Klein's conjecture for contact automorphisms of the three-dimensional affine space

Marat Gizatullin

Abstract. In [**Kl**], §75, p.300, F. Klein formulated a conjecture on generators of the group of birational contact transformations of the projective plane. These transformations act birationally on the three-dimensional projective space. We solve the problem for the subgroup of contact transformations acting biregularly on the three-dimensional affine space. A description of the structure of the group of polynomial contact automorphisms of multidimensional affine space is given in the final section of the text.

1. Introduction

The ground field k is of characteristic zero.

Let (x, y, p) be three affine coordinates. The Pfaffian form

$$\omega = dy - pdx \tag{1.1}$$

is said to be a contact form of the three-dimensional space. A Cremona transformation T of the three-dimensional (x, y, p)-space defined by

$$x' = f(x, y, p), \quad y' = g(x, y, p), \quad p' = h(x, y, p) \tag{1.2}$$

is said to be a contact Cremona transformation of the (x, y)-plane if the image $T^*(\omega)$ of the contact form (1.1) is proportional to this form:

$$T^*(\omega) = \rho(x, y, p) \cdot \omega, \tag{1.3}$$

where $\rho(x, y, p)$ is a non-zero rational function. We will say that $\rho(x, y, p)$ is the *multiplier* of T. The contact transformation T is said to be a contact affine transformation if T and its inverse T^{-1} are polynomial. For a contact affine transformation, the multiplier is a non-zero constant (see Corollary 3.2 of Lemma 3.1 below).

Example 1.1. Let

$$x' = f(x, y) \quad y' = g(x, y) \tag{1.4}$$

be a Cremona transformation of the (x, y)-plane. It is possible to extend (1.4) to a contact transformation

$$x' = f(x, y), \quad y' = g(x, y), \quad p' = h(x, y, p), \tag{1.5}$$

1991 *Mathematics Subject Classification.* Primary 14E07; Secondary 14R10, 53D10.

Key words and phrases. Cremona transformation, Contact transformation, Polynomial automorphism, Symplectic transformation.

where

$$h(x,y,p) = \frac{pg'_y + g'_x}{pf'_y + f'_x}.$$

See a description of such an extension for a one-parameter group in [**In**], n^o 4.5, page 103. According to a tradition, we will say that (1.5) is *a contact extension of point transformation* (1.4), or shorter that (1.5) is a *point transformation.*

EXAMPLE 1.2. It is not hard to verify that the following transformation

$$L: \quad x' = p, \quad y' = xp - y, \quad p' = x \tag{1.6}$$

is involutive and contact. It is the *Legendre transformation.* See [**In**], n^{o}2.5, pages 40-41. The Legendre transformation belongs to the set of duality transformations. All the duality transformations are conjugate by extended plane projective collineations to the Legendre transformation. See [**Kl**], §62 for a description of space duality transformations as contact transformations. According to [**Po**], page 125, the connection between the reciprocity defined by a quadric and the Legendre transformation was observed by Michel Chasles.

In [**Kl**], §75.1, page 300, one can find the following conjecture.
Klein's conjecture. *The group of contact Cremona transformation of the projective plane is generated by the subgroup of point contact transformation and by the Legendre transformation.*

REMARK 1.3. The above formulation of the conjecture is more explicit than Klein's original description of his principle. In the mentioned place of [**Kl**], Klein comments an example of decomposition of a contact transformation (it was the pedal transformation) and writes the following.

Wir entnehmen aus unserem Beispiel daher das folgende allgemeine Prinzip: Um Beispiele ein eideutiger Berührungstransformation herzustellen, braucht man nur eine beliebige dualistische Transformation mit einer beliebigen Cremona Transformation verbunden .

Later authors stated the conjecture without a reference to Klein. For example, Ott-Heinrich Keller in [**Ke**], page 651 wrote that he does not know a birational contact transformation of the plane which is not presentable as a composition of point Cremona transformations and duality transformations:
Korrelationen und Cremona Transformationen und alles daraus Zusammengestzen sind Berührungstransformationen. Eine birationale Berührungstransformation die nicht dieser Gruppe angehört, ist nur nicht bekannt.

