A Matrix Poincaré Formula For Holomorphic Automorphisms Of Real Associative Quadrics

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A MATRIX POINCARÉ FORMULA FOR HOLOMORPHIC AUTOMORPHISMS OF REAL ASSOCIATIVE QUADRICS

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ABSTRACT. We introduce the class of real associative Hermitian quadrics of codimension n in $\mathbb{C}^{2n}$ (RAQ) having the property that their automorphisms can be written by means of a matrix version of the Poincaré formula for $\operatorname{Aut} S^3$.

1. INTRODUCTION

Let $z = (z^j)$, $j = 1, \dots, n$, $w = u + iv = (w^j) = (u^j + iv^j)$, $j = 1, \dots, k$ be coordinates in $\mathbb{C}^n \times \mathbb{C}^k$, $k \leq n$;

$$\langle z, z\rangle = (\langle z, z\rangle^1, \dots, \langle z, z\rangle^k)$$

a $\mathbb{R}^k$-valued Hermitian form.

The following set is called a quadric $Q = \{(z, w) \in \mathbb{C}^{n+k} : v = \langle z, z\rangle\}$. Q is presumed to be nondegenerate, i.e.

i) $\langle z, b\rangle^j = 0$ for all $j = 1, \dots, k, z \in \mathbb{C}^n$ implies $b = 0$

ii) $\langle z, z\rangle^j$ are linearly independent $j = 1, \dots, k$.

Since Q is a homogeneous manifold ($\operatorname{Aut} Q$ acts transitively via the transformations $z \mapsto p + z$, $w \mapsto q + w + 2i\langle z, p\rangle$ with $(p, q) \in Q$) then $\operatorname{Aut} Q \cong Q \times \operatorname{Aut}_0 Q$, where $\operatorname{Aut}_0 Q$ is the isotropy group of a fixed point, say the origin.

$\operatorname{Aut}_0 Q$ is a finite dimensional Lie group iff Q is nondegenerate (see [Bel89]).

The problem of the description of $\operatorname{Aut}_0 Q$ dates back to the work of H. Poincaré 1907 [Poi07], where he discoverd the formula of the holomorphic automorphisms of the hypersphere $S^3 \subset \mathbb{C}^2$. N.Tanaka [Tan62] generalized the results of Poincaré for arbitrary nondegenerate hyperquadrics $Q^{2n-1} \subset \mathbb{C}^n$.

The case of codimension $k > 1$ has not been completed yet. The question about the description of $\operatorname{Aut} Q$ has been recently formulated once again by F. Forstnerič [For92].

Below we list some known facts concerning $\operatorname{Aut}_0 Q$.

Using the reflection principle G.Henkin and A.Tumanov [HT83] proved that $\operatorname{Aut}_0 Q$ consists of rational transformations.

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V. Beloshapka [Bel90] gave a description of the Lie algebra of the infinitesimal automorphisms of Q and he proved also that the quadrics of codimension $k > 2$ in general position are rigid, i.e. their isotropy group consists of trivial automorphisms $z \mapsto az$, $w \mapsto |a|^2 w$ for some complex number a (see [Bel91]).

For $k = 2$ A.Abrosimov [Abr92] discovered a sufficient condition for $\mathrm{Aut}_0 Q$ to consist of linear transformations: if in some coordinates the operator $(H^1)^{-1}H^2$ (H^j - the Hermitian matrix related to $\langle z, z\rangle^j$) has more than two different eigenvalues.

However, this result does not contribute anything to the case $n = k = 2$ as well as the "rigidity" property of Beloshapka.

The case $n = k = 2$ happened to be rather interesting since the quadrics of codimension 2 in $\mathbb{C}^4$ (there are three different types of such quadrics, namely the hyperbolic, elliptic and parabolic) have relatively rich automorphism groups consisting of rational transformations of degree ≤ 2 ([ES92a]).

The method we used was based on the fact that the quadrics in $\mathbb{C}^4$ admit a "matrix representation": there exist real, commutative matrix algebras $\mathfrak{A}$ with unit of dimension two realizing the quadric in the following sense: There is a linear isomorphism $\tau : \mathbb{R}^2 \longrightarrow \mathfrak{A}$, $\tau(x) = X$, such that $\tau(\langle x, y\rangle) = \tau(x)\tau(y)$, and therefore, $Q = \{(Z, W) \in (\mathfrak{A} \otimes \mathbb{C})^2 : \mathrm{Im}\, W = Z\bar{Z}\}$. Inserting matrices of this form into the automorphism formula of the Heisenberg sphere in $\mathbb{C}^2$ one immediately obtains an automorphism formula for the given quadric. An analogue of Chern Moser's normalization procedure shows that these are all automorphisms.

