Notes on the Airy function

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ABSTRACT. It is proved the addition formula involving the Airy function and its derivatives and primitives of arbitrary order. The formula is applied to initial value problem for linearized Korteweg–de Vries equation and to the Airy operators.

Introduction. Our purpose in these notes is to point out on some addition formulae for the Airy function which may have interesting consequences in connection with certain differential equations. The Airy function $Ai\colon \mathbf{C} \to \mathbf{C}$ we talk about is an entire function defined for $z \in \mathbf{R}$ by the classical Airy integral, [1],

$$Ai(z) = \frac{1}{2\pi}\int\limits_{\mathbf{R}} \exp\bigl(it^3/3 + izt\bigr)\, dt. \tag{1}$$

The addition formula stated in Theorem 1 below has been founded by the author in [4], [5]. Not going into the details which are irrelevant in here, let us notice that the Airy function arises in [4] as the radial part of the Whittaker function attached to author's cubic theta function Θ on the group $\mathrm{SL}(3, \mathbf{C})$. In the present work we give more general addition formulae, stated in Theorem 2, and apply them to the the initial value problem for linearized Korteweg–de Vries equation and to the Airy operators. This work is partially supported by RFFR grant 02-01-00087.

Preliminaries. Let us begin with some notations. We will write $\mathbf{Z}$, $\mathbf{C}$, $\mathbf{R}$, $\mathbf{R}_+$, fo the ring of the rational integers, the field of complex numbers, the field of real numbers, the additive group of positive real numbers. By F', F'', ... and $F^{(n)}$ we mean the first, the second ... and the n-th order derivatives of the function F, $n > 0$.

The Airy function satisfies well known differential equation

$$Ai''(z) = zAi(z) \tag{2}$$

and has a lot of other pleasant properties; for instance, see [3]. We will need some of them. First of all, we have

Ai is bounded on $\mathbf{R}$,

Ai is positive and decreases on $\mathbf{R}_+$,

Ai' is negative and increases on $\mathbf{R}_+$,

$|Ai|$ and $|Ai'|$ decrease on $\mathbf{R}_+$.

The Airy integral above does not converge absolutely. However, one can apply Cauchy integral theory to find that

$$Ai(z) = \frac{1}{2\pi i}\int\limits_{\mathcal{L}} \exp\bigl(v^3/3 - zv\bigr)\,dv, \tag{3}$$

where $\mathcal{L}$ is any contour that begins at a point at infinity in the sector $-\pi/2 \le \arg v \le -\pi/6$ and ends at infinity in the sector $\pi/6 \le \arg v \le \pi/2$, and the integral in (3) converges absolutely for all $z \in \mathbf{C}$. Moreover, if δ is an arbitrary small positive number and $\mathcal{L}$ begins in the sector $-\pi/2 + \delta \le \arg v \le -\pi/6 - \delta$ and ends at infinity in the sector $\pi/6 + \delta \le \arg v \le \pi/2 - \delta$ then the integral (3) converges absolutely and uniformly on compacts in $\mathbf{C}$. So, (3) suplies the analytic continuation of Ai to the whole of $\mathbf{C}$. The same statements on convergence are valid for the integrals representing the derivatives of Ai, that is for

$$Ai^{(n)}(z) = \frac{1}{2\pi i}\int\limits_{\mathcal{L}} \exp\bigl(v^3/3 - zv\bigr)(-v)^n\,dv \tag{4}$$

with the contour $\mathcal{L}$ as above. The integral in (4) has sense for all $n \in \mathbf{Z}$, and not only for $n \ge 0$. Clearly, $Ai^{(n)}$ with negative $n \in \mathbf{Z}$ is the $(-n)$-th order primitive of Ai. Assuming the contour $\mathcal{L}$ located in the right half plane, we have $Ai^{(n)}(z) \to 0$ as $z \to \infty$, $z \in \mathbf{R}$. So, we have the series of functions $Ai^{(n)}$, $n \in \mathbf{Z}$, uniquely determined by

(a) $Ai^{(0)} = Ai$,

(b) $Ai^{(n+1)}$ is the derivative of $Ai^{(n)}$,

(c) $Ai^{(n)}(z) \to 0$ as $z \to \infty$, $z \in \mathbf{R}$.

