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Indecomposable Homotopy Types with at most two non–trivial Homology Groups

Hans–Joachim Baues and Yuri Drozd

Abstract. We classify indecomposable spaces in the stable range with at most two non–trivial finitely generated homology groups.

It is a classical result of Brown–Copeland [**BC**] and Eckmann–Hilton [**EH**] that a 1–connected homotopy type X with at most two non trivial homology groups $H_mX = A$ and $H_nX = B, 2 \leq m < n$, is a mapping cone of a map

$$k_X : M(B, n-1) \to M(A, m) \tag{1}$$

Here $M(A, m)$ denotes the Moore space with homology A in degree m.

Let p be a prime and let $\mathbb{Z}_{(p)} \subset \mathbb{Q}$ be the smallest subring of $\mathbb{Q}$ containing $1/q$ for all primes $q \neq p$. We consider finitely generated $\mathbb{Z}_{(p)}$–modules A and B in the *stable range* $n < 2m-2$. Hence X is a p–local space with at most two non–trivial homology groups in a stable range. Then the homotopy type of X admits a decomposition as a one point union

$$X \simeq X_1 \vee \ldots \vee X_j \tag{2}$$

where all X_i with $1 \leq i \leq j$ are indecomposable and this decomposition is unique up to permutation. We classify in this paper the indecomposable summands in (2).

Theorem (A). *Let A and B be finitely generated free $\mathbb{Z}_{(p)}$–modules. Then the classification of indecomposable summands in (2) is*

$$\begin{cases} \textit{finite} & \textit{if } \pi_{n-1}(S^m) \otimes \mathbb{Z}_{(p)} \textit{ is cyclic,} \\ \textit{tame} & \textit{if } \pi_{n-1}(S^m) \otimes \mathbb{Z}_{(p)} \cong \mathbb{Z}/p \oplus \mathbb{Z}/p^k, k \geq 1, \textit{ and} \\ \textit{wild} & \textit{otherwise.} \end{cases}$$

Here $\pi_{n-1}(S^m) = \pi^s_{n-m-1}$ is given by stable homotopy group of spheres since we assume $n < 2m-2$. For example the wild case appears for $n-m-1 = 9$ since $\pi^s_9 = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$ and tame cases appear for $n - m - 1 = 8, 15$ since $\pi^s_8 = \mathbb{Z}/2 \oplus \mathbb{Z}/2$ and $\pi^s_{15} = \mathbb{Z}/2 \oplus \mathbb{Z}/480$. The proof of theorem (A) is based on the representation theory of matrices with entries in a fixed $\mathbb{Z}_{(p)}$–module M. For the finite and tame cases we describe the indecomposable summands explicitly.

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The space X in (2) is $(m-1)$–connected and we write $\mathsf{conn}(X) = m-1$ if $H_m X \neq 0$. Moreover if $H_n X \neq 0$ the *p–local dimension* of X is

$$\mathsf{dim}_{(p)}(X) = \begin{cases} n & \text{if } H_n X \text{ is a free } \mathbb{Z}_{(p)}\text{–module} \\ n+1 & \text{otherwise} \end{cases}$$

We say that X is trivial if X is a one point union of Moore spaces or equivalently if k_X in (1) is null homotopic.

Theorem (B). *Let A and B be finitely generated $\mathbb{Z}_{(p)}$–modules and let $\mathsf{dim}_{(p)}(X) - m \leq 4p-5$. Then the classification of indecomposable summands in* (2) *is*

$$\begin{cases} \textit{tame} & \textit{if } n-m = 2p-2, \\ \textit{essentially finite} & \textit{if } n-m = 2p-3, 2p-1, \textit{ and} \\ \textit{trivial} & \textit{otherwise.} \end{cases}$$

For odd primes p this result can be deduced from the more general result of Henn [**H**]. For the prime $p = 2$ and $\mathsf{dim}_{(p)}(X) - m \leq 2$ the result is due to J. H. C. Whitehead [**W1**] and Chang [**Ch**]. Therefore we give a proof only for the highly sophisticated case $p = 2$ and $\mathsf{dim}_{(p)}(X) - m = 3$. This case was also treated by Baues–Hennes [**BH**]; see also [**B1**] and [**B2**]. Our approach here using representation theory yields a new proof and confirms the intricate computation in [**BH**].

Theorem (C). *Let $p = 2$ and let A and B be finitely generated $\mathbb{Z}/2$–vector spaces and let $n = m+3 \geq 9$. Then the classification of indecomposable summands in* (2) *is wild.*

This result solves an old question of homotopy theory: Let $\mathbf{A}^k = \mathbf{A}^k_m$ be the homotopy category of $(m-1)$–connected $(m+k)$–dimensional finite CW–spaces with $m \geq k+1$. Since the spaces of theorem (C) are objects in $\mathbf{A}^4$ we get:

Corollary (D). *$\mathbf{A}^k$ has wild representation type for $k \geq 4$.*

It was shown in Baues–Drozd [**BD1**] that $\mathbf{A}^k$ has wild representation type for $k \geq 6$. On the other hand J. H. C. Whitehead [**W1**] and Chang [**Ch**] computed the indecomposable objects of $\mathbf{A}^2$; compare also the books of Hilton [**H1**], [**H2**]. Moreover Baues–Hennes computed all indecomposable objects of $\mathbf{A}^3$. Hence since $\mathbf{A}^k$ is wild for $k \geq 4$ the representation type of $\mathbf{A}^k$ is now known for all k. This answers a classification problem started by J. H. C. Whitehead 50 years ago.

In this paper a space is a CW–complex. Let $\mathbf{Top}^*/\simeq$ be the homotopy category of pointed spaces and pointed maps. For pointed spaces X, Y let $[X, Y]$ be the set of homotopy classes of pointed maps $X \to Y$; this is the set of morphism $X \to Y$ in the category $\mathbf{Top}^*/\simeq$.

1. The torsion free case

A space X is *decomposable* if there exists a homotopy equivalence $X \simeq A \vee B$ where $A \vee B$ is the one point union of non–contractible spaces A and B; otherwise X is *indecomposable*.

Definition 1.1. We say that X is a *p–local (m,n)–atom* if X is a 1–connected indecomposable space for which the homology groups $H_m(X)$ and $H_n(X)$ are non trivial finitely generated $\mathbb{Z}_{(p)}$–modules and $\widetilde{H}_i(X) = 0$ for $i \neq n, m$. We say that X is *torsion free* if the homology $H_m(X), H_n(X)$ are free $\mathbb{Z}_{(p)}$–modules.

The indecomposable summands of the space X in the introduction are either indecomposable Moore spaces or (m,n)–atoms as in (1.1).

For a prime p let $S^n_{(p)}$ be the p–local sphere. An element $\alpha \in \pi_{n-1}(S^m) \otimes \mathbb{Z}_{(p)}$ yields a map $\alpha : S^{n-1}_{(p)} \to S^m_{(p)}$ and the mapping cone of this map is denoted by

$$C_\alpha = S^m_{(p)} \cup_\alpha e^n_{(p)}. \tag{1.1}$$

For $n \in \mathbb{Z}$ with $1/n \in \mathbb{Z}_{(p)} \subset \mathbb{Q}$ there is a homotopy equivalence $C_\alpha \simeq C_{n\alpha}$.

Theorem 1.2. *If $\pi_{n-1}(S^m) \otimes \mathbb{Z}_{(p)} = \mathbb{Z}/p^k$ is a cyclic group with generator ξ then the spaces C_α with $\alpha = \xi, p\xi, \dots, p^{k-1}\xi$ form a complete list of torsion free p–local (n,m)–atoms.*

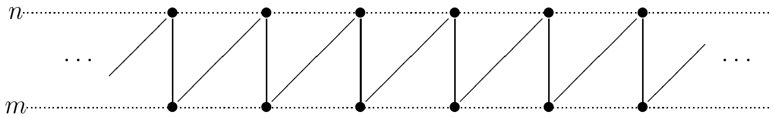
This result yields the finite case of theorem (A). The next result yields the tame case of theorem (A).

