Heegaard splittings of Graph manifolds

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Abstract

Let M be a totally orientable graph manifold with characteristic submanifold $\mathcal{T}$ and let $M = V \cup_S W$ be a Heegaard splitting. We prove that there is an isotopy of S so that for the completion C of a component of $M - \mathcal{T}$, $S \cap C$ is either horizontal (according to one of three possible definitions) or vertical.

1 Introduction

The subject of this investigation is the structure of Heegaard splittings of graph manifolds. This investigation continues the work begun in [17], [18], [11] and [19]. Since the publication of those papers, new techniques have been added to the repertoire of those interested in classifying Heegaard splittings. These include the idea of untelescoping a weakly reducible Heegaard splitting into a generalized strongly irreducible Heegaard splitting (for definitions, see below) and the variation on Cerf-theoretic techniques developed by J.H. Rubinstein and M. Scharlemann. These insights have not left the investigation here unaffected. We hope that their role may be a tribute to the value of proper affinage. (The structural theorem given has been promised for rather a long time.) A similar theorem was proven by J.H.Rubinstein using other methods.

The graph manifolds considered are totally orientable, that is, they are orientable 3-manifolds and for each Seifert fibered submanifold the underlying surface of the orbit space is orientable. It follows from [9, VI.34] that an incompressible surface can be isotoped to be either horizontal or vertical in each Seifert fibered submanifold of a totally orientable graph manifold. In conjunction with the notion of untelescoping a weakly reducible Heegaard splitting into a strongly irreducible generalized Heegaard splitting, this fact reduces the investigation at hand to the investigation of strongly irreducible Heegaard splittings of generalized graph manifolds (for definitions, see below).

In the investigation of strongly irreducible Heegaard splittings of generalized graph manifolds, the nice properties of strongly irreducible Heegaard splittings often reduce this investigation to a study of the behaviour of the Heegaard splittings near the characteristic submanifolds. In this context, the techniques developed by J. H. Rubinstein and M. Scharlemann provide the necessary structure for a complete description of this behaviour, see Theorem 5.1 and Corollary 5.5. This theorem may be viewed as a "twin" to Theorem 4.2 in [6].

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2 Untelescoping and amalgamation

For standard definitions pertaining to knot theory see for instance [4], [10] or [13]. For 3-manifolds see [8] or [9].

Definition 2.1. *A compression body is a 3-manifold W obtained from a connected closed orientable surface S by attaching 2-handles to $S \times \{0\} \subset S \times I$ and capping off any resulting 2-sphere boundary components. We denote $S \times \{1\}$ by $\partial_+ W$ and $\partial W - \partial_+ W$ by $\partial_- W$. Dually, a compression body is a connected orientable 3-manifold obtained from a (not necessarily connected) closed orientable surface $\partial_- W \times I$ or a $3 - ball$ by attaching 1-handles.*

In the case where $\partial_- W = \emptyset$ (i.e., in the case where a $3 - ball$ was used in the dual construction of W), we also call W a handlebody. If $W = \partial_- W \times I$, we say that W is a trivial compression body.

Definition 2.2. *A set of defining disks for a compression body W is a set of disks $\{D_1, \dots, D_n\}$ properly imbedded in W with $\partial D_i \subset \partial_+ W$ for $i = 1, \dots, n$ such that the result of cutting W along $D_1 \cup \dots \cup D_n$ is homeomorphic to $\partial_- W \times I$ or to a 3-ball in the case that W is a handlebody.*

Definition 2.3. *A Heegaard splitting of a 3-manifold M is a decomposition $M = V \cup_S W$ in which V, W are compression bodies, $V \cap W = \partial_+ V = \partial_+ W = S$ and $M = V \cup W$. We call S the splitting surface or Heegaard surface.*

The notion of strong irreducibility of a Heegaard splitting was introduced by A. Casson and C. McA. Gordon in [5] and has proven extremely useful.

Definition 2.4. *A Heegaard splitting $M = V \cup_S W$ is strongly irreducible if for any pair of essential disks $D \subset V$ and $E \subset W$, $\partial D \cap \partial E \neq \emptyset$.*

Recall also the following related definitions:

Definition 2.5. *A Heegaard splitting $M = V \cup_S W$ is reducible if there exists a pair of essential disks $D \subset V$ and $E \subset W$ such that $\partial D = \partial E$. If $M = V \cup_S W$ is not reducible, then it is irreducible.*

A Heegaard splitting $M = V \cup_S W$ is stabilized if there exists a pair of essential disks $D \subset V$ and $E \subset W$ such that $|\partial D \cap \partial E| = 1$.

Though all compact 3-manifolds admit Heegaard splittings, many do not admit strongly irreducible Heegaard splitting. This fact prompted M. Scharlemann and A. Thompson to introduce the following notion of generalized Heegaard splittings.

Definition 2.6. *A* <u>*generalized Heegaard splitting*</u> *of a compact orientable* 3*-manifold* M *is a decomposition* $M = (V_1 \cup_{S_1} W_1) \cup_{F_1} (V_2 \cup_{S_2} W_2) \cup_{F_2} \cdots \cup_{F_{m-1}} (V_m \cup_{S_m} W_m)$ *such that each of the* V_i *and* W_i *is a union of compression bodies with* $\partial_+ V_i = S_i = \partial_+ W_i$ *(i.e.,* $V_i \cup_{S_i} W_i$ *is a union of Heegaard splittings of a union of submanifolds of* M*) and* $\partial_- W_i = F_i = \partial_- V_{i+1}$*.*

We say that a generalized Heegaard splitting is <u>*strongly irreducible*</u> *if each Heegaard splitting* $V_i \cup_{S_i} W_i$ *is strongly irreducible and each* F_i *is incompressible in* M*. We will denote* $\cup_i F_i$ *by* $\mathcal{F}$ *and* $\cup_i S_i$ *by* $\mathcal{S}$*. The surfaces in* $\mathcal{F}$ *are called the* <u>*thin levels*</u> *and the surfaces in* $\mathcal{S}$ *the* <u>*thick levels*</u>*.*

Let $M = V \cup_S W$ *be an irreducible Heegaard splitting. We may think of* M *as being obtained from* $\partial_- V \times I$ *by attaching all* 1*-handles in* V *(dual definition of compression body) followed by all* 2*-handles in* W *(standard definition of compression body), followed, perhaps, by* 3*-handles. An* <u>*untelescoping*</u> *of* $M = V \cup_S W$ *is a rearrangement of the order in which the* 1*-handles of* V *and the* 2*-handles of* W *are attached yielding a generalized Heegaard splitting. A* <u>*weak reduction*</u> *of* $M = V \cup_S W$ *is a untelescoping that is strongly irreducible.*

The Main Theorem in [15] implies the following:

Theorem 2.7. *For any irreducible* 3*-manifold* M*, any Heegaard splitting* $M = V \cup_S W$ *has a weak reduction.*

Definition 2.8. *Let* N, L *be* 3*-manifolds with* R *a closed subsurface of* ∂N*, and* S *a closed subsurface of* ∂L*, such that* R *is homeomorphic to* S *via a homeomorphism* h*. Further, let* $(U_1, U_2), (V_1, V_2)$ *be Heegaard splittings of* N, L *such that* $\eta(R) \subset U_1, \eta(S) \subset V_1$*. Then, for some* $R' \subset \partial N - R$ *and* $S' \subset \partial L - S$*,* $U_1 = \eta(R \cup R') \cup (1-$ *handles) and* $V_1 = \eta(S \cup S') \cup (1 - handles)$*. Thus* $\eta(R)$ *is homeomorphic to* $R \times I$ *via a homeomorphism* f *and* $\eta(S)$ *is homeomorphic to* $S \times I$ *via a homeomorphism* g*. Let* $\sim$ *be the equivalence relation on* $N \cup L$ *generated by*

(1) $x \sim y$ *if* $x, y \epsilon \eta(R)$ *and* $p_1 \cdot f(x) = p_1 \cdot f(y)$*,*