Some other authors, for example, Manfred Hermann [**He**], attributed the conjecture to O-H. Keller but not to F. Klein.

I add final Klein's sorrowful remark at the end of the mentioned §75.1 in [**Kl**]. He says that so far we do not have a general theory of one-to-one algebraic contact transformations. (*Eine allgemeine Theorie der eindeutigen und albebraischen Berührungstransformationen sheint noch nicht entwickelt zu sein.*) Moreover, I add that for the first time Klein stated his conjecture in his lithographic lectures on higher geometry (the first publication of [**Kl**]) in 1893.

Our result is the following theorem

THEOREM 1.4. *Any polynomial contact automorphism of the affine (x, y, p)-space is a composition of some extended point polynomial automorphisms of the (x, y)-plane and of some number of Legendre transformations.*

In the final section of our paper, we present a description of the structure of polynomial contact transformation of multidimensional affine space.

Acknowledgement.

I would like to thank the organizers of the Oberwolfach Conference on Affine Algebraic Geometry (7th-13th of January 2007), where I had a happy opportunity to meet my geometric colleagues and to discuss the topics relevant to the paper.

The main theorem was a subject of my talk at Max-Planck-Institut für Mathematik (11.00-12.00 ante meridiem of 16th of January 2007). I would like to thank a few listeners of the talk. I am much obliged to the administration of MPIM for having made possible a short but productive visit to Bonn.

I would like to thank Igor Dolgachev for linguistic corrections of my English text.

2. Some examples, the notions of potential and lift

EXAMPLE 2.1. Let $P(x)$ be a polynomial of x.

$$x' = x, \quad y' = y + P(x), \quad p' = p + P'(x),$$

where $P'(x)$ is the derivative of $P(x)$. According to example 1.1, this transformation is a contact extension of a triangular point transformation

$$x' = x, \quad y' = y + P(x). \tag{2.1}$$

EXAMPLE 2.2. Let us take a polynomial automorphism of the affine (x, p)-plane

$$x' = f(x, p), \quad p' = h(x, p). \tag{2.2}$$

The Jacobian of the automorphism is a non-zero constant r,

$$f'_x(x, p)h'_p(x, p) - h'_x(x, p)f'_p(x, p) = r.$$

The Pffafian form of two variables (x, p)

$$\Theta = (hf'_x - rp)dx + (hf'_p)dp$$

is closed, that is $d\Theta = 0$, hence Θ is the differential of some polynomial $U(x, p)$, $dU = \Theta$. Thus we have identities

$$U'_x = hf'_x - rp, \quad U'_p = hf'_p. \tag{2.3}$$

The polynomial $U(x, p)$ is defined up to an additive constant.

It is not difficult to verify that

$$x' = f(x, p), \quad y' = ry + U(x, p), \quad p' = h(x, p) \tag{2.4}$$

is a contact affine transformation.

We will say that $U(x, p)$ is a *potential* of the polynomial affine automorphism (2.2) of the (x, p)-plane and that transformation (2.4) is a *contact lift* of (2.2).

If

$$f(x, p) = x, \quad h(x, p) = p + P'(x), \quad U = P(x),$$

then by lift we obtain the transformation of example 2.1.

Legendre transformation (1.6) is a lift of the transposition of variables x, p.

Example 2.3. More generally, the linear transformation

$$x' = Ap, \quad p' = Bx, \tag{2.5}$$

where $A \in k^*, B \in k^*$, has the following lift

$$x' = Ap, \quad y' = AB(px - y), \quad p' = Bx.$$

Example 2.4. One of the lifts of the following triangular transformation

$$x' = x + F(p), \quad p' = p, \tag{2.6}$$

where $F(p) \in k[p]$, is

$$x' = x + F(p), \quad y' = y + U(p), \quad p' = p,$$

where $U(p) \in k[p], U' = pF$. The latter transformation is a composition of two Legendre transformations L from (1.6) and of a point transformation. Indeed, if R is the following point transformation

$$x' = x, \quad y' = y + U(x), \quad p' = p + F(x),$$

then the lift coincides with LRL.

