On Gelfand Duality

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Introduction: In [GGPS] the name duality theorem is given on the one hand to something which nowadays is called Frobenius reciprocity for (co–)induced representations and on the other hand to the interplay between automorphic forms and automorphic representations, which up to now has kept this name. In the classical version it is only given under a suitable multiplicity one and dimension one assumption. In [Takase] there is given a version for multiplicity one $K-$ types which need not be one dimensional. In this paper we indicate, how the multyplicity one condition can be removed. For this, the characterization of automorphic forms by an eigenequation has to be substituted by saying that they span a finite dimensional module of a certain Hecke algebra. For an application see also [Be-Bö].

The first two sections are devoted to the relation of the restriction of a representation π of a locally compact group G to a compact subgroup K on the one hand and associated representations of certain convolution algebras on the other. The key lemma for our duality is Lemma 2.1 which characterizes a $K-$ isotype within a $G-$ isotype by its behavior under a certain convolution algebra. In section 3 the duality theorem is proven.

We mention a differential version of the duality for Lie groups and give a general reciprocity theorem which allows a reinterpretation of the dimension formula given by the duality theorem.

0 Notation All vector spaces are over $\mathbb{C}$. For a vector space V we will write V^* for the dual space, so $V^* = \mathrm{Hom}(V, \mathbb{C})$. All topological vector spaces will be assumed locally compact and Hausdorff. For topological vector spaces V and W we write $\mathrm{Hom}^c(V, W)$ for the set of continuous linear maps from V to W. We write V' for the topological dual, i.e. $V' = \mathrm{Hom}^c(V, \mathbb{C})$. A representation on a topological group G means a weakly continuous representation on a topological vector space. It will be called irreducible if it admits only the trivial invariant closed subspaces.

A $G-$ module is a vector space with G acting by linear maps. There is a functor from the category $Rep(G)$ of $G-$ representations and continuous linear $G-$ maps to the category Mod (G) of $G-$ modules and linear $G-$ maps by forgetting the topology. The morphisms in Rep (G) will be denoted $\mathrm{Hom}^c_G(A, B)$,$A, B \in \mathrm{Ob}(\mathrm{Rep}G)$ and those of Mod (G) simply by $\mathrm{Hom}_G(A, B)$. If not otherwise stated the term unitary representation will imply the topological vector space to be complete, i.e. a Hilbert space.

1.1

Let G denote a locally compact unimodular group and Γ a closed unimodular subgroup. Let (ρ, U) denote a representation of Γ. The continuously induced representation $C^G_\Gamma(\rho)$ of G is defined on the space $C^G_\Gamma U$ of all continuous functions $f : G \to U$ with $f(\gamma x) = \rho(\gamma) f(x)$ for all $\gamma \in \Gamma$. This space is endowed with the topology of locally uniform convergence and the representation of G by right shifts, i.e. $R_g f(x) = f(xg)$.

Assuming (ρ, U) to be a unitary representation and fixing Haar measures on G and Γ we define the unitarily induced representation $\mathrm{I}^G_\Gamma(\rho)$ to live on the space $\mathrm{I}^G_\Gamma(U)$ of measurable functions $f : G \to U$ with $f(\gamma x) = \rho(\gamma) f(x)$ and $\|f\|$ square integrable. On $\mathrm{I}^G_\Gamma(U)$ we have the scalar product $(f, g) = \int_{\Gamma\backslash G} \langle f(x), g(x)\rangle dx$. The group G acts as above by right shifts.

1.2.

For any locally compact group L the set of isomorphism classes of irreducible unitary representations is denoted by $\hat{L}$. Concerning elements of $\hat{L}$ we do not distinguish between a class and a representative. For $(\pi, V_\pi) \in \hat{L}$ and any representation (ρ, V) of L we denote by $V(\pi)$ The $\pi-$ isotopic component of V, i.e. the closure of the sum of all $T(V_\pi)$ with $T \in \mathrm{Hom}^c_G(V_\pi, V)$. If the space $\mathrm{Hom}^c_G(V_\pi, V)$ is finite dimensional we denote its dimension by $[\rho : \pi]$ and call it the multiplicity of π in ρ.

