Theory of Submanifolds, Associativity Equations in 2D Topological Quantum Field Theories, and Frobenius Manifolds[1]

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To the memory of my wonderful mother
Maya Nikolayevna Mokhova (04.05.1926 – 12.09.2006)

Abstract

We prove that the associativity equations of two-dimensional topological quantum field theories are very natural reductions of the fundamental nonlinear equations of the theory of submanifolds in pseudo-Euclidean spaces and give a natural class of *potential* flat torsionless submanifolds. We show that all potential flat torsionless submanifolds in pseudo-Euclidean spaces bear natural structures of Frobenius algebras on their tangent spaces. These Frobenius structures are generated by the corresponding flat first fundamental form and the set of the second fundamental forms of the submanifolds (in fact, the structural constants are given by the set of the Weingarten operators of the submanifolds). We prove in this paper that each N-dimensional Frobenius manifold can locally be represented as a potential flat torsionless submanifold in a $2N$-dimensional pseudo-Euclidean space. By our construction this submanifold is uniquely determined up to motions. Moreover, in this paper we consider a nonlinear system, which is a natural generalization of the associativity equations, namely, the system describing all flat torsionless submanifolds in pseudo-Euclidean spaces, and prove that this system is integrable by the inverse scattering method.

[1]This research was supported by the Max-Planck-Institut für Mathematik (Bonn, Germany), the Russian Science Support Foundation, the Russian Foundation for Basic Research (Grant No. 05-01-00170), and the Program of Support for Leading Scientific Schools (Grant No. NSh-4182.2006.1).

Key words and phrases: Frobenius manifold, submanifold in a pseudo-Euclidean space, flat submanifold, submanifold with flat normal bundle, flat submanifold with zero torsion, associativity equations in two-dimensional topological quantum field theories, integrable system.

AMS 2000 Mathematics Subject Classification: 53B25, 53D45, 35Q58, 81T45.

1 Introduction. Associativity equations and Frobenius structures

In this paper we prove that the associativity equations of two-dimensional topological quantum field theories (the Witten–Dijkgraaf–Verlinde–Verlinde and Dubrovin equations, see [1]) for a function (a *potential* or *prepotential*) $\Phi = \Phi(u^1, \ldots, u^N)$,

$$\sum_{k=1}^{N} \sum_{l=1}^{N} \frac{\partial^3 \Phi}{\partial u^i \partial u^j \partial u^k} \eta^{kl} \frac{\partial^3 \Phi}{\partial u^l \partial u^m \partial u^n} = \sum_{k=1}^{N} \sum_{l=1}^{N} \frac{\partial^3 \Phi}{\partial u^i \partial u^m \partial u^k} \eta^{kl} \frac{\partial^3 \Phi}{\partial u^l \partial u^j \partial u^n}, \tag{1.1}$$

where η^{ij} is an arbitrary constant nondegenerate symmetric matrix, $\eta^{ij} = \eta^{ji}$, $\eta^{ij} =$ const, $\det(\eta^{ij}) \neq 0$, are very natural reductions of the fundamental nonlinear equations of the theory of submanifolds in pseudo-Euclidean spaces and give a natural class of *potential* flat torsionless submanifolds. All potential flat torsionless submanifolds in pseudo-Euclidean spaces bear natural structures of Frobenius algebras on their tangent spaces. These Frobenius structures are generated by the corresponding flat first fundamental form and the set of the second fundamental forms of the submanifolds (in fact, the structural constants are given by the set of the Weingarten operators of the submanifolds). We recall that each solution $\Phi(u^1, \ldots, u^N)$ of the associativity equations (1.1) gives N-parametric deformations of Frobenius algebras, i.e., commutative associative algebras equipped by nondegenerate invariant symmetric bilinear forms. Indeed, consider algebras $A(u)$ in an N-dimensional vector space with basis $e_1, \ldots, e_N$ and multiplication (see [1])

$$e_i \circ e_j = c^k_{ij}(u) e_k, \qquad c^k_{ij}(u) = \eta^{ks} \frac{\partial^3 \Phi}{\partial u^s \partial u^i \partial u^j} e_k. \tag{1.2}$$