Remark 2.5. Sophus Lie preferred the following writing of the lift of 2.6

$$x' = x + \frac{dW(p)}{dp}, \quad y' = y - W(p) + p\frac{dW(p)}{dp}, \quad p' = p.$$

Certainly, Lie considered $W(p)$ more general than a rational function. He proved in [**Lie**], Chap. 2, Theorem 11, page 60 that such a commutative subgroup of contact transformations coincides with his own centralizer in the group of all contact transformations.

The following lemma is obvious.

Lemma 2.6. *Polynomial contact transformation* (1.2) *of the affine* (x, y, p)*-space is a contact lift of some polynomial automorphism of the affine* (x, p)*-plane if and only if polynomials* f *and* h *do not depend on* y*.*

3. The proof of Theorem 1.4

Lemma 3.1. *Let* T *be a contact Cremona transformation* (1.2)*,* $\rho(x, y, p)$ *be its multiplier. The square of the multiplier coincides with the Jacobian* $J(T)$ *of the transformation:*

$$J(T) = \rho^2.$$

Proof. The contact condition (1.3) is equivalent to the following three identities

$$g'_p - hf'_p = 0, \tag{3.1}$$

$$g'_y - hf'_y = \rho, \tag{3.2}$$

$$g'_x - hf'_x = -\rho p. \tag{3.3}$$

Taking the difference of the partial derivatives of the first identity (3.1) by y and of the second identity (3.2) by p, we obtain

$$\begin{vmatrix} f'_y & f'_p \\ h'_y & h'_p \end{vmatrix} = -\rho'_p. \tag{3.4}$$

Taking the difference of the partial derivatives of the first identity (3.1) by x and of the third identity (3.3) by p, we obtain

$$\begin{vmatrix} f'_x & f'_p \\ h'_x & h'_p \end{vmatrix} = \rho + p\rho'_p. \tag{3.5}$$

Taking the difference of the partial derivatives of the second identity (3.2) by x of the third identity (3.3) by y we obtain

$$\begin{vmatrix} f'_x & f'_y \\ h'_x & h'_y \end{vmatrix} = \rho'_x + p\rho'_y. \tag{3.6}$$

If we multiply the first row of the Jacobian matrix

$$\begin{pmatrix} f'_x & f'_y & f'_p \\ g'_x & g'_y & g'_p \\ h'_x & h'_y & h'_p \end{pmatrix} \tag{3.7}$$

by $(-h)$ and add the produced row to the the second row, then using the above three identities (3.1),(3.2),(3.3), we obtain new matrix

$$\begin{pmatrix} f'_x & f'_y & f'_p \\ -\rho p & \rho & 0 \\ h'_x & h'_y & h'_p \end{pmatrix}$$

with the same determinant. Taking the expansion of the determinant of the latter matrix along the second row, we obtain that

$$J(T) = \rho(p\begin{vmatrix} f'_y & f'_p \\ h'_y & h'_p \end{vmatrix} + \begin{vmatrix} f'_x & f'_p \\ h'_x & h'_p \end{vmatrix}).$$

According to the above two determinantal identities (3.4), (3.5), the expression in the parentheses coincides with ρ, hence the Jacobian $J(T)$ is equal to ρ^2. □

COROLLARY 3.2. *The multiplier of any polynomial contact transformation of the affine (x, y, p)-space is a nonzero constant.*

PROOF. The Jacobian of any polynomial affine automorphism is a constant. □

LEMMA 3.3. *Any polynomial contact transformation of the affine (x, y, p)-space is a contact lift of some polynomial automorphism of the affine (x, p)-plane, that is such a transformation is representable as* (2.4)*, where U satisfies* (2.3)*.*

