Tutte decomposition for graphs, weighted graphs and symmetric matrices

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Abstract

Deleting and contracting a link in a graph determines an equivalence relation in the Abelian group freely generated by graphs: modulo this relation a graph is equivalent to the sum of itself with deleted edge and itself with contracted edge. Tutte's theorem states that any graph is equivalent to the unique linear combination of graphs without links. We call this linear combination the *Tutte decomposition* of the original graph. Recent developments in the knot theory and the theory of Abelian avalanches and sandpiles led to the Tutte decomposition for graphs with weighted vertices and edges. We prove that this decomposition is also unique.and discuss the Hopf algebra structures underlying it.

A class of graph invariants introduced by Tutte in early fourties had been thouroughly investigated since that time. It had proved that a large amount of natural functions on graphs satisfies Tutte relations. The chromatic polynomial, the dichromat, the number of spanning trees and so on are among them.

Recent developments in knot invariants [1] and the theory of avalanches and sandpiles [2, 3] led to functions on graphs with additional structure. This additional structure is a weight assigned either to vertices of the graph

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(in the case of knot invariants) or both to vertices and edges (in the case of avalanches). The functions arising appear to be very close to the original Tutte invariants.

This progress produced a new point of view on the Tutte relation. It had been understood, for example, that there exists a deep connection between the Tutte relation and the Jacoby identity for Lie algebras [1].

In the first three sections we describe Tutte decomposition for all three types of objects. In Sec. 2 also a Negami-like extension of the weighted invariants is given. The construction of the universal Tutte invariant for symmetric matrices in Sec. 3 is new. In Sec. 4 we discuss Hopf algebras underlying the Tutte decomposition in all three cases. Sec. 5 is devoted to the description of a model unifying the Tutte case of Sec. 1 and the weighted case of Sec. 2.

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1 Tutte decomposition for graphs

1.1 Definitions

A*graph* may contain *loops* (edges whose ends coincide) and multiple edges. An edge which is not a loop is called a *link*. A *Tutte invariant* (or, a *V-function*, following [6]) is a function on graphs satisfying some relations. The function takes values in some commutative associative ring K with unity.

Let A denote a link in a graph Γ. The graph Γ'_A is obtained from the graph Γ by removing the edge A.

The graph Γ''_A is obtained from the graph Γ by contracting the edge A. After contracting an edge its ends become one and the same vertex. All other edges whose both ends coincide with the ends of A become loops.

A loop may not be deleted or contracted.

Both deletion and contraction decrease the number of edges in the graph by one.

We say that a function f *satisfies the Tutte relation* if

$$f(\Gamma) = f(\Gamma'_A) + f(\Gamma''_A)$$

for any graph Γ and any link A in it.

Consider the free Abelian group G generated by all the graphs as free generators. Elements of G are linear combinations of graphs with integer coefficients. Any function on graphs with values in K can be extended to a linear function $G \to K$.

Consider the subgroup $GT \subset G$ generated by all expressions of the form $\Gamma - \Gamma'_A - \Gamma''_A$ for every graph Γ and for every link A in it. Denote by x_k the graph consisting of one vertex and k loops. Any element of G is equivalent, modulo GT, to a linear combination of graphs x_k and their disjoint unions. Indeed, according to the Tutte relation, any graph is equivalent modulo GT to a sum of two graphs with less number of links, and we can proceed by induction.

There is a huge number of ways for subsequent deletion and contraction of links in the graph. Different ways can provide, in principle, different decompositions. The main theorem by Tutte (see below) states, however, that the decomposition is unique. It is called *the Tutte decomposition* of a graph. The theorem is proved by constructing the universal Tutte invariant.

1.2 The universal Tutte invariant

Let $\sqcup$ denote the operation of disjoint union of graphs. A function f is called *multiplicative* if

$$f(\Gamma_1 \sqcup \Gamma_2) = f(\Gamma_1) f(\Gamma_2)$$

for any pair Γ_1, Γ_2 of graphs.

Definition 1.1 A function f is called *a Tutte invariant* if it is multiplicative and satisfies the Tutte relation.

Each Tutte invariant can be obtained from the universal one which we are going to construct now.