1.3.

Back to our previous setting let (π, V_π) denote an unitary representation of G. The group homomorphism $\pi : G \to GL(V_\pi)$ gives rise to a representation, also denoted π, of the convolution algebra $C_c(G)$ of compactly supported continuous function G.

Let K denote some fixed compact subgroup of G. Fix some $(\tau, V_\tau) \in \hat{K}$ such that τ occurs in $\pi|_K$. Such a τ is called K-type of π. If τ has finite multiplicity in $\pi|_K$ then the pair (π, τ) will be called an admissible pair. If π is irreducible we say (π, τ) is an irreducible admissible pair. Since K is compact the space V_τ is finite dimensional and we have an isomorphism

$$(1.3.1.) \qquad \begin{array}{ccc} \mathrm{Hom}_K(V_\delta, V_\pi) & \otimes & V_\delta \tilde{\to} V_\pi(\delta) \\ T & \otimes & v \mapsto T(V). \end{array}$$

Define a function $e_\tau \in C(K)$ by $e_\tau(k) = (\dim \tau)\overline{tr\ \tau(k)}$. Then for any unitary representation (σ, V_σ) of K the operator $\sigma(e_\tau)$ is just the orthogonal projection onto $V_\sigma(\tau)$.

For any $h \in C(K)$ and $f \in C(G)$ we write

$$\begin{aligned} h * f(x) &= \int_K h(k) f(k^{-1}x) dk, \\ f * h(x) &= \int_K h(k) f(xk) dk \quad . \end{aligned}$$

1.4.

Let (π, τ) be an admissible pair and set $\psi_{\pi,\tau}(x) := tr\pi(e_\tau)\pi(x)$ for $x \in G$. It is known [Gaal,p.475], that $\psi_{\pi,\tau} = \psi_{\pi',\tau'}$ implies $(\pi, \tau) = (\pi', \delta')$ if $\pi, \pi' \in \hat{G}$.

For $(\pi, V_\pi) \in \hat{G}$ the Hilbert space V_π decomposes under K as a direct sum of isotypic components. The algebra $C_c(G)$ does not respect this decomposition, but the subalgebra

$$C_c(G)^K := \{f \in C_c(G) | f(xk) = f(kx) \text{ for } k \in K, x \in G\}$$

does so, since for $f \in C_c(G)^K$ and $k \in K$ we have $\pi(f)\pi(k) = \pi(k)\pi(f)$. By this we conclude that $C_c(G)^K$ acts on $\mathrm{Hom}_K(V_\tau, V_\pi)$ for any $\tau \in \hat{K}$. Denote this representation by $\alpha_{\pi,\tau}$. A vector v in V_π is called cyclic if V_π is the closure of the span of $\pi(G)v$.

Proposition *Let (π, τ) be an irreducible admissible pair. Then the representation $\alpha_{\pi,\tau}$ is irreducible. Given a second admissible pair (π', τ') such that $V_{\pi'}(\tau')$ contains a cyclic vector for $V_{\pi'}$ and $\alpha_{\pi,\tau} \equiv \alpha_{\pi',\tau'}$ then $(\pi, \tau) = (\pi', \tau')$.*

Proof: The first claim follows from Proposition 12,p.471 in [Gaal]. An irreducible finite dimensional representation α of the algebra $C_c(G)^K$ is given by its character χ_α. We get by (1.3.1.)

$$(1.6.1.)\quad \chi_{\alpha_{\pi,\tau}}(f) = \frac{1}{\dim\tau}\int_G f(x)\psi_{\pi,\tau}(x)dx.$$