Given any $\varepsilon > 0$, one has the asymptotics

$$\begin{aligned} Ai(z) &\sim \frac{1}{2\pi^{1/2} z^{1/4}} \exp(-Z), \\ Ai'(z) &\sim -\frac{1}{2\pi^{1/2}} z^{1/4} \exp(-Z), \end{aligned} \tag{5}$$

as $|z| \to \infty$, $|\arg z| \le \pi - \varepsilon$, $z \in \mathbf{C}$; here $Z = (2/3)z^{3/2}$.

One can follow Copson approach (see [3], ch. 4, section 4) to find asymptotics for all of $Ai^{(n)}$ or the estimates like

$$|Ai^{(n)}(z)\exp(Z)| < C_n \quad \text{for all} \quad z \in \mathbf{C} \quad \text{with} \quad |\arg z| < \pi/3, \tag{6}$$

where $n \in \mathbf{Z}$, $n < 0$, $Z = (2/3)z^{3/2}$, and C_n is a constant depending on n only.

Integral addition formulae for Airy function. The following addition formula has been proved by the author in [5]. Two different proofs were given in [4].

Theorem 1. *Given* $a, b \in \mathbf{C}$ *under the conditions* $|\arg a| < \pi/3$, $|\arg b| < \pi/3$, *one has*

$$Ai(a+b) = \int_0^1 Ai'\Big(\frac{a}{x^{1/3}}\Big) Ai'\Big(\frac{b}{(1-x)^{1/3}}\Big) \frac{dx}{x^{2/3}(1-x)^{2/3}}. \tag{7}$$

This theorem serves as the starting point for all what follows. The main purpose of the present work is to generalize it as it is given in Theorem 2 below.
To recall shortly the proof given in [5], let us define the double Mellin transform of the function $F\colon \mathbf{R}_+ \times \mathbf{R}_+ \to \mathbf{C}$ to be

$$M(s,t;F) = \int_0^\infty\!\!\int_0^\infty F(x,y) x^{s-1} y^{t-1}\, dx\, dy, \qquad s,t \in \mathbf{C}.$$

If F is continuously differentiable function and the integral above converges absolutely at least for all s, t with real parts in some intervals in $\mathbf{R}$ then F can be expressed as a certain integral transform of $M(\cdot,\cdot;F)$. Thus, this case F is determined uniquely by its double Mellin transform. Using the known formulae for the Mellin transform of Ai and Ai' (see [3], ch.2 and ch.7) one can evaluate the double Mellin transform of the left hand side of (7); then one can evaluate the double Mellin transform of the right hand side of (7), and apply the triplication formula for the Γ-function to find that is the same. In such a way we obtain the formula (7) at least for $a, b \in \mathbf{R}_+$. Then it can be extended to a, b as stated above by analytic continuation.
Recall once more that $Ai^{(0)}$ denotes the Airy function Ai, and that $Ai^{(n)}$ denotes the n-th order derivatives of Ai if $n > 0$, and $(-n)$-th order primitive of Ai if $n < 0$.

Theorem 2. *Given* a, b *under the conditions* $|\arg a| < \pi/3$, $|\arg b| < \pi/3$ *(i.e. as in Theorem 1), and* $\tau \in \mathbf{R}_+$, *one has*

$$\frac{1}{\tau^{(1+l)/3}} Ai^{(l)}\Big(\frac{a+b}{\tau^{1/3}}\Big) = \int_0^\tau Ai^{(n)}\Big(\frac{a}{t^{1/3}}\Big) Ai^{(m)}\Big(\frac{b}{(\tau-t)^{1/3}}\Big) \frac{dt}{t^{(n+1)/3}(\tau-t)^{(m+1)/3}}, \tag{8}$$

where $l = n+m-2$, $n, m \in \mathbf{Z}$.

