Non-resonance D-modules over arrangements of hyperplanes

Sergei Khoroshkin and Vadim Schechtman

1 Introduction

The aim of this note is a combinatorial description of a category of D-modules over an affine space, smooth along the stratification defined by an arrangement of hyperplanes. These D-modules are assumed to satisfy certain *non-resonance* condition. The main result, see Theorem 4.1, generalizes [3].

2 Quivers of an arrangement

2.1. Let $\mathcal{A} = \{H_i\}, i \in I$ be a collection of affine hyperplanes in a complex affine space $X = \mathbf{C}^N$. Hyperplanes H_i define on X the structure of a stratified space which we denote by the same letter $\mathcal{A}$ or by $X_{\mathcal{A}}$. The closure of a stratum $X_\alpha \subset \mathbf{C}^N$ is an intersection of some hyperplanes

$$\overline{X}_\alpha = \bigcap_{i \in J \subset I} H_i \ , \tag{2.1}$$

and the stratum X_α is an interiour of $\overline{X}_\alpha$:

$$X_\alpha = \overline{X}_\alpha \setminus (\bigcup_{\overline{X}_\beta \subset \overline{X}_\alpha, \overline{X}_\beta \neq \overline{X}_\alpha} \overline{X}_\beta) \ .$$

The combinatorial structure of the adjacements of the strata can be described by a connected oriented graph $\Gamma_{\mathcal{A}}$. The vertices α of the graph correspond to the strata X_α, the set of the vertices we denote by $\mathrm{ver}(\mathcal{A})$; an arrow $\alpha \to \beta \in \mathrm{arr}(\mathcal{A})$ corresponds to the adjacement in codimension 1:

$$\alpha \to \beta \qquad \overline{X}_\alpha \supset \overline{X}_\beta \ , \quad \mathrm{codim}\,_{\overline{X}_\alpha} \overline{X}_\beta = 1 \ .$$

Inclusions of closed strata define the partial order in the set $\mathrm{ver}(\mathcal{A})$: $\alpha > \beta \Leftrightarrow \overline{X}_\alpha \supseteq \overline{X}_\beta \Leftrightarrow$ there exists an oriented path from α to β in graph $\Gamma_{\mathcal{A}}$. This partial order is in agreement with integervalued function codim $: \Gamma_{\mathcal{A}} \Rightarrow \mathbf{N}$, $\mathrm{codim}\ \alpha = \mathrm{codim}\,_{\mathbf{C}^N} \overline{X}_\alpha$ which means that $\alpha > \beta \Rightarrow \mathrm{codim}\ \alpha < \mathrm{codim}\ \beta$. The partial order allows to attach to any vertice α of the graph $\Gamma_{\mathcal{A}}$ the full subgraph $\Gamma^\alpha_{\mathcal{A}} \subset \Gamma_{\mathcal{A}}$ whose vertices are the vertices $\beta \in \mathrm{ver}(\mathcal{A})$ which satisfy the condition $\beta < \alpha$. One can easily see that the graph $\Gamma^\alpha_{\mathcal{A}}$ describes the combinatorial structure of affine subspace $\overline{X}_\alpha$ with induced stratification. By $X_\emptyset$ we denote the the complement to all hyperplanes H_i which is an open in $\mathbf{C}^N$ stratum.

Definition 2.1 *A quiver* $V_{\mathcal{A}} = \{V_\alpha, A_{\alpha,\beta}\}$ *of the arrangement* $\mathcal{A}$ *is a collection of linear spaces* V_α *over* $\mathbf{C}$, $\alpha \in \mathrm{ver}(\mathcal{A})$ *together with a collection of linear maps* $A_{\alpha,\beta} : V_\beta \to V_\alpha$, $\alpha, \beta \in \mathrm{ver}(\mathcal{A})$ *such that* $A_{\alpha,\beta} = 0$ *unless* α *and* β *are connected by an arrow (* $A_{\alpha,\beta} \neq 0 \Rightarrow \alpha \to \beta$ *or* $\beta \to \alpha$*) , and which satisfy the following quadratic relations:*

$$\sum_\beta A_{\alpha,\beta} A_{\beta,\gamma} = 0 \qquad \textit{for any } \alpha, \gamma \in \mathrm{ver}(\mathcal{A}),\ \alpha \neq \gamma\ . \tag{2.2}$$

Remark. One can see that the relation (2.2) is nontrivial only in two cases:

(i) $|\,\mathrm{codim}\ \alpha - \mathrm{codim}\ \gamma\,| = 2$ and $\alpha > \gamma$ or $\alpha < \gamma$;

(ii) $\mathrm{codim}\ \alpha = \mathrm{codim}\ \gamma$, $\mathrm{codim}\ _{X_\alpha} X_\alpha \cap X_\gamma = 1$.

In the last case there are one or two summands in the relation (2.2).

The quivers $V_{\mathcal{A}}$ compose the category $Qui(\mathcal{A})$ with natural morphisms. In other words, $Qui(\mathcal{A})$ is the category of finite-dimensional modules over graded algebra $\mathcal{B}_{\mathcal{A}}$. The algebra $\mathcal{B}_{\mathcal{A}}$ is generated by orthogonal idempotents e_α, $\alpha \in \mathrm{ver}(\mathcal{A})$ of degree 0, $\sum e_\alpha = 1$ and by degree one generators $a_{\alpha,\beta}$, $\alpha, \beta \in \mathrm{ver}(\mathcal{A})$ such that $a_{\alpha,\beta} = 0$ if α and β are connected by an arrow in $\Gamma_{\mathcal{A}}$ and, moreover,

$$a_{\alpha,\beta} e_\gamma = \delta_{\beta,\gamma} a_{\alpha,\beta}, \qquad e_\gamma a_{\alpha,\beta} = \delta_{\gamma,\alpha} a_{\alpha,\beta}\ ,$$

$$\sum_\beta a_{\alpha,\beta} a_{\beta,\gamma} = 0, \qquad \alpha \neq \gamma.$$

Let $V_{\mathcal{A}} = \{V_\alpha, A_{\alpha,\beta}\}$ be a quiver of the arragement $\mathcal{A}$. We denote by $V^t_{\mathcal{A}} = \{V^t_\alpha, A^t_{\alpha,\beta}\}$ the following dual quiver:

$$V^t_\alpha = V^*_\alpha, \qquad A^t_{\alpha,\beta} = (-1)^{(\mathrm{codim}\ \alpha + \mathrm{codim}\ \beta - 1)/2} A^*_{\beta,\alpha}\ . \tag{2.3}$$

The correspondence $V_{\mathcal{A}} \to V^t_{\mathcal{A}}$ defines the contravariant functor of duality

$$^t : Qui(\mathcal{A}) \to Qui(\mathcal{A})\ .$$

3 Quiver D-modules

3.1. For any stratum X_α we denote by Ω_α one-dimensional space of codim X_α-forms $\omega_\alpha \in \det T^*_{X_\alpha}$, which are constant along X_α, and by $\hat{\Omega}_\alpha$ denote one-dimensional space of top degree forms of the conormal bundle of X_α, $\hat{\omega}_\alpha \in T^*_{(X_\alpha|X)}$ constant along X_α.

Let $V_{\mathcal{A}}$ be a quiver of an arrangement $\mathcal{A}$. We relate to it a D_X-module $EV_{\mathcal{A}}$ by the following rules:

(i) $EV_{\mathcal{A}}$ is generated by the linear space $\bigoplus_\alpha (V_\alpha \otimes \Omega_a)$ as D_X-module,

(ii) for any α and for any vectorfield L_α along X_α with constant coefficients

$$L_\alpha(v_\alpha \otimes \omega_\alpha) = \sum_{\beta:\alpha\to\beta} \frac{L_\alpha(\hat{\omega}_\beta) \wedge \omega_\alpha}{\hat{\omega}_\beta \wedge \omega_\beta} A_{\beta,\alpha} v_\alpha \otimes \omega_\beta, \qquad v_\alpha \in V_\alpha, \omega_\gamma \in \Omega_\gamma, \hat{\omega}_\gamma \in \hat{\Omega}_\gamma\ , \tag{3.1}$$

(iii) for any α and for any affine function f_α, $f_\alpha|_{X_\alpha}=0$

$$f_\alpha(v_\alpha\otimes w_\alpha)=\sum_{\beta:\beta\to\alpha}\frac{df_\alpha\wedge\hat\omega_\beta\wedge\omega_\alpha}{\hat\omega_\beta\wedge\omega_\beta}A_{\beta,\alpha}v_\alpha\otimes\omega_\beta,\qquad v_\alpha\in V_\alpha,\omega_\gamma\in\Omega_\gamma,\hat\omega_\gamma\in\hat\Omega_\gamma\ , \tag{3.2}$$

In other words, $EV_{\mathcal{A}}$ is a factor of free D_X-module $D_X\otimes(\oplus_\alpha V_\alpha\otimes\Omega_\alpha)$ modulo the relations (3.1)–(3.2).