Theorem 1.3. *If $\pi_{n-1}(S^m) \otimes \mathbb{Z}_{(p)} = \mathbb{Z}/p \oplus \mathbb{Z}/p^k$ with generators $\eta \in \mathbb{Z}/p$ and $\xi \in \mathbb{Z}/p^k$ then a complete list of torsion free p–local (n,m)–atoms $A(g)$ and $A(g,\varphi)$ is given as follows.*

We consider a finite connected non empty subword g of the infinite word $(i \in \mathbb{Z})$

$$\dots \xi_i \eta \xi_{i+1} \eta \xi_{i+2} \eta \dots \tag{1}$$

where $\xi_i \in \{\xi, p\xi, \dots, p^{k-1}\xi\}$. Hence g corresponds to a connected subgraph of the infinite graph



(2)

in which the vertical edges denote η and the diagonal edges denote elements in $\{\xi, p\xi, \dots, p^{k-1}\xi\}$. According to the graph g we attach p–local cells $e^n_{(p)}$ to a one point union of p–local spheres $S^n_{(p)}$. Here each vertex of level n in the graph g is a cell and each vertex of level m in g is a sphere and the cell is attached according to the edges of the graph g. More precisely, let B be the set of vertices of level m of g and let T be the set of vertices of level n of g and consider the one pont unions of p–local spheres

$$S^{n-1}_T = \bigvee_{e \in T} S^{n-1}_e \qquad (T\text{=top cells})$$

$$S^m_B = \bigvee_{e \in B} S^m_e \qquad (B\text{=bottom spheres})$$

with $S^{n-1}_e = S^n_{(p)}, S^m_e = S^m_{(p)}$. Then the *atom* $A(g)$ is the mapping cone of a map

$$\alpha_g : S^{n-1}_T \to S^m_B \tag{3}$$

Here α_g is defined by the graph g, in particular, if $e \in T$ is the top vertex of $\xi_i\eta$ in g then the coordinate α^e_g is given by the sum $i_a\xi_i + i_b\eta : S^{n-1}_e \to S^m_a \vee S^m_b \subset S^m_B$ where a is the bottom vertex of ξ_i and b is the bottom vertex of η and i_a, i_b are the inclusions. Hence the space $A(g)$ is the p–local form of a "lightning flash space" defined in Baues–Drozd [**BD3**].

Next we define a *cyclic word* (g,φ) where g is a word as above of the form $g=\xi_1\eta\dots\xi_{c-1}\eta\xi_c$ with $c\geq 1$ and φ is an automorphism of a finite dimensional $\mathbb{Z}/p$–vector space $V=V(\varphi)$. Two cyclic words $(g,\varphi),(g',\varphi')$ are *equivalent* if g' is a cyclic permutation of g, that is $(1\leq i\leq c)$

$$g'=\xi_i\eta\xi_{i+1}\eta\dots\xi_c\eta\xi_1\eta\dots\xi_{i-1}\eta,$$

and there is an isomorphism $\psi:V(\varphi)\to V(\varphi')$ with $\varphi=\psi^{-1}\varphi'\psi$. A cyclic word (g,φ) is a *special cyclic word* if g is not of the form $g=g'g'\dots g'$ where the right hand side is a k–fold power of a word g' with $k>1$ and if φ is an indecomposable automorphism. Here we say that an automorphism of a homomorphism f between vector spaces is decomposable if f is a non–trivial direct sum of homomorphism.

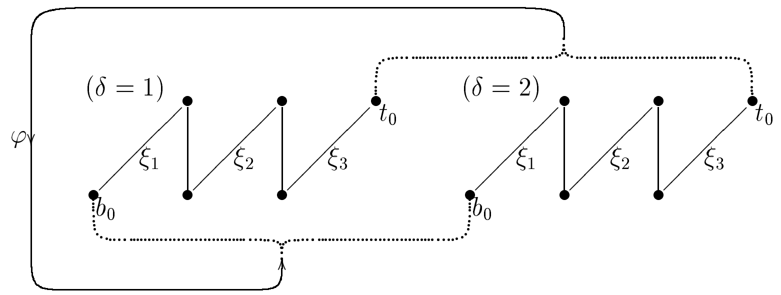
For a special cyclic word $(g=\xi_1\eta\dots\eta\xi_c,\varphi)$ we define the p–local (m,n)*–atom* $A(g,\varphi)$ by the mapping cone of a map

$$\alpha_{g,\varphi}:\bigvee^d S^{n-1}_T\to\bigvee^d S^m_B \tag{4}$$

Here $d=\mathsf{dim}_{\mathbb{Z}/p}(V(\varphi))$ and $\bigvee^d Y$ denotes the d–fold connected sum of the space Y with inclusion $j_\delta:S\subset\bigvee^d S$ for $\delta=1,\dots,d$. Let $b_0\in B$ be the bottom vertex of ξ_1 and $t_0\in T$ be the top vertex of ξ_c. Then the map $\alpha_{g,\varphi}$ on $j_\delta S^n_e$ with $e\in T-\{t_0\}$ is defined as α_g above compatible with the inclusion j_δ. The map $\alpha_{g,\varphi}$ restricted to $\bigvee^d S^{n-1}_{t_0}$, however, is a sum of the map $\bigvee^d_{\delta=1}j_\delta\xi_c$ and of the map

$$\bigvee^d S^{n-1}_{t_0}\xrightarrow{\varphi}\bigvee^d S^m_{b_0}$$

given by η and φ. We sketch the map $\alpha_{g,\varphi}$ for $d=2$ and $g=\xi_1\eta\xi_2\eta\xi_3$ as follows



Now the atoms $A(g,\varphi)$ and $A(g',\varphi')$ given by special cyclic words are homotopy equivalent if and only if (g,φ) and (g',φ') are equivalent.

2. Proof of theorem (A)

For $1<m<n$ and $R\subset\mathbb{Q}$ let

$$\mathbf{CW}(m,n)_R\subset\mathbf{Top}^*/\simeq \tag{2.1}$$

be the full subcategory consisting of 1–connected spaces X with at most two non trivial homology groups $H_mX=A$ and $H_nX=B$ which are finitely generated R–modules. Hence A and B are direct sums of the cyclic R–modules R and $\mathbb{Z}/p^k,k\geq$

$1, \frac{1}{p} \notin R$. We associate with X an element

$$k_X \in [M(B, n-1), M(A, m)] \tag{2.2}$$

obtained by the homology decomposition of X. Here $M(A, m)$ is the Moore space of A in degree m and X is homotopy equivalent to the mapping cone of a map representing k_X. Now let $\mathbf{M}(m, n)_R$ be the following category. Objects are triple $k_X = (A, B, k_X)$ with k_X as in (2.2) and morphisms $k_X \to k_Y$ are commutative diagrams

$$\begin{tikzcd} M(B, n-1) \arrow[r, "\beta"] \arrow[d, "k_X"'] & M(B', n-1) \arrow[d, "k_Y"] \\ M(A, m) \arrow[r, "\alpha"] & M(A', m) \end{tikzcd} \tag{2.3}$$

in $\mathbf{Top}^*/\simeq$. Using the homology decomposition of spaces we get the following result.

Lemma 2.1. *For $n \geq m+3$ there is a functor*

$$k : \mathbf{CW}(m, n)_R \to \mathbf{M}(m, n)_R$$

which carries X to k_X in (2.2). *This functor reflects isomorphism, is full and representative.*

The lemma shows that k induces a bijection on equivalence classes of objects. For $n = m+2$ there is slightly more delicate lemma which we describe in (3.1) below. For $n < 2m-2$ the categories in (2.1) are additive categories with the direct sum defined by one point union of spaces and maps respectively. The functor k is additive since $k_{X \vee Y} = k_X \vee k_Y$.

Next we define for a finitely generated R–module M the following *category of matrices with entries* in M denoted by $\mathbf{Mat}_R(M)$. Objects are triple (A, B, k) where A and B are finitely generated free R–modules and $k \in \mathsf{Hom}(B, A \otimes M)$. Morphisms are pairs $\beta : B \to B', \alpha : A \to A'$ for which the diagram

$$\begin{tikzcd} B \arrow[r, "\beta"] \arrow[d, "k"'] & B' \arrow[d, "k'"] \\ A \otimes M \arrow[r, "\alpha \otimes 1"] & A' \otimes M \end{tikzcd} \tag{2.4}$$

commutes. Let $\mathbf{M}(m, n)_R^{free}$ be the full subcategory of $\mathbf{M}(m, n)_R$ consisting of objects (A, B, k_X) for which A and B are finitely generated free R–modules. Then the following result is an easy application of standard facts of homotopy theory.

Lemma 2.2. *Let $M = \pi_{n-1}(S^m) \otimes R$. Then for $n < 2m-2$ one has an isomorphism of additive categories*

$$\mathbf{M}(m, n)_R^{free} \cong \mathbf{Mat}_R(M)$$

Proof. The isomorphism carries k_X to the induced morphism

$$k : B = \pi_{n-1} M(B, n-1) \xrightarrow{(k_X)_*} \pi_{n-1} M(A, m) = A \otimes M$$

and carries the morphism (α, β) in (2.3) to $(\pi_{n-1}(\beta), \pi_m(\alpha))$. q.e.d.

