ON THE MODULAR EMBEDDIBGS

FOR BASIC P-EXTENSIONS

by

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1. Introduction

Let p be an odd prime number. We call a field F the basic p-extension if F is a cyclic extension of the rational number field Q of degree p and only p ramifies in F. By the class field theory, such a field F is uniquly determined as the subfield with the discriminant $p^{2(p-1)}$ of the cyclotomic field $Q(\zeta)$, where ζ is a primitive p^2th root of unity. Let F be the basic p-extension, o the ring of integers of F and g the galois group of F/Q. We fix a generator σ of g. As F is a totally real numer field, we consider Hilbert modular group $SL_2(o)$ over F, which acts on the product H_1^p of p copies of the upper half plane H_1 by the standard way. Now, according to Hammond [1], we call a couple (Ξ, E) consisting of a homomorphism Ξ of $SL_2(o)$ into Siegel modular group $Sp(2p,Z)$ of degree $2p$ over the ring Z of rational integers and a holomorphic map E of $H_1{}^p$ into the generalized Siegel upper half space H_p of degree p, on which $Sp(2p,Z)$ acts by the fractional transformation, a modular embedding for F if it satisfies the following properties : for every element g of $SL_2(o)$ and every point z of $H_1{}^p$,

(1) $\Xi(g)(E(z)) = E(g(z))$; (2) $j(g,z) = J(\Xi(g),E(z))$, where j and J are the standard automorphic factors of $SL_2(o)$ and $Sp(2p,Z)$, respectively(see the section 4).

In this paper we shall construct a modular embedding for the basic p-field F for each p explicitly. To obtain it, we put

$$\omega = Tr_{Q(\zeta)/F}(\zeta),$$

$$\omega_\mu = \omega^{\sigma^{\mu-1}} \quad (\mu=1,2,\cdots,p),$$

$$\Omega_\mu = \frac{1+\omega_\mu}{p} \quad (\mu=1,2,\cdots,p),$$

$$a = \Sigma_{\mu=1}^{p} Z\Omega_\mu ,$$

where $Tr_{K/k}$ denotes the trace of K over k for a field extension K/k. After studying the arithemetic of o in section 2, we can show in section 3 that a is a fractional ideal of F and $Tr_{F/Q}(\Omega_\mu\Omega_\nu)$

$= \delta_{\mu\nu}$, where

$$\delta_{\mu\nu} = \begin{cases} 1 & (\mu=\nu) ; \\ 0 & (\mu\neq\nu) . \end{cases}$$

Let us consider the regular representation ξ of F with respect to $\{ \Omega_1, \cdots, \Omega_p \}$. Then the above facts show that every element of $\xi(o)$ is a symmetric matrix over Z. Thus we obtain that Ξ is a homomorphism of $SL_2(o)$ into $Sp(2p,Z)$ if we put for each element g of $SL_2(o)$

$$\Xi(g) = \begin{pmatrix} \xi(\alpha) & \xi(\beta) \\ \xi(\gamma) & \xi(\delta) \end{pmatrix} \qquad (g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}).$$

Furthermore we can naturally get a map E of $H_1{}^p$ into H_p such that (Ξ, E) becomes a modular embedding for F.

We should note the following three remarks. Firstly, our method to construct the above modular embedding which comes from a certain representation of o by rational integral symmetric matrices is analogy to Hammond's one given in [1]. Next, for $p=3$, our result is the special case of Oka's result [4] where he constructed a modular embedding for arbitrary cyclic cubic fields. Finally, our homomorphism Ξ is of full Hilbert modular group $SL_2(o)$ into

Sp(2p,Z). On the other hand for each totally real number field Shimura gave in [5] a homomorphism of certain congruence subgroups of Hilbert modular group such that it is compatible with a imbedding between the spaces and the standard automorphic factors.

Notations. We use the following notations in this paper, adding notations used in section 1 :

For each set X , $|X|$ means the cardinarity of X . For galois extension K/k , $Gal(K/k)$ means galois group of K/k . For $a(\neq 0)$, $b \in Z$, we define $\delta_{a|b}$ by

$$\delta_{a|b} = \begin{cases} 1 & \text{if } b \equiv 0 \ (\mathrm{mod.}\, a), \\ 0 & \text{otherwise.} \end{cases}$$

For a ring R with unity, $R^{\times}$ means the multiplicative group consisting of all invertible elements of R , and we denote by $M(n,R)$ the total matrix ring over R and 1_n the unity of $M(n,R)$ for each positive integer n .