This method was very helpful to describe $\mathrm{Aut}_0 Q$ for another three different types of quadrics in $\mathbb{C}^6$, $k = 3$ [ES92b].

In the present paper we consider the case $k = n$ and describe $\mathrm{Aut}_0 Q$ for a certain class of quadrics, we call them "real associative quadrics" (RAQ).

Definition 1. *A quadric Q is called real associative (RAQ) iff there are coordinates in which the Hermitian form defines a real, associative product, i.e.*

$$\langle x, y\rangle \text{ is real for all real vectors } x, y \tag{1}$$

$$\langle x, \langle y, z\rangle\rangle = \langle\langle x, y\rangle, z\rangle \text{ for all } x, y, z \tag{2}$$

It occurs that this type of quadrics is the only one that admits a "matrix representation", and we prove that the formula for the automorphisms is a generalized Poincaré formula.

On the contrary to the mentioned above results of V.Beloshapka and A.Abrosimov indicating the automorphism groups being either poor or trivial, the RAQ class provides us with the relatively rich groups consisting of rational transformations of degree $\leq n$.

The RAQ construction gives us a hope to find the answer to the following open questions:

i) Are RAQ quadrics the only irreducible ones having nonlinear automorphisms?
ii) Is the degree of an element of $\mathrm{Aut}_0 Q$ always $\leq n$?
iii) Does the quadric which provides the maximal dimension of $\mathrm{Aut}_0 Q$ belong to the RAQ class?

2. Results

Before we formulate the results we introduce two types of elementary quadrics with matrix representations:

Definition 2. *Let $\mathfrak{A}_{r_1\dots r_s}$ be the m-dimensional ($m = 1+\sum_{i=1}^{s} r_s$) real, commutative, associative algebra spanned by $(1, n_1, \dots, n_1^{r_1}, \dots, n_s, \dots, n_s^{r_s})$ where 1 is the unit and $n_i^j n_i^l = n_i^{j+l}$ if $j+l \leq r_i$ and 0 if $j+l > r_i$, $n_i^j n_m^l = 0$ if $i \neq m$.*

By $\mathfrak{A}^{\mathbb{C}}_{r_1\dots r_s}$ we denote the 2m-dimensional real algebra $\mathfrak{A}_{r_1\dots r_s} \otimes_{\mathbb{R}} \mathbb{C}$.

Then the elementary quadric $Q_{r_1\dots r_s}$ of first type with parameters $(r_1, \dots, r_s)$ is the quadric in $\mathbb{C}^{2m} = \mathbb{C}^m_w \oplus \mathbb{C}^m_z = \mathfrak{A}_{r_1\dots r_s} \otimes_{\mathbb{R}} \mathbb{C} \oplus \mathfrak{A}_{r_1\dots r_s} \otimes_{\mathbb{R}} \mathbb{C}$ defined by $\mathrm{Im}\, w = z\bar{z}$.

The elementary quadric $Q^{\mathbb{C}}_{r_1\dots r_s}$ of second type with paramters $(r_1, \dots, r_s)$ is the quadric in $\mathbb{C}^{4m} = \mathbb{C}^{2m}_w \oplus \mathbb{C}^{2m}_z = \mathfrak{A}^{\mathbb{C}}_{r_1\dots r_s} \otimes_{\mathbb{R}} \mathbb{C} \oplus \mathfrak{A}^{\mathbb{C}}_{r_1\dots r_s} \otimes_{\mathbb{R}} \mathbb{C}$ defined by $\mathrm{Im}\, w = z\bar{z}$. This type is only for even n defined.

The simpliest quadrics of this form corresponding to $\mathfrak{A} = \mathbb{R}$ are the Heisenberg sphere $Q_H = S^3$ in $\mathbb{C}^2$ and the elliptic quadric $Q_H^{\mathbb{C}}$ in $\mathbb{C}^4$. The hyperbolic quadric in $\mathbb{C}^4$ is equivalent to $Q_H \times Q_H$, and the parabolic quadric to Q_1.