The closure $\overline{X}_\alpha$ of any stratum is affine subspace of $X=\mathbf{C}^N$. We can choose local affine coordinates $\{\overline{z}^\alpha,\overline{w}^\alpha\}=\{z_1^\alpha,\ldots,z^\alpha_{\mathrm{dim}X_\alpha},w_1^\alpha,\ldots,w^\alpha_{\mathrm{codim}X_\alpha}\}$ such that $\overline{X}_\alpha$ is defined by the equations:

$$w_1^\alpha=w_2^\alpha=\ldots=w^\alpha_{\mathrm{codim}X_\alpha}=0\ . \tag{3.3}$$

Proposition 3.1 (i) *$EV_{\mathcal{A}}$ is holonomic D_X-module with regular singularities flat along the stratification $\mathcal{A}$ (that is, $ss.EV_{\mathcal{A}}\subset\bigcup_\alpha T^*_{(X_\alpha|X)}$);*

(ii) *The global sections of $EV_{\mathcal{A}}$ are:*

$$\Gamma(EV_{\mathcal{A}})\equiv\oplus_\alpha\mathbf{C}[\{\overline{z}^\alpha\},\{\frac{\partial}{\overline{w}^\alpha}\}]V_\alpha\otimes\Omega_\alpha\ . \tag{3.4}$$

Proof . *(ii)* Due to the relaions (3.1) and (3.2) the space of global sections is isomorphic to a factorspae of linear space

$$U=\oplus_\alpha U^\alpha=\oplus_\alpha\mathbf{C}[\{\overline{z}^\alpha\},\{\frac{\partial}{\overline{w}^\alpha}\}]V_\alpha\otimes\Omega_\alpha\ .$$

So it is sufficient to define the structure of D_X module in linear space U. We can equip U with a structure of graded space by the relations deg $(v_\alpha\otimes w_a)=0$, deg $(z_i^\alpha)=$ deg $(\partial_{w_j^\alpha})=1$ $(\partial_{w_j^\alpha}=\frac{\partial}{\overline{w}^\alpha})$ and define an action of the generators x_i,∂_{x_i} of the ring D_X by induction over the grading. The generators $\partial_{z_i^\alpha}$ and w_j^α act on the elements of degree 0 $v_\alpha\otimes\omega_\alpha$ according to the relations (3.1) and (3.2), whereas the generators $\partial_{w_j^\alpha}$ and z_i^α act on them by free multiplication. If we already define the action of the generators x_i,∂_{x_i} on the elements of degree $\le(n-1)$, the action of x_i,∂_{x_i} on the elements of degree n from U_α is uniquely defined by the following conditions:
(a) the elements $\partial_{w_j^\alpha}$ and z_i^α act on subspace U^α by free multiplication, in particular,

$$[z_i^\alpha,z_k^\alpha]=[z_i^\alpha,\partial_{w_j^\alpha}]=[\partial_{w_j^\alpha},\partial_{w_l^\alpha}]=0\ , \tag{3.5}$$

(b) the operators $\partial_{z_i^\alpha}$ and w_j^α commute with $\partial_{w_j^\alpha}$ and z_i^α according to the relations in D_X, that is,

$$[\partial_{z_i^\alpha},\partial_{w_j^\alpha}]=[w_j^\alpha,z_i^\alpha]=0\ ,[\partial_{z_i^\alpha},z_k^\alpha]=\delta_{i,k},\ [w_j^\alpha,\partial_{w_l^\alpha}]=-\delta_{j,l}\ . \tag{3.6}$$

We have to prove that this action satisfied all the relation in D_X, so we are to check the relations

$$[\partial_{z_i^\alpha},\partial_{z_k^\alpha}]=[w_j^\alpha,w_l^\alpha]=[\partial_{z_i^\alpha},w_j^\alpha]=0\ . \tag{3.7}$$

The relations (3.5) and (3.6) allow to reduce the check of (3.7) to the space of degree zero, where it is valid due to (3.1), (3.2) and (2.1) (this is a simple but crucial calculation).

(i) The description of the global sections of the module $EV_{\mathcal{A}}$ enables one to describe the concrete good filtration of $EV_{\mathcal{A}}$. Remind that a filtration M_k of D_X-module X is good, if it is an argeement with a filtration D_k of D_X over the degree of differential operator: $D_k M_l \subset M_{k+l}$; $D_k M_l = M_{k+l}$ for all $l \geq l_0, k \geq 0$; and all M_k are finite O_X-modules.

Let, as before$U^\alpha = \oplus_\alpha \mathbf{C}[\{\overline{z}^\alpha\}, \{\frac{\partial}{\overline{w}^\alpha}\}] V_\alpha \otimes \Omega_\alpha$ .. We showed that, globally, $\Gamma(EV_\mathcal{A}) = U = \oplus_\alpha U_\alpha$. We equip U_α with a filtration putting $\deg(V_\alpha \otimes \Omega_\alpha) = \operatorname{codim} \alpha$, $\deg z_i^\alpha = 0$, $\deg \partial_{\omega_i^\alpha} = 1, U_k^\alpha = \{m \in U_\alpha, \deg m \leq k\}$ and let $U_k = \oplus_\alpha U_k^\alpha$. Then, due to the relations (3.1) and (3.2), the filtration $0 \subset U_0 \subset \ldots \subset U$ is a good filtration. Moreover, the same relations show that the product

$$\prod_\alpha \left(\prod_i p_i^\alpha \cdot \prod_j w_j^\alpha \right)$$

annihilate gr $EV_\mathcal{A}$ (here $p_i^\alpha = \operatorname{gr} \frac{\partial}{\partial_{z_i^\alpha}}$). It means that $EV_\mathcal{A}$ is holonomic D_X module with singular support contained in $\cup_\alpha T^*_{(X_\alpha | \mathbf{C}^N)}$. It has regular singularities due to (3.1), (3.2)

Thus we have a functor

$$E : Qui(\mathcal{A}) \longrightarrow \mathcal{M}(\mathcal{A}) \ , \tag{3.8}$$

where $\mathcal{M}(\mathcal{A})$ is a category of holonomic RS D_X-modules flat anong the stratification $\mathcal{A}$. Note that the presentation of the module $EV_\mathcal{A}$ allows one in addition easily describe the sections over the open sets

$$X_H = X \setminus H$$

where H is a hyperplane. Let, for instance, H_β be a generic hyperplane which contains the stratum X_β. Then the global sections over X_{H_β} constitue the linear space

$$U_\beta = \oplus_{\alpha : \beta \not\to \alpha} U^\alpha = \oplus_\alpha \mathbf{C}[\{\overline{z}^\alpha\}, \{\frac{\partial}{\overline{w}^\alpha}\}] V_\alpha \otimes \Omega_\alpha \ ,$$

where the polynomial over z_i^α should be replaced by the regular functions of $\overline{X}_\beta \cap U$.