For all values of the parameters $u = (u^1, \ldots, u^N)$ the algebras $A(u)$ are commutative, $e_i \circ e_j = e_j \circ e_i$, and the associativity condition

$$(e_i \circ e_j) \circ e_k = e_i \circ (e_j \circ e_k) \tag{1.3}$$

in the algebras $A(u)$ is equivalent to equations (1.1). The matrix η_{ij} inverse to the matrix η^{ij}, $\eta^{is}\eta_{sj} = \delta^i_j$, defines a nondegenerate invariant symmetric bilinear form on the algebras $A(u)$,

$$\langle e_i, e_j \rangle = \eta_{ij}, \qquad \langle e_i \circ e_j, e_k \rangle = \langle e_i, e_j \circ e_k \rangle. \tag{1.4}$$

Recall that locally the tangent space at every point of any Frobenius manifold (see [1]) bears the structure of Frobenius algebra (1.2)–(1.4), which is determined by a solution of the associativity equations (1.1) and smoothly depends on the point. Besides, one should also impose additional conditions on Frobenius manifolds, but we do not consider these conditions here. We prove in this paper that each N-dimensional Frobenius manifold can locally be represented as a potential flat torsionless submanifold in a $2N$-dimensional pseudo-Euclidean space. By our construction this submanifold is uniquely determined up to motions. Moreover, in this paper we consider a nonlinear system, which is a natural generalization of the associativity equations (1.1), namely, the system describing all flat torsionless submanifolds in pseudo-Euclidean spaces, and prove that this system is integrable by the inverse scattering method. The connection of the construction with integrable hierarchies, nonlocal Hamiltonian operators of hydrodynamic type with flat metrics, Poisson pencils and recursion operators can be found in [2]. The results of applications of this construction to the theory of Frobenius manifolds will be published in a separate paper.

2 Fundamental nonlinear equations of the theory of submanifolds in Euclidean spaces

Let us consider an arbitrary smooth N-dimensional submanifold M^N in an $(N+L)$-dimensional Euclidean space E^{N+L}, $M^N \subset E^{N+L}$, and introduce the standard classic notation. Let the submanifold M^N be given locally by a smooth vector-function $r(u^1, \ldots, u^N)$ of N independent variables $(u^1, \ldots, u^N)$ (some independent parameters on the submanifold), $r(u^1, \ldots, u^N) = (z^1(u^1, \ldots, u^N), \ldots, z^{N+L}(u^1, \ldots, u^N))$, where $(z^1, \ldots, z^{N+L})$ are coordinates in the Euclidean space E^{N+L}, $(z^1, \ldots, z^{N+L}) \in E^{N+L}$, $(u^1, \ldots, u^N)$ are local coordinates (parameters) on M^N, $\operatorname{rank}(\partial z^i / \partial u^j) = N$ (here $1 \le i \le N+L$, $1 \le j \le N$). Then $\partial r / \partial u^i = r_{u^i}$, $1 \le i \le N$, are tangent vectors at an arbitrary point $u = (u^1, \ldots, u^N)$ on M^N. Let $\mathbf{N}_u$ be the normal space of the submanifold M^N at an arbitrary point $u = (u^1, \ldots, u^N)$ on M^N, $N_u = \langle n_1, \ldots, n_L \rangle$, where n_α, $1 \le \alpha \le L$, is an orthonormalized basis of the normal space (orthonormalized normals), $(n_\alpha, r_{u^i}) = 0$, $1 \le \alpha \le L$, $1 \le i \le N$, $(n_\alpha, n_\beta) = 0$, $1 \le \alpha, \beta \le L$,

$\alpha \neq \beta$, and $(n_\alpha, n_\alpha) = 1$, $1 \leq \alpha \leq L$.

Then $\mathbf{I} = ds^2 = g_{ij}(u) du^i du^j$, $g_{ij}(u) = (r_{u^i}, r_{u^j})$, is the first fundamental form and $\mathbf{II}_\alpha = \omega_{\alpha,ij}(u) du^i du^j$, $\omega_{\alpha,ij}(u) = (n_\alpha, r_{u^i u^j})$, $1 \leq \alpha \leq L$, are the second fundamental forms of the submanifold M^N.

