*F'(\sigma) \to F(\sigma)$ to be the composition:

$$s_i^*F'(\sigma) = F\big((d_i \circ s_i)(\sigma)\big) = F(d_k\tau) \xrightarrow{F(d_k)} F(\tau) \xrightarrow{F(d_{k+1})^{-1}} F(d_{k+1}\tau) = F(\sigma)$$

One can check that these morphisms induce the desired natural transformation $s_i^*F' \xrightarrow{\simeq} F$. $\square$

13.2. **Proposition**. *Let $f : L \to K$ be a map. Consider the following commutative diagram:*

$$\begin{tikzcd} \Delta[n+1] \arrow[r, "s_i"] \arrow[d, "\tau"'] & \Delta[n] \arrow[d, "\xi"] \arrow[dr] & \\ L \arrow[r, "h"] \arrow[rr, bend right, "f"'] & M \arrow[r, "g"] & K \end{tikzcd}$$

where $M = colim\big(L \xleftarrow{\tau} \Delta[n+1] \xrightarrow{s_i} \Delta[n]\big)$.

Then:

1. *A diagram $F : L \to \mathcal{C}$ is f-bounded if and only if there exists a g-bounded diagram $F' : M \to \mathcal{C}$ for which the composition $L \xrightarrow{h} M \xrightarrow{F'} \mathcal{C}$ is isomorphic to F. Any such F' is unique up to isomorphism.*
2. *The following diagram commutes:*

$$\begin{tikzcd} Fun^b_g(M,\mathcal{C}) \arrow[rr, "h^*"] \arrow[dr, "colim_M"'] & & Fun^b_f(L,\mathcal{C}) \arrow[dl, "colim_L"] \\ & \mathcal{C} & \end{tikzcd}$$

Proof of (1). Let $F : L \to \mathcal{C}$ be an f-bounded diagram. It follows that the composition $\Delta[n+1] \xrightarrow{\tau} L \xrightarrow{F} \mathcal{C}$ is bounded relative to $s_i : \Delta[n+1] \to \Delta[n]$. Thus Propositions 8.1 and 13.1 imply that there exists a diagram $F' : M \to \mathcal{C}$ for which the composition $L \xrightarrow{h} M \to \mathcal{C}$ is isomorphic to F. Since $h : L \to M$ is an epimorphism we can conclude two things. First, $F' : M \to \mathcal{C}$ is g-bounded (see Corollary 12.5). Second, F' is unique up to isomorphism.

Proof of (2). Let $G : M \to \mathcal{C}$ be g-bounded. Since $M = colim(L \xleftarrow{\tau} \Delta[n+1] \xrightarrow{s_i} \Delta[n])$, according to Corollary 8.4.(1):

$$colim_M G = colim\big(colim_L G \leftarrow colim_{\Delta[n+1]} G \to colim_{\Delta[n]} G\big)$$

By simplifying we get $colim_M G = colim\big(colim_L G \leftarrow G(h(\tau)) \xrightarrow{G(s_i)} G(\xi)\big)$. The boundness condition on G implies that $G(s_i) : G(h(\tau)) \to G(\xi)$ is an isomorphism. Hence so is $colim_L G \to colim_M G$. $\square$

The inverse for h^* in Proposition 13.2 can be identified with the Kan extension h^k.

13.3. **Proposition**. *Assume that we are in the same setting as in Proposition 13.2. Let $G : M \to \mathcal{C}$ be g-bounded. Then, for any simplex $\sigma : \Delta[m] \to M$, the map $dh(\sigma) \to \Delta[m]$ induces an isomorphism $h^k h^* G(\sigma) = colim_{dh(\sigma)} G \to G(\sigma)$. In other words $h^k h^* G$ is isomorphic to G.*

Proof. By pulling back $\sigma : \Delta[m] \to M$ along the diagram given in Proposition 13.2 we get the following commutative cube, where all the side squares are pull-backs and the top and bottom squares are push-outs:

$$\begin{tikzcd}
 & P_2 \arrow[rr] \arrow[dl] \arrow[dd] & & P_1 \arrow[dl] \arrow[dd] \\
dh(\sigma) \arrow[rr] \arrow[dd] & & \Delta[m] \arrow[dd, "\sigma"] & \\
 & \Delta[n+1] \arrow[rr, "s_i"] \arrow[dl, "\tau"'] & & \Delta[n] \arrow[dl, "\xi"] \\
L \arrow[rr, "h"] & & M &
\end{tikzcd}$$

Let $G : M \to \mathcal{C}$ be g-bounded. According to Corollary 8.4.(1):

$$G(\sigma) = colim_{\Delta[m]} G = colim\big(colim_{dh(\sigma)} G \leftarrow colim_{P_2} G \to colim_{P_1} G\big)$$

Since G is g-bounded thus in particular it is a bounded diagram. Lemma 11.2 implies therefore that $colim_{P_2} G \to colim_{P_1} G$ is an isomorphism. It follows that so is $colim_{dh(\sigma)} G \to G(\sigma)$. □

13.4. **Definition**. *We say that a map $f : L \to K$ is reduced if it sends non-degenerate simplices in L to non-degenerate simplices in K.*

13.5. **Example**. For any map of spaces $f : L \to K$, the induced maps on nerves $N(f) : N(L) \to N(K)$ and $N(f^{op}) : N(L^{op}) \to N(K^{op})$ are always reduced.

Reduced maps can be characterized in terms of lifting properties with respect to degeneracy maps.

13.6. **Proposition**. *A map $f : L \to K$ is reduced if and only if in any commutative diagram of the form:*

$$\begin{tikzcd}
\Delta[n+1] \arrow[r] \arrow[d, "s_i"'] & L \arrow[d, "f"] \\
\Delta[n] \arrow[r] & K
\end{tikzcd}$$

there is a lift, i.e. a map $\Delta[n] \to L$, such that the resulting diagram with five arrows commutes. Observe that if it exists such a lift is necessarily unique.

13.7. **Proposition**. *For any map $f : L \to K$, there is a functorial factorization $red(f) = (L \xrightarrow{f_{red}} red(f) \to K)$ such that:*

(1) *$red(f) \to K$ is reduced.*
(2) *If $F = (L \to X \to K)$ is another factorization of f, where $X \to K$ is reduced, then there exists a unique map of factorizations $red(f) \to F$, i.e. there exists a unique map of spaces $red(f) \to X$ for which the following diagram commutes:*

$$\begin{tikzcd} & red(f) \arrow[d] \arrow[dr] & \\ L \arrow[ur, "f_{red}"] \arrow[r] \arrow[rr, bend right, "f"'] & X \arrow[r] & K \end{tikzcd}$$

Proof. Let J be the set of all commutative diagrams of the form:

$$\begin{tikzcd} \Delta[n+1] \arrow[r] \arrow[d, "s_i"'] & L \arrow[d, "f"] \\ \Delta[n] \arrow[r] & K \end{tikzcd}$$

If d is such a diagram, then we use the same symbol to denote the number i. Define $f_{red} : L \to red(f)$ to be a map that fits into the following push-out square:

$$\begin{tikzcd} \coprod_{d\in J} \Delta[n+1] \arrow[r] \arrow[d, "\coprod_{d\in J} s_d"'] & L \arrow[d, "f_{red}"] \\ \coprod_{d\in J} \Delta[n] \arrow[r] & red(f) \end{tikzcd}$$

Observe that f_{red} is an epimorphism. Define $red(f) \to K$ to be induced by $f : L \to K$ and the "evaluation" map $\coprod_J \Delta[n] \to K$.

We are going to show that the factorization $L \xrightarrow{f_{red}} red(f) \to K$ satisfies conditions (1) and (2) of the proposition. Consider a commutative diagram:

$$\begin{tikzcd} \Delta[n+1] \arrow[r] \arrow[d, "s_i"'] & red(f) \arrow[d] \\ \Delta[n] \arrow[r] & K \end{tikzcd}$$

The map $red(f) \to K$ was constructed precisely in such a way so we can find a lift in the above square in the case when $\Delta[n+1] \to red(f)$ can be factored as $\Delta[n+1] \to L \xrightarrow{f_{red}} red(f)$. Since $f_{red} : L \to red(f)$ is an epimorphism such a factorization always exists. This shows that $red(f) \to K$ is reduced.

Let $F = (L \to X \to K)$ be another factorization of f, where $X \to K$ is reduced. It follows that in any commutative diagram of the form:

$$\begin{tikzcd} \Delta[n+1] \arrow[r] \arrow[d, "s_i"'] & L \arrow[r] & X \arrow[d] \\ \Delta[n] \arrow[rr] & & K \end{tikzcd}$$

there is always a lift. Such a lift is necessarily unique. By "summing up" over these diagrams we get a unique map of factorizations $red(f) \to F$. □

13.8. **Definition**. *Let $f : L \to K$ be a map. The factorization $L \xrightarrow{f_{red}} red(f) \to K$ of f, constructed in Proposition 13.7, is called the reduction of f.*

Using the reduction process, checking the condition for relative boundness can always be reduced to checking the condition for absolute boundness.

13.9. **Corollary**. *Let $f : L \to K$ be a map. A diagram $F : L \to \mathcal{C}$ is f-bounded if and only if there exists a bounded diagram $F' : red(f) \to \mathcal{C}$ for which the composition $L \xrightarrow{f_{red}} red(f) \xrightarrow{F'} \mathcal{C}$ is isomorphic to F. Any such F' is isomorphic to $(f_{red})^k F$.*

Proof. The map $red(f) \to K$ sends non-degenerate simplices in $red(f)$ to non-degenerate simplices in K (it is reduced). Thus a diagram is bounded relative to $red(f) \to K$ if and only if it is a bounded diagram. The corollary follows now from Propositions 13.2 and 13.3, since to construct $f_{red} : L \to red(f)$ we glued to L maps of the form $s_i : \Delta[n+1] \to \Delta[n]$. □

¿From this section on we start discussing the homotopy theoretical aspects of bounded diagrams. One of the main goals is to show that there is an appropriate model category structure on $Fun^b(K, \mathcal{C})$.

14. Cofibrations

Let $f : L \to K$ be a map, $F : L \to \mathcal{C}$ and $G : L \to \mathcal{C}$ be diagrams and $\Psi : F \to G$ be a natural transformation. For any $\sigma : \Delta[n] \to L$, let us pull-back $\Psi : F \to G$ along $\partial(\Delta[n]) \overset{i}{\hookrightarrow} \Delta[n] \xrightarrow{\sigma} K$, take the colimits and define:

$$M_\Psi(\sigma) := colim \left(colim_{\Delta[n]} F \longleftarrow colim_{\partial(\Delta[n])} F \xrightarrow{colim_{\partial(\Delta[n])}\Psi} colim_{\partial(\Delta[n])} G \right)$$

where we use the letter F to denote also the pull-backs $\sigma^* F$ and $(\sigma \circ i)^* F$. We can then form the following commutative diagram:

$$\begin{tikzcd}
colim_{\partial(\Delta[n])}F \arrow[r] \arrow[d] & colim_{\partial(\Delta[n])}G \arrow[d] \arrow[dr] & \\
colim_{\Delta[n]}F \arrow[r] \arrow[d, equal] & M_\Psi(\sigma) \arrow[r] & colim_{\Delta[n]}G \arrow[d, equal] \\
F(\sigma) \arrow[rr, "\Psi_\sigma"] & & G(\sigma)
\end{tikzcd}$$

Observe that in this way we get a functor $M_\Psi : L \to \mathcal{C}$ and natural transformations $F \to M_\Psi \to G$ whose composition equals Ψ. In the case $dim(\sigma) = 0$, $M_\Psi(\sigma) = F(\sigma)$ and the morphisms $F(\sigma) \to M_\Psi(\sigma) \to G(\sigma)$ coincide with $F(\sigma) \xrightarrow{id} F(\sigma) \xrightarrow{\Psi_\sigma} G(\sigma)$.

14.1. **Definition**. *Let $f : L \to K$ be a map, $F : L \to \mathcal{C}$ and $G : L \to \mathcal{C}$ be f-bounded diagrams and $\Psi : F \to G$ be a natural transformation.*

- *We say that $\Psi : F \to G$ is an (acyclic) f-cofibration, or that it is an (acyclic) cofibration relative to f if, for any simplex $\sigma \in L$ such that $f(\sigma)$ is non-degenerate in K, the morphism $M_\Psi(\sigma) \to G(\sigma)$ is an (acyclic) cofibration in $\mathcal{C}$.*
- *We say that $\Psi : F \to G$ is a cofibration, or an absolute cofibration, if it is a cofibration relative to id.*
- *Let $\emptyset : L \to \mathcal{C}$ be the constant diagram whose value is the initial object $\emptyset$ in $\mathcal{C}$. We say that F is f-cofibrant, or that it is a cofibration relative to f, if the natural transformation $\emptyset \to F$ is an f-cofibration. We say that F is cofibrant, or absolutely cofibrant, if it is cofibrant relative to the identity.*

Let $\Psi : F \to G$ be an (acyclic) f-cofibration. In the case $dim(\sigma) = 0$, since the morphism $M_\Psi(\sigma) \to G(\sigma)$ coincides with $\Psi_\sigma : F(\sigma) \to G(\sigma)$, Definition 14.1 tells us that Ψ_σ is an (acyclic) cofibration. Later on we will show that in general, for any $\sigma \in L$, $\Psi_\sigma : F(\sigma) \to G(\sigma)$ is an (acyclic) cofibration (see Corollary 16.5.(1)). The reverse implication of course is not true.

Cofibrant diagrams relative to f can be explicitly characterized as follows:

14.2. **Proposition**. *Let $F : L \to \mathcal{C}$ be bounded relative to $f : L \to K$. Then F is f-cofibrant if, for any simplex $\sigma : \Delta[n] \to L$ such that $f(\sigma)$ is non-degenerate in K, the morphism $colim_{\partial(\Delta[n])}F \to F(\sigma)$ is a cofibration in $\mathcal{C}$.*

Here are some examples of absolutely cofibrant diagrams:

14.3. **Example**. A diagram $F : \Delta[0] \to \mathcal{C}$ in $Fun^b(\Delta[0], \mathcal{C})$ is cofibrant if and only if the object $F(0)$ is cofibrant in $\mathcal{C}$.

14.4. **Example**. Let X be an object in $\mathcal{C}$ different from the initial one. Then a constant diagram $X : K \to \mathcal{C}$ with value X is cofibrant if and only if X is cofibrant in $\mathcal{C}$ and K does not have any non-degenerate simplices of dimension 1.

14.5. **Example**. A diagram $F(0) \xrightarrow{F(d_1)} F(0,1) \xleftarrow{F(d_0)} F(1)$ in $Fun^b(\Delta[1], \mathcal{C})$ (see Example 10.6) is cofibrant if the objects $F(0)$ and $F(1)$ are cofibrant and the morphism $F(d_1) \coprod F(d_0) : F(0) \coprod F(1) \to F(0,1)$ is a cofibration.

14.6. **Example**. A diagram $F(0) \xrightarrow{F(d_1)} F(0,1) \xleftarrow{F(d_0)} F(1) \xrightarrow{F(d_1)} F(1,2) \xleftarrow{F(d_0)} F(2)$ in $Fun^b(\Delta[2,1], \mathcal{C})$ (see Example 10.7) is cofibrant if, in addition to objects $F(0)$, $F(1)$, and $F(2)$ being cofibrant, the morphisms $F(d_1) \coprod F(d_0) : F(0) \coprod F(1) \to F(0,1)$ and $F(d_1) \coprod F(d_0) : F(1) \coprod F(2) \to F(1,2)$ are cofibrations. In particular a diagram $\emptyset \to B \leftarrow A \to C \leftarrow \emptyset$ is cofibrant if A is cofibrant and the morphisms $A \to B$ and $A \to C$ are cofibrations.

14.7. **Example**. Let $n > 1$. A diagram $F(0) \xrightarrow{F(\alpha)} F(\tau)$ in $Fun^b(S^n, \mathcal{C})$ (see Example 10.8) is cofibrant if $F(0)$ is cofibrant and the morphism $F(\alpha)$ is a cofibration. In particular a constant diagram in $Fun^b(S^n, \mathcal{C})$, i.e. a diagram associated with the identity morphism $id : X \to X$, is cofibrant if and only if the object X is cofibrant in $\mathcal{C}$ (compare with Example 14.4).

15. Cofibrations and the pull-back process

In this section we discuss the way cofibrations between bounded diagrams behaves with respect to the pull-back process.

Maps between spaces can send non-degenerate simplices to degenerate ones. Thus in general absolute cofibrations are *not* preserved by the pull-back process.

15.1. **Example**. Consider the map $\Delta[1] \to \Delta[0]$. Let X be a cofibrant object in $\mathcal{C}$ different from the initial one. The constant diagram $X : \Delta[0] \to \mathcal{C}$ is clearly cofibrant in $Fun^b(\Delta[0], \mathcal{C})$. However its pull-back along $\Delta[1] \to \Delta[0]$, the constant diagram $X : \Delta[1] \to \mathcal{C}$, is not cofibrant in $Fun^b(\Delta[1], \mathcal{C})$ (see Example 14.4).

15.2. **Proposition**. *Let $\Psi : F \to G$ be a cofibration in $Fun^b(K, \mathcal{C})$. If $f : L \to K$ is reduced, then the pull-back $f^*\Psi : f^*F \to f^*G$ is a cofibration in $Fun^b(L, \mathcal{C})$.*

The property of a natural transformation being an absolute cofibration is *not* a local property. The main reason we have introduced the notion of a relative cofibration is to capture those properties of absolute cofibrations which are local.