Proof. Beginning with the formula (7) in Theorem 1 with $a\tau^{-1/3}, b\tau^{-1/3}$ instead of a, b, and changing variable as $x = \tau^{-1}t$ we obtain

$$\frac{1}{\tau^{1/3}} Ai\Big(\frac{a+b}{\tau^{1/3}}\Big) = \int_0^\tau Ai'\Big(\frac{a}{t^{1/3}}\Big) Ai'\Big(\frac{b}{(\tau-t)^{1/3}}\Big) \frac{dt}{t^{2/3}(\tau-t)^{2/3}}. \tag{9}$$

Notice that (9) is nothing but (8) for $n = m = 1$. To ensure the integral in the right hand side (8) converges, let us remark that equation (2) combined with the asymptotic (5) imply

$$Ai^{(h)}(z) = P_h(z) Ai(z) + Q_h(z) Ai'(z) \tag{10}$$

with some polinomyals P_h, Q_h, $h \geq 0$, and then

$$\sup\{\,|z^k Ai^{(h)}(z)| \;\big|\; |z| > R,\ |\arg z| < \pi/3 - \varepsilon\} \to 0 \quad \text{as} \quad R \to \infty.$$

for all $k > 0$ and $\pi/3 > \varepsilon > 0$. This remark and the standard theorems (like the one given in ch.2, section 3.5 in [2]) allow to differentiate under the integral sign in (8), (9). Applying repeatedly ∂_a and ∂_b to both sides of the formula (9) we obtain (8) for all $n \geq 1$, $m \geq 1$.
To prove (8) for the case $n \leq 0$ and/or $m \leq 0$, let us write $R_{n,m}(a,b)$ for the left hand side, and $Q_{n,m}(a,b)$ for the right hand side of (8). So, we have two families of functions $R_{n,m}$ and $Q_{n,m}$ satisfying

$$R_{n,m} = Q_{n,m} \quad \text{for} \quad n, m \geq 1,$$

and we are interested to prove the same still holds for all $n, m \in \mathbf{Z}$ as well. To this end it is sufficient to show that, given $n, m \in \mathbf{Z}$,

$$\text{if} \quad R_{n,m} = Q_{n,m} \quad \text{then} \quad R_{n-1,m} = Q_{n-1,m}, \quad R_{n,m-1} = Q_{n,m-1}.$$

We find easily that

$$R'_{n-1,m} = R_{n,m}, \quad Q'_{n-1,m} = Q_{n,m}, \quad \text{and hence} \quad R'_{n-1,m} = Q'_{n-1,m},$$

where $'$ denotes differentiation over the first variable, and also

$$R'_{n,m-1} = R_{n,m}, \quad Q'_{n,m-1} = Q_{n,m-1} \quad \text{and hence} \quad R'_{n,m-1} = Q'_{n,m-1},$$

where $'$ denotes differentiation over the second variable. It follows then that $R_{n-1,m} - Q_{n-1,m}$ and $R_{n,m-1} - Q_{n,m-1}$ are constants. On the other hand it is clear that all of $R_{n,m}(a,b)$, $Q_{n,m}(a,b)$ tends to 0 as $a \to \infty$ or $b \to \infty$. Hence the constants above equal to 0, as required.

Examples. Let us end this section with two examples. First, take $l = 0$ to obtain possible expressions for the Airy function itself; that are

$$\frac{1}{\tau^{1/3}} Ai\Big(\frac{a+b}{\tau^{1/3}}\Big) = \int_0^\tau Ai^{(1+k)}\Big(\frac{a}{t^{1/3}}\Big) Ai^{(1-k)}\Big(\frac{b}{(\tau-t)^{1/3}}\Big) \frac{dt}{t^{(2+k)/3}(\tau-t)^{(2-k)/3}}, \tag{11}$$

$k \in \mathbf{Z}$. For the second example take $n = m = 2$ We have

$$\frac{1}{\tau} Ai''\Big(\frac{a+b}{\tau^{1/3}}\Big) = \int_0^\tau Ai''\Big(\frac{a}{t^{1/3}}\Big) Ai''\Big(\frac{b}{(\tau-t)^{1/3}}\Big) \frac{dt}{t(\tau-t)}. \tag{12}$$

and this is the only case with $l = n = m$.