Since the set of vectors $(r_{u^1}(u), \ldots, r_{u^N}(u), n_1(u), \ldots, n_L(u))$ forms a basis in E^{N+L} at each point of the submanifold M^N, we can decompose each of the vectors $n_{\alpha,u^i}(u)$, $1 \leq \alpha \leq L$, $1 \leq i \leq N$, with respect to this basis, namely, $n_{\alpha,u^i}(u) = A^k_{\alpha,i}(u) r_{u^k}(u) + \varkappa_{\alpha\beta,i}(u) n_\beta(u)$, where $A^k_{\alpha,i}(u)$ and $\varkappa_{\alpha\beta,i}(u)$ are some coefficients depending on u (the *Weingarten decomposition*). It is easy to prove that $A^k_{\alpha,i}(u) = -\omega_{\alpha,ij}(u) g^{jk}(u)$, where $g^{jk}(u)$ is the contravariant metric inverse to the first fundamental form $g_{ij}(u)$, $g^{is}(u) g_{sj}(u) = \delta^i_j$. The coefficients $\varkappa_{\alpha\beta,i}(u)$ are said to be the *coefficients of torsion of the submanifold M^N*, $\varkappa_{\alpha\beta,i}(u) = (n_{\alpha,u^i}(u), n_\beta(u))$. It is also easy to prove that the coefficients $\varkappa_{\alpha\beta,i}(u)$ are skew-symmetric with respect to the indices α and β, $\varkappa_{\alpha\beta,i}(u) = -\varkappa_{\beta\alpha,i}(u)$, and form covariant tensors (1-forms) with respect to the index i on the submanifold M^N. The 1-forms $\varkappa_{\alpha\beta,i}(u) du^i$ are said to be the *torsion forms of the submanifold M^N*.

It is well known that for each submanifold M^N the forms $g_{ij}(u)$, $\omega_{\alpha,ij}(u)$ and $\varkappa_{\alpha\beta,i}(u)$ satisfy the Gauss equations, the Codazzi equations and the Ricci equations, which are the fundamental equations of the theory of submanifolds. In our case, the Gauss equations have the form

$$R_{ijkl}(u) = \sum_{\alpha=1}^{L} \left(\omega_{\alpha,jl}(u) \omega_{\alpha,ik}(u) - \omega_{\alpha,jk}(u) \omega_{\alpha,il}(u) \right), \tag{2.1}$$

where $R_{ijkl}(u)$ is the tensor of Riemannian curvature of the first fundamental form $g_{ij}(u)$, the Codazzi equations have the form

$$\nabla_k(\omega_{\alpha,ij}(u)) - \nabla_j(\omega_{\alpha,ik}(u)) = \varkappa_{\alpha\beta,k}(u) \omega_{\beta,ij}(u) - \varkappa_{\alpha\beta,j}(u) \omega_{\beta,ik}(u), \tag{2.2}$$

where ∇_k is the covariant differentiation generated by the Levi-Civita connection of the first fundamental form $g_{ij}(u)$, the Ricci equations have the form

$$\begin{aligned} \nabla_k(\varkappa_{\alpha\beta,i}(u)) - \nabla_i(\varkappa_{\alpha\beta,k}(u)) + \sum_{\gamma=1}^{L} \left(\varkappa_{\alpha\gamma,i}(u) \varkappa_{\gamma\beta,k}(u) - \varkappa_{\alpha\gamma,k}(u) \varkappa_{\gamma\beta,i}(u) \right) + \\ + \left(\omega_{\alpha,kl}(u) \omega_{\beta,ji}(u) - \omega_{\alpha,il}(u) \omega_{\beta,jk}(u) \right) g^{lj}(u) = 0. \end{aligned} \tag{2.3}$$

The Bonnet theorem. *Let K^N be an arbitrary smooth N-dimensional Riemannian manifold with a metric $g_{ij}(u) du^i du^j$. Let some 2-forms $\omega_{\alpha,ij}(u) du^i du^j$,*

$1 \leq \alpha \leq L$, *and some 1-forms* $\varkappa_{\alpha\beta,i}(u)$, $1 \leq \alpha, \beta \leq L$, *be given in a simply connected domain of the manifold* K^N. *If* $\omega_{\alpha,ij}(u) = \omega_{\alpha,ji}(u)$, $\varkappa_{\alpha\beta,i}(u) = -\varkappa_{\beta\alpha,i}(u)$, *and the Gauss equations* (2.1), *the Codazzi equations* (2.2), *and the Ricci equations* (2.3) *are satisfied for the forms* $g_{ij}(u)$, $\omega_{\alpha,ij}(u)$ *and* $\varkappa_{\alpha\beta,i}(u)$, *then there exists a unique (up to motions) smooth* N*-dimensional submanifold* M^N *in an* $(N+L)$*-dimensional Euclidean space* E^{N+L} *with the first fundamental form* $ds^2 = g_{ij}(u)du^i du^j$, *the second fundamental forms* $\omega_{\alpha,ij}(u)du^i du^j$ *and the torsion forms* $\varkappa_{\alpha\beta,i}(u)du^i$.