PROOF. Let (1.2) be a contact polynomial transformation, ρ be its multiplier. By the corollary 3.2 of Lemma 3.1, ρ is a constant, hence all the partial derivatives of ρ vanish. Therefore we may rewrite three determinantal identities (3.4),(3.5),(3.6) as

$$\begin{vmatrix} f'_y & f'_p \\ h'_y & h'_p \end{vmatrix} = 0, \tag{3.8}$$

$$\begin{vmatrix} f'_x & f'_p \\ h'_x & h'_p \end{vmatrix} = \rho, \tag{3.9}$$

$$\begin{vmatrix} f'_x & f'_y \\ h'_x & h'_y \end{vmatrix} = 0. \tag{3.10}$$

Using the three latter identities (3.8), (3.9),(3.10) and the expansion of the Jacobian determinant of $J(T)$ by the second row of matrix (3.7), we see that the Jacobian determinant coincides with $g'_y\rho$. By Lemma 3.1, the Jacobian determinant of any polynomial transformation is the square of the multiplier, hence

$$g'_y = \rho, \quad g(x, y, p) = \rho y + U(x, p)$$

for some polynomial $U(x, p)$. Because of identity (3.2), we obtain

$$hf'_y = 0,$$

hence f does not depend on y,

$$f(x, y, p) = f(x, p).$$

Let us consider the following two subcases.

$$\text{First}: \qquad f'_x(x,p) = 0.$$

$$\text{Second}: \qquad f'_x(x,p) \neq 0.$$

If we have the first subcase, then $f(x, p) = f(p)$, f'_p is a divisor of the Jacobian determinant, therefore f'_p =const, $f = Ap + B$, where A, B are constants, $A \neq 0$, $-Ah'_x = \rho$, whence

$$h = -A^{-1}x + G(p),$$

where $G(p)$ is a polynomial, therefore using Lemma 2.6 we obtain a polynomial automorphism of type (2.2), and a potential exists.

Let us consider the second subcase. According to (3.10), we obtain

$$h'_y = 0,$$

that is h is independent on y, $h = h(x, p)$, and again by Lemma 2.6 we obtain a polynomial automorphism type (2.2).

The Lemma is proved. □

The end of the proof of Theorem 1.4.

It is enough to find a set of generators of the group of contact polynomial automorphisms of the affine 3-space such that any generator is decomposable into a composition of some extended point transformations and some number of the Legendre transformations.

According to well-known Jung's and Van der Kulk's theorem [**Ju**],[**Ku**], the group of polynomial automorphisms of the (x, p)-plane is generated by transformations (2.5) (2.6) from Examples 2.3, 2.4 respectively. We saw that some contact lifts of the transformations exist. Any contact lift is defined up to a translation parallel to the y-axis

$$x' = x, \quad y' = y + b, \quad p' = p. \tag{3.11}$$

By Lemma 3.3, the lifted transformations (2.6) and (2.5) together with extended (2.1) and the translations (3.11) generate the group of contact polynomial automorphisms of the affine 3-space. It is clear that any translation is a point transformation,. Moreover, we saw that the lifts of (2.6) and (2.5) satisfy Klein's conjecture.

4. Some generalizations

We begin with multidimensional generalizations of the basic definitions. Here we consider
odd-dimensional affine space $\mathbb{A}^{2n+1}$ with point coordinates

$$(\mathbf{x}, y, \mathbf{p}) = (x_1, ..., x_n, y, p_1, ..., p_n),$$

its even-dimensional (sub/quotient)space $\mathbb{A}^{2n}$ with coordinates

$$(\mathbf{x}, \mathbf{p}) = (x_1, ..., x_n, p_1, ..., p_n),$$

two differential form on $\mathbb{A}^{2n+1}$

$$\omega(\mathbf{x}, y, \mathbf{p}) = dy - p_1 dx_1 - ... - p_n dx_n \tag{4.1}$$

and the differential

$$\Omega = d\omega = dx_1 \wedge dp_1 + ... + dx_n \wedge dp_n. \tag{4.2}$$

Certainly, (4.1) generalizes (1.1). A Cremona transformation T of the $(2n+1)$-dimensional affine space $\mathbb{A}^{2n+1}$ defined by

$$\begin{aligned} x'_i &= f_i(\mathbf{x}, y, \mathbf{p}), \\ y' &= g(\mathbf{x}, y, \mathbf{p}), \\ p'_i &= h_i(\mathbf{x}, y, \mathbf{p}), \end{aligned} \tag{4.3}$$

where $1 \leq i \leq n$, is said to be a contact Cremona transformation of the space if the image $T^*(\omega)$ of the contact form (4.1) is proportional to this form:

$$T^*(\omega) = \rho(\mathbf{x}, y, \mathbf{p}) \cdot \omega, \tag{4.4}$$

where $\rho(\mathbf{x}, y, \mathbf{p})$ is a non-zero rational function.
The function $\rho(\mathbf{x}, y, \mathbf{p})$ is the *multiplier* of T. The contact Cremona transformation T is said to be a contact affine transformation if T and its inverse T^{-1} are polynomial. The multiplier of any contact affine transformation is a non-zero constant. A Cremona transformation S of the $2n$-dimensional affine space $\mathbb{A}^{2n}$ defined by

$$\begin{aligned} x'_i &= f_i(\mathbf{x}, \mathbf{p}), \\ p'_i &= h_i(\mathbf{x}, \mathbf{p}) \end{aligned} \tag{4.5}$$

is said to be a *conformally symplectic Cremona transformation* of the space if the image $S^*(\Omega)$ of the symplectic form (4.2) is proportional to this form:

$$S^*(\Omega) = \sigma(\mathbf{x}, \mathbf{p}) \cdot \Omega, \tag{4.6}$$

where $\sigma(\mathbf{x}, \mathbf{p})$ is a non-zero rational function. The function σ is the *conformal multiplier* of S.

There exists the following generalization of Lemma 3.1. (According to [**Po**], page 138, such an assertion as Lemma 4.1 for a multidimensional case was proved by Sophus Lie, the proof is reproduced in [**Ca**], page 109. Caratheodory did not use exterior differential forms. E. M. Polistchuk writes that the idea of application of differential forms in a proof is due to F. Frobenius and E. Cartan. We use the idea below.)

LEMMA 4.1. *For a contact Cremona transformation T defined by (4.3), the determinant $J(T)$ of the Jacobian matrix*

$$M(T) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_n} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial p_1} & \dots & \frac{\partial f_1}{\partial p_n} \\ . & \dots & . & . & . & \dots & . \\ \frac{\partial f_n}{\partial x_1} & \dots & \frac{\partial f_n}{\partial x_n} & \frac{\partial f_n}{\partial y} & \frac{\partial f_n}{\partial p_1} & \dots & \frac{\partial f_n}{\partial p_n} \\ \frac{\partial g}{\partial x_1} & \dots & \frac{\partial g}{\partial x_n} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial p_1} & \dots & \frac{\partial g}{\partial p_n} \\ \frac{\partial h_1}{\partial x_1} & \dots & \frac{\partial h_1}{\partial x_n} & \frac{\partial h_1}{\partial y} & \frac{\partial h_1}{\partial p_1} & \dots & \frac{\partial h_1}{\partial p_n} \\ . & \dots & . & . & . & \dots & . \\ \frac{\partial h_n}{\partial x_1} & \dots & \frac{\partial h_n}{\partial x_n} & \frac{\partial h_n}{\partial y} & \frac{\partial h_n}{\partial p_1} & \dots & \frac{\partial h_n}{\partial p_n} \end{pmatrix}. \tag{4.7}$$

is equal to $(n+1)$-th power of the multiplier:

$$J(T) = \rho^{n+1}.$$

PROOF. Indeed, if T is contact, then

$$T^*(\Omega) = T^*(d\omega) = dT^*(\omega) = \tag{4.8}$$

$$d\rho \wedge \omega + \rho\Omega,$$

therefore

$$J(T)dx_1 \wedge \dots \wedge dx_n \wedge dy \wedge dp_1 \wedge \dots \wedge dp_n =$$
$$T^*(dx_1 \wedge \dots \wedge dx_n \wedge dy \wedge dp_1 \wedge \dots \wedge dp_n) =$$
$$\frac{1}{n!}T^*(\omega \wedge \Omega^{\wedge n}) = \frac{1}{n!}\rho^{n+1}\omega \wedge \Omega^{\wedge n} =$$
$$\rho^{n+1}dx_1 \wedge \dots \wedge dx_n \wedge dy \wedge dp_1 \wedge \dots \wedge dp_n.$$

□

A parallel similar assertion with almost the same proof takes place for conformally symplectic transformations.