Since $\alpha_{\pi,\tau}(f) = \alpha_{\pi,\tau}(e_\tau * f * e_\tau)$ we get $\tau = \tau'$. By Lemma 5,p.484 in [Gaal] it follows that $\psi_{\pi,\tau}$ is determined by $\alpha_{\pi,\tau}$. Now let $C_c(G)_\tau$ denote the convolution algebra of all $f \in C_c(G)$ that satisfy $e_\tau * f * e_\tau = f$. This algebra acts on $V_\pi(\tau)$ and the trace of this action is also given by the right hand side of (1.5.1.)Hence the spaces $V_\pi(\tau)$ and $V_{\pi'}(\tau)$ are equivalent under $C_c(G)_\tau$. Let v' be a cyclic vector of $V_\pi(\tau)$ and v its image in $V_\pi(\tau)$ under some $C_c(G)_\tau-$ isomorphism. As in the proof of Thm.18,p.475 [Gaal] we conclude

$$\langle\pi(x)v, v\rangle = \langle\pi'(x)v', v'\rangle \quad \text{for } x \in G.$$

By Proposition 17,p.455 [Gaal] we get the claim. ☺

2.1. Lemma: *Let (ρ, E) be an unitary representation of G. Let (π,τ) be an irreducible admissible pair. We denote by $E(\pi)(\tau)$ the τ–isotypic component of $(\rho|_K, E(\pi))$. Then $E(\pi)(\tau) = E(\alpha_{\pi,\tau})$, where on the right hand side we take the $\alpha_{\pi,\tau}$-isotypic component with respect to the action of $C_c(G)^K$.*

Proof: The inclusion "$\subset$" is clear by 1.4. For "$\supset$" let v be some nonzero vector in $E(\alpha_{\pi,\tau})$. Then Proposition 1.4. applied to π and the cyclic space generated by v gives the claim. ☺

2.2. Lemma *Let (π, V_π) be an unitary representation of G. Any finite dimensional $C_c(G)^K$-submodule of V_π is semisimple*

Proof: Consider the involution $f^*(x) := \overline{f(x^{-1})}$ on $C_c(G)^K$. Then π includes a $*$-representation of the $*$-algebra $C_c(G)^K$. So let $W \subset V \subset V_\pi$ be $C_c(G)^K$-modules then $W^\perp \cap V$ also is $C_c(G)^K$-stable, hence a complementary to W in V. ☺

2.3. Lemma *Let (π, V_π) be an unitary representation of G. $(\tau, V_\tau) \in \hat{K}$ and $0 \neq v \in V_\pi(\tau)$ such that $\pi\left(C_c(G)^K\right)V$ is a finite dimensional irreducible $C_c(G)^K$-module. Then v generates an irreducible G-representation.*

Proof: Let $V := \overline{\pi(L^1(G))v} \subset V_\pi$. Let $L^1(\tau) := e_\tau * L^1(G) * e_\tau$. For the K-type τ we get

$$\begin{aligned} V(\tau) &= \overline{\pi(L^1(G))v\ (\tau)} \\ &= \overline{\pi(L^1(\tau))v} \\ &= \overline{K - \text{span}\pi\left(L^1(\tau)^0\right)v} \\ &= K - \text{span}C_c(G)^K v \end{aligned}$$

by Proposition 4,p. 483 in [Gaal]. Here $L^1(\tau)^0$ means the set of all $f \in L^\wedge(\tau)^0$ that satisfy $f(kx) = f(xk)$ for $k \in K, x \in G$. It follows that $V(\tau)$ is irreducible under the action of

$C(K) \otimes C_c(G)^K$. Hence for any proper submodule $W \subset V$ we have $W(\tau) = 0$. Let $\widetilde{W}$ denote the closure of the sum of all proper submodules of V then $\widetilde{W}(\tau) = 0$, hence $\widetilde{W}$ is proper again. Thus $\widetilde{W}^{\perp} \sim V$ is a nontrivial submodule since it contains $V(\tau)$. But $V(\tau)$ generates V, so $\widetilde{W} = 0$ and V is irreducible. ☺

3. Automorphic forms

3.1.

We recall the space $I_\Gamma^G V_\rho = L^2\left(\Gamma^{\backslash G}, \rho\right)$ of functions $f : G \to V_\rho$ such that

$$1) \quad f(\gamma x) = \rho(\gamma) f(x) \quad \text{for } \gamma \in \Gamma$$

$$2) \quad \int_G \|f(x)\|^2 dx < \infty.$$

An irreducible representation π of G that occurs as subrepresentation of $I_\rho^G \rho$ is called automorphic.