(2) $x \sim y$ *if* $x, y \epsilon \eta(S)$ *and* $p_1 \cdot g(x) = p_1 \cdot g(y)$*,*

(3) $x \sim y$ *if* $x \epsilon R$*,* $y \epsilon S$ *and* $h(x) = y$*,*

where p_1 *is projection onto the first coordinate. Perform isotopies so that for* D *an attaching disk for a* 1*-handle in* U_1, D' *an attaching disk for a* 1*-handle in* V_1*,* $[D] \cap [D'] = \emptyset$*. Set* $M = (N \cup L)/\sim, W_1 = (U_1 \cup V_2)/\sim$*, and* $W_2 = (U_2 \cup V_1)/\sim$*. In particular,* $\eta(R) \cup \eta(S)/\sim \cong R, S$*. Then* $W_1 = V_2 \cup \eta(R') \cup (1 - handles)$*, where the* 1*-handles are attached to* $\partial_+ V_2$ *and connect* $\partial \eta(R')$ *to* $\partial_+ V_2$*, and hence* W_1 *is a compression body. So* (W_1, W_2) *is a Heegaard splitting of* M*. The splitting* (W_1, W_2) *is called the* <u>*amalgamation of* (U_1, U_2) *and* (V_1, V_2) *along* R*,* S *via* h</u>*.*

Theorem 2.7 together with [17, Proposition 2.8] implies the following:

Theorem 2.9. *Suppose* $M = V \cup_S W$ *is an irreducible Heegaard splitting and* $M = (V_1 \cup_{S_1} W_1) \cup_{F_1} (V_2 \cup_{S_2} W_2) \cup_{F_2} \cdots \cup_{F_{m-1}} (V_m \cup_{S_m} W_m)$ *a weak reduction of* $M = V \cup_S W$*. Then the amalgamation of* $M = (V_1 \cup_{S_1} W_1) \cup_{F_1} (V_2 \cup_{S_2} W_2) \cup_{F_2} \cdots \cup_{F_{m-1}} (V_m \cup_{S_m} W_m)$ *(first along* F_1*, then* F_2*, then* $F_3, \dots$*) is* $M = V \cup_S W$*.*

One of the nice properties of strongly irreducible Heegaard splittings is apparent in the following lemma which is a deep fact and is proven, for instance, in [20, Lemma 6].

Lemma 2.10. *Suppose $M = V \cup_S W$ is a strongly irreducible Heegaard splitting and $P \subset M$ an essential incompressible surface. Then S can be isotoped so that $S \cap P$ consists only of curves essential in both S and P.*

3 Incompressible surfaces and generalized graph manifolds

In the following, we will make certain orientability assumptions on our graph manifolds. These will ensure that, after isotopy, an incompressible surface is either horizontal or vertical in any given Seifert fibered submanifold. Generalized graph manifolds are the 3-manifolds obtained from cutting a graph manifold along an incompressible surface.

Definition 3.1. *A graph manifold M is totally orientable if M is orientable and if for each Seifert fibered submanifold N of M, the base orbifold of N is orientable.*

Lemma 3.2. *Let F be an incompressible surface in a totally orientable graph manifold M. Let $\mathcal{T}$ be the characteristic submanifold of M. Then F may be isotoped so that for each component N of $M - \mathcal{T}$, $F \cap N$ is either horizontal or vertical.*

Proof: Since F and $\mathcal{T}$ are incompressible, F may be isotoped so that $F \cap \mathcal{T}$ consists only of curves essential in both F and $\mathcal{T}$. We may assume that this has been done in such a way that the number of components in $F \cap \mathcal{T}$ is minimal. Let N be a component of $M - \mathcal{T}$, then $F \cap N$ is incompressible.

Suppose $F \cap N$ is boundary compressible in N. Let $\hat{D}$ be a boundary compressing disk for $F \cap N$. Then $\partial \hat{D} = a \cup b$, with $a \subset \partial N$ and $b \subset F$. Since $F \cap \mathcal{T}$ consists only of curves essential in both F and $\mathcal{T}$, the component A of $\partial N - (F \cap \partial N)$ that contains a is an annulus. Let $B(\hat{D})$ be a bicollar of $\hat{D}$. Then $\partial B(\hat{D})$ has two components, $\hat{D}_0, \hat{D}_1$. Consider the disk $D = (A - (A \cap B(\hat{D})) \cup \hat{D}_0 \cup \hat{D}_1$. Since F is incompressible, D must be parallel to a disk in F, but this implies that the number of components of $F \cap \mathcal{T}$ is not minimal, a contradiction. Thus, $F \cap N$ is boundary incompressible in N.

Here [9, VI.34] allows three possibilities for F: 1) $F \cap N$ is vertical; 2) $F \cap N$ is horizontal; or 3) $F \cap N$ is the boundary of a twisted I-bundle over a horizontal surface $\hat{F}^N$ in N. For a boundary incompressible surface in N this latter possibility would imply that there is a nonorientable horizontal surface $\hat{F}^N$ in C. In particular, $\hat{F}^N$ would be a cover of the base orbifold. But this is impossible. Hence $F \cap N$ is either horizontal or vertical. □

Definition 3.3. *A generalized graph manifold is a manifold M such that there is a collection $\mathcal{T}$ of incompressible annuli and tori in M such that each component of $M - \mathcal{T}$ is either a Seifert fibered submanifold or is homeomorphic to (compact*

$surface$)
$\times I$. *Here $\mathcal{T}$ is called the characteristic submanifold of M.*

A generalized graph manifold is totally orientable if each component of $M - \mathcal{T}$ is either a totally orientable Seifert fibered submanifold or homeomorphic to (compact orientable $surface$) $\times I$.

It follows from Lemma 3.2 that cutting a totally orientable graph manifold along an incompressible surface yields totally orientable generalized graph manifolds.

4 Horizontal and vertical Heegaard splittings

In this section we describe the structure of Heegaard splittings for graph manifolds and generalized graph manifolds. In accordance with traditional usage, we use the adjectives "horizontal" and "vertical". The reader is cautioned to observe that these terms have caused a certain amount of confusion as they describe surfaces that are not "horizontal" or "vertical" in the true sense in which incompressible surfaces are horizontal (everywhere transverse to fibers) or vertical (consisting of fibers). We begin by giving a few motivating examples of Heegaard splittings of graph manifolds.

Definition 4.1. *Let F be a surface in a 3-manifold M and α an arc with interior in $M - F$ and endpoints on F. Let $C(\alpha)$ be a collar of α in M. The boundary of $C(\alpha)$ consists of an annulus A together with two disks D_1, D_2, which we may assume to lie in F. We call the process of replacing F by $(F - (D_1 \cup D_2)) \cup A$ performing ambient 1-surgery on F along α.*

Example 4.2. *Let Q be a closed orientable surface. The standard Heegaard splitting of $Q \times S^1$ may be constructed in more than one way. In particular, consider a small disk $D \subset Q$ and a collection Γ of arcs that cut $Q - D$ into a disk. Let S be the result of performing ambient 1-surgery on $\partial D \times S^1$ along $\Gamma \times \{point\}$. Then S is the splitting surface of a Heegaard splitting $Q \times S^1 = V \cup_S W$ (see for instance [17]). (From this point of view, this Heegaard splitting will be called a vertical Heegaard splitting below.)*

The same Heegaard splitting may be obtained in another way. Partition S^1 into two intervals I_1, I_2 that meet in their endpoints. Consider two distinct points $p, q \subset Q$. Then the surface obtained by performing ambient 1-surgery on $Q \times (I_1 \cap I_2)$ along $p \times I_1 \cup q \times I_2$ is isotopic to S above. (From this point of view, this Heegaard splitting will be called a horizontal (h1) Heegaard splitting below.)