Using (2.2), (2.1) we see that theorem (A) and (1.2) and (1.3) are consequences of the representation theory in the category $\mathbf{Mat}_R(M)$.

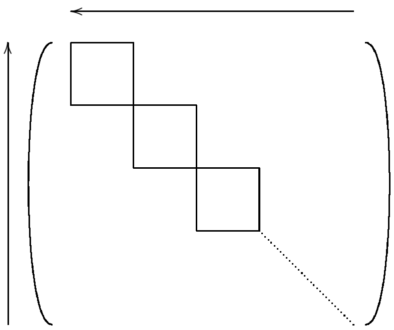
Proof of (1.3). Let $R = \mathbb{Z}_{(p)}$ be given by a prime p. Let $M = \mathbb{Z}/p \oplus \mathbb{Z}/p^k$ be generated by elements $\eta \in \mathbb{Z}/p$ and $\xi \in \mathbb{Z}/p^k$. Then an object k in $\mathbf{Mat}_R(M)$. is given by

$$k = K\eta + L\xi$$

where K, L are $a \times b$–matrices with entries in $\mathbb{Z}/p$ and $\mathbb{Z}/p^k$ respectively. We may suppose that L is of block–diagonal form

$$L = \begin{pmatrix} I_1 & & & 0 \\ & pI_2 & & \\ & & p^2 I_3 & \\ 0 & & & \ddots \end{pmatrix}$$

where the identity matrices $I_1, I_2, I_3, \ldots$ may have different sizes. Then for K one gets the known matrix problem



where the squares denote matrices which are transformed by conjugation and where the arrows indicate that we may add rows (columns) of the lower (right) stripes to those of the upper (left) ones. The answer of this problem is known by Bondarenko [**B**] or Drozd [**D**] and this yields the indecomposable objects described in (1.3). q.e.d.

We now recall some notation from representation theory. The *wild quiver* W consists of one vertex v and two arrows $a, b : v \to v$. Let $\mathbf{Vec}$ be the category of finite dimensional vector spaces over a field k and let $\mathbf{Wild}$ be the category of representations of W in $\mathbf{Vec}$ (i. e. objects are functors $A : W \to \mathbf{Vec}$ and morphisms $A \to B$ are natural transformations). The direct sum of vector spaces yields a direct sum $A \oplus B$ in $\mathbf{Wild}$. The *universal problem of representation theory* is the computation of all objects in $\mathbf{Wild}$ which are indecomposable with respect to direct sum. We call a classification problem *wild* if it requires the solution of the universal problem of representation theory.

Proposition 2.3. *Let $R = \mathbb{Z}_{(p)}$ and let M be a finitely generated $\mathbb{Z}_{(p)}$–module which is not cyclic and not of the form $\mathbb{Z}/p \oplus \mathbb{Z}/p^l$. Then the classification of indecomposable objects in $\mathbf{Mat}_R(M)$ is a wild problem.*

This result proves by (2.2) and (2.1) the wild case of theorem (A).

Proof of (2.3). The proposition is well known if M has 3 cyclic summands. Hence one only has to prove wildness for $M = \mathbb{Z}/p^2 \oplus \mathbb{Z}/p^2$. Then an object in $\mathbf{Mat}(M)$ is a pair of $a \times b$–matrices (K, L) over $\mathbb{Z}/p^2$. We now consider special matrices K, L

of the following shape. We choose $r \times r$–matrices X, Y over $\mathbb{Z}/p$ and we choose $a = b = 7r$. Then let $K = I_a$ be the identity matrix and let $L = L(X,Y)$ be the matrix

$$L = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_r & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_r & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_r & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I_r \\ pI_r & 0 & 0 & pX & pY & 0 & 0 \\ 0 & 0 & 0 & pI_r & 0 & 0 & 0 \end{bmatrix}$$

Then a simple though tedious calculation shows that $(K = I_a, L)$ is isomorphic in $\mathbf{Mat}_R(M)$ to $(K = I_a, L')$ with $L' = L(X', Y')$ if and only if there is an invertible $r \times r$–matrix R with $RX = X'R$ and $RY = Y'R$. This shows that the classification of indecomposable objects in $\mathbf{Mat}_R(M)$ for $M = \mathbb{Z}/p^2 \oplus \mathbb{Z}/p^2$ is wild. q.e.d.

3. Proof of theorem (B)

We consider the following diagram of functors $(m > 4)$

(3.1)
$$\begin{tikzcd} \mathbf{CW}(m, m+2)_{\mathbb{Z}} \arrow[dr, "k"'] & & \mathbf{M}(m, m+2)_{\mathbb{Z}} \arrow[dl, "q"] \\ & \mathbf{M}(m, m+2)_{\mathbb{Z}}/\sim & \end{tikzcd}$$

Here $\mathbf{CW}(m, m+2)_{\mathbb{Z}}$ and $\mathbf{M}(m, m+2)_{\mathbb{Z}}$ are defined as in (2.2) and (2.3) for $R = \mathbb{Z}$. The functor k is defined as in (2.1) though this functor maps only to a quotient category of $\mathbf{M}(m, m+2)$. The functor q is the quotient functor for the following natural quivalence relation $\sim$ in $\mathbf{M}(m, m+2)_{\mathbb{Z}}$. Consider the diagram

$$\begin{tikzcd} M(B, m+1) \arrow[r, "\beta, \beta_1"] \arrow[d, "k_X"'] & M(B', m+1) \arrow[d, "k_Y"] \\ M(A, m) \arrow[r, "\alpha, \alpha_1"'] \arrow[ur, "\xi"'] & M(A', m) \end{tikzcd}$$

where $(\alpha, \beta), (\alpha_1, \beta_1)$ are morphisms $k_X \to k_Y$ in $\mathbf{M}(m, m+2)_{\mathbb{Z}}$. Then we set $(\alpha, \beta) \sim (\alpha_1, \beta_1)$ if there is

$$\xi \in [M(A, m), M(B', m+1)] = \mathsf{Ext}(A, B')$$

with $\beta_1 = \beta$ and $\alpha_1 = \alpha + k_Y \xi$. Since $m > 4$ we know that all categories and functors in (3.1) are additive.

Proposition 3.1. *The functor k in* (3.1) *is well defined. Moreover k reflects isomorphisms, is full and representative.*

This is a special case of the classification result 6.8.2 in Baues [**B1**]. Since also the quotient functor q reflects isomorphisms, is full and representative we see that the indecomposable objects in $\mathbf{CW}(m, m+2)$ coincide with the mapping cones of indecomposable objects in $\mathbf{M}(m, m+2)_{\mathbb{Z}}$.

In order to obtain an algebraic description of the category $\mathbf{M}(m, m+2)_{\mathbb{Z}}$ we recall first some general notation on bimodule problems. Let $\mathcal{A}$, $\mathcal{B}$ be two categories (fully additive, i. e., such additive categories where every idempotent splits), U be an $\mathcal{A}$–$\mathcal{B}$–bimodule, i. e., a functor $\mathcal{A}^{op} \times \mathcal{B} \to \mathbf{Ab}$. Define the category $\mathbf{El}(U)$ of elements of this bimodule as follows:

$$\operatorname{Ob}\mathbf{El}(U) = \coprod_{\substack{A \in \operatorname{Ob}\mathcal{A} \\ B \in \operatorname{Ob}\mathcal{B}}} U(A,B) \tag{3.2}$$

A morphism $u \to v$, where $u \in U(A,B), v \in U(A',B')$, is a pair (α,β), where $\alpha : A' \to A, \beta : B \to B'$, such that $\beta u = v\alpha$ (Here we write $v\alpha$ for $U(\alpha,1)v$ and βu for $U(1,\beta)u$).

We have the following example of a category of elements in a bimodule. Let $\mathbf{M}^m$ be the full subcategory of $\mathbf{Top}^*/\simeq$ consisting of Moore spaces $M(A,m)$ of finitely generated abelian groups. Then the suspension $\Sigma : \mathbf{M}^m \to \mathbf{M}^{m+1}$ is an equivalence of categories for $m \geq 3$ and $\mathbf{M}^m$ is an additive category for $m \geq 3$. We have a bimodule

$$U : (\mathbf{M}^{m+1})^{op} \times \mathbf{M}^m \to \mathbf{Ab} \tag{3.3}$$

defined by the abelian group $[M(A,m+1), M(H,m)]$. Now the category $\mathbf{El}(U)$ coincides with $\mathbf{M}(m,m+2)_{\mathbb{Z}}$.