15.3. **Proposition**. *Let $f : L \to K$ be a map and $\Psi : F \to G$ be a natural transformation between f-bounded diagrams $F : L \to \mathcal{C}$ and $G : L \to \mathcal{C}$. The following are equivalent:*

(1) *Ψ is an (acyclic) f-cofibration.*
(2) *For any simplex $\sigma : \Delta[n] \to L$, the pull-back of Ψ along σ is an (acyclic) cofibration relative to the composition $\Delta[n] \xrightarrow{\sigma} L \xrightarrow{f} K$.*
(3) *For any simplex $\sigma : \Delta[n] \to L$ such that $f(\sigma)$ in non-degenerate in K, the pull-back of Ψ along σ is an (acyclic) cofibration relative to $\Delta[n] \xrightarrow{\sigma} L \xrightarrow{f} K$.*

As a corollary we get a useful procedure to check inductively on the cell decomposition of L whether a diagram $F : L \to \mathcal{C}$ is f-cofibrant (the same can be used for detecting if a natural transformation is a f-cofibration). Each step of this procedure consists of the following:

15.4. **Corollary**. *Let $\sigma : \Delta[n] \to K$ be a simplex of K and $F : \Delta[n] \to \mathcal{C}$ be a σ-bounded diagram.*

(1) *Assume that σ is degenerate. Then F is σ-cofibrant if and only if, for any face map $d_i : \Delta[n-1] \to \Delta[n]$, the composition $\Delta[n-1] \xrightarrow{d_i} \Delta[n] \xrightarrow{F} \mathcal{C}$ is cofibrant relative to $\Delta[n-1] \xrightarrow{d_i} \Delta[n] \xrightarrow{\sigma} K$.*
(2) *Assume that σ is non-degenerate. Then F is σ-cofibrant if and only if, in addition to $\Delta[n-1] \xrightarrow{d_i} \Delta[n] \xrightarrow{F} \mathcal{C}$ being cofibrant relative to $d_i\sigma$ for any i, the morphism $colim_{\partial(\Delta[n])}F \to F(\sigma)$ is a cofibration in $\mathcal{C}$.*

In general the way relative cofibrations behave with respect to the pull-back process can be described as follows:

15.5. **Corollary**. *Let $L \xrightarrow{h} M \xrightarrow{g} K$ be maps of spaces, $F : M \to \mathcal{C}$ and $G : M \to \mathcal{C}$ be g-bounded diagrams and $\Psi : F \to G$ be a natural transformation.*

(1) *If Ψ is an (acyclic) cofibration relative to g, then its pull-back along h is an (acyclic) cofibration relative to the composition $g \circ h$.*
(2) *If $h : L \to M$ is an epimorphism, then Ψ is an (acyclic) g-cofibration if and only if its pull-back along h is an (acyclic) $(h \circ g)$-cofibration.*

Using the reduction process (see Definition 13.8) checking the condition for a relative cofibration can always be reduced to checking the condition for an absolute cofibration. Corollaries 13.9 and 15.5.(2) imply:

15.6. **Proposition**. *Let $L \xrightarrow{f_{red}} red(f) \to K$ be the reduction of $f : L \to K$. Let $F : L \to \mathcal{C}$ and $G : L \to \mathcal{C}$ be f-bounded diagrams. A natural transformation $\Psi : F \to G$ is a (acyclic) f-cofibration if and only if $(f_{red})^k\Psi$ is an (acyclic) cofibration in $Fun^b\big(red(f), \mathcal{C}\big)$. In particular F is f-cofibrant if and only if $(f_{red})^kF$ is absolutely cofibrant.*

16. Cofibrations and colimits

In this section we generalize Propositions 3.5 and 3.6 to more complicated diagrams. We show that the colimit converts relative (acyclic) cofibrations in $Fun^b_f(L,\mathcal{C})$ into (acyclic) cofibrations in $\mathcal{C}$.

A natural transformation in $Fun^b_f(L,\mathcal{C})$ is a (acyclic) f-cofibration if, for all simplices $\sigma : \Delta[n] \to L$ such that $f(\sigma)$ is non-degenerate, the colimits of its pull-backs along $\partial(\Delta[n]) \hookrightarrow \Delta[n] \xrightarrow{\sigma} L$ satisfy a certain condition (see Definition 14.1). This information can be assembled to give a similar condition for more general maps $B \hookrightarrow A \to L$. The following can be shown by induction on the dimension of the relative space (A,B), applying Proposition 3.6.(3) in the case when this dimension is infinite:

16.1. **Proposition**. *Let $B \hookrightarrow A \xrightarrow{g} L$, $f : L \to K$ be maps and $\Psi : F \to G$ be an (acyclic) f-cofibration in $Fun^b_f(L,\mathcal{C})$. Consider the following commutative square:*

$$\begin{tikzcd}
colim_B F \arrow[r] \arrow[d, "colim_B \Psi"'] & colim_A F \arrow[d, "colim_A \Psi"] \\
colim_B G \arrow[r] & colim_A G
\end{tikzcd}$$

Let $M = colim\big(colim_B G \leftarrow colim_B F \to colim_A F\big)$ and $M \to colim_A G$ be the morphism induced by the commutativity of the above square. If for any non-degenerate simplex $\sigma \in A \setminus B$, $(f \circ g)(\sigma)$ is non-degenerate in K, then $M \to colim_A G$ is an (acyclic) cofibration in $\mathcal{C}$.

16.2. **Corollary**. *Let $L \hookrightarrow K$ be a monomorphism. If $F : K \to \mathcal{C}$ is an absolutely cofibrant diagram, then $colim_L F \to colim_K F$ is a cofibration in $\mathcal{C}$.*

The following generalizes Proposition 3.6:

16.3. **Theorem**. *Let $F : L \to \mathcal{C}$ and $G : L \to \mathcal{C}$ be bounded relative to $f : L \to K$ and $\Psi : F \to G$ be a natural transformation.*

(1) *If Ψ is an (acyclic) f-cofibration, then the morphism $colim_L \Psi : colim_L F \to colim_L G$ is an (acyclic) cofibration in $\mathcal{C}$.*
(2) *If Ψ is an f-cofibration and, for any simplex $\sigma \in L$, $\Psi_\sigma : F(\sigma) \to G(\sigma)$ is a weak equivalence, then $colim_L \Psi : colim_L F \to colim_L G$ is an acyclic cofibration.*

Since the proof are analogous we show only (1). We start with discussing the absolute case.

16.4. **Lemma**. *Let $F : K \to \mathcal{C}$ and $G : K \to \mathcal{C}$ be bounded. If Ψ is an (acyclic) absolute cofibration, then the morphism $colim_K \Psi : colim_K F \to colim_K G$ is an (acyclic) cofibration in $\mathcal{C}$.*

Proof. Assume first that K is finite dimensional. In this case we prove the lemma by induction on the dimension of K.

If $dim(K) = 0$, then the morphism $colim_K\Psi : colim_K F \to colim_K G$ coincides with:

$$\coprod_{\sigma\in K_0}\Psi_\sigma : \coprod_{\sigma\in K_0} F(\sigma) \to \coprod_{\sigma\in L_0} G(\sigma)$$

Since by assumption each Ψ_σ is an (acyclic) cofibration, Proposition 3.6.(1) implies that so is their coproduct.

Let us assume that the lemma is true for those spaces whose dimension is less than n. Let $dim(K) = n$. For simplicity let us assume that K has only one non-degenerate simplex of dimension n, i.e. K fits into the following push-out square:

$$\begin{tikzcd} \partial(\Delta[n]) \arrow[r, hook] \arrow[d] & \Delta[n] \arrow[d, "\sigma"] \\ N \arrow[r, hook] & K \end{tikzcd} \qquad \text{where } dim(N) < n.$$

Consider the following commutative diagram:

$$\begin{tikzcd} colim_K F \arrow[d, "colim_K\Psi"'] & = & colim\Big(colim_N F \arrow[d, "colim_N\Psi"'] & colim_{\partial(\Delta[n])} F \arrow[l] \arrow[r] \arrow[d, "colim_{\partial(\Delta[n])}\Psi"] & F(\sigma)\Big) \arrow[d, "\Psi_\sigma"] \\ colim_K G & = & colim\Big(colim_N G & colim_{\partial(\Delta[n])} G \arrow[l] \arrow[r] & G(\sigma)\Big) \end{tikzcd}$$

We apply Proposition 3.6.(2) to show that $colim_K\Psi$ is an (acyclic) cofibration. The pull-back of Ψ along $N \hookrightarrow L$ is an (acyclic) cofibration, and thus by the inductive assumption so is $colim_N\Psi$.

Since the map $\partial(\Delta[n]) \to K$ can send non-degenerate simplices to degenerate ones, the pull-back of Ψ along $\partial(\Delta[n]) \to K$ is not an (acyclic) cofibration. We can not therefore apply the inductive assumption directly to argue that $colim_{\partial(\Delta[n])}\Psi$ is an (acyclic) cofibration. However, according to Corollary 15.5.(1) this pull-back is a *relative* (acyclic) cofibration with respect to $\partial(\Delta[n]) \to K$. Let us denote this map by g and consider its reduction $\partial(\Delta[n]) \xrightarrow{g_{red}} red(g) \to K$. According to Proposition 15.6 the natural transformation $(g_{red})^k\Psi : (g_{red})^k F \to (g_{red})^k G$ is an (acyclic) absolute cofibration whose pull-back along g is isomorphic to Ψ. Since $\partial(\Delta[n]) \to red(g)$ is an epimorphism, the dimension of $red(g)$ is less than n. Hence, by the inductive assumption, $colim_{red(g)}(g_{red})^k\Psi$ is an (acyclic) cofibration. As the Kan extension process does not modify colimits (see Proposition 9.2), it follows that $colim_{\partial(\Delta[n])}\Psi$ is also an (acyclic) cofibration.

The cofibrancy assumption on Ψ implies that $M_\Psi(\sigma) \to G(\sigma)$ is an (acyclic) cofibration. The assumptions of Proposition 3.6.(2) are therefore satisfied, and hence $colim_K\Psi$ is an (acyclic) cofibration.

So far we have proven the lemma in the case when K is finite dimensional. If K is infinite dimensional, by considering the skeleton filtration of K and applying Proposition 3.6.(3), Corollary 8.4.(2) and Proposition 16.1, we can conclude that the lemma is also true in this case. □

Proof of Theorem 16.3. Consider the reduction $L \xrightarrow{f_{red}} red(f) \to K$ of f. Since $\Psi : F \to G$ is an (acyclic) f-cofibration, according to Proposition 15.6, $(f_{red})^k\Psi : (f_{red})^kF \to (f_{red})^kG$ is an absolute cofibration. Thus Lemma 16.4 implies that $colim_{red(f)}(f_{red})^k\Psi$ is an (acyclic) cofibration in $\mathcal{C}$. As the Kan extension process does not modify colimits (see Proposition 9.2), it follows that $colim_L\Psi$ is also an (acyclic) cofibration. □

16.5. **Corollary**. *Let $f : L \to K$ be a map.*

(1) *If $\Psi : F \to G$ is an (acyclic) f-cofibration, then, for all the simplices $\sigma \in L$, $\Psi_\sigma : F(\sigma) \to G(\sigma)$ is an (acyclic) cofibration in $\mathcal{C}$.*
(2) *If $G : L \to \mathcal{C}$ is f-cofibrant, then $colim_LG$ is a cofibrant object in $\mathcal{C}$ and, for any simplex $\sigma \in L$, $G(\sigma)$ is cofibrant in $\mathcal{C}$.*
(3) *A natural transformation $\Psi : F \to G$ is an acyclic f-cofibration if and only if Ψ is an f-cofibration and, for any simplex $\sigma \in L$, the morphism $\Psi_\sigma : F(\sigma) \to G(\sigma)$ is a weak equivalence in $\mathcal{C}$.*

Proof of (1). Let $\sigma : \Delta[n] \to L$ be a simplex. The pull-back of Ψ along $\sigma : \Delta[n] \to L$ is an (acyclic) cofibration relative to $\Delta[n] \xrightarrow{\sigma} L \xrightarrow{f} K$. Thus, by Theorem 16.3, $colim_{\Delta[n]}F = F(\sigma) \xrightarrow{\Psi_\sigma} G(\sigma) = colim_{\Delta[n]}G$ is an (acyclic) cofibration.

Proof of (3). If Ψ is an acyclic f-cofibration, then in particular it is an f-cofibration. Thus part (1) of the corollary shows that, for any simplex $\sigma \in L$, Ψ_σ is a weak equivalence.

Assume that Ψ is an f-cofibration and, for any $\sigma \in L$, Ψ_σ is a weak equivalence. For any simplex $\sigma : \Delta[n] \to L$, consider the following commutative diagram associated with σ:

$$\begin{tikzcd}
colim_{\partial(\Delta[n])}F \arrow[r] \arrow[d] & colim_{\partial(\Delta[n])}G \arrow[d] \arrow[dr] & \\
F(\sigma) \arrow[r] \arrow[rr, bend right, "\Psi_\sigma"'] & M_\Psi(\sigma) \arrow[r] & G(\sigma)
\end{tikzcd}$$

To prove the second implication, we have to show that $M_\Psi(\sigma) \to G(\sigma)$ is a weak equivalence. The pull-back of Ψ along $\partial(\Delta[n]) \to L$ is a cofibration relative to $\partial(\Delta[n]) \to L \xrightarrow{f} K$. As it is also an objectwise weak equivalence, Theorem 16.3.(2) implies that $colim_{\partial(\Delta[n])}F \to colim_{\partial(\Delta[n])}G$ is an acyclic cofibration. Since Ψ_σ is a weak equivalence, it is clear now that so is $M_\Psi(\sigma) \to G(\sigma)$. □

We have seen that the functor $colim_L : Fun^b_f(L,\mathcal{C}) \to \mathcal{C}$ converts (acyclic) f-cofibrations in $Fun^b_f(L,\mathcal{C})$ into (acyclic) cofibrations in $\mathcal{C}$ (see Theorem 16.3). In the following we consider a similar problem. Instead of acyclic cofibrations however, we look at general objectwise weak equivalences assuming though that the involved diagrams are cofibrant. The following generalizes Proposition 3.5:

16.6. **Theorem**. *Let $f : L \to K$ be a map, $F : L \to \mathcal{C}$ and $G : L \to \mathcal{C}$ be f-cofibrant diagrams and $\Psi : F \to G$ be a natural transformation. If Ψ is a weak equivalence (objectwise weak equivalence), then the morphism $colim_L\Psi : colim_L F \to colim_L G$ is a weak equivalence in $\mathcal{C}$.*

Using K. Brown Lemma (see [7] and [28, Lemma 9.9]), Theorem 16.6 could be seen as a corollary of Theorems 16.3 and 17.3. However, for the sake of completeness we include its elementary proof here. As in the case of Theorem 16.3 we first look at the absolute case.

16.7. **Lemma**. *Let $F : K \to \mathcal{C}$ and $G : K \to \mathcal{C}$ be cofibrant diagrams and $\Psi : F \to G$ be a natural transformation. If Ψ is a weak equivalence (objectwise weak equivalence), then the morphism $colim_L\Psi : colim_L F \to colim_L G$ is a weak equivalence in $\mathcal{C}$.*

Proof. Assume first that K is finite dimensional. In this case we prove the lemma by induction on the dimension of K.

Proposition 3.5.(1) implies that the lemma is true in the case $dim(K) = 0$. Let us assume that the lemma holds for those spaces whose dimension is less than n. Let $dim(K) = n$. For simplicity assume that K has only one non-degenerate simplex of dimension n, i.e. K fits into the following push-out square:

$$\begin{tikzcd} \partial(\Delta[n]) \arrow[r, hook] \arrow[d] & \Delta[n] \arrow[d, "\tau"] \\ N \arrow[r, hook] & K \end{tikzcd} \qquad \text{where } dim(N) < n.$$

The cofibrancy assumption on F and G imply that we have the following commutative diagram in $\mathcal{C}$, where indicated morphisms are cofibrations:

$$\begin{tikzcd} colim_K F \arrow[d, "colim_K\Psi"'] & = & colim\ \big(& colim_N F \arrow[d, "colim_N\Psi"'] & colim_{\partial(\Delta[n])} F \arrow[l] \arrow[r, hook] \arrow[d, "colim_{\partial(\Delta[n])}\Psi"] & F(\sigma) \arrow[d, "\wr"', "\Psi_\sigma"] & \big) \\ colim_K G & = & colim\ \big(& colim_N G & colim_{\partial(\Delta[n])} G \arrow[l] \arrow[r, hook] & G(\sigma) & \big) \end{tikzcd}$$

We apply Proposition 3.5.(2) to prove that $colim_K\Psi$ is a weak equivalence. The pull-backs of F and G along $N \hookrightarrow K$ are cofibrant diagrams. Hence, by the inductive assumption, $colim_N\Psi$ is a weak equivalence.

Since the map $\partial(\Delta[n]) \to K$ can send non-degenerate simplices to degenerate ones, the pull-backs of F and G along $\partial(\Delta[n]) \to K$ are no longer absolutely cofibrant. However, these pull-backs are cofibrant relative to $\partial(\Delta[n]) \to K$. Let us denote this map by g, and consider its reduction $\partial(\Delta[n]) \xrightarrow{g_{red}} red(g) \to K$. According to Proposition 15.6 $(g_{red})^k F$ and $(g_{red})^k G$ are absolutely cofibrant. The map $g_{red} : \partial(\Delta[n]) \to red(g)$ is an epimorphism and therefore the dimension of $red(g)$ is less than n. This epimorphism condition implies also that $(g_{red})^k \Psi$ is an objectwise weak equivalence as its pull-back along g_{red} is one (it is isomorphic to Ψ). Therefore, by the inductive assumption, we can conclude that $colim_{red(g)}(g_{red})^k\Psi$ is a weak equivalence. As the Kan extension process does not modify colimits (see Proposition 9.2), $colim_{\partial(\Delta[n])}\Psi$ is also a weak equivalence. Thus the assumptions of Proposition 3.5.(2) are satisfied and $colim_K \Psi$ is a weak equivalence.

So far we have proven the lemma in the case when K is finite dimensional. If K is infinite dimensional, by considering the skeleton filtration of K and applying Proposition 3.5.(3), Corollary 8.4.(2) and Proposition 16.1, we can conclude that the lemma is also true in this case. □

Proof of Theorem 16.6. Consider the reduction $L \xrightarrow{f_{red}} red(f) \to K$ of f. Since F and G are f-cofibrant, according to Proposition 15.6, $(f_{red})^k F$ and $(f_{red})^k G$ are absolutely cofibrant. It is also clear that $(f_{red})^k \Psi$ is an objectwise weak equivalence as its pull-back along f_{red} is one (it is isomorphic to Ψ). Lemma 16.7 implies therefore that $colim_{red(f)}(f_{red})^k\Psi$ is a weak equivalence. Since the Kan extension process does not modify colimits (see Proposition 9.2), $colim_L \Psi$ is a weak equivalence as well. □

17. $Fun^b_f(L, \mathcal{C})$ AS A MODEL CATEGORY

We are now ready to prove that f-bounded diagrams form a model category.

17.1. **Definition**. *Let $f : L \to K$ be a map and $\Psi : F \to G$ be a natural transformation in $Fun^b_f(L, \mathcal{C})$.*
Weak equivalences. *Ψ is called a weak equivalence if it is an objectwise weak equivalence, i.e. if, for any $\sigma \in L$, $\Psi(\sigma) : F(\sigma) \to G(\sigma)$ is a weak equivalence in $\mathcal{C}$.*
Fibrations. *Ψ is called a fibration if it is an objectwise fibration, i.e. if, for any simplex $\sigma \in L$, $\Psi(\sigma) : F(\sigma) \to G(\sigma)$ is a fibration in $\mathcal{C}$.*
Cofibrations. *Ψ is called a cofibration if it is a cofibration relative to f (see Definition 14.1).*

17.2. **Remark**. Observe that the notions of weak equivalences, fibrations and cofibrations in $Fun^b_f(L, \mathcal{C})$ are local (see Definition 7.1). A natural transformation $\Psi \in Fun^b_f(L, \mathcal{C})$ is a weak equivalence, fibration or a cofibration if and only if, for any simplex $\sigma : \Delta[n] \to L$, its pull-back $\sigma^*\Psi$ is so in $Fun^b_{f\circ\sigma}(\Delta[n], \mathcal{C})$.

17.3. **Theorem**. *The category $Fun^b_f(L, \mathcal{C})$ together with the choice of weak equivalences, fibrations and cofibrations, given in Definition 17.1, satisfies the axioms of a model category.*

We look first at the absolute case.

17.4. **Lemma**. *The category $Fun^b(K,\mathcal{C})$ together with the choice of weak equivalences, fibrations and cofibrations, given in Definition 17.1, satisfies the axioms of a model category.*

Proof. **MC1**, **MC2**. These axioms are obviously satisfied.

MC3. Weak equivalences and fibrations are clearly closed under retracts. Since the construction M_Ψ is natural with respect to Ψ, cofibrations are also preserved by retracts.

