Max-Planck-Institut für Mathematik
Bonn

Distribution theorems for Saito-Kurokawa lifts

by

Bernhard Heim

Max-Planck-Institut für Mathematik
Preprint Series 2010 (77)

Distribution theorems for Saito-Kurokawa lifts

Bernhard Heim

Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
Germany

German University of Technology in Oman
Way No. 36
Building No. 331
North Ghubrah, Muscat
Sultanate of Oman

MPIM 10-77

[1]

DISTRIBUTION THEOREMS FOR SAITO-KUROKAWA LIFTS

BERNHARD HEIM

Abstract. A Siegel modular cusp form F of weight k and degree 2 is a Saito-Kurokawa lift if and only if F satisfies the Maass relations. Recently it has been shown that this is equivalent to the requirement that F satisfies the Hecke duality relations. In this paper we prove that if the Hecke duality relations are fulfilled for all primes outside a set of primes with Dirichlet density smaller 1/8 then the cusp form is a Saito-Kurokawa lift. Further, for every weight k, there exists an effective constant c such that, if for all prime $p \leq c$ the Hecke duality relation is satisfied, then the form is a Saito-Kurokawa lift. Moreover we indicate further perspectives suggested by results of M. Manickam, B. Ramakrishnan and T.C. Vasudevan.

1. Introduction and statement of results

Let S_k^2 be the space of Siegel modular cusp forms of weight k and degree 2 with respect to the Siegel modular group $\Gamma^2 := Sp_2(\mathbb{Z})$. Let $A(T)$ be the Fourier coefficients of a cusp form $F \in S_k^2$, where $T = [n, r, m] = \begin{pmatrix} n & \frac{r}{2} \\ \frac{r}{2} & m \end{pmatrix}$ runs through the set of half-integral positive definite matrices T. Then F is a Saito-Kurokawa lift if and only if the Fourier coefficients satisfy the Maass relations

$$A[n,r,m] = \sum_{d|(n,r,m)} d^{k-1}\, A\left[\frac{nm}{d^2}, \frac{r}{d}, 1\right]. \tag{1.1}$$

Here d is summed over positive integers dividing the greatest common divisor of n, r and m, and we take the usual convention that $A(T) = 0$ for T not half-integral positive definite. See [3], [22], [10], [11] for more details and a summary. In [5],[6] we proved that F is a Saito-Kurokawa lift if and only if

$$A\,[n,r,pm] - A\,[np,r,m] \;=\; p^{k-1}\left(A\left[\frac{n}{p}, \frac{r}{p}, m\right] - A\left[n, \frac{r}{p}, \frac{m}{p}\right]\right) \tag{1.2}$$

[1]This work is partly supported by the *Max-Planck-Institut Mathematik Bonn* and *GUtech*, the German University of Technology in Oman .

2000 *Mathematics Subject Classification.* 11F.

for all $T = [n, r, m]$ and primes p. For a fixed prime p, the collection of such relations for all T we call the *p-Hecke duality* relation, since it reflects a duality property of Hecke operators for elliptic modular forms ([5], [4]). If F satisfies the p-Hecke duality relation for all primes p, say that F *satisfies the Hecke duality relations.* Employing a deep result of Chai-Faltings [2] on the Ramanujan conjecture for Siegel modular forms, Pitale and Schmidt [16] have recently extended this result. They have shown among other things that finitely many p-Hecke duality relations can be omitted. In this paper we extend these results in the Hecke duality aspect in two directions. First, we show that the p-Hecke duality conditions for a set of primes with Dirichlet density smaller than 1/8 can be omitted. Further, we show that, for fixed weight k, the p-Hecke duality relations need be checked only for finitely many primes p, which can be explicitly determined.

Theorem 1.1. *Let $\mathcal{R}$ be any set of primes with Dirichlet density smaller than 1/8. Let k be a positive even integer. Then we have the following property. For $F \in S_k^2$, F is a Saito-Kurokawa lift if and only if for all primes p outside $\mathcal{R}$ the p-Hecke duality relation (1.2) is satisfied.*

Finite sets have Dirichlet density 0. Hence this theorem contains the already mentioned results of [5] and [16].

Complementary to this result, we have:

Theorem 1.2. *For a positive even integer k there exists a constant $c(k)$, depending only upon k, such that $F \in S_k^2$ is a Saito-Kurokawa lift if and only if the p-Hecke duality relation is satisfied for all primes p with $p \leq c(k)$.*

The method used in this paper makes it possible to transfer multiplicity one theorems [18], [20], [21], [15] from elliptic modular forms to the space of Siegel modular forms of degree 2. Of course once this new method is established the main ingredient is in this case the beautiful and strong result of Ramakrishnan [20]. The second theorem indicates that there are sets $\mathcal{R}$ of density 1 (depending on k). The constant $c(k)$ is effective.