We describe $\mathbf{M}^m$ and the bifunctor U algebraically as follows. For an abelian group A, consider the short exact sequence

$$A \otimes \mathbb{Z}/2 \overset{\eta^A}{\rightarrowtail} G(A) \twoheadrightarrow A * \mathbb{Z}/2 \tag{1}$$

coresponding to the composition

$$A * \mathbb{Z}/2 \to A \to A \otimes \mathbb{Z}/2.$$

Applying $\mathsf{Hom}(\_,\mathbb{Z}/4)$, one gets the exact sequence

$$\begin{array}{ccccc} \mathsf{Hom}(A * \mathbb{Z}/2, \mathbb{Z}/4) & \rightarrowtail & \mathsf{Hom}(G(A), \mathbb{Z}/4) & \twoheadrightarrow & \mathsf{Hom}(A \otimes \mathbb{Z}/2, \mathbb{Z}/4) \\ \wr\vert & & & & \wr\vert \\ \mathsf{Ext}(A, H \otimes \mathbb{Z}/2) & & & & \mathsf{Hom}(A, \mathbb{Z}/2) \end{array} \tag{2}$$

Now, applying $\_ \otimes H$, one gets

$$\begin{array}{ccccccc} \mathsf{Ext}(A,\mathbb{Z}/2) \otimes H & \overset{\delta}{\to} & \mathsf{Hom}(G(A),\mathbb{Z}/4) \otimes H & \to & \mathsf{Hom}(A,\mathbb{Z}/2) \otimes H & \to & 0 \\ \wr\vert & & & & \wr\vert & & \\ \mathsf{Ext}(A, H \otimes \mathbb{Z}/2) & & & & \mathsf{Hom}(A, H \otimes \mathbb{Z}/2) & & \end{array} \tag{3}$$

(all groups are supposed to be finitely generated).

Now define $G(A,H)$ by the push–down diagram:

(4)

$$\begin{tikzcd} \mathsf{Ext}(A, H \otimes \mathbb{Z}/2) \arrow[r, "\Delta"] \arrow[d, "\eta^H_*"'] & \mathsf{Hom}(G(A), \mathbb{Z}/4) \otimes H \arrow[r, "\mu"] \arrow[d] & \mathsf{Hom}(A, H \otimes \mathbb{Z}/2) \arrow[r] \arrow[d, equal] & 0 \\ \mathsf{Ext}(A, G(H)) \arrow[r, "\Delta"'] & G(A,H) \arrow[r, "\mu"'] & \mathsf{Hom}(A, H \otimes \mathbb{Z}/2) \arrow[r] & 0 \end{tikzcd}$$

Define the category $\mathfrak{I}$ where objects are abelian groups (finitely generated) and morphisms are pairs $(\varphi, \widetilde{\varphi}), \varphi : A \to B, \widetilde{\varphi} : G(A) \to G(B)$ such that the following diagram commutes:

$$\begin{tikzcd} A\otimes\mathbb{Z}/2 \arrow[r] \arrow[d, "\varphi\otimes 1"'] & G(A) \arrow[r] \arrow[d, "\widetilde{\varphi}"'] & A*\mathbb{Z}/2 \arrow[d, "\varphi*1"] \\ B\otimes\mathbb{Z}/2 \arrow[r] & G(B) \arrow[r] & B*\mathbb{Z}/2 \end{tikzcd} \tag{5}$$

Then $G(A,H)$ can be consider as $\mathfrak{I}$–$\mathfrak{I}$–bimodule, the (right) action of $(\varphi,\widetilde{\varphi}) : A \to A'$ induced by

$$(\varphi,\widetilde{\varphi})^* = \mathsf{Ext}(\varphi, G(H)) \oplus \mathsf{Hom}(\widetilde{\varphi}, \mathbb{Z}/4)\otimes H$$

and the (left) action of $(\psi,\widetilde{\psi}) : H \to H'$ induced by

$$(\psi,\widetilde{\psi})_* = \mathsf{Ext}(A,\widetilde{\psi}) \oplus \mathsf{Hom}(G(A), \mathbb{Z}/4)\otimes \psi.$$

Hence, one can consider the category $\mathbf{El}(G)$ of the elements of this bimodule.

PROPOSITION 3.2. *For $m \geq 3$ there is an isomorphism of categories $\mathfrak{I} \cong \mathbf{M}^m$ and one has a natural isomorphism*

$$[M(B,m+1), M(H,m)] \cong G(B,H)$$

Hence the category $\mathbf{M}(m,m+2)$ is isomorphic to the category $\mathbf{El}(G)$.

This result is proved in (8.2.10) Baues [**B1**]. In order to obtain the indecomposable objects in $\mathbf{CW}(m,m+2)_{\mathbb{Z}}$ we thus have to classify the indecomposable objects in the category $\mathbf{El}(G)$.

Note that if $A = \bigoplus_j A_j$ and $H = \bigoplus_i H_i$, the elements of $G(A,H)$ can be naturally written as the matrices (g_{ij}) with $g_{ij} \in G(A_j, H_i)$. Hence, we only have to calculate $G(A,H)$ with indecomposable A, H and the action of the morphisms $A \to A'$ and $H \to H'$ for indecomposable $A, A'; H, H'$. Thus, we are interested in the cases $A = \mathbb{Z}/2^a, H = \mathbb{Z}/2^h$ (putting $\mathbb{Z}/2^\infty = \mathbb{Z}$). First note that, if $A = \mathbb{Z}^a$ with $a > 1$, the sequence (1) splits and $A\otimes\mathbb{Z}/2 = A*\mathbb{Z}/2 = \mathbb{Z}/2$. Moreover, given $\varphi : A' \to A$, the homomorphisms $\widetilde{\varphi} : G(A') \to G(A)$ such that $(\varphi,\widetilde{\varphi})$ is a morphism from $\mathfrak{I}$, are given by the homomorphisms $\varphi' : A*\mathbb{Z}/2 \to A\otimes\mathbb{Z}/2$; namely, w. r. t. the decomposition $G(A) = A\otimes\mathbb{Z}/2 \oplus A*\mathbb{Z}/2$, $\widetilde{\varphi}$ is given by the matrix $\begin{pmatrix} \varphi\otimes 1 & \varphi' \\ 0 & \varphi*1 \end{pmatrix}$. Certainly, in this case (2) and (3) are also split sequences, hence, the lower row of the push–down (4) also splits, i. e.

$$G(A,H) = \mathsf{Ext}(A, G(H)) \oplus \mathsf{Hom}(A, H\otimes\mathbb{Z}/2).$$

If $h > 1$, one also has that $\mathsf{Ext}(A,G(H)) = \mathsf{Ext}(A, H*\mathbb{Z}/2) \oplus \mathsf{Ext}(A, H\otimes\mathbb{Z}/2)$, so $G(A,H) = \mathsf{Ext}(A, H*\mathbb{Z}/2) \oplus \mathsf{Ext}(A, H\otimes\mathbb{Z}/2) \oplus \mathsf{Hom}(A, H\otimes\mathbb{Z}/2)$ and all three direct summands here are $\mathbb{Z}/2$. We denote them, respectively, by $G(*^a,_h *), G(*^a, \otimes_h)$ and $G(^a\otimes, \otimes_h)$. One easily sees that a homomorphism $(\varphi,\widetilde{\varphi})$, where $\widetilde{\varphi} = \begin{pmatrix} \varphi\otimes 1 & \varphi' \\ 0 & \varphi*1 \end{pmatrix}$ maps a triple (g_1, g_2, g_3) from this direct sum to $(\varphi^* g_1, \varphi^* g_2 + \hat{\varphi}' g_3, \varphi^* g_3)$, where $\hat{\varphi}'$ is the composition

$$\begin{aligned} \mathsf{Hom}(A, H\otimes\mathbb{Z}/2) \simeq \mathsf{Hom}(A\otimes\mathbb{Z}/2, H\otimes\mathbb{Z}/2) &\xrightarrow{(\varphi')^*} \mathsf{Hom}(A'*\mathbb{Z}/2, H\otimes\mathbb{Z}/2) \simeq \\ &\simeq \mathsf{Ext}(A'*\mathbb{Z}/2, H\otimes\mathbb{Z}/2) \simeq \mathsf{Ext}(A', H\otimes\mathbb{Z}/2). \end{aligned}$$