2. Arithemetic of basic p-extension

Let p be an odd prime number and F the basic p-extension (see section 1). We denote by g the galois group of F/Q and fix a generator σ of g. Let ζ be a primitive p^2 th root of unity and put

$$L = Q(\zeta), \quad G = \mathrm{Gal}(L/Q) \quad \text{and} \quad H = \mathrm{Gal}(L/F) .$$

Then we have a coset decomposition $G = \cup_{\mu=0}^{p-1} H\sigma^{\mu}$. We also denote by R and r the residue rings Z/p^2Z and Z/pZ , respectively, and put

$$N_p = \{ 1,2,\cdots,p-1 \} \quad , \quad N_p^+ = N_p \cup \{ p \} .$$

Then we obtain the natural projection π of R to r and the canonical group homomorphism ψ of G into $R^{\times}$ by the class field theory. For each element μ of N_p^+ we define a subset A_μ of R by

$$A_\mu = \{ \psi(h\sigma^{\mu-1}) \mid h\in H \}.$$

It is easily seen that $|A_\mu| = |\pi(A_\mu)| = p$ for each μ of N_p^+. For a positive integer m and $\mu_1,\cdots,\mu_m \in N_p^+$, we define three sets $X_0(\mu_1,\cdots,\mu_m)$, $X_1(\mu_1,\cdots,\mu_m)$ and $Y^{(m)}$ by

$$X_0(\mu_1,\cdots,\mu_m) = \{(x_1,\cdots,x_m) \mid x_i \in A_{\mu_i} (i=1,\cdots,m),\ \Sigma_{i=1}^m x_i^p = 0\},$$

$$X_1(\mu_1,\cdots,\mu_m) = \{(x_1,\cdots,x_m) \mid x_i \in A_{\mu_i} (i=1,\cdots,m),\ \Sigma_{i=1}^m x_i^p \notin R^\times\},$$

$$Y^{(m)} = \{(y_1,\cdots,y_m) \mid y_i \in r^\times (i=1,\cdots,m),\ \Sigma_{i=1}^m y_i = 0\}$$

and put

$$x_j(\mu_1,\cdots,\mu_m) = |X_j(\mu_1,\cdots,\mu_m)| \qquad (j=0,1),$$

$$y^{(m)} = |Y^{(m)}|.$$

We note that $x_1(\mu_1,\cdots,\mu_m) = y^{(m)}$.

Now we define ω as section 1 by

$$\omega = Tr_{L/F}(\zeta)$$

and put

$$\omega_\mu = \begin{cases} 1 & (\mu=0), \\ \omega^{\sigma^{\mu-1}} & (\mu \in N_p^+). \end{cases}$$

It is clear that ω_μ is an integer of F and $Tr_{F/Q}(\omega_\mu) = 0$ for each μ of N_p^+.

LEMMA 1 (1) For m elements $\mu_1,\cdots,\mu_m$ of N_p^+, we have

$$Tr_{F/Q}(\Pi_{i=1}^m \omega_{\mu_i}) = \frac{p^2}{p-1} x_0(\mu_1,\cdots,\mu_m) - \frac{p}{p-1} x_1(\mu_1,\cdots,\mu_m).$$

(2) For $\mu,\nu \in N_p^+$, we have

$$Tr_{F/Q}(\ \omega_\mu\omega_\nu\ ) = p(\ p\delta_{\mu\nu} - 1\ ).$$

(3) For $\lambda,\mu,\nu \in N_p^+$, we have

$$Tr_{F/Q}(\ \omega_\lambda\omega_\mu\omega_\nu\ ) \equiv 2p\ (\ \mathrm{mod}.p^2).$$

Proof. (1) Since $\omega_\mu = \Sigma_{\tau\in H}\ \zeta^{\sigma^{\mu-1}\tau}$, it is enough to notice following two facts (I),(II) ;

(I) $\quad Tr_{F/Q}(\alpha) = \frac{1}{p-1}\ Tr_{L/Q}(\alpha) \qquad (\alpha\in F)$;

(II) $\quad Tr_{L/Q}(\ \zeta^a\ ) = p^2\delta_{p^2|a} - p\delta_{p|a} \qquad (a\in Z)$;

(2) Since $Y^{(2)} = \{(s,-s)\mid s \in r^\times\}$, we obtain that $x_1(\mu,\nu) = p-1$.