The following proposition characterizes RAQ quadrics in terms of "matrix representation".

Proposition 1. *A quadric Q admits a matrix representation, i.e. for some n-dimensional commutative real matrix algebra $\mathfrak{A}_Q$ $Q = \{(w, z) \in \mathfrak{A}_Q \otimes \mathbb{C} \times \mathfrak{A}_Q \otimes \mathbb{C} : \mathrm{Im}\, w = z\bar{z}\}$, if and only if it belongs to the RAQ class.*

Theorem 1. *A quadric Q is contained in the RAQ class if and only if it is equivalent to the direct product of elementary quadrics.*

The theorem below gives the description of the isotropy groups of RAQ quadrics:

Theorem 2. *Let $Q \in RAQ$, then any automorphism preserving the origin is of the form*

$$\begin{aligned} z &\mapsto C(z + aw)(1 - 2i\bar{a}z - (r + ia\bar{a})w)^{-1} \\ w &\mapsto \rho w(1 - 2i\bar{a}z - (r + ia\bar{a})w)^{-1} \end{aligned}$$

where $a \in \mathfrak{A} \otimes \mathbb{C}, r \in \mathfrak{A}, C \in \mathrm{GL}(n, \mathbb{C}), \rho \in \mathrm{GL}(n, \mathbb{R})$, such that $\langle Cz, Cz\rangle = \rho\langle z, z\rangle$.

3. RAQ Criteria

We start with the proof of Proposition 1.

Proof. It is clear, that (1) and (2) are necessary.

We show that they also are sufficient for having a matrix representation. In fact, if (1) is satisfied, one can define the map

$$\begin{array}{rcl} \tau : \mathbb{R}^n & \longrightarrow & \mathfrak{gl}(n, \mathbb{R}) \\ x & \mapsto & \langle x, \cdot \rangle . \end{array}$$

Then τ is a homomorphism if and only if (2) is satisfied. □

We will show that any nondegenerate quadric Q satisfying (1) and (2) is a direct product of some elementary quadrics. At first we translate the nondegeracy condition into the language of $\mathfrak{A}_Q$.

Proposition 2. *Let Q be a quadric satisfying (1) and (2) (with respect to some coordinates). Then Q is nondegenerate if and only if there exists a real vector e such that $\langle e, x\rangle = x$ for all real vectors x.*

Remark. We will see in the proof that under the conditions of the proposition the two properties in the definition of nondegeracy are equivalent.

Proof. Assume that there exists a vector e with the mentioned property. Consider a linear combination of the Hermitian forms

$$\sum \lambda_\kappa \langle , \rangle^\kappa \equiv 0$$

Then

$$\sum \lambda_\kappa \langle e, x\rangle^\kappa = \sum \lambda_\kappa x^\kappa \equiv 0$$

for all x. Hence $\lambda_\kappa = 0$ for all κ.

The second property also follows trivially: If $\langle x, y\rangle = 0$ for all vectors y then $\langle x, e\rangle = x = 0$.

We prove by induction with respect to the dimension n that from the nondegeracy condition follows the existance of a vector e with the desired properties.

If $n = 1$ both of the nondegeneracy properties imply that for a nonvanishing vector x there exists some $\lambda \neq 0$ such that $\langle x, x\rangle = \lambda x$. Hence, $\lambda^{-1}x$ is a unit.

Assume that the assertion is proved for $n < n_0$.

Let $x \in \mathfrak{A}_Q$ and $\tau(x) = X$ the corresponding linear operator. Then one of the following is true: either any X has only one eigenvalue or there is some X with two different eigenvalues. Since the algebra is commutative, in the second case it spells into two subspaces being invariant with respect to all $Y = \tau(y)$. The dimension of these subalgebras are less than n_0, and both nondegeneracy conditions hold true for both subalgebras, hence the assertion is proved. In the first case we consider two subcases. If for some X the unique eigenvalue is different from 0 then X is invertible and $X^{-1}(x)$ is the unit. Indeed, for any y $X^{-1}(x)y = X^{-1}Xy = y$.