Example 4.3. *Let N be a Seifert fibered space fibered over the disk with two exceptional fibers e_1, e_2. Consider $T_j = \partial N(e_j)$ and let α be an arc connecting T_1 and T_2 that projects to an imbedded arc in N and whose interior is disjoint from $T_1 \cup T_2$. Let S be the result of performing ambient 1-surgery on $T_1 \cup T_2$ along α. Then S is the splitting surface of a Heegaard splitting of $N = V \cup_S W$ (see [11]). Now consider two copies N_1, N_2 of N with Heegaard splittings $N_i = V_i \cup_{S_i} W_i$. We may assume that $\partial M_1 \subset V_1$ and $\partial M_2 \subset W_2$. Now amalgamate M_1 and M_2 along $\partial M_1, \partial M_2$. (The Heegaard splittings of the Seifert fibered submanifolds of the manifold $M = N_1 \cup N_2$ will be called vertical Heegaard splittings below. Heegaard splittings of graph manifolds that result from amalgamations of vertical Heegaard splittings of the Seifert* *fibered submanifolds are in some sense the canonical Heegaard splitting of a graph manifold. They can be constructed for all graph manifolds, whereas some of the other construction require very special circumstances, see for instance the example below.)*

Example 4.4. *Let N be a Seifert fibered space over the sphere with four exceptional fibers $e_1, \dots, e_4$ with carefully chosen invariants: $\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{l}{2l+1}$. Here $N - \eta(e_4)$ is a Seifert fibered manifold with boundary and hence fibers over the circle. More specifically, it fibers as a once punctured torus bundle over the circle. By partitioning the circle into two intervals I_1, I_2 that meet in their endpoints, we obtain a decomposition $N - \eta(e_4) = V' \cup_{S'} W'$ with $V' =$ (once punctured torus) $\times I_1$, $W' =$ (once punctured torus) $\times I_2$ and $S' =$ once punctured torus. Note that this decomposition is not a Heegaard splitting in the sense used here as S' is not closed. Now the carefully chosen invariants guarantee that the boundary of a meridian disk of $N(e_4)$ meets $\partial V'$ in a single arc. In particular, $V = V' \cup N(e_4)$ is also a handlebody (of genus two). Setting $W = W'$, this defines a Heegaard splitting $N = V \cup_S W$. (This type of Heegaard splitting will be called a horizontal (h3) Heegaard splitting below. Note that the splitting surface of this Heegaard splitting is a horizontal incompressible surface away from $N(e_4)$. $S \cap N(e_4)$ is an annulus that wraps once around e.)*

Example 4.5. *Let Q be a once punctured torus and consider $M_i = Q \times S^1$ for $i = 1, 2$. Partition S^1 into two intervals I_1, I_2 meeting in their endpoints. Set $V_i = Q \times I_1 \subset M_i$ and $W_i = Q \times I_2 \subset M_i$. Let $T_i = \partial M_i$ and $c_i = \partial Q \times \{point\} \subset T_i$. Glue T_1 to T_2 so that $|c_1 \cap c_2| = 1$. Then V_1 and V_2 meet in a (square) disk, hence $V = V_1 \cup V_2$ is a handlebody. Similarly, $W = W_1 \cup W_2$ is a handlebody. Set $S = \partial V = \partial W$, then $M = V \cup_S W$ is a Heegaard splitting of $M = M_1 \cup_{T_1 = T_2} M_2$.*

The conventions below will produce a slightly different positioning of S with respect to $T = T_1 = T_2$. (In what follows, it will be advantageous to consider a pushoff of T into M_1 (or into M_2). It is then the case that on one side of T, S is a horizontal incompressible surface, whereas on the other side of T, S is what will be called h2 below.)

Definition 4.6. *Let M be a graph manifold or generalized graph manifold and $\mathcal{T}$ its characteristic submanifold. Let $M = V \cup_S W$ be a Heegaard splitting of M and let C be the completion of a component of $M - \mathcal{T}$. We say that $S \cap C$ is <u>h1</u> if it is a horizontal incompressible surface with an even number of components or if it is obtained from such a surface $\hat{S}$ by ambient 1-surgery along vertical arcs connecting two adjacent components of $\hat{S}$. (Note that if ambient 1-surgery is performed along more than one vertical arc between two adjacent components of $\hat{S}$, then the Heegaard splitting is stabilized, so one need only consider the case in which there is at most one such arc.)*

Suppose C meets a component T of $\mathcal{T}$. Let $C(T)$ be a collar of T in C. Then $\partial C(T) = T_0 \cup T_1$ (where $T_0 = T$). If there are curves $c, c' \subset T$ such that $c \cap c'$ consists of a single point p, and $S \cap C(T)$ is isotopic to the boundary of a collar in $C(T)$ of $(c \times \{0\}) \cup (c' \times \{1\}) \cup (p \times I)$ and so that $S \cap (C - C(T))$ is a (two component) horizontal incompressible surface, then we say that $S \cap C$ is <u>h2</u>.

If there is an exceptional fiber e in C, such that $e \subset S$ and $S \cap (C - \eta(e))$ is a (two sheeted) horizontal incompressible surface, and such that $S \cap N(e)$ is an annulus *that wraps once around e, then we say that $S \cap C$ is h3. If $S \cap C$ is either h1, h2, or h3, we say that $S \cap C$ is horizontal.*

We say that $S \cap C$ is vertical if it is a (2-sided) vertical incompressible surface or if it is obtained from such a surface $\tilde{S}$ by ambient 1-surgery along a collection of horizontal arcs in C that projects to a collection of disjoint imbedded arcs. Note that in order for such a surface to be a Heegaard splitting or part of a Heegaard splitting, it must be true that the complement of the vertical incompressible surface can be partitioned into two subsets (those components that lie predominantly in V and those components that lie predominantly in W); these subsets must be such that along the vertical incompressible surface, components from distinct subsets lie on the two sides of the surface.

We say that $M = V \cup_S W$ is standard if for each component C of $M - \mathcal{T}$, $S \cap C$ is either horizontal or vertical.

5 Submanifolds homeomorphic to $(torus) \times I$

We here consider imbedded submanifolds homeomorphic to $(torus) \times I$. We are interested in the way a strongly irreducible Heegaard splitting intersects this submanifold. The theorem proven here applies in very general contexts. The techniques used are those of [14] and are inspired by the application of the central argument in [14] to the setting of [6, Theorem 4.2].

Theorem 5.1. *Suppose $M = V \cup_S W$ is a strongly irreducible Heegaard splitting and $T \times I$ an imbedding of $(torus) \times I$ in the interior of M. Further suppose that $T \times \{0\}$ (and thus also $T \times \{1\}$) is essential in M. If $S \cap (T \times \partial I)$ consists of curves essential in both S and $(T \times \partial I)$ and of a minimal number of such curves, then either*

1) there is a union of essential curves c in T such that $S \cap (T \times I)$ is isotopic to $c \times I \subset T \times I$;

or

2) there is a union of essential curves $c \subset T$ and an arc $\alpha \subset T$ connecting two adjacent components of c such that $S \cap (T \times I)$ is isotopic to the result of performing ambient 1-surgery of $c \times I \subset T \times I$ along $\alpha \times \frac{1}{2}$;

or

3) there are curves $c, c' \subset T$ such that $c \cap c'$ consists of a single point p, and $S \cap (T \times I)$ is isotopic to the boundary of a collar in $T \times I$ of $(c \times \{0\}) \cup (c' \times \{1\}) \cup (p \times I)$.

Proof: Much of the argument here is identical to that of [6, Theorem 2.4] but there are a few subtle differences. We provide a sketch. In V there is a properly imbedded 1-complex Σ_V such that V collapses to $\partial_- V \cup \Sigma_V$. Σ_V is called a spine of V. Similarly, there is a spine Σ_W for W. Now $M - \overline{(\partial_- V \cup \Sigma_V \cup \partial_- W \cup \Sigma_W)}$ is homeomorphic to $S \times I$. This product foliation is called a sweepout. Here to each point (s, t) in $I \times I$ we may associate $S_s = S \times \{s\}$ and $T_t = \overline{T \times \{t\}}$. We are interested in $S_s \cap T_t$.