Let $r_0, r_1, r_2, \ldots$ denote an infinite set of commuting formal variables. The universal Tutte invariant we are going to describe is a function on graphs taking values in the ring $\mathbb{Z}[r_0, r_1, \ldots]$ of polynomials in these variables with integer coefficients.

Let Γ be a connected graph with the first Betti number (= number of independent cycles = cyclomatic number) $k = b_1(\Gamma)$. We set $t(\Gamma) = r_k$. For example, $t(x_k) = r_k$. For an arbitrary graph Γ we set $t(\Gamma) = \prod_{\Gamma_i} t(\Gamma_i)$, where the product is taken over all connected components Γ_i of the graph Γ. The function t thus takes any graph to a monomial, which we call the *Tutte monomial.*

A *spanning subgraph* of a graph Γ is an arbitrary subgraph of Γ containing all vertices of Γ.

Definition 1.2 The *universal Tutte invariant* T is the function defined by

$$T(\Gamma) = \sum_{\gamma} t(\gamma),$$

where the sum is taken over all spanning subgraphs γ of the graph Γ.

For example,

$$T(x_k) = \sum_{i=0}^{k} \binom{n}{k} r_k.$$

Theorem 1.1 [6]

1. *The universal Tutte invariant is a Tutte invariant.*

2. *Let K be an associative commutative ring with unity. Substituting arbitrary elements of K instead of each element $r_0, r_1, r_2, \ldots$ in the function T one obtains a Tutte invariant with values in K.*

3. *Any Tutte invariant with values in K can be obtained in this way.*

The Tutte theorem means in particular that a Tutte invariant may take arbitrary values on the set $\{x_k\}$ of graphs, and it is uniquely determined by this set of values.

Corollary 1.1 *The Tutte decomposition is unique.*

2 Tutte decomposition for weighted graphs

2.1 Weighted invariants

A theory we are going to construct now is parallel to the theory of Tutte invariants, but it works with slightly different objects. It appeared in [1] in connection with Vassiliev knot invariants.

Definition 2.1 A *weighted graph* is a graph Γ having no loops or multiple edges endowed with a mapping $w : V(\Gamma) \to \mathrm{N} \sqcup \{0\}$ to be called a *weight* taking each vertex of the graph Γ to a non-negative integer. The *weight* $w(\Gamma)$ *of the graph* Γ is the sum of the weights of all its vertices, $w(\Gamma) = \sum_{v \in V(\Gamma)} w(v)$.

For an edge A of the graph Γ the graph Γ'_A is obtained from Γ by removing the edge A just as above. The weights of the vertices do not change.

The *contraction* Γ''_A of an edge A is defined as follows:

1) the edge A is contracted into a vertex A of the graph Γ''_A;

2) if multiple edges arise they are replaced by unique edges;

3) the weight $w(A)$ of the new vertex A is set to be equal to the sum of the weights of the ends of the edge A in the original graph Γ; weights of other vertices do not change.

Definition 2.2 A function f on weighted graphs *satisfies the weighted Tutte relation* if

$$f(\Gamma) = f(\Gamma'_A) + f(\Gamma''_A)$$

for any weighted graph Γ and any edge A in it.

Consider the free Abelian group W generated by all weighted graphs as free generators. Let $WT \subset W$ be the subgroup generated by all the expressions of the form $\Gamma - \Gamma'_A - \Gamma''_A$. Denote by $y_n, n = 0, 1, 2, \ldots$ the weighted graph consisting of one vertex of weight n. Since both deletion and contraction decrease the number of edges in the graph, any weighted graph is equivalent, modulo WT, to a linear combination of graphs y_n and there disjoint unions. In fact, this linear combination is unique (see the theorem below). We call it the *Tutte decomposition for weighted graphs.*

2.2 Homogeneity of weighted Tutte relations

Fix an integer n and consider a weighted graph Γ of weight n. Then for any edge A of Γ both graphs Γ'_A and Γ''_A will be of the same weight n.

This allows one to consider the homogeneous component $W_n \subset W$ generated by weighted graphs of weight n and the homogeneous component $WT_n \subset W_n$ generated by all Tutte expressions for graphs of weight n.