LEMMA 4.2. *For a conformally symplectic Cremona transformation S defined by 4.5, the determinant $J(S)$ of the Jacobian matrix*

$$M(S) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_n} & \frac{\partial f_1}{\partial p_1} & \dots & \frac{\partial f_1}{\partial p_n} \\ . & \dots & . & . & \dots & . \\ \frac{\partial f_n}{\partial x_1} & \dots & \frac{\partial f_n}{\partial x_n} & \frac{\partial f_n}{\partial p_1} & \dots & \frac{\partial f_n}{\partial p_n} \\ \frac{\partial h_1}{\partial x_1} & \dots & \frac{\partial h_1}{\partial x_n} & \frac{\partial h_1}{\partial p_1} & \dots & \frac{\partial h_1}{\partial p_n} \\ . & \dots & . & . & \dots & . \\ \frac{\partial h_n}{\partial x_1} & \dots & \frac{\partial h_n}{\partial x_n} & \frac{\partial h_n}{\partial p_1} & \dots & \frac{\partial h_n}{\partial p_n} \end{pmatrix} \tag{4.9}$$

is equal to n-th power of the conformal multiplier:

$$J(S) = \sigma^n.$$

PROOF. We omit it. □

We will say that for a conformally symplectic Cremona transformation (4.5) there exists a potential $U = U(\mathbf{x}, \mathbf{p}) \in k(x_1, ..., x_n, p_1, ..., p_n)$ if the following identities are fulfilled:

$$\frac{\partial U}{\partial p_i} = \sum_{k=1}^{n} h_k \frac{\partial f_k}{\partial p_i},$$

$$\frac{\partial U}{\partial x_i} = \sum_{k=1}^{n} h_k \frac{\partial f_k}{\partial x_i} - p_i\sigma,$$

where $1 \le i \le n$, σ is the conformal multiplier of the transformation. It is clear that for any symplectic polynomial automorphism of $\mathbb{A}^{2n}$ the conformal multiplier σ is a constant and a potential exists.

If σ is a constant and a potential for (4.5) exists, then

$$x'_i = f_i(\mathbf{x}, \mathbf{p}), \tag{4.10}$$
$$y' = \sigma y + U(\mathbf{x}, \mathbf{p}),$$
$$p'_i = h_i(\mathbf{x}, \mathbf{p})$$

is a contact transformation. We will say that the latter transformation (4.10) is a *contact lift* of (4.5). For such a lift (4.10), the multiplier ρ coincides with σ. Any potential is defined up to an additive constant, therefore any lift is defined up to an element from group $\mathbf{T}_y(\mathbb{A}^{2n+1})$ of translations parallel to the y-axis

$$x'_i = x_i, \quad y' = y + b, \quad p'_i = p_i, \tag{4.11}$$

where b is an element of the ground field k.

Lemma 3.3 has the following multidimensional analog.

LEMMA 4.3. *Any polynomial contact transformation of the affine* $(2n+1)$-*space is a contact lift of some polynomial conformally symplectic automorphism of the affine* $2n$-*space, that is such a transformation is representable as* (4.10).

PROOF. For a polynomial contact transformation T defined by by (4.3), one can write

$$\det(M(T)) = \frac{\partial g}{\partial x_1}F_1 + ... + \frac{\partial g}{\partial x_n}F_n + \frac{\partial g}{\partial y}G + \frac{\partial g}{\partial p_1}H_1 + ... + \frac{\partial g}{\partial p_n}H_n,$$

where $F_1, ...F_n, G, H_1, ..., H_n$ are the co-factors of elements of $(n+1)$-th row of matrix $M(T)$ in (4.7).

The plan of the proof is as follows.

First, we have to show that if the multiplier ρ is a constant, then all the co-factors (with the exception of G) of the $(n+1)$-th row of $M(T)$ vanish, but the co-factor G is equal to the n-th power of the multiplier, therefore, according to Lemma 4.2, the $(n+1)$-th element $\frac{\partial g}{\partial y}$ of the $(n+1)$-th row of matrix is equal to the multiplier:

$$\frac{\partial g}{\partial y} = \rho.$$

Second, we have to show that if the multiplier ρ of transformation (4.3) is a constant, then the right hand sides f_i and g_i of (4.3) do not depend on y, that is the corresponding lines of formulas for (4.3) have the same form as the lines of (4.10).