On the other hand, we have

(#) $$X_0(\mu,\nu) = \begin{cases} X_1(\mu,\nu) & (\mu=\nu), \\ \phi & (\mu\neq\nu). \end{cases}$$

In fact, we put

$$X = \{(x,y)\mid x,y \in R^\times,\ x^p+y^p=0\ \}.$$

Then we see that $X = \{(x,-x)\mid x\in R^\times\}$. Since x and $-x$ are contained in the same A_μ , we obtain (#). Thus we have that $x_0(\mu,\nu) = \delta_{\mu\nu}(p-1)$. therfore by (1) we get (2). (3) Since $Y^{(3)} = \{(s,t-s,-t)\mid s \in r^\times,\ t \in r^\times,\ s\neq t\}$, we have that $x_1(\lambda,\mu,\nu)=(p-1)(p-2)$.

Hence by (1)

$$Tr_{F/Q}(\omega_\lambda\omega_\mu\omega_\nu) = 2p - p^2 + \frac{p^2}{p-1}x_0(\lambda,\mu,\nu)$$. Since $\omega_\lambda\omega_\mu\omega_\nu$ is integer in F , $Tr_{F/Q}(\omega_\lambda\omega_\mu\omega_\nu) \in Z$. Therefore $\frac{p^2}{p-1}x_0(\lambda,\mu,\nu) \in Z$. This implies that $\frac{x_0(\lambda,\mu,\nu)}{p-1} \in Z$. Thus we have (3).

Remark 1. Lemma 2-(1) shows that $Tr_{F/Q}(\omega_\mu\omega_\nu)$ are the same value for μ,ν of N_p^+ $(\mu\neq\nu)$ though any two elements of $\{\omega_\mu\omega_F \mid \mu\in N_p\}$ do not conjugate each other. On the other hand, it is not true that $|\{Tr_{F/Q}(\omega_\lambda\omega_\mu\omega_\nu) \mid \lambda,\mu,\nu \in N_p^+ \ (\lambda\neq\mu,\mu\neq\nu,\nu\neq\lambda)\}| = 1$. In fact, we take $p = 5$ and $\zeta = \exp(2\pi i/25)$. Then we know by calculation that $\omega_1\omega_2\omega_3 = -3 + \omega_1 - 2\omega_2 - \omega_3$ and $\omega_1\omega_2\omega_4 = 2 - \omega_1 - \omega_4$, hence $Tr_{F/Q}(\omega_1\omega_2\omega_3) = -15$ and $Tr_{F/Q}(\omega_1\omega_2\omega_4) = 10$.

We denote by o the ring of integers of F .

PROPOSITION 1. $\{1,\omega_1,\cdots,\omega_{p-1}\}$ is Z-basis of o .

Proof. Let o' be the Z-module generated by $\{1,\omega_1,\cdots,\omega_{p-1}\}$. Then it is clear that $o' \subset o$. From Lemma 1-(2), it is easily shown that the discriminant of o' is equal to $p^{2(p-1)}$. On the other hand, the discriminant of the basic p-extension F is equal to $p^{2(p-1)}$ as stated in section 1 . Therefore we obtain our assertion.

Remark 2. Since F is an abelian field, Proposition 1 is obtained from the main theorem of Leopoldt[2], that used gauss sums, by combining with the result of Odoni [3].

PROPOSITION 2. Put

$$\omega_\mu^* = \begin{cases} \dfrac{1}{p} & (\mu = 0), \\ \dfrac{\omega_\mu - \omega_p}{p^2} & (\mu \in N_p). \end{cases}$$

(1) $\{\omega_0^*, \omega_1^*, \cdots, \omega_{p-1}^*\}$ is the dual basis of $\{1, \omega_1, \cdots, \omega_{p-1}\}$ with respect to $Tr_{F/Q}$.

(2) $\Sigma_{\mu=1}^{p-1} \omega_\mu^* = \dfrac{-\omega_p}{p}$.

Proof. (1) It is enough to show that $Tr_{F/Q}(\omega_\mu^* \omega_\nu) = \delta_{\mu\nu}$ for $\mu, \nu \in N_p$. By Lemma 1-(2),

$$\begin{aligned} & Tr_{F/Q}(\omega_\mu^* \omega_\nu) \\ &= \frac{1}{p^2} \{Tr_{F/Q}(\omega_\mu \omega_\nu) - Tr_{F/Q}(\omega_p \omega_\nu)\} \\ &= \frac{1}{p^2} \{p(p\delta_{\mu\nu} - 1) - (-p)\} = \delta_{\mu\nu} . \end{aligned}$$

(2) It is obvious from the definition of ω_μ^* .

