THE SHRINKING TARGET PROBLEM FOR MATRIX TRANSFORMATIONS OF TORI

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The Shrinking Target Problem for Matrix Transformations of Tori

Richard Hill   Sanju L. Velani

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Abstract

Let T be a $d \times d$ matrix with integral coefficients. Then T determines a self-map of the d-dimensional torus $X = \mathbb{R}^d/\mathbb{Z}^d$. We shall consider the following question. Choose for each natural number n a ball $B(n)$ in X and suppose that $B(n+1)$ has smaller radius than $B(n)$ for all n. Now let W be the set of points $x \in X$ such that $T^n(x) \in B(n)$ for infinitely many $n \in \mathbb{N}$. What is the Hausdorff dimension of W? This question arises from analogies with Diophantine approximation, in particular Jarník–Besicovitch's description of the Hausdorff dimensions of the sets of well-approximable real numbers. The answer depends on the quantity

$$\tau = \liminf_{n\to\infty} \frac{-\log(\text{Radius of } B(n))}{n}.$$

We are able to give a complete description only when the matrix is diagonalizable over $\mathbb{Q}$. In other cases we obtain a result for sufficiently large τ. Our results, in as far as they go, show that the Hausdorff dimension of W is a strictly decreasing, continuous function of τ which is piecewise of the form $\frac{A\tau+B}{C\tau+D}$. The numbers A, B, C and D which arise in this way are typically sums of logarithms of the absolute values of eigenvalues of T.

1 Introduction

1.1 The Shrinking Target Problem and its Connection with Diophantine Approximation

Let X be a metric space and $T : X \to X$ a transformation. Suppose that X is equipped with a Borel probability measure m which is preserved by T. We shall also assume that T is ergodic with respect to m. It is know that for any ball B in X of positive measure the subset

$$\{x \in X : T^n x \in B \text{ for infinitely many } n \in \mathbb{N}\}$$

of X has full m-measure. This means that the trajectories of almost all points will go through the ball B infinitely often. In general one can ask the question what happens if the ball B shrinks with time and moves around. More precisely if at time n one has a ball $B(n) = B(z_n, \mathrm{rad}(n))$ of radius $\mathrm{rad}(n)$ ($\mathrm{rad}(n) \to 0$ as $n \to \infty$), then what kind of properties does the set of points z have, whose images $T^n(z)$ are in $B(n)$ for infinitely many n?

These points can be thought of as trajectories which hit a shrinking, moving target infinitely often. We shall call such points "well approximable" in analogy with the classical theory of metric Diophantine approximation [1, 10] and its more recent extensions to the theory of discrete hyperbolic groups (see [2, 7, 8, 9, 11, 12]). In the classical theory, the projective real line, $\mathbb{R} \cup \{\infty\}$ is identified with the unit tangent space at a point of the modular surface $\mathbb{H}/\mathrm{SL}_2(\mathbb{Z})$ (the modular group $\mathrm{SL}_2(\mathbb{Z})$ acts on the upper half plane $\mathbb{H}$ by fractional linear transformations). The "well approximable" real numbers in the classical sense (see [10]) correspond to geodesics which enter a shrinking neighbourhood of the only cusp of $\mathbb{H}/\mathrm{SL}_2(\mathbb{Z})$ infinitely often.

1.2 Results

In this paper we shall consider only a special case of the above general problem, in which T is a matrix transformation of a d-dimensional torus $X := \mathbb{R}^d/\mathbb{Z}^d$. For simplicity we suppose that the determinant of T is non-zero. For any sequence of balls $B(n) = B(z_n, \mathrm{rad}(n))$ ($\mathrm{rad} : \mathbb{N} \to \mathbb{R}^{\geq 0}$ being a decreasing function), we shall examine the set

$$W := \{z \in X \ : \ T^n(z) \in B(z_n, \mathrm{rad}(n)) \text{ for infinitely many } n \in \mathbb{N}\}. \tag{1}$$

Our results will involve the eigenvalues of T and the number τ defined by

$$\tau = \liminf_{n\to\infty} \frac{-\log \mathrm{rad}(n)}{n}. \tag{2}$$

We shall prove the following.

Theorem 1 *Let $T : X \to X$ be a matrix transformation of the torus $X := \mathbb{R}^d/\mathbb{Z}^d$. Let $e_1, \ldots, e_d$ be the absolute values of the eigenvalues of T (with multiplicity). Suppose these are ordered: $e_1 \leq e_2 \leq \ldots \leq e_d$. Then for $\tau \geq \log(e_d/e_1)$ one has*

$$\dim W = \min_{i=1,\ldots,d} \left\{ \frac{i \log e_i + \sum_{j=i+1}^{d} \log e_j}{\tau + \log e_i} \right\}. \tag{3}$$

If the matrix T is diagonalizable over $\mathbb{Q}$ then we obtain the following stronger result, which fills the gap in the graph of $\dim W$ against τ.

Theorem 2 *Let $T : X \to X$ be diagonalizable over $\mathbb{Q}$, and let $e_1, \ldots, e_d \in \mathbb{Z}$ be as in Theorem 1. Then one has*

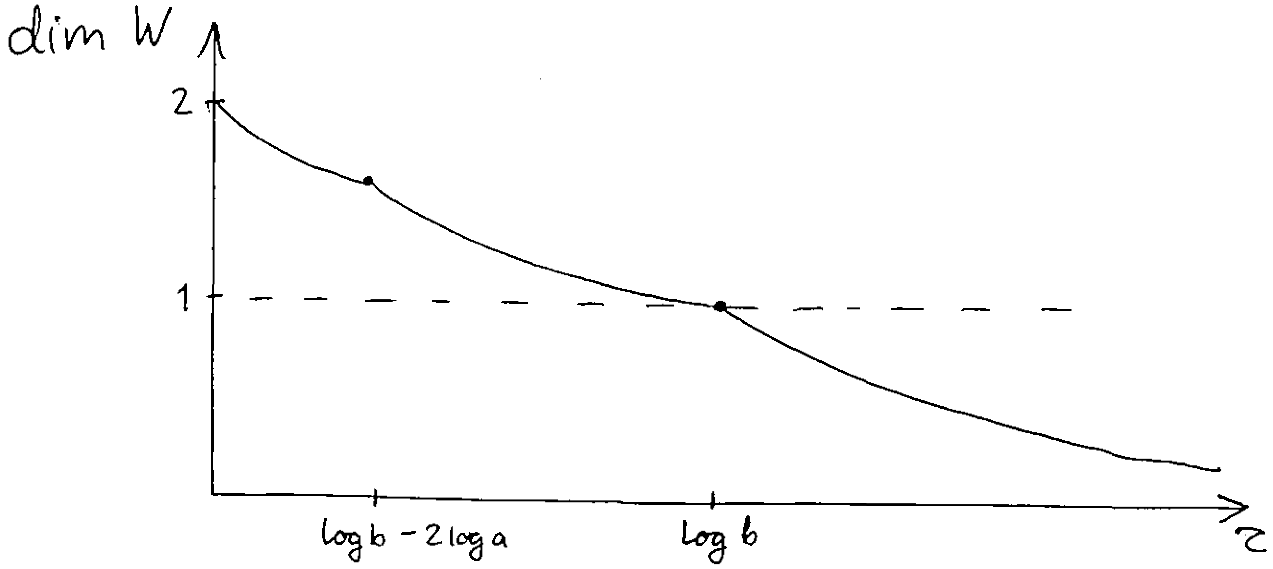
$$\dim W = \min_{i=1,\ldots,d} \left\{ \frac{i \log e_i - \sum_{j \,:\, e_j > e_i e^{\tau}} (\log e_j - \log e_i - \tau) + \sum_{j>i} \log e_j}{\tau + \log e_i} \right\}. \tag{4}$$