Similar fundamental equations and the Bonnet theorem are true for all *totally nonisotropic submanifolds in pseudo-Euclidean spaces* (we recall that if we have a submanifold in an arbitrary pseudo-Euclidean space E^m_n, then the metric induced on the submanifold from the ambient pseudo-Euclidean space E^m_n is nondegenerate if and only if this submanifold is totally nonisotropic, i.e., it is not tangent to isotropic cones of the ambient pseudo-Euclidean space E^m_n at its points).

3 Flat submanifolds with zero torsion in pseudo-Euclidean spaces

Let us consider totally nonisotropic smooth N-dimensional submanifolds with *zero torsion* in an $(N+L)$-dimensional pseudo-Euclidean space, i.e., the torsion forms of submanifolds of this class vanish, $\varkappa_{\alpha\beta,i}(u) = 0$. In the normal spaces N_u we will also use bases n_α, $1 \leq \alpha \leq L$, with arbitrary admissible Gram matrices $\mu_{\alpha\beta}$, $(n_\alpha, n_\beta) = \mu_{\alpha\beta}$, $\mu_{\alpha\beta} = \text{const}$, $\mu_{\alpha\beta} = \mu_{\beta\alpha}$, $\det \mu_{\alpha\beta} \neq 0$ (the signature of the metric $\mu_{\alpha\beta}$ is determined by the signature of the first fundamental form of the submanifold and the signature of the ambient pseudo-Euclidean space).

For torsionless N-dimensional submanifolds in an $(N+L)$-dimensional pseudo-Euclidean space we obtain the following system of fundamental equations, the Gauss equations

$$R_{ijkl}(u) = \sum_{\alpha=1}^{L} \sum_{\beta=1}^{L} \mu^{\alpha\beta}(\omega_{\alpha,ik}(u)\omega_{\beta,jl}(u) - \omega_{\alpha,il}(u)\omega_{\beta,jk}(u)), \tag{3.1}$$

where $\mu^{\alpha\beta}$ is the inverse to the matrix $\mu_{\alpha\beta}$, $\mu^{\alpha\gamma}\mu_{\gamma\beta} = \delta^\alpha_\beta$, the Codazzi equations

$$\nabla_k(\omega_{\alpha,ij}(u)) = \nabla_j(\omega_{\alpha,ik}(u)), \tag{3.2}$$

and the Ricci equations

$$g^{ij}(u)\,(\omega_{\alpha,ik}(u)\omega_{\beta,jl}(u) - \omega_{\alpha,il}(u)\omega_{\beta,jk}(u)) = 0. \tag{3.3}$$

Now let $g_{ij}(u)$ be a flat metric, i.e., we consider flat torsionless N-dimensional submanifolds M^N in an $(N+L)$-dimensional pseudo-Euclidean space. Then we can consider that $u = (u^1, \ldots, u^N)$ are certain flat coordinates of the metric $g_{ij}(u)$ on M^N. In flat coordinates the metric is a constant nondegenerate symmetric matrix η_{ij}, $\eta_{ij} = \eta_{ji}$, $\eta_{ij} = \text{const}$, $\det(\eta_{ij}) \neq 0$, and the Codazzi equations (3.2) have the form

$$\frac{\partial \omega_{\alpha,ij}}{\partial u^k} = \frac{\partial \omega_{\alpha,ik}}{\partial u^j}. \tag{3.4}$$

Thus there exist locally some functions $\chi_{\alpha,i}(u)$, $1 \leq \alpha \leq L$, $1 \leq i \leq N$, such that

$$\omega_{\alpha,ij}(u) = \frac{\partial \chi_{\alpha,i}}{\partial u^j}. \tag{3.5}$$

¿From symmetry of the second fundamental forms $\omega_{\alpha,ij}(u) = \omega_{\alpha,ji}(u)$ we have

$$\frac{\partial \chi_{\alpha,i}}{\partial u^j} = \frac{\partial \chi_{\alpha,j}}{\partial u^i}. \tag{3.6}$$

Therefore, there exist locally some functions $\psi_\alpha(u)$, $1 \leq \alpha \leq L$, such that

$$\chi_{\alpha,i}(u) = \frac{\partial \psi_\alpha}{\partial u^i}, \qquad \omega_{\alpha,ij}(u) = \frac{\partial^2 \psi_\alpha}{\partial u^i \partial u^j}. \tag{3.7}$$

Thus we have proved the following important lemma.