If $(\psi,\psi') : H' \to H$, then it maps (g_1, g_2, g_3) to $(\psi_* g_1, \psi_* g_2 + (\psi')_* g_1, \psi_* g_3) \in G(A, H')$. If $h = 1$, the sequence (1) for H is indeed

$$\mathbb{Z}/2 \rightarrowtail \mathbb{Z}/4 \twoheadrightarrow \mathbb{Z}/2,$$

hence, for $a > 1$, one still has the exact sequence

$$\mathsf{Ext}(A, H \otimes \mathbb{Z}/2) \rightarrowtail \mathsf{Ext}(A, G(H)) \twoheadrightarrow \mathsf{Ext}(A, H * \mathbb{Z}/2);$$

which is indeed

$$\mathbb{Z}/2 \rightarrowtail \mathbb{Z}/4 \twoheadrightarrow \mathbb{Z}/2.$$

Again a homomorphism $(\varphi, \widetilde{\varphi}) : A' \to A$ maps a pair (g_1, g_3) with $g_1 \in \mathsf{Ext}(A, G(H)), g_2 \in G({}^a\otimes, \otimes_1) = \mathsf{Hom}(A, H \otimes \mathbb{Z}/2)$ to $(\varphi^* g_1 + \hat{\varphi}' g_3, \varphi^* g_3)$. On the other hand, to define a homomorphism $(\psi, \widetilde{\psi}) : H \to H'$, where $H = \mathbb{Z}/2$, one only has to define $\widetilde{\psi} : \mathbb{Z}/4 \to H'$: then $\psi = \widetilde{\psi} \otimes \mathbb{Z}/2 : \mathbb{Z}/2 \to H'$. Such a homomorphism maps a pair (g_1, g_3) as above to $(\widetilde{\psi}_* g_1, \psi_* g_3)$.

Suppose now that $a = 1$. Then (1) is indeed

$$\mathbb{Z}/2 \rightarrowtail \mathbb{Z}/4 \twoheadrightarrow \mathbb{Z}/2.$$

If $h > 1$, the tensor multiplication by H does not change this exact sequence; η_*^H in (4) is split monomorphism, hence, $G(A, H) \simeq \mathbb{Z}/4 \oplus \mathsf{Ext}(A, H * \mathbb{Z}/2)$, and we have an exact sequence

$$\mathbb{Z}/2 = G(*^1, \otimes_h) \to \mathbb{Z}/4 \to G({}^1\otimes, \otimes_h) = \mathbb{Z}/2.$$

A morphism $(\varphi, \widetilde{\varphi})$ is given by any homomorphism $\widetilde{\varphi} : G(A') \to \mathbb{Z}/4$: then $\varphi = \widetilde{\varphi}|_{A' \otimes \mathbb{Z}/2} \in \mathsf{Hom}(A' \otimes \mathbb{Z}/2, \mathbb{Z}/2) \simeq \mathsf{Hom}(A', \mathbb{Z}/2)$. It maps a pair $(g_1, g_2) \in \mathsf{Hom}(G(A), \mathbb{Z}/4) \otimes H \oplus \mathsf{Ext}(A, H * \mathbb{Z}/2)$ to $((\widetilde{\varphi}^* \otimes 1) g_1, \varphi^* g_2)$. A morphism $(\psi, \widetilde{\psi}) : H \to H'$, where $\widetilde{\psi} = \begin{pmatrix} \psi\otimes 1 & \psi' \\ 0 & \psi * 1 \end{pmatrix}$, maps (g_1, g_2) to $((1 \otimes \psi)_* g_1 + \psi'_* g_2, (\psi * 1)_* g_2)$. At last, if $a = 1 = h$, the sequence (3) is

$$\mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2 \xrightarrow{1} \mathbb{Z}/2 \to 0 \text{ and } \eta_*^H = 0$$

Hence, $G(A, H) = \mathsf{Ext}(A, G(H)) \oplus \mathsf{Hom}(A, H \otimes \mathbb{Z}/2)$, both summand being $\mathbb{Z}/2$ (we denote them by $G({}_1*, *^1)$ and $G(\otimes_1, {}^1\otimes)$ respectively, putting $G({}_1*, {}^1\otimes) = 0$).

Denote by $\widetilde{G}$ the sub–bimodule of G such that, for $A = \mathbb{Z}/2^a$, $H = \mathbb{Z}/2^h$, $\widetilde{G}(A, H) = G({}_h*, {}^a\otimes)$. In the definition of the weak equivalence (6), one always has $\Delta(\mathrm{Im}\varphi^* \eta_*^H, \mu(\beta)_*) \subseteq \widetilde{G}(A, H)$. We will use this remark later to show that indeed the weak equivalence coincide with the isomorphism in $\mathbf{El}(G)$.

Now we are going to reformulate our bimodule problem in the matrix form.

This bimodule problem can obviously be rewritten in the matrix form as follws. We consider the "striped" matrices M with the horizontal stripes marked by the set $\mathcal{E} = \{\otimes_h, {}_h* | h \in \mathbb{N}\} \cup \{\otimes_\infty\}$ and the vertical stripes marked by the set $\mathcal{F} = \{{}^a\otimes, *^a | a \in \mathbb{N}\} \cup \{{}^\infty\otimes\}$. We denote by $M(x, y)$ the block placed at the intersection of the horizontal stripe x and the vertical stripe y. These matrices should satisfy the following conditions:

(1) The number of rows in the stripes $\otimes_h$ and ${}_h*$ is the same, as well as the number of columns in the stripes ${}^a\otimes$ and $*_a$.
(2) $M({}_h*, {}^a\otimes) = 0; M(\otimes_1, *^a) = 0; M(\otimes_h, *^1) = 0$. (We consider them as "matrices over the zero ring $\mathbb{Z}/1$").
(3) The matrices $M(\otimes_h, {}^1\otimes)$ with $h > 1$ and $M({}_1*, *^a)$ with $a > 1$ are with the entries from $\mathbb{Z}/4$, all other ones are with the entries from $\mathbb{Z}/2$.

Here is the picture describing such matrices (we always indicate the ring, where the entries of the blocks are from):

	${}^1\otimes$	${}^2\otimes$	${}^3\otimes$	$\dots$	${}^\infty\otimes$	$\dots$	$*^3$	$*^2$	$*^1$
$\otimes_1$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\dots$	$\mathbb{Z}/2$	$\dots$	0	0	0
$\otimes_2$	$\mathbb{Z}/4$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\dots$	$\mathbb{Z}/2$	$\dots$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0
$\otimes_1$	$\mathbb{Z}/4$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\dots$	$\mathbb{Z}/2$	$\dots$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0
$\vdots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
$\otimes_\infty$	$\mathbb{Z}/4$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\dots$	$\mathbb{Z}/2$	$\dots$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0
$\vdots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
${}_3*$	0	0	0	$\dots$	0	$\dots$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
${}_2*$	0	0	0	$\dots$	0	$\dots$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
${}_1*$	0	0	0	$\dots$	0	$\dots$	$\mathbb{Z}/4$	$\mathbb{Z}/4$	$\mathbb{Z}/2$

The following transformations of the matrix M are called "admissible transformations" (in what follows we denote by $M(x,\_)$ and by $M(\_,y)$, respectively, the horizontal stripe marked by x and the vertical stripe marked by y).

(a) Replacing the stripes $M(\otimes_h,\_)$ and $M({}_h*,\_)$ by $XM(\otimes_h)$ and $XM({}_h*)$.
(a') Replacing the stripes $M(\_,{}^a\otimes)$ and $M(\_,*^a)$ by $M(\_,{}^a\otimes)X$ and $M(\_,*^a)X$.
(b) Replacing $M(\otimes_h,\_)$ by $M(\otimes_h,\_)+XM(\otimes_{h'},\_)+YM({}_k*,\_)$, where $h'>h,k$ any.
(b') Replacing $M(\_,*^a)$ by $M(\_,*^a)+M(\_,*^{a'})X+M(\_,{}^b\otimes)$, where $a'>a,b$ any.
(c) Replacing $M({}_h*,\_)$ by $M({}_h*,\_)+XM({}_{h'}*,\_)$ and $M(\otimes_h,\_)Y$ by $M(\otimes_h,\_)+2^{h-h'}XM(\otimes_{h'},\_)$, where $h'<h$.
(c') Replacing $M(\_,{}^a\otimes)$ by $M(\_,{}^a\otimes)+M(\_,{}^{a'}\otimes)X$ and $M(\_,*^a)$ by $M(\_,*^a)+2^{a-a'}M(\_,*^{a'})X$, where $a'<a$.
(d) Replacing $M({}_1*,\_)$ by $M_1(*,\_)+2XM({}_h*,\_)+2YM(\otimes_k,\_)$ for arbitrary h,k.
(d') Replacing $M(\_,{}^1\otimes)$ by $M(\_,{}^1\otimes)+2M(\_,{}^a\otimes)X+2M(\_,*^b)Y$ for arbitrary a,b.
(e) Replacing $M(\otimes_h,{}^1\otimes), h>1$, by $M(\otimes_h,{}^1\otimes)+2XM({}_{h'}*,*^1)$, h' any.
(e') Replacing $M({}_1*,*^a), a>1$, by $M({}_1*,*^a)+2M(\otimes_1,{}^{a'}\otimes)X$, a' any.