It remains to consider the case when all X have only the eigenvalue 0. Then the algebra $\mathfrak{A}_Q$ would be nilpotent. This contradicts to both of the nondegeneracy properties. □

Now Theorem 1 will be a corollary of the following

Proposition 3. *Let $\mathfrak{A}$ be a real n-dimensional, commutative, associative algebra with unit then $\mathfrak{A}$ is isomorphic to the direct sum of some algebras $\mathfrak{A}_{r_1 \dots r_s}$ and $\mathfrak{A}^{\mathbf{C}}_{r_1 \dots r_s}$.*

Proof. Since $\mathfrak{A}$ is an Artinian ring it splits into local Artinian rings. We have to show that any local Artinian ring which is a n-dimensional algebra over $\mathbf{R}$ is isomorphic to $\mathfrak{A}_{r_1 \dots r_s}$ or $\mathfrak{A}^{\mathbf{C}}_{r_1 \dots r_s}$.

We will prove this by induction with respect to the dimension of the maximal ideal $\mathfrak{m}$ of $\mathfrak{A}$.

If dim $\mathfrak{m} = 0$ then $\mathfrak{A}$ is a field and therefore isomorphic to $\mathbf{R}$ or to $\mathbb{C}$.

Assume, that the Proposition is proved for dim $\mathfrak{m} < m$ and that dim $\mathfrak{m} = m$. We have to consider the two cases: $\mathfrak{A}/\mathfrak{m} \cong \mathbb{R}$ or $\mathfrak{A}/\mathfrak{m} \cong \mathbb{C}$. In both cases, the maximal ideal consists of nilpotent elements. Take a nilpotent element a of maximal order r. We consider a as linear operator A on $\mathfrak{m}$. There exists a basis of $\mathfrak{m}$ $\{a, a^2, \dots, a^{r-1}, b_1, \dots, b_{m-r}\}$ in the first case and $\{a, ia, a^2, ia^2 \dots, a^{r-1}, ia^{r-1}, b_1, \dots, b_{m-2r}\}$ in the second case, such that the matrix of the operator A in this basis has Jordan normal form. Then $\mathfrak{A}_1$ being span of $\{1, b_1, \dots, b_{m-r}\}$ resp. $\{1, i, b_1, \dots, b_{m-2r}\}$ forms a commutative, associative subalgebra with unit such that the dimension of the maximal ideal is less than m, and $ax = 0$ for any x from the maximal ideal of $\mathfrak{A}_1$. It follows now by induction that $\mathfrak{A}$ is of the first type in the case $\mathfrak{A}/\mathfrak{m} \cong \mathbb{R}$, and of the second type if $\mathfrak{A}/\mathfrak{m} \cong \mathbb{C}$. □

Corollary 1. *The number of pairwise nonequivalent irreducible RAQ quadrics in $\mathbb{C}^{2n}$ is $\pi(n-1)$ for odd n and $\pi(n-1) + \pi(\frac{n}{2} - 1)$ for even n (where $\pi(s)$ is the number of partitions of s).*

An important class of quadrics is the class of strictly pseudoconvex quadrics, i.e. the quadrics Q with the property that there exists a positive definite linear combination of the Hermitian forms defining Q.

These quadrics are automatically nondegerate. Any strictly pseudoconvex quadric Q is the Shilov boundary of some Siegel domain of second kind. It was proved by Tumanov [Tum89] that the automorphisms Q extend holomorphically to automorphisms of the corresponding Siegel domain. We will give the description of strictly pseudoconvex RAQ quadrics.

Corollary 2. *Any strictly pseudoconvex RAQ quadric Q is equivalent to the direct product of n spheres in $\mathbb{C}^2$.*

Proof. Since Q is strictly pseudoconvex, $\langle x, x\rangle \neq 0$ for all $x \neq 0$. Therefore, the algebra $\mathfrak{A}_Q$ does not contain nilpotent elements.

Hence Q is, according to Theorem 1, the direct product of some copies of Q_H and $Q_H^{\mathbb{C}}$.