One obviously obtains

Statement 2.1 *The groups W and WT can be represented as direct sums of their homogeneous components:*

$$W = W_0 \oplus W_1 \oplus W_2 \oplus \ldots$$

$$WT = WT_0 \oplus WT_1 \oplus WT_2 \oplus \ldots$$

The homogeneity of Tutte relations for weighted graphs is their main difference from the situation with ordinary graphs.

2.3 The universal weighted Tutte invariant

The definition of multiplicativity for a function on weighted graphs coincides with that for ordinary graphs.

Definition 2.3 A function f on weighted graphs is called a *weighted Tutte invariant* if it is multiplicative and satisfies the weighted Tutte relation.

The universal weighted Tutte invariant acts from the set of weighted graphs into the ring $Z[s_0, s_1, s_2, \ldots]$.

For a connected weighted graph Γ with the first Betti number $k = b_1(\Gamma)$ and the weight $w = w(\Gamma)$ we set $wt(\Gamma) = (-1)^k s_w$. For an arbitrary weighted graph Γ we set $wt(\Gamma) = \prod_{\Gamma_i} wt(\Gamma_i)$, where the product is taken over all connected components Γ_i of the graph Γ. We call the function wt the *weighted Tutte monomial.*

Definition 2.4 The *universal weighted Tutte invariant* wT is defined as follows. For any weighted graph Γ set

$$wT(\Gamma) = \sum_{\gamma} wt(\gamma),$$

where the sum is taken over all spanning subgraphs γ of the graph Γ.

Theorem 2.2 [1]

1. *The function wT is a weighted Tutte invariant.*
2. *Let K be an associative commutative ring with unity. Substituting arbitrary elements of K for each element $s_1, s_2, s_3, \ldots$ in the function wT one obtains a weighted Tutte invariant with values in K.*
3. *Any weighted Tutte invariant with values in K can be obtained in this way.*

Corollary 2.1 *Any weighted graph admits the unique Tutte decomposition.*

2.4 A Negami-like extension of the weighted invariant

S. Negami [4] introduced a modification of the Tutte relation for ordinary graphs. The relation of Negami looks like

$$f(\Gamma) = xf(\Gamma_A') + yf(\Gamma_A'').$$

As a result, the invariant under consideration becomes a polynomial in two additional variables, x and y.

The decomposition of weighted graphs described in the present section admits a similar extension, namely, we can consider functions satisfying

$$f(\Gamma) = f(\Gamma_A') + yf(\Gamma_A'')$$

for weighted graphs. Modulo the corresponding equivalence relation on graphs any weighted graph becomes equivalent to a linear combination of graphs y_n and their disjoint unions with coefficients in $\mathbf{Z}[y]$.

Note that preserving the factor x in the first term would lead to a non-unique decomposition. The simplest example is given by the triangle with vertices of weights $1, 1, 2$.

Theorem 2.3 *Any weighted graph admits the unique decomposition with respect to the extended Tutte relation.*

The proof is also achieved by constructing the universal invariant similar to that of Sec. 2.3. The only difference is in the definition of the weighted Tutte monomial. We set $wt(\Gamma) = (-1)^k s_w y^{E(\Gamma)-k}$ for a connected weighted graph Γ of weight w with $E(\Gamma)$ edges. Here k is the first Betti number of the graph G. This universal invariant takes values in the ring $\mathbf{Z}[y; s_0, s_1, \ldots]$ and is obviously more subtle than that of Sec. 2.3.

3 Tutte decomposition for symmetric matrices

3.1 Symmetric matrices and *ve*-weighted graphs

In this section we will consider symmetric square matrices $g = (g_{ij}), i, j = 1, \ldots, m; g_{ij} = g_{ji}$ with non negative integer entries. The dimension m of a matrix may take arbitrary values $m = 1, 2, 3, \ldots$. Matrices are considered up to a common permutation of columns and rows.

Tutte decomposition for such matrices has been introduced by A. Gabrielov [2, 3] in connection with the theory of Abelian avalanches and sandpiles. We construct the universal Tutte invariant for these matrices and prove thus the uniqueness of the Tutte decomposition.