After a small isotopy, we may assume that the spines and the sweepout are in general position with respect to the foliation of $T \times I$. There is then a 1-dimensional complex Γ in $I \times I$ called the graphic. At a point (s, t) on an edge of Γ the surfaces

S_s and T_t have a single point of tangency. At a vertex of Γ there are either two points of tangency or a "birth-death" singularity.

For (s,t) in a region of $I \times I - \Gamma$, $S_s \cap T_t$ is topologically rigid. If there is a curve in $S_s \cap T_t$ that is essential in S_s but bounds a disk in $T_t \cap V$ then we label the region V. Similarly, if there is a curve in $S_s \cap T_t$ that is essential in S_s but bounds a disk in $T_t \cap W$ then we label the region W. Since $M = V \cup_S W$ is strongly irreducible no region can be labelled both V and W. Furthermore, since $M = V \cup_S W$ is strongly irreducible and since S_s is isotopic to S for $0 < s < 1$, there cannot be regions containing (s, t_0) and (s, t_1) that are labelled V and W respectively.

Section 5 in [14] contains key results pertaining to the labelling of the graphic associated to a strongly irreducible Heegaard splitting. Most notably, the labels V and W cannot both appear in regions adjacent along an edge ([14, Corollary 5.5]). This fact plays a crucial role in the argument below.

Notice that for s small, $S_s \cap (T \times I)$ is the boundary of a small regular neighborhood of $\Sigma_V \cap (T \times I)$. Thus every region along the left edge of $I \times I$ is labelled V. Similarly, every region along the right edge of $I \times I$ is labelled W. (It is here that we must assume that the leaves of the foliation of $T \times I$ are essential, for otherwise a spine could miss $T \times I$.)

Claim 1: We may assume that $S_{\frac{1}{2}} = S$.

Begin with $S \cap (T \times I)$ as given. Then shrink first to one side, to obtain the spine Σ_V. Then shrink to the other side, to obtain Σ_W.

In particular, there are unlabelled regions R_0 and R_1 such that R_0 abuts the middle of the bottom edge of $I \times I$, R_1 abuts the middle of the top edge of $I \times I$ and such that for $(s,t) \epsilon R_i$, $S_{\frac{1}{2}} \cap T_t$ is topologically equivalent to $S \cap (T \times t)$.

Consider the effect of traversing an edge from one unlabelled region of $I \times I - \Gamma$ to another. Since compression bodies do not contain essential surfaces, $S_s \cap T_t \neq \emptyset$ for all $(s,t) \epsilon I \times I$. Furthermore, this intersection may not consist entirely of curves inessential in S. As the edge is traversed, curves of intersection inessential in both S_s and T_t may appear or disappear. If the single point of tangency represented by e corresponds to a saddle singularity, then this could conceivably affect curves of intersection that are essential in both S_s and T_t. But note that since $S_s \cap T_t$ is separating in T_t, such curves come in parallel pairs. Passing through e can be visualized as a band sum of either a curve with itself or of two curves with each other. If the saddle singularity affects only one of these curves, then it does not affect the slope of the curve. If it affects two adjacent curves, then these two curves are banded together to form a curve that is inessential in T_t. Since the region entered is unlabelled, it must also be inessential in S_s.

Now suppose we start out in R_0 and traverse only unlabelled regions, and edges between such regions.

Claim 2: As an edge is traversed, the slope of essential curves $S_s \cap T_t$ in T_t is unchanged.

If the slope is affected, then consider the first instance in which it changes. By the above, two adjacent essential curves are banded together as the edge is traversed. The resulting inessential curve may be capped off with the disk it bounds in S_s. This yields an annulus A that is parallel into $T \times \{0\}$. But this contradicts our assumption that $|S \cap (T \times \{0\})|$ is minimal.

Thus, essential curves remain and must have the same slope when traversing an edge from one unlabelled region of $I \times I - \Gamma$ to another. It follows that if there is a path in $I \times I$ from R_0 to R_1 that passes only through unlabelled regions and across edges between such regions, then option 1) above holds.

Suppose now that there is no path in $I \times I$ from R_0 to R_1 that passes only through unlabelled regions and edges between such regions. Then there must be a vertex $v_0 = (s_0, t_0)$ of Γ at which four regions meet with one region labelled V, one region labelled W and two regions unlabelled. Futhermore, the regions labelled V and W must lie opposite each other. This implies that $S_{s_0} \cap T_{t_0}$ consists of a graph G in T_{t_0} that has two valence four vertices.

This setup places enormous restrictions on G. See the delicate arguments preceeding [14, Lemma 5.6]. Suppose, for instance, that the two valence four vertices lie on distinct components of G. In the four quadrants defined by v the valence four vertices break apart in four possible combinations. See figure 1:

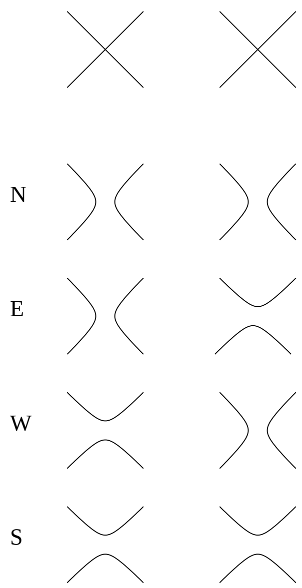

Figure 1: *How S_s meets T_t in the four regions N, E, W, S meeting v_0*

The labelling of the regions around the vertex (s_0, t_0) implies that two of these combinations involve only essential curves, two involve essential disks one giving rise to the label V, the other to the label W. If, however, the two valence four vertices of G lie on distinct components, two of these configurations would involve essential disks in V and two would involve essential disks in W. Thus the two valence four vertices lie on the same component of $S_{s_0} \cap T_{t_0}$. Hence, there are four edges connecting the two valence four vertices in such a way as to constitute a connected graph on T_{t_0}. For each possible such graph, there will be one quadrant in which this component breaks into a single connected curve.

Denote the component of G containing the valence four vertices by $\tilde{G}$. Denote the two vertices by v_1, v_2. If an edge, e_1, of $\tilde{G}$ begins and ends at a single vertex, say v_1, of $\tilde{G}$ then there are a priori two possibilities. The first possibility is that e_1 cuts out an inessential monogon. Then the associated disk survives in two quadrants where it gives rise to the same label. But this is impossible. The other possibility is that e_1 is essential in T_{t_0}. Since $\tilde{G}$ is connected, there must then be exactly two edges, e_2, e_3, emanating from v_1 that end on v_2. Therefore, on v_2, there must also be an edge, e_4, that begins and ends on v_2. It too must be essential in T_{t_0}. In particular, e_1 and e_4 must be parallel.

There are then (again a priori) exactly two possibilities for $\tilde{G}$. If e_2 and e_3 are in the same portion of $T_{t_0} - (e_1 \cup e_4)$ then $\tilde{G}$ must be as in figure 2:

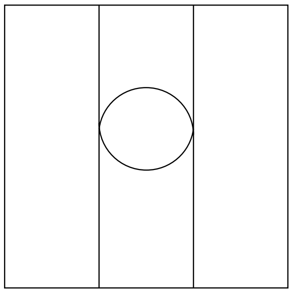

Figure 2: *$\tilde{G}$ contains a bigon*

This implies that as v_0 is traversed from one unlabelled region to another keeping s_0 fixed, $S_s \cap T_t$ changes as in figure 3:

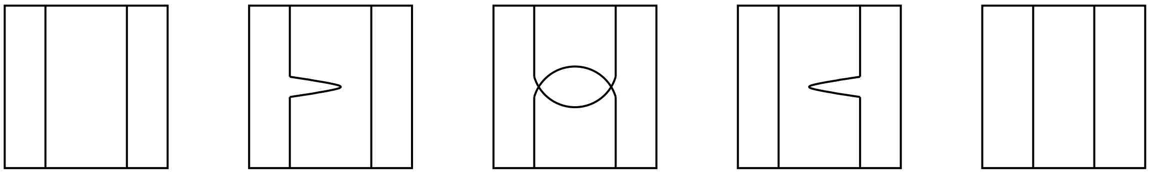

Figure 3: *Adding a 1-handle*

But this corresponds to the addition of a 1-handle. The core of the 1-handle can be isotoped into T_{t_0}. Since there are essential disks for both V and W cut out by $\tilde{G}$, there can be no other components of $S_{s_0} \cap T_t$ that are inessential in T_t and essential in S_{s_0} for t. In particular, there must be a path from (s_0, t_0) to R_0 and to R_1 respectively that avoids labelled regions and vertices of Γ. Thus $S \cap (T \times I) = S_{s_0} \times (T \times I)$ is as in option two.