If the direct product contained some $Q_H^{\mathbb{C}}$, then the cone $C = \{\langle z, z\rangle : z \in \mathbb{C}^n\}$ would contain an entire line. This contradicts to the strict pseudoconvexity. It follows that Q is equivalent to the direct product of n copies of Q_H. $\square$

4. The isotropy groups

Let Q be a RAQ quadric. It is easy to verify that then the mappings

$$\begin{aligned} z &\mapsto C(z+aw)(1-2i\bar{a}z-(r+ia\bar{a})w)^{-1} \\ w &\mapsto \rho w(1-2i\bar{a}z-(r+ia\bar{a})w)^{-1} \end{aligned}$$

with $a \in \mathfrak{A}\otimes\mathbb{C}, r \in \mathfrak{A}, C \in \mathrm{GL}(n,\mathbb{C}), \rho \in \mathrm{GL}(n,\mathbb{R})$, such that $\langle Cz, Cz\rangle = \rho\langle z, z\rangle$, are automorphisms.

We prove that any holomorphic automorphism preserving 0 has this form. Therfore, we show that any automorphism Φ with

$$\left.\frac{\partial\Phi}{\partial z}\right|_{T_0^{\mathbb{C}}Q} = \mathrm{id}$$

has the form

$$\begin{aligned} z &\mapsto (z+aw)(1-2i\bar{a}z-(r+ia\bar{a})w)^{-1} \\ w &\mapsto w(1-2i\bar{a}z-(r+ia\bar{a})w)^{-1} \end{aligned}$$

with $a \in \mathfrak{A}\otimes\mathbb{C}, r \in \mathfrak{A}$.

We recall that any $\Phi \in \mathrm{Aut}_0 S^3$ with identical CR projection can be represented as a composition $\Phi = \Phi^3 \circ \Phi^2 \circ \Phi^1$ ($\Phi^j, j = 1,2,3$ are the steps of the "normalization" of the equation) with

$$\begin{aligned} (\Phi^1)^{-1} : z &\mapsto z + p(w) + 2it(z,w) \\ w &\mapsto q(w) + 2ig(z,w), \end{aligned}$$

where the choice of p, q, t, g provides the property that the image of S^3 via Φ^1 does not contain terms of degree $(k,0), (k+1,1)$ and $(3,2)$ with respect to $(z,\bar{z})$ for $k \geq 1$.

$$\begin{aligned}\Phi^2 : z^* &= e^{i\theta(w)}z\\ w^* &= w,\end{aligned}$$

where $\theta(u) \in \mathbb{R}$. The choice of θ provides the property that the term of degree $(2,2)$ in $(z,\bar{z})$ is vanishing.

$$\begin{aligned}\Phi^3 : z^* &= \sqrt{h'(w)}z\\ w^* &= h(w),\end{aligned}$$

where $h(u) \in \mathbb{R}$ and $h'(0) > 0$.

The map Φ_3 can be represented as composition of two maps of this form corresponding to functions h_1, h_0, where the function h_1 is chosen to eliminate the term of degree $(3,3)$ in $(z,\bar{z})$ from the equation of S^3. The function h_0 will be chosen so that in the equation of S^3 do not occur new terms of degree $(3,3)$.

The computations show that

$$\begin{aligned}
p(u) &= \frac{1}{2i\bar{a}}\left(e^{2ia\bar{a}u}-1\right),\\
q(u) &= \frac{1}{2ia\bar{a}}\left(e^{2ia\bar{a}u}-1\right), \quad a \in \mathbb{C},\\
z + 2it(z,w) &= \frac{z}{1-2i\bar{p}'(w)},\\
g(z,w) &= \frac{\bar{p}(w)z}{1-2i\bar{p}'(w)},\\
\theta(u) &= -3|a|^2u,\\
h_1(w) &= \frac{\tan(|a|^2w)}{|a|^2},\\
h_0(w) &= \frac{w}{1-rw},
\end{aligned}$$

The only free parameters in Φ_1, Φ_2, Φ_3 are $a = \frac{dp}{du}(0) \in \mathbb{C}$ and $r = \frac{dh_0}{du}(0) \in \mathbb{R}$. These parameters have the following geometric sense.

A real curve on S^3 is called chain (according to Chern Moser) if it is the biholomorphic image of the standard chain $z = 0, \operatorname{Im} w = 0$. Thus any holomorphic automorphism of S^3 corresponds to some chain. For any hyperquadric chains are exactly the intersections of the hyperquadric with complex lines $z = aw$. The parameter $a = \frac{dp}{du}(0)$ determines the chain corresponding to the automorphism. The parameter $r = \frac{dh_0}{du}(0)$ determines the parametrization of the chain. Therefore we call the map Φ_3 with $h = h_0$ reparametrization map.