3.2. One can also localize the whole construction to the complement of certain hyperlanes, that is, take as an initial space X the complement to some hyperplanes instead of affine space. For instance, the configuration $\{\mathcal{A}, H\}$ of hyperplanes $\mathcal{A} = \{H_i\}, i \in I$ in $\mathbf{C}^N \setminus H$ defines the quivers which consist of linear spaces $V_\alpha, \alpha \in \Gamma_\mathcal{A}$, together with linear maps $A_{\alpha,\beta} : V_\beta \to V_\alpha$, which are zero unless α and β are connected by an arrow, and linear maps $N_\alpha : V_\alpha \to V_\alpha$ which are not zero only for the strata X_α which intersect with H over codimension one. These operators are subjected to the relations:

$$\sum_\beta A_{\alpha,\beta} A_{\beta,\gamma} = 0 \qquad \text{for } \textit{any } \alpha, \gamma \in \operatorname{ver}(\mathcal{A}), \ \alpha \neq \gamma \ . \tag{3.9}$$

$$N_\beta A_{\beta,\alpha} = A_{\beta,\alpha} \left(N_\alpha + \sum_{\substack{\gamma \neq \beta : \\ \overline{X}_\gamma \cap H = \overline{X}_\beta \cap H}} A_{\alpha,\gamma} A_{\gamma,\alpha} \right) \ , \qquad \alpha \to \beta \ , \tag{3.10}$$

$$A_{\alpha,\beta} N_\beta = \left(N_\alpha + \sum_{\substack{\gamma \neq \beta : \\ \overline{X}_\gamma \cap H = \overline{X}_\beta \cap H}} A_{\alpha,\gamma} A_{\gamma,\alpha} \right) A_{\alpha,\beta} \ , \qquad \alpha \to \beta \ , \tag{3.11}$$

The corresponding D_X-module is described by analogous relations:

$$L_\alpha(v_\alpha \otimes \omega_\alpha) = \sum_{\beta:\alpha\to\beta} \frac{L_\alpha(\hat\omega_\beta)\wedge\omega_\alpha}{\hat\omega_\beta\wedge\omega_\beta} A_{\beta,\alpha} v_\alpha \otimes \omega_\beta +$$

$$+\phi_{L_\alpha}(v_\alpha\otimes\omega_\alpha), \qquad v_\alpha\in V_\alpha, \omega_\gamma\in\Omega_\gamma, \hat\omega_\gamma\in\hat\Omega_\gamma\ , \tag{3.12}$$

for any α and for any vectorfield L_α along X_α with constant coefficients

$$f_\alpha(v_\alpha\otimes w_\alpha) = \sum_{\beta:\beta\to\alpha} \frac{df_\alpha\wedge\hat\omega_\beta\wedge\omega_\alpha}{\hat\omega_\beta\wedge\omega_\beta} A_{\beta,\alpha} v_\alpha\otimes\omega_\beta, \qquad v_\alpha\in V_\alpha, \omega_\gamma\in\Omega_\gamma, \hat\omega_\gamma\in\hat\Omega_\gamma\ , \tag{3.13}$$

for any α and for any affine function f_α, $f_\alpha|_{X_\alpha}=0$.

The summand $\phi_{L_\alpha}(v_\alpha\otimes\omega_\alpha)$ is nonzero only for those $\overline{X}_\alpha$ which intersect with H over in the codimension 1 and looks like

$$\phi_{L_\alpha}(v_\alpha\otimes\omega_\alpha) = \frac{L_\alpha(f)}{f}\Big(N_\alpha v_\alpha\otimes\omega_\alpha -$$

$$\sum_{\gamma:\overline{X}_\gamma\cap H=\overline{X}_\alpha\cap H} \frac{df\wedge\hat\omega_\gamma\wedge\omega_{H\alpha}}{\hat\omega_\gamma\wedge\omega_\gamma}\cdot\frac{\hat\omega_\alpha\wedge\omega_\alpha}{df\wedge\hat\omega_\alpha\wedge\omega_{H\alpha}} A_{\gamma,\delta}A_{\delta,\alpha} v_\alpha\otimes\omega_\gamma\Big)$$

Here $\omega_{H\alpha}$ is an arbitrary top form on $H\cap X_\alpha$ with constant coefficients, the symbol δ stands for a stratum which coincides with $X_\alpha+X_\gamma$; if $X_\alpha+X_\gamma$ is not a stratum then the corresponding summand in r.h.s. of the above formula is zero ($X_\alpha+X_\gamma$ means a sum of linear subspaces in linear space with an origin in $X_\alpha\cap X_\gamma$). The statement of Proposition remains true for such D-modules.

3.3. A free resolution of quiver D-modules.

Let $T_\alpha = T'_\alpha\oplus T''_\alpha$, $dim\ T_\alpha = N$ be a direct sum of the space of the sections of tangent bundle T_{X_α} to a stratum X_α constant along X_α and of the space of sections of conormal bundle $T^*_{(X_\alpha|\mathbf{C}^N)}$ constant along X_α. In local coordinates $\overline{z}^\alpha, \overline{w}^\alpha$ (3.3) the spaces T'_α and T''_α are: $T'_\alpha = \mathbf{C} < \partial_{z_i^\alpha} >$, $T''_\alpha = \mathbf{C} < dw_j^\alpha >$. In other way the space T_α can be identify with the space of constant sections of tangent bundle $T_{T^*_{(X_\alpha|\mathbf{C}^N)}}$ to the space of conormal bundle $T^*_{(X_\alpha|\mathbf{C}^N)}\subset T^*_{\mathbf{C}^N}$.

There is a natural embedding I_α of the space T_α into the space of first order differential operators:

$$I_\alpha : T_\alpha \hookrightarrow D_{\le 1}\ .$$

In local coordinates (3.3) $I_\alpha(\partial_{z_i^\alpha} = \frac{\partial}{\partial z_i^\alpha}$, $I_\alpha(dw_j^\alpha = w_j^\alpha$. The map I_α does not depend on ths choise of affine coordinates $\overline{z}^\alpha, \overline{w}^\alpha$ satisfying (3.3).

For any quiver $V_{\mathcal{A}}$ the relations (3.1) and (3.2) define also a natural map

$$Q_\alpha :\ T_\alpha\otimes V_\alpha\otimes\Omega_\alpha \to \oplus_\beta V_\beta\otimes\Omega_\beta$$

which looks as follows:

$$Q_\alpha(\xi\otimes v_\alpha\otimes\omega_\alpha) = \sum_{\beta:\alpha\to\beta} \frac{\xi(\hat\omega_\beta)\wedge\omega_\alpha}{\hat\omega_\beta\wedge\omega_\beta} A_{\beta,\alpha} v_\alpha\otimes\omega_\beta\ , \qquad \xi\in T'_\alpha, \hat\omega_\beta\in\hat\Omega_\beta\ , \tag{3.14}$$

$$Q_\alpha(df_\alpha \otimes v_\alpha \otimes \omega_\alpha) = \sum_{\beta:\beta\to\alpha} \frac{df_\alpha \wedge \hat{\omega}_\beta \wedge \omega_\alpha}{\hat{\omega}_\beta \wedge \omega_\beta} A_{\beta,\alpha} v_\alpha \otimes \omega_\beta \ , \qquad df_\alpha \in T''_\alpha, \hat{\omega}_\beta \in \hat{\Omega}_\beta \ . \tag{3.15}$$

Let $E^n V_{\mathcal{A}}, n = 1, 2, \dots, N$ be a free left D-module

$$E^n V_{\mathcal{A}} = \oplus_\alpha D \otimes \left(\bigwedge^n (T_\alpha) \otimes V_\alpha \otimes \Omega_\alpha \right)$$

and $d^{(n)} : \ E^n V_{\mathcal{A}} \to E^{n-1} V_{\mathcal{A}}$ be the following D-linear map of free D-modules:

$$d^{(n)}(1 \otimes t_1 \wedge \dots \wedge t_n \otimes v_\alpha \otimes \omega_\alpha) = \sum_{k=1}^{n} (-1)^{k+1} I_\alpha(t_k) \otimes t_1 \wedge \dots \wedge \hat{t_k} \wedge \dots \wedge t_n \otimes v_\alpha \otimes \omega_\alpha -$$

$$- \sum_{k=1}^{n} (-1)^{k+1} 1 \otimes t_1 \wedge \dots \wedge \hat{t_k} \wedge \dots \wedge t_n \otimes Q_\alpha(t_k \otimes v_\alpha \otimes \omega_\alpha) \ . \tag{3.16}$$

We state that $d^{(n-1)} d^{(n)} = 0$ and

Proposition 3.2 $\left(E^\bullet V_{\mathcal{A}}, d^{(\bullet)} \right)$ *is a free resolution of D-module $EV_{\mathcal{A}}$.*

Proof. The relation $d^{(n-1)} d^{(n)} = 0$ is equivalent to basic relations (2.2) on the operators $A_{\alpha,\beta}$. The proof of the exactness of the complex is the same as for standard Koszul resolution of the ring $\mathcal{O}_X$.