3. Ideal with self dual basis

In this section we shall give an explicit fractional ideal, which has a self dual basis, of each basic p-extension. We use the same notations as in section 2. Now we define p elements Ω_1, $\cdots$, Ω_p of F by

$$\Omega_\mu = \frac{1 + \omega_\mu}{p} \qquad (\mu \in N_p^+) .$$

We note that $\Omega_\mu{}^\sigma = \Omega_{\mu+1}$ $(\mu \in N_p)$, $\Omega_p{}^\sigma = \Omega_1$ and $\Sigma_{\mu=1}^p \Omega_\mu = 1$.

LEMMA 2. $\{\Omega_1, \Omega_2, \cdots, \Omega_p\}$ is a self dual basis of F, or a basis of F satisfying

$$\mathrm{Tr}_{F/Q}(\Omega_\mu \Omega_\nu) = \delta_{\mu\nu} \qquad (\mu, \nu \in N_p^+).$$

Proof. For $\mu, \nu \in N_p^+$, by Lemma 1-(2)

$$\begin{aligned} &\mathrm{Tr}_{F/Q}(\Omega_\mu \Omega_\nu) \\ &= \frac{1}{p^2} \mathrm{Tr}_{F/Q}(1 + \omega_\mu + \omega_\nu + \omega_\mu \omega_\nu) \\ &= \frac{1}{p^2} \{p + p(p\delta_{\mu\nu} - 1)\} \\ &= \delta_{\mu\nu} . \end{aligned}$$

We denote by a the Z-module generated by $\{\Omega_1, \cdots, \Omega_p\}$. From Lemma 2 we see that a has rank p and $a \ni 1$.

PROPOSITION 3. a is a fractional ideal of F .

Proof. It is enough to show that

$$\omega_\lambda \Omega_\mu \in a \qquad (\lambda \in N_p, \mu \in N_p^+) .$$

By Proposition 2-(1) and Lemma 2 ,

$$\begin{aligned} \omega_\lambda \omega_\mu &= \Sigma_{\nu=0}^{p-1} \mathrm{Tr}_{F/Q}(\omega_\lambda \omega_\mu \omega_\nu^*)\omega_\nu \\ &= \frac{1}{p}\mathrm{Tr}_{F/Q}(\omega_\lambda \omega_\mu) + \Sigma_{\nu=1}^{p-1}\mathrm{Tr}_{F/Q}(\omega_\lambda \omega_\mu \omega_\nu^*) \\ &= p\delta_{\lambda\mu} - 1 + p\ \Sigma_{\nu=1}^{p-1}\mathrm{Tr}_{F/Q}(\omega_\lambda \omega_\mu \omega_\nu^*)\Omega_\nu + \frac{1}{p}\mathrm{Tr}_{F/Q}(\omega_\lambda \omega_\mu \omega_p) . \end{aligned}$$

Hence by Proposition 2-(2)

$$\begin{aligned} \omega_\lambda \Omega_\mu &= \Omega_\lambda + \frac{\omega_\lambda \omega_\mu - 1}{p} \\ &= \Omega_\lambda + \Sigma_{\nu=1}^{p-1}\mathrm{Tr}_{F/Q}(\omega_\lambda \omega_\mu \omega_\nu^*)\Omega_\nu + \delta_{\lambda\mu} + \frac{1}{p^2}\mathrm{Tr}_{F/Q}(\omega_\lambda \omega_\mu \omega_p) - \frac{1}{p} . \end{aligned}$$

Since $\omega_\lambda \omega_\mu \in o$, we have $\mathrm{Tr}_{F/Q}(\omega_\lambda \omega_\mu \omega_\nu^*) \in Z$. Finally by Lemma 1-(3) we have that $\omega_\lambda \Omega_\mu \in a$.

COROLLARY. pa is the unique integral ideal of F with norm p .