MC4. Let the following be a commutative square in $Fun^b(K,\mathcal{C})$:

$$\begin{tikzcd} F \arrow[r] \arrow[d, "\Psi"'] & E \arrow[d, "\Phi"] \\ G \arrow[r] & B \end{tikzcd}$$

where Ψ and Φ are respectively either a cofibration and acyclic fibration, or an acyclic cofibration and fibration. We need to show that in the above diagram there exists a lift $\Omega : G \to E$. We are going to construct the morphisms $\Omega_\sigma : G(\sigma) \to E(\sigma)$ by induction on the dimension of σ.

Let $dim(\sigma) = 0$. Since $\Psi_\sigma : F(\sigma) \to G(\sigma)$ is an (acyclic) cofibration in $\mathcal{C}$, by the lifting axiom, there exists $h : G(\sigma) \to E(\sigma)$ which makes the following diagram commutative:

$$\begin{tikzcd} F(\sigma) \arrow[r] \arrow[d, "\Psi_\sigma"'] & E(\sigma) \arrow[d, "\Phi_\sigma"] \\ G(\sigma) \arrow[r] \arrow[ur, "h"] & B(\sigma) \end{tikzcd}$$

Define $\Omega_\sigma := h$.

Let us assume that the appropriate morphisms $\Omega_\sigma : G(\sigma) \to E(\sigma)$ have been constructed for all the simplices $\sigma \in K$ such that $dim(\sigma) < n$.

Let $dim(\tau) = n$. If τ is degenerate, i.e. if $\tau = s_i\xi$, we define Ω_τ as a composition:

$$G(\tau) \xrightarrow{G(s_i)} G(\xi) \xrightarrow{\Omega_\xi} E(\xi) \xrightarrow{E(s_i)^{-1}} E(\tau)$$

Using simplicial identities one can show that Ω_τ does not depend on the choice of ξ.

If τ is non-degenerate, let us consider the following commutative diagram induced by $\tau : \Delta[n] \to K$:

$$\begin{tikzcd} & colim_{\partial(\Delta[n])}E \arrow[rr] & & E(\tau) \arrow[d, "\Phi_\tau"] \\ colim_{\partial(\Delta[n])}F \arrow[ur] \arrow[r] \arrow[d, "colim(\Psi)"'] & F(\tau) \arrow[rru] \arrow[d] & & \\ colim_{\partial(\Delta[n])}G \arrow[r] \arrow[uur, dotted, "a"] & M_\Psi(\tau) \arrow[r] \arrow[uurr, dotted, "b"] & G(\tau) \arrow[r] \arrow[uur, dotted, "c"'] & B(\tau) \end{tikzcd}$$

where:

- $a : colim_{\partial(\Delta[n])}G \to colim_{\partial(\Delta[n])}E$ is induced by $\{\Omega_\sigma : G(\sigma) \to E(\sigma)\}_{dim(\sigma)<n}$,
- $b : M_\Psi(\tau) \to E(\tau)$ is then constructed by the universal property of the push-out $M_\Psi(\tau)$ using the morphism a.
- Since Ψ is a (acyclic) cofibration in $Fun^b(K, \mathcal{C})$, by Definition 14.1, $M_\Psi(\tau) \to G(\tau)$ is a (acyclic) cofibration in $\mathcal{C}$. The morphism $c : G(\tau) \to E(\tau)$ is then constructed using b and the lifting axiom in $\mathcal{C}$.

We define $\Omega_\tau := c$.

The family of morphisms $\{\Omega_\sigma : G(\sigma) \to E(\sigma)\}_{\sigma\in K}$ forms a natural transformation $\Omega : G \to E$ which is the desired lift.

MC5. Let $\Psi : F \to G$ be a natural transformation in $Fun^b(K, \mathcal{C})$. We need to show that Ψ can be expressed as compositions: $F \overset{\sim}{\hookrightarrow} F' \twoheadrightarrow G$ and $F \hookrightarrow G' \overset{\sim}{\twoheadrightarrow} G$. As in the previous case, to construct diagrams $F' : K \to \mathcal{C}$, $G' : K \to \mathcal{C}$ and appropriate natural transformations we argue by induction on the dimension of simplices in K.

Let $dim(\sigma) = 0$. We define $F'(\sigma)$ and $G'(\sigma)$ to be any objects that fit into the following factorizations of Ψ_σ in $\mathcal{C}$:

$$\begin{tikzcd} F(\sigma) \arrow[r, hook, "\sim"] \arrow[rr, bend right, "\Psi_\sigma"'] & F'(\sigma) \arrow[r, two heads] & G(\sigma) \end{tikzcd} \qquad \begin{tikzcd} F(\sigma) \arrow[r, hook] \arrow[rr, bend right, "\Psi_\sigma"'] & G'(\sigma) \arrow[r, two heads, "\sim"] & G(\sigma) \end{tikzcd}$$

Let us assume that we have constructed $F'(\sigma)$, $G'(\sigma)$ and appropriate maps for all the simplices $\sigma \in K$ whose dimension is less than n.

Let $dim(\tau) = n$. If τ is degenerate, i.e. if $\tau = s_i\xi$, we define $F'(\tau) := F'(\xi)$ and $G'(\tau) := G'(\xi)$.

Let us assume that $\tau : \Delta[n] \to K$ is non-degenerate. Consider the composition $\partial(\Delta[n]) \hookrightarrow \Delta[n] \xrightarrow{\tau} K$. By Theorem 16.3.(1), the inductive assumption implies that $colim_{\partial(\Delta[n])}F \overset{\sim}{\hookrightarrow} colim_{\partial(\Delta[n])}F'$ is an acyclic cofibration and $colim_{\partial(\Delta[n])}F \hookrightarrow colim_{\partial(\Delta[n])}G'$ is a cofibration.

Let:

$$M_1 := colim\big(colim_{\partial(\Delta[n])}F' \overset{\sim}{\hookleftarrow} colim_{\partial(\Delta[n])}F \to F(\tau)\big)$$

$$M_2 := colim\big(colim_{\partial(\Delta[n])}G' \hookleftarrow colim_{\partial(\Delta[n])}F \to F(\tau)\big)$$

This data can be arranged into commutative diagrams:

$$\begin{tikzcd} colim_{\partial(\Delta[n])}F \arrow[r] \arrow[d, hook, "\wr"'] & F(\tau) \arrow[d, hook, "\wr"] & \\ colim_{\partial(\Delta[n])}F' \arrow[r] \arrow[d] & M_1 \arrow[d] & \\ colim_{\partial(\Delta[n])}G \arrow[r] & M_\Psi(\tau) \arrow[r] & G(\tau) \end{tikzcd} \qquad \begin{tikzcd} colim_{\partial(\Delta[n])}F \arrow[r] \arrow[d, hook] & F(\tau) \arrow[d, hook] & \\ colim_{\partial(\Delta[n])}G' \arrow[r] \arrow[d] & M_2 \arrow[d] & \\ colim_{\partial(\Delta[n])}G \arrow[r] & M_\Psi(\tau) \arrow[r] & G(\tau) \end{tikzcd}$$

Define $F'(\tau)$ and $G'(\tau)$ to be any objects that fit into the following factorizations of the morphisms $M_1 \to G(\tau)$ and $M_2 \to G(\tau)$:

$$M_1 \overset{\sim}{\hookrightarrow} F'(\tau) \twoheadrightarrow G(\tau) \quad , \quad M_2 \hookrightarrow G'(\tau) \overset{\sim}{\twoheadrightarrow} G(\tau)$$

In this way we get bounded diagrams $F' : K \to \mathcal{C}$, $G' : K \to \mathcal{C}$ and the desired natural transformations $F \overset{\sim}{\hookrightarrow} F' \twoheadrightarrow G$, $F \hookrightarrow G' \overset{\sim}{\twoheadrightarrow} G$. □

Proof of Theorem 17.3. Consider the reduction $f \xrightarrow{f_{red}} red(f) \to K$ of f. Checking the axioms of a model category on $Fun^b_f(L, \mathcal{C})$ can be reduced to checking these axioms for $Fun^b\big(red(f), \mathcal{C}\big)$, and hence the theorem follows from Lemma 17.4. □

Let $f : L \to K$ be a map. In the absolute case, we have associated with f a pair of adjoint functors: the pull-back process $f^* : Fun^b(K, \mathcal{C}) \to Fun^b(L, \mathcal{C})$ and the Kan extension $f^k : Fun^b(L, \mathcal{C}) \to Fun^b(K, \mathcal{C})$ (see Corollary 11.4). Clearly f^* converts (acyclic) fibrations in $Fun^b(K, \mathcal{C})$ into (acyclic) fibrations in $Fun^b(L, \mathcal{C})$. By adjointness this implies:

17.5. **Proposition**. *The Kan extension functor $f^k : Fun^b(L, \mathcal{C}) \to Fun^b(K, \mathcal{C})$ converts (acyclic) cofibrations in $Fun^b(L, \mathcal{C})$ into (acyclic) cofibrations in $Fun^b(K, \mathcal{C})$.*

A pair of adjoint functors which satisfies properties given in Proposition 17.5 is said to form a Quillen pair (see [16]). Thus we can conclude that, for any map of spaces $f : L \to K$, the pair of functors $f^k : Fun^b(L, \mathcal{C}) \to Fun^b(K, \mathcal{C})$ and $f^* : Fun^b(K, \mathcal{C}) \to Fun^b(L, \mathcal{C})$ is a Quillen pair.

18. Ocolimit of bounded diagrams

In this section we discuss a homotopy meaningful approximation of the colimit of bounded diagrams, i.e. its total left derived functor as described in Section 5.

18.1. **Definition**. *The total left derived functor of $colim_K : Fun^b(K, \mathcal{C}) \to \mathcal{C}$ is denoted by $ocolim_K : Fun^b(K, \mathcal{C}) \to Ho(\mathcal{C})$.*

In Lemma 16.7 we showed that $colim_K : Fun^b(K, \mathcal{C}) \to \mathcal{C}$ is homotopy invariant on cofibrant objects. Thus Proposition 4.4 implies:

18.2. **Proposition**. *The functor $ocolim_K : Fun^b(K, \mathcal{C}) \to \mathcal{C}$ exists. It can be constructed by choosing a cofibrant replacement Q in $Fun^b(K, \mathcal{C})$ and assigning to a diagram $F \in Fun^b(K, \mathcal{C})$ the colimit $colim_K QF \in Ho(\mathcal{C})$. The natural transformation $ocolim_K \to colim_K$ is induced by $colim_K(QF \overset{\sim}{\twoheadrightarrow} F)$.*

What is the intuition behind the construction of the ocolimit of a bounded diagram? It should be seen as a homotopy meaningful process which involves three basic steps: coproducts, push-outs and telescopes. Let $F : K \to \mathcal{C}$ be a bounded diagram. We can build its ocolimit by induction on the cell decomposition of K.

For all 0-dimensional simplices σ we take cofibrant replacements $QF(\sigma)\overset{\sim}{\twoheadrightarrow}F(\sigma)$ and sum them up: $\coprod QF(\sigma)$. We then go on by attaching generalized cells along their boundaries. Let $\tau : \Delta[n] \to K$ be a non-degenerate simplex. Assume that we already know how to construct the ocolimit of F on a subcomplex $N \hookrightarrow K$ containing the boundary of τ. We then turn the morphism $colim_{\partial(\Delta[n])}QF \to F(\tau)$ into a cofibration $colim_{\partial(\Delta[n])}QF \hookrightarrow QF(\tau) \overset{\sim}{\twoheadrightarrow} F(\tau)$ and glue a generalized cell $QF(\tau)$ to $ocolim_N F$ along its boundary $colim_{\partial(\Delta[n])}QF$, i.e. we take a push-out:

$$ocolim_{(N\cup_{\partial(\Delta[n])}\Delta[n])}F = colim\big(ocolim_N F \leftarrow colim_{\partial(\Delta[n])}QF \hookrightarrow QF(\tau)\big)$$

This push-out process is homotopy meaningful since the objects involved are cofibrant and the morphism $colim_{\partial(\Delta[n])}QF \hookrightarrow QF(\tau)$ is a cofibration. In the case K in infinite dimensional we finish the construction by taking the telescope.

18.3. **Remark**. Recall that the total left derived functor of $colim_K : Fun(K, \mathcal{C}) \to \mathcal{C}$ is denoted by $hocolim_K$ (see Definition 5.1). Its construction will be given in Section 21. Let $F : K \to \mathcal{C}$ be a bounded diagram. We can perform two constructions on F. Take its ocolimit $ocolim_K F$, or take its hocolimit $hocolim_K F$. These two constructions are both homotopy meaningful and map naturally to the colimit. However, in general $ocolim_K F$ is *NOT* equivalent to $hocolim_K F$. Consider a constant diagram $\Delta[0] : S^2 \to Spaces$ with value $\Delta[0]$. Since it is cofibrant in $Fun^b(S^2, Spaces)$, $ocolim_{S^2}\Delta[0] \simeq colim_{S^2}\Delta[0] = \Delta[0]$ (see Examples 10.8 and 14.7). On the other hand $hocolim_{S^2}\Delta[0]$ is weakly equivalent to the classifying space of the category S^2. Therefore $hocolim_{S^2}\Delta[0] \simeq S^2$.

The main property that distinguishes between hocolimit and ocolimit constructions is the additivity with respect to the indexing spaces. By the same arguments as in Proposition 25.1 one can show that for any functor of spaces $H : I \to Spaces$ and for any bounded diagram $H : colim_I H \to \mathcal{C}$, the morphism $colim_I hocolim_H F \to hocolim_{colim_I H} F$ is an isomorphism, i.e. hocolimit is additive with respect to the indexing spaces. The ocolimit functor does not have this property (as shown by the above example since $S^2 = colim(* \leftarrow \partial\Delta[2] \hookrightarrow \Delta[2]*)$). This is due to the fact that absolute cofibrations are not invariant under the pull-back process (see Example 15.1).

18.4. **Remark**. Let us assume that $\mathcal{C}$ has a functorial factorization of morphisms by cofibrations followed by acyclic fibrations. This functorial factorization can be used to construct a functorial cofibrant replacement $Q : Fun^b(K, \mathcal{C}) \to Fun^b(K, \mathcal{C})$. We can then construct a "rigid" ocolimit by taking $colim_K Q(-) : Fun^b(K, \mathcal{C}) \to \mathcal{C}$ (by rigid we mean a functor with values in the category $\mathcal{C}$ rather than in its homotopy category $Ho(\mathcal{C})$). The natural transformation $colim_K Q(-) \to colim_K$ is induced by $QF \overset{\sim}{\twoheadrightarrow} F$.

19. Cones

In this section we discuss constructions of cones in the category of spaces and the category of small categories.

A cone over a space K is a space CK which is built by adding an extra vertex to K and joining all the simplices of K with this vertex. Here is its precise definition:

19.1. **Definition**. *Let K be a space. The cone over K is the simplicial set CK whose set of n-dimensional simplices is given by:*

$$(CK)_n = \{e^{n+1}\} \coprod \Big(\coprod_{i=0}^{n} K_i \times \{e^{n-i}\} \Big)$$

The simplicial operators d_i and s_i are given by:

$$s_0 : (CK)_0 \to (CK)_1 \ , \ s_0(e^1) := e^2 \ , \ s_0(\sigma, e^0) := (s_0\sigma, e^0)$$

For $n > 0$, $dim(\sigma) = i$ and $0 \leq j \leq n$ the maps $(CK)_{n+1} \overset{s_j}{\leftarrow} (CK)_n \overset{d_j}{\rightarrow} (CK)_{n-1}$ are defined as:

$$d_j(e^{n+1}) := e^n \ , \ d_j(\sigma, e^{n-i}) := \begin{cases} e^n & \text{if } \ i = j = 0 \\ \big(d_j(\sigma), e^{n-j}\big) & \text{if } \ j \leq i \text{ and } i > 0 \\ (\sigma, e^{n-j-1}) & \text{if } \ j > i \end{cases}$$

$$s_j(e^{n+1}) := e^{n+2} \ , \ s_j(\sigma, e^{n-i}) := \begin{cases} \big(s_j(\sigma), e^{n-i}\big) & \text{if } \ j \leq i \\ (\sigma, e^{n+1-i}) & \text{if } \ j > i \end{cases}$$

A simplex (σ, e^k) is non-degenerate in CK if σ is non-degenerate in K and either $k = 0$ or $k = 1$.

The cone construction is natural, i.e. a map $f : L \to K$ induces a map of cones $Cf : CL \to CK$. It sends $e^i \in CL$ and $(\sigma, e^i) \in CL$ to $e^i \in CK$ and $(f(\sigma), e^i) \in CK$ respectively. Observe that there is a natural inclusion $K \hookrightarrow CK$, $K \ni \sigma \mapsto (\sigma, e^0) \in CK$.

Since the colimit commutes with sums, the cone functor commutes with colimits. Explicitly, the map $colim_I CF \to C(colim_I F)$, induced by the natural transformation $C(F \to colim_I F)$, is an isomorphism for any $F : I \to Spaces$. In particular if $K = colim\big(L \leftarrow \partial(\Delta[n]) \hookrightarrow \Delta[n]\big)$, then $CK = colim\big(CL \leftarrow C\partial(\Delta[n]) \hookrightarrow C\Delta[n]\big)$. This can be used to build CK inductively on the cell decomposition of K.

19.2. **Example**. The cone over $C\Delta[n]$ can be identified with $\Delta[n+1]$ where the inclusion $\Delta[n] \hookrightarrow C\Delta[n]$ corresponds to the map $d_{n+1} : \Delta[n] \to \Delta[n+1]$. The cone $C\partial(\Delta[n])$ is isomorphic to $\Delta[n+1, n+1]$ and the map $C(\partial(\Delta[n]) \hookrightarrow \Delta[n])$ corresponds to the inclusion $\Delta[n+1, n+1] \hookrightarrow \Delta[n+1]$.

19.3. **Remark**. The opposite category $(CK)^{op}$ has a more transparent interpretation as a Grothendieck construction:

$$(CK)^{op} = Gr\big(K^{op} \xleftarrow{pr_1} K^{op} \times \Delta[0]^{op} \xrightarrow{pr_2} \Delta[0]^{op}\big)$$

We will take advantage of this presentation in Proposition 26.1.

The cone construction in *Spaces* has its analogue in the category of small categories. In *Cat* the construction is much simpler. To a small category we just add a terminal object.

19.4. **Definition**. *Let I be a small category. The cone over I is a category CI defined as follows:*

$$ob(CI) = ob(I) \coprod \{e\}$$

$$mor_{CI}(a,b) = \begin{cases} mor_I(a,b) & \textit{if } \ a \neq e, \ b \neq e \\ \{e_a : a \to e\} & \textit{if } \ b = e \\ \emptyset & \textit{if } \ a = e, \ b \neq e \end{cases}$$

The object $e \in CI$ is terminal. As in the case of *Spaces*, the cone construction in *Cat* is natural, i.e. a functor $f : I \to J$ induces a functor of cones $Cf : CI \to CJ$. Observe that there is a natural inclusion $I \to CI$.

On the level of nerves $N(CI)$ can be identified with $CN(I)$. Under this identification the object $e \in CI$ corresponds to the vertex $e^1 \in CN(I)$ and the map $N(I \to CI)$ corresponds to the natural inclusion $N(I) \hookrightarrow CN(I)$.