In fact the methods we use show that for any T we always have

$$\min_{i=1,\ldots,d} \frac{i \log e_i - \sum_{j\,:\,e_j > e_i e^\tau} (\log e_j - \log e_i - \tau) + \sum_{j>i} \log e_j}{\tau + \log e_i} \leq \dim W,$$

$$\dim W \leq \min_{i=1,\ldots,d} \frac{i \log e_i + \sum_{j>i} \log e_j}{\tau + \log e_i}.$$

To show what this looks like we give an example. Suppose $T = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ with $|a|^2 \leq |b|$ and $a, b \in \mathbb{Z} \setminus \{0\}$. Then the graph of $\dim W$ against τ is as follows.



This paper this the first step towards obtaining similar kinds of results for pseudo-Anasov diffeomorphisms. In other papers [4, 5] we have considered this and related questions for expanding rational maps acting on their Julia sets and in a forthcoming paper [6] we shall describe a partial solution for Markov maps of the interval (including the case of infinite Markov partitions). An analogous problem for geodesic flows on surfaces of constant negative curvature has been handled in [2, 7, 8, 9, 11, 12]. In the special case of the surface $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ (where $\mathrm{SL}_2(\mathbb{Z})$ acts on the upper half plane $\mathbb{H}$ by fractional linear transformations) this reduces to a theorem of Jarník and Besicovitch describing the Hausdorff dimension of the classical set of well approximable real numbers (see [10]).

The main difficulty in this paper is that W is a limsup set of a collection of subsets of X which are not close to being circular. There is therefore no "obvious" cover of W by balls as there is in the case of rational maps or of maps of the interval. In fact the notches in the graph of $\dim W$ against τ are consequences of a change in the kind of cover used.

Notation. To simplify notation the symbols $\ll$ and $\gg$ will be used to indicate an inequality with an unspecified positive multiplicative constant. If $a \ll b$ and $a \gg b$ both hold, then we write $a \asymp b$ and say that the quantities a and b are comparable. Similarly we shall write $a = O(b)$ if $a \ll b$ and $a = o(b)$ if a/b tends to zero. For $z \in X$ and $r > 0$ we shall write $B(z,r)$ for the ball with centre z and radius r (with respect to the usual metric on $\mathbb{R}^d/\mathbb{Z}^d$). The set of non-negative real numbers will be written $\mathbb{R}^{\geq 0}$. For a real number x we shall write $[x]$ for the largest integer less than or equal to x. The cardinality of a finite set S will be written $\#S$.

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1.3 Hausdorff Measure and Dimension

The Hausdorff dimension of a metric space X is an aspect of the size of X which can discriminate between sets of Lebesgue measure zero. The upper bounds on the Hausdorff dimensions of the sets of well approximable points will follow from the definition of this dimension, which we include in order to establish some notation.

The diameter $\sup\{|\mathbf{x} - \mathbf{y}| : \mathbf{x}, \mathbf{y} \in V\}$ of a non-empty subset V of a metric space will be denoted by $d(V)$. A collection $\{V_i\}$ such that $d(V_i) \leq \rho$ for each i and $X \subset \bigcup_i V_i$ is called a ρ-cover of X.

Let s be a non-negative number and for any positive ρ define,

$$\mathcal{H}^s_\rho(X) := \inf\left\{\sum_{i=1}^{\infty} d(V_i)^s \ : \{V_i\} \text{ is a countable } \rho\text{--cover of } X\right\}.$$

The *s-dimensional Hausdorff measure* $\mathcal{H}^s(X)$ of X is defined by

$$\mathcal{H}^s(X) := \lim_{\rho\to 0} \mathcal{H}^s_\rho(X) = \sup_{\rho>0} \mathcal{H}^s_\rho(X)$$

and the *Hausdorff dimension* $\dim X$ of X by

$$\dim X := \inf\{s : \mathcal{H}^s(X) = 0\} = \sup\{s : \mathcal{H}^s(X) = \infty\}.$$

Further details and alternative definitions of Hausdorff measure and dimension can be found in [3].

In order to produce an upper bound $\dim \leq s$ on the Hausdorff dimension of a given set it is sufficient to exhibit covers of the set, and to prove convergence of the sum of the diameters raised to the power s. Producing lower bounds on Hausdorff dimensions is not as easy. We shall use the following classical lemma.

Lemma 1 (Mass Distribution Principle) *Let W be a metric space with a Borel probability measure μ. Suppose there are constants $r_o, s, C > 0$ such that for all $x \in W$, $0 < r < r_o$ one has*

$$\mu(B(x,r)) < C \cdot r^s.$$

Then the following holds

$$\dim W \geq s.$$

Proof. Suppose one has a ρ-cover $\{V_i\}$ of W with $\rho < r_o$. Then one has $\sum_i d(V_i)^s \geq \sum_i 2C^{-1}\mu(V_i) \geq 2C^{-1}\mu(W) > 0$. Therefore $\mathcal{H}^s(W) > 0$, which implies the lower bound on the dimension.

2 Geometry of Misshaped Balls

In the proofs of Theorems 1 and 2 we shall be interested in what we will call misshaped balls. For example let D be the unit ball centred at the origin in $\mathbb{R}^d$. Then a misshaped ball is something of the form $\mathcal{A}D$ where $\mathcal{A}$ is an affine transformation of $\mathbb{R}^d$ (ie. a linear bijection composed with a translation).

Lemma 2 *Let T be a $d \times d$ matrix all of whose eigenvalues have absolute value 1. Then the entries of T^n are bounded by a polynomial in n. In particular there are $\alpha_1, C_1 \in \mathbb{R}^{\geq 0}$ depending on T, such that for all $n \in \mathbb{N}$,*

$$B_0(C_1^{-1}n^{-\alpha_1}) \subset T^n(D) \subset B_0(C_1 n^{\alpha_1})$$

and $T^n D$ can be covered by $C_1 n^{\alpha_1 d}$ balls of radius $n^{-\alpha_1}$.