Remark. The constructions below can be easily generalized to the case of non symmetric matrices as well. We do not present such a generalization for the sake of clarity.

A symmetric matrix can be presented as a graph with weighted vertices and edges (a *ve-weighted graph*), diagonal elements corresponding to the weight of vertices. If $g_{ij} = 0, i \neq j$, there is no edge between vertices i and j. Otherwise the corresponding edge has the weight $2g_{ij}$ (or there are g_{ij} edges of weight 2 each between vertices i and j). We will use further both the language of matrices and that of graphs.

Definition 3.1 [2] The *deletion* of the edge ij provides the $m \times m$ matrix $g'(ij)$ given by the following formulas:

$$g'(ij)_{ii} := g_{ii} + g_{ij}; g'(ij)_{jj} := g_{jj} + g_{ji}$$

$$g'(ij)_{ij} := 0; g'(ij)_{ji} := 0$$

with all other elements not changed.

The *contraction* of the edge ij provides the $(m-1) \times (m-1)$ matrix $g''(ij)$ given by the following formulas:

$$g''(ij)_{ii} := g_{ii} + g_{jj} + g_{ij} + g_{ji}$$

$$g''(ij)_{ik} := g_{ik} + g_{jk}; g''(ij)_{ki} := g_{ki} + g_{kj}$$

for $k \neq i, j$ with j-th row and column removed and all other elements not changed.

A function f on symmetric matrices *satisfies the Tutte relation* if

$$f(g) = f(g'(ij)) + g_{ij} f(g''(ij))$$

for any matrix g and any edge ij in it.

Consider the free Abelian group S generated by all symmetric matrices as free generators. The subgroup $ST \subset S$ is generated by all expressions of the form $g - g'(ij) - g_{ij} g''(ij)$. Both deletion and contraction decrease the number of nonzero nondiagonal elements in the matrix ("number of edges"). It means that any symmetric matrix (or any element of S) is equivalent modulo ST to a linear combination of diagonal matrices. In fact, such linear combination is unique. We call it the *Tutte decomposition of the symmetric matrix.*

3.2 Weight invariancy of contraction and deletion

We may set the weight $w(g)$ of a symmetric matrix g to be equal to the sum of all elements of g. The weight $w(i)$ of the i-th row is equal to the sum of elements in the row. Both contraction and deletion obviously preserve the weight of the matrix.

Denote by $S_n \subset S$ the subgroup generated by all matrices of weight n; the subgroup $ST_n \subset S_n$ is generated by all Tutte expressions for matrices of weight n.

Similarly to the weighted case we have

Statement 3.1 *The groups S and ST are represented as direct sums of the corresponding weight homogeneous subgroups:*

$$S = S_0 \oplus S_1 \oplus S_2 \oplus \dots$$

$$ST = ST_0 \oplus ST_1 \oplus ST_2 \oplus \dots$$

3.3 Constructing the universal invariant for symmetric matrices

Denote by $g \sqcup h$ the direct sum of symmetric matrices g and h. A function f on symmetric matrices is *multiplicative* if $f(g \sqcup h) = f(g)f(h)$ for any two matrices g and h.

Definition 3.2 A function f on symmetric matrices is called a *ve-weighted Tutte invariant* if it is multiplicative and satisfies the Tutte relation for any matrix g and any edge ij in it.

We are going to construct the universal ve-weighted Tutte invariant veT with values in the ring $Z[s_0, s_1, s_2, \ldots]$. Let g be a symmetric matrix and d be an arbitrary partition of the set of rows of g into a disjoint union of subsets, $d = d_1 \sqcup d_2 \sqcup \ldots \sqcup d_k$. We set $vet_d(g) = c_{d_1} c_{d_2} \ldots c_{d_k} s_{w(d_1)} s_{w(d_2)} \ldots s_{w(d_k)}$. Here c_{d_i} is an integer constant. For the subset d_i this constant is defined as follows. Associate with d_i the square matrix obtained by intersecting the rows from d_i with the corresponding columns. Consider the set of all elements of the matrix lying above the diagonal. We set c_{d_i} to be equal to the symmetric function of degree $\#(d_i) - 1$ in these elements equal to the sum of all products of different elements in number $|d_i| - 1$. Here $|d_i|$ denotes the number of rows in d_i.