It is interesting to ask if the p-Hecke duality relation (1.2) for fixed prime number p itself, which contains infinitely many relations and is related to infinitely many Fourier coefficients, has an effective finiteness property. Of course since the vector space of Siegel modular forms of fixed weight k has a dimension asymptotic to k^3, we would somehow expect this, but it is not clear how to design such an algorithm in accordance to the p-Hecke duality relation. It is maybe interesting to note that the p- Hecke Duality relation of a cusp form F can be encoded in saying that F is in the kernel of an operator $|_k \bowtie T(p)$. Here $T(l)$ ($l \in \mathbb{N}$) is related to the Hecke operator of elliptic cusp forms. We prove

$$(F|_k \bowtie T(l)) = 0 \Leftrightarrow \Big((F|_k \bowtie T(l)) \Big)|_k \bowtie T(l) = 0. \tag{1.3}$$

Then we have

Theorem 1.3. *Let k, l be positive integers. Let k be even and $F \in S_k^2$. Then*

$$\left((F|_k \bowtie T(l)) \right)|_k \bowtie T(l) \tag{1.4}$$

is a Siegel modular form with respect to a congruence subgroup

$$\Gamma^2(N) := \left\{ \gamma \in Sp(4)(\mathbb{Z}) | \gamma \equiv \left(\begin{smallmatrix} I_2 & 0 \\ 0 & I_2 \end{smallmatrix} \right) \ mod\ N \right\},$$

where the level $N = N(l)$ is effectively computable.

This has applications towards Siegel modular forms F and the functions $(F|_k \bowtie T(p))$. In this case we can choose the level to be equal to p^2.

Theorem 1.4. *Let k be a positive even integer and let p be a prime number. Let*

$$M := \Big\lfloor \frac{k \cdot [\Gamma^2 : \Gamma^2(N(p))]}{10} \Big\rfloor. \tag{1.5}$$

Let $F \in S_k^2$ and let $F|_k \bowtie T(p) \neq 0$. Then there exists a $\nu_0 \leq M$ such that the Taylor expansion is given by

$$(F|_k \bowtie T(p)) \left(\begin{smallmatrix} \tau & z \\ z & \tilde{\tau} \end{smallmatrix} \right) = \sum_{\nu=\nu_0}^{\infty} \chi_\nu^{F|_k \bowtie T(p)}(\tau, \tilde{\tau})\ z^{2\nu} \tag{1.6}$$

with $\chi_{\nu_0}^{F|_k \bowtie T(p)} \neq 0$.

From this we obtain by involving the differential operators $\mathcal{D}_{2\nu}$ (see (2.20) for a precise definition) the following application.

Corollary 1.5. *Let $F \in S_k^2$ be a Siegel modular form of even weight k and degree 2. Then F satisfies the p-Hecke duality relation if and only if for $\nu = 0, 1, \ldots, M$ the image of F with respect to $\mathcal{D}_{2\nu}$ is contained in the kernel of the operator $T(p) \otimes id - id \otimes T(p)$, i.e,*

$$(F|_k \bowtie T(p)) = 0 \Leftrightarrow \bigoplus_{\nu=0}^{M} \mathcal{D}_{2\nu}(F)|_{k+2\nu} \Big(T(p) \otimes id - id \otimes T(p) \Big) = 0. \tag{1.7}$$

Let $p_{k,\underline{2\nu}}(a,b)$ be the ultraspherical polynomial as defined in (2.19). Then

$$\mathcal{D}_{2\nu}(F)(\tau, \tilde{\tau}) = \sum_{n,m=1}^{\infty} \sum_{r, r^2 < 4nm} p_{k,\underline{2\nu}}(r, nm)\ A[n,r,m]\ e^{2\pi\, i\, n \tau} e^{2\pi\, i\, m \tau} \tag{1.8}$$

is an element of $\mathrm{Sym}^2\left(S_{k+2\nu}\right)$. Let $(f_j)_j$ be the primitive Hecke eigenbasis of $S_{k+2\nu}$ then

$$\mathcal{D}_{2\nu}(F) = \sum_{i,j=1}^{\dim S_k} \alpha_{i,j}\ f_i \otimes f_j \quad (\alpha_{i,j} \in \mathbb{C}). \tag{1.9}$$

Let d_k be the dimension of S_k. Let

$$L(\tau, \tilde{\tau}) = \sum_{n,m=1}^{\infty} B(n,m)\ e^{2\pi\, in}\ e^{2\pi\, im} \in \mathrm{Sym}^2\left(S_{k+2\nu}\right).$$

Then $L = 0$ if for all $B(n,m) = 0$ for $n, m \leq d_{k+2\nu}$. Hence there are only finitely many conditions to check if $\mathcal{D}_{k,2\nu}(F)|_k \bowtie T(p) \equiv 0$. Combining this with the condition (1.8) demonstrates that the p-Hecke duality condition is actually a criteria with finitely many condition in praxis.