Proof. It is sufficient to show that the matrix entries of T^n are bounded by a polynomial in n. Assume without loss of generality that T is in Jordan canonical form. One can then show by induction that T^n satisfies

$$T^n = \begin{pmatrix} O(1) & O(n) & O(n^2) & \dots & O(n^{d-1}) \\ 0 & O(1) & O(n) & \dots & O(n^{d-2}) \\ 0 & 0 & O(1) & \dots & O(n^{d-3}) \\ \vdots & & & \ddots & \vdots \\ 0 & & & & O(1) \end{pmatrix}.$$

Lemma 3 (Decomposition Lemma) *Let T be a real, non-singular $d \times d$ matrix. Then there is an expression*

$$T = T_1 \cdot T_2$$

such that all the eigenvalues of T_1 have absolute value 1; T_2 is diagonalizable over $\mathbb{R}$, and T_1 and T_2 commute.

Proof. To prove this one decomposes $\mathbb{R}^d$ into irreducible T-subspaces, and then proves the lemma independently for any such subspace. Assuming V to be (real-) irreducible under the action of T, it follows that all eigenvalues of T in V have the same absolute value. We shall call this absolute value t. Now let T_2 be scalar multiplication on V by t, and let $T_1 := T \cdot T_2^{-1}$. Then T_1 and T_2 satisfy the lemma.

Lemma 4 (Covering/Squeezing Lemma) *Let T be a non-singular real matrix. There are constants $C_2, \alpha_2 \in \mathbb{R}^{\geq 0}$ depending on T with the following property. Let $e_1 \leq \ldots \leq e_d$ be the absolute values of the eigenvalues of T counting multiplicity. Then for $r > 0$ and $n \in \mathbb{N}$, $T^n(D)$ can be covered by*

$$C_2 n^{\alpha_2 d} \prod_{j \,:\, e_j^n > r n^{\alpha_2}} \frac{e_j^n}{r n^{\alpha_1}}$$

balls of radius r. Furthermore if $r < C_2^{-1} n^{-\alpha_2} e_1^n$ then $T^n(D)$ contains a collection of

$$C_2^{-1} r^{-d} |\det T|^n$$

disjoint balls of radius r.

Proof. We begin by decomposing $T = T_1 \cdot T_2$ by the previous lemma, where all the eigenvalues of T_1 have absolute value 1; T_2 is diagonalizable over $\mathbb{R}$, and T_1 and T_2 commute. The eigenvalues of T_2 are $e_1, \ldots, e_d$. Now note that since T_1 and T_2 commute we have

$$T^n = T_1^n T_2^n.$$

We assume without loss of genarality that T_2 is diagonal. Then $T_2^n(D)$ is contained in a rectangle whose sides have lengths $2e_1^n, \ldots, 2e_d^n$. From this we see that $T_2^n(D)$ can be covered by

$$\prod_{j \,:\, 2e_j^n > r n^{\alpha_1}} \left\lceil \frac{4e_j^n}{r n^{\alpha_1}} \right\rceil \ll \prod_{j \,:\, e_j^n > r} \frac{e_j^n}{r n^{\alpha_1}}$$

balls of radius $r n^{\alpha_1}$.

Now let B be any ball of radius $r n^{\alpha_1}$. By Lemma 2, $T_1^n(B)$ can be covered by $C_1 n^{\alpha_1 d}$ balls of radius r. Therefore $T_1^n(T_2^n(D))$ can be covered by

$$C_2 n^{\alpha_1 d} \prod_{j \,:\, e_j > r} \frac{e_j^n}{r n^{\alpha_1}}$$

balls of radius r, with a suitably chosen C_2. This proves the first part of the lemma.

For the second part let $r < C_2^{-1} n^{-\alpha} e_1^n$. Note that $T_2^n(D)$ contains a rectangle whose sides have lengths $e_1^n, \ldots, e_d^n$. Therefore $T_2^n(D)$ contains a collection of

$$C_2^{-1} n^{-\alpha_1 d} |\det T|^n r^{-d}$$

disjoint balls of radius $C_1 n^{\alpha_1} r$, where C_1 is as in Lemma 2 and C_2 is suitably chosen. Then by the previous lemma, each of these, when transformed by T_1^n will contain $\gg n^{\alpha_1 d}$ balls of radius r. This proves the second part of the lemma.

We shall also need the following, which can be thought of as a local counting result (see [9, 5]).

Lemma 5 (Local Counting Result) *Let T be a non-singular matrix transformation of $X = \mathbb{R}^d/\mathbb{Z}^d$ and let $e_1 \leq \ldots \leq e_d$ be the absolute values of the eigenvalues of T counting multiplicity. Then there are constants $C_3, \alpha_3 > 0$ with the following property. For any ball $B = B(x, r)$ in X and any $n \in \mathbb{N}$, one has*

$$\#\{y \in B : T^n(y) = z_n\} \leq C_3 n^{\alpha_3 d} \prod_{j: re_j^n > n^{\alpha_3}} \frac{re_j^n}{n^{\alpha_3}}.$$

Furthermore if $r > C_3 n^{\alpha_3} e_1^{-n}$ then one has

$$C_3^{-1} r^d |\det T|^n \leq \#\{y \in B : T^n(y) = z_n\} \leq C_3 r^d |\det T|^n.$$

Proof. We transfer the problem to $\mathbb{R}^d$, where it is more easily dealt with. Let $\tilde{B}$ be a lift of B in $\mathbb{R}^d$, ie. $\tilde{B} = B(\tilde{x}, r)$ where $\tilde{x} \in \mathbb{R}^d$ projects onto x. Furthermore choose $\tilde{z}_n \in \mathbb{R}^d$ which projects onto z_n. Then one has

$$\#\{y \in B : T^n(y) = z_n\} = \#\{y \in \tilde{B} : T^n(y) - \tilde{z}_n \in \mathbb{Z}^d\} = \#\left((T^n(\tilde{B}) - \tilde{z}_n) \cap \mathbb{Z}^d\right).$$

Our notation means that $T^n(\tilde{B}) - \tilde{z}_n = \{T^n(y) - \tilde{z}_n : y \in \tilde{B}\}$. By the previous lemma, $T^n(\tilde{B}) - \tilde{z}_n$ can be covered by $C_2 n^{\alpha_1 d} \prod_{j: re_j^n > \frac{1}{2} n^{\alpha_1}} 2re_j^n n^{-\alpha_1}$ balls of radius $1/2$, each of which may contain at most one point of $\mathbb{Z}^d$. This proves the first part of the lemma. The other half is proved using the "squeezing" part of the Covering/Squeezing Lemma.

By modifying the arguments in the proof of Lemma 4 we can prove the following.