Lemma 3.1 *All the second fundamental forms of each flat torsionless submanifold in a pseudo-Euclidean space are Hessians in any flat coordinates in any simply connected domain on the submanifold.*

Moreover, in any flat coordinates the Gauss equations (3.1) have the form

$$\sum_{\alpha=1}^{L} \sum_{\beta=1}^{L} \mu^{\alpha\beta} \left(\frac{\partial^2 \psi_\alpha}{\partial u^i \partial u^k} \frac{\partial^2 \psi_\beta}{\partial u^j \partial u^l} - \frac{\partial^2 \psi_\alpha}{\partial u^i \partial u^l} \frac{\partial^2 \psi_\beta}{\partial u^j \partial u^k} \right) = 0 \tag{3.8}$$

and the Ricci equations (3.3) have the form

$$\sum_{i=1}^{N} \sum_{j=1}^{N} \eta^{ij} \left(\frac{\partial^2 \psi_\alpha}{\partial u^i \partial u^k} \frac{\partial^2 \psi_\beta}{\partial u^j \partial u^l} - \frac{\partial^2 \psi_\alpha}{\partial u^i \partial u^l} \frac{\partial^2 \psi_\beta}{\partial u^j \partial u^k} \right) = 0, \tag{3.9}$$

where η^{ij} is the inverse to the matrix η_{ij}, $\eta^{is}\eta_{sj} = \delta^i_j$.

Theorem 3.1 *The class of N-dimensional flat torsionless submanifolds in $(N+L)$-dimensional pseudo-Euclidean spaces is described (in flat coordinates) by the system of nonlinear equations* (3.8), (3.9) *for functions $\psi_\alpha(u)$, $1 \leq \alpha \leq L$. Here η^{ij} and $\mu^{\alpha\beta}$ are arbitrary constant nondegenerate symmetric matrices, $\eta^{ij} = \eta^{ji}$, $\eta^{ij} = \mathrm{const}$, $\det(\eta^{ij}) \neq 0$, $\mu^{\alpha\beta} = \mathrm{const}$, $\mu^{\alpha\beta} = \mu^{\beta\alpha}$, $\det \mu^{\alpha\beta} \neq 0$, the signature of the ambient $(N+L)$-dimensional pseudo-Euclidean space is the sum of the signatures of the metrics η^{ij} and $\mu^{\alpha\beta}$, $\mathbf{I} = ds^2 = \eta_{ij} du^i du^j$ is the first fundamental form, where η_{ij} is the inverse to the matrix η^{ij}, $\eta^{is}\eta_{sj} = \delta^i_j$, $\mathbf{II}_\alpha = (\partial^2 \psi_\alpha / (\partial u^i \partial u^j)) du^i du^j$, $1 \leq \alpha \leq L$, are the second fundamental forms given by the Hessians of the functions $\psi_\alpha(u)$, $1 \leq \alpha \leq L$.*

According to the Bonnet theorem any solution $\psi_\alpha(u)$, $1 \leq \alpha \leq L$, of the nonlinear system (3.8), (3.9) determines a unique (up to motions) N-dimensional flat torsionless submanifold of the corresponding $(N+L)$-dimensional pseudo-Euclidean space with the first fundamental form $\eta_{ij} du^i du^j$ and the second fundamental forms $\omega_\alpha(u) = (\partial^2 \psi_\alpha / (\partial u^i \partial u^j)) du^i du^j$, $1 \leq \alpha \leq L$, given by the Hessians of the functions $\psi_\alpha(u)$, $1 \leq \alpha \leq L$. It is obvious that we can always add arbitrary terms linear in the coordinates $(u^1, \ldots, u^N)$ to any solution of the system (3.8), (3.9), but the set of the second fundamental forms and the corresponding submanifold will be the same. Moreover, any two sets of the second fundamental forms of the shape $\omega_{\alpha,ij}(u) = \partial^2 \psi_\alpha / (\partial u^i \partial u^j)$, $1 \leq \alpha \leq L$, coincide if and only if the corresponding functions $\psi_\alpha(u)$, $1 \leq \alpha \leq L$, coincide up to terms linear in the coordinates, so we must not distinguish solutions of the nonlinear system (3.8), (3.9) up to terms linear in the coordinates $(u^1, \ldots, u^N)$.