To generalize this work to Siegel modular forms with level structure seems to be a non trivial task. Nevertheless by the work of M. Manickam, B. Ramakrishnan and T.C. Vasudevan [13] which analyses the space of Saito-Kurokawa lifts inside the whole space of Siegel modular forms and also examines the micro structure (newforms, old forms) inside the Saito-Kurokawa space, there is hope to extend our knowledge on this very interesting subject, which has many applications in the field of modern number theory.

Acknowledgements

This paper was partly written during the authors stay at the MPIM in Bonn (Germany) and at the GUtech in Muscat (Oman). It is a pleasure to thank the MPIM and GUtech for its excellent working atmosphere and financial support.

2. Proofs of the results

Siegel modular forms of degree 2 can be described in terms of tensor products of elliptic modular forms. This is one of the main results of [5]. In this paper we demonstrate that this viewpoint can be used to obtain strong results on the distribution of certain local properties, enabling us to decide whether or not a Siegel modular form is a Saito-Kurokawa lift. Hecke eigenforms which are not lifts are expected to satisfy the generalized Ramanujan conjecture.

We recall some notation and facts about modular forms. For a positive integer n, elements $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ of the symplectic group $\mathrm{GSp}_n(\mathbb{R})$ with positive similitudes, act on the Siegel upper half-space $\mathfrak{H}_n$ of degree n by $\gamma(\tau) = (a\tau + b)\,(c\tau + d)^{-1}$. Here Sp_n denotes the symplectic group. For a positive integer k, denote by S_k^n the space of Siegel modular cusp forms of degree n and weight k. Elements are holomorphic functions F on $\mathfrak{H}_n$ which, for all $\gamma \in \Gamma_n := Sp_n(\mathbb{Z})$, satisfy the functional equation

$$F(\gamma(\tau)) = \det(c\tau + d)^k \, F(\tau). \tag{2.10}$$

Then F has a Fourier expansion

$$F(\tau) = \sum_{T \in \mathbb{A}_n} A(T) \; e^{2\pi i \, tr(T\tau)}, \tag{2.11}$$

where T runs over the set of all half-integral positive definite matrices. We let $|_k$ be the Petersson slash operator with the normalization:

$$F|_k\gamma(\tau) := \det(\gamma)^h \det(c\tau + d)^{-k} \; F(\gamma(\tau)), \tag{2.12}$$

where $\gamma \in \mathrm{GSp}_n(\mathbb{R})$ and $h = \frac{k}{2}$. We also could put $h = k - 1$. Then in the case $n = 1$ the eigenvalues and Fourier coefficients of primitive Hecke eigenforms coincide. But in this work we want to have the center of $\mathrm{GSp}_n(\mathbb{R})$ to act trivially. Then (2.10) is $F|_k\gamma = F$ for all $\gamma \in \Gamma_n$.

For convenience, we drop the index $n = 1$ in the case of elliptic modular forms. The vector space S_k is equipped with an action of *Hecke operators* T_m, indexed by non-negative integers m, defined by

(2.13)
$$T_m f(\tau) := m^{1-\frac{k}{2}} \sum_{n=1}^{\infty} \left(\sum_{d|(n,m)} d^{k-1} a\left(\frac{nm}{d^2}\right) \right) q^n, \text{ where } f(\tau) = \sum_{n=1}^{\infty} a(n) q^n.$$

with $q = e^{2\pi i\tau}$. These operators commute and are self-adjoint with respect to the Petersson inner product

$$\langle f, g \rangle := \int_{\Gamma \backslash \mathfrak{H}} f(\tau) \, \overline{g(\tau)} \; \mathrm{Im}(\tau)^{k-2} d\tau. \tag{2.14}$$

Hence, there exists a basis of simultaneous Hecke eigenforms. It is easy to see that Hecke eigenforms can be normalized such that the first Fourier coefficient is 1. Then the m-Fourier coefficient coincides with the m-th eigenvalue. A basis with this normalization property is denoted a basis of *primitive newforms*. The eigenvalues determine a newform uniquely. Since the algebra of Hecke operators at p is locally generated by the Hecke operators T_p for p prime, the associated eigenvalues $\lambda(p)$ of an eigenform uniquely determine the cusp form. This is the *multiplicity one* property of the space S_k.

A subset $\mathcal{R}$ of the set of primes is said to have Dirichlet density δ when the limit

$$\lim_{s \mapsto 1+} \frac{\sum_{p \in \mathcal{R}} p^{-s}}{\log\left(\frac{1}{s-1}\right)} \tag{2.15}$$

exists and is equal to δ. Then we ask the following question.

Let $f, g \in S_k$ two primitive newforms such that the eigenvalues for the operators T_p with p outside $\mathcal{R}$ are equal. For which sets $\mathcal{R}$ does this imply $f = g$?