20. Diagrams indexed by cones I

In this section we calculate ocolimits of certain bounded diagrams indexed by cones. Let K be a space. For any simplex of the form $(\sigma, e^1) \in CK$ there is a unique morphism $e^1 \to (\sigma, e^1)$ corresponding to the inclusion of the vertex e^1 into (σ, e^1).

20.1. **Proposition**. *Let $F : CK \to \mathcal{C}$ be a cofibrant diagram in $Fun^b(CK, \mathcal{C})$. If for every simplex of the form $(\sigma, e^1) \in CK$ the morphism $F(e^1) \to F(\sigma, e^1)$ is a weak equivalence, then so is $F(e^1) \to colim_{CK} F$.*

20.2. **Lemma**. *Let $f : L \to K$ be a map. The reduction of $Cf : CL \to CK$ can be identified with $CL \xrightarrow{C(f_{red})} Cred(f) \to CK$ (see Proposition 13.7).*

Proof. The lemma follows from the fact that $Cs_i : C\Delta[n+1] \to C\Delta[n]$ can be identified with $s_i : \Delta[n+2] \to \Delta[n+1]$. ☐

Proof of Proposition 20.1. The strategy is the same as in the proof of Lemma 16.4. We first assume that K is finite dimensional. In this case we argue by induction on the dimension of K.

Let $dim(K) = 0$. For simplicity assume $K = \Delta[0]$. The general case can be proven analogously. Since $K = \Delta[0]$, $CK = \Delta[1]$. The non-degenerate and 1-dimensional simplex in $\Delta[1]$ corresponds to $(0, e^1) \in CK$. Hence, according to the assumption, $F(e^1) \to colim_{\Delta[1]} F = F(0, e^1)$ is a weak equivalence.

Let us assume that the proposition holds for those spaces whose dimension is less than n. Let $dim(K) = n$. For simplicity assume that K has only one non-degenerate simplex of dimension n, i.e. K fits into the following push-out square:

$$\begin{tikzcd} \partial(\Delta[n]) \arrow[r, hook] \arrow[d] & \Delta[n] \arrow[d, "\sigma"] \\ N \arrow[r, hook] & K \end{tikzcd} \quad \text{where } dim(N) < n.$$

By applying the cone construction we get another push-out diagram:

$$\begin{tikzcd} \Delta[n+1, n+1] = C\partial(\Delta[n]) \arrow[r, hook] \arrow[d] & C\Delta[n] = \Delta[n+1] \arrow[d, "C\sigma"] \\ CN \arrow[r, hook] & CK \end{tikzcd}$$

By Corollary 8.4.(1) this induces yet another push-out:

$$\begin{tikzcd} colim_{C\partial(\Delta[n])} F \arrow[r] \arrow[d] & F(\sigma, e^1) \arrow[d] \\ colim_{CN} F \arrow[r] & colim_{CK} F \end{tikzcd}$$

Since $CN \hookrightarrow CK$ is a monomorphism, the pull-back $CN \hookrightarrow CK \xrightarrow{F} \mathcal{C}$ is a cofibrant diagram. The inductive assumption implies therefore that $F(e^1) \to colim_{CN} F$ is a weak equivalence. Thus the proposition would be proved once we show that $colim_{C\partial(\Delta[n])} F \to F(\sigma, e^1) \simeq F(e^1)$ is an acyclic cofibration. By Proposition 3.1 this would imply that so is $colim_{CN} F \to colim_{CK} F$, and we could conclude that the composition $F(e^1) \to colim_{CN} F \to colim_{CK} F$ is a weak equivalence.

The map $C\sigma : C\Delta[n] \to CK$ sends non-degenerate simplices in $C\Delta[n] \setminus C\partial(\Delta[n])$ to non-degenerate simplices in CK. Hence cofibrancy of F implies that the morphism $colim_{C\partial(\Delta[n])} F \to F(\sigma, e^1)$ is a cofibration (see Corollary 16.2).

We are going to show that this morphism is also a weak equivalence. As the map $C\partial(\Delta[n]) \to CK$ can send non-degenerate simplices to degenerate ones, the pull-back $C\partial(\Delta[n]) \to CK \xrightarrow{F} \mathcal{C}$ is not necessarily cofibrant. However, this diagram is cofibrant relative to $C\partial(\Delta[n]) \to CK$. Let us denote the map $\partial(\Delta[n]) \to K$ by g. Consider the reduction $C\partial(\Delta[n]) \xrightarrow{Cg_{red}} Cred(g) \to CK$ of Cg (see Lemma 20.2). According to Proposition 15.6 $(Cg_{red})^k F$ is a cofibrant diagram. Its pull-back along Cg_{red} is isomorphic to F, and hence $(Cg_{red})^k F$ satisfies the assumption of the proposition. Thus, by the inductive assumption, $F(e^1) \to colim_{Cred(g)} (Cg_{red})^k F$ is a weak equivalence. As the Kan extension does not modify colimits we can conclude that $F(e^1) \to colim_{C\partial(\Delta[n])}F$ is also a weak equivalence.

So far we have proven the proposition in the case when K is finite dimensional. The infinite dimensional case can be proven by considering skeleton filtration of K and using Proposition 3.5.(3), Corollary 8.4.(2) and Corollary 16.2. □

20.3. **Corollary**. *Let $F : CK \to \mathcal{C}$ be a bounded diagram. If for every simplex of the form $(\sigma, e^1) \in CK$, $F(e^1) \to F(\sigma, e^1)$ is a weak equivalence, then $ocolim_{CK}F$ is weakly equivalent to $F(e^1)$.*

20.4. **Corollary**. *Let I be a small category and $F : N(CI) \to \mathcal{C}$ be a bounded and cofibrant diagram. Assume that F sends any morphism of the form $d_n \circ \cdots \circ d_1 : e \to (i_n \to \cdots \to i_1 \to e)$ in $N(CI)$ to a weak equivalence in $\mathcal{C}$. Then the morphism $F(e) \to colim_{N(CI)}F$ is also a weak equivalence.*

Corollary 20.4 can be generalized to categories with a terminal object.

20.5. **Proposition**. *Let I be a category with a terminal object t and $F : N(I) \to \mathcal{C}$ be a bounded and cofibrant diagram. Assume that F sends any morphism of the form $d_n \circ \cdots \circ d_1 : t \to (i_n \to \cdots \to i_1 \to t)$ in $N(I)$ to a weak equivalence. Then the morphism $F(t) \to colim_{N(I)}F$ is also a weak equivalence.*

Proof. Even though I has a terminal object, in general, $N(I)$ is not isomorphic to a cone. Thus we can not apply Corollary 20.4 directly. However, using the existence of a terminal object $t \in I$ we can construct a retraction $CI \to I$ of $I \hookrightarrow CI$:

$$CI \to I,\ I \ni i \mapsto i,\ e \mapsto *$$

$$I \ni (i \to j) \mapsto (i \to j),\ (e_i : i \to e) \mapsto (i \to t)$$

The composition $I \hookrightarrow CI \to I$ is the identity.

By pulling back F along $N(CI) \to N(I)$ we get a diagram which is no longer cofibrant. Let $QF : N(CI) \to \mathcal{C}$ together with $QF \overset{\sim}{\twoheadrightarrow} F$ be a cofibrant replacement of the composition $N(CI) \to N(I) \xrightarrow{F} \mathcal{C}$ such that the pull-back $N(I) \hookrightarrow N(CI) \xrightarrow{QF} \mathcal{C}$ coincides with $F : N(I) \to \mathcal{C}$. We can form now the following commutative diagram:

$$\begin{tikzcd}
F(t) \arrow[r, "id"] \arrow[d] & F(t) \arrow[r, "id"] \arrow[d] & F(t) \arrow[d] \\
colim_{N(I)}F \arrow[r] \arrow[rr, bend right, "id"'] & colim_{N(CI)}QF \arrow[r] & colim_{N(I)}F
\end{tikzcd}$$

The diagram $QF : N(CI) \to \mathcal{C}$ satisfies the assumption of Corollary 20.4, and hence $QF(e) \to colim_{N(CI)}QF$ is a weak equivalence. It follows that so is $F(t) \to colim_{N(CI)}QF$. According to axiom **MC3** for the model structure on $\mathcal{C}$ the morphism $F(t) \to colim_{N(I)}F$ is also a weak equivalence. This proves the proposition. □

20.6. **Corollary**. *Let I be a small category with a terminal object t. Consider the forgetful functor $\epsilon : N(I) \to I$ (see Section 6). Let $F : N(I) \to \mathcal{C}$ be bounded and cofibrant. Assume that there exists $F' : I \to \mathcal{C}$ and a weak equivalence $F \xrightarrow{\sim} \epsilon^* F'$ in $Fun^b\big(N(I), \mathcal{C}\big)$. Then the morphism:*

$$colim_{N(I)} F \to colim_{N(I)} \epsilon^* F' = colim_I F' = F'(t)$$

is a weak equivalence.

Since $N(I^{op}) \cong N(I)$, Proposition 20.5 dualizes to categories with an initial object.

20.7. **Proposition**. *Let I be a category with an initial object e and $F : N(I) \to \mathcal{C}$ be a bounded and cofibrant diagram. Assume that F sends any morphism of the form $d_{n-1} \circ \cdots \circ d_0 : e \to (e \to i_{n-1} \to \cdots \to i_0)$ in $N(I)$ to a weak equivalence. Then the morphism $F(e) \to colim_{N(I)} F$ is also a weak equivalence.*

21. The homotopy colimit

In this section we construct a homotopy invariant approximation of the functor $colim_I : Fun(I, \mathcal{C}) \to \mathcal{C}$, where I is an arbitrary small category. To do so we use the forgetful functor $\epsilon : N(I) \to I$, $(i_0 \to \cdots \to i_n) \mapsto i_n$ (see Section 6). We apply it to compare the category $Fun(I, \mathcal{C})$ with $Fun^b\big(N(I), \mathcal{C}\big)$. In this way we can take advantage of the model structure on $Fun^b\big(N(I), \mathcal{C}\big)$ and find an analogue of a cofibrant replacement in $Fun(I, \mathcal{C})$.

21.1. **Theorem**. *The composition:*

$$Fun(I, \mathcal{C}) \xrightarrow{\epsilon^*} Fun^b(N(I), \mathcal{C}) \xrightarrow{ocolim_{N(I)}} Ho(\mathcal{C})$$

together with a natural transformation induced by the morphisms:

$$ocolim_{N(I)} \epsilon^* F \to colim_{N(I)} \epsilon^* F = colim_I F$$

is the total left derived functor of $colim_I$.

Proof. We are going to show that the following diagram satisfies the assumptions of Proposition 5.4:

$$\begin{tikzcd} Fun(I,\mathcal{C}) \arrow[rr, shift left, "\epsilon^*"] \arrow[dr, "colim_I"'] & & Fun^b\big(N(I),\mathcal{C}\big) \arrow[ll, shift left, "\epsilon^k"] \arrow[dl, "colim_{N(I)}"] \\ & C & \end{tikzcd}$$

Since the forgetful functor $\epsilon : N(I) \to I$ is cofinal with respect to taking colimits, the above diagram does commute.

Lemma 16.7 implies that $colim_{N(I)} : Fun^b\big(N(I), \mathcal{C}\big) \to \mathcal{C}$ is homotopy meaningful on cofibrant objects. It is also clear that ϵ^* is homotopy meaningful. Thus to prove the theorem we only need to check that, for any $F' : I \to \mathcal{C}$, if $F : N(I) \to \mathcal{C}$

is a cofibrant diagram and $F \to \epsilon^* F'$ is a weak equivalence, then so is its adjoint $\epsilon^k F \to F'$.

By definition $\epsilon^k F : I \to \mathcal{C}$ assigns to $i \in I$ the colimit $colim_{\epsilon\downarrow i} F$ (see A.3). The category $\epsilon\downarrow i$ can be identified with the simplex category $N(I\downarrow i)$. Under this identification the functor $\epsilon\downarrow i \to N(I)$ corresponds to the map $N(I\downarrow i) \to N(I)$ (see Example 6.6). Since $N(I\downarrow i) \to N(I)$ is reduced, i.e. it sends non-degenerate simplices to non-degenerate ones, the composition $N(I\downarrow i) \to N(I) \xrightarrow{F} \mathcal{C}$ is a cofibrant diagram. As the category $I\downarrow i$ has a terminal object, we can apply Corollary 20.6 to conclude that the following is a weak equivalence:

$$colim_{N(I\downarrow i)} F \to colim_{N(I\downarrow i)} \epsilon^* F' = colim_{I\downarrow i} F' = F'(i)$$

This shows that the natural transformation $\epsilon^k F \to F'$ is a weak equivalence in $Fun(I, \mathcal{C})$. $\square$

21.2. **Remark**. The proof of Theorem 21.1 implies the following fact: even though the category $Fun(I, \mathcal{C})$ does not admit a (known) model category structure, there is a good candidate for a "cofibrant replacement". Let us choose a cofibrant replacement Q in $Fun^b\big(N(I), \mathcal{C}\big)$. For any diagram $F : I \to \mathcal{C}$, define $QF := \epsilon^k Q \epsilon^* F$ and $QF \to F$ to be the adjoint of $Q\epsilon^* F \overset{\sim}{\twoheadrightarrow} \epsilon^* F$. The homotopy colimit of F can be then computed by taking the colimit $colim_I QF$. The natural transformation $hocolim_I \to colim_I$ is induced by $QF \to F$.

¿From the fact that the map $N(I\downarrow i) \to N(I)$ is reduced we can deduce that the Kan extension ϵ^k is homotopy meaningful on cofibrant object. Thus the proof of Theorem 21.1 and Proposition 5.4.(2) imply:

21.3. **Corollary**. *For any small category I, the localization of $Fun(I, \mathcal{C})$ with respect to weak equivalences exists. Its homotopy category can be constructed by taking the full subcategory of $Ho\big(Fun^b\big(N(I), \mathcal{C}\big)\big)$ generated by the image of the composition $Fun(I, \mathcal{C}) \xrightarrow{\epsilon^*} Fun^b\big(N(I), \mathcal{C}\big) \to Ho\big(Fun^b\big(N(I), \mathcal{C}\big)\big)$.*

21.4. **Remark**. In the case C has a functorial factorization of morphisms by cofibrations followed by acyclic fibrations, we can choose a functorial cofibrant replacement $Q : Fun^b\big(N(I), \mathcal{C}\big) \to Fun^b\big(N(I), \mathcal{C}\big)$. This can be used to define a "rigid" hocolimit by taking $hocolim_I : Fun(I, \mathcal{C}) \to \mathcal{C}$ to be $colim_{N(I)} Q\epsilon^*(-)$ (compare with Remark 18.4). The natural transformation $hocolim_I \to colim_I$ is induced by $Q\epsilon^* F \overset{\sim}{\twoheadrightarrow} \epsilon^* F$.

So far we have looked at diagrams indexed by a single space. In sections 22–24 we consider diagrams indexed by a parameterized collection of spaces. We start with the simplest case.

22. Fubini theorem

In this section we look at diagrams indexed by the product of two simplex categories. The aim of this section is to prove the so called Fubini theorem for ocolimits and hocolimits, i.e. that the ocolimit, respectively hocolimit, commutes with itself.

Recall that $K\tilde{\times}N$ denotes the product of the simplex categories of K and N (see 6.9). Via the standard exponential map $Fun^b\big(K, Fun^b(N,\mathcal{C})\big)$ can be embedded, as a full subcategory, into $Fun(K\tilde{\times}N,\mathcal{C})$. Its objects can be characterized as those functors $F : K\tilde{\times}N \to \mathcal{C}$ where, for any degeneracy morphisms $s_i\sigma \to \sigma$ in K and $s_j\tau \to \tau$ in N, the natural transformations $F(s_i\sigma,-) \to F(\sigma,-)$ and $F(-,s_i\tau) \to F(-,\tau)$ are isomorphisms. By symmetry we can identify $Fun^b\big(K, Fun^b(N,\mathcal{C})\big)$ with $Fun^b\big(N, Fun^b(K,\mathcal{C})\big)$. This full subcategory of $Fun(K\tilde{\times}N,\mathcal{C})$ is denoted by $Fun^b(K\tilde{\times}N,\mathcal{C})$. Its objects are called bounded diagrams.

Bounded diagrams with values in a model category form a model category (see Lemma 17.4). The identifications:

$$Fun^b(K\tilde{\times}N,\mathcal{C}) = Fun^b\big(K, Fun^b(N,\mathcal{C})\big)\ ,\ Fun^b(K\tilde{\times}N,\mathcal{C}) = Fun^b\big(N, Fun^b(K,\mathcal{C})\big)$$

can be therefore used to induce two model structures on $Fun^b(K\tilde{\times}N,\mathcal{C})$. We leave to the reader to verify:

22.1. **Proposition**. *The two model structures on $Fun^b(K\tilde{\times}N,\mathcal{C})$ coincide.*

Weak equivalences and fibrations in $Fun^b(K\tilde{\times}N,\mathcal{C})$ are easy to describe. They are simply objectwise weak equivalences and fibrations. Cofibrations however are more subtle. The following two lemmas describe their crucial properties.

22.2. **Lemma**. *Let $F : K\tilde{\times}N \to \mathcal{C}$ be a bounded and cofibrant diagram. Then:*

(1) *For all $\sigma \in K$ and $\tau \in N$, the diagrams $F(\sigma,-) : N \to \mathcal{C}$ and $F(-,\tau) : K \to \mathcal{C}$ are bounded and cofibrant.*
(2) *The diagrams $colim_{\sigma\in K}F(\sigma,-) : N \to \mathcal{C}$ and $colim_{\tau\in N}F(-,\tau) : K \to \mathcal{C}$ are bounded and cofibrant.*

Proof. Cofibrancy of $F : K \to Fun^b(N,\mathcal{C})$ means that, for any non-degenerate simplex $\gamma : \Delta[n] \to K$, the natural transformation $colim_{\xi\in\partial(\Delta[n])}F(\xi,-) \to F(\gamma,-)$ is a cofibration in $Fun^b(N,\mathcal{C})$. Using Corollary 16.5.(1) we can conclude that, for any $\tau \in L$, the morphism $colim_{\xi\in\partial(\Delta[n])}F(\xi,\tau) \to F(\gamma,\tau)$ is a cofibration in $\mathcal{C}$. This means exactly that $F(-,\tau) : K \to \mathcal{C}$ is cofibrant. By symmetry $F(\sigma,-) : N \to \mathcal{C}$ is also cofibrant.