On the Airy equation. In this section we will apply the addition formula to find certain qualitative property of solutions of the Airy differential equation. By the Airy equation we mean

$$\partial_t u + \partial_x^3 u = 0. \tag{13}$$

It is known also as the linearized Korteweg–de Vries equation. The classical formula

$$u(x,t) = \frac{1}{(3t)^{1/3}} \int\limits_{\mathbf{R}} Ai\Big(\frac{x-\alpha}{(3t)^{1/3}}\Big) f(\alpha)\, d\alpha, \tag{14}$$

gives a unique solution of the initial value problem

$$\partial_t u + \partial_x^3 u = 0, \quad u(x,0) = f(x), \qquad x,t \in \mathbf{R}, \ \ t \geq 0.$$

Here $f\colon \mathbf{R} \to \mathbf{R}$ is an infinitely differentiable function which we assume (for further needs) to be a function of moderate growth and bounded support in the following sense. We say that $P\colon \mathbf{R} \to \mathbf{R}$ is a function of moderate growth with bounded support if it satisfies

(a) Given any $\varepsilon > 0$, there is C_ε such that $|P(z)| < C_\varepsilon \exp\big(\varepsilon |z|^{3/2}\big)$ for all z;

(b) There is c such that $P(z) = 0$ for all $z > c$.

Clearly, if $f, g\colon \mathbf{R} \to \mathbf{R}$ satisfy (a),(b) then so does the convolution

$$z \mapsto \int\limits_{\mathbf{R}} f(z-\xi) g(\xi)\, d\xi.$$

Theorem 3. *Given infinitely differentiable functions $f, g\colon \mathbf{R} \to \mathbf{R}$ of moderate growth and bounded support, let u, v, w be the solutions of the initial value problems for the Airy equation, —*

$$\begin{aligned}
&\partial_t u + \partial_x^3 u = 0, \qquad u(x,0) = f(x),\\
&\partial_t v + \partial_y^3 v = 0, \qquad v(y,0) = g(y),\\
&\partial_\tau w + \partial_z^3 w = 0, \quad w(z,0) = f{*}g(z), \quad x,y,z,t,\tau \in \mathbf{R}, \ \ t,\tau \geq 0,
\end{aligned}$$

where $f{*}g$ is the convolution of f and g, so that

$$f{*}g(z) = \int\limits_{\mathbf{R}} f(z-\xi) g(\xi)\, d\xi.$$

Let us write u' for $\partial_x u$, and v' for $\partial_y v$. Then, for $z = x + y$ with sufficiently large x and y one has

$$w(z,\tau) = 3\int\limits_0^\tau u'(x, \tau - t) v'(y,t)\, dt. \tag{15}$$

More precisely, if $f(x) = g(y) = 0$ for all $x \geq \varepsilon$, $y \geq \delta$ then the formula (15) *holds for all $z = x + y$, $x \geq \varepsilon$, $y \geq \delta$.*

Proof. Differentiation under the integral sign in (15) gives rise to

$$\begin{aligned}
u'(x,t) &= \frac{1}{(3t)^{2/3}} \int\limits_{\mathbf{R}} Ai'\Big(\frac{x-\alpha}{(3t)^{1/3}}\Big) f(\alpha)\, d\alpha,\\
v'(y,t) &= \frac{1}{(3t)^{2/3}} \int\limits_{\mathbf{R}} Ai'\Big(\frac{y-\beta}{(3t)^{1/3}}\Big) g(\beta)\, d\beta;
\end{aligned} \tag{16}$$

the justification required will be given below. Let us write $I(x,y,\tau)$ for the right-hand side of

(15). Now let us substitute the expressions in (16) to the right-hand side of (15) and invert the order of integration, postponing justification to the end of the proof. We find that $I(x,y,\tau)$ equals

$$3\int_0^\tau \Big[\frac{1}{(3(\tau-t))^{2/3}}\int_{\mathbf{R}} Ai'\Big(\frac{x-\alpha}{(3(\tau-t))^{1/3}}\Big) f(\alpha)\,d\alpha\Big]\Big[\frac{1}{(3t)^{2/3}}\int_{\mathbf{R}} Ai'\Big(\frac{y-\beta}{(3t)^{1/3}}\Big) g(\beta)\,d\beta\Big]\,dt$$

$$= \frac{1}{3^{1/3}}\int_{\mathbf{R}}\int_{\mathbf{R}}\int_0^\tau Ai'\Big(\frac{x-\alpha}{(3(\tau-t))^{1/3}}\Big) Ai'\Big(\frac{y-\beta}{(3t)^{1/3}}\Big)\,\frac{dt}{t^{2/3}(\tau-t)^{2/3}} f(\alpha)g(\beta)\,d\alpha\,d\beta. \quad (17)$$