Finite sets $\mathcal{R}$ have this property [14],[8]. This is the strong multiplicity one theorem . In this case the set $\mathcal{R}$ has Dirichlet density 0. But much more is known. Ramakrishnan [19] has proven that every set $\mathcal{R}$ with Dirichlet density at most 1/8 satisfies the strong multiplicity one condition. Certain improvements have also given by Rajan [17], [18]. Ramakrishnan's proof is based on analytic techniques. Rajan uses the Galois representation attached to a modular forms and invokes the Chebotarev density theorem.

On the other hand, given a weight k, there exists a constant c_k such that for all non-zero $f \in S_k$ there is at least one Fourier coefficient $a(m) \neq 0$ for $1 \leq m \leq c_k$. This follows from the finite-dimensionality of S_k. Non-trivial bounds are known (see for example [15]).

In the case of Siegel modular forms of degree 2 it is convenient to identify the index $T = \begin{pmatrix} n & \frac{r}{2} \\ \frac{r}{2} & m \end{pmatrix}$ for the Fourier coefficients with the quadratic form $[n, r, m]$.

2.1. **Proofs of Theorem 1.1 and Theorem 1.2.** As indicated in the introduction, $F \in S_k^2$ is a Saito-Kurokawa lift if and only if F satisfies the Maass relations (1.1). We have shown in [5] that this is equivalent to the requirement that F satisfies the infinitely many duality relations attached to the classical Hecke operators $T(l)$, $(l \in \mathbb{N})$ acting on elliptic modular forms. This can be further developed. Let $j : Sl_2(\mathbb{R}) \times Sl_2(\mathbb{R}) \hookrightarrow Sp_2(\mathbb{R})$ be the standard imbedding

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \mapsto \begin{pmatrix} a & 0 & b & 0 \\ 0 & a' & 0 & b' \\ c & 0 & d & 0 \\ 0 & c' & 0 & d' \end{pmatrix}. \tag{2.16}$$

Identify j_1 and j_2 with the separate embeddings of $Sl_2(\mathbb{R})$ given by $j_1(\gamma) = j(\gamma \times I_2)$ and $j_2(\gamma) = j(I_2 \times \gamma)$. If $\gamma_1, \gamma_2 \in \Gamma^2 \in GSp_1(\mathbb{R})$ have the same factor of similitude then it also makes sense to consider $\gamma_1 \times \gamma_2$ as an element of $GSp_2(\mathbb{R})$. Let $\mathrm{Mat}(2,\mathbb{Z})_l$ be the set of all integral 2×2 matrices with determinant l. Then $\mathrm{Mat}(2,\mathbb{Z})_l$ is the union of finitely many left cosets $\Gamma\,\gamma_j$, where $\Gamma = Sp_1(\mathbb{Z})$. Let $\widetilde{\gamma_j} := l^{-\frac{1}{2}}\,\gamma_j$, and define, for $i=1,2$,

$$j_i\left(T(l)\right)(F) := \sum_j F|_k j_i(\widetilde{\gamma_j}). \tag{2.17}$$

In [5] we have proven that $F\in S_k^2$ is a Saito-Kurokawa lift if and only if

$$j_1\left(T(l)\right)(F) = j_2\left(T(l)\right)(F) \qquad \text{for all } l\in\mathbb{N}. \tag{2.18}$$

Since the algebra of operators containing $T(l)$ is generated by $T(p)$, the property (2.18) is equivalent to the p-Hecke duality $j_1\left(T(p)\right)(F) = j_2\left(T(p)\right)(F)$ for all primes p. A straightforward calculation [6] translates this property into an identity among Fourier coefficients. We have $j_1(T(p)(F)) = j_2(T(p)(F))$ if and only if (1.2) is satisfied. Ultraspherical polynomials $p_{k,\underline{2\nu}}$ are defined as follows. Let $\nu\in\mathbb{N}_0$ and a and b be elements of a commutative ring. Then

$$p_{k,\underline{2\nu}}\,(a,b) := \sum_{\mu=0}^{\nu}(-1)^{\mu}\frac{(2\nu)!}{\mu!(2\nu-2\mu)!}\frac{(k+2\nu-\mu-2)!}{(k+\nu-2)!}\,a^{2\nu-2\mu}\,b^{\mu}. \tag{2.19}$$

Specializing the parameters, we have $p_{k,\underline{0}}\,(a,b) = 1$ and $p_{k,\underline{2\nu}}\,(0,0) = 0$ for $\nu\in\mathbb{N}$. Write $(\tau,z,\tilde{\tau})$ for the point $\left(\begin{smallmatrix}\tau & z\\ z & \tilde{\tau}\end{smallmatrix}\right)\in\mathfrak{H}_2$. Define differential operators

$$\mathcal{D}_{k,\underline{2\nu}}F\,(\tau,\tilde{\tau}) \;:=\; p_{k,\underline{2\nu}}\left(\frac{1}{2\pi i}\frac{\partial}{\partial z},\left(\frac{1}{2\pi i}\right)^2\frac{\partial}{\partial\tau}\frac{\partial}{\partial\tilde{\tau}}\right)F\Bigg|_{z=0}(\tau,\tilde{\tau}). \tag{2.20}$$