On the other hand, by Lemma 16.4, we can also conclude that the morphism $colim_{\tau\in N}\big(colim_{\xi\in\partial(\Delta[n])}F(\xi,\tau) \to F(\gamma,\tau)\big)$ is a cofibration in $\mathcal{C}$. This is equivalent to $colim_{\tau\in N}F(-,\tau) : K \to \mathcal{C}$ being cofibrant. By symmetry $colim_{\sigma\in K}F(\sigma,-) : N \to \mathcal{C}$ is also cofibrant. $\square$

22.3. **Lemma**. *The functor $colim_{K\tilde{\times}N} : Fun^b(K\tilde{\times}N, \mathcal{C}) \to \mathcal{C}$ is homotopy invariant on cofibrant objects.*

Proof. Let $F : K\tilde{\times}N \to \mathcal{C}$ and $G : K\tilde{\times}N \to \mathcal{C}$ be bounded and cofibrant diagrams and $\Phi : F \xrightarrow{\sim} G$ be a weak equivalence. According to Lemma 22.2.(1), for any $\sigma \in K$, $F(\sigma, -)$ and $G(\sigma, -)$ are cofibrant. Hence the following morphism is a weak equivalence (see Lemma 16.7):

$$colim_N\Phi_{(\sigma,-)} : colim_N F(\sigma, -) \to colim_N G(\sigma, -)$$

As the diagrams $K \ni \sigma \mapsto colim_N F(\sigma, -) \in \mathcal{C}$ and $K \ni \sigma \mapsto colim_N G(\sigma, -) \in \mathcal{C}$ are also cofibrant (see Lemma 22.2.(2)), $colim_K colim_N \Phi$ is a weak equivalence as well. Since $colim_{K\tilde{\times}N}\Phi$ coincides with $colim_K colim_N \Phi$, the lemma is proven. □

22.4. **Proposition**. *The total left derived functor of $colim_{K\tilde{\times}N} : Fun^b(K\tilde{\times}N, \mathcal{C}) \to \mathcal{C}$ exists. It can be constructed by choosing a cofibrant replacement Q in $Fun^b(K\tilde{\times}N, \mathcal{C})$ and assigning to $F \in Fun^b(K\tilde{\times}N, \mathcal{C})$ the colimit $colim_{K\tilde{\times}N}QF \in Ho(\mathcal{C})$.*

Proof. The proposition follows from Proposition 4.4 and Lemma 22.3. □

The total left derived functor of $colim_{K\tilde{\times}N} : Fun^b(K\tilde{\times}N, \mathcal{C}) \to \mathcal{C}$ is denoted by $ocolim_{K\tilde{\times}N} : Fun^b(K\tilde{\times}N, \mathcal{C}) \to Ho(\mathcal{C})$.

In the case when $\mathcal{C}$ has a functorial factorization of morphisms by cofibrations followed by acyclic fibrations one can construct a functorial cofibrant replacement $Q : Fun^b(K\tilde{\times}N, \mathcal{C}) \to Fun^b(K\tilde{\times}N, \mathcal{C})$. This can be used to construct a rigid ocolimit by taking $colim_{K\tilde{\times}N}Q(-) : Fun^b(K\tilde{\times}N, \mathcal{C}) \to \mathcal{C}$ (compare with Remarks 18.4 and 21.4).

The following proposition describes one of the crucial global feature of the ocolimit construction, the so called Fubini theorem.

22.5. **Proposition**. *Assume that $\mathcal{C}$ has a functorial factorization of morphisms by cofibrations followed by acyclic fibrations. Then, for any $F \in Fun^b(K\tilde{\times}N, \mathcal{C})$, we have the following weak equivalences in $\mathcal{C}$:*

$$ocolim_{K\tilde{\times}N}F \simeq ocolim_K ocolim_N F \simeq ocolim_N ocolim_K F$$

Proof. Let $Q : Fun^b(K\tilde{\times}N, \mathcal{C}) \to Fun^b(K\tilde{\times}N, \mathcal{C})$ be a functorial cofibrant replacement. Lemma 22.2.(1) tells us that, for any $\sigma \in K$ and $\tau \in N$, $QF(\sigma, -) : N \to \mathcal{C}$ and $QF(-, \tau) : K \to C$ are cofibrant replacements of $F(\sigma, -) : N \to \mathcal{C}$ and $F(-, \tau) : K \to C$ respectively. Hence $colim_N QF(\sigma, -) \simeq ocolim_N F(\sigma, -)$ and $colim_K QF(-, \tau) \simeq ocolim_K F(-, \tau)$. Since the diagrams $K \ni \sigma \mapsto colim_N QF(\sigma, -)$ and $N \ni \tau \mapsto colim_K QF(-, \tau)$ are also cofibrant (see Lemma 22.2.(2)), they are cofibrant replacements of $K \ni \sigma \mapsto ocolim_N F(\sigma, -)$ and $N \ni \tau \mapsto ocolim_K F(-, \tau)$ respectively. The proposition clearly follows. □

Let I and J be small categories. To construct the homotopy colimit $hocolim_{I\times J}$ Theorem 21.1 tells us that we should use the forgetful functor $\epsilon : N(I \times J) \to I \times J$

to approximate $Fun(I\times J,\mathcal{C})$ by the model category $Fun^b\big(N(I\times J),\mathcal{C}\big)$. In this way we can take advantage of a cofibrant replacement in $Fun^b\big(N(I\times J),\mathcal{C}\big)$. In the next theorem we are going to prove that to construct $hocolim_{I\times J}$ we could also take the product of the forgetful functors $\epsilon : N(I)\to I$ and $\epsilon : N(J)\to J$ (denoted by the same symbol $\epsilon : N(I)\tilde{\times}N(J)\to I\times J$) and approximate $Fun(I\times J,\mathcal{C})$ by the model category $Fun^b\big(N(I)\tilde{\times}N(J),\mathcal{C}\big)$. In this way we can use a cofibrant replacement in $Fun^b\big(N(I)\tilde{\times}N(J),\mathcal{C}\big)$.

22.6. **Theorem**. *The composition:*

$$Fun(I\times J,\mathcal{C})\xrightarrow{\epsilon^*} Fun^b\big(N(I)\tilde{\times}N(J),\mathcal{C}\big)\xrightarrow{ocolim_{N(I)\tilde{\times}N(J)}} Ho(\mathcal{C})$$

together with a natural transformation induced by the morphisms:

$$ocolim_{N(I)\tilde{\times}N(J)}\epsilon^*F\to colim_{N(I)\tilde{\times}N(J)}\epsilon^*F = colim_{I\times J}F$$

is the total left derived functor of $colim_{I\times J}$.

Proof. The strategy is the same as in the proof of Theorem 21.1. We are going to show that the following diagram satisfies the assumptions of Proposition 5.4:

$$\begin{tikzcd} Fun(I\times J,\mathcal{C}) \arrow[rr, shift left, "\epsilon^*"] \arrow[dr, "colim_{I\times J}"'] & & Fun^b\big(N(I)\tilde{\times}N(J),\mathcal{C}\big) \arrow[ll, shift left, "\epsilon^k"] \arrow[dl, "colim_{N(I)\tilde{\times}N(J)}"] \\ & C & \end{tikzcd}$$

Since $\epsilon : N(I)\to I$ and $\epsilon : N(J)\to J$ are cofinal with respect to taking colimits, then so is their product $\epsilon : N(I)\tilde{\times}N(J)\to I\times J$, and hence the above diagram does commute.

Lemma 22.3 implies that $colim_{N(I)\tilde{\times}N(J)}$ is homotopy meaningful on cofibrant objects. It is also clear that ϵ^* is homotopy meaningful. Thus to prove the theorem we need to show that, for any $F' : I\times J\to\mathcal{C}$, if $F : N(I)\tilde{\times}N(J)\to\mathcal{C}$ is cofibrant and $F\to\epsilon^*F'$ is a weak equivalence, then so is its adjoint $\epsilon^kF\to F'$.

Let (i,j) be an object in $I\times J$. By definition $\epsilon^kF(i,j)=colim_{\epsilon\downarrow(i,j)}F$. The category $\epsilon\downarrow(i,j)$ can be identified with $N(I\downarrow i)\tilde{\times}N(J\downarrow j)$. Under this identification the functor $\epsilon\downarrow(i,j)\to N(I)\tilde{\times}N(J)$ corresponds to the product map $N(I\downarrow i)\tilde{\times}N(J\downarrow j)\to N(I)\tilde{\times}N(J)$. The maps $N(I\downarrow i)\to N(I)$ and $N(J\downarrow j)\to N(J)$ are reduced, i.e. they send non-degenerate simplices to non-degenerate ones. Thus the composition $N(I\downarrow i)\tilde{\times}N(J\downarrow j)\to N(I)\tilde{\times}N(J)\xrightarrow{F}\mathcal{C}$ is a cofibrant diagram. Using Lemma 22.2.(1) we can conclude that, for any simplex $\sigma=(i_n\to\cdots\to i_0, i_0\to i)\in N(I\downarrow i)$, $F(\sigma,-) : N(J\downarrow j)\to\mathcal{C}$ is also cofibrant. Since $F(\sigma,-)\to\epsilon^*F'(i_0,-)$ is a weak equivalence and $J\downarrow j$ has a terminal object, according to Corollary 20.6, the following morphism is a weak equivalence in $\mathcal{C}$:

$$colim_{N(J\downarrow j)}F(\sigma,-)\to colim_{N(J\downarrow j)}\epsilon^*F'(i_0,-) = colim_{J\downarrow j}F'(i_0,-) =$$

$$= F'(i_0, j) = \epsilon^* F'(\sigma, j)$$

Thus the natural transformation between $N(I\downarrow i) \ni \sigma \mapsto colim_{N(J\downarrow j)}F(\sigma, -)$ and $\epsilon^* F'(-, j)$ is a weak equivalence. As the first diagram is cofibrant (see Lemma 22.2.(2)) and the category $I\downarrow i$ has a terminal object, again according to Corollary 20.6, the morphism:

$$colim_{N(I\downarrow i)} colim_{N(J\downarrow j)} F \to colim_{N(I\downarrow i)} \epsilon^* F'(-, j) = colim_{(I\downarrow i)} F'(-, j) = F'(i, j)$$

is a weak equivalence. The theorem has been proven. □

As a corollary of Proposition 22.5 and Theorem 22.6 we get the so called Fubini Theorem for the homotopy colimits.

22.7. **Corollary**. *Assume that $\mathcal{C}$ has a functorial factorization of morphisms by cofibrations followed by acyclic fibrations. Then, for any $F : I \times J \to \mathcal{C}$, we have the following weak equivalences in $\mathcal{C}$:*

$$hocolim_{I\times J} F \simeq hocolim_I hocolim_J F \simeq hocolim_J hocolim_I F$$

23. Bounded diagrams indexed by Grothendieck constructions

In section 22 we introduced the notion of boundness for diagrams indexed by the product of simplex categories. In this section we generalize this notion to diagrams indexed by Grothendieck constructions.

If $H : K \to Spaces$ is a diagram, then we can think about its values as simplex categories, and hence form its Grothendieck construction $Gr_K H$ (see Definition A.5). We would like to understand the local data needed to describe a functor indexed by $Gr_K H$.

23.1. **Definition**. *Let $H : K \to Spaces$ be a functor. We say that a family of functors $F = \{F_\sigma : \Delta[n] \to Fun\big(H(\sigma), \mathcal{C}\big)\}_{\sigma:\Delta[n]\to K}$ is compatible over H if, for any simplices $\sigma : \Delta[n] \to K$ and $\xi : \Delta[m] \to K$ and for any morphism $\alpha : \sigma \to \xi$ in K, the following diagram commutes:*

$$\begin{tikzcd} \Delta[n] \arrow[r, "F_\sigma"] \arrow[d, "\alpha"'] & Fun\big(H(\sigma), \mathcal{C}\big) \arrow[d, "H(\alpha)^k"] \\ \Delta[m] \arrow[r, "F_\xi"] & Fun\big(H(\xi), \mathcal{C}\big) \end{tikzcd}$$

where $H(\alpha)^k$ is the Kan extension of $H(\alpha) : H(\sigma) \to H(\xi)$.

Let F and G be compatible families of functors over H. A family of natural transformations $\Psi = \{\Psi_\sigma : F_\sigma \to G_\sigma\}_{\sigma\in K}$ is called a morphism between F and G if, for any $\alpha : \sigma \to \xi$ in K, the pull-back $\alpha^*\Psi_\xi$ coincide with $H(\alpha)^k(\Psi_\sigma)$.

Compatible families over H, together with morphisms between them as defined above, clearly form a category.

Let ι be the only non-degenerate n-dimensional simplex in $\Delta[n]$, i.e. ι corresponds to the map $id : \Delta[n] \to \Delta[n]$. With any compatible family $F = \{F_\sigma\}_{\sigma \in K}$ over H we can associate a functor which is denoted by the same symbol $F : Gr_K H \to \mathcal{C}$. It assigns to $(\sigma, \tau) \in Gr_K H$ an object $F_\sigma(\iota)(\tau) \in \mathcal{C}$. In this way we get a functor between the category of compatible families over H and $Fun(Gr_K H, \mathcal{C})$. One can show that this functor is an isomorphism of categories. Thus we do not distinguish between diagrams indexed by $Gr_K H$ and compatible families over H. We also use the same symbol $Fun(Gr_K H, \mathcal{C})$ to denote both the category of compatible family of functors over H and the category of functors indexed by $Gr_K H$.

A compatible family over H carries a lot of redundant data just to describe a functor indexed by $Gr_K H$. However, this way of thinking about diagrams over $Gr_K H$ helps in keeping tracks of their local data. This local information is important to describe a notion of boundness for diagrams indexed by $Gr_K H$ as well as a model structure on such diagrams.

23.2. **Definition**. *Let $H : K \to Spaces$ be bounded. A functor $F : Gr_K H \to \mathcal{C}$ is called bounded if, for any simplex $\sigma : \Delta[n] \to K$:*

- $F_\sigma : \Delta[n] \to Fun\big(H(\sigma), \mathcal{C}\big)$ *has bounded values, i.e.* $F_\sigma : \Delta[n] \to Fun^b\big(H(\sigma), \mathcal{C}\big)$,
- $F_\sigma : \Delta[n] \to Fun^b\big(H(\sigma), \mathcal{C}\big)$ *is bounded relative* $\sigma : \Delta[n] \to K$.

The full subcategory of $Fun(Gr_K H, \mathcal{C})$ consisting of bounded diagrams is denoted by $Fun^b(Gr_K H, \mathcal{C})$.

An object $(\sigma, \tau) \in Gr_K H$ is called non-degenerate if σ is non-degenerate in K and τ is non-degenerate in $H(\sigma)$. A bounded diagram $F : Gr_K H \to \mathcal{C}$ is determined, up to isomorphism, by the values it takes on the non-degenerate objects in $Gr_K H$.

In the case $N : K \to Spaces$ is a constant diagram with value N the boundness condition on a diagram indexed by $Gr_K N = K \tilde{\times} N$ coincide with the one given in Section 22.

In the case of a constant diagram $N : K \to Spaces$ to get a model structure on $Fun^b(Gr_K N, \mathcal{C})$ we used the symmetry of the product $Gr_K N = K \tilde{\times} N = Gr_N K$ (see Section 22). For a general bounded diagram $H : K \to Spaces$ such a symmetry does not hold. To find an appropriate model structure on $Fun^b(Gr_K H, \mathcal{C})$ we need to use other methods. In this case the crucial role is played by the local data. As in the case of boundness (see Definition 23.2) definitions of weak equivalences, fibration and cofibrations in $Fun^b(Gr_K H, \mathcal{C})$ have local nature.

23.3. **Definition**. *Let $\Psi : F \to G$ be a natural transformation in $Fun^b(Gr_K H, \mathcal{C})$.*
Weak equivalences. *Ψ is called a weak equivalence if, for any $\sigma : \Delta[n] \to K$, $\Psi_\sigma : F_\sigma \to G_\sigma$ is a weak equivalence in $Fun^b_\sigma\big(\Delta[n], Fun^b(H(\sigma), \mathcal{C})\big)$.*
Fibrations. *Ψ is called a fibration if, for any $\sigma : \Delta[n] \to K$, $\Psi_\sigma : F_\sigma \to G_\sigma$ is a fibration in $Fun^b_\sigma\big(\Delta[n], Fun^b(H(\sigma), \mathcal{C})\big)$.*
Cofibrations. *Ψ is called a cofibration if, for any $\sigma : \Delta[n] \to K$, $\Psi_\sigma : F_\sigma \to G_\sigma$ is a cofibration in $Fun^b_\sigma\big(\Delta[n], Fun^b(H(\sigma), \mathcal{C})\big)$.*

Weak equivalences and fibrations in $Fun^b(Gr_K H, \mathcal{C})$ are easy to describe in terms of diagrams indexed by $Gr_K H$. They are simply objectwise weak equivalences and fibrations. The analogous description for cofibrations is more complicated. For example, a bounded diagram $F : Gr_K H \to \mathcal{C}$ is cofibrant if and only if, for any non-degenerate simplex $\sigma : \Delta[n] \to K$, the natural transformation $colim_{\tau \in \partial\Delta[n]} F(\tau, -) \to F(\sigma, -)$ is a cofibration in $Fun^b\big(H(\sigma), \mathcal{C}\big)$.

Using exactly the same arguments as in the proof of Lemma 17.4 one can show:

23.4. **Theorem**. *Let $H : K \to Spaces$ be bounded. The category $Fun^b(Gr_K H, \mathcal{C})$, together with the choice of weak equivalences, fibrations and cofibrations given in Definition 23.3, satisfies the axioms of a model category.*

In the case of a constant diagram $N : K \to Spaces$ with value N the model structure on $Fun^b(Gr_K N, \mathcal{C}) = Fun^b(K \tilde{\times} N, \mathcal{C})$ given in Definition 23.3 coincides with the one given in Section 22.

24. Thomason Theorem

In this section we prove the so called Thomason Theorem ([39]). It is a generalization of Fubini Theorem (see Proposition 22.5 and Corollary 22.7). The strategy is the same as in Section 22. The following lemmas are generalizations of Lemmas 22.2 and 22.3.

24.1. **Lemma**. *Let $F : Gr_K H \to \mathcal{C}$ be a bounded and cofibrant diagram. Then:*

(1) *For all $\sigma : \Delta[n] \to K$, the diagram $F(\sigma, -) : H(\sigma) \to \mathcal{C}$ is bounded and cofibrant.*
(2) *The diagram $colim_{H(-)} F : K \to \mathcal{C}$, $\sigma \mapsto colim_{H(\sigma)} F(\sigma, -)$, is bounded and cofibrant.*

Proof of (1). Cofibrancy of F tells us that, for any $\sigma : \Delta[n] \to K$, the diagram $F_\sigma : \Delta[n] \to Fun^b\big(H(\sigma), \mathcal{C}\big)$ is cofibrant relative to σ. By Corollary 16.5.(1), it follows that $F_\sigma(\iota)$ is a cofibrant object in $Fun^b\big(H(\sigma), \mathcal{C}\big)$. Since $F_\sigma(\iota) = F(\sigma, -)$ the first part of the lemma is proven.

Proof of (2). The boundness of $colim_{H(-)} F$ is clear. To check that it is a cofibrant diagram we have to show that, for any non-degenerate simplex $\sigma : \Delta[n] \to K$, the morphism $colim_{\sigma' \in \partial(\Delta[n])} colim_{H(\sigma')} F \to colim_{H(\sigma)} F$ is a cofibration.