Lemma 6 *Let T be a non-singular matrix transformation of $X = \mathbb{R}^d/\mathbb{Z}^d$ and let $e_1 \leq \ldots \leq e_d$ be be absolute values of the eigenvalues of T counting multiplicity. Then there are constants C_4, α_4 depending only on T with the following property. For any ball $B \subset \mathbb{R}^d$ of radius r and any $0 < s \leq r$, the intersection $B \cap T^n(D)$ can be covered by*

$$C_4 n^{\alpha_4 d} \prod_{j \,:\, e_j^n > r} \frac{r}{s} \prod_{j \,:\, s < e_j^n < r} \frac{e_j^n}{s}.$$

balls of radius $sn^{-\alpha_4}$.

Corollary 1 *Let T be a non-singular matrix transformation of $X = \mathbb{R}^d/\mathbb{Z}^d$ and let $e_1 \leq \ldots \leq e_d$ be the absolute values of the eigenvalues of T counting multiplicity. Then there is a constants C_5 depending only on T with the following property. For any ball $B \subset \mathbb{R}^d$ of radius r one has*

$$m(B \cap T^n(D)) \leq C_5 r^d \prod_{j \,:\, e_j^n < r} \frac{e_j^n}{r}.$$

Proof. Let s tend to zero in the previous lemma and set $C_5 = C_4 m(D)$.

Now let $C = \max\{C_1, C_2, C_3, C_4, C_5\}$ and $\alpha = \max\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$. Each lemma continues to hold with C in place of C_i and α in place of α_i, so from now on we shall save on notation by just writing C and α instead of C_i and α_i.

3 Proof of Theorem 1

Let $T : X \to X$ be a matrix transformation of a torus as described above. Suppose that $e_1, \ldots, e_d$ are the absolute values of the eigenvalues of T listed in ascending order counting multiplicity. Let τ be as in (2) and suppose that it satisfies the condition $\tau > \log(e_d/e_1)$. Define

$$\delta_i := \frac{i \log e_i + \sum_{j=i+1}^{d} \log e_j}{\tau + \log e_i}, \qquad \delta := \min_{i=1,\ldots,d} \delta_i.$$

We begin by proving in §3.1 the upper bound $\dim W \leq \delta$. After that we shall describe in §3.2 the lower bound $\dim W \geq \delta$.

3.1 The Upper Bound

We fix an $i = 1, \ldots, d$. For this i we show that $\dim W \leq \delta_i$. Since i is arbitrary the upper bound follows. Our technique will be to find a cover of W, and to show that the sum of the diameters of the elements of the cover raised to the power $\delta_i + \epsilon$ converges.

Choose $\rho > 0$. We shall describe a ρ-cover of W. Let $N \in \mathbb{N}$ be sufficiently large so that one has

$$\mathrm{rad}(N) e_1^{-N} < \rho.$$

Here we have used the condition that $\tau \geq -\log e_1$. Let D be the unit ball in $\mathbb{R}^d$ and let $\tilde{P}(n) \subset \mathbb{R}^d$ be given by

$$\tilde{P}(n) := \mathrm{rad}(n) T^{-n} D.$$

This will be a small, misshaped ball at the origin. Now denote by $P(n)$ its projection in $X = \mathbb{R}^d/\mathbb{Z}^d$. Finally, for any $y \in X$ we shall write $P_y(n)$ for the translation by y of $P(n)$. Thus $P_y(n)$ will be a small misshaped ball in X around the point y. Then the preimage $T^{-n} B(z_n, \mathrm{rad}(n))$ is given by

$$T^{-n} B(z_n, \mathrm{rad}(n)) = \bigcup_{y \,:\, T^n y = z_n} P_y(n).$$

It therefore follows that

$$W \subset \bigcup_{n \geq N} \; \bigcup_{y \,:\, T^n y = z_n} P_y(n).$$

This is a cover of W. However it is not necessarily the one that we want. It may be necessary to take a finer cover. To do this we first cover $\tilde{P}(n)$ by balls of radius $e_i^{-n} \mathrm{rad}(n)$, centred at points in some finite set $S(n)$:

$$\tilde{P}(n) \subset \bigcup_{z \in S(n)} B_z(e_i^{-n} \mathrm{rad}(n)).$$

Note that this is a ρ-cover. We therefore have a corresponding ρ-cover of W:

$$W \subset \bigcup_{n \geq N} \; \bigcup_{y \,:\, T^n y = z_n} \; \bigcup_{z \in S(n)} B_{y+z}(e_i^{-n} \mathrm{rad}(n)).$$

If e_i is the smallest eigenvalue then $P(n)$ itself has diameter only $2e_i^{-n}\mathrm{rad}(n)$, and therefore $S(n)$ can be taken to have only one element. In general by Lemma 4 the cardinality of $S(n)$ can be bounded as follows:

$$\#S(n) \ll n^{\alpha} \prod_{j=1}^{i} (e_i/e_j)^n.$$

Using this we may now bound the s-dimensional Hausdorff measure of W for $s \in \mathbb{R}^{\geq 0}$:

$$\begin{aligned}
\mathcal{H}^s_\rho(W) &\leq \sum_{n\geq N} \sum_{y\,:\,T^n y = z_n} \sum_{z\in S(n)} \left(e_i^{-n}\mathrm{rad}(n)\right)^s \\
&\leq \sum_{n\geq N} \sum_{y\,:\,T^n y = z_n} \#S(n)\left(e_i^{-n}\mathrm{rad}(n)\right)^s \\
&\ll \sum_{n\geq N} |\det T|^n \left(e_i^{-n}\mathrm{rad}(n)\right)^s n^{\alpha} \prod_{j=1}^{i}(e_i/e_j)^n
\end{aligned}$$

This converges as long as

$$|\det T|(e_i^{-1}e^{-\tau})^s \prod_{j=1}^{i} \frac{e_i}{e_j} < 1,$$

which is equivalent to the condition

$$s > \frac{\log|\det T| - \sum_{j\leq i}\log e_j + i\log e_i}{\tau + \log e_i}.$$

Using the fact that $\log|\det T| = \sum_{j=1}^{d} \log e_j$, this reduces to

$$s > \frac{\sum_{j>i}\log e_j + i\log e_i}{\tau + \log e_i} = \delta_i.$$

Thus for $s > \delta_i$ we have

$$\mathcal{H}^s_\rho(W) \leq \sum_{n=1}^{\infty} |\det T|^n \left(e_i^{-n}\mathrm{rad}(n)\right)^s n^{\alpha} \prod_{j=1}^{i}(e_i/e_j)^n$$

However the right hand side of this is independent of ρ. We therefore have

$$\mathcal{H}^s(W) \leq \sum_{n=1}^{\infty} |\det T|^n \left(e_i^{-n}\mathrm{rad}(n)\right)^s n^{\alpha} \prod_{j=1}^{i}(e_i/e_j)^n < \infty.$$

This implies $\dim W \leq \delta_i$.