All considerations from the hypersphere case can be applicated to Q if we prove two things:

First, that any mapping

$$\begin{aligned} F : z &\mapsto z + p(w) + 2i \sum_{k=2}^{\infty} t_k(z, w), \\ w &\mapsto q(w) + 2i \sum_{k=1}^{\infty} g_k(z, w), \end{aligned}$$

with the property that the equation of the image of Q via F does not contain terms of degree $(1,0), (k,0), (k,1)$ for $k > 1, \ldots$ has the the form:

$$\begin{aligned} F : z^* &= \sum_{m=0}^{\infty} a_m(w) z^m \\ w^* &= \sum_{m=0}^{\infty} b_m(w) z^m, \end{aligned}$$

with multiplication in sense of the algebra $\mathfrak{A}$ and $a_n(w), b_n(w)$ can be represented as

$$\begin{aligned} a_m &= \sum_{l=0}^{\infty} a_{m,l} w^l \\ b_m &= \sum_{l=0}^{\infty} b_{m,l} w^l. \end{aligned}$$

The second thing what we have to show is that in the mapping Φ_3 there does not occur additional freedom from the fact, that the group of linear automorphisms can be bigger than the group which is generated by $\mathfrak{A}^{\mathbf{C}}$.

We need the following

Lemma 1. *Let $\mathfrak{A}^{\mathbf{C}}$ be a complex, commutative, associative algebra with unit, and $G : \mathfrak{A}^{\mathbf{C}} \to \mathfrak{A}^{\mathbf{C}}$ be a holomorphic map defined in some neighbourhood of $w \in \mathfrak{A}^{\mathbf{C}}$ with the property:*

$$G(w + h) = G(w) + G'(w)h + o(|h|),$$

where $G'(w)$ is some $\mathfrak{A}^{\mathbf{C}}$-valued map (in fact, $G'(w)$ is then the partial derivative of G with respect to the direction of the unit.)

then

$$G(w + h) = \sum_{m=0}^{\infty} \frac{1}{m!} G^{(m)}(w) h^m,$$

where $G^{(m)}(w)$ are the partial derivatives of G of order m with respect to the direction of the unit.

Proof. We have to prove that for all $m \in \mathbb{N}$ the differential of order m takes the form

$$\frac{1}{m!}G^{(m)}(w)h^m$$

We prove this by induction. For $m = 1$ this is the condition of the Lemma. Assume that the assertion is proved for $m = m_0$

Let $\sigma_1, \dots, \sigma_n$ be a basis of $\mathfrak{A}$ where σ_1 is the unit. Let $(w^1, \dots, w^n)$ be corresponding coordinates. Then

$$\frac{\partial G}{\partial w^j} = G'\sigma_j, \tag{3}$$

in particular,

$$\frac{\partial G}{\partial w^1} = G'.$$

Let

$$D^{(m)}G(w,h) = \sum_{m_1+\dots m_n=m} \frac{\partial^m G}{(\partial w^1)^{m_1}\dots(\partial w^n)^{m_n}}(w)\frac{(h^1)^{m_1}\dots(h^n)^{m_n}}{m_1!\dots m_n!}$$

be the m-th differential of G. Then $D^{(m+1)}G(w,z)$ can be obtained from $D^{(m)}G(w,z)$ by means of the formula

$$D^{(m+1)}G(w,h) = \sum_{i=1}^{n}\int_0^{h^i}\frac{\partial}{\partial w^i}D^{(m)}G(w,0,\dots,0,\chi,h^{i+1},\dots,h^n)d\chi$$

It follows from 3 that

$$\frac{\partial}{\partial w^j}\frac{\partial^m G}{(\partial w^1)^m} = \frac{\partial^{(m+1)}G}{(\partial w^1)^{(m+1)}}\sigma_j, \tag{4}$$

On the other hand, one easily shows that

$$\int_0^{h^i}\frac{(0,\dots,0,\chi,h^{i+1},\dots,h^n)^m}{m!}\sigma_i d\chi =$$
$$\frac{(0,\dots,0,h^i,\dots,h^n)^{(m+1)}}{(m+1)!} - \frac{(0,\dots,0,h^{i+1},\dots,h^n)^{(m+1)}}{(m+1)!}$$