In the case $\nu=0$ we get the pullback $F(\tau,0,\tilde{\tau})$ of F on $\mathfrak{H}\times\mathfrak{H}$. Then we have the imbedding

$$\mathcal{D}_k := \bigoplus_{\nu=0}^{\lfloor\frac{k}{10}\rfloor}\mathcal{D}_{k,\underline{2\nu}} :\; S_k^2 \hookrightarrow \bigoplus_{\nu=0}^{\lfloor\frac{k}{10}\rfloor}\mathrm{Sym}^2\left(S_{k+2\nu}\right). \tag{2.21}$$

Here $\lfloor *\rfloor$ is the greatest integer less than or equal $*$. It follows from [5], that F is a Saito-Kurokawa lift if and only if for all $\nu\in\{0,1,\ldots\lfloor\frac{k}{10}\rfloor\}$ and for all primes p we have

$$\left(T_p\otimes\mathrm{id}\right)\left(\mathcal{D}_{k,\underline{2\nu}}\,(F)\right) = \left(\mathrm{id}\otimes T_p\right)\left(\mathcal{D}_{k,\underline{2\nu}}\,(F)\right). \tag{2.22}$$

For every ν let $\left(f_i^{\nu}\right)_i$ be a basis of primitive newforms in $S_{k+2\nu}$. Let $\lambda_i^{\nu}(p)$ be the eigenvalues of f_i^{ν} with respect to the Hecke operator T_p and $\mathcal{D}_{k,\underline{2\nu}}F = \sum_{i,j}\alpha_{i,j}^{\nu}\,f_i^{\nu}\otimes f_j^{\nu}$ with $\alpha_{i,j}^{\nu}\in\mathbb{C}$. Let $\mathcal{R}_{k,\nu}\subseteq\mathcal{P}$ be any subset of the set of primes of $\mathbb{Z}$ with the property that, if $\lambda_i^{\nu}(p) = \lambda_j^{\nu}(p)$ for all primes p outside $\mathcal{R}_{k,\nu}$, then $i=j$. This is for example fulfilled for the set given in [19].

If (2.22) is satisfied for all primes p outside $\mathcal{R}_{k,\nu}$, then

$$\mathcal{D}_{k,\underline{2\nu}}F = \sum_i \alpha_{i,i}^{\nu}\ f_i^{\nu} \otimes f_i^{\nu}. \tag{2.23}$$

Hence, F is a Saito-Kurokawa lift. Let $\mathcal{R}_k := \cap_{\nu=0}^{\lfloor \frac{k}{10} \rfloor}\mathcal{R}_{k,\nu}$, where $\mathcal{R}_{k,\nu}$ satisfies the properties just described. As noted in the introduction of section 2, every set $\mathcal{R}$ of primes with Dirichlet density at most 1/8 satisfies the properties of $\mathcal{R}_{k,\nu}$ (not depending on k and ν), hence we can also choose for $\mathcal{R}_k$ this $\mathcal{R}$. Hence we obtain: F is a Saito-Kurokawa lift if and only if $j_1(T(p))(F) = j_2(T(p))(F)$ for all primes outside $\mathcal{R}$. This proves Theorem 1.1. The eigenvalues of f_i^{ν}, f_j^{ν} are the same for all $0 \leq \nu \leq \lfloor \frac{k}{10} \rfloor$ and primes $2 \leq p \leq \dim S_{k+2\lfloor \frac{k}{10} \rfloor}$. Hence Theorem 1.2 is obvious.

2.2. **Properties of the Hecke duality operator.** Let $F \in S_k^2$ and let p be a prime number. Then the p-Hecke duality relation is given by

$$A\left[n,r,pm\right] - A\left[np,r,m\right] \ = \ p^{k-1}\left(A\left[\frac{n}{p},\frac{r}{p},m\right] - A\left[n,\frac{r}{p},\frac{m}{p}\right],\right)$$

where $T = [n,r,m]$ runs through all positive definite binary quadratic forms $nx^2+rxy+my^2$. This relation is satisfied if and only if $j_1T(p)(F) = j_2T(p)(F)$. This can be put together to the following useful definition.