Here X,Y always denote arbitrary matrices of the appropriate size with the entries from $\mathbb{Z}/4$; in the transformations of type (a) and (a') the matrix X should be invertible. Certainly, if an original block was with the entries from $\mathbb{Z}/2$, the resulting one should also be calculated modulo 2; if an original block was over zero ring, so is the resulting one.

Two matrices, M,M' (of the same size) are called *equivalent* if they can be transformed to each other by a sequence of admissible transformations. Call M,M' *equivalent modulo 2* if M can be transformed by a sequence of admissible transformations to a matrix M'' such that $M''\equiv M' \mod 2$. Of course, considering the equivalence modulo 2, we may reduce the stripes $M(\_,{}^1\otimes)$ and $M({}_1*,\_)$ modulo 2 (thus forgetting the transformations of types (d), (d')), as well as always suppose X,Y being over $\mathbb{Z}/2$.

One can easily see that for the equivalence modulo 2 we get a sort of representations of a bunch of chains in the sense of [**D**]. Therefore, one can write down all indecomposable matrices. Namely, we have the chain $\mathcal{E}$ for rows, with the order:

$$\otimes_1 > \otimes_2 > \dots > \otimes_\infty > \dots > {}_3* > {}_2* > {}_1*,$$

and the chain $\mathcal{F}$ for columns, with the order:

$${}^1\otimes < {}^2\otimes < \dots < {}^\infty\otimes < \dots < *^3 < *^2 < *^1.$$

The equivalence relation $\sim$ is given by

$$\otimes_h \sim {}_h*(h \in \mathbb{N}); {}^a\otimes \sim *^a(a \in \mathbb{N}).$$

Remind the corresponding combinatorics. Put $\mathcal{X} = \mathcal{E} \cup \mathcal{F}$ and write $x - y$ if $x \in \mathcal{E}, y \in \mathcal{F}$ or vice versa. Then $\mathcal{X}$–word is a sequence $w = x_1r_2x_2r_3 \dots r_nx_n$, where $x_i \in \mathcal{X}, r_i \in \{\sim, -\}$, such that $r_i \neq r_{i+1}(i = 2\dots, n-1), x_ir_{i+1}x_{i+1}$ in $\mathcal{X}(i = 1, \dots, n-1)$. Such a word is said to be *full* if:
either $r_2 =\sim$ or $x_1 \nsim y$ for any $y \neq x_1$;
either $r_n =\sim$ or $x_n \nsim y$ for any $y \neq x_n$.
w is called *cyclic* if $r_2 = r_n = -$ and $x_n \sim x_1$; it is called *aperiodic* if it is not of the type $v \sim v \sim \dots \sim v$ for a shorter word v.

Call a polynomial $\pi(t) \in \mathbb{Z}/2[t]$ *primitive* if it is a power of an irreducible polynomial (with the leading coefficient 1). Then the indecomposable representations of this bunch of chains are in 1–1 correspondence with the set $\mathcal{S} \cup \mathcal{B}$, where $\mathcal{S}$ consists of all full words and $\mathcal{B}$ consists of the pairs $(w, \pi(t))$, where w is an aperiodic cyclic word and $\pi(t) \neq t^d$ is a primitive polynomial. More precisely, we should identify any word w with its inverse and any cycle with its cyclic shift!

We call the representations corresponding to $\mathcal{S}$ "strings" and those corresponding to $\mathcal{B}$ "bands".

The condition (2) from the definition of the matrix M imposes some restrictions on the representations, which can occur as equivalence classes modulo 2 for such matrices. In terms of the $\mathcal{X}$–words, they mean that the corresponding word contains no subword ${}^a\otimes -{}_h*, \otimes_1 - *^a, *^1 - \otimes_h$ or their inverse. (Call such words admissible.) One can easily check that such a word can only contain at most one subword of the form $\otimes_h - *^a$ (or its inverse). To simplify the notations, we replace the subword ${}^a\otimes - \otimes_h$ by ${}^a\otimes_h$, $\otimes_h - {}^a\otimes$ by ${}_h\otimes^a$; ${}_h * - *^a$ by ${}_h*^a$, $*^a - {}_h*$ by ${}^a*_h$; $\otimes_h - *^a$ by ${}_h\theta^a$. We also omit all $\sim$ signs and alway replace a superscript ^{aa} by ^{a} as well as a subscript ${}_{hh}$ by ${}_h$. At last, we omit x_1 if $r_2 =\sim$ and x_n if $r_n =\sim$. Certainly, one can always restore the original $\mathcal{X}$–word having the result of all these simplifications.

Now, any admissible word (or its inverse) can be written as a subword of

(i) $${}^{a_1}\otimes_{h_1} *^{a_2} \otimes_{h_2} * \dots {}^{a_n}\otimes_{h_n} \quad (\text{"usual word"})$$

or

(ii) $${}_{h_{-m}}\otimes \dots {}^{a_{-2}} *_{h_{-2}} \otimes^{a_{-1}} *_{h_{-1}} \theta^{a_1} \otimes_{h_1} *^{a_2} \otimes_{h_2} \dots *^{a_n} \quad (\text{"}\theta\text{–word"})$$

The strings correponding to θ–words are called "θ–strings", all other ones are called "usual string". Any admissible cyclic word (or its shift) can be written in the form

$${}^{a_1}\otimes_{h_1} *^{a_2} \otimes_{h_2} * \dots {}^{a_n}\otimes_{h_n} *^{a_1}$$

Moreover, the following conditions hold:

(1) $a_i = \infty$ or $h_j = \infty$ can only occur at the end of a word as ${}^\infty\otimes\,(\otimes^\infty)$ or $\otimes_\infty({}_\infty\otimes)$;
(2) In any θ–word, $h_{-1} \neq 1$ and $a_1 \notin \{1, \infty\}$.

Note that the description of the representations of a bunch of chains [**D**] implies:

- Any row (column) of a string contains at most 1 non–zero element.
- There are at most 2 zero rows or columns in a string, namely, they are in the following stripes:
  - $M({}_h*,\_)$ if w has an end $\otimes_h$ (or ${}_h\otimes$), $h \neq \infty$;
  - $M(\_,*^a)$ if w has an end ${}^a\otimes$, $a \neq \infty$;
  - $M(\otimes_h,\_)$ if w has an end ${}_h*$;
  - $M(\_,{}^a\otimes)$ if w has an end $*^a$ (or ${}^a*$).
- The horizontal and vertical stripes of a band can be subdivided in such a way that every new horizontal or vertical sub–stripe contains exactly 1 non–zero block, which is invertible.

Corollary 3.3. *Let $\overline{M}$ be the reduction modulo 2 of an admissible matrix M, $\overline{N}$ be its indecomposable direct summand. If $\overline{N}$ is a band or a θ–string, then $M \simeq N \oplus M'$ such that $\overline{N} \equiv N \mod 2$.*

So, from now on, we may only consider such M that $\overline{M}$ is a direct sum of usual strings. Denote $\widetilde{M}$ the matrix obtained from M by replacing all invertible entries by zeroes (in particular, all its elements are of the form $2c$). Let $\overline{M} = \bigoplus_{i=1}^r \overline{M_i}$, where $\overline{M_i}$ is the usual string corresponding to a word w_i.

Corollary 3.4. *$M \simeq M'$, where $\overline{M'} = \overline{M}$ and the only non–zero rows and columns of $\widetilde{M'}$ may be the following:*

1. *the columns of $\widetilde{M'}(\_,{}^1\otimes)$ corresponding to the ends ${}^1\otimes$ occurring in the word w_i;*
2. *the rows of $\widetilde{M'}({}_1*,\_)$ corresponding to the ends ${}_1*$ occurring in the words w_i;*
3. *the rows of $\widetilde{M'}(\_,{}^1\otimes)$ corresponding to the ends ${}_h*$ $(h > 1)$;*
4. *the columns of $\widetilde{M'}({}_1*,\_)$ corresponding to the ends ${}^a\otimes$ $(1 < a < \infty)$.*

We call all these ends the "distinguished ends".

