Vertex algebras and the space of formal loops

M. Kapranov*

One of the salient mathematical features of string theory is the importance of vertex algebras. Their role in the theory can be compared to that of Lie algebras in the "ordinary" physics of point particles.

Mathematically, the approach of string theory can be cast in terms of analysis on the space of free loops, i.e., smooth maps $S^1 \to X$ where X is a given "spacetime" manifold. Accordingly, one has the folklore principle that constructions involving the space of free loops lead to vertex algebras. One class of such constructions is provided by the spaces of highest weight representations of loop groups. Another is Ω_X^{ch}, the chiral de Rham complex of an algebraic variety X, introduced by Malikov, Schechtman and Vaintrob [MSV]. Heuristically, this complex should be interpreted in terms of LX, the space of free loops and its subvariety L^0X consisting of loops extending holomorphically into the unit disk. More precisely, Ω_X^{ch} can be thought of as the semiinfinite de Rham complex with coefficients in the space of distributions on LX supported on L^0X. This is not, however, the way Ω_X^{ch} has been defined mathematically. The definition given in [MSV] is of more computational nature and proceeds by constructing the action of the group of diffeomorphisms on the irreducible module over the Heisenberg algebra. In that approach it seems miraculous that such an action exists at all.

The aim of this work is twofold. First, to give a precise mathematical theorem underlying the above folklore principle about vertex algebras. For this, we introduce an algebro-geometric version of the free loop space $\mathcal{L}(X)$ for any scheme X of finite type over a field. This is an ind-scheme containing $\mathcal{L}^0(X)$, the scheme of formal germs of curves on X studied in [DL]. We prove that both $\mathcal{L}(X)$ and $\mathcal{L}^0(X)$ themselves possess a non-linear version of the vertex algebra structure (which makes it clear that any natural linear construction applied to them should give a vertex algebra in the usual sense). More precisely, we use the geometric approach to vertex algebras developed by Beilinson and Drinfeld [BD] and based on the concepts of chiral algebras and factorization algebras. The latter concept has a natural nonlinear version, that of a factorization semigroup. What we prove is that natural "global"versions of $\mathcal{L}(X)$, $\mathcal{L}^0(X)$ have natural structures of factorization semigroups. An earlier known example of a factorization semigroup is given by the affine Grassmannian [G], and this explains why the spaces of representations of loop groups are vertex algebras. Our construction is similar in spirit.

To give a good definition of the algebraic analog of the full loop space LX one has to overcome a certain subtlety. Namely, a natural approach would be to try to ind-represent a

---

* **The talk is based on a joint work with Eric Vasserot.**

functor which to any commutative ring R associates the set of $R((t))$-points of X. (This is exactly how one defines the scheme $\mathcal{L}^0(X)$, with $R[[t]]$ instead of $R((t))$.) If X is affine, this indeed gives a good ind-scheme which we denote $\tilde{\mathcal{L}}(X)$. But when X is, say, projective, then (for R a field) there is no difference between $R[[t]]$-points and $R((t))$-points of X (valuative criterion of properness), so it may seem that nothing is gained by allowing Laurent series. To state this phenomenon differently, the ind-schemes $\tilde{\mathcal{L}}(U)$ for affine $U \subset X$ do not glue together well. This is in fact understandable on general grounds: the loop space LX is not the union of the LU since a loop need not spend all its time in any given U.

To get around this difficulty we adopt the following strategy. For an affine X we consider $\mathcal{L}(X)$, the formal neighborhood of $\mathcal{L}^0(X)$ in $\tilde{\mathcal{L}}(X)$. So we are dealing with formal loops which are "infinitesimal in the Laurent direction".Then, we prove that the $\mathcal{L}(U), U \subset X$, do indeed possess the right gluing properties. This is due to the infinitesimal nature of our loops.

Our second goal is to give a direct geometric construction of Ω_X^{ch} (for smooth X) in terms of our model for the loop space. By the above, this construction explains also the fact that Ω_X^{ch} is a sheaf of vertex algebras. In order to achieve this, we represent $\mathcal{L}(X)$ as an ind-pro-object in the category of schemes of finite type and then show that the shifted de Rham complexes of the terms of this ind-pro-system arrange naturally into a double inductive system whose inductive limit is identified with Ω_X^{ch}.

As with the study of formal arcs and motivic integration [DL], one can view our considerations as algebro-geometric analogs of the basic constructions of p-adic analysis. The difference between our ind-scheme $\mathcal{L}(X)$ and the more familiar scheme $\mathcal{L}^0(X)$ is similar to the difference between $\mathbf{Q}_p$ and $\mathbf{Z}_p$: while the latter is a pro-object in the category of schemes of finite type (resp. finite sets), the former is an ind-pro-object. Further, our approach to Ω_X^{ch} is similar to the construction of the space of locally constant functions with compact support on $\mathbf{Q}_p = \varinjlim_i \varprojlim_j p^{-i}\mathbf{Z}_p/p^j\mathbf{Z}_p$ as the double inductive limit of the spaces of functions on the finite sets $p^{-i}\mathbf{Z}_p/p^j\mathbf{Z}_p$, cf [P]. Notice that the reason that these spaces of functions indeed form a double inductive system (with respect to the maps of inverse image in the j-direction and direct image in the i-direction) is that the commutative squares in the ind-pro-system $p^{-i}\mathbf{Z}_p/p^j\mathbf{Z}_p$ are Cartesian (so that we have base change). This is an algebraic counterpart of the property of the local compactness of $\mathbf{Q}_p$, see [Kat]. In our situation it is equally important that the ind-scheme $\mathcal{L}(X)$ satisfies a certain formal analog of local compactness.

References

[BD] Beilinson, A., Drinfeld, V., Chiral algebras, preprint.

[DL] Denef, J., Loeser, F., Germs of arcs on singular algebraic varieties and motivic integration, *Invent. Math.* **135** (1999), 201-232.

[D] Gaitsgory, D., Notes on 2d conformal field theory and string theory in: "Quantum fields and strings: a course for mathematicians", vol 2 (P. Deligne et al. Eds.), 1017-1089 Amer. Math. Soc., Providence RI

[Kat] Kato, K., Existence theorem for higher local class field theory, preprint Math.AG/0012150.

[MSV] Malikov, A., Schechtman, V., Vaintrob, A., Chiral De Rham complex, *Comm. Math. Phys.* **204** (1999), 439-473.

[P] Parshin, A.N., Higher-dimensional local fields and L-functions, preprint Math.AG/0012151.