3.2

Let V be a module of the algebra A. A vector $v \in V$ is called <u>A-finite</u>, if Av is finite dimensional space. The same terminology is used for group actions.
The space of automorphic forms $A\left(\Gamma^{\backslash G}, \rho\right)$ is by definition the space of all $f \in L^2\left(\Gamma^{\backslash G}, \rho\right)$ such that

1) f is K-finite
2) f is $C_c(G)^K$-finite.

3.3.

We have a direct sum decomposition into K-types

$$A\left(\Gamma^{\backslash G}, \rho\right) = \bigoplus_{\tau \in \hat{K}} A_\tau\left(\Gamma^{\backslash G}, \rho\right)$$

Since $C_c(G)^K$ respects this decomposition we get by 2.2. and 2.3.

$$A\left(\Gamma^{\backslash G}, \rho\right) = \bigoplus_{\tau = \hat{K}} \bigoplus_{\substack{\pi \in \hat{G} \\ (\pi, \tau) \text{ admissible}}} A_\tau\left(\Gamma^{\backslash G}, \rho, \pi\right)$$

Here $A_\tau\left(\Gamma^{\backslash G}, \rho, \pi\right)$ is the space of all $\psi \in A_\tau\left(\Gamma^{\backslash G}, \rho\right)$ that lie in the π-isotopic component of $L^2\left(\Gamma^{\backslash G}, \rho\right)$. The elements of $A_\tau\left(\Gamma^{\backslash G}, \rho\right)$ are called automorphic forms of weight τ and level ρ. Summarizing we can state this as our Gelfand Duality:

Theorem *Let* $C_c(G)_\tau^K := e_\tau * C_c(G)^K$, *then the convolution algebra* $C_c(G)_\tau^K$ *acts on the automorphic forms of weight* τ *and level* ρ. *The* $C_c(G)_\tau^K$*-isotypic components are precisely the spaces of automorphic forms that belong to the same automorphic representation. The multiphicity of a* $C_c(G)_\tau^K$*-module equals the multiphicity of the corresponding automorphic representation. We have*

$$\dim A_\pi\left(\Gamma^{\backslash G}, \rho, \pi\right) = \left[I_\Gamma^G(\rho) : \pi\right][\pi|_K : \tau] \dim \tau.$$

3.4.

For applications, the algebra $C_c(G)_\tau^K$ often is too large. One seeks for finitely generated algebras that suffice to do the job. This can be achieved in the case of a real reductive Lie group G. So let K be a maximal compact subgroup of G, denote by $\mathfrak{g}$ the Lie algebra of G and by $U(\mathfrak{g})$ its universal enveloping algebra over $\mathbb{C}$. The algebra $U(\mathfrak{g})^K$ of $Ad(K)$-invariants in $U(\mathfrak{g})$ is finitely generated since its graded version is so.

Proposition: *The space* $A\left(\Gamma^{\backslash G}, \rho\right)$ *equals the space of* K*-finite,* $U(\mathfrak{g})^K$*-finite differentiable vectors in* $L^2\left(\Gamma^{\backslash G}, \rho\right)$. *Theorem 3.3. holds with* $C_c(G)_\tau^K$ *replaced by* $U(\mathfrak{g})^K$.

Proof: The analogue of Lemma 2.1. is in Proposition 3.5.4. in [Wall]. The analogue of Lemma 2.2. holds because also $U(\mathfrak{g})^K$ is a $*$- algebra with $X^* = -\overline{X}$ for $X \in \mathfrak{g} \otimes \mathbb{C}$. The analogue of Lemma 2.3 holds because of Lemma 3.5.3 in [Wall]. ☺

3.5. Remark. As in [Takase] it is possible to define a different space of automorphic forms $A'_\tau\left(\Gamma^{\backslash G}, \rho, \pi\right)$ consisting of $\mathrm{Hom}_{\mathbb{C}}(V_\tau, V_\rho)$-valued functions on G such that