If $f(\alpha)g(\beta)\neq 0$ then $\alpha<\varepsilon$, $\beta<\delta$. Hence, for such α,β one has $x-\alpha>0$, $y-\beta>0$ for all $x\geq\varepsilon$, $y\geq\delta$, and one can apply Theorem 2 with $n=m=1$ and

$$a=\frac{x-\alpha}{3^{1/3}},\quad b=\frac{y-\beta}{3^{1/3}}$$

to evaluate the interior integral in (17). In such a way we find

$$I(x,y,\tau)=\frac{1}{(3\tau)^{1/3}}\int_{\mathbf{R}}\int_{\mathbf{R}} Ai\Big(\frac{x+y-\alpha-\beta}{(3\tau)^{1/3}}\Big) f(\alpha)g(\beta)\,d\alpha\,d\beta$$

for $x\geq\varepsilon$, $y\geq\delta$. Changing variables as $r=\alpha+\beta$, $\xi=\beta$, we obtain the final formula

$$I(x,y,\tau)=\frac{1}{(3\tau)^{1/3}}\int_{\mathbf{R}} Ai\Big(\frac{x+y-r}{(3\tau)^{1/3}}\Big) h(r)\,dr \quad \text{with} \quad h(r)=\int_{\mathbf{R}} f(r-\xi)g(\xi)\,d\xi.$$

Comparing with (15) we find $w(z,\tau)=I(x,y,\tau)$ for all $z=x+y$ with $x\geq\varepsilon$, $y\geq\delta$, $\tau>0$, as required.

To complete the proof it remains now to justify differentiation under the integral sign in (14) and inversion of integration in (17). Since we need the formula in (16) for $x>\varepsilon$ only, and $f(\alpha)=0$ for $\alpha\geq\varepsilon$, we may assume that $x-\alpha>0$ in (14) and to apply the asymptotic (5) to ensure that with the function f of moderate growth we are under the conditions of the standard theorem (see [2], ch.2, section 3.5) which imply (16). To justify (17) it is sufficient to prove convergence of the integral

$$\int_{-\infty}^{\delta}\int_{-\infty}^{\varepsilon} J(x-\alpha,y-\beta)\,|f(\alpha)g(\beta)|\,d\alpha\,d\beta, \quad \text{where} \quad x\geq\varepsilon,\ y\geq\delta, \quad \text{and} \quad (18)$$

$$J(p,q)=\int_0^\tau |Ai'\Big(\frac{p}{(3(\tau-t))^{1/3}}\Big) Ai'\Big(\frac{q}{(3t)^{1/3}}\Big)|\,\frac{dt}{t^{2/3}(\tau-t)^{2/3}}.$$

To this end, let

$$N=\int_0^\tau \frac{dt}{t^{2/3}(\tau-t)^{2/3}},\quad D(z)=|Ai'\Big(\frac{z}{(3\tau)^{1/3}}\Big)|.$$

Then, taking into account that $|Ai'|$ decreases on $\mathbf{R}_+$, and applying asymptotic (5) to majorate Ai' we find

$$J(p,q)\leq ND(p)D(q)\leq CN\exp\big(-R(p^{3/2}+q^{3/2})\big) \quad \text{if} \quad p\geq 0,\ q\geq 0. \quad (19)$$

where C and R are positive constants. Since f, g are functions of bounded supports and of moderate growth, (19) is sufficient to ensure the integral (18) converges.

On the Airy transforms. Following [3], ch.9, we define the Airy transform of the function $f\colon \mathbf{R}_+ \to \mathbf{R}$, denoted $\hat{A}i(\cdot; f)$, as

$$\hat{A}i(z; f) = \int_0^\infty Ai(zu) f(u)\, du, \qquad z \in \mathbf{R}_+. \tag{20}$$

More generally, given $n \in \mathbf{Z}$, we can define n-th order Airy transform as

$$\hat{A}i^{(n)}(z; f) = \int_0^\infty Ai^{(n)}(zu) f(u)\, du, \qquad z \in \mathbf{R}_+, \tag{21}$$

with $Ai^{(n)}$ as defined in Preliminaries. We can apply our addition formula in Theorem 2 to prove the theorem below.
Firstly, let us agree to say that $P\colon \mathbf{R}_+ \to \mathbf{R}$ is a function of moderate growth with support separated from zero if it satisfies

(a) Given any $\varepsilon > 0$, there is C_ε such that $|P(z)| < C_\varepsilon \exp\big(\varepsilon |z|^{3/2}\big)$ for all z;

(b) There is c such that $P(z) = 0$ for all $z < c$.