For example, if d is the set of all rows of the square matrix

$$\begin{pmatrix} a_1 & c_1 & c_2 \\ c_1 & a_2 & c_3 \\ c_2 & c_3 & a_3 \end{pmatrix},$$

then $vet_d = (c_1c_2 + c_2c_3 + c_3c_1)s_{a_1+a_2+a_3+2c_1+2c_2+2c_3}$.

Definition 3.3 We call the monomial vet_d the *universal ve-weighted Tutte monomial* corresponding to the partion d. The function

$$veT(g) = \sum_d vet_d$$

is called the *universal ve-weighted Tutte invariant.*

Theorem 3.2 *1. The universal ve-weighted Tutte invariant is a ve-weighted Tutte invariant.*

2. *Let K be an associative commutative ring with unity. Substituting arbitrary elements of K for each element $s_0, s_1, s_2, s_3, \dots$ in the function veT one obtains a ve-weighted Tutte invariant with values in K.*

3. *Any ve-weighted Tutte invariant with values in K can be obtained in this way.*

Proof.

The universal invariant constructed above is obviously multiplicative. We need to prove that it satisfies the Tutte relation, i.e. that

$$veT(g) = veT(g^{'}(ij)) + g_{ij}veT(g^{''}(ij))$$

for any symmetric matrix g and any edge ij in it.

All partitions d of the set of rows of the matrix g can be separated into two classes: those for which rows i and j belong to one an the same set of the partition; and those for which rows i and j belong to different sets.

Let a partition d belong to the first class. Then it corresponds to a partition $d^{'}$ of the set of rows of the matrix $g^{'}$, and one obviously has $vet_d = vet_{d^{'}}$.

In the case the partition d belongs to the second class it corresponds to a pair $d^{'}, d^{''}$ of the partitions of the sets of rows of matrices $g^{'}$ and $g^{''}$ correspondingly. Let $d_k, d^{'}_k, d^{''}_k$ be the sets of the corresponding partitions containing both rows i and j. Then the constants $c_{d_k}, c_{d^{'}_k}, c_{d^{''}_k}$ in the definition of the ve-weighted Tutte monomial satisfy the equality

$$c_{d_k} = c_{d^{'}_k} + g_{ij}c_{d^{''}_k},$$

what ends the proof.

The second and the third statements of the theorem follow from the fact that any symmetric matrix is equivalent modulo the subgroup ST to a linear combination of diagonal matrices.

Corollary 3.1 *Any symmetric matrix admits the unique Tutte decomposition.*

I am obliged to Yu. Volvovskii for the following remark.

Remark. Theorem 2.2 for weighted graphs follows from the proof of Theorem 3.2 for ve-weighted graphs by the following trick. We can consider a weighted graph as a *ve-weighted graph with edges of weight zero.* Such a graph can be described by a symmetric matrix with two different kind of zeroes: those coming from existing edges, and those coming from non-existing ones. After that the summing in the definition of the universal *ve*-weighted invariant is taken not over *all* partitions d of the set of rows, but over all partions d such that *each component of the underlying graph is connected.*

4 The Hopf algebra structures

In his original paper [7] Tutte introduced a ring of graphs. In the terms of the present paper this ring is simply the group G equipped with the multiplication induced by the disjoint union of graphs. This ring is commutative and the empty graph serves the unity for it. Theorem 1.1 shows that the quotient ring G/GT is isomorphic to the ring of polynomials in an infinite set of variables.

Similar statements are valid for the other two situations. It means, in particular that all the rings $G/GT, W/WT, S/ST$ are mutually isomorphic. Unfortunately, the isomorphism between G/GT and the other two rings does not seem to be a natural one, since no natural weight can be defined in G/GT.

Besides multiplication all the groups $G/GT, W/WT, S/ST$ can be endowed with a comultiplication operation making them not just rings, but bialgebras (Hopf algebras, in fact, see definitions in [5]). We describe the comultiplication for the ring W/WT, for other rings it is defined similarly. Our description follows that of [1].