- $f(\gamma x k) = \rho(\gamma) f(x) \tau(k) \qquad$ for $\gamma \in \Gamma, k \in K$.
- f is $C_c(G)^K$-finite of type $\alpha_{\pi,\tau}$
- f is L^2-integrable

and it clearly follows

$$\dim A'_\tau\left(\Gamma^{\backslash G}, \rho, \pi\right) = \left[I_\Gamma^G \rho : \pi\right][\pi|_K : \tau].$$

3.6. The reciprocity law.

In this section we assume the quotient $\Gamma^{\backslash G}$ to be compact. For any unitary representation (π, V_π) of G let V_π^c denote the G-span of the K-finite vectors in V_π. The representation π is called <u>admissible</u> if every $\tau \in \hat{K}$ occurs in $\pi|_K$ with finite multiplicity, i.e. (π, τ) is an admissible pair.

Theorem *Assume* $(\pi, V_\pi) \in \hat{G}$ *admissible. We have an isomorphism of vector spaces*

$$\mathrm{Hom}_G^c\left(V_\pi, I_\Gamma^G U\right) \xrightarrow{\sim} \mathrm{Hom}_\Gamma(V_\pi^c, U).$$

Especially for trivial ρ *we get*

$$\left[I_\Gamma^G 1 : \pi\right] = \dim \left(V_\pi^{c,*}\right)^\Gamma$$

Proof: At first note that by restriction we get

$$\mathrm{Hom}_G^c\left(V_\pi, I_\Gamma^G U\right) \xrightarrow{\sim} \mathrm{Hom}_G^c\left(V_\pi^c, I_\Gamma^G U\right)$$

Since V_π is admissible we get for the K-finite vectors $V_{\pi,K}$ in V_π that $V_{\pi,K} = (C_c(G)V_\pi)_K$. Hence any G-morphism from V_π to $I_\Gamma^G U$ will map V_π^c into the space of continuous functions $C_\Gamma^G U$ in $I_\Gamma^G U$. So it makes sense to define

$$\phi : \mathrm{Hom}_G^c\left(V_\pi^c, I_\Gamma^G U\right) \to \mathrm{Hom}_\Gamma(V_\pi^c, U)$$
$$\phi(F)(v) := F(v)(1)$$

and

$$\psi : \mathrm{Hom}_\Gamma(V_\pi^c, U) \to \mathrm{Hom}_\Gamma\left(V_\pi^c, I_\Gamma^G U\right)$$
$$\psi(f)(u)(g) = f(\pi(g)u)$$

The only thing we have to show is that $\psi(f)$ is bounded for any f since then it automatically follows that ϕ and ψ are inverse to each other. So let $f \in \mathrm{Hom}_\Gamma\left(V_\pi^C, U\right)$. Consider $\psi(f)$ first as a map from $V_{\pi,K}$ to $I_\Gamma^G U$. The space $V_{\pi,K}$ splits into an orthogonal sum of finite dimensional spaces. Let $\langle\cdot,\cdot\rangle$ denote the scalar product on $V_{\pi,K}$. Let $(\cdot,\cdot)$ denote the second scalar product defined by $(x,y) = \langle\psi(f)x, \psi(f)y\rangle$. Since also under $(,)$ the decomposition of $V_{\pi,K}$ is orthogonal there is a linear operator B such that

$$(x,y) = \langle x, By\rangle.$$

The algebra $C_c(G)_K$ of on both sides K-finite functions acts irreducibly on $V_{\pi,K}$ and B commutes with this action hence B is a scalar and we conclude that $\psi(f)$ is bounded on $V_{\pi,K}$ and G equivariant, hence bounded on V_π^C. ☺

3.7. Remark The map of the theorem extends to an isomorphism of bifunctors on the categories of admissible unitary semisimple representations of G and finite dimensional unitary representations of Γ.

3.8. Corollary *For the space of automorphic forms we get*

$$\dim A_\tau\left(\Gamma^{\backslash G}, \rho, \pi\right) = \dim\left(V_\pi^{C,*} \otimes U\right)^\Gamma [\pi|_K : \tau] \dim \tau.$$

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