Theorem 3.2 *The nonlinear system* (3.8), (3.9) *is integrable by the inverse scattering method.*

Consider the following linear problem for vector-functions $\partial a(u)/\partial u^i$ and $b_\alpha(u)$, $1 \leq \alpha \leq L$:

$$\frac{\partial^2 a}{\partial u^i \partial u^j} = \lambda \, \mu^{\alpha\beta} \omega_{\alpha,ij}(u) b_\beta(u), \qquad \frac{\partial b_\alpha}{\partial u^i} = \rho \, \eta^{kj} \omega_{\alpha,ij}(u) \frac{\partial a}{\partial u^k}, \tag{3.10}$$

where η^{ij}, $1 \leq i, j \leq N$, and $\mu^{\alpha\beta}$, $1 \leq \alpha, \beta \leq L$, are arbitrary constant nondegenerate symmetric matrices, $\eta^{ij} = \eta^{ji}$, $\eta^{ij} = \mathrm{const}$, $\det(\eta^{ij}) \neq 0$, $\mu^{\alpha\beta} = \mathrm{const}$, $\mu^{\alpha\beta} = \mu^{\beta\alpha}$, $\det \mu^{\alpha\beta} \neq 0$; it is obvious that here the coefficients $\omega_{\alpha,ij}(u)$, $1 \leq \alpha \leq L$, must be symmetric matrix functions, $\omega_{\alpha,ij}(u) = \omega_{\alpha,ji}(u)$; λ and ρ are arbitrary constants (parameters).

The consistency conditions for the linear system (3.10) are equivalent to the nonlinear system (3.8), (3.9) describing the class of N-dimensional flat torsionless submanifolds in $(N+L)$-dimensional pseudo-Euclidean spaces. Indeed, we have

$$\begin{aligned}
\frac{\partial^3 a}{\partial u^i \partial u^j \partial u^k} &= \lambda\, \mu^{\alpha\beta} \frac{\partial \omega_{\alpha,ij}}{\partial u^k} b_\beta(u) + \lambda\, \mu^{\alpha\beta} \omega_{\alpha,ij}(u) \frac{\partial b_\beta}{\partial u^k} = \\
&= \lambda\, \mu^{\alpha\beta} \frac{\partial \omega_{\alpha,ij}}{\partial u^k} b_\beta(u) + \lambda\, \mu^{\alpha\beta} \omega_{\alpha,ij}(u) \rho\, \eta^{ls} \omega_{\beta,ks}(u) \frac{\partial a}{\partial u^l} = \\
&= \lambda\, \mu^{\alpha\beta} \frac{\partial \omega_{\alpha,ik}}{\partial u^j} b_\beta(u) + \lambda\, \mu^{\alpha\beta} \omega_{\alpha,ik}(u) \rho\, \eta^{ls} \omega_{\beta,js}(u) \frac{\partial a}{\partial u^l},
\end{aligned} \tag{3.11}$$

whence we obtain

$$\frac{\partial \omega_{\alpha,ij}(u)}{\partial u^k} = \frac{\partial \omega_{\alpha,ik}(u)}{\partial u^j} \tag{3.12}$$

and

$$\mu^{\alpha\beta} \omega_{\alpha,ij}(u) \omega_{\beta,ks}(u) = \mu^{\alpha\beta} \omega_{\alpha,ik}(u) \omega_{\beta,js}(u). \tag{3.13}$$