Proof. Since $pa = \oplus_{\mu=1}^{p} Z(1+\omega_\mu)$, pa is an integral ideal of F and

$$(\mathfrak{o}:p\mathfrak{a}) = \det \begin{pmatrix} 1 & & \\ 1 & & \\ \vdots & -1_{p-1} & \\ 1 & & \\ 1 & -1 \ -1 \ \cdots & -1 \end{pmatrix} = p\ .$$

THEOREM 1. Let ξ the regular representation of F with respect to the basis $\{\Omega_1,\cdots,\Omega_p\}$. Then we have ;

(1) ξ is a Q-algebra homomorphism of F into $M(p,Q)$;

(2) ${}^t\xi(\alpha) = \xi(\alpha)$ $(\alpha \in F)$;

(3) $\xi(\alpha) \in M(p,Z)$ $(\alpha \in \mathfrak{o})$;

(4) $\xi(\alpha^\sigma) = C^{-1}\xi(\alpha)C$ $(\alpha \in F)$, where C is the cyclic matrix given by

$$C = \begin{pmatrix} 0 & 1_{p-1} \\ 1 & 0 \end{pmatrix} .$$

Proof. (1) It is obvious by the definition of ξ . (2) It is an easy consequence from Lemma 2 since the regular representation with respect to the dual basis coincides the transposed of original regular representation. (3) It is clear by Proposition 3 . (4) It is implied directly from the following relation

$$\begin{pmatrix} \Omega_1^\sigma \\ \vdots \\ \Omega_p \end{pmatrix} = \begin{pmatrix} \Omega_2 \\ \vdots \\ \Omega_p \\ \Omega_1 \end{pmatrix} = C \begin{pmatrix} \Omega_1 \\ \vdots \\ \Omega_p \end{pmatrix} .$$

Example. We take $\zeta = \exp(2\ i/p^2)$. (1) When $p = 3$, we get

$$\xi(1) = 1_3 , \quad \xi(\omega_1) = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & -1 \\ 0 & -1 & 0 \end{pmatrix} , \quad \xi(\omega_2) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix} .$$

(2) When $p = 5$, we get

$$\xi(1) = 1_5 ,$$

$$\xi(\omega_1) = \begin{pmatrix} 1 & 2 & 1 & 0 & 0 \\ 2 & -1 & -1 & 0 & -1 \\ 1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & -1 & 1 \end{pmatrix} , \quad \xi(\omega_2) = \begin{pmatrix} 1 & 0 & -1 & 0 & -1 \\ 0 & 1 & 2 & 1 & 0 \\ -1 & 2 & -1 & -1 & 0 \\ 0 & 1 & -1 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix} ,$$

$$\xi(\omega_3) = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & -1 & 2 & -1 & -1 \\ 0 & 0 & 1 & -1 & -1 \end{pmatrix} , \quad \xi(\omega_4) = \begin{pmatrix} -1 & 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 & -1 \\ 1 & 0 & 0 & 1 & 2 \\ -1 & 0 & -1 & 2 & -1 \end{pmatrix} .$$

4. Modular embedding for basic p-extension

Let F be the basic p-extension and o the ring of integers of F. In this section we consider a modular embedding for F (see section 1). We let ξ the regular representation of F defined in Theorem 1 and define a map Ξ of $SL_2(F)$ into $M(2p,Q)$ by

$$\Xi(g) = \begin{pmatrix} \xi(\alpha) & \xi(\beta) \\ \xi(\gamma) & \xi(\delta) \end{pmatrix} \qquad \left(g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \right).$$

PROPOSITION 4. (1) Ξ is a group homomorphism of $SL_2(F)$ into $Sp(2p,Q)$.

(2) $\Xi(SL_2(o)) \subset Sp(2p,Z)$.

Proof. (1) From Theorem 1-(1),(2), it is clear. (2) From (1) and Theorem 1-(3), it is obvious.

We put for an element α of F and an element g of $SL_2(F)$

$$\alpha^{(\nu)} = \alpha^{\sigma^{\nu-1}} \qquad (\nu \in N_p^+),$$

$$\xi_1(\alpha) = \mathrm{diag}(\alpha^{(1)},\alpha^{(2)},\cdots \alpha^{(p)})$$

$$\Xi_1(g) = \begin{pmatrix} \xi_1(\alpha) & \xi_1(\beta) \\ \xi_1(\gamma) & \xi_1(\delta) \end{pmatrix} \qquad \left(g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \right).$$

Then there exists an orthogonal matrix V of degree p such that

$V\xi(\alpha)V^{-1} = \xi_1(\alpha)$ for any element α of F :

LEMMA 3. $CV = VC$:

Proof. From the definition, for any $\alpha \in F$

$$V\xi(\alpha)V^{-1} = \xi_1(\alpha) \ , \qquad V\xi(\alpha^\sigma)V^{-1} = \xi_1(\alpha^\sigma) \ .$$

On the other hand, by Theorem 1-(4)

$$\xi(\alpha^\sigma) = C^{-1}\xi(\alpha)C \ :$$

By the same reason, $\xi_1(\alpha^\sigma) = C^{-1}\xi_1(\alpha)$. Therefore

$$VCV^{-1}C^{-1}\xi_1(\alpha) = \xi_1(\alpha)VCV^{-1}C^{-1}$$

for any $\alpha \in F$. This implies that $VCV^{-1}C^{-1} = 1_p$.

Now we let $SL_2(F)$ and $Sp(2p,Q)$ act on $H_1{}^p$ and H_p , respectively, by the standard way. Put

$$E_1(z) = \mathrm{diag}(z_1,\cdots,z_p) \qquad (z=(z_1,\cdots,z_p)\in H_1{}^p) ,$$

$$E_V(z) = VE_1(z)V^{-1} \ .$$

THEOREM 2. $(\ \Xi,\ E_V\ )$ is a modular embedding for F .

Proof. We have to show the following three statements ;

(1) $\Xi(g)(E_V(z)) = E_V(gz)$ for $g \in SL_2(F)$ and $z \in H_1{}^p$;

(2) $\Xi(SL_2(o)) \subset Sp(2p,Z)$; (3) for $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL_2(o)$

$$\Pi_{\nu=1}^{p}\left(\gamma^{(\nu)}z_\nu + \delta^{(\nu)}\right) = \det(\xi(\gamma)E_V(z) + \xi(\delta)) .$$

(1) Put $\Lambda = \begin{pmatrix} V & 0 \\ 0 & V \end{pmatrix}$. Then we see that

$$(\Delta) \qquad \Xi(g) = V\Xi_1(g)v^{-1} .$$

This implies our assertion. (2) This is already shown in Proposition 4.

(3) It is also clear by (Δ).

We define the standard automorphic factors j of $SL_2(o)$ on $H_1{}^p$ and J of $Sp(2p,Z)$ on H_p by

$$j(g,z) = \Pi_{\nu=1}^{p}\left(\gamma^{(\nu)}z_\nu + \delta^{(\nu)}\right)$$

$$\left(g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL_2(o),\ z = (z_1,\cdots,z_p) \in H_1{}^p \right),$$

$$J(\Gamma,Z) = \det(CZ+D)$$

$$\left(\Gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2p,Z),\ Z \in H_p \right) ,$$

respectively. We call a meromorphic function f on H_p (resp. $H_1{}^p$) a standard Siegel (resp. Hilbert) modular form of weight k if $f(gZ) = J(g,Z)^k f(Z)$ (resp. $f(gZ) = j(g,Z)^k f(Z)$) for all elements $g \in Sp(2p,Z)$ (resp. $SL_2(o)$) and $Z \in H_p$ (resp. $H_1{}^p$)

Now for each $z = (z_1,\cdots,z_p) \in H_1{}^p$, we put

$$z^{\sigma} = (z_2, z_3, \cdots, z_p, z_1).$$

Then we have that $E_1(z^{\sigma}) = CE_1(z)C^{-1}$, hence $E_V(z^{\sigma}) = CE_V(z)C^{-1}$ by Lemma 3. A standard Hilbert modular form f of weight k over F is called symmetric if $f(z^{\sigma}) = f(z)$.

THEOREM 3. Let f be a standard Siegel form of weight k with respect to $Sp(2p,Z)$ on H_p. We put

$$\tilde{f}(z) = f(E_V(z)) \qquad (z \in H_1^p).$$

Then $\tilde{f}$ is a symmetric standard Hilbert modular form of weight k over F.

Proof. It is clear by Theorem 2 that $\tilde{f}$ is a standard Hilbert modular form of weight k over F.

$$\begin{aligned} \tilde{f}(z^{\sigma}) &= f(E_V(z^{\sigma})) \\ &= f(CE_V(z)C^{-1}) \\ &= f\left(\begin{pmatrix} C & 0 \\ 0 & C^{-1} \end{pmatrix} E_V(z)\right) \\ &= f(E_V(z)) \\ &= \tilde{f}(z). \end{aligned}$$

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