Definition. *Let $k,l \in \mathbb{N}$. Let F be a holomorphic function on $\mathfrak{H}$ which has the property $(F|_k j_1(\gamma_1))\,|_k j_2(\gamma_2) = F$ for all $\gamma_1, \gamma_2 \in \Gamma$. Then we define the Hecke duality operator $|_k \bowtie$ by*

$$F|_k \bowtie T(l) := j_1\left(T(l)\right)(F) - j_2\left(T(l)\right)(F). \tag{2.24}$$

By abuse of notation we say that $|_k \bowtie T(l)$ is an operator on the space of modular forms. Let $F \in S_k^2$. Then $F \in S_k^2$ satisfies the p-Hecke duality property if and only if F is in the kernel of $|_k \bowtie T(p)$. At this point we note that in general we can not assume that $F|_k \bowtie T(l)$ is a Siegel modular form for some congruence subgroup of Γ^2. This makes it somehow difficult to determine the vanishing order with respect to the variable z. Here we assume the standard parametrization of an element $Z = \left(\begin{smallmatrix}\tau & z\\ z & \tilde{\tau}\end{smallmatrix}\right) \in \mathfrak{H}_2$ in the Siegel upper half-space of degree 2. Nevertheless we have the following surprising result.

Proposition 2.6. *Let $k,l \in \mathbb{N}$. Let k be even and $F \in S_k^2$. Then*

$$F|_k \bowtie T(l) = 0 \Leftrightarrow \left(F|_k \bowtie T(l)\right)|_k \bowtie T(l) = 0. \tag{2.25}$$

Proof. Note that the iteration of the operators $\bowtie T(l)$ is well-defined, since $H := F|_k \bowtie T(l)$ is invariant with respect to $|_k j_1(\tilde{\gamma})$ and $|_k j_2(\tilde{\gamma})$ for $\gamma \in \Gamma$.

Let G be a holomorphic function on the Siegel upper half-space of degree 2. Then $G \equiv 0$ if and only if $\mathcal{D}_{k,2\nu}(G) \equiv 0$ for one even positive integer k and all $\nu \in \mathbb{N}_0$. If $F \in S_k^2$ then we have

$$\begin{aligned} F|_k \bowtie T(l) \equiv 0 \quad &\Leftrightarrow \quad \bigoplus_{\nu=0}^{\infty} \mathcal{D}_{k,2\nu}(F|_k \bowtie T(l)) \equiv 0 \\ &\Leftrightarrow \quad \bigoplus_{\nu=0}^{\infty} \mathcal{D}_{k,2\nu}(F)|_{k+2\nu} \left(T(l) \otimes \mathrm{id} - \mathrm{id} \otimes T(l)\right) \equiv 0. \end{aligned}$$

Let $\left(f_i^{(\nu)}\right)_i$ be a Hecke eigenbasis of $S_{k+2\nu}$ with $T(l)$ eigenvalues $\lambda_i^{(\nu)}(l)$. Let $\mathcal{D}_{k,2\nu}(F) = \sum_{i,j=1}^{\dim S_{k+2\nu}} \alpha_{i,j}^{(\nu)} \; f_i^{(\nu)} \otimes f_j^{(\nu)}$. Then

$$\begin{aligned} F|_k \bowtie T(l) \equiv 0 \quad &\Leftrightarrow \quad \bigoplus_{\nu=0}^{\infty} \sum_{i,j=1}^{\dim S_{k+2\nu}} \alpha_{i,j}^{(\nu)} \left(\lambda_i^{(\nu)}(l) - \lambda_j^{(\nu)}(l)\right) \; f_i^{(\nu)} \otimes f_j^{(\nu)} \equiv 0 \\ &\Leftrightarrow \quad \alpha_{i,j}^{(\nu)} \left(\lambda_i^{(\nu)}(l) - \lambda_j^{(\nu)}(l)\right) = 0 \\ &\qquad \text{for all } \nu, i, j. \end{aligned}$$

Now we put $H := F|_k \bowtie T(l)$. Then

$$\begin{aligned} H|_k \bowtie T(l) \equiv 0 \quad &\Leftrightarrow \quad \bigoplus_{\nu=0}^{\infty} \mathcal{D}_{k,2\nu}(H)|_{k+2\nu} \left(T(l) \otimes \mathrm{id} - \mathrm{id} \otimes T(l)\right) \equiv 0 \\ &\Leftrightarrow \quad \bigoplus_{\nu=0}^{\infty} \sum_{i,j=1}^{\dim S_{k+2\nu}} \alpha_{i,j}^{(\nu)} \left(\lambda_i^{(\nu)}(l) - \lambda_j^{(\nu)}(l)\right) \; f_i^{(\nu)} \otimes f_j^{(\nu)} \\ &\qquad |_{k+2\nu} \left(T(l) \otimes \mathrm{id} - \mathrm{id} \otimes T(l)\right) \equiv 0 \\ &\Leftrightarrow \quad \bigoplus_{\nu=0}^{\infty} \sum_{i,j=1}^{\dim S_{k+2\nu}} \alpha_{i,j}^{(\nu)} \left(\lambda_i^{(\nu)}(l) - \lambda_j^{(\nu)}(l)\right)^2 \; f_i^{(\nu)} \otimes f_j^{(\nu)} \equiv 0 \end{aligned}$$

□

Once the tool of invertible differential operators [5] is available the proof is straightforward. The main point here is the existence of the property and some consequences.