3.2 The Lower Bound

To prove the lower bound we construct a "Cantor-like" subset K of W and a measure μ on K. We then show that for any $\epsilon > 0$ there is an $r_0 > 0$ such that for all $x \in K$, $r < r_0$ one has

$$\mu(B_x(r)) < r^{\delta - \epsilon}.$$

This will imply Theorem 1 via the Mass Distribution Principle (Lemma 1). Note that the above inequality is equivalent to the condition

$$\exists i \in \{1, \ldots, d\} \text{ such that } \mu(B_x(r)) < r^{\delta_i - \epsilon}.$$

3.2.1 The Cantor Subset

The Cantor subset $K \subset W$ will be defined to be the intersection of sets $K(l)$, $l \in \mathbb{N}$, where one has $K(l+1) \subset K(l)$ for each l. We begin by defining $K(1) := X$. Then $K(l)$ is defined recursively by the formula

$$K(l+1) = \bigcup_{y \in \mathbb{I}(l+1)} P_y(N(l+1)).$$

Here $N(l)$ is a rapidly increasing sequence of natural numbers and the union is taken over all y in the set

$$\mathbb{I}(l+1) := \{y \in X : T^{N(l+1)} y = z_{N(l+1)} \text{ and } P_y(N(l+1)) \subset K(l)\}.$$

The subsequence $N(l)$ will be chosen so as to satisfy the following three conditions:

1. $$e^{-(\tau + o(1))N(l)} < \mathrm{rad}(N(l)) < e^{-(\tau - o(1))N(l)},$$

   by which we mean that
   $$\lim_{l \to \infty} \frac{-\log \mathrm{rad}(N(l))}{N(l)} = \tau.$$

2. For all l one has

   $$e_1^{-N(l)} < \frac{1}{2} \mathrm{rad}(N(l-1)) e_d^{-N(l-1)} N(l-1)^{-\alpha}.$$

3. For all l one has

   $$N(l) > \exp\left(\sum_{j=1}^{l-1} N(j)\right).$$

3.2.2 The Measure

We construct the measure μ as a limit of measures μ_l on the sets $K(l)$. From this it follows imediately that μ is supported on K. The set $K(l)$ is made up of misshaped balls $P_y(N(l))$. We give each of these a weight $\mu(y,l)$ given recursively by the formulae

$$\mu(y,l+1)) := \frac{\mu(z,l)}{\#(\mathbb{I}(l+1)\cap P_z(N(l)))}, \qquad \mu(y,1) := 1, \tag{5}$$

in which z is the unique element of $\mathbb{I}(l)$ satisfying $y \in P_z(N(l))$. We define μ_l on $P_y(N(l))$ to be distributed like Lebesgue measure but with $\mu_l(P_y(N(l))) = \mu(y,l)$.

From the way that we've set things up it follows that $\mu_l(P_y(N(l))) = \mu_{l+n}(P_y(N(l)))$ for all $n \in \mathbb{N}$. Since the $P_y(N(l))$ generate the sigma algebra of K, this implies that the measures μ_l converge to a limit μ on K, which is what we want.

The definition of the numbers $\mu(y,l)$ is rather uninformative, so we deduce a more useful approximation of these numbers.

Lemma 7 *As $l \to \infty$ one has*

$$\log\mu(y,l) = -N(l)\log|\det T| + \tau d\sum_{i=1}^{l-1} N(i) + O(l)$$

Proof. We shall prove this by induction. It certainly holds for $l = 1$. Now suppose it holds for l. From the recursive definition of $\mu(y,l+1)$ we have

$$\log\mu(y,l+1) = \log\mu(z,l) - \log\#\Big(\mathbb{I}(l+1)\cap P_z(N(l))\Big).$$

The inductive hypothesis implies

$$\log\mu(y,l+1) = -N(l)\log|\det T| + \tau d\sum_{i=1}^{l-1} N(i) + O(l) - \log\#\Big(\mathbb{I}(l+1)\cap P_z(N(l))\Big). \tag{6}$$

We must estimate $\#\Big(\mathbb{I}(l+1)\cap P_z(N(l))\Big)$. By Lemma 4, $P_z(N(l))$ contains a collection of $e_d^{dN(l)}N(l)^{\alpha d}|\det T|^{N(l)}$ disjoint balls of radius $C^{-1}N(l)^{-\alpha}e_d^{-N(l)}\mathrm{rad}(N(l))$, and can be covered by a comparable number of such balls. By Lemma 5 and the second condition on sequence $N(l)$, any ball with this radius which is contained in $P_z(N(l))$ must contain $\asymp N(l)^{-d\alpha}\mathrm{rad}(N(l))^d e_d^{-N(l)d}|\det T|^{N(l+1)}$ points of $\mathbb{I}(l+1)$. We therefore have (after some cancellation)

$$\log\#\Big(\mathbb{I}(l+1)\cap P_z(N(l))\Big) = (N(l+1) - N(l))\log|\det T| + d\log\mathrm{rad}(N(l)) + O(1) \tag{7}$$

Putting formulae (6) and (7) together we get

$$\begin{aligned}\log\mu(y,l+1) &= -N(l)\log|\det T| + \tau d\sum_{i=1}^{l-1} N(i) + O(l)\\ &\qquad -(N(l+1) - N(l))\log|\det T| - d\log\mathrm{rad}(N(l)) + O(1),\\ &= -N(l+1)\log|\det T| + O(l+1)\end{aligned}$$

This proves the lemma.

3.2.3 The Sting

We now choose any ball $B = B_x(r)$ centred on a point of K and find an upper bound on $\mu(B)$. This is trickier than in previous calculations of the same kind of thing, since the $P(n)$ are far from being round. We start by introducing some notation.

Since x is in K, we know that $x \in K(l)$ for all $l \in \mathbb{N}$. Thus for each l there is a unique $y \in \mathbb{I}(l)$ such that $x \in P_y(N(l))$. We shall refer to this unique y as $y(l)$. We shall also use the abbreviation $Q_l := P_{y(l)}(N(l))$. Note that one has

$$Q_l \supset Q_{l+1} \supset \ldots \ni x.$$

There is one particular value of l which will be of special interest to us. Let ℓ be the smallest l for which no ball of radius r is contained in Q_l. By the second condition on the sequence $N(l)$, the diameter of $P(N(\ell+1))$ is less than r.

As a starting point for our calculation of $\mu(B)$ we take

$$\mu(B) \leq \sum_{y \in \mathbb{I}(\ell+1)\,:\, P_y(N(\ell+1)) \cap B \neq \emptyset} \mu(y, \ell+1).$$

Since the diameter of $P_y(N(\ell+1))$ is less than r, we have

$$\mu(B) \leq \sum_{y \in \mathbb{I}(\ell+1) \cap 2B} \mu(y, \ell+1), \tag{8}$$

where $2B$ is the ball whose centre is that of B but whose radius is $2r$.