The comultiplication is a linear mapping

$$\mu : W/WT \to W/WT \otimes W/WT$$

defined on a weighted graph Γ as

$$\mu : \Gamma \mapsto \sum_{V(\Gamma)=V(\Gamma_1)\sqcup V(\Gamma_2)} \Gamma_1 \otimes \Gamma_2,$$

where the sum is taken over all partitions of the set of vertices $V(\Gamma)$ into a disjoint union of sets $V(\Gamma_1), V(\Gamma_2)$, the graphs Γ_1 and Γ_2 being complete subgraphs of Γ with the corresponding sets of vertices. Proving that G becomes a Hopf algebra is a technical exercise.

The ring $Z[s_0, s_1, s_2, \ldots]$ has a natural structure of a Hopf algebra as well. The comultiplication is defined on a monomial $s_{k_1} s_{k_2} \ldots s_{k_m}, k_1 \leq k_2 \leq \ldots \leq k_m$ as follows:

$$\begin{aligned}\mu(s_{k_1} s_{k_2} \ldots s_{k_m}) &= 1 \otimes s_{k_1} s_{k_2} \ldots s_{k_m} + s_{k_1} \otimes s_{k_2} \ldots s_{k_m} \\ &+ s_{k_2} \otimes s_{k_1} s_{k_3} \ldots s_{k_m} + \ldots \\ &+ s_{k_1} s_{k_2} \otimes s_{k_3} \ldots s_{k_m} + \ldots + s_{k_1} s_{k_2} \ldots s_{k_m} \otimes 1.\end{aligned}$$

The homomorphism given by the universal invariant is, in fact, an isomorphism of Hopf algebras. Thus we have.

Theorem 4.1 *The groups $G/GT, W/WT, S/ST$ carry a (commutative and cocommutative) Hopf algebra structure making them isomorphic to the Hopf algebra of polynomials in an infinite set of variables, one variable of each order.*

5 Graphs and weighted graphs, a unified approach

Let us unify graphs and weighted graphs by considering graphs (loops and multiple edges allowed) with weighted vertices. The deletion of a link is defined as usual, and the contraction of a link is the combination of two contractions: the topology of the underlying graph is defined as in the case of ordinary graphs, while the weight of the resulting vertex becomes the sum of the weights of two link ends.

Denote the Abelian group freely generated by these objects by U. The definition of the Tutte relation and the Tutte decomposition in U is standard. A multiplicative function satisfying the Tutte relation will be called a *unified Tutte invariant.* The result of a Tutte decomposition is a linear combination of the graphs $x_{k,m}$ having one vertex of weight m and k loops and their disjoint unions.

We are going to show that the Tutte decomposition in this case is unique.

The universal invariant looks like

$$uT(\Gamma) = \sum_{\gamma} ut(\gamma),$$

where the sum in the right hand side is taken over all spanning subgraphs γ of (the underlying graph of) Γ, and $ut(\gamma) = \prod_{\gamma'}(-1)^{b_1(\gamma')}\rho_{b_1(\gamma')}\sigma_{w(\gamma')}$. This invariant takes its values in the ring $\mathbf{Z}[\rho_0, \rho_1, \ldots, \sigma_1, \sigma_2, \ldots]$. The universal invariant for ordinary graphs can be obtained from it by setting $\rho_i = (-1)^i r_i, i = 0, 1, 2, \ldots$; $\sigma_j = 1, j = 1, 2, 3, \ldots$. Similarly, the universal invariant for weighted graphs is the result of specializing $\rho_i = 1, i = 0, 1, 2, \ldots$; $\sigma_j = s_j, j = 1, 2, 3, \ldots$.

The same methods as above lead to the following theorem.

Theorem 5.1 *1. The function uT is a unified Tutte invariant.*

2. Let K be an associative commutative ring with unity. Substituting arbitrary elements of K for each element $\rho_0, \rho_1, \ldots, \sigma_1, \sigma_2, \ldots$ in the function uT one obtains a unified Tutte invariant with values in K.

3. Any unified Tutte invariant with values in K can be obtained in this way.

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