This implies that, if $D^{(m)}G(w,h)$ has the desired form then so has $D^{(m+1)}G(w,h)$. The proof is complete. $\square$

Now we deduce that F has the mentioned form.
From the condition that the terms $(0,k)$ in the new equation of Q vanish we derive

$$\begin{aligned} g_1(z,w) &= z\bar{p}(w) \\ g_k(z,w) &= t_k(z,w)\bar{p}(w) \text{ for } k>1. \end{aligned}$$

Here g_k, t_k and p are $\mathfrak{A}^{\mathbf{C}}$-valued functions.
The vanishing of terms $(1,k)$ gives

$$t_2(z,w)\bar{z} = z\frac{\partial \bar{p}}{\partial u}(iz\bar{z}) \tag{5}$$

$$t_k(z,w)\bar{z} = 2t_{k-1}\frac{\partial \bar{p}}{\partial u}(iz\bar{z}) \text{ for } k>2 \tag{6}$$

From (5) we obtain

$$t_2(z_1,w)z_2 = z_1\frac{\partial \bar{p}}{\partial u}(iz_1z_2),$$

and, setting $z_2 = e$,

$$t_2(z,w) = z\frac{\partial \bar{p}}{\partial u}iz.$$

Since t_2 is a bilinear form with respect to z, we have

$$z_1\frac{\partial \bar{p}}{\partial u}iz_2 = z_2\frac{\partial \bar{p}}{\partial u}iz_1.$$

Setting here $z_2 = e$ we obtain

$$\frac{\partial \bar{p}}{\partial u}iz = z\frac{\partial \bar{p}}{\partial u}(e).$$

Thus $\bar{p}$ satisfies the condition of Lemma 1.
It follows that (6) takes the form

$$t_k(z,w) = 2it_{k-1}\bar{p}'(w)z.$$

Since, according to Lemma 1, p can be represented as a power series in $\mathfrak{A}$, this implies immediately that F has the desired form.

It remains to prove that in the "reparametrization map"

$$\begin{aligned} \Phi^3 : z^* &= C(w)z \\ w^* &= h(w), \end{aligned}$$

with $C(w)C(w) = \frac{\partial h}{\partial w}(w)$ does not occur additional freedom.

Therefore, we have to study the group of linear automorphisms G_Q. This group consists of mappings $z \mapsto Cz$, $w \mapsto \rho w$, where $C \in \mathrm{GL}(n, \mathbb{C})$ and $\rho \in \mathrm{GL}(n, \mathbb{R})$, such that $\langle Cz, Cz \rangle = \rho \langle z, z \rangle$ for all $z \in \mathbb{C}^n$.

G_Q containes a subgroup G_Q^0 of (C, ρ) corresponding to the action of the invertible elements $c \in \mathfrak{A}_Q \otimes \mathbb{C}$ in the z-component, and of $c\bar{c}$ in the w-component.

Now any (C, ρ) can be uniquely represented as composition of some element of G_Q^0 and some (C', ρ') preserving the unit of $\mathfrak{A}_Q$. This property implies immediately that $C' = \rho'$. In particular, this means that C' is real.

It follows then, that any holomorphic map $C(w)$ with $C(0) = \mathrm{id}$ has values only in $\mathfrak{A}_Q^* \otimes \mathbb{C}$ - the group of invertible elements of $\mathfrak{A}_Q \otimes \mathbb{C}$.

This completes the proof of Theorem 2.

For any quadric one can define an analogue to Chern-Moser chains. Let Q be a quadric in $\mathbb{C}^{n+k}$ of codimension k. Then a chain is the k-dimensional real submanifold of Q being the image of $z = 0, \mathrm{Im}\, w = 0$ under some holomorphic automorphism.

As a corollary from Theorem 2, we obtain the following description of the chains for RAQ quadrics.

Corollary 3. *Let Q be a RAQ quadric in $\mathbb{C}^{2n}$. Then a chain is the intersection of Q with a "matrix line", i.e. a complex n-plane $z = \langle w, a \rangle$.*

We return to the group G_Q. We have already seen that any element of G_Q can be represented as a composition of some $(C, C\bar{C})$, where $C = \tau(c)$ for some $c \in \mathfrak{A}_Q^{\mathbb{C}}$ and some (ρ, ρ), where $\rho \in \mathrm{GL}(n, \mathbb{R})$ such that $\langle \rho x, \rho x \rangle = \rho \langle x, x \rangle$. We denote the subgroup of these (ρ, ρ) by G_Q^1.