The point now is that since we are assuming $\tau + \log e_1 > \log e_d$, we may deduce that $2B$ does not intersect any $P_y(N(\ell))$ except Q_ℓ (actually this is only true for sufficiently large ℓ, but this is enough for our purposes). This means that each y arising in the above sum must be contained in $Q_\ell \cap 2B$. We therefore have

$$\mu(B) \leq \sum_{y \in \mathbb{I}(\ell+1) \cap 2B \cap Q_\ell} \mu(y, \ell+1).$$

The recurrence relation (5) now gives

$$\mu(B) \leq \frac{\#(\mathbb{I}(\ell+1) \cap 2B \cap Q_\ell)}{\#(\mathbb{I}(\ell+1) \cap Q_\ell)} \mu(y(\ell), \ell).$$

By Lemma 6, $2B \cap Q_\ell$ may be covered by

$$CN(\ell)^{\alpha d} \prod_{j\,:\, e_j^{-N(\ell)} \mathrm{rad}(N(l)) > r} \frac{r e_d^{N(\ell)}}{\mathrm{rad}(N(\ell))} \prod_{j\,:\, e_j^{-N(\ell)} \mathrm{rad}(N(l)) \leq r} \frac{e_d^{N(\ell)}}{e_j^{N(\ell)}}$$

balls of radius $N(\ell)^{-\alpha} \mathrm{rad}(N(\ell)) e_d^{-N(\ell)}$. Each of these can contain by Lemma 5 at most

$$CN(\ell)^{-\alpha d} \left(\mathrm{rad}(N(\ell)) e_d^{-N(\ell)}\right)^d |\det T|^{N(\ell+1)}$$

points of $\mathbb{I}(\ell+1)$. We therefore have

$$\begin{aligned}\#(\mathbb{I}(\ell+1)\cap 2B\cap Q_l) \quad &\ll \prod_{j\,:\,e_j^{-N(\ell)}\mathrm{rad}(N(\ell))>r} \frac{re_d^{N(\ell)}}{\mathrm{rad}(N(\ell))}\\ &\times \prod_{j\,:\,e_j^{-N(\ell)}\mathrm{rad}(N(\ell))\le r} \frac{e_d^{N(\ell)}}{e_j^{N(\ell)}}\\ &\times \left(\mathrm{rad}(N(\ell))e_d^{-N(\ell)}\right)^d |\det T|^{N(\ell+1)}.\end{aligned}$$

On the other hand, we have by a similar argument but squeezing rather than covering

$$\#(\mathbb{I}(\ell+1)\cap Q_l) \gg \left(\mathrm{rad}(N(\ell))e_d^{-N(\ell)}\right)^d |\det T|^{N(\ell+1)} \prod_{j=1}^{d} \frac{e_d^{N(\ell)}}{e_j^{N(\ell)}}.$$

The last two formulae combine to give

$$\frac{\#(\mathbb{I}(\ell+1)\cap 2B\cap Q_\ell)}{\#(\mathbb{I}(\ell+1)\cap Q_\ell)} \ll \prod_{j\,:\,e_j^{-N(\ell)}\mathrm{rad}(N(\ell))>r} re_j^{N(\ell)}\mathrm{rad}(N(\ell))^{-1}.$$

Substituting this into the above formula for $\mu(B)$ we obtain

$$\mu(B) \ll \mu(y(\ell),\ell) \prod_{j\,:\,e_j^{-N(\ell)}\mathrm{rad}(N(\ell))>r} \frac{re_j^{N(\ell)}}{\mathrm{rad}(N(\ell))}. \tag{9}$$

We remark that this can be reformulated as follows:

$$\mu(B) \ll \frac{m(Q_\ell\cap 2B)}{m(Q_\ell)}\mu(y(\ell),\ell).$$

This formula has analogues in our other papers (see especially [5]) in which Lebesgue measure m is replaced by a suitable conformal measure. The result now follows with just a bit more calculus.

3.2.4 The Calculus

We shall now fix a value of i. If $r < e_d^{-N(\ell)}\mathrm{rad}(N(\ell))$ then let $i := d$. Otherwise, choose i so as to satisfy

$$e_{i+1}^{-N(\ell)}\mathrm{rad}(N(\ell)) < r \le e_i^{-N(\ell)}\mathrm{rad}(N(\ell)).$$

We shall show that $\frac{\log\mu(B)}{\log r} \ge \delta_i + o(1)$.

First note that by 9, we have

$$\mu(B) \ll \mu(y(\ell),\ell)\prod_{j=1}^{i} \frac{re_j^{N(\ell)}}{\mathrm{rad}(N(\ell))}.$$

Taking the logarithm of this, we obtain

$$\log \mu(B) \leq \log \mu(y(\ell), \ell) + \sum_{j=1}^{i} \Big(\log r + N(\ell) \log e_j - \log \mathrm{rad}(N(\ell)) \Big) + O(1).$$

Applying the estimate of Lemma 7 we get

$$\log \mu(B) \leq N(\ell) \log |\det T| + \sum_{j=1}^{i} \Big(\log r + N(\ell) \log e_j - \log \mathrm{rad}(N(\ell)) \Big) + o(N(\ell)).$$

By the first condition on the sequence $N(l)$, we may replace $\log \mathrm{rad}(N(\ell))$ by $-N(\ell)(\tau + o(1))$:

$$\log \mu(B) \leq N(\ell) \log |\det T| + \sum_{j=1}^{i} (\log r + N(\ell)(\log e_j + \tau)) + o(N(\ell)).$$

Dividing by the negative number $\log r$ gives

$$\frac{\log \mu(B)}{\log r} \geq i + \frac{N(\ell)}{\log r} \left(\log |\det T| + \sum_{j=1}^{i} \log e_j + \tau \right) + o\left(\frac{N(\ell)}{\log r} \right).$$

By our choice of i, we have

$$0 \geq \frac{N(\ell)}{\log r} \geq \frac{-1}{\tau + \log e_i}.$$

The last two formulae give

$$\frac{\log \mu(B)}{\log r} \geq i - \frac{1}{\tau + \log e_i} \left(\log |\det T| + \sum_{j=1}^{i} \log e_j + \tau \right) + o(1),$$

which we can tidy up to obtain

$$\frac{\log \mu(B)}{\log r} \geq \delta_i + o(1), \tag{10}$$

3.2.5 The Error Term

We now handle the error term $o(1)$. This tends to zero as $\ell \to \infty$. On the other hand, by making r very small we can ensure that ℓ is very big. Thus for any $\epsilon > 0$ there is an $r_0 > 0$ such that for all $r < r_0$ the error term is less than ϵ. Assume $r < r_o$. Then by (10) there is some i such that

$$\mu(B) \leq r^{\delta_i - \epsilon}$$

Thus we always have

$$\mu(B) \leq r^{\delta - \epsilon}$$

This implies by the Mass Distribution Principle (letting ϵ tend to zero) that

$$\dim W \geq \delta.$$

4 Proof of Theorem 2

As before $T : X \to X$ is a matrix transformation of a torus. Assume in addition that T is diagonalizable over $\mathbb{Q}$. Suppose that $e_1, \dots, e_d \in \mathbb{Z}$ are the absolute values of the eigenvalues of T, listed in ascending order. Let τ be as in (2) and define

$$\delta_i := \frac{i \log e_i - \sum_{j\,:\,e_j > e_i e^\tau} (\log e_j - \log e_i - \tau) + \sum_{j>i} \log e_j}{\tau + \log e_i}, \qquad \delta := \min_{i=1,\dots,d} \delta_i.$$

We begin by proving the upper bound $\dim W \leq \delta$. After that we shall describe the lower bound $\dim W \geq \delta$.