In what follows, we always suppose that already $M = M'$.

To prove Corollaries 3.3, 3.4, one only has to use the transformations (d), (d') and (e), (e').

The following Lemma is now decisive.

Lemma 3.5. *Suppose that two words w_i, w_j have a common distinguished end. There is a sequence of admissible transformations of $M = M'$ which does not change the matrix $\overline{M}$ and the resulting transformation of $\widetilde{M}$ adds the row (column) corresponding to the end of w_i to that corresponding to the end of w_j or vice versa.*

Corollary 3.6. *There is a sequence of admissible transformations of $M = M'$, which does not change $\overline{M}$ and transform $\widetilde{M}$ to a matrix having at most one non–zero element in every row and every column.*

Proof of Lemma. We consider the case of the end ${}^1\otimes$, all other cases being quite analogous. So let $w_i = {}^1\otimes_{h_1} *^{a_2} \otimes_{h_2} \dots$, while $w_j = {}^1\otimes_{k_1} *^{b_2} \otimes_{k_2} \dots$. Suppose that $h_1 \leq k_1$. If indeed $h_1 < k_1$, one can add the column corresponding to the end ${}^1\otimes_{h_1}$ of w_i to that corresponding to the end ${}^1\otimes_{k_1}$ of w_j and then subtract the row corresponding to the latter one from that corresponding to the first one to restore $\overline{M}$. If $h_1 = k_1$, compare a_2 and b_2. If $a_2 < b_2$, one can perform the same transformations as above and afterwards subtract the column corresponding to ${}_{h_1}*^{b_2}$ from that corresponding to ${}_{h_1}*^{a_2}$, to restore $\overline{M}$. Continuing these considerations, one sees that we can add the column corresponding to the end of w_i to that corresponding to the end of w_j if $(h_1, a_2, h_2, \dots) < (k_1, b_2, k_2, \dots)$ lexicographically. As the lexicographical order is linear, it proves the lemma.

Corollary 3.7. *Suppose M an indecomposable matrix such that $\widetilde{M'} \neq 0$ for any matrix M' equivalent to M. Let $\overline{M} = \bigoplus_{i=1}^r \overline{M_i}$, where $\overline{M_i}$ are usual strings. Then, up to equivalence, there are but the following possibilities:*

(1) *$r = 1, \overline{M} = \overline{M_1}$ corresponds to a word $w_1 = {}^1 \otimes_{h_1} *^{a_2} \otimes_{h_2} \dots$, $\widetilde{M}$ has one non–zero element in the matrix $M(\otimes_h, {}^1\otimes)$. We denote this case by the word $w = {}_h\, \theta^1 \otimes_{h_1} *^{a_2} \otimes_{h_2} \dots$*

(2) *$r = 1, \overline{M} = \overline{M_1}$ corresponds to a word ${}_1 *^{a_1} \otimes_{h_1} *^{a_2} \dots$; $\widetilde{M}$ has one non–zero element in the matrix $M({}_1*, *^a)$. We denote this case by the word $w = \dots^{a_2} *_{h_1} \otimes^{a_1} *_1 \theta^a$.*

(3) *$r = 1, \overline{M} = \overline{M_1}$ corresponds to a word ${}^{a_1} \otimes_{h_1} *^{a_2} \dots$; $\widetilde{M}$ has one non–zero element in the matrix $M({}_1*, *^{a_1})$. We denote this case by the word $w = {}_1\, \theta^{a_1} \otimes_{h_1} *^{a_2} \dots$*

(4) *$r = 1, \overline{M} = \overline{M_1}$ corresponds to a word ${}_{h_1} *^{a_1} \otimes_{h_2} * \dots$; $\widetilde{M}$ has one non–zero element in the matrix $M(\otimes_{h_1}, {}^1\otimes)$. We denote this case by the word $w = \dots *_{h_2} \otimes^{a_1} *_{h_1} \theta^1$.*

(5) *$r = 2, \overline{M_1}$ corresponds to a word ${}^1 \otimes_{h_1} *^{a_1} \otimes_{h_2} \dots$, $\overline{M_2}$ corresponds to a word ${}_{h_{-1}} *^{a_{-1}} \otimes_{h_{-2}} * \dots$; the unique non–zero element of $\widetilde{M}$ is in the matrix $M(\otimes_{h_{-1}}, {}^1\otimes)$. We denote this case by the word $w = \dots *_{h_{-1}} \otimes^{a_{-1}} *_{h_{-1}} \theta^1 \otimes_{h_1} *^{a_1} \otimes_{h_2} \dots$*

(6) *$r = 2, \overline{M_1}$ corresponds to a word ${}^{a_1} \otimes_{h_1} *^{a_2} \otimes_{h_2} \dots$; $\overline{M_2}$ corresponds to a word ${}_1 *^{a_{-1}} \otimes_{h_{-1}} *^{a_{-2}} \dots$; the unique non–zero element of $\widetilde{M}$ is in the matrix $M({}_1*, *^{a_1})$. We denote this case by the word $w = \dots^{a_{-2}} \otimes_{h_{-1}} \otimes^{a_{-1}} *_1 \theta^{a_1} \otimes_{h_1} *^{a_2} \otimes_{h_2} \dots$*

Obviously, the cases (1)–(4) always give indecomposable representations. On the other hand, in the case (5), if $h_1 \geq h_{-1}$, the representation M evidently decomposes as $M_1 \oplus M_2$; the same is with the case (6) and $a_{-1} \geq a_1$. If $h_1 < h_{-1} - 1$ in the case (5) or $a_{-1} < a_1 - 1$ in the case (6), the corresponding representation is evidently indecomposable. Suppose, in the case (5), $h_1 = h_{-1} - 1$. Then we can delete the non–zero element of $\widetilde{M}(\otimes_{h_{-1}}, {}^1\otimes)$ using the transformation (c), but it changes the zero entry in $M({}_{h_{-1}}*, *^{a_1})$. To restore it, we need that $a_{-1} \geq a_1$, moreover, if $a_{-1} = a_1$, we also need that $h_2 \geq h_{-2}$, etc. Thus, such a representation remains indecomposable if and only if $(h_1 + 1, a_{-1}, h_2, a_{-2}, \dots) < (h_{-1}, a_1, h_{-1}, a_2, \dots)$ lexicographically. Just in the same way, the representation of type (6) is indecomposable if and only if $(a_{-1} + 1, h_1, a_{-2}, h_2, \dots) < (a_1, h_{-1}, a_2, h_{-2}, \dots)$ lexicographically.

Therefore, the complete list of indecomposable matrices looks like follows.

Theorem 3.8. *Indecomposable objects in $\mathbf{El}(G)$ are in 1–1 correspondence with the following types of data:*

(1) *usual words, i. e. subword of the words ${}^{a_1} \otimes_{h_1} *^{a_2} \otimes_{h_2} * \dots^{a_n} \otimes_{h_n}$, where $a_i, h_j \in \mathbb{N} \cup \{\infty\}, \infty$ being only possible for a_1 or h_n.*

(2) *θ–words, i. e. subwords containing θ of the words ${}_{h_{-m}} \otimes \dots^{a_{-2}} *_{h_{-2}} \otimes^{a_{-1}} *_{h_{-1}} \theta^{a_1} \otimes_{h_1} *^{a_2} \otimes_{h_2} \dots *^{a_n} \otimes_{h_n}$, where $a_i, h_j \in \mathbb{N} \cup \{\infty\}, \infty$ being only possible for h_{-m} or $h_n, (a_1, h_{-1}) \neq (1, 1)$ and,*
if $h_{-1} = 1$,

then $(a_{-1}+1, h_1, a_{-2}, h_2, \dots) < (a_1, h_{-2}, a_2, h_{-3}, \dots)$ *lexicographically, and, if* $a_1 = 1$,
then $(h_1+1, a_{-2}, h_2, a_{-3}, \dots) < (h_{-1}, a_2, h_{-2}, a_3, \dots)$ *lexicographically. Moreover the* θ*–word does not coincide with* ${}_\infty\theta^1\otimes_\infty$.

(3) *Pairs* $(w, \pi(t))$, *where* w *is an aperiodic cyclic word:* $w = {}^{a_1}\otimes_{h_1} *^{a_2} \otimes_{h_2} \dots {}^{a_n}\otimes_{h_n} *^{a_1}$ *(up to shift), and* $\pi(t) \neq t^d$ *is a primitive polynomial over* $\mathbb{Z}/2$.