Proposition 3.3 *The dual complex*

$$\bigwedge^N (T^*_{\mathbf{C}^N})[-N] \otimes_{\mathcal{O}} \mathrm{Hom}_D \left(E^\bullet V_{\mathcal{A}}, d^{(\bullet)} \right)$$

is a free resolution of the module $EV^t_{\mathcal{A}}$.

Proof. One can see that

$$\mathrm{Hom}_D(D \otimes \bigwedge^n (T_\alpha) \otimes V_\alpha \otimes \Omega_\alpha, D) \approx D \otimes \bigwedge^{N-n} (T_\alpha) \otimes V^*_\alpha \otimes \Omega_\alpha \otimes \bigwedge^N (T_{\mathbf{C}^N}) \tag{3.17}$$

and the dual maps $d^{(n)*}$ of right D-modules $D \otimes \bigwedge^{N-n} (T_\alpha) \otimes V^*_\alpha \otimes \Omega_\alpha \otimes \bigwedge^N (T_{\mathbf{C}^N})$ are of the form analogous to (3.16):

$$d^{(n)*}(1 \otimes t_1 \wedge \dots \wedge t_n \otimes v^*_\alpha \otimes \omega_\alpha \otimes \pi_\emptyset) = \sum_{k=1}^{n} (-1)^{k+1} I_\alpha(t_k) \otimes t_1 \wedge \dots \wedge \hat{t_k} \wedge \dots \wedge t_n \otimes v^*_\alpha \otimes \omega_\alpha \otimes \pi_\emptyset -$$

$$- \sum_{k=1}^{n} (-1)^{k+1} 1 \otimes t_1 \wedge \dots \wedge \hat{t_k} \wedge \dots \wedge t_n \otimes Q^*_\alpha(t_k \otimes v^*_\alpha \otimes \omega_\alpha) \otimes \pi_\emptyset \ ,$$

where Q^*_α is given by the relations (3.14), (3.15) with v^*_α, v^*_β and $A^*_{\beta,\alpha}$ substituted instead of v_α, v_β and $A_{\alpha,\beta}$, $\pi_\emptyset$ is an N-vector on $\mathbf{C}^N$. The passage from the right D-modules to the left D-modules gives the sign $(-1)^{(\mathrm{codim}\ \alpha - \mathrm{codim}\ \beta - 1)/2}$ before $A^*_{\beta,\alpha}$.

Corollary 3.1 *The D-module $EV^t_{\mathcal{A}}$ is the dual to $EV_{\mathcal{A}}$.*

Note that the appearence of N-vector in (3.17) instead of the top form shows that it is more natural to treate the quiver's D-modules as right D-modules but the presense of affine structure in the stratified space allows one to pass from the left to right D-modules just by simple change of signes in their defining relations.

4 Weighted quivers and weighted D-modules

4.1. The Category $Qui_\lambda(\mathcal{A})$ of weighted quivers.

Let us assign to any hyperplane H_i of the arrangement $\mathcal{A}$ a complex number (weight) λ_i, and set

$$\lambda_\alpha = \sum_{i:H_i\supset\overline{X}_\alpha} \lambda_i \tag{4.1}$$

for any stratum X_α of the arrangement, and

$$\lambda_{\alpha,\beta} = \sum_{i:H_i\supset\overline{X}_\beta, H_i\not\supset X_\alpha} \lambda_i \tag{4.2}$$

for any arrow $\alpha \to \beta$ of the graph $\Gamma_{\mathcal{A}}$. Let us fix a stratum $\overline{X}_\alpha$. Then we can attach to this stratum a subalgebra $\mathcal{B}^0_\alpha \subset \mathcal{B}_{\mathcal{A}}$ of the path algebra $\mathcal{B}_{\mathcal{A}}$ (see section 1) generated by the elements $A^\beta_\alpha = A_{\alpha,\beta}A_{\beta,\alpha}$, where β runs all the vertices of $\Gamma_{\mathcal{A}}$ satisfying the condition $\alpha \to \beta$. The algebra $\mathcal{B}^0_\alpha$ can be described as an algebra with generators $A^\beta_\alpha, \alpha \to \beta$ subject to quadratic relations

$$[\sum_{\beta:\alpha\to\beta\to\gamma} A^\beta_\alpha, A^\delta_\alpha] = 0 \tag{4.3}$$

which are enumerated by the flags $\alpha \to \delta \to \gamma$. One can easily see that for any quiver $V_{\mathcal{A}}$ the linear space V_α is equiped with a structure of $\mathcal{B}^0_\alpha$ module. We define a subcategory $Qui_\lambda(\mathcal{A}) \subset Qui(\mathcal{A})$ as follows:

Definition 4.1 *A quiver $V_{\mathcal{A}} \in Qui(\mathcal{A})$ belongs to the subcategory $Qui_\lambda(\mathcal{A})$ if for any stratum X_α, the linear space V_α considered as a $\mathcal{B}^0_\alpha$-module admits a finite filtration by $\mathcal{B}^0_\alpha$-submodules with the factors isomorphic to $\mathbf{C}_{\lambda,\alpha}$, where $\mathbf{C}_{\lambda,\alpha}$ is one-dimensional $\mathcal{B}^0_\alpha$-module in which the generators A^β_α act by multiplication over $\lambda_{\alpha,\beta}$:*

$$\rho_{\lambda,\alpha}(A^\beta_\alpha) = \lambda_{\alpha,\beta} \cdot \mathrm{Id}. \tag{4.4}$$

In other words, the action of the operators $A_{\alpha,\beta}$ in the space V_α can be given in some basis by triangular matrices with diagonal $\lambda_{\alpha,\beta} \cdot \mathrm{Id}$. The proposition below describes the properties of the category $Qui_\lambda(\mathcal{A})$.

Proposition 4.1 (i) *The category $Qui_\lambda(\mathcal{A})$ is a full abelian subcategory of $Qui(\mathcal{A})$ closed under extensions;*
(ii) *The category $Qui_\lambda(\mathcal{A})$ is stable with respect to the involution ^{t};*
(iii) *The category $Qui_\lambda(\mathcal{A})$ has finite number of irreducibles; the irreducibles $L_{\lambda,\alpha}$ are labeled by the vertices α of the graph $\Gamma_{\mathcal{A}}$ and are supported on the subgraph $\Gamma^\alpha_{\mathcal{A}}$ of the closure of the stratum X_α, that is $(L_{\lambda,\alpha})_\beta \neq 0 \Rightarrow \alpha > \beta$ and $(L_{\lambda,\alpha})_\alpha \neq 0$.*

The conjecture that the category $Qui_\lambda(\mathcal{A})$ containes the projective covers and has homological dimension at most N also makes sense but we did not try to prove it.

Proof. The statements *(i)* and *(ii)* follow directly from the definitions and from the description of the involution. The most nontrivial statement *(iii)* is based on the explicit construction of an analog of Verma modules $M_{\lambda,\alpha} \subset Qui_\lambda(\mathcal{A})$ which we give below. The quiver $M_\lambda = M_{\lambda,\emptyset}$ has the unique irreducible factor $L_\lambda = L_{\lambda,\emptyset}$ and is characterized by the property

$$\mathrm{Hom}_{Qui(\mathcal{A})}\left(M_\lambda, V_\mathcal{A}\right) \approx \mathrm{Hom}_{\mathrm{B}^0_\emptyset}\left(\mathbf{C}_{\lambda,\emptyset}, (V_\mathcal{A})_\emptyset\right) \tag{4.5}$$

for any quiver $V_\mathcal{A} \in Qui(\mathcal{A})$. The $\mathrm{B}^0_\emptyset$ character of $\mathbf{C}_{\lambda,\emptyset}$ is defined by the relation (4.4):

$$\rho_{\lambda,\emptyset}(A_{\emptyset,i}A_{i,\emptyset}) = \lambda_i \cdot \mathrm{Id}\ . \tag{4.6}$$

The construction of the quivers $M_{\lambda,\alpha}$ and of their irreducible factors uses the inclusions $j_\alpha : \overline{X}_\alpha \hookrightarrow \mathbf{C}^N$. The inclusion j_α defines the induced stratification $\mathcal{A}^\alpha$ in affine space $\overline{X}_\alpha$ which is described by the graph $\Gamma^\alpha_\mathcal{A}$. It also defines the full exact functor (which we denote by the same letter)

$$j_\alpha :\ Qui(\mathcal{A}^\alpha) \to Qui(\mathcal{A})\ .$$

A collection of weights $\lambda = \{\lambda_i\}$ for the stratification $\mathcal{A}$ naturally defines a collection of weights $\lambda^\alpha = \{\lambda_{\alpha,\beta}\}$ where β runs all the vertices of $\Gamma^\alpha_\mathcal{A}$ satisfying the condition $\alpha \to \beta$. Simple linear algebra calculations show that the functor j_α maps the category $Qui_{\lambda^\alpha}(\mathcal{A}^\alpha)$ into $Qui_\lambda(\mathcal{A})$:

$$j_\alpha\left(Qui_{\lambda^\alpha}(\mathcal{A}^\alpha)\right) \subset Qui_\lambda(\mathcal{A})\ . \tag{4.7}$$

Thus the irreducible quiver $L_{\lambda,\alpha}$ with a support $\Gamma^\alpha_\mathcal{A}$ is

$$L_{\lambda,\alpha} = j_\alpha(L_{\lambda^\alpha})$$

and the proof of the statement *(iii)* of the proposition follows by induction over the codimension of the support of the irreducibles using (4.5). Let us describe now the quiver M_λ.