Since $\tau(\rho x) = \rho \tau(x) \rho^{-1}$, G_Q^1 consists of all $\rho \in \mathrm{GL}(n, \mathbb{R})$ preserving $\mathfrak{A}_Q$ with respect to the adjungation. Thus, G_Q^1 is the factor group of the normalizer $N(\mathfrak{A}_Q)$ by its trivially acting normal subgroup.

Examples. 1. The algebra $\mathfrak{A}_{1\ldots1}$ defines for $n = 2$ the parabolic quadric, and for $n > 2$ a nullquadric (i.e. any linear combination of the Hermitian forms defining Q is degenerate). The group G_Q^1 for this quadric consists of all nondegenrate linear transformations of the maximal ideal in $\mathfrak{A}_{1\ldots1}$. Thus, it has dimension $(n-1)^2$. Hence, the dimension of the isotropy group of the nullquadric in $\mathbb{C}^{2n}$ is $n^2 + 3n + 1$. It is easy to see, that these nullquadrics have the automorphism groups of maximal dimension in the RAQ class.

2. Another example is the elementary quadric Q_{n-1} in $\mathbb{C}^{2n}$. In this case, the subset of nilpotent vectors of order n is invariant for all elements of G_Q^1.

Let x_n be such a vector. Then any vector with this property has the form $y_n = \alpha_1 x_n + \alpha_2 x_n^2 + \cdots + \alpha_{n-1} x_n^{n-1}$ with $\alpha_1 \neq 0$, α_i - real. It is easy to verify that there exists a uniquely determined element of G_Q^1 mapping x_n to y_n. Thus, the isotropy group of these quadrics has the dimension $6n - 1$. (For n=2 this quadric is the parabolic quadric as in the previous example.)

Remark. According to Theorem 1, in $\mathbb{C}^6$ any irreducible RAQ quadric is equivalent either to Q_{11} from Example 1, or to Q_2 from Example 2.

5. Linear representation of $\mathrm{Aut}_0 Q$

Let $\mathfrak{A} = \mathfrak{A}_Q \otimes \mathbb{C}$ for some RAQ quadric Q, and $\mathfrak{A}^3$ be the $\mathfrak{A}$ module of triples $(\theta_0, \theta_1, \theta_2)$ with $\theta_i \in \mathfrak{A}$. By $\mathfrak{A}^*$ we denote the group of invertible elements of $\mathfrak{A}$ and by $\hat{\mathfrak{A}}^3$ the factor space under the action of $\mathfrak{A}^*$: $a(\theta_0, \theta_1, \theta_2) = (a\theta_0, a\theta_1, a\theta_2)$ for $a \in \mathfrak{A}^*$. $\hat{\mathfrak{A}}^3$ is a compact manifold which can be considered as a compactification of $\mathbb{C}^{2n} = \mathfrak{A}^2$ by the embedding

$$(z, w) \mapsto (\mathrm{id}, z, w).$$

Now, any automorphism of Q can be represented as a linear transformation of $\mathbb{C}^{3n}$ in the following way:

Let Q be given in the form $\mathrm{Im}\, w = z\bar{z}$. Then the automorphisms can be written as a composition of

$$\begin{aligned} z &\mapsto (\mathrm{id} - 2i\bar{a}z - (r + ia\bar{a})w)^{-1}(z + aw) \\ w &\mapsto (\mathrm{id} - 2i\bar{a}z - (r + ia\bar{a})w)^{-1}w, \end{aligned}$$

where $a, r \in \mathfrak{A}$, with $r = \bar{r}$, and a linear (C, ρ) transformation.

Together these automorphisms induce the following mapping on $\mathfrak{A}^3$:

$$\begin{aligned} \theta_0 &\mapsto \theta_0 - 2i\bar{A}\theta_1 - (R + iA\bar{A})\theta_2, \\ \theta_1 &\mapsto C\theta_1 + CA\theta_2, \\ \theta_2 &\mapsto \rho\theta_2. \end{aligned}$$

This representation is an analogue to the representation of the fractional linear automorphisms of the hyperquadrics in the projective space.

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