If e_2 and e_3 lie in distinct portions of $T_{t_0} - (e_1 \cup e_4)$, then $\tilde{G}$ must be as in figure 4.

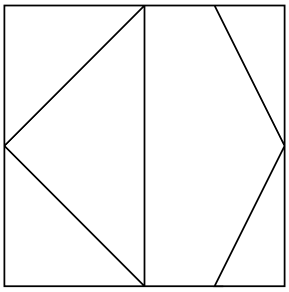

Figure 4: *$\tilde{G}$ contains a triangle*

Again, since $S_{s_0} \cap T_{t_0}$ bounds essential disks in $T_{t_0} - (S_{s_0} \cap T_{t_0})$ that lie on either side of S_{s_0}, there must be a path from (s_0, t_0) to R_0 and to R_1 respectively that avoids labelled regions and vertices of Γ. Thus $S_{s_0} \cap (T \times I)$ is isotopic to an even number of spanning annuli together with one component that is obtained as follows: an annulus that is parallel into $\partial(T \times I)$ is tubed to a spanning annulus. See figure 5.

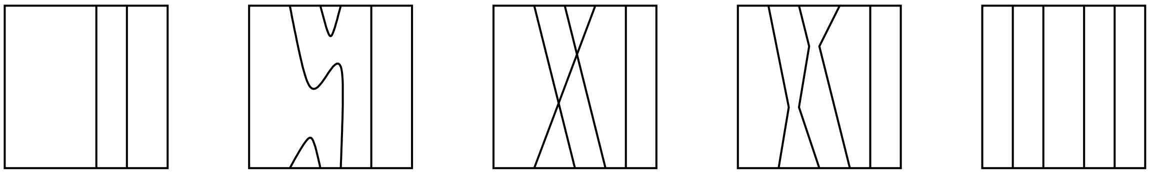

Figure 5: *Adding an "essential" 1-handle*

But this contradicts the assumed minimality of the number of components of $S \cap (T \times \{0,1\})$. Hence this case does not occur.

This allows for only one more possibility, namely that $\tilde{G}$ is as below. See figure 6:

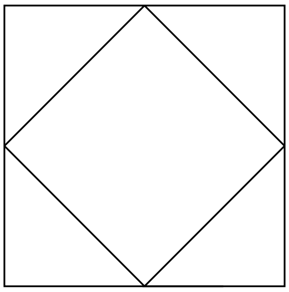

Figure 6: $\tilde{G}$ *contains a square*

This corresponds to option 3). See figure 7:

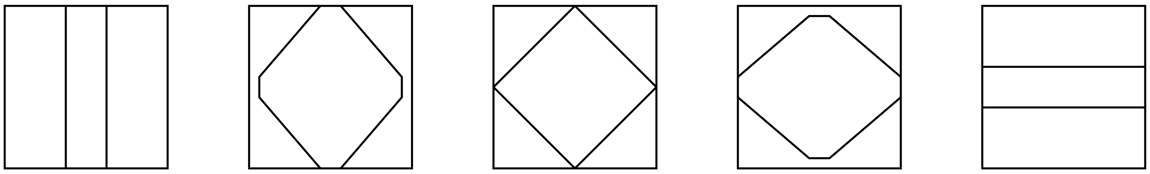

Figure 7: *Adding a 1-handle*

□

Remark 5.2. *Those familiar with the argument in [6, Theorem 4.2] will note that in the setting there, options 1) and 2) are ruled out since the slope of* $S \cap (T \times \{0\})$ *may not equal the slope of* $S \cap (T \times \{1\})$.

In fact, the proof of Theorem 5.1 also proves the results below. These corollaries will be used in the next section.

Corollary 5.3. *Suppose* $M = V \cup_S W$ *is a strongly irreducible Heegaard splitting and* $T \times I$ *an imbedding of* (*torus*) $\times I$ *in the interior of* M. *Further suppose that*

$T \times \{0\}$ *(and thus also* $T \times \{1\}$*) is essential in* M*. If* $S \cap (T \times \partial I)$ *consists of curves essential in both* S *and* $(T \times \partial I)$ *and if* $|S \cap (T \times \{0\})|$ *is minimal, then either*

1) there is a union of essential curves c *in* T *such that* $S \cap (T \times I)$ *is isotopic to* $c \times I \subset T \times I$ *and annuli parallel into* $T \times \{1\}$*;*

or

2a) there is a union of essential curves $c \subset T$ *and an arc* $\alpha \subset T$ *connecting two adjacent components of* c *such that* $S \cap (T \times I)$ *is isotopic to the result of performing ambient* 1*-surgery on* $c \times I \subset T \times I$ *along* $\alpha \times \frac{1}{2}$*;*

or

2b) there is a union of essential curves $c \subset T$*, a collection of annuli* $\mathcal{B} \subset T \times I$ *parallel into* $T \times \{1\}$*, an arc* α *that has one endpoint on* $c \times I$*, one endpoint on* $\mathcal{B}$*, that is otherwise disjoint from* $(c \times I) \cup \mathcal{B}$ *and that is parallel into* $T \times \{1\}$*, such that* $S \cap (T \times I)$ *is isotopic to the result of performing ambient* 1*-surgery on* $(c \times I) \cup \mathcal{B}$ *along* α*;*

or

3) there are curves $c, c' \subset T$ *such that* $c \cap c'$ *consists of a single point* p*, and* $S \cap (T \times I)$ *is isotopic to the boundary of a collar in* $T \times I$ *of* $(c \times \{0\}) \cup (c' \times \{1\}) \cup (p \times I)$*.*

Proof: The argument proceeds analogously to the proof of Theorem 5.1. The proof here diverges from the one above when we consider the effect of traversing an edge from one unlabelled region to another. Note that it is still true that essential curves of $S_s \cap T_t$ remain and have the same slope as the edge is traversed in the direction of increasing t. However, new curves essential in both S_s and T_t may now appear if a component of $S_s \cap T_t$ that is inessential in both S_s and T_t is banded to itself. These new essential curves will then remain unaffected as t increases. (Note that they must be parallel to the preexisting essential curves.) This phenomenon hence yields an annulus that is parallel into $T \times \{1\}$. This may happen any number of times. Note however, that this cannot happen in conjunction with option 3).

The rest of the argument in the proof of Theorem 5.1 then remains in effect except that the situation described in figures 4 and 5 can occur. This option is now included as option 2b) above. □

Corollary 5.4. *Suppose* $M = V \cup_S W$ *is a strongly irreducible Heegaard splitting and* $T \times I$ *an imbedding of* $(torus) \times I$ *in the interior of* M*. If* $S \cap (T \times \partial I)$ *consists of curves essential in both* S *and* $T \times \partial I$*, then either*

1) there is a union of essential curves c *(possibly empty) in* T *such that* $S \cap (T \times I)$ *is isotopic to* $c \times I \subset T \times I$ *and annuli parallel into* $T \times \{1\}$*;*

or

2a) there is a union of essential curves $c \subset T$ *and an arc* $\alpha \subset T$ *connecting two adjacent components of* c *such that* $S \cap (T \times I)$ *is isotopic to the result of performing ambient* 1*-surgery on* $c \times I \subset T \times I$ *along* $\alpha \times \frac{1}{2}$*;*

or

2b) there is a union of essential curves $c \subset T$*, a collection of annuli* $\mathcal{B} \subset T \times I$ *parallel into* $T \times \{1\}$*, an arc* α *that has one endpoint on* $c \times I$*, one endpoint on* $\mathcal{B}$*, that is otherwise disjoint from* $(c \times I) \cup \mathcal{B}$ *and that is parallel into* $T \times \{1\}$*, such that*

$S \cap (T \times I)$ is isotopic to the result of performing ambient 1-surgery on $(c \times I) \cup \mathcal{B}$ along α;

or

3) there are curves $c, c' \subset T$ such that $c \cap c'$ consists of a single point p, and $S \cap (T \times I)$ is isotopic to the boundary of a collar in $T \times I$ of $(c \times \{0\}) \cup (c' \times \{1\}) \cup (p \times I)$.