Construction of the quiver M_λ.

The quiver M_λ which represents the functor

$$\mathcal{F}(V_\mathcal{A}) = \mathrm{Hom}_{\mathrm{B}^0_\emptyset}\left(\mathbf{C}_{\lambda,\emptyset}, (V_\mathcal{A})_\emptyset\right)$$

can be defined in terms of the flag complex [4].

Definition 4.2 *A (complete) flag $L_{\alpha_0,\ldots,\alpha_n}$ in the stratified space $X_\mathcal{A}$ is a chain of the closures of strata*

$$\overline{X}_{\alpha_0} \supset \overline{X}_{\alpha_1} \supset \ldots \supset \overline{X}_{\alpha_n}\ , \qquad \mathrm{codim}\overline{X}_{\alpha_j} = j\ . \tag{4.8}$$

In terms of the graph $\Gamma_\mathcal{A}$, the complete flag $L_{\alpha_0,\ldots,\alpha_n}$ is a chain $\alpha_0 \to \ldots \to \alpha_n$, $\alpha_0 = \emptyset$. The linear space $(M_\lambda)_\gamma$ of the quiver M_λ we identify with the factorspace of linear space with a basis given by all flags $L_{\alpha_0,\ldots,\alpha_n}$, $\alpha_n = \gamma$ modulo the relations

$$\sum_{\beta:\alpha_{k-1}\to\beta\to\alpha_{k+1}} L_{\alpha_0,\ldots,\alpha_{k-1},\beta,\alpha_{k+1},\ldots\alpha_n} = 0 \tag{4.9}$$

valid for all partial flags $\alpha_0 \to \ldots \to \alpha_{k-1} > \alpha_{k+1} \to \ldots \to \alpha_n$, $\alpha_0 = \emptyset, \alpha_n = \gamma$, codim $\alpha_{k+1} =$ codim $\alpha_{k-1} + 2$. For any arrow $\alpha \to \beta$ the linear operators

$$A_{\beta,\alpha} : (M_\lambda)_\alpha \to (M_\lambda)_\beta, \qquad A_{\alpha,\beta} : (M_\lambda)_\beta \to (M_\lambda)_\alpha$$

have the following form:

$$A_{\beta,\alpha}(L_{\alpha_0,\dots,\alpha_{n-1},\alpha}) = L_{\alpha_0,\dots,\alpha_{n-1},\alpha,\beta} \ , \tag{4.10}$$

$$A_{\alpha,\beta}L_{\alpha_0,\dots,\alpha_n,\beta} =$$

$$\sum_{k=0}^{n}(-1)^k\lambda_{\alpha_{n-k},\alpha_{n-k+1}} \sum_{\alpha'_{n-k+1},\dots,\alpha'_n:\alpha'_j\to\alpha_{j+1},\alpha'_j\neq\alpha_j,\alpha'_n=\beta} L_{\alpha_0,\dots,\alpha_{n-k},\alpha'_{n-k+1},\dots,\alpha'_n} \ . \tag{4.11}$$

One can verify by linear algebra calculations that the operators (4.10) and (4.11) are correctly determined maps of corresponding factorspaces and define a quiver from $Qui_\lambda(\mathcal{A})$ satisfying the condition (4.5).

4.2. The category $\mathcal{M}_\lambda(\mathcal{A})$ of weighted D-modules.

For any collection $\lambda = \{\lambda_i, i \in I\}$ of weights we define also a category $\mathcal{M}_\lambda(\mathcal{A}) \subset \mathcal{M}(\mathcal{A})$ of weighted D-modules.

Let f_γ, $\gamma \in \Gamma_{\mathcal{A}}$ stands for a generic linear (affine) function over $\mathbf{C}^N$, such that $f_\gamma|_{\overline{X}_\gamma} = 0$. Let $\Phi_{f_\alpha}(M)|_{X_\alpha}$ be the restriction to the open part X_α of affine subspace $\overline{X}_\alpha$ of D-module of vanishing cycles $\Phi_{f_\alpha}(M)$. If $M \in \mathcal{M}(\mathcal{A})$ then $\Phi_{f_\alpha}(M)|_{X_\alpha}$ is a flat D_{X_α}-module which one can describe by a flat connection in trivial bundle over X_α with regular singularities on $\overline{X}_\alpha$. Let us fix for a moment a collection of generic functions f_α, $\alpha \in \Gamma_{\mathcal{A}}$.

Definition 4.3 *A D-module M from $\mathcal{M}(\mathcal{A})$ belongs to the category $\mathcal{M}_\lambda(\mathcal{A})$, if for any stratum X_α D_{X_α}-module $\Phi_{f_\alpha}(M)|_{X_\alpha}$ admits D_{X_α}-filtration with factors isomorphic to the modules given by a flat connection in one-dimensional bundle with a form*

$$\nu_\alpha = \sum_{\beta:\alpha\to\beta} \lambda_{\alpha,\beta} d\log f_\beta$$

In other words, for some affine coordinates D-module $\Phi_{f_\alpha}(M)|_{X_\alpha}$ is given by a connection in trivial bundle with a form

$$\nu_\alpha = \sum_{\beta:\alpha\to\beta} L_{\alpha,\beta} d\log f_\beta$$

where $L_{\alpha,\beta}$ is a triangular matrix with constant coefficients and diagonal entries being equal to $\lambda_{\alpha,\beta}$. One can see that the definition of the category $\mathcal{M}_\lambda(\mathcal{A})$ does not depend on the choise of generic linear functions f_α, $\alpha \in \Gamma_{\mathcal{A}}$ and depends only on the projection $\tilde{\pi}(\lambda)$ of the collections of weights to $\mathbf{C}/\mathbf{Z}$, $\tilde{\pi}(\lambda) = \exp(2\pi i\lambda)$.

Definition 4.4 *A collection $\{\lambda\}$ of weights is called* **non-resonance** *if for any $\alpha, \beta, \gamma, \delta \in \Gamma_{\mathcal{A}}$, the following conditions hold:*

$$\lambda_{\alpha,\beta} \notin \mathbf{Z} \setminus \{0\} \qquad \text{and} \qquad \lambda_{\alpha,\beta} - \lambda_{\gamma,\delta} \notin \mathbf{Z} \ . \tag{4.12}$$

The non-resonance property also essentially depends only on the exponents of weights in a sense that one can or cannot get the condition (4.12) by shifting some of λ_i by an integer. One of the main properties of weighted D-modules is given in the following theorem.