Consider $F_\sigma : \Delta[n] \to Fun^b\big(H(\sigma), \mathcal{C}\big)$. Since $\sigma : \Delta[n] \to K$ is non-degenerate and F_σ is cofibrant relative σ, by definition, $colim_{\partial(\Delta[n])} F_\sigma \to F_\sigma(\iota)$ is a cofibration in $Fun^b\big(H(\sigma), \mathcal{C}\big)$. Hence, by Lemma 16.4, we can conclude that the morphism $colim_{H(\sigma)} colim_{\partial(\Delta[n])} F_\sigma \to colim_{H(\sigma)} F_\sigma(\iota)$ is a cofibration in $\mathcal{C}$. Since this morphism can be identified with $colim_{\sigma' \in \partial(\Delta[n])} colim_{H(\sigma')} F \to colim_{H(\sigma)} F$ the second part of the lemma is proven. $\square$

24.2. **Lemma**. *The functor $colim_{Gr_K H} : Fun^b(Gr_K H, \mathcal{C}) \to \mathcal{C}$ is homotopy invariant on cofibrant objects.*

Proof. Let $F : Gr_K H \to \mathcal{C}$, $G : Gr_K H \to \mathcal{C}$ be bounded and cofibrant diagrams and $\Psi : F \xrightarrow{\sim} G$ be a weak equivalence. According to Lemma 24.1.(1), for any simplex $\sigma \in K$, $F(\sigma, -)$ and $G(\sigma, -)$ are cofibrant diagrams. It follows that the morphism $colim_{H(\sigma)}\Psi_{(\sigma,-)} : colim_{H(\sigma)}F(\sigma, -) \to colim_{H(\sigma)}G(\sigma, -)$ is a weak equivalence (see Lemma 16.7). As the diagrams $colim_{H(-)}F : K \to \mathcal{C}$ and $colim_{H(-)}G : K \to \mathcal{C}$ are also cofibrant (see Lemma 24.1.(2)), $colim_K colim_{H(-)}\Psi$ is a weak equivalence as well. Since $colim_{Gr_K H}\Psi$ coincide with $colim_K colim_{H(-)}\Psi$, the lemma is proven. □

24.3. **Proposition**. *The total left derived functor of $colim_{Gr_K H} : Fun^b(Gr_K H, \mathcal{C}) \to \mathcal{C}$ exists. It can be constructed by choosing a cofibrant replacement Q in $Fun^b(Gr_K H, \mathcal{C})$ and assigning to $F \in Fun^b(Gr_K H, \mathcal{C})$ the colimit $colim_{Gr_K H}QF \in Ho(\mathcal{C})$.*

Proof. The proposition follows from Proposition 4.4 and Lemma 24.2. □

The total left derived functor of $colim_{Gr_K H} : Fun^b(Gr_K H, \mathcal{C}) \to \mathcal{C}$ is denoted by $ocolim_{Gr_K H} : Fun^b(Gr_K H, \mathcal{C}) \to Ho(\mathcal{C})$.

Assume that $\mathcal{C}$ has a functorial factorization of morphisms by cofibrations followed by acyclic fibrations. In this case one can construct a functorial cofibrant replacement $Q : Fun^b(Gr_K H, \mathcal{C}) \to Fun^b(Gr_K H, \mathcal{C})$. This can be used to construct a rigid ocolimit by taking $colim_{Gr_K H}Q(-) : Fun^b(Gr_K H, \mathcal{C}) \to \mathcal{C}$ (compare with remarks 18.4 and 21.4).

We are now ready to prove Thomason theorem for ocolimits.

24.4. **Proposition**. *Assume that $\mathcal{C}$ has a functorial factorization of morphisms by cofibrations followed by acyclic fibrations. Then, for any $F \in Fun^b(Gr_K H, \mathcal{C})$, we have a weak equivalence in $\mathcal{C}$:*

$$ocolim_{Gr_K H}F \simeq ocolim_K ocolim_{H(-)}F$$

Proof. Let $Q : Fun^b(Gr_K H, \mathcal{C}) \to Fun^b(Gr_K H, \mathcal{C})$ be a functorial cofibrant replacement. Lemma 24.1.(1) tells us that, for any $\sigma \in K$, $QF(\sigma, -) : H(\sigma) \to \mathcal{C}$ is a cofibrant replacement of $F(\sigma, -) : H(\sigma) \to \mathcal{C}$. Hence $colim_{H(\sigma)}QF(\sigma, -) \simeq ocolim_{H(\sigma)}F(\sigma, -)$. By Lemma 24.1.(2) the diagram $colim_{H(-)}QF : K \to \mathcal{C}$, $\sigma \mapsto colim_{H(\sigma)}QF(\sigma, -)$, is cofibrant. Thus it is a cofibrant replacements of $ocolim_{H(-)}F : K \to \mathcal{C}$. The proposition clearly follows, since $colim_K colim_H QF$ coincides with $colim_{Gr_K H}QF \simeq ocolim_{Gr_K H}F$ (see Proposition A.10). □

Let I be an arbitrary small category and $H : I \to Cat$ be a functor. Consider the forgetful functors $\epsilon : N(I) \to I$ and $\epsilon : N\big(H(i)\big) \to H(i)$. We use the same symbol $\epsilon : Gr_{N(I)}N(H) \to Gr_I H$ to denote the induced functor on the level of Grothendieck construction. Since the above forgetful functors are cofinal with respect to taking colimits, then so is $\epsilon : Gr_{N(I)}N(H) \to Gr_I H$.

We are going to use this functor to approximate the category $Fun(Gr_I H, \mathcal{C})$ by a model category $Fun^b\big(Gr_{N(I)}N(H), \mathcal{C}\big)$. In this way we get a more convenient construction of $hocolim_{Gr_I H}$.

24.5. **Theorem**. *The composition:*

$$Fun(Gr_I H, \mathcal{C}) \xrightarrow{\epsilon^*} Fun^b\big(Gr_{N(I)}N(H), \mathcal{C}\big) \xrightarrow{ocolim_{Gr_{N(I)}N(H)}} Ho(\mathcal{C})$$

together with a natural transformation induced by the morphisms:

$$ocolim_{Gr_{N(I)}N(H)}\epsilon^* F \to colim_{Gr_{N(I)}N(H)}\epsilon^* F = colim_{Gr_I H} F$$

is the total left derived functor of $colim_{Gr_I H}$.

Proof. The strategy is the same as in the proof of Theorems 21.1 and 22.6. We are going to show that the following diagram satisfies the assumptions of Proposition 5.4.

$$\begin{tikzcd} Fun(Gr_I H, \mathcal{C}) \arrow[rr, shift left, "\epsilon^*"] \arrow[dr, "colim_{Gr_I H}"'] & & Fun^b\big(Gr_{N(I)}N(H), \mathcal{C}\big) \arrow[ll, shift left, "\epsilon^k"] \arrow[dl, "colim_{Gr_{N(I)}N(H)}"] \\ & C & \end{tikzcd}$$

Since $\epsilon : Gr_{N(I)}N(H) \to Gr_I H$ is cofinal with respect to taking colimits, the above diagram does commute.

Lemma 24.2 implies that $colim_{Gr_{N(I)}N(H)}$ is homotopy meaningful on cofibrant objects. It is also clear that ϵ^* is homotopy meaningful. Thus to prove the theorem we need to show that, for any diagram $F' : Gr_I H \to \mathcal{C}$, if $F : Gr_{N(I)}N(H) \to \mathcal{C}$ is bounded and cofibrant and $F \to \epsilon^* F'$ is a weak equivalence, then so is its adjoint $\epsilon^k F \to F'$.

Let (i, x) be an object in $Gr_I H$. By definition $\epsilon^k F(i, x) = colim_{\epsilon\downarrow(i,x)} F$. The category $\epsilon\downarrow(i, x)$ is isomorphic to the Grothendieck construction $Gr_{N(I\downarrow i)}N(H\downarrow x)$. Under this identification $\epsilon\downarrow(i, x) \to Gr_{N(I)}N(H)$ corresponds to $Gr_{N(I\downarrow i)}N(H\downarrow x) \to Gr_{N(I)}N(H)$. Since the maps $N(I\downarrow i) \to N(I)$ and $N(H\downarrow x) \to N(H)$ are reduced, i.e. they send non-degenerate simplices to non-degenerate ones, the composition:

$$Gr_{N(I\downarrow i)}N(H\downarrow x) \to Gr_{N(I)}N(H) \xrightarrow{F} \mathcal{C}$$

is a cofibrant diagram. It follows that so is $colim_{N(H(-)\downarrow x)} F : N(I\downarrow i) \to \mathcal{C}$ (see Lemma 24.1.(2)).

We are going to show that this functor satisfies the assumptions of Corollary 20.6. It is clear that the category $I\downarrow i$ has a terminal object $t = (i, i \xrightarrow{id} i)$. Let us consider a simplex $N(I\downarrow i) \ni \sigma = (i_n \to \cdots \to i_1 \to i, i \xrightarrow{id} i)$ and a morphism $\alpha = d_n \circ \cdots \circ d_1 : t \to \sigma$ in $N(I\downarrow i)$. We have to show that α is sent via $colim_{N(H(-)\downarrow x)} F$ to a weak equivalence. Since F is weakly equivalent to $\epsilon^* F'$ the natural transformation $F(t, -) \to F(\sigma, -)$, induced by α, is a weak equivalence.

These diagrams are cofibrant and thus, on the colimits, we get a weak equivalence $colim_{N(H(t)\downarrow x)}F(t,-) \xrightarrow{\sim} colim_{N(H(\sigma)\downarrow x)}F(\sigma,-)$. The assumptions of Corollary 20.6 are therefore satisfied and we can conclude that the following morphism is a weak equivalence as well:

$$colim_{N(H(i)\downarrow x)}F(i,-) \xrightarrow{\sim} colim_{N(I\downarrow i)}colim_{N(H(-)\downarrow x)}F = \epsilon^k F(i,j)$$

The category $H(i)\downarrow x$ has a terminal object, the diagram $F(i,-)$ is bounded and cofibrant, and there is a weak equivalence $F(i,-) \to \epsilon^* F'(i,-)$. Thus using again Corollary 20.6, we can conclude that the morphisms:

$$colim_{N(H(i)\downarrow x)}F(i,-) \to colim_{N(H(i)\downarrow x)}\epsilon^* F'(i,-) = colim_{H(i)\downarrow x}F'(i,-) = F'(i,x)$$

is a weak equivalence. The theorem has been proven. □

As a corollary of Proposition 24.4 and Theorem 24.5 we get the so called Thomason Theorem for homotopy colimits.

24.6. **Corollary**. *Let $H : I \to Cat$ be a diagram of small categories. Assume that $\mathcal{C}$ has a functorial factorization of morphisms by cofibrations followed by acyclic fibrations. Then, for any $F : Gr_I H \to \mathcal{C}$, we have a weak equivalence in $\mathcal{C}$:*

$$hocolim_{Gr_I H}F \simeq hocolim_I hocolim_H F$$

24.7. **Corollary** (R. W. Thomason [39, Theorem 1.2]). *Let $H : I \to Cat$ be a diagram of small categories. Then there is a weak equivalence*

$$N(Gr_I H) \simeq hocolim_I N(H)$$

Proof. Apply Corollary 24.6 to the constant diagram $\Delta[0] : Gr_I H \to Spaces$. □

25. Étale spaces of presheaves

In this section we discuss homotopy colimits of contravariant functors indexed by simplex categories, i.e. we look at functors of the form $F : K^{op} \to \mathcal{C}$. We take advantage of the fact that after applying the nerve functor to a map of spaces $f : L \to K$ we get a reduced map $N(f^{op}) : N(L^{op}) \to N(K^{op})$, i.e. a map which sends non-degenerate simplices in $N(L^{op})$ to non-degenerate simplices in $N(K^{op})$ (see Example 13.5). The key property of such maps is that the pull-back process along them preserves absolute cofibrancy of bounded diagrams (see Proposition 15.2).

Throughout this section let us fix a functor $F : K^{op} \to \mathcal{C}$. Let us also choose and fix a cofibrant replacement $QF \overset{\sim}{\twoheadrightarrow} \epsilon^* F$ in $Fun^b\big(N(K^{op}), \mathcal{C}\big)$ of the composition $N(K^{op}) \xrightarrow{\epsilon} K^{op} \xrightarrow{F} \mathcal{C}$, where $\epsilon : N(K^{op}) \to K^{op}$ is the forgetful functor.

Let $f : L \to K$ be a map. The colimit $colim_{L^{op}}(f^{op})^* F$ is denoted simply by $colim_{L^{op}}F$. The colimit $colim_{N(L^{op})}\big(N(f^{op})^* QF\big)$ is denoted by $hocolim_{L^{op}}F$ and called the étale space of the pull-back $(f^{op})^* F$. In this way we get two functors:

$$colim_{\_}F : Spaces\downarrow K \to \mathcal{C}\ ,\ \ (f : L \to K) \mapsto colim_{L^{op}}F$$

$$hocolim_{\_}F : Spaces{\downarrow}K \to \mathcal{C}\ ,\ \ (f : L \to K) \mapsto hocolim_{L^{op}}F$$

The morphisms:

$$colim_{N(L^{op})}\big(N(f^{op})^*QF\big) \to colim_{N(L^{op})}\big((N(f^{op})\circ\epsilon)^*F\big) = colim_{L^{op}}(f^{op})^*F$$

form a natural transformation $hocolim_{\_}F \to colim_{\_}F$.

The notation $hocolim_{L^{op}}F$ for the étale space of $(f^{op})^*F$ is justified by the fact that $hocolim_{L^{op}}F$ is weakly equivalent to the homotopy colimit of the pull-back $(f^{op})^*F : L^{op} \to \mathcal{C}$.

The étale space construction is "additive" with respect to the indexing space. This additivity property distinguishes the ocolimit construction from the hocolimit one. The ocolimit is not additive with respect to the indexing space (see Remark 18.3).

25.1. **Proposition**. *Let $H : I \to Spaces{\downarrow}K$ be a functor. Then the natural morphism $colim_I hocolim_{H^{op}}F \to hocolim_{(colim_I H)^{op}}F$ is an isomorphism.*

Proof. The proposition is a consequence of Proposition 8.2.(2) and the fact that $N\big((colim_I H)^{op}\big)$ is naturally isomorphic to $colim_I N(H^{op})$ (see 6.10). □

The étale space construction converts cofibrations of spaces into cofibrations in $\mathcal{C}$.

25.2. **Proposition**.
(1) *For any map $f : L \to K$, $hocolim_{L^{op}}F$ is a cofibrant object.*
(2) *Let $h : N \to L$ be a map in $Spaces{\downarrow}K$. If h is a monomorphism, then $hocolim_{N^{op}}F \to hocolim_{L^{op}}F$ is a cofibration in $\mathcal{C}$.*

Proof. Part (1) follows immediately from Corollary 16.5.(2). Part (2) is a consequence of Corollary 16.2. □

25.3. **Corollary**.
(1) *Let the following be a push-out square of spaces over K:*

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d] & B \arrow[d] & \\ C \arrow[r, hook] & D \arrow[r] & K \end{tikzcd} \qquad \textit{where } A \hookrightarrow B \textit{ is a cofibration.}$$

Then the following is a push-out square in $\mathcal{C}$:

$$\begin{tikzcd} hocolim_{A^{op}}F \arrow[r, hook] \arrow[d] & hocolim_{B^{op}}F \arrow[d] \\ hocolim_{C^{op}}F \arrow[r, hook] & hocolim_{D^{op}}F \end{tikzcd}$$

(2) *Let the following be a telescope diagram of spaces over K:*

$$\begin{tikzcd} A_0 \arrow[r, hook] \arrow[dr] & A_1 \arrow[r, hook] \arrow[d] & A_2 \arrow[r, hook] \arrow[dl] & \cdots \\ & K & & \end{tikzcd}$$

where $A_i \hookrightarrow A_{i+1}$ is a cofibration for $i \geq 0$. Let $A = colim(A_0 \hookrightarrow A_1 \hookrightarrow \cdots)$. Then $hocolim_{A^{op}} F$ is isomorphic to $colim\big(hocolim_{A_0^{op}} F \hookrightarrow hocolim_{A_1^{op}} F \hookrightarrow \cdots\big)$.

We say that a functor $F : I \to \mathcal{C}$ if homotopically constant if any morphism in I is sent, via F, to a weak equivalence in $\mathcal{C}$. We are going to apply Corollary 25.3 to calculate the étale space of a homotopically constant functor indexed by a contractible space.

25.4. **Proposition**. *If $F : K^{op} \to \mathcal{C}$ is homotopically constant, then, for any map $f : L \to K$ such that L is contractible and for any simplex $\sigma \in L$, the morphism $F\big(f(\sigma)\big) \simeq QF\big(f(\sigma)\big) \to colim_{N(L^{op})} N(f^{op})^* QF = hocolim_{L^{op}} F$ is a weak equivalence.*

Proof. It is enough to show that $QF\big(f(\sigma)\big) \to hocolim_{L^{op}} F$ is a weak equivalence for *some* simplex $\sigma \in L$.

Step 1. We show that the proposition holds in the case when $L = \Delta[n]$. As F is homotopically constant, then so is the diagram $QF : N(K^{op}) \to \mathcal{C}$. Thus the composition $N(\Delta[n]^{op}) \xrightarrow{N(f^{op})} N(K^{op}) \xrightarrow{QF} \mathcal{C}$ satisfies the assumptions of Proposition 20.7 since the category $\Delta[n]^{op}$ has an initial object ι. We can therefore conclude that the following morphism is a weak equivalence:

$$QF\big(f(\iota)\big) \to colim_{N(\Delta[n]^{op})} N(f^{op})^* QF = hocolim_{\Delta[n]^{op}} F$$

Step 2. We say that a space L is basic if it can be built inductively starting with $\Delta[0]$ and gluing simplices along horns, i.e. one can decompose L as a telescope $colim(L_0 \hookrightarrow L_1 \hookrightarrow \cdots)$ with $L_0 = \Delta[0]$ and L_{n+1} is obtained from L_n as a push-out $colim(L_n \leftarrow \coprod \Delta[n+1,k] \hookrightarrow \coprod \Delta[n+1])$. A basic space is necessarily contractible.

We show that the proposition holds in the case when L is basic and finite dimensional. We prove it by induction on the dimension of L.

In the case $dim(L) = 0$, $L = \Delta[0]$ and Step 2 follows from Step 1. Let us assume that the proposition holds for those basic spaces whose dimension is less than n. Let L be basic and $dim(L) = n$. For simplicity assume that L has only one non-degenerate simplex of dimension n, i.e. L fits into a push-out square:

$$\begin{tikzcd} \Delta[n,k] \arrow[r, hook] \arrow[d] & \Delta[n] \arrow[d] & \\ N \arrow[r, hook] & L \arrow[r] & K \end{tikzcd}$$

where N is basic and $dim(N) < n$.

By Corollary 25.3.(1) we get a push-out diagram of étale spaces:

$$\begin{tikzcd} hocolim_{\Delta[n,k]^{op}}F \arrow[r, hook] \arrow[d] & hocolim_{\Delta[n]^{op}}F \arrow[d] \\ hocolim_{N^{op}}F \arrow[r, hook] & hocolim_{L^{op}}F \end{tikzcd}$$

where the horizontal morphisms are cofibrations as indicated.

Since $\Delta[n,k]$ is basic and $dim(\Delta[n,k]) < n$, the inductive assumption implies that $QF\big(f(\sigma)\big) \to hocolim_{\Delta[n,k]^{op}}F$ is a weak equivalence for any $\sigma \in \Delta[n,k]$. Therefore, according to Step 1, $hocolim_{\Delta[n,k]^{op}}F \to hocolim_{\Delta[n]^{op}}F$ is an *acyclic* cofibration. It follows that so is $hocolim_{N^{op}}F \to hocolim_{L^{op}}F$ (see Proposition 3.1). As by inductive assumption $QF\big(f(\sigma)\big) \to hocolim_{N^{op}}F$ is a weak equivalence, we can conclude that $QF\big(f(\sigma)\big) \to hocolim_{L^{op}}F$ is a weak equivalence as well.