In the proof of Lemma 4.2, we have seen identity (4.8). Here the multiplier is a constant, therefore

$$T^*(\Omega) = \rho\Omega, \tag{4.12}$$

whence

$$T^*(\Omega^{\wedge n}) = \rho^n\Omega^{\wedge n},$$
$$T^*(dx_1 \wedge ... \wedge dx_n \wedge dp_1 \wedge ... \wedge dp_n) = \rho^n \cdot dx_1 \wedge ... \wedge dx_n \wedge dp_1 \wedge ... \wedge dp_n.$$

The latter identity means that the co-factor G is equal to ρ^n, and that other co-factors vanish.

Second, identity (4.12) implies that

$$\Sigma_{i=1}^n \begin{vmatrix} \frac{\partial f_i}{\partial x_m} & \frac{\partial f_i}{\partial y} \\ \frac{\partial h_i}{\partial x_m} & \frac{\partial h_i}{\partial y} \end{vmatrix} = 0, \qquad m = 1, ..., n,$$

or

$$\Sigma_{i=1}^n \frac{\partial h_i}{\partial y} \frac{\partial f_i}{\partial x_m} - \Sigma_{i=1}^n \frac{\partial f_i}{\partial y} \frac{\partial h_i}{\partial x_m} = 0, \quad m = 1, ..., n.$$

One can consider the latter identities as a system of homogeneous linear equations with unknown quantities $\frac{\partial h_i}{\partial y}, \frac{\partial f_i}{\partial y}$. The determinant of the system is equal (up to a sign) to G (or to the determinant of matrix of form (4.9)). Because of non-vanishing of the determinant, the solution of the linear system is trivial.

We add that a comparison of (4.12) and (4.6) implies equality $\rho = \sigma$. $\square$

Remark 4.4. About the second step of our proof. The vanishing of the partial derivatives by y admits some interpretation with the point of view of a general theory of contact varieties, one can see such a theory in chap. 4 of [**Hu**]. On a general contact variety, the structure contact form ω defines a vector field V_ω by the following condition

$$V_\omega(f) \cdot \omega \wedge (d\omega)^{\wedge n} = df \wedge (d\omega)^{\wedge n}.$$

For the case of our standard ω (see (4.1)), Cartan used notation $\{f\}$ instead of $V_\omega(f)$ (see [**Car**], chap. XIII, n^{o}131). Certainly, for the standard case, the vector field is parallel to the y-axis, $\{f\} = \partial f / \partial y$. The vanishing of $V_\omega(f)$ means that f is a constant along the trajectories of the vector field V_ω, and f is a lift of a function defined on a symplectic quotient of the contact variety.

Remark 4.5. We would like to say a few final words about the structure of the group of contact polynomial automorphisms.

Let $\mathbf{CSAut}(\mathbb{A}^{2n})$ denote the group of all conformally symplectic polynomial automorphisms of $\mathbb{A}^{2n}$ (see (4.5) and (4.6)), let $\mathbf{ContAut}(\mathbb{A}^{2n+1})$ denote the group of all contact polynomial automorphisms of $\mathbb{A}^{2n+1}$ (see (4.3) and (4.4)), let $\mathbf{T}_y(\mathbb{A}^{2n+1})$ be the group of translations parallel to the y-axis(see (4.11)). The group of such translations is a subgroup $\mathbf{ContAut}(\mathbb{A}^{2n+1})$. Lemma 4.3 says that if we omit the middle line in (4.3), then we obtain formulas of type (4.5). Thus we have a homomorphism of $\mathbf{ContAut}(\mathbb{A}^{2n+1})$ to $\mathbf{CSAut}(\mathbb{A}^{2n})$. The latter homomorphism is surjective. Hence we obtain the following

Theorem 4.6. *The sequence of homomorphisms*

$$\{1\} \to \boldsymbol{T}_y(\mathbb{A}^{2n+1}) \to \boldsymbol{ContAut}(\mathbb{A}^{2n+1}) \to \boldsymbol{CSAut}(\mathbb{A}^{2n}) \to \{1\} \tag{4.13}$$

is exact.