Proof: The important difference here is that the spines Σ_V, Σ_W need no longer intersect $T \times (\partial I)$. If they intersect neither component of $T \times (\partial I)$, then the components of $T \times (\partial I)$ are boundary parallel. After relabelling, $T \times \{0\}$ lies between a component of ∂M and $T \times \{1\}$. In this case $S \cap (T \times I)$ option 1) holds.

More interesting is the case in which Σ_V, Σ_W are disjoint from $T \times \{0\}$ but not from $T \times \{1\}$. The important difference here is that in the analysis of the graphic, all regions of $I \times I - \Gamma$ meeting the bottom edge of $I \times I$ are unlabelled. The assumption on the spines of V and W does, however, quarantee that the regions of $I \times I - \Gamma$ meeting some portion of the left edge of $I \times I$ are labelled V and those meeting some portion of the right edge of $I \times I$ are labelled W. The analysis then proceeds along exactly the same lines as in the proof of Corollary 5.3, but it is more fruitful to consider decreasing t.

Since $M = V \cup_S W$ is strongly irreducible, the two spines must either both meet or both be disjoint from $T \times I$. If the two spines meet both components of $T \times (\partial I)$, then the argument proceeds as in the proof of Theorem 5.1. □

Corollary 5.5. *Suppose $M = V \cup_S W$ is a strongly irreducible Heegaard splitting and $A \times I$ an imbedding of (annulus) $\times I$ in the interior of M with $A \times \{0, 1\}$ essential in M. If both components of $S \cap (A \times \partial I)$ consist of curves essential in both S and $(A \times \partial I)$ and of a minimal number of such curves, then either*

1) there is a union of essential curves c in A such that $S \cap (A \times I)$ is isotopic to $c \times I \subset A \times I$;

or

2) there is a union of essential curves $c \subset A$ and an arc $\alpha \subset A$ connecting two adjacent components of c such that $S \cap (A \times I)$ is isotopic to the result of performing ambient 1-surgery on $c \times I \subset A \times I$ along $\alpha \times \frac{1}{2}$.

Proof: The proof of Theorem 5.1 carries over verbatim, except that here option 3) is ruled out. □

6 Strongly irreducible Heegaard splittings of generalized graph manifolds

In this section we consider a generalized graph manifold M with characteristic submanifold $\mathcal{T}$. We show that if $M = V \cup_S W$ is a strongly irreducible Heegaard splitting, then S may be isotoped so that for C the completion of a component of $M - \mathcal{T}$, then either $S \cap C$ is horizontal, in any one of the three ways, or there is a fiber in C that is a core in either V or W.

Definition 6.1. *Let M be a generalized graph manifold with characteristic submanifold $\mathcal{T}$. By a Seifert component of M we mean the completion of a component of $M - \mathcal{T}$ that is a Seifert fibered space. By a product component we mean the completion of a component of $M - \mathcal{T}$ that is homeomorphic to $(surface) \times I$.*

Lemma 6.2. *Let M be a generalized graph manifold and $M = V \cup_S W$ a strongly irreducible generalized Heegaard splitting. Let $\mathcal{D}$ be a collection of defining disks for V and $\mathcal{E}$ a collection of defining disks for W. There is a Seifert or product component C of M so that, after isotopy, each outermost disk component of both $\mathcal{D} - (\mathcal{T} \cap \mathcal{D})$ and of $\mathcal{E} - (\mathcal{T} \cap \mathcal{E})$ lies in C.*

Proof: Isotope S so that $\mathcal{T} \cap S$ consists only of curves essential in both $\mathcal{T}$ and S. Furthermore, assume that the isotopy has been chosen so that $|\mathcal{T} \cap S|$ is minimal subject to this condition. Suppose that D' is an outermost subdisk of $\mathcal{D} - (\mathcal{T} \cap \mathcal{D})$. Let C be the Seifert component of M containing D'. Then either D' is a disk properly imbedded in V or D' meets an annular component A of $V \cap \partial C$. In case of the latter, D' meets A in a single arc, a. Let D'' be the disk obtained by cutting A along a, adding two copies of D' and isotoping the result to be a properly imbedded disk in V. The assumption that $|\mathcal{T} \cap S|$ be minimal guarantees that D'' is an essential disk in V. Thus in both cases, there is an essential disk properly imbedded in V that lies entirely in C, we refer to this disk as D.

Similarly, consider an outermost subdisk of $\mathcal{E} - (\mathcal{T} \cap \mathcal{E})$. The above argument shows that in some Seifert component C' of M, there is an essential disk E properly imbedded in W. Since $M = V \cup_S W$ is strongly irreducible, D must meet E, hence the Seifert component C' must coincide with the Seifert component C. □

Lemma 6.3. *Let M be a generalized graph manifold and $M = V \cup_S W$ a strongly irreducible generalized Heegaard splitting. There is a Seifert or product component C of M so that, after isotopy, $S \cap \tilde{C}$ consists of horizontal or vertical incompressible surfaces for all Seifert or product components $\tilde{C} \neq C$.*

Proof: By Lemma 6.2, $S \cap C'$ must be incompressible for all components $\tilde{C} \neq C$. The result thus follows. □

Definition 6.4. *If M is a generalized graph manifold, with a strongly irreducible Heegaard splitting $M = V \cup_S W$, then the Seifert or product component C as in Lemma 6.3 is called the active component of $M = V \cup_S W$.*

6.1 The active component meets the boundary of the generalized graph manifold

We first consider the case in which the active component C of $M = V \cup_S W$ meets the boundary of M. There are two cases, that in which the active component is a product and that in which it is Seifert fibered. In the former case, the required work has already been done by M. Scharlemann and A. Thompson. It turns out that in the latter case, $S \cap C$ is necessarily vertical. This mirrors the fact that a Heegaard splitting of a Seifert fibered space with non empty boundary has only vertical Heegaard splittings.

Lemma 6.5. *Let M be a generalized graph manifold and let $M = V \cup_S W$ be a strongly irreducible Heegaard splitting. Suppose the active component C of $M = V \cup_S W$ is a product. Then $S \cap C$ is h1.*

Proof: The result of compressing S as much as possible in C yields an incompressible surface S^* consisting of parallel sheets in the product C. Consider a collar $\tilde{C}$ of a component of $C - S^*$. The main argument of [16] applies directly to show that $S \cap \tilde{C}$ consists either of two parallel sheets of the product $\tilde{C}$ or is obtained from such sheets via ambient 1-surgery along a vertical arc. (I.e., "is of Type II".) □

Lemma 6.6. *Suppose that M is a generalized graph manifold and $M = V \cup_S W$ is a Heegaard splitting. Suppose further that the active component C is a Seifert fibered submanifold of M and that ∂C meets ∂M. Then each exceptional fiber of C is a core of either V or W.*

Proof: This is [18, Lemma 4.1], the central argument in [18]. Note that the isotopy performed in [18, Lemma 4.1] takes place within a small regular neighborhood of a saturated annulus. This means that in the setting here, the isotopy can be performed entirely within C. □

More generally:

Lemma 6.7. *Suppose that M is a generalized graph manifold and $M = V \cup_S W$ is a Heegaard splitting. Suppose further that the active component C is a Seifert fibered submanifold of M and a regular fiber of C can be isotoped to lie on $S \cap C$. Then each exceptional fiber of C is a core of either V or W.*

Proof: In fact, in the central argument in [18], the existence of a boundary component is used exclusively to produce a regular fiber that lies on a level surface of a Morse function corresponding to the Heegaard splitting. The existence of a regular fiber that lies on the splitting surface itself guarantees the existence of such a fiber. □

This more general lemma will be used in the next subsection. A consequence of Lemma 6.6 is that in this setting we may restrict our attention to the case in which C does not contain any exceptional fibers. See the lemma below.