Step 3. We show that the proposition holds for any basic space L (not necessarily finite dimensional). In this case L can be represented as a colimit $L = colim(L_0 \hookrightarrow L_1 \hookrightarrow \cdots)$, where L_n is basic and n-dimensional. Step 3 follows now from Step 2 and Corollary 25.3.(2).

Step 4. Let L be contractible. In this case $f : L \to K$ is a retract in $Spaces{\downarrow}K$ of a map $PL \to K$, where PL is basic. Explicitly, there are maps $L \to PL$ and $PL \to L$ in $Spaces{\downarrow}K$ for which the composition $L \to PL \to L$ is the identity.

Let us choose a simplex $\sigma \in L$ and form the following commutative diagram:

$$\begin{tikzcd} QF\big(f(\sigma)\big) \arrow[r, "id"] \arrow[d] & QF\big(f(\sigma)\big) \arrow[r, "id"] \arrow[d] & QF\big(f(\sigma)\big) \arrow[d] \\ hocolim_{L^{op}}F \arrow[r] \arrow[rr, bend right, "id"'] & hocolim_{PL^{op}}F \arrow[r] & hocolim_{L^{op}}F \end{tikzcd}$$

According to Step 3 the morphism $QF\big(f(\sigma)\big) \to hocolim_{PL^{op}}F$ is a weak equivalence. It follows that $QF\big(f(\sigma)\big) \to hocolim_{L^{op}}F$ is a weak equivalence as well. ☐

¿From this section on we assume that the category $\mathcal{C}$ has a functorial factorization of morphisms by cofibrations followed by acyclic fibrations. Although unnecessary, this assumption is going to simplify the statements and make the exposition more clear.

26. Diagrams indexed by cones II

In this section we consider contravariant functors indexed by cones. We prove analogous statements to those in Section 19.

Let K be a space and CK be its cone (see Definition 19.1). The opposite category $(CK)^{op}$ can be represented as a Grothendieck construction (see Remark 19.3):

$$(CK)^{op} = Gr\big(K^{op} \xleftarrow{p_1} K^{op} \times \Delta[0]^{op} \xrightarrow{p_2} \Delta[0]^{op}\big)$$

where:

- K^{op} corresponds to the full subcategory of $(CK)^{op}$ consisting of simplices of the form (σ, e^0).
- $K^{op} \times \Delta[0]^{op}$ corresponds to the full subcategory of $(CK)^{op}$ consisting of simplices of the form (σ, e^l), where $l \geq 1$.
- $\Delta[0]^{op}$ corresponds to the full subcategory of $(CK)^{op}$ consisting of simplices of the form e^l.

26.1. **Proposition**. *Let $F : (CK)^{op} \to \mathcal{C}$ be a functor. Assume that F sends morphisms of the form: $d_j : (\sigma, e^l) \to (\sigma, e^{l-1})$ (for $j > dim(\sigma)$) and $d_j : e^l \to e^{l-1}$ (for $j \geq 0$) to weak equivalences in $\mathcal{C}$. Then $hocolim_{(CK)^{op}} F$ is weakly equivalent to $F(e)$.*

Proof. To prove the proposition we use Thomason Theorem (see Corollary 24.6). It gives a weak equivalence:

$$hocolim_{(CK)^{op}} F \simeq hocolim\big(hocolim_{K^{op}} F \leftarrow hocolim_{K^{op} \times \Delta[0]^{op}} F \to hocolim_{\Delta[0]^{op}} F\big)$$

According to Fubini Theorem (see 22.7):

$$hocolim_{K^{op} \times \Delta[0]^{op}} F \simeq hocolim_{K^{op}} hocolim_{\Delta[0]^{op}} F$$

Let $\sigma \in K^{op}$ be an object. We denote by $F(\sigma, -) : \Delta[0]^{op} \to \mathcal{C}$ the following composition:

$$\{\sigma\} \times \Delta[0]^{op} \subset K^{op} \times \Delta[0]^{op} \subset (CK)^{op} \xrightarrow{F} \mathcal{C}$$

The assumptions on F imply that $F(\sigma, -) : \Delta[0]^{op} \to \mathcal{C}$ is a homotopically constant functor. Thus, by Proposition 25.4, $hocolim_{\Delta[0]^{op}} F(\sigma, -)$ is weakly equivalent to $F(\sigma, e^0)$. We can conclude that the functor $K^{op} \ni \sigma \mapsto hocolim_{\Delta[0]^{op}} F(\sigma, -)$ is weakly equivalent to the composition $K^{op} \subset (CK)^{op} \xrightarrow{F} \mathcal{C}$. The morphism:

$$hocolim_{K^{op}} hocolim_{\Delta[0]^{op}} F \to hocolim_{K^{op}} F$$

is therefore a weak equivalence. This argument shows that $hocolim_{(CK)^{op}} F$ is weakly equivalent to $hocolim_{\Delta[0]^{op}} F$. However, since $F : \Delta[0]^{op} \to \mathcal{C}$ is also homotopically constant (it is a consequence of the assumptions on F), we get a weak equivalence $F(e^1) \simeq hocolim_{(CK)^{op}} F$. □

We are going to apply Proposition 26.1 to certain contravariant functors indexed by nerves of categories with a terminal object.

26.2. **Corollary**. *Let $F : N(CI)^{op} \to \mathcal{C}$ be a functor. Assume that there exists $F' : CI \to \mathcal{C}$ and a weak equivalence $F \xrightarrow{\sim} \epsilon^* F'$ in $Fun\big(N(CI)^{op}, \mathcal{C}\big)$, where $\epsilon : N(CI)^{op} \to CI$, $(i_n \to \cdots \to i_0) \mapsto i_n$ is the forgetful functor (see Section 6). Then the composition:*

$$hocolim_{N(CI)^{op}} F \to hocolim_{N(CI)^{op}} \epsilon^* F' \to colim_{N(CI)^{op}} \epsilon^* F' \to colim_{CI} F' = F'(e)$$

is a weak equivalence. In particular $hocolim_{N(CI)^{op}} F \simeq F(e)$.

Corollary 26.2 can be generalized to an arbitrary small category with a terminal object.

26.3. **Proposition**. *Let I be a category with a terminal object t and $F : N(I)^{op} \to \mathcal{C}$ be a functor. Assume that there exists $F' : I \to \mathcal{C}$ and a weak equivalence $F \xrightarrow{\sim} \epsilon^* F'$. Then the composition*

$$hocolim_{N(I)^{op}} F \to hocolim_{N(I)^{op}} \epsilon^* F' \to colim_{N(I)^{op}} \epsilon^* F' \to colim_I F' = F'(t)$$

is a weak equivalence. In particular $hocolim_{N(I)^{op}} F \simeq F(t)$.

Proof. The proposition can be proven using Corollary 26.2 and the fact that in the case I has a terminal object, it is a retract of its cone CI (compare with the proof of Proposition 20.5). $\square$

27. Homotopy colimits as étale spaces

In this section we show that the forgetful functor $\epsilon : N(I)^{op} \to I$, $(i_n \to \cdots \to i_0) \mapsto i_n$, is cofinal with respect to taking homotopy colimits. This reduces calculating homotopy colimits indexed by any small category to calculating étale spaces of presheaves.

27.1. **Theorem**. *The composition:*

$$Fun(I, \mathcal{C}) \xrightarrow{\epsilon^*} Fun\big(N(I)^{op}, \mathcal{C}\big) \xrightarrow{hocolim_{N(I)^{op}}} Ho(\mathcal{C})$$

together with a natural transformation induced by the morphisms:

$$hocolim_{N(I)^{op}} \epsilon^* F \to colim_{N(I)^{op}} \epsilon^* F = colim_I F$$

is the total left derived functor of $colim_I$.

Proof. It is clear that this composition is homotopy invariant. Thus to prove the theorem we have to show that, if $\mathcal{G} : Fun(I, \mathcal{C}) \to Ho(\mathcal{C})$ is also a homotopy invariant functor, then any natural transformation $\mathcal{G} \to colim_I$ factors uniquely as $\mathcal{G} \to hocolim_{N(I)^{op}} \epsilon^*(-) \to colim_I$. For any functor $F \in Fun(I, \mathcal{C})$ we need to define a morphism $\mathcal{G}(F) \to hocolim_{N(I)^{op}} \epsilon^* F$ in $Ho(\mathcal{C})$.

Let $F : I \to \mathcal{C}$ be a functor. The nerve $N(I)$ can be expressed as a colimit $colim_I N(I\downarrow -)$ (see Example 6.6). Since the étale space construction is additive with respect to the indexing space (see Proposition 25.1), we have an isomorphism:

$$hocolim_{N(I)^{op}} \epsilon^* F = colim_I hocolim_{N(I\downarrow -)^{op}} \epsilon^* F$$

Consider a diagram $I \ni i \mapsto hocolim_{N(I\downarrow i)^{op}} \epsilon^* F$. As the category $I\downarrow i$ has a terminal object, Proposition 26.3 tells us that we have a weak equivalence:

$$hocolim_{N(I\downarrow i)^{op}} \epsilon^* F \to colim_{N(I\downarrow i)^{op}} \epsilon^* F = colim_{I\downarrow i} F = F(i)$$

Let $\Psi : hocolim_{N(I\downarrow -)^{op}} \epsilon^* F \to F$ denote the natural transformation induced by these morphisms. Since it is a weak equivalence and $\mathcal{G}$ is homotopy invariant, $\mathcal{G}(\Psi)$ is an isomorphism in $Ho(\mathcal{C})$. We can now define $\mathcal{G}(F) \to hocolim_{N(I)^{op}} \epsilon^* F$ to be the composition:

$$\mathcal{G}(F) \xrightarrow{\mathcal{G}(\Psi)^{-1}} \mathcal{G}(hocolim_{N(I\downarrow -)^{op}} \epsilon^* F) \to colim_I hocolim_{N(I\downarrow -)^{op}} \epsilon^* F = hocolim_{N(I)^{op}} \epsilon^* F$$

In this way we get the desired natural transformation $\mathcal{G} \to hocolim_{N(I)^{op}} \epsilon^*(-)$. Its uniqueness is clear. $\square$

27.2. **Corollary**. *Let $F : I \to \mathcal{C}$ be a homotopically constant functor. If I is contractible, then $hocolim_I F$ is weakly equivalent to $F(i)$ for any $i \in I$.*

Proof. The corollary is a consequence of Theorem 27.1 and Proposition 25.4. $\square$

28. Cofinality

In this section we discuss cofinality properties of the homotopy colimit construction. We generalize [6, Theorem XI.9.2] to arbitrary model categories. A similar result is also proven in [16, 10.7].

We give a sufficient condition for a functor $f : I \to J$ to induce a weak equivalence $hocolim_I f^* F \to hocolim_J F$ for any diagram $F : J \to \mathcal{C}$.

28.1. **Definition**. *Let I and J be small categories. A functor $f : I \longrightarrow J$ is said to be terminal if the space $N(j\downarrow f)$ (see A.2) is contractible for every $j \in J$.*

Terminal functors are cofinal with respect to taking colimits, i.e. if $f : I \to J$ is terminal, then, for any diagram $F : J \to \mathcal{C}$, the morphism $colim_I f^* F \to colim_J F$ is an isomorphism (see A.4).

28.2. **Example**. Let I be a contractible category. The functor $I \to *$ is terminal.

28.3. **Example**. Let J be a category with a terminal object t. The functor $* \to J$, $* \mapsto t$, is terminal. Indeed, for any $j \in J$, the category $j\downarrow *$ is the trivial one.

28.4. **Theorem**. *Let $f : I \to J$ be terminal. Then the composition:*

$$Fun(J,\mathcal{C}) \xrightarrow{f^*} Fun(I,\mathcal{C}) \xrightarrow{hocolim_I} Ho(\mathcal{C})$$

together with a natural transformation induced by the morphisms:

$$hocolim_I f^* F \to colim_I f^* F = colim_J F$$

is the total left derived functor of $colim_J$. Explicitly, there is a natural weak equivalence $hocolim_I f^* F \simeq hocolim_J F$.

Proof. We start with describing two functors:

- $H : N(J)^{op} \to Cat$ is defined as $H(j_n \to \cdots \to j_0) := j_0 \downarrow f$.
- $G : I \to Cat$ is defined as $G(i) := N\big(J \downarrow f(i)\big)^{op}$.

Consider their Grothendieck constructions $Gr_{N(J)^{op}} H$ and $Gr_I G$. Observe that the following is an isomorphism of categories:

$$Gr_{N(J)^{op}} H \to Gr_I G$$

$$\big((j_n \to \cdots \to j_0); (i, j_0 \overset{\alpha}{\to} f(i))\big) \;\longmapsto\; \big(i\,;\, j_n \to \cdots \to j_0 \overset{\alpha}{\to} f(i)\big)$$

Let us denote the category $Gr_{N(J)^{op}} H = Gr_I G$ by Λ and the composition $\Lambda = Gr_{N(J)^{op}} H \to N(J)^{op} \overset{\epsilon}{\to} J$ by λ, where $\epsilon : N(I)^{op} \to I$, $(i_n \to \cdots \to i_0) \mapsto i_n$, is the forgetful functor. Let $F : J \to \mathcal{C}$ be a functor. We are going to apply Thomason Theorem (see Corollary 24.6) to calculate the homotopy colimit of the pull-back $\lambda^* F : \Lambda \to \mathcal{C}$. We can do it using the two presentations of the category Λ as Grothendieck constructions.

Case $\Lambda = Gr_{N(J)^{op}} H$. Corollary 24.6 tells us that there is a weak equivalence:

$$hocolim_\Lambda \lambda^* F \simeq hocolim_{N(J)^{op}} hocolim_H \lambda^* F$$

For any $\sigma = (j_n \to \cdots \to j_0) \in N(J)^{op}$, the diagram $\lambda^* F : H(\sigma) = j_0 \downarrow f \to \mathcal{C}$ is constant with value $F(j_n)$. Since by assumption the category $j_0 \downarrow f$ is contractible, Proposition 25.4 implies that the following morphism is a weak equivalence:

$$hocolim_{H(\sigma)} \lambda^* F \to colim_{j_0 \downarrow f} F(j_n) = F(j_n)$$

The object $F(j_n)$ can be identified with $\epsilon^* F(\sigma)$, where $\epsilon : N(I)^{op} \to I$ is the forgetful functor. Thus the induced natural transformation $hocolim_{H(-)} \lambda^* F \to \epsilon^* F$ is a weak equivalence, and hence so is the morphism:

$$hocolim_{N(J)^{op}} hocolim_H \lambda^* F \to hocolim_{N(J)^{op}} \epsilon^* F$$

However, since $hocolim_{N(J)^{op}} \epsilon^* F \simeq hocolim_J F$ (see Theorem 27.1), we can conclude that $hocolim_\Lambda \lambda^* F$ is naturally weakly equivalent to $hocolim_J F$.

Case $\Lambda = Gr_I G$. Corollary 24.6 tells us that there is a weak equivalence:

$$hocolim_\Lambda \lambda^* F \simeq hocolim_I hocolim_G \lambda^* F$$

For any $i \in I$, the diagram $\lambda^* F : G(i) = N\big(J{\downarrow}f(i)\big)^{op} \to \mathcal{C}$ coincides with the composition $N\big(J{\downarrow}f(i)\big)^{op} \xrightarrow{\epsilon} J{\downarrow}f(i) \to J \xrightarrow{F} \mathcal{C}$. Since the category $J{\downarrow}f(i)$ has a terminal object $f(i) \xrightarrow{id} f(i)$, according to Proposition 26.3, the following morphism is a weak equivalence:

$$hocolim_{G(i)}\lambda^* F \to colim_{N(J\downarrow f(i))^{op}}\epsilon^* F = colim_{J\downarrow f(i)} F = F\big(f(i)\big) = f^*F(i)$$

Thus the induced natural transformation $hocolim_{G(-)}\lambda^* F \to f^*F$ is also a weak equivalence, and hence so is the morphism $hocolim_I hocolim_G \lambda^* F \to hocolim_I f^* F$.

The theorem has been proven since we have found natural weak equivalences:

$$hocolim_J F \simeq hocolim_\Lambda \lambda^* F \simeq hocolim_I f^* F \qquad \square$$

Appendix A. Categorical preliminaries

A.1. **Categories over and under an object** (see [32, Section II.6]). Let I be a small category, i be an object in I and $\alpha : i \to j$ be a morphism in I.

By $I{\downarrow}i$ we denote a category whose objects are morphisms $l \to i$, and a morphism in $I{\downarrow}i$ between $l_0 \to i$ and $l_1 \to i$ is a commutative triangle:

$$\begin{tikzcd} l_0 \arrow[rr] \arrow[dr] & & l_1 \arrow[dl] \\ & i & \end{tikzcd}$$

The morphism $id : i \to i$ is a terminal object in $I{\downarrow}i$.

By $I{\downarrow}\alpha$ we denote a functor $I{\downarrow}\alpha : I{\downarrow}i \to I{\downarrow}j$ which sends $l \to i$ to the composition $l \to i \xrightarrow{\alpha} j$. Clearly this construction defines a functor $I{\downarrow}- : I \to Cat$.

For any $i \in I$, there is a forgetful functor $(I{\downarrow}i) \to I$ which sends an object $l \to i$ in $I{\downarrow}i$ to l in I. These functors induce a natural transformation $(I{\downarrow}-) \to I$ between $I{\downarrow}-$ and the category I.

Dually, by $i{\downarrow}I$ we denote a category whose objects are morphisms $i \to l$, and a morphism in $i{\downarrow}I$ between $i \to l_0$ and $i \to l_1$ is a commutative triangle

$$\begin{tikzcd} & i \arrow[dl] \arrow[dr] & \\ l_0 \arrow[rr] & & l_1 \end{tikzcd}$$

The morphism $id : i \to i$ is an initial object in $i{\downarrow}I$.

By $\alpha{\downarrow}I$ we denote a functor $\alpha{\downarrow}I : j{\downarrow}I \to i{\downarrow}I$ which sends $j \to l$ to the composition $i \xrightarrow{\alpha} j \to l$. Clearly this construction defines a functor $-{\downarrow}I : I^{op} \to Cat$.

A.2. **Relative version of categories over and under an object** (see [32, Section II.6]). Let $f : J \to I$ be a functor. By $f{\downarrow}i$ and $i{\downarrow}f$ we denote the categories that fit into the following pull-back squares in Cat:

$$\begin{tikzcd} f{\downarrow}i \arrow[r] \arrow[d] & J \arrow[d, "f"] & i{\downarrow}f \arrow[l] \arrow[d] \\ I{\downarrow}i \arrow[r] & I & i{\downarrow}I \arrow[l] \end{tikzcd}$$

The categories $f{\downarrow}i$ and $i{\downarrow}f$ are called respectively over and under categories of f. Explicitly, we can think about $f{\downarrow}i$ as a category whose objects are pairs $\big(l, f(l) \to i\big)$ consisting of an object l in J and a morphism $f(l) \to i$ in I. The functor $f{\downarrow}i \to I{\downarrow}i$ maps such an object to $\big(f(l) \to i\big)$. There is a similar description of $i{\downarrow}f$.

In the case $f = id_I$, $id{\downarrow}i$ and $i{\downarrow}id$ coincide respectively with $I{\downarrow}i$ and $i{\downarrow}I$. If $f : J \to I$ is fixed, we denote $f{\downarrow}i$ and $i{\downarrow}f$ simply by $J{\downarrow}i$ and $i{\downarrow}J$.