We shall require the following refinemint of Lemma 5.

Lemma 8 *Let $B = B(x,r)$, $r < 1$. Then*

$$\#\{y \in B : T^n y = z_n\} \ll \prod_{j\,:\,e_j^{-n} < r} r e_j^n.$$

If in addition one assumes that $T^n x = z_n$ then one has

$$\#\{y \in B : T^n x = z_n\} \gg \prod_{j\,:\,e_j^{-n} < r} r e_j^n.$$

The first part of the lemma will be used for the lower bound and the second part for the upper bound.

Proof. Let $v_1, \dots, v_d \in \mathbb{Z}^d$ be eigenvectors of T corresponding to the eigenvalues $e_1, \dots, e_d$. These eigenvectors span a sublattice L of $\mathbb{Z}^n$ of finite index $\mathrm{ind} := [\mathbb{Z}^d : L]$.

For $v \in X$ consider the parallelopiped

$$\mathcal{P}(v; e_1^{-n} v_1, \dots, x_n e_d^{-n} v_d) := \{v + x_1 e_1^{-n} v_1 + \dots + x_n e_d^{-n} v_d : \forall i,\ 0 \leq x_i < 1\} \subset X.$$

Each such parallelopiped contains exactly ind points y satisfying $T^n y = z_n$. To obtain the upper bound one covers B by these parallelopipeds and estimates the number of parallelopipeds required in such a cover.

For the lower bound note that there is a constant $c > 0$ depending only on $v_1, \dots, v_d$ such that

$$\mathcal{P}(x; cv_1 r, \dots, cv_d r) \subset B(x,r).$$

We fix such a c. Now let L' be the subgroup of $\mathbb{R}^d$ generated by those $e_j^{-n} v_j$ for which $e_j^{-n} < cr$. One has

$$S := \{x + l : l \in L'\} \subset \{y : T^n y = z_n\}$$

The lower bound will therefore follow from a lower bound for $\#(S \cap \mathcal{P}(x; cv_1 r, \dots, cv_d r))$. The latter is easily seen to be at least $\prod_{j : e_j^{-n} < cr} [cre_j^n]$, where the square bracket notation indicates the integral part. This proves the lemma.

4.1 The Upper Bound

We proceed very much as in §2.1. We fix an i and show that

$$\dim W \leq \delta_i.$$

Since this is true for any i we have $\dim W \leq \delta$. Choose $\rho > 0$. We shall describe a ρ-cover of W. Let $N \in \mathbb{N}$ be sufficiently large so that one has

$$\mathrm{rad}(N) e_i^{-N} < \rho.$$

Here we have used the fact that $e_i \geq 1$. We define $\tilde{P}(n)$, $P(n)$ and $P_y(n)$ as in §2.1. Then the preimage $T^{-n}B(z_n, \mathrm{rad}(n))$ is given by

$$T^{-n}B(z_n, \mathrm{rad}(n)) = \bigcup_{y \,:\, T^n y = z_n} P_y(n).$$

It therefore follows that

$$W \subset \bigcup_{n \geq N} \bigcup_{y \,:\, T^n y = z_n} P_y(n).$$

As before we shall refine this cover. To do this we first cover $\tilde{P}(n)$ by balls of radius $e_i^{-n}\mathrm{rad}(n)$, centred at points in a finite set $S(n)$:

$$\tilde{P}(n) \subset \bigcup_{z \in S(n)} B_z(e_i^{-n}\mathrm{rad}(n)).$$

We therefore have a corresponding cover of W:

$$W \subset \bigcup_{n \geq N} \bigcup_{y \,:\, T^n y = z_n} \bigcup_{z \in S(n)} B_{y+z}(e_i^{-n}\mathrm{rad}(n)).$$

As in §2.1 we have

$$\#S(n) \ll n^{\alpha} \prod_{j \,:\, e_j < e_i} (e_i/e_j)^n.$$

The problem with this new cover is that there will be a lot of unnecessary overlapping. To be more precise, the inner union

$$\bigcup_{z \in S(n)} B_{y+z}(e_i^{-n}\mathrm{rad}(n))$$

may cover not just $P_y(n)$, but also several other $P_w(n)$, as long as the distance $|y - w|$ is less than $e_i^{-n}\mathrm{rad}(n)$. Thus by Lemma 8 the number of $P_w(n)$ covered is $\gg M(n)$, where

$$M(n) := \prod_{j \,:\, e_j^n \mathrm{rad}(n) > e_i^n} \frac{e_j^n \mathrm{rad}(n)}{e_i^n}. \tag{11}$$

After pruning out the unnecessary elements of the new cover we obtain the following bound on the Hausdorff measure of W.

$$\begin{aligned}\mathcal{H}^s_\rho(W) &\ll \sum_{n\geq N} M(n)^{-1} \sum_{y\,:\,T^n y = z_n} \sum_{z\in S(n)} (e_i^{-n}\mathrm{rad}(n))^s \\ &\ll \sum_{n\geq N} M(n)^{-1} \sum_{y\,:\,T^n y = z_n} \#S(n)(e_i^{-n}\mathrm{rad}(n))^s \\ &\ll \sum_{n\geq N} M(n)^{-1}|\det T|^n (e_i^{-n}e^{-n(\tau+o(1))})^s n^\alpha \prod_{j=1}^{i}(e_i/e_j)^n\end{aligned}$$

Let

$$M := \exp\left(\liminf_{n\to\infty}\frac{\log M(n)}{n}\right) = \prod_{j\,:\,e_j>e_ie^\tau}\frac{e_j}{e_ie^\tau}.$$

The above sum converges as long as

$$M^{-1}|\det T|(e_i^{-1}e^{-\tau})^s\prod_{j=1}^{i}\frac{e_i}{e_j} < 1,$$

which is equivalent to the condition

$$s > \frac{i\log e_i - \sum_{j\,:\,e_j>e_ie^\tau}(\log e_j - \log e_i - \tau) + \sum_{j>i}\log e_j}{\tau + \log e_i} = \delta_i.$$

We therefore have as in §3.1,

$$\dim W \leq \delta_i.$$

Since this holds for every i, we get

$$\dim W \leq \delta.$$

4.2 The Lower Bound

We construct the Cantor subset $K \subset W$ and the measure μ on K exactly as in §2.2. However the estimate on the measure will be different.

Choose any ball $B = B_x(r)$ centred on a point of K. We shall find an upper bound on $\mu(B)$. We define $y(l)$ and Q_l exactly as in the previous section. Again we shall be interested in ℓ, which as before is defined to be the smallest l for which no ball of radius r is contained in Q_l. This implies that $e_1^{-N(\ell+1)} < r$.