Lemma 6.8. *Suppose that M is a generalized graph manifold and $M = V \cup_S W$ is a Heegaard splitting. Suppose further that the active component C is a Seifert fibered submanifold of M, that e is an exceptional fiber of C and that e is a core of V. Then $M - \eta(e) = (V - \eta(e)) \cup_S W$ is a Heegaard splitting. Furthermore, if $S \cap (C - \eta(e))$ is vertical, then $S \cap C$ is vertical. The same holds if e is a core of W.*

Proof: Since e is a core of V, $V - \eta(e)$ is still a compression body. Thus $M - \eta(e) = (V - \eta(e)) \cup_S W$ is a Heegaard splitting. Now if $S \cap (C - \eta(e))$ is vertical, then by the definition of vertical Heegaard splitting, $S \cap C$ is vertical. □

Theorem 6.9. *Suppose that M is a generalized graph manifold and $M = V \cup_S W$ is a strongly irreducible Heegaard splitting. Suppose further that the active component C of $M = V \cup_S W$ is Seifert fibered and that ∂C meets ∂M. Then $S \cap C$ is vertical.*

Proof: The proof is by induction on the number of exceptional fibers in C. Suppose first that C contains no exceptional fibers. Let $\mathcal{A}$ be a collection of disjoint vertical annuli in C that cut C into a regular neighborhood of the components of ∂C. After an isotopy, $S \cap \mathcal{A}$ consists of closed curves essential in both S and $\mathcal{A}$ and in the minimal possible number of such curves. In particular, it consists of regular fibers of C. Now apply Corollary 5.4 to the components of the complement of a small regular neighborhood of $\mathcal{A}$ in C. Since S intersects $T \times \{0\}$, say, in fibers of C, options 1), 2a) and 2b) of Corollary 5.4 complete $S \cap \eta(\mathcal{A})$ so that $S \cap C$ is vertical. Note that option 3) can only occur in a regular neighborhood of a component of ∂C that coincides completely with a component T of $\mathcal{T}$. If this occurs, isotope T across the collar of T in C containing the compressing disks for both V and W in option 3) of Corollary 5.4. Let $\tilde{C}$ be the completion of the component of $M - \mathcal{T}$ adjacent to C along T. After the isotopy, $S \cap \mathcal{C}$ is h2. Since $\tilde{C}$ is now the active component of $M = V \cup_S W$, it follows that $S \cap C$ is a vertical incompressible surface.

To prove the inductive step, suppose e is an exceptional fiber of C, then by Lemma 6.6, e is a core of either V or W, say of V. The inductive hypothesis in conjunction with Lemma 6.8 then shows that $S \cap C$ is vertical. $\square$

6.2 The active component does not meet the boundary of the generalized graph manifold

We here consider the remaining case.

Definition 6.10. *Let $h : M \to [0,1]$ be a Morse function and γ a 1-manifold properly imbedded in M that is transverse to h. Let $c_1, \ldots, c_n$ be the critical values of $h|_\gamma$ listed in increasing order so that $h(c_1) < \cdots < h(c_n)$. Choose $r_1, \ldots, r_{n-1}$ so that $c_i < r_i < c_{i+1}$ and set $R_i = h^{-1}(r_i)$. The width of γ relative to h, denoted by $w(\gamma, h)$, is $\sum_i |\gamma \cap R_i|$. The width of γ, denoted by $w(K)$, is the minimum of this relative width over all height functions. We say that K is in thin position if it is presented with respect to a height function realizing its width.*

Lemma 6.11. *Let M be a generalized graph manifold and let $M = V \cup_S W$ be a strongly irreducible Heegaard splitting. Suppose the active component C of $M = V \cup_S W$ is a Seifert fibered submanifold. Then $S \cap C$ is either horizontal or vertical.*

Proof: The argument here is a variation on the central argument in [11]. Abusing notation slightly, let $\mathcal{T}$ denote the subset of the characteristic submanifold of M that meets C. If $\mathcal{T} = \emptyset$, the result is the main theorem of [11], so we may assume, in the following, that $\mathcal{T} \neq \emptyset$. Let $\mathcal{A}$ be a 2-complex comprised of disjoint vertical annuli whose ends all lie on a single regular fiber γ of C that is chosen so that $\mathcal{A}$ cuts C into a regular neighborhood of $\mathcal{T} \cup (exceptional\ \ fibers\ \ of\ \ C)$. Choose a Morse function $h : M \to [0,1]$ corresponding to $M = V \cup_S W$ for which γ is in thin position with respect to $h|_C$.

Let B be a component of ∂C. If B is also a component of ∂M, then the result follows from Theorem 6.9. We may thus assume that B is not a component of ∂M. Thus Corollary 5.3 applies to the component $N(B)$ of $C - \mathcal{A}$. Recall that if $S \cap N(B)$ is as in option 2a) or 3), then there are compressing disks for S into both V and W

contained in $N(B)$. So in these cases, since $M = V \cup_S W$ is strongly irreducible, $S \cap (C - \eta(B))$ must be incompressible. Thus $S \cap C$ is horizontal if $S \cap (C - \eta(B))$ is a horizontal incompressible surface and vertical if $S \cap (C - \eta(B))$ is a vertical incompressible surface. The same holds if $S \cap N(B)$ is as in option 2b) and $c \times I$ (and thus also $\mathcal{B}$) is vertical.

The possibility that $S \cap N(B)$ is as in option 2b) and $c \times I$ is horizontal can be ruled out because it violates thin position of γ. Indeed, the arc α in option 2b) can be isotoped to be parallel to a subarc of γ. It can then be used to define a disk in, say, V, that is cobounded by a subarc of γ and an arc in S. The component of $\mathcal{B}$ meeting one end of α similarly defines a disk, now in W, that is cobounded by a subarc of γ and an arc in S. These disks can be isotoped to be disjoint except at one endpoint. By isotoping γ across these disks and then tilting slightly so that it is transverse with respect to h, the width of γ is reduced. But this violates the assumption that γ is in thin position with respect to $h|_C$.

We may thus assume in the following that $S \cap N(B)$ is as in option 1) for all components B of ∂C.

Suppose e is an exceptional fiber of C, two cases must be considered.

Case 1: $S \cap (\partial N(e))$ bounds meridians in $N(e)$.

Case 2: $S \cap (\partial N(e))$ does not bound meridians in $N(e)$.

Suppose that Case 1 holds for all exceptional fibers e of C. Then for each exceptional fiber e of C, $S \cap N(e)$ may be compressed to obtain meridians. Thus S compresses to a surface $\tilde{S}$ such that $\tilde{S} \cap C$ is horizontal. Here S can be reconstructed from $\tilde{S}$ by reversing the compressions. Here the proof of Lemma 6.5 applies, so $S \cap C$ is h1.

In Case 2, either Lemma 6.7 (in the case that $S \cap N(e)$ is vertical) or the standard argument in [1] or [11] shows that e can be isotoped within $N(e)$ to lie in S. Consider a shrunk copy $\tilde{N}(e)$ of $N(e)$ such that $S \cap \tilde{N}(e)$ consists of an annulus A with e as its core. Compress S as much as possible in $C - \tilde{N}(e)$ to obtain an incompressible surface $S^* \subset M - \tilde{N}(e)$. By Haken's Theorem, each compressing disk may be chosen to lie entirely on one side of S. Since $M = V \cup_S W$ is strongly irreducible, it then follows that S^* lies either in V or in W. There are three options for $S^* \cap C$:

Subcase 2.1: $S^* \cap (C - N(e))$ contains an annulus that is parallel into $\partial N(e)$.