A.3. **Pull-back process and Kan extension** (see [32, Section X]). Let $\mathcal{C}$ be a category closed under colimits. Consider a functor $f : J \to I$. With f we can associate two other functors:

- *The pull-back process.* It is a functor $f^* : Fun(I, \mathcal{C}) \to Fun(J, \mathcal{C})$ which assigns to $H : I \to \mathcal{C}$ the composition $J \xrightarrow{f} I \xrightarrow{H} \mathcal{C}$.
- *The Kan extension.* It is a functor "going in the other direction" $f^k : Fun(J, \mathcal{C}) \to Fun(I, \mathcal{C})$. Here is its construction. Let $H : J \to \mathcal{C}$ be a functor. For every $i \in I$ pull-back H along $f{\downarrow}i \to J$ and take the colimit $colim_{f{\downarrow}i} H$. In this way we get a functor $colim_{f{\downarrow}-} H : I \to \mathcal{C}$. The assignment:

  $$Fun(J, \mathcal{C}) \ni H \mapsto colim_{f{\downarrow}-} H \in Fun(I, \mathcal{C})$$

  in natural in H, and hence it induces a functor $Fun(J, \mathcal{C}) \to Fun(I, \mathcal{C})$. This functor is called the Kan extension of f.

The Kan extension and the pull-back process are closely related. For any $f : J \to I$, $f^k : Fun(J, \mathcal{C}) \to Fun(I, \mathcal{C})$ is left adjoint to $f^* : Fun(I, \mathcal{C}) \to Fun(J, \mathcal{C})$.

A.4. **Cofinality for colimits** (see [32, Theorem IX.3.1]). The main application of relative categories under an object is in cofinality statements for colimits. We say that a functor $f : J \to I$ is cofinal with respect to taking colimits if, for any $F : I \to \mathcal{C}$, the induced morphism $colim_J f^* F \to colim_I F$ is an isomorphism.

If for any $i \in I$, the under category $i{\downarrow}f$ is non empty and connected (the nerve $N(I)$ is a non empty and connected space), then $f : J \to I$ is cofinal with respect to taking colimits.

A.5. **Grothendieck construction** (see [39, Definition 1.1]). *Let $H : I \to Cat$ be a functor. The Grothendieck construction on H is the category $Gr_I H$, where:*

- *an object in $Gr_I H$ is a pair (i, a) consisting of an object $i \in I$ and an object $a \in H(i)$;*
- *a morphism $(\alpha, h) : (i, a) \to (j, b)$ in $Gr_I H$ is a pair (α, h) consisting of a morphism $\alpha : i \to j$ in I and a morphism $h : H(\alpha)(a) \to b$ in $H(j)$;*
- *the composition of $(\alpha, h) : (i, a) \to (j, b)$ and $(\beta, g) : (j, b) \to (l, c)$ is defined to be $\big(\beta \circ \alpha, g \circ H(\beta)(h)\big) : (i, a) \to (l, c)$.*

The construction $Gr_I H$ is natural. Let $f : J \to I$, $H_1 : J \to Cat$ and $H_0 : I \to Cat$ be functors and $\Psi : H_1 \to H_0$ be a natural transformation. This data induces a functor $Gr_f \Psi : Gr_J H_1 \to Gr_I H_0$ which sends an object (j, a) to $\big(f(j), \Psi_j(a)\big)$.

A.6. **Example**. If $H : I \to Cat$ is a constant functor with value J, then the Grothendieck construction $Gr_I H$ is isomorphic to the product category $I \times J$.

A.7. **Example**. Let us consider the category given by the following graph:

$$\begin{tikzcd}
(0,1,2) \arrow[rr] \arrow[dr] \arrow[dd] & & (0,1) \arrow[dr] \arrow[dd] & \\
& (0,2) \arrow[rr] \arrow[dd] & & (0) \\
(1,2) \arrow[rr] \arrow[dr] & & (1) & \\
& (2) & &
\end{tikzcd}$$

This category is isomorphic to the following Grothendieck construction:

$$Gr\left((2) \leftarrow \left(\begin{tikzcd} (0,1,2) \arrow[r] \arrow[d] & (0,2) \\ (1,2) & \end{tikzcd} \right) \to \left(\begin{tikzcd} (0,1) \arrow[r] \arrow[d] & (0) \\ (1) & \end{tikzcd} \right) \right)$$

A.8. **Grothendieck construction & the pull-back process.** The category $Gr_I H$ is equipped with a functor $Gr_I H \to I$, $(i, a) \mapsto i$. This functor behaves well with regard to the pull-back process. Let $f : J \to I$ be a functor and $i \in I$ be an object.

The following are pull-back squares in Cat:

$$\begin{tikzcd}
Gr_J H \arrow[r] \arrow[d] & Gr_I H \arrow[d] & H(i) \arrow[l] \arrow[d] \\
J \arrow[r, "f"] & I & \{i\} \arrow[l]
\end{tikzcd}$$

In particular, by pulling back $Gr_I H \to I$ along $I{\downarrow}i \to I$ we get a natural isomorphism $Gr_{I\downarrow i}H \cong (Gr_I H){\downarrow}i$.

A.9. **Functors indexed by Grothendieck constructions.** To describe a functor $F : Gr_I H \to \mathcal{C}$ it is sufficient and necessary to have the following data:

(1) for every object i in I, a functor $F_i : H(i) \to \mathcal{C}$;

(2) for every morphism $\alpha : i \to j$ in I, a natural transformation $F_\alpha : F_i \to F_j$ between $F_i : H(i) \to \mathcal{C}$ and the composition $H(i) \xrightarrow{H(\alpha)} H(j) \xrightarrow{F_j} \mathcal{C}$;

(3) if $\alpha : i \to j$ and $\beta : j \to l$ are composable morphisms in I, then the natural transformations $F_{\beta\circ\alpha}$ and $F_\beta \circ F_\alpha$ should coincide.

Let $F : Gr_I H \to \mathcal{C}$ be a diagram. The functor $F_i : H(i) \to \mathcal{C}$ is given by the composition $H(i) \hookrightarrow Gr_I H \xrightarrow{F} \mathcal{C}$. The natural transformation $F_\alpha : F_i \to F_j$ is induced by the morphisms $F(\alpha, id) : F(i, a) \to F\big(j, G(\alpha)(a)\big)$. By applying the colimit to this data we get a functor:

$$colim_{H(-)}F : I \to \mathcal{C}\ ,\ i \mapsto colim_{H(i)}F\ ,\ (\alpha : i \to j) \mapsto colim\, F_\alpha$$

Let $colim_{H(-)}F \to colim_{Gr_I H}F$ be the natural transformation induced by the morphisms $colim_{H(i)}F \to colim_{Gr_I H}F$.

A.10. **Proposition**. *The natural transformation $colim_{H(-)}F \to colim_{Gr_I H}F$ satisfies the universal property of the colimit of $colim_{H(-)}F : I \to \mathcal{C}$. The induced morphism $colim_{i\in I}colim_{H(i)}F \to colim_{Gr_I H}F$ is therefore an isomorphism.*

Appendix B. Degeneracy map

In this section we investigate the degeneracy map $s_i : \Delta[n+1] \to \Delta[n]$. We discuss the functor $ds_i : \Delta[n] \to Spaces$ (see 9).

Observe that any map $\sigma : \Delta[m] \to \Delta[n]$ can be realized uniquely as the nerve of a functor $[m] \to [n]$, which is also denoted by σ. For example, $s_i : \Delta[n+1] \to \Delta[n]$ is the nerve of a functor $s_i : [n+1] \to [n]$ determined by the assignment:

$$\begin{tikzcd}
{[n+1]} \arrow[dd, "s_i"'] & *_{n+1} \arrow[r] \arrow[dd, dotted, mapsto] & \cdots \arrow[r] & *_{i+2} \arrow[r] \arrow[dd, dotted, mapsto] & *_{i+1} \arrow[d] & & & \\
& & & & *_i \arrow[r] \arrow[d, dotted, mapsto] & *_{i-1} \arrow[r] \arrow[d, dotted, mapsto] & \cdots \arrow[r] & *_0 \arrow[d, dotted, mapsto] \\
{[n]} & *_n \arrow[r] & \cdots \arrow[r] & *_{i+1} \arrow[r] & *_i \arrow[r] & *_{i-1} \arrow[r] & \cdots \arrow[r] & *_0
\end{tikzcd}$$

Since the nerve functor converts pull-backs in Cat into pull-backs in $Spaces$ (see 6.7), for any $\sigma : \Delta[m] \to \Delta[n]$, the space $ds_i(\sigma)$ can be identified with the nerve of the pull-back $lim([m] \xrightarrow{\sigma} [n] \xleftarrow{s_i} [n+1])$ in Cat.

We are going to identify values of the functor $d\big(\Delta[n+1] \xrightarrow{s_i} \Delta[n]\big) : \Delta[n] \to Spaces$ for various simplices $\sigma : \Delta[m] \to \Delta[n]$. This is done on the level of categories first.

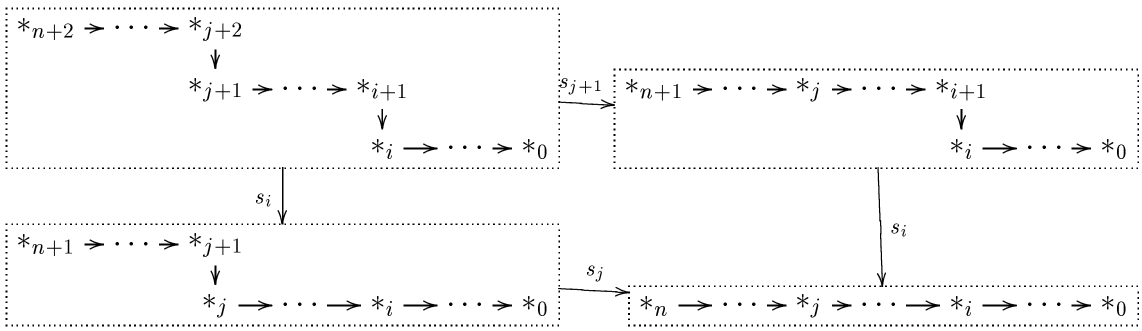
B.1. Let $\sigma : [m] \to [n]$ be a *monomorphism*. If the image of σ does not contain the object $*_i$, then the following is a pull-back square in Cat:

$$\begin{tikzcd} {[m]} \arrow[r] \arrow[d, "id"'] & {[n+1]} \arrow[d, "s_i"] \\ {[m]} \arrow[r, "\sigma"] & {[n]} \end{tikzcd}$$

If the image of σ does contain the object $*_i$, then, for a certain j we get a pull-back square in Cat of the form:

$$\begin{tikzcd} {[m+1]} \arrow[r] \arrow[d, "s_j"'] & {[n+1]} \arrow[d, "s_i"] \\ {[m]} \arrow[r, "\sigma"] & {[n]} \end{tikzcd}$$

B.2. Let us now look at a degeneracy $s_j : [n+1] \to [n]$ where, for simplicity, we assume that $j < i$. Observe that the following is a pull-back square in Cat:



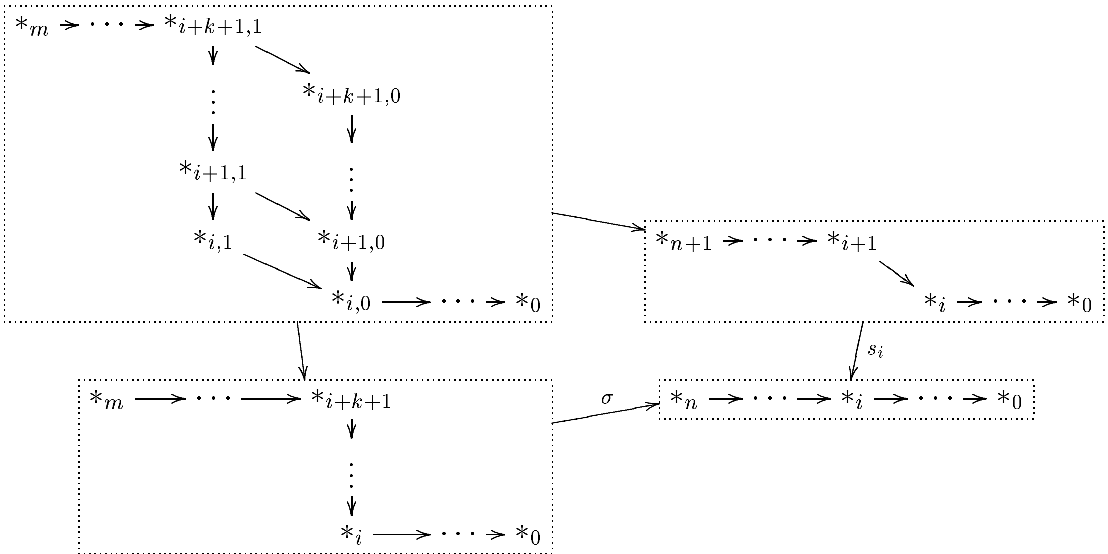
In short we have a pull-back square:

$$\begin{tikzcd} {[n+2]} \arrow[r, "s_{j+1}"] \arrow[d, "s_i"'] & {[n+1]} \arrow[d, "s_i"] \\ {[n+1]} \arrow[r, "s_j"] & {[n]} \end{tikzcd}$$

B.3. Assume that $\sigma : \Delta[m] \to \Delta[n]$ is given by a composition $s_i \circ \cdots \circ s_{i+k-1} \circ s_{i+k} = (s_i)^{k+1}$. On the level of small categories this map corresponds to a functor given by the assignment:

$$
\begin{tikzcd}
 & *_m \arrow[r] \arrow[ddd, mapsto, dotted] & \cdots \arrow[r] & *_{i+k+2} \arrow[r] \arrow[ddd, mapsto, dotted] & *_{i+k+1} \arrow[d] & & & \\
{[m]} \arrow[dd, "\sigma"'] & & & & \vdots \arrow[d] & & & \\
 & & & & *_i \arrow[r] \arrow[d, mapsto, dotted] & *_{i-1} \arrow[r] \arrow[d, mapsto, dotted] & \cdots \arrow[r] & *_0 \arrow[d, mapsto, dotted] \\
{[n]} & *_n \arrow[r] & \cdots \arrow[r] & *_{i+1} \arrow[r] & *_i \arrow[r] & *_{i-1} \arrow[r] & \cdots \arrow[r] & *_0
\end{tikzcd}
$$

Observe that the following is a pull-back square in Cat:



By P let us denote the category that sits in the top left corner of the above diagram. Consider subspaces of $N(P)$ which correspond to subcategories of P given by the graphs:

$$
C \qquad
\begin{tikzcd}
*_m \arrow[r] & \cdots \arrow[r] & *_{i+k+1,1} \arrow[d] \arrow[dr] & \\
 & & \vdots \arrow[d] & *_{i+k+1,0} \arrow[d] \\
 & & *_{i+1,1} \arrow[dr] & \vdots \arrow[d] \\
 & & & *_{i+1,0} \arrow[d] \\
 & & & *_{i,0} \arrow[r] & \cdots \arrow[r] & *_0
\end{tikzcd}
$$

$$
A \qquad
\begin{tikzcd}
*_m \arrow[r] & \cdots \arrow[r] & *_{i+k+1,1} \arrow[d] & \\
 & & \vdots \arrow[d] & \\
 & & *_{i+1,1} \arrow[dr] & \\
 & & & *_{i,0} \arrow[r] & \cdots \arrow[r] & *_0
\end{tikzcd}
$$

$$
B \qquad
\begin{tikzcd}
*_m \arrow[r] & \cdots \arrow[r] & *_{i+k+1,1} \arrow[d] & \\
 & & \vdots \arrow[d] & \\
 & & *_{i+1,1} \arrow[d] & \\
 & & *_{i,1} \arrow[dr] & \\
 & & & *_{i,0} \arrow[r] & \cdots \arrow[r] & *_0
\end{tikzcd}
$$

We can identify $N(B)$ with $\Delta[m+1]$, $N(A)$ with $\Delta[m]$ and the map $N(A) \to N(B)$ (induced by the inclusion $A \hookrightarrow B$) with $d_{i+1} : \Delta[m] \to \Delta[m+1]$. By induction we can identify $N(C)$ with a space that fits into the pull-back square:

$$
\begin{tikzcd}
N(C) \arrow[r] \arrow[d] & \Delta[n+2] \arrow[d, "s_{i+1}"] \\
\Delta[m] \arrow[r, "\sigma'"] & \Delta[n+1]
\end{tikzcd}
$$

where $\sigma' : \Delta[m] \to \Delta[n+1]$ is the composition $s_{i+1} \circ \cdots \circ s_{i+k-1} \circ s_{i+k}$. Notice that $N(B)$ and $N(C)$ cover $N(P)$, and hence since they intersect along $N(A)$:

$$N(P) = colim\big(N(B) \leftarrow N(A) \to N(C)\big)$$

We can now summarize the above discussion:

B.4. **Proposition**. *Let $\sigma : \Delta[m] \to \Delta[n]$ be a simplex.*

(1) *If σ does not contain the vertex i, then the following is a pull-back square:*

$$
\begin{tikzcd}
\Delta[m] \arrow[r] \arrow[d, "id"'] & \Delta[n+1] \arrow[d, "s_i"] \\
\Delta[m] \arrow[r, "\sigma"] & \Delta[n]
\end{tikzcd}
$$

(2) *If the preimage* $\sigma^{-1}(i)$ *consists of only one element, then for some* j *the following is a pull-back square:*

$$\begin{tikzcd} \Delta[m+1] \arrow[r] \arrow[d, "s_j"'] & \Delta[n+1] \arrow[d, "s_i"] \\ \Delta[m] \arrow[r, "\sigma"] & \Delta[n] \end{tikzcd}$$

(3) *Assume the preimage* $\sigma^{-1}(i)$ *consists of more than one element, i.e.* σ *can be expressed as a composition* $\Delta[m] \xrightarrow{\sigma'} \Delta[n+1] \xrightarrow{s_i} \Delta[n]$, *where the image of* σ' *contains vertices* i *and* $i+1$. *Let* P' *be the space that fits into the pull-back square:*

$$\begin{tikzcd} P' \arrow[r] \arrow[d] & \Delta[n+2] \arrow[d, "s_{i+1}"] \\ \Delta[m] \arrow[r, "\sigma'"] & \Delta[n+1] \end{tikzcd}$$

Then we can choose a map $\Delta[m] \to P'$ *with the following two properties. First, the composition* $\Delta[m] \to P' \to \Delta[m]$ *is the identity. Second, let us define a space* P *and a map* $P \to \Delta[m]$ *by the push-out:*

$$\begin{tikzcd} P \arrow[d] & = & colim \Big(& \Delta[m+1] \arrow[drr, "s_i"] & & \Delta[m] \arrow[ll, "d_{i+1}"'] \arrow[r] \arrow[d, "id"] & P' \arrow[dl] \Big) \\ \Delta[m] & & = & & & \Delta[m] & \end{tikzcd}$$

Then this map fits into the following pull-back square:

$$\begin{tikzcd} P \arrow[r] \arrow[d] & \Delta[n+1] \arrow[d, "s_i"] \\ \Delta[m] \arrow[r, "\sigma"] & \Delta[n] \end{tikzcd}$$

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