Corollary 3.9. *Let* α *be an indecomposable element of the bimodule* G. *Denote by* $\widetilde{\alpha}$ *its component belonging to the sub–bimodule* $\widetilde{G}$ *and by* $\widetilde{G_\alpha}$ *the sub–bimodule of* $\widetilde{G}$ *generated by* $\widetilde{\alpha}$ *Then* $\widetilde{G_\alpha} \cap \mathrm{Im}\eta_*\mu(\alpha)_* = 0$.

Proof. Follows immediately from the description above.

Here are examples of indecomposable representations.

(1) Usual string corresponding to the word ${}_2 *^3 \otimes_2 *^4 \otimes_1 *^3 \otimes_4$

	${}^3\otimes$	${}^3\otimes$	${}^4\otimes$	$*^4$	$*^3$	$*^3$
$\otimes_1$	0	0	1	0		
$\otimes_2$	0	0	0			
	1	0	0			
$\otimes_4$	0	1	0			
${}_4*$				0	0	0
${}_2*$				0	1	0
				1	0	0
${}_1*$				0	0	1

$$A = 2\cdot\mathbb{Z}/8 \oplus \mathbb{Z}/16$$
$$H = \mathbb{Z}/2 \oplus 2\cdot\mathbb{Z}/4 \oplus \mathbb{Z}/16$$

(2) θ–string corresponding to the word ${}_\infty \otimes^4 *_3 \otimes^2 *_2\theta^4 \otimes_3 *^3$

	${}^2\otimes$	${}^3\otimes$	${}^4\otimes$	${}^4\otimes$	$*^4$	$*^4$	$*^3$	$*^2$
$\otimes_2$	0	0	0	0	0	1	0	0
$\otimes_3$	1	0	0	0	0	0	0	0
	0	0	0	1	0	0	0	0
$\otimes_\infty$	0	0	1	0	0	0	0	0
${}_3*$					1	0	0	0
					0	0	1	0
${}_2*$					0	0	0	1

$$A = \mathbb{Z}/4 \oplus \mathbb{Z}/8 \oplus 2\cdot\mathbb{Z}/16$$
$$H = \mathbb{Z}/4 \oplus 2\cdot\mathbb{Z}/8 \oplus \mathbb{Z}$$

(3) θ–string corresponding to the word ${}^2 *_4 \otimes^1 *_3 \theta^1 \otimes_2 *^3\otimes_1$

	${}^1\otimes$	${}^1\otimes$	${}^2\otimes$	${}^3\otimes$	$*^3$	$*^2$	$*^1$	$*^1$
$\otimes_1$	0	0	0	1	0			
$\otimes_2$	0	1	0	0				
$\otimes_3$	0	2	0	0				
$\otimes_4$	1	0	0	0				
${}_1*$					0	0	0	0
${}_2*$					1	0	0	0
${}_3*$					0	0	1	0
${}_4*$					0	1	0	0

$$A = 2 \cdot \mathbb{Z}/2 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/8$$
$$H = \mathbb{Z}/2 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/8 \oplus \mathbb{Z}/16$$

(4) band corresponding to the word ${}^2\otimes_4 *^3 \otimes_1 *^3 \otimes_2 *^2$ and the polynomial $\pi(t) = t^2 + t + 1$

$$\begin{array}{c|c|c|c|c|}
 & {}^2\otimes & {}^3\otimes & {}^3* & {}^2* \\
\hline
\otimes_1 & & \begin{matrix}1 & 0\\ 0 & 1\end{matrix} & & \\
\otimes_2 & & \begin{matrix}1 & 0\\ 0 & 1\end{matrix} & 0 & \\
\otimes_4 & \begin{matrix}0 & 1\\ 1 & 1\end{matrix} & & & \\
\hline
{}_4* & & & \begin{matrix}1 & 0\\ 0 & 1\end{matrix} & \\
{}_2* & & & & \begin{matrix}1 & 0\\ 0 & 1\end{matrix} \\
{}_1* & & & \begin{matrix}1 & 0\\ 0 & 1\end{matrix} & \\
\hline
\end{array}$$

Frobenius matrix corresponding to $\pi(t)$

$$A = 2 \cdot \mathbb{Z}/4 \oplus 4 \cdot \mathbb{Z}/8$$
$$H = 2 \cdot \mathbb{Z}/2 \oplus 2 \cdot \mathbb{Z}/4 \oplus 2 \cdot \mathbb{Z}/16$$

4. Proof of theorem (C)

Let $\mathbf{M}^m(\mathbb{Z}/2)$ be the homotopy category of Moore spaces $M(A,m)$ where A is a finitely generated $\mathbb{Z}/2$–vector space. Then we know by Baues [**B1**] 1.6.7 that $\mathbf{M}^m(\mathbb{Z}/2)$ for $m \geq 3$ is isomorphic to the category $\mathbf{mod}(\mathbb{Z}/4)$ of finitely generated free $\mathbb{Z}/4$–modules. Using this result we show:

Theorem 4.1. *Let $m \geq 6$. Then the bimodule $\mathbf{M}^{m+3}(\mathbb{Z}/2)^{op} \times \mathbf{M}^m(\mathbb{Z}/2) \to$ $\mathbf{Ab}$ given by $[M(a, m+3), M(B,m)]$ is natural isomorphic to the bimodule*

$$\mathbf{mod}(\mathbb{Z}/4)^{op} \times \mathbf{mod}(\mathbb{Z}/4) \to \mathbf{Ab}$$

which carries $(\overline{A}, \overline{B})$ to $\mathsf{Hom}(\overline{A} \otimes \mathbb{Z}/2, \overline{B} \otimes (\mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2))$.

Since the quiver

$$\bullet \overset{\longrightarrow}{\underset{\longrightarrow}{\longrightarrow}} \bullet$$

is wild we see that theorem (4.1) implies theorem (C) in the introduction.

Proof of (4.1). Since the bimodule is biadditive it suffices to compute for $A = B = \mathbb{Z}/2$ the $\mathbb{Z}/4$–bimodule $\pi = [M(A, m+3), M(B,m)]$. Here the $\mathbb{Z}/4$–module structure is already determined by the structure of π as an abelian group. We know that

$$\pi_{m+2}M(B,m) = \mathbb{Z}/4$$
$$\pi_{m+3}M(B,m) = \mathbb{Z}/2 \oplus \mathbb{Z}/2$$

and that the Hopf map $\eta : S^{m+3} \to S^{m+2}$ induces the map

$$\eta^* : \pi_{m+2}M(B,m) \to \pi_{m+3}M(B,m)$$

corresponding to the composite

$$\eta^* : \mathbb{Z}/4 \twoheadrightarrow \mathbb{Z}/2 \xrightarrow{i_1} \mathbb{Z}/2 \oplus \mathbb{Z}/2$$

of the inclusion i_1 and the quotient map. This follows for example from Baues–Goerss [**BG**] 5.2. Moreover using Baues [**B1**] 1.6.11 we have the push out diagram of abelian groups

$$\begin{tikzcd}
\mathbb{Z}/2 \oplus \mathbb{Z}/2 \arrow[r, equal] & \mathsf{Ext}(A, \pi_{m+3} M(B,m)) \arrow[r] & \pi \\
\mathbb{Z}/2 \arrow[r, equal] & \mathsf{Ext}(A, \pi_{m+2}(B,m) \otimes \mathbb{Z}/2) \arrow[r, "j"] \arrow[u, "(\eta^*)_*"] & \mathbf{G}(A, \pi_{m+2} M(B,m)) \arrow[u]
\end{tikzcd}$$

Here $(\eta^*)_*$ is induced by η^* above so that $(\eta^*)_*$ is split injective with cokernel $\mathbb{Z}/2$. Moreover we have by definition of the category $\mathbf{G}$ in Baues [**B1**] the isomorphism

$$\mathbf{G}(A, \pi_{m+2}(B,m)) = \mathbf{G}(\mathbb{Z}/2, \mathbb{Z}/4) = \mathbb{Z}/2 \oplus \mathbb{Z}/2$$

with j above corresponding to the inclusion $i_1 : \mathbb{Z}/2 \subset \mathbb{Z}/2 \oplus \mathbb{Z}/2$. This shows $\pi = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$. q.e.d.

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Hans–Joachim Baues
Max–Planck–Institut für Mathematik
P.O.Box: 7280
D–53072 Bonn
Germany
E-mail address: baues@mpim-bonn.mpg.de

Yuri A. Drozd
Faculty of Mechanics and Mathematics
Kiev Taras Shevchenko University
RUS–252033 Kiev
Ukraine