Moreover,

$$\begin{aligned}
\frac{\partial^2 b_\alpha}{\partial u^i \partial u^l} &= \rho\, \eta^{kj} \frac{\partial \omega_{\alpha,ij}}{\partial u^l} \frac{\partial a}{\partial u^k} + \rho\, \eta^{kj} \omega_{\alpha,ij}(u) \frac{\partial^2 a}{\partial u^k \partial u^l} = \\
&= \rho\, \eta^{kj} \frac{\partial \omega_{\alpha,ij}}{\partial u^l} \frac{\partial a}{\partial u^k} + \rho\, \eta^{kj} \omega_{\alpha,ij}(u) \lambda\, \mu^{\gamma\beta} \omega_{\gamma,kl}(u) b_\beta(u) = \\
&= \rho\, \eta^{kj} \frac{\partial \omega_{\alpha,lj}}{\partial u^i} \frac{\partial a}{\partial u^k} + \rho\, \eta^{kj} \omega_{\alpha,lj}(u) \lambda\, \mu^{\gamma\beta} \omega_{\gamma,ki}(u) b_\beta(u),
\end{aligned} \tag{3.14}$$

whence we have

$$\frac{\partial \omega_{\alpha,ij}}{\partial u^l} = \frac{\partial \omega_{\alpha,lj}}{\partial u^i} \tag{3.15}$$

and

$$\eta^{kj} \omega_{\alpha,ij}(u) \omega_{\gamma,kl}(u) = \eta^{kj} \omega_{\alpha,lj}(u) \omega_{\gamma,ki}(u). \tag{3.16}$$

It follows from (3.12) and (3.15) that there exist locally some functions $\psi_\alpha(u)$, $1 \le \alpha \le L$, such that

$$\omega_{\alpha,ij}(u) = \frac{\partial^2 \psi_\alpha}{\partial u^i \partial u^j} \tag{3.17}$$

and then the relations (3.13) and (3.16) are equivalent to the nonlinear system (3.8), (3.9) for the functions $\psi_\alpha(u)$, $1 \le \alpha \le L$.

In arbitrary local coordinates, we obtain the following integrable description of all N-dimensional flat torsionless submanifolds in $(N+L)$-dimensional pseudo-Euclidean spaces.

Theorem 3.3 *For each N-dimensional flat torsionless submanifold in an $(N+L)$-dimensional pseudo-Euclidean space with a flat first fundamental form $g_{ij}(u)$ there locally exist functions $\psi_\alpha(u)$, $1 \le \alpha \le L$, such that the second fundamental forms have the form*

$$(w_\alpha)_{ij}(u) = \nabla_i \nabla_j \psi_\alpha, \tag{3.18}$$

where ∇_i is the covariant differentiation defined by the Levi-Civita connection generated by the metric $g_{ij}(u)$. The class of N-dimensional flat torsionless submanifolds in $(N+L)$-dimensional pseudo-Euclidean spaces is described by the following integrable system of nonlinear equations for functions $\psi_\alpha(u)$, $1 \le \alpha \le L$:

$$\sum_{n=1}^{N} \nabla^n \nabla_i \psi_\alpha \nabla_n \nabla_l \psi_\beta = \sum_{n=1}^{N} \nabla^n \nabla_i \psi_\beta \nabla_n \nabla_l \psi_\alpha, \tag{3.19}$$

$$\sum_{\alpha=1}^{L} \sum_{\beta=1}^{L} \mu^{\alpha\beta} \nabla_i \nabla_j \psi_\alpha \nabla_k \nabla_l \psi_\beta = \sum_{\alpha=1}^{L} \sum_{\beta=1}^{L} \mu^{\alpha\beta} \nabla_i \nabla_k \psi_\alpha \nabla_j \nabla_l \psi_\beta, \tag{3.20}$$

where ∇_i is the covariant differentiation defined by the Levi-Civita connection generated by a flat metric $g_{ij}(u)$, $\nabla^i = g^{is}(u)\nabla_s$, $g^{is}(u) g_{sj}(u) = \delta^i_j$. Moreover, in this case the systems of hydrodynamic type

$$u^i_{t_\alpha} = \left(\nabla^i \nabla_j \psi_\alpha\right) u^j_x, \qquad 1 \le \alpha \le L, \tag{3.21}$$

are commuting integrable bi-Hamiltonian systems of hydrodynamic type.

Now we will also find some natural and very important integrable reductions of the nonlinear system (3.8), (3.9).