Theorem 4.1 *For any non-resonance collection $\{\lambda\}$ of weights, the functor E establishes an equivalence of categories $Qui_\lambda(\mathcal{A})$ and $\mathcal{M}_\lambda(\mathcal{A})$.*

The statement of the theorem is a direct generalization of the results of [3]. Let us describe the scheme of the proof. It consists of the inductive (over the codimension of strata) construction of an inverse functor $\widetilde{E} : \mathcal{M}_\lambda(\mathcal{A}) \to Qui_\lambda(\mathcal{A})$ with simultenious proof of basic statement that the functor E coincides with Beilinson's functor [1] of glueing the D-modules. The functor $\widetilde{E}$ is as follows. The linear space $(\widetilde{E}M)_\alpha$ for $M \in \mathcal{M}_\lambda(\mathcal{A})$ is isomorphic to the space of flat sections of $\Phi_{f_\alpha}(M)|_{X_\alpha}$ and as $\mathcal{B}^0_\alpha$-module the space $(\widetilde{E}M)_\alpha$ is equivalent to $\Phi_{f_\alpha}(M)|_{X_\alpha}$ as $\pi_1(\overline{X}_\alpha \setminus X_\alpha)$-module, the operators $A_{\alpha,\beta}$ and $A_{\beta,\alpha}$, $\alpha \to \beta$ are proportional to canonical maps *can* and *var* between D_{X_α}-modules $\Phi_{f_\alpha}(M)|_{X_\alpha}$ and $\Psi_{f_\alpha}(M)|_{X_\alpha}$. Thus, the crucial calculations are the calculation of the vanishing cycles functor $\Psi_{f_\alpha}(M)|_{X_\alpha}$ with the action of monodromy operator on it. These calculations were made in [3], under unipotent monodromy assumption, that is, for zero weights λ_i.

As a result of a more general calculation of the specialization functors, we show in the next section that the vanishing cycles calculations remain valid also under a weaker non-resonance assumption. This allows to establish an equivalence of categories for non-resonance weighted D-modules as well.

5 Specialization of quivers and D-modules

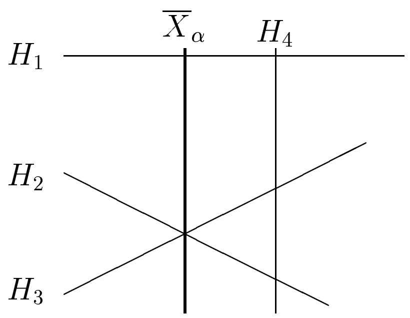
5.1. Specialization of quivers.

Let X_α be a stratum of stratification $X_\mathcal{A}$. Its closure $\overline{X}_\alpha$ is an affine subspace of $X = \mathbf{C}^N$, and the total space of normal bundle $T_{\overline{X}_\alpha}X$ inherits the affine structure and can be naturally identified with a product of the affine space $\overline{X}_\alpha$ and the linear space $X/\overline{X}_\alpha$: $T_{\overline{X}_\alpha}X \approx \overline{X}_\alpha \times X/\overline{X}_\alpha$. Both $\overline{X}_\alpha$ and $X/\overline{X}_\alpha$ are stratified spaces: the strata of $\overline{X}_\alpha$ are $X_\beta, X_\beta \subset \overline{X}_\alpha$, $\beta \in \Gamma_\mathcal{A}$, and the strata of $X/\overline{X}_\alpha$ are the projections $p_\alpha(X_\beta)$, $\beta \in \Gamma_\mathcal{A}$, $X_\beta \cap \overline{X}_\alpha \neq 0$. Here $p_\alpha : \mathbf{C}^N = X \to X/\overline{X}_\alpha$ is a natural projection. Let us denote strata of $\overline{X}_\alpha$ as $X_{\beta'}$ and strata of $X/\overline{X}_\alpha$ as $X_{\beta''}$. Note that the stratification of $X/\overline{X}_\alpha$ is linear: the closure of any stratum contains zero vector.

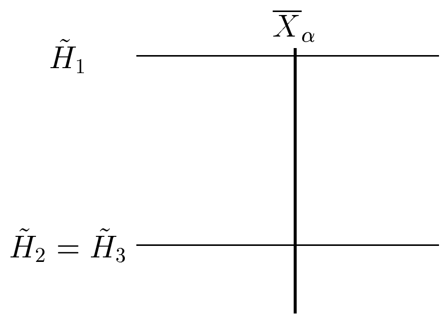
The stratification $T_{\overline{X}_\alpha}\mathcal{A}$ of the space $T_{\overline{X}_\alpha}X$ identified as affine space with $\overline{X}_\alpha \times X/\overline{X}_\alpha$ is described by certain direct products of strata of $\overline{X}_\alpha$ and $X/\overline{X}_\alpha$: $(X_{\beta'} \times X_{\beta''}) \in T_{\overline{X}_\alpha}\mathcal{A}$ if there exists such a stratum X_β of stratification $\mathcal{A}$ for $\mathbf{C}^N$ that

$$X_{\beta'} = \overline{X}_\alpha \cap X_\beta, \qquad X_{\beta''} = p_\alpha(X_\beta) \ .$$

One can see that the stratification $T_{\overline{X}_\alpha}\mathcal{A}$ is also described by intersections of hyperplanes $\tilde{H}_i, i \in I : \tilde{H}_i = (H_i \cap \overline{X}_\alpha) \times p_\alpha(H_i)$. The passage from stratification $\mathcal{A}$ to $T_{\overline{X}_\alpha}\mathcal{A}$ is illustrated by a two-dimensional example below.



$\mathcal{A}$



$T_{\overline{X}_\alpha}\mathcal{A}$

Correspondingly, the graph $\Gamma_{T_{\overline{X}_\alpha}\mathcal{A}}$ is a subgraph of direct product of the graph $\Gamma'_{\mathcal{A}}$ of the stratification of the space $\overline{X}_\alpha$ and of the graph $\Gamma''\mathcal{A}$ of the stratification of the space $X/\overline{X}_\alpha$: $\Gamma_{T_{\overline{X}_\alpha}\mathcal{A}} \in \Gamma'_{\mathcal{A}} \times \Gamma''_{\mathcal{A}}$. It means that the vertices of $\Gamma_{T_{\overline{X}_\alpha}\mathcal{A}}$ are numerated by some pairs (β', β'') of the vertices of $\Gamma'\mathcal{A}$ and of $\Gamma''_{\mathcal{A}}$, and the arrows could be of two possible types:

$$(\beta', \beta'') \to (\gamma', \beta'')\ , \qquad \text{if } \beta' \to \gamma' \in \operatorname{arr}(\Gamma'_{\mathcal{A}})\ ,$$

$$(\beta', \beta'') \to (\beta', \gamma'')\ , \qquad \text{if } \beta'' \to \gamma'' \in \operatorname{arr}(\Gamma''_{\mathcal{A}})\ .$$

Since the stratification $T_{\overline{X}_\alpha}\mathcal{A}$ is given by intersections of hyperplanes $\tilde{H}_i$, we have by definition 1.1 the quiver category $Qui(T_{\overline{X}_\alpha}\mathcal{A})$.

Let now $V_{\mathcal{A}} = \{V_\beta, A_{\beta,\gamma}\}$ be a quiver from the category $Qui(\mathcal{A})$. We define a quiver $Sp_\alpha V_{\mathcal{A}} = \{V_{(\beta',\beta'')}, A_{(\gamma',\gamma''),(\beta',\beta'')}\}$ as follows. The space $V_{(\beta',\beta'')}$ is the following direct sum of linear spaces V_β:

$$V_{(\beta',\beta'')} = \oplus_{\beta: X_\beta \cap \overline{X}_\alpha = X_{\beta'}, p_\alpha(X_\beta) = X_{\beta''}} V_\beta\ . \tag{5.1}$$

The operators $A_{(\beta',\beta''),(\gamma',\gamma'')}$ coincide with operators from the quiver $V_{\mathcal{A}}$:

$$A_{(\gamma',\beta''),(\beta',\beta'')}(v_\beta) = \oplus_{\gamma: X_\gamma \cap \overline{X}_\alpha = X_{\gamma'}, p_\alpha(X_\gamma) = X_{\beta''}} A_{\gamma,\beta}(v_\beta)\ , \tag{5.2}$$

$$A_{(\beta',\gamma''),(\beta',\beta'')}(v_\beta) = \oplus_{\gamma: X_\gamma \cap \overline{X}_\alpha = X_{\beta'}, p_\alpha(X_\gamma) = X_{\gamma''}} A_{\gamma,\beta}(v_\beta)\ , \tag{5.3}$$

where the vector v_β belongs to linear space V_β such that $X_\beta \cap \overline{X}_\alpha = X_{\beta'}, p_\alpha(X_\beta) = X_{\beta''}$. A map Sp_α is also defined on the weights. For a collection of weights $\{\lambda\} = \lambda_i = \lambda(H_i)$ of stratification $\mathcal{A}$ the collection $\{Sp_\alpha(\lambda)\}$ is defined by the rule $Sp_\alpha(\lambda)_i = \sum_{j: \tilde{H}_j = \tilde{H}_i} \lambda(H_i)$.