As a starting point for our calculation of $\mu(B)$ we take the formula (8):

$$\mu(B) \leq \sum_{y\in 2B\cap \mathbf{I}(\ell+1)} \mu(y,\ell+1).$$

This gives us analogously to (9)

$$\mu(B) \ll \frac{m(K(\ell)\cap 2B)}{m(K(\ell))}, \tag{12}$$

where m is Lebesgue measure. We now have to do some more calculus. First note that

$$m(K(\ell)) = m(Q_\ell)\#\mathbb{I}(l).$$

By Lemma 7 this gives us

$$m(K(\ell)) \asymp \mathrm{rad}(N(\ell))^d |\det T|^{-N(\ell)} |\det T|^{N(\ell)} \exp(o(N(\ell))) \asymp \mathrm{rad}(N(\ell))^d \exp(o(N(\ell))).$$

Together with (12) and the first condition on the sequence $N(l)$, this gives

$$\log \mu(B) \leq \log m(K(\ell) \cap 2B) + N(\ell)d\tau + o(N(\ell)). \tag{13}$$

We shall now fix a value of i. If $r < e_d^{-N(\ell)}\mathrm{rad}(N(\ell))$ then let $i := d$. Otherwise, choose i so as to satisfy

$$e_{i+1}^{-N(\ell)}\mathrm{rad}(N(\ell)) < r \leq e_i^{-N(\ell)}\mathrm{rad}(N(\ell)).$$

We then have for any $w \in X$ by Corollary 1

$$m(2B \cap P_w(N(\ell))) \ll r^i \prod_{j=i+1}^{d} e_i^{-N(\ell)}\mathrm{rad}(N(\ell)).$$

However, $2B$ will intersect other things in $K(\ell)$ than just Q_ℓ. Using Lemma 8 the number of pieces $P_y(N(\ell))$ of $K(\ell)$ which B may intersect is $\ll M'(N(l))$, where

$$M'(n) := \prod_{j \,:\, e_j^{-n} < r} r e_j^n.$$

Since $r \leq e_i^{-N(\ell)}\mathrm{rad}(N(\ell))$, we have $M'(N(\ell)) \leq M(N(\ell))$, where $M(n)$ is as defined in §3.1. We therefore have

$$\begin{aligned} m(2B \cap K(\ell)) &\ll M'(N(\ell))r^i \prod_{j=i+1}^{d} e_i^{-N(\ell)}\mathrm{rad}(N(\ell)) \\ &\ll M(N(\ell))r^i \prod_{j=i+1}^{d} e_i^{-N(\ell)}\mathrm{rad}(N(\ell)). \end{aligned}$$

Together with (13) and the first condition on the sequence $N(l)$ this implies

$$\log \mu(B) \leq \log M(N(\ell)) + i \log r - N(\ell) \sum_{j=i+1}^{d} (\tau + \log e_i) + N(\ell)d\tau + o(N(\ell)).$$

We therefore have

$$\frac{\log \mu(B)}{\log r} \geq \frac{\log M(N(\ell))}{\log r} + i + \frac{N(\ell)}{\log r}\left(i\tau - \sum_{j=i+1}^{d} \log e_i\right) + o\left(\frac{N(\ell)}{\log r}\right).$$

Substitutuing the definition (11) of $M(n)$, we get

$$\frac{\log \mu(B)}{\log r} \geq \frac{1}{\log r} \sum_{j\ :\ e_j^{N(\ell)}\mathrm{rad}(N(\ell))>e_i^{N(\ell)}} \Big(N(\ell)(\log e_j - \log e_i) + \log \mathrm{rad}(N(\ell))\Big)$$
$$+i + \frac{N(\ell)}{\log r}\left(i\tau - \sum_{j=i+1}^{d} \log e_i\right) + o\left(\frac{N(\ell)}{\log r}\right).$$

From our first condition on the subsequence $N(l)$ we get

$$\frac{\log \mu(B)}{\log r} \geq \frac{N(\ell)}{\log r} \sum_{j\ :\ e_j e^{-\tau}>e_i} \Big(\log e_j - \log e_i - \tau\Big)$$
$$+ i + \frac{N(\ell)}{\log r}\left(i\tau - \sum_{j=i+1}^{d} \log e_i\right) + o\left(\frac{N(\ell)}{\log r}\right).$$

Using the fact that $r \leq e_i^{-N(\ell)}\mathrm{rad}(N(\ell))$ this becomes

$$\frac{\log \mu(B)}{\log r} \geq i - \frac{1}{\tau + \log e_i}\left(i\tau - \sum_{j=i+1}^{d} \log e_i + \sum_{j\ :\ e_j>e_i e^{\tau}} (\log e_j - \log e_i - \tau)\right) + o(1),$$

which we can tidy up to obtain

$$\frac{\log \mu(B)}{\log r} \geq \delta_i + o(1). \tag{14}$$

4.2.1 The End of the Proof

The proof finishes like the proof of Theorem 1. By (14) we have for small emough r,

$$\mu(B) \leq r^{\delta-\epsilon}$$

This implies by the Mass Distribution Principle that

$$\dim W \geq \delta.$$

References

[1] J.W.S.Cassels : *An Introduction to Diophantine Approximation*, (C.U.P., 1957).

[2] M.M.Dodson, M.Melián, D.Pestana and S.L.Velani : "Patterson measure and Ubiquity" Mathematica Gottingensis Heft 3, Jan 1993 (preprint).

[3] K.J.Falconer : *Fractal Geometry - Mathematical Foundations and Applications*, (J. Wiley, Chichester. 1990).

[4] R.Hill and S.L.Velani : "The Ergodic Theory of Shrinking Targets" to appear in Invent.Math.

[5] R.Hill and S.L.Velani : "Diophantine Approximation in the Julia Sets of Expanding Rational Maps" preprint of the Max Planck Institute.

[6] R.Hill and S.L.Velani : "A Generalization of the Jarnik-Besicovitch Theorem" in preparation.

[7] R.Hill and S.L.Velani : "Diophantine Approximation in Kleinean groups of the second kind with cusps" in preparation.

[8] M.V.Melián and D.Pestana : "Geodesic excursions into cusps in finite volume hyperbolic manifolds" in Michigan Math.Journal. 40 (1993) 77-93.

[9] M.V.Melián and S.L.Velani : "Geodesic excursions into cusps in infinite volume hyperbolic manifolds" Goettingen Preprint (1993).

[10] V.G.Sprindžuk : *Metric theory of Diophantine approximation* (translated by R. A. Silverman), V. H. Winston & Sons, Washington D.C. (1979).

[11] S.L.Velani : "Diophantine approximation and Hausdorff dimension in Fuchsian groups" in Math.Proc.Camb.Philos.Soc. 113 (1993), 343-354.

[12] S.L.Velani : Doctoral Thesis, York 1991.

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