4 Reduction to the associativity equations of two-dimensional topological quantum field theories and potential flat torsionless submanifolds in pseudo-Euclidean spaces

Theorem 4.1 *If we take $L = N$, $\mu^{ij} = c\eta^{ij}$, $1 \le i, j \le N$, c is an arbitrary nonzero constant, and $\psi_\alpha(u) = \partial \Phi / \partial u^\alpha$, $1 \le \alpha \le N$, where $\Phi = \Phi(u^1, \ldots, u^N)$, then the Gauss equations* (3.8) *coincide with the Ricci equations* (3.9) *and both of them coincide with* *the associativity equations of two-dimensional topological quantum field theories for the potential* $\Phi(u)$:

$$\sum_{i=1}^{N}\sum_{j=1}^{N}\eta^{ij}\left(\frac{\partial^3\Phi}{\partial u^i\partial u^m\partial u^k}\frac{\partial^3\Phi}{\partial u^j\partial u^n\partial u^l}-\frac{\partial^3\Phi}{\partial u^i\partial u^m\partial u^l}\frac{\partial^3\Phi}{\partial u^j\partial u^n\partial u^k}\right)=0, \qquad (4.1)$$

Theorem 4.2 *The associativity equations of two-dimensional topological quantum field theories describe a special class of N-dimensional flat submanifolds without torsion in $2N$-dimensional pseudo-Euclidean spaces* (*a class of* potential flat torsionless submanifolds)*.*

Definition 4.1 *A flat torsionless N-dimensional submanifold in a $2N$-dimensional pseudo-Euclidean space with a flat first fundamental form $g_{ij}(u)du^idu^j$ is called potential if locally there always exist a certain function $\Phi(u)$ in a neighborhood on the submanifold such that locally, in this neighborhood, the second fundamental forms of this submanifold have the form*

$$(\omega_i)_{jk}(u)du^jdu^k=\left(\nabla_i\nabla_j\nabla_k\Phi(u)\right)du^jdu^k, \qquad 1\le i\le N, \qquad (4.2)$$

where ∇_i is the covariant differentiation defined by the Levi-Civita connection generated by the flat metric $g_{ij}(u)$.

According to the Bonnet theorem any solution $\Phi(u)$ of the associativity equations (with the corresponding constant metric η_{ij}) determines a unique (up to motions) N-dimensional potential flat torsionless submanifold of the corresponding $2N$-dimensional pseudo-Euclidean space with the first fundamental form $\eta_{ij}du^idu^j$ and the second fundamental forms $\omega_n(u)=(\partial^3\Phi/(\partial u^n\partial u^i\partial u^j))du^idu^j$ given by the third derivatives of the potential $\Phi_\alpha(u)$. Here, we do not distinguish solutions of the associativity equations up to terms quadratic in the coordinates u.

Theorem 4.3 *On each potential flat torsionless submanifold in a pseudo-Euclidean space there is a structure of a Frobenius algebra given* (*in flat coordinates*) *by the flat first fundamental form η_{ij} and by the Weingarten operators $(A_s)^i_j(u)=-\eta^{ik}(\omega_s)_{kj}(u)$,*

$$\langle e_i,e_j\rangle=\eta_{ij}, \qquad e_i\circ e_j=c^k_{ij}(u)e_k, \qquad e_i=\frac{\partial}{\partial u^i},$$
$$c^k_{ij}(u^1,\dots,u^N)=-(A_i)^k_j(u)=\eta^{ks}(\omega_i)_{sj}(u^1,\dots,u^N). \qquad (4.3)$$

In arbitrary local coordinates, this Frobenius structure has the form

$$\begin{aligned}
&\langle e_i, e_j \rangle = g_{ij}, \qquad e_i \circ e_j = c^k_{ij}(u) e_k, \qquad e_i = \frac{\partial}{\partial u^i}, \\
&c^k_{ij}(u^1, \dots, u^N) = -(A_i)^k_j(u) = g^{ks}(u^1, \dots, u^N)(\omega_i)_{sj}(u^1, \dots, u^N),
\end{aligned} \tag{4.4}$$

where $g^{ij}(u)$ is the contravariant metric inverse to the first fundamental form $g_{ij}(u)$, $g^{is}(u)g_{sj}(u) = \delta^i_j$, $(\omega_k)_{ij}(u)du^i du^j$, $1 \le k \le N$, are the second fundamental forms.

Theorem 4.4 *Each N-dimensional Frobenius manifold can locally be represented as a potential flat torsionless N-dimensional submanifold in a $2N$-dimensional pseudo-Euclidean space. This submanifold is uniquely determined up to motions.*

References

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