Proposition 5.1 (i) *The relations (5.1)–(5.3) define an exact functor*

$$Sp_\alpha \ : Qui(\mathcal{A}) \to Qui(T_{\overline{X}_\alpha}\mathcal{A})$$

which commutes with the duality ^{t}*;*

(ii) *For any collection* λ *of weights for the configuration* $\mathcal{A}$*,*

$$Sp_\alpha(M_\lambda) = M_{Sp_\alpha(\lambda)}\ . \tag{5.4}$$

Proof. In order to check *(i)* it is suficient to check that the operators defined by the relations (5.2) and (5.3) satisfy the relations (2.2) for the configuration $T_{\overline{X}_\alpha}\mathcal{A}$. These are linear algebra calculations. Analogously, for the proof of *(ii)* one has to check the formulas (4.10), (4.11) or to use the functorial definition of the modules M_λ and $M_{Sp_\alpha(\lambda)}$.

Let now $\overline{X}_{\alpha_1} \subset \overline{X}_{\alpha_2} \subset X = \mathbf{C}^N$ be a flag of inclusions of closed strata to X. Then the spaces $T_{\overline{X}_{\alpha_1}}(T_{\overline{X}_{\alpha_2}})$ and $T_{T_{\overline{X}_{\alpha_1}}\overline{X}_{\alpha_2}}(T_{\overline{X}_{\alpha_1}}X)$ are canonically isomorphic to a direct product $\overline{X}_{\alpha_1} \times \overline{X}_{\alpha_2}/\overline{X}_{\alpha_1} \times X/\overline{X}_{\alpha_2}$. Let us call it the space of flags of normal covectors and denote by $T_{\overline{X}_{\alpha_2} \subset \overline{X}_{\alpha_2}}$. It is a stratified space with a stratification analogous to described above. We denote it by $T_{\overline{X}_{\alpha_2} \subset \overline{X}_{\alpha_2}}\mathcal{A}$. One can easily see that the compositions

$$Sp_{\alpha_1}Sp_{\alpha_2} : Qui(\mathcal{A}) \to Qui(T_{\overline{X}_{\alpha_2} \subset \overline{X}_{\alpha_2}}\mathcal{A})$$

and

$$Sp_{\alpha_2}Sp_{\alpha_1} : Qui(\mathcal{A}) \to Qui(T_{\overline{X}_{\alpha_2} \subset \overline{X}_{\alpha_2}}\mathcal{A}) \ ,$$

are canonically isomorphic. Here the functor Sp_{α_2} in the second composition is the functor of specialization to the stratum $\overline{X}_{\alpha_1} \times \overline{X}_{\alpha_2}/\overline{X}_{\alpha_1}$.

Analogously, for a three-flag $\overline{X}_{\alpha_1} \subset \overline{X}_{\alpha_2} \subset \overline{X}_{\alpha_3} \subset X = \mathbf{C}^N$ the morphism of associativity is identical on the linear spaces of a quiver, so for any flag

$$\overline{X}_{\alpha_1} \subset \overline{X}_{\alpha_2} \subset \ldots \subset \overline{X}_{\alpha_n} \subset X = \mathbf{C}^N$$

of closed strata we have a specialization functor

$$Sp_{\alpha_1,\ldots\alpha_n} : Qui(\mathcal{A}) \to Qui(T_{\overline{X}_{\alpha_1} \subset \ldots \subset \overline{X}_{\alpha_n}})$$

which is described by itereted relations (5.1)–(5.3) and satisfy the properties mentioned in Proposition 4.1.

5.2. Specialization of quiver D-modules.

Let us first consider the case of *linear* stratification in linear space $\mathbf{C}^N$, that is we suppose that all hyperplanes H_i contain an origin. Let I_α be an ideal of $\overline{X}_\alpha$: $f \in I_\alpha \Leftrightarrow f|_{\overline{X}_\alpha} = 0$ and let $F^\bullet_\alpha(D_X)$ be a filtration of the ring D_X corresponding to ideal I_α [2]:

$$F^k_\alpha(D_X) = \{P \in D_X : P(I^j_\alpha \subset I^{j+k}_\alpha \text{for any } j\} \ .$$

Here $I^k_\alpha = \mathbf{C}[X]$ for $k \leq 0$. Then $F^k_\alpha(D_X)/F^{k+1}_\alpha(D_X)$ is isomorphic to a space of differential operators over $T_{\overline{X}_\alpha}X$ homogenuous of order k and the ring $Gr\ F_\alpha(D_X)$ is isomorphic to the ring $D_{T_{\overline{X}_\alpha}X}$.

Let now $V_\mathcal{A}$ be a quiver from $Qui(\mathcal{A})$ and $EV_\mathcal{A}$ be corresponding D_X-module. We define a filtration $F^\cdot_\alpha EV_\mathcal{A}$ of $EV_\mathcal{A}$ in the following way:

$$F^n_\alpha EV_\mathcal{A} = \sum_{k=0}^{N} \sum_{\beta : codim_{X_\beta} X_\beta \cap \overline{X}_\alpha = k} F^{n-k}_\alpha(D_X) \cdot (V_\beta \otimes \Omega_\beta) \ . \tag{5.5}$$

One can check that the filtration $F^\bullet_\alpha EV_\mathcal{A}$ is good, that is satisfy the following conditions:

(i) $F^k_\alpha(D_X) \cdot F^j_\alpha EV_\mathcal{A} \subset F^{k+j}_\alpha EV_\mathcal{A}$;

(ii) $F^k_\alpha(D_X) \cdot F^j_\alpha EV_\mathcal{A} \ = F^{k+j}_\alpha EV_\mathcal{A}$ for $j, k \geq 0$ and for $j \leq -N, k \leq 0$;

(iii) $\cup_j F^j_\alpha EV_{\mathcal{A}} = EV_{\mathcal{A}}$;

(iv) $F^j_\alpha EV_{\mathcal{A}}$ is coherent $F^0_\alpha(D_X)$-module.

The following proposition describes $Gr\ F_\alpha(EV_{\mathcal{A}})$ as $D_{T_{\overline{X}_\alpha}X}$- module. It follows directly from the definition of quiver's D-module $EV_{\mathcal{A}}$ and from the definition of the filtration.

Proposition 5.2 *There is a canonical isomorphism of $D_{T_{\overline{X}_\alpha}X}$-modules:*

$$Gr\ F_\alpha(EV_{\mathcal{A}}) \approx ESp_\alpha\ V_{\mathcal{A}}\ . \tag{5.6}$$

As it follows from the relations (5.1)–(5.3), the isomorphism (5.6) means that $D_{T_{\overline{X}_\alpha}X}$-module $Gr\ F_\alpha(EV_{\mathcal{A}})$ is generated by vectors $v_\alpha \otimes \omega_{\beta'} \wedge \omega_{\beta''}$ where $\beta \in \mathrm{ver}(\Gamma_{\mathcal{A}})$, $v_\beta \in V_\beta$, $\omega_{\beta'}$ is a top form for $X_\beta \cap \overline{X}_\alpha$, $\omega_{\beta''}$ is a top form for $X_\beta / X_\beta \cap \overline{X}_\alpha = p_\alpha(X_\beta)$ with the following relations:

$$L_{\beta'}(v_\beta \otimes \omega_{\beta'} \wedge \omega_{\beta''}) = \sum_{\substack{\gamma:\beta\to\gamma \\ X_\gamma\cap\overline{X}_\alpha \neq X_\beta\cap\overline{X}_\alpha}} \frac{L_{\beta'}(\hat{\omega}_{\gamma'}) \wedge \omega_{\beta'}}{\hat{\omega}_{\gamma'} \wedge \omega_{\gamma'}} A_{\gamma,\beta}(v_b) \otimes \omega_{\gamma'} \wedge \omega_{\beta''} \tag{5.7}$$

where $L_{\beta'}$ is linear vectorfied along stratum $X_\beta \cap \overline{X}_\alpha$, $\hat{\omega}_{\gamma'}$ is a top form for $\overline{X}_\alpha / \overline{X}_\gamma \cap \overline{X}_\alpha$, $\omega_{\gamma'}$ is a top form for $\overline{X}_\gamma \cap \overline{X}_\alpha$;