One application one might be interested in is given by the following observation. Let $l \in \mathbb{N}$ be square-free and larger then one. Then it is not possible to embed $T(l)$ via j_1 or j_2 into $\mathrm{Sp}_2(\mathbb{Q})$ or even $\mathrm{GSp}_2(\mathbb{Q})$. This has been the reason why we had to take roots in our definition of $j_1(T(l)$ and $j_1(T(l))$. This has the consequence that we can not assume in general that $|_k \bowtie T(l)$ maps a Siegel modular form to a Siegel modular form (for some congruence subgroup).

Now let $F \in S_k^2$ and let p be a prime. Then we fix a disjoint Γ-left coset decomposition of $\Gamma \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right) \Gamma$ with representatives $(\gamma_i)_i$. Then we have

$$\begin{aligned} H &:= (F|_k \bowtie T(p))\,|_k \bowtie T(p) \\ &= \sum_{i,j} F|_k j_1 \left(\begin{pmatrix} p^{-1} & 0 \\ 0 & p^{-1} \end{pmatrix} \gamma_i \gamma_j \right) + j_2 \left(\begin{pmatrix} p^{-1} & 0 \\ 0 & p^{-1} \end{pmatrix} \gamma_i \gamma_j \right) \qquad (2.26) \\ & \quad -2 \sum_{i,j} F|_k \left(\gamma_i \times \gamma_j \right). \end{aligned}$$

But this shows that $H = \sum_t F|_k g_t$ is a finite sums with $g_t \in \mathrm{GSp}_2(\mathbb{Q})$. Each summand is a Siegel modular form with respect to $\Gamma^2(N_i)$, for $N_i \in \mathbb{N}$ suitable. We put $N = \max_i N_i$. Then H is a Siegel modular form with respect to $\Gamma^2(N)$. It is obvious that N can be effectively computed. Hence this proves Theorem 1.3.

2.3. **Vanishing orders.**

Theorem 2.7. *Let k be a positive even integer. Let $F \in S_k^2(\Gamma^2(N))$ with Taylor expansion*

$$F \left(\begin{smallmatrix} \tau & z \\ z & \tilde{\tau} \end{smallmatrix}\right) = \sum_{\nu=0}^{\infty} \chi_{2\nu}(\tau, \tilde{\tau})\; z^{2\nu}. \qquad (2.27)$$

Let $\mathcal{D}_{k,2\nu}(F) \in \; Sym^2(S_{k+2\nu}(\Gamma^2(N))$ be the associated modular forms. If

$$\mathcal{D}_{k,2\nu}^{\kappa}(F) := \oplus_{\nu=0}^{\kappa} \mathcal{D}_{k,2\nu}(F) = 0, \qquad (2.28)$$

where $\kappa = \lfloor \frac{k \cdot \kappa(N)}{10} \rfloor$ with $\kappa(N)$ the index of $\Gamma^2(N)$ in Γ^2, then $F = 0$.

Proof. If $h \in \Gamma^2(N)$ then $F|_k h = F$ and if g_i runs through a representative system of $\Gamma^2(N) \backslash \Gamma^2$ then $F|_k g_i$ is well-defined. We put

$$G := \prod_{i=1}^{\kappa(N)} F|_k g_i \in M^2_{k \cdot \kappa(N)}(\Gamma^2). \qquad (2.29)$$

In [5], section 2 we have proven the following: let $H \in S_k^2(\Gamma^2)$ and $m \in \mathbb{N}_0$ then

$$\left(\chi_{2\nu}^H(\tau, \tilde{\tau})\right)_{\nu}^{m} = 0 \text{ if and only if } \left(\mathcal{D}_{k,2\nu}(\tau, \tilde{\tau})\right)_{\nu=0}^{m} = 0. \qquad (2.30)$$

Moreover let $m \geq \lfloor \frac{k}{10} \rfloor$ and let $\left(\chi_{2\nu}^{H}(\tau, \tilde{\tau})\right)_{\nu}^{m} = 0$ then $H = 0$. Hence we can deduce that $G = 0$ if and only if all the Taylor coefficients

$$\left(\chi_{2\nu}^{G}(\tau, \tilde{\tau}\right)_{\nu=0}^{\lfloor \frac{k \cdot \kappa(N)}{10} \rfloor} = 0. \tag{2.31}$$

It is obvious that $G = 0$ if and only $F = 0$. Now we assume that (2.28) is satisfied and deduce that $G = 0$, which implies the claim of the theorem. Let $G_i := F|_k g_i$. Then

$$\chi_{2n}^{G} = \sum_{\substack{(n_i)_i \in \mathbb{N}, \\ \sum_i n_i = n}} \prod_{i=1}^{\kappa(N)} \chi_{2n_i}^{G_i}.$$