Note that $S \cap N(e)$ is also parallel into $\partial N(e)$. Thus $(S^* \cap (C - N(e))) \cup (S \cap N(e))$ is a torus bounding a solid torus in either V or W and with a meridian disk that meets e once. Thus e is a core of either V or W. It then follows from Lemma 6.8 and Theorem 6.9 that $S \cap C$ is vertical.

Subcase 2.2: $S^* \cap (C - N(e))$ is a vertical incompressible surface.

Here Lemma 6.7 shows that e is a core of either V or W and the result then follows as in Subcase 2.1.

Subcase 2.3: $S^* \cap (C - N(e))$ is a horizontal incompressible surface.

There are two components of $S^* \cap (C - N(e))$. The proof of Lemma 6.5 applies to collars of the two components of $(C - N(e)) - (S^* \cap (C - N(e)))$. Suppose that ambient 1-surgery is performed on $S^* \cap (C - N(e))$ along a vertical arc α to reconstruct

$S \cap (C - N(e))$. Then note that α is parallel into $(S^* \cap (C - N(e))) \cup A$. Let $\hat{D}_1$ be the disk that defines this parallelism and let D_1 be a slightly shrunken version of $\hat{D}_1$. Let D_2 be a disk such that compressing S along ∂D_2 reverses the ambient 1-surgery performed along α. Then ∂D_1 meets ∂D_2 in exactly one point, illustrating that $M = V \cup_S W$ is stabilized. Since $M = V \cup_S W$ is strongly irreducible, this is impossible. Therefore $S \cap C = (S^* \cap (C - N(e))) \cup A$, whence $S \cap C$ is h3. □

Theorem 6.12. *Suppose M is a totally orientable generalized graph manifold with characteristic submanifold $\mathcal{T}$. If $M = V \cup_S W$ is a strongly irreducible Heegaard splitting, then S may be isotoped so that for C the completion of any component of $M - \mathcal{T}$, $S \cap C$ is standard.*

Proof: By Lemma 6.3, it suffices to consider the active component, hence the theorem follows from Lemma 6.5, Theorem 6.9 and Lemma 6.11. □

7 The main theorem

We here prove the main theorem. It follows readily from Theorem 6.12 along with the constructions of untelescoping and amalgamation.

Theorem 7.1. *Suppose M is a totally orientable graph manifold with characteristic submanifold $\mathcal{T}$. If $M = V \cup_S W$, then S can be isotoped so that if N is the completion of any component of $M - \mathcal{T}$, then $S \cap N$ is horizontal or vertical.*

Proof: Let $M = (V_1 \cup_{S_1} W_1) \cup_{F_1} \cdots \cup_{F_{n-1}} (V_n \cup_{S_n} W_n)$ be a weak reduction of $M = V \cup_S W$. Then each $M_i = V_i \cup_{S_i} W_i$ is a strongly irreducible Heegaard splitting of a totally orientable generalized graph manifold. Furthermore, $\mathcal{T}_i = \mathcal{T} \cap M_i$ is the characteristic submanifold of M_i. Thus by Theorem 6.12, S_i may be isotoped so that for N_i the completion of any component of $M_i - \mathcal{T}_i$, $S_i \cap N_i$ is standard. Moreover, if $\partial N_i \neq \emptyset$, then $S_i \cap N_i$ is h1 if N_i is a product component and vertical if N_i is a Seifert component.

Consider the completion N of a component of $M - \mathcal{T}$. There are three possibilities:

Case 1: N is entirely contained in $V_i \cup_{S_i} W_i$ for some i.

In this case $S \cap N = S_i \cap N$ and is thus either standard by Theorem 6.12.

Case 2: N is comprised of Seifert fibered submanifolds of a union of $V_i \cup_{S_i} W_i$'s.

In this case $S \cap N$ can be reconstructed by considering the process of amalgamation restricted to N. By Theorem 6.9 we know that $S_i \cap N$ is vertical for each i (for which this intersection is non empty). It then follows from the definition of amalgamation that $S \cap N$ is vertical.

Case 3: N is is comprised of product submanifolds of a union of $V_i \cup_{S_i} W_i$'s.

In this case again $S \cap N$ can be reconstructed by considering the process of amalgamation restricted to N. By Lemma 6.5 we know that $S_i \cap N$ is h1 for each i (for which this intersection is non empty). It then follows from the definition of amalgamation that $S \cap N$ is h1. □

References

[1] M. Boileau, J.P. Otal: *Groupe des diffeotopies de certaines varietes de Seifert*, C. R. Acad. Sc. Paris, t. 303, Serie 1, no 1, 1986

[2] Boileau, M., Otal, J.P.: *Scindements de Heegaard et groupes des homotopies des petites varietes de Seifert*, Invent. Math., 1991, 106, 85–107

[3] F. Bonahon, J.P. Otal: *Scindements de Heegaard des espaces Lenticulaires*, Ann. scient. Ec. Norm. Sup. 4^e série, t. 16, 451-466

[4] G. Burde; H. Zieschang: *Knots*, de Gruyter Studies in Mathematics, 5. Walter de Gruyter & Co., Berlin, 1985

[5] A. Casson and C. Gordon: *Reducing Heegaard splittings*, Topology and its Applications 27 (1987) 275-283

[6] D. Cooper; M. Scharlemann: *The structure of a solvmanifold's Heegaard splittings* Proceedings of 6th Gkova Geometry-Topology Conference. Turkish J. Math. 23 (1999), no. 1, 1–18.

[7] D. Gabai: *Foliations and the topology of 3-manifolds III*, J. Differential Geom. 26 (1987) 479-536

[8] J. Hempel: 3-*manifolds*, Annals of Mathematics Studies, Princeton University Press, Study 86, Princeton, NJ, 1976

[9] W. Jaco: *Lectures on Three-Manifold Topology*, Regional conference series in mathematics; no. 43

[10] Lickorish, W. B. Raymond: An introduction to knot theory. Graduate Texts in Mathematics, 175. Springer-Verlag, New York, 1997.

[11] Y. Moriah and J. Schultens: *Irreducible Heegaard splittings of Seifert fibered spaces are horizontal or vertical*, Topology, 37 (5) 1089–1112

[12] J. Nielsen: *Abbildungsklassen endlicher Ordnung*, Acta Mathematica 75, (1943) 23-115

[13] Rolfsen, Dale: *Knots and links*, Corrected reprint of the 1976 original. Mathematics Lecture Series, 7. Publish or Perish, Inc., Houston, TX, 1990.

[14] J. H. Rubinstein; M. Scharlemann: *Comparing Heegaard splittings of non-Haken* 3-*manifolds* Topology 35 (1996), no. 4, 1005–1026.

[15] M. Scharlemann, A. Thompson: *Thin position for* 3-*manifolds* AMS Contemporary Math. 164 (1994) 231–238

[16] M. Scharlemann, A. Thompson: *Heegaard splittings of* $(surface) \times I$ *are standard*, Math. Ann. 295 (1993), 549-564

[17] J. Schultens: *The Classification of Heegaard splittings for* (*closed orientable* $surface$) $\times S^1$, London Math. Soc. (3) 67 (1993) 425-448

[18] J. Schultens: *Heegaard splittings of Seifert fibered spaces with boundary*, Trans. Amer. Math. Soc. 347 (1995), no. 7, 2533–2552.

[19] J. Schultens: *Weakly reducible Heegaard splittings of Seifert manifolds are vertical*, Topology Appl. 100 (2000), no. 2-3, 219–222.

[20] J. Schultens: *Additivity of Tunnel Number for Small Knots*, Comment. Math. Helv. 75 (2000), no. 3, 353–367.

[21] W. P. Thurston. Edited by S. Levy. *Three-dimensional geometry and topology. Vol. 1* Princeton Mathematical Series, 35. Princeton University Press, Princeton, NJ, 1997. x+311 pp. ISBN: 0-691-08304-5

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