$$L_{\beta''}(v_\beta \otimes \omega_{\beta'} \wedge \omega_{\beta''}) = \sum_{\substack{\gamma:\beta\to\gamma \\ X_\gamma\cap\overline{X}_\alpha = X_\beta\cap\overline{X}_\alpha}} \frac{L_{\beta''}(\hat{\omega}_{\gamma''}) \wedge \omega_{\beta''}}{\hat{\omega}_{\gamma''} \wedge \omega_{\gamma''}} A_{\gamma,\beta}(v_b) \otimes \omega_{\beta'} \wedge \omega_{\gamma''} \tag{5.8}$$

where $L_{\beta''}$ is linear vectorfied in $X/\overline{X}_\alpha$ along $X_\beta / X_\beta \cap \overline{X}_\alpha$, $\hat{\omega}_{\gamma''}$ is a top form for $(X/\overline{X}_\alpha)/(X_\gamma/\overline{X}_\gamma \cap \overline{X}_\alpha)$, $\omega_{\gamma''}$ is a top form for $X_\gamma / \overline{X}_\gamma \cap \overline{X}_\alpha$;

$$f_{\beta'}(v_\beta \otimes \omega_{\beta'} \wedge \omega_{\beta''}) = \sum_{\substack{\gamma:\gamma\to\beta \\ X_\gamma\cap\overline{X}_\alpha \neq X_\beta\cap\overline{X}_\alpha}} \frac{df_{\beta'} \wedge \hat{\omega}_{\gamma'} \wedge \omega_{\beta'}}{\hat{\omega}_{\gamma'} \wedge \omega_{\gamma'}} A_{\gamma,\beta}(v_b) \otimes \omega_{\gamma'} \wedge \omega_{\beta''} \tag{5.9}$$

where $f_{\beta'}$ is linear function on $\overline{X}_\alpha$ equal to zero on $X_\beta \cap \overline{X}_\alpha$, with the same $\hat{\omega}_{\gamma'}$, $\omega_{\gamma'}$ as for (5.7);

$$f_{\beta''}(v_\beta \otimes \omega_{\beta'} \wedge \omega_{\beta''}) = \sum_{\substack{\gamma:\gamma\to\beta \\ X_\gamma\cap\overline{X}_\alpha = X_\beta\cap\overline{X}_\alpha}} \frac{df_{\beta''} \wedge \hat{\omega}_{\gamma''} \wedge \omega_{\beta''}}{\hat{\omega}_{\gamma''} \wedge \omega_{\gamma''}} A_{\gamma,\beta}(v_b) \otimes \omega_{\beta'} \wedge \omega_{\gamma''} \tag{5.10}$$

where $f_{\beta''}$ is linear function on $X/\overline{X}_\alpha$ equal to zero on $X_\beta / X_\beta \cap \overline{X}_\alpha$ with the same $\hat{\omega}_{\gamma''}$ and $\omega_{\gamma''}$ as in (5.8).

We can drop the assumption of linearity for the stratification if we pass first to analytic D-modules, and thus kill the strata which do not intersext with $\overline{X}_\alpha$, and then use the well known equivalence between analytic and algebraic D-modules. Alternatively, we could first localize the initial D-module to a Zariski open set which does not contain strata not intersecting with $\overline{X}_\alpha$, and then use the construction above.

In order to compare $D_{T_{\overline{X}_\alpha} X}$- module $Gr\ F_\alpha(EV_{\mathcal{A}})$ with the Malgrange-Kashiwara specialization of the D_X-module $EV_{\mathcal{A}}$, we should investigate the eigenvalue of the target vectorfiled θ^α which satisfy the relations $\theta^\alpha|_{I_\alpha/I_\alpha^2} = id$ [2]. In local coordinates $\{\overline{z}^\alpha, \overline{w}^\alpha\} = \{z_1^\alpha, \ldots, z_{dim\ X_\alpha}^\alpha, w_1^\alpha, \ldots, w_{codim\ X_\alpha}^\alpha\}$ satisfying (3.3) it looks like $\theta^\alpha = \sum_j w_j^\alpha \frac{\partial}{\partial w_j^\alpha}$. Let also $\lambda = \{\lambda_1, \ldots, \lambda_n\}$ be a collection of weights and M_λ be a quiver constructed in section 3.1 (see (4.5)).

Proposition 5.3

$$\theta^\alpha | Gr^k F_\alpha E M_\lambda = \lambda_\alpha + k\ . \tag{5.11}$$

The proof follows from the relations (5.7), (5.8).

Correspondingly, for D_X-module $E(M_{\lambda,\beta})$ supported on the stratum $\overline{X}_\beta$ ($\overline{X}_\beta \cap \overline{X}_\alpha \neq 0$), we have

$$\theta^\alpha | Gr^k F_\alpha E M_{\lambda,\alpha} = \lambda_{\beta',\beta} + k\ ,$$

where $X_{\beta'} = X_\beta \cap \overline{X}_\alpha$.

Corollary 5.1 (i) *Let the number* $\lambda_\alpha = \sum_{i: H_i \supset X_\alpha} \notin \mathbf{Z} \setminus 0$. *Then*

$$Sp_{\overline{X}_\alpha} E(M_\lambda) \approx E\ Sp_\alpha(M_\lambda)\ ; \tag{5.12}$$

(ii) *Let the nonresonance condition takes place:* $\lambda_{\beta,\gamma} - \lambda_{\nu,\varepsilon} \notin \mathbf{Z} \setminus 0$. *Then*

$$Sp_{\overline{X}_\alpha} E(M) \approx E\ Sp_\alpha(M) \tag{5.13}$$

for any $M \in Qui(\mathcal{A})$.

Here $Sp_{\overline{X}_\alpha}$ denotes the Malgrange-Kashiwara specialization functor, [2].

Proof of Theorem 4.1. Following the scheme of [3], we choose a collection of generic linear functions f_α, $f_\alpha|_{\overline{X}_\alpha} = 0$ and perform the glueing procedure starting from a local system on the open stratum $X_\emptyset$ and adding step by step strata of bigger codimension. In order to glue the stratum X_α we add hyperplane $f_\alpha = 0$ to the stratification with zero weight and with a condition that the support of D-module intersects with hyperplane $f_\alpha = 0$ only by stratum $\overline{X}_\alpha$. Part *(ii)* of the Corollary 4.1 allows to calculate the linear algebra data

$$\Psi_{f_\alpha}(M) \overset{\leftarrow}{\underset{\rightarrow}{}} \Phi_{f_\alpha}(M)$$

in terms of Malgrange-Kashiwara speciaization functor assuming we start from quiver's D-modules with nonresonance condition.

Corollary 4.1 guarantees that they coincide with the linear algebra data calculated by means of specialization functors in category $Qui_\lambda(\mathcal{A})$ (and thus coincide with those calculated by Beilinson's technique for the unipotent case in [3]). Then we can refer to [3] since we drop unipotent restriction. Alternatively, one can get an independent proof after the formulation of an analog of the Beilinson-Kashiwara equivalence theorem for the category $Qui(\mathcal{A})$ and functors Sp_α from section 4.1, which is purely combinatorial.

References

[1] A. Beilinson, How to glue perverse sheaves, in: K-theory, arithmetic and geometry, Yu.I. Manin (Ed.), *Lect. Notes in Math.*, **1289**, Springer-Verlag, Berlin et al., 1987; 42-51.

[2] M. Kashiwara, Vanishing cycles and holonomic systems of differential equations, in: Algebraic Geometry (Tokyo/Kyoto, 1982), M. Raynaud, T. Shioda (Eds.), *Lect. Notes in Math.*, **1016**, Springer-Verlag, Berlin et al., 1983; 134-142.

[3] S. Khoroshkin, D-modules over arrangements of hyperplanes, *Comm. in Algebra* **23(9)** (1995), 3481-3504.

[4] V. Schechtman, A. Varchenko, Arrangements of hyperplanes and Lie algebra homology, *Inv. Math.,* **106** (1991), 134-194.

S.K.: Institute for Theorerical and Experimental Physics, Cheremushkinskaya 25, 117259 Moscow, Russia; khor@heron.itep.ru

V.S.: Max-Planck-Institut für Mathematik, Gottfried-Claren-Straße 26, 53225 Bonn, Germany; vadik@mpim-bonn.mpg.de