Proof. The theorem is a reformulation of Lemma 4.3. $\square$

Remark 4.7. In (4.13), the middle group is an extension of abelian invariant subgroup $\mathbf{T}_y(\mathbb{A}^{2n+1})$ by $\mathbf{CSAut}(\mathbb{A}^{2n})$. One can describe such an extension with the help of an action of the quotient group on the kernel together with a system of factors, see [**Ku**], §48. In our case, the action coincides with the multiplication by the Jacobian determinant, that is if $\alpha \in \mathbf{CSAut}(\mathbb{A}^{2n})$ is the image of $g_\alpha \in \mathbf{ContAut}(\mathbb{A}^{2n+1})$, and t is the translation (4.11), then $g_\alpha t g_\alpha^{-1}$ is defined by

$$x_i' = x_i, \quad y' = y + J(\alpha)b, \quad p_i' = p_i,$$

where $J(\alpha)$ is the Jacobian determinant of α. For a general extension, the system of factors is the function $m_{\alpha,\beta}$ of pairs of elements α, β of the quotient group with values in the kernel, the function is defined (after a fixation of some representatives g_α) by the following identity

$$g_\alpha g_\beta = m_{\alpha,\beta} g_{\alpha\beta}.$$

For our case, we can fix the representatives by the condition of vanishing of potentials at the origin ($\mathbf{0}$). By such a fixation,

$$m_{\alpha,\beta} = U_\alpha(\mathbf{0}),$$

where $U_\alpha((\mathbf{x},\mathbf{p}))$ is the potential of α vanishing at ($\mathbf{0}$), is the system of factors defining the extension (4.13).

References

[Ca] C. Caratheodory, *Variationsrechnung und partielle differential Gleichungen erster Ordnung,* Band I, Zweiter Auflage, Leipzig, 1956, English translation, *Calculus of variations and partial differential equations of the first order,* Chelsea, 1965,1967,1982.

[Car] E. Cartan, *Leçons sur les invariants intégraux,* Paris, 1922. Russian translation 1940.

[He] M. Hermann, *Über birationale Berührungstransformationen zweiter Ordnung,* Wissenschaftliche Zeitschrift der Martin-Luther-Universität, Halle-Wittenberg, **8**, 3,(1959), 375–386.

[Hu] N.E. Hurt, *Geometric quantization in action,* D. Reidel Publishing Company ,1983. Russian translation 1985.

[In] E.L.Ince, *Ordinary differential equations,* Dover Publications, New York, 1926, 1956.

[Ju] H.W.Jung, *Über ganze birationale Transformationen der Ebene,* Journal für die reine und angewandte Mathematik, **184**, (1942), 161–174.

[Ke] O.H. Keller,*Zur Theorie der ebenen Berührungstransformationen I*, Mathematische Annalen, **120** (1947), 650–675.

[Kl] F. Klein, *Vorlesungen über höhere Geometrie,* dritte Auflage, bearbeitet und herausgegeben von W. Blashke, Berlin, Verlag von Julius Springer, 1926. Russian translation 1939. The first lithographic edition, *Einleitung in die höhere Geometrie,*

[Ku] A.G. Kurosh, *The theory of groups,* Third Russian edition, Moscow, 1967. Second English edition in two volumes, Chelsea, 1960.

[Lie] Sophus Lie, *Geometrie der Berührungstransformationen,* Leipzig, 1896, Chelsea, 1977.

[Ku] W. van der Kulk, *On polynomial rings in two variables,* Nieuw Archief voor Wiskunde (**3**) **1** (1953), 33–41.

[Po] E.M. Polistchuk, *Sophus Lie,* in Russian, Leningrad, 1983.

Max-Planck-Institut für Mathematik, Vivatsgasse 7, 53111, Bonn, Deutschland

E-mail address: `gizmarat@yandex.ru`

Current address: Chair of Mathematical Analysis, The Institute of Mathematics, Physics and Informatics, Samara State Pedagogical University, Maxim Gorki Street, building 65/67, 443099 Samara, RUSSIA