Let $G_1 = F$ then $\chi_{2l}^{G_1} = 0$ for $0 \leq l \leq \lfloor \frac{k \cdot \kappa(N)}{10} \rfloor$. But this leads to $\chi_{2n}^{G} = 0$ for

$$0 \leq n \leq \lfloor \frac{k \cdot \kappa(N)}{10} \rfloor.$$

Hence we obtain $G = 0$. □

Proof of Theorem 1.4
Let $G := F|_k \bowtie T(p)$. We have shown that $G \neq 0$ if and only if $H := G|_k \bowtie T(p) \neq 0$. Moreover H is a Siegel modular form of degree 2 and weight k with respect to the congruence group $\Gamma^2(N(p))$. Therefore we can apply the result of Theorem 2.7. Since the vanishing order of G with respect to the variable z is not larger than the vanishing order of H with respect to the same variable we have proven the theorem.

References

[1] K. Bringmann, B. Heim: *Hecke duality and Jacobi forms.* arXiv:0711.2532v2 [math.NT] 15.11.2007

[2] Ch.-L. Chai, G. Faltings: *Degeneration of Abelian Varieties.* **22**. Ergebnisse der Mathematik, Berlin,Heidelberg, New York: (1990).

[3] M. Eichler, D. Zagier: *The theory of Jacobi forms. Progress in Mathematics.* **Vol. 55**. Boston-Basel-Stuttgart: Birkhäuser (1985).

[4] P. Garrett, B. Heim: *Hecke duality of Ikeda lifts.* Preprint.

[5] B. Heim: *On the Spezialschar of Maass.* ArXiv: 0801.1804v1 [math.NT].

[6] B. Heim: *A strong symmetry of Eisenstein series.* ArXiv: 0801.18104v1 [math.NT].

[7] T. Ikeda: *On the lifting of elliptic cusp forms to Siegel cusp forms of degree* $2n$*.* Ann. of Math. **154** no. 3 (2001), 641-681.

[8] H. Jacquet, J. Shalika: *On Euler products and a classification of automorphic representations, I and II.* Amer. J. Math. **103** (1981), 499-558 and 777-815.

[9] N. Kurokawa: *Examples of eigenvalues of Hecke operators on Siegel cusp forms of degree two.* Inventiones Math. **49** (1978), 149-165.

[10] H. Maass: *Über eine Spezialschar von Modulformen zweiten Grades I.* Invent. Math. **52** (1979), 95-104.

[11] H. Maass: *Über eine Spezialschar von Modulformen zweiten Grades II.* Invent. Math. **53** (1979), 249-253.

[12] H. Maass: *Über eine Spezialschar von Modulformen zweiten Grades III.* Invent. Math. **53** (1979), 255-265.

[13] M. Manickam, B. Ramakrishnan, and T.C. Vasudevan: *On the Saito-Kurokawa descent for congruence subgroups.* Manuscripta Math. **81** (1993), 161–182.

[14] T. Miyawaki: *On automorphic forms on* $GL(2)$ *and Hecke operators.* Ann. of Math. **94** (1971), 174-189.

[15] C. Moreno: *Analytic proof of the strong multiplicity one theorem.* Amer. J. Math **107** no.1 (1985), 163-206.

[16] A. Pitale, R. Schmidt: *Ramanujan-type results for Siegel cusp forms of degree* 2. Homepapage Ralf Schmidt

[17] C.S. Rajan: *Refinement of strong multiplicity one for automorphic representations of Gl(n).* Proc. Amer. Math. Soc. **128** (1998), 691-700.

[18] C.S. Rajan: *On strong multiplicity one for l-adic representations.* IMRN **3** (1998), 161-172.

[19] D. Ramakrishnan: *Refinement of the strong multiplicity one theorem for* $GL(2)$. Invent. math. **116** (1994), 645-649.

[20] D. Ramakrishnan: *Modularity of the Rankin-Selberg L-series, and multiplicity one for SL(2).* Ann. math. **152** (2000), 45-111.

[21] D. Ramakrishnan: *Recovering representation form square.* Appendix to: W. Duke and E. Kowalski, A problem of Linnik for elliptic curves and mean-value estimates for automorphic representations. Invent. Math. **139** no.1 (2000), 1-39.

[22] D. Zagier: *Sur la conjecture de Saito-Kurokawa (d'après H. Maass.* Sém. Delange-Pisot-Poitou 1979/1980, Progress in Math. **12** (1980), 371 -394.

German University of Technology in Oman, Way No. 36, Building No. 331, North Ghubrah, Muscat, Sultanate of Oman

E-mail address: bernhard.heim@gutech.edu.om