Clearly, if $f, g\colon \mathbf{R} \to \mathbf{R}$ satisfy (a) and (b) then so does the convolution

$$z \mapsto \int_0^r f(x) g(z - x)\, dx.$$

Theorem 4. *Let $f, g\colon \mathbf{R}_+ \to \mathbf{R}$ be continuous functions of moderate growth with supports separated from zero, as it is defined above, and let $h\colon \mathbf{R}_+ \to \mathbf{R}$ be their convolution,*

$$h(r) = \int_0^r f(u) g(r - u)\, du.$$

Then, for all $n, m \in \mathbf{Z}$ and $z \in \mathbf{R}_+$ one has

$$\hat{A}i^{(l)}(z; h) = \int_0^1 \hat{A}i^{(n)}\Big(\frac{z}{t^{1/3}}; f\Big)\, \hat{A}i^{(m)}\Big(\frac{z}{(1-t)^{1/3}}; g\Big) \frac{dt}{t^{(n+1)/3}(1-t)^{(m+1)/3}},$$

where $l = n + m - 2$. More generally, if $\tau \in \mathbf{R}_+$, one has

$$\frac{1}{\tau^{(1+l)/3}} \hat{A}i^{(l)}\Big(\frac{z}{\tau^{1/3}}; h\Big) = \int_0^\tau \hat{A}i^{(n)}\Big(\frac{z}{t^{1/3}}; f\Big)\, \hat{A}i^{(m)}\Big(\frac{z}{(\tau - t)^{1/3}}; g\Big) \frac{dt}{t^{(n+1)/3}(\tau - t)^{(m+1)/3}}.$$

Proof. Let us write $I(z,\tau)$ for the right hand side of the last formula. By definition (21),

$$\begin{aligned}\hat{A}i^{(n)}\Big(\frac{z}{t^{1/3}};f\Big) &= \int_0^\infty Ai^{(n)}\Big(\frac{zu}{t^{1/3}}\Big) f(u)\,du,\\ \hat{A}i^{(m)}\Big(\frac{z}{(\tau-t)^{1/3}};g\Big) &= \int_0^\infty Ai^{(m)}\Big(\frac{zv}{(\tau-t)^{1/3}}\Big) g(v)\,dv.\end{aligned} \tag{22}$$

Substituting the expressions in (22) to $I(z,\tau)$ and inverting the order of integration we find that $I(z,\tau)$ equals to

$$\int_0^\infty\int_0^\infty\Big[\int_0^\tau Ai^{(n)}\Big(\frac{zu}{t^{1/3}}\Big)Ai^{(m)}\Big(\frac{zv}{(\tau-t)^{1/3}}\Big)\frac{dt}{t^{(n+1)/3}(\tau-t)^{(m+1)/3}}\Big] f(u)g(v)\,du\,dv. \tag{23}$$

By Theorem 2, the interior integral in here equals to

$$\frac{1}{\tau^{(1+l)/3}}Ai^{(l)}\Big(\frac{z(u+v)}{\tau^{1/3}}\Big), \quad \text{where} \quad l = n+m-2.$$

Hence,

$$I(z,\tau) = \frac{1}{\tau^{(1+l)/3}}\int_0^\infty\int_0^\infty Ai^{(l)}\Big(\frac{z(u+v)}{\tau^{1/3}}\Big) f(u)g(v)\,du\,dv.$$

Changing variables as $r = u+v$, $x = v$, we obtain

$$\begin{aligned}I(z,\tau) &= \frac{1}{\tau^{(1+l)/3}}\int_0^\infty\int_0^r Ai^{(l)}\Big(\frac{zr}{\tau^{1/3}}\Big) f(r-x)g(x)\,dx\,dr\\ &= \frac{1}{\tau^{(1+l)/3}}\int_0^\infty Ai^{(l)}\Big(\frac{zr}{\tau^{1/3}}\Big)h(r)\,dr = \frac{1}{\tau^{(1+l)/3}}\hat{A}i^{(l)}\Big(\frac{z}{\tau^{1/3}};h\Big),\end{aligned}$$

as required. It remains to justify inversion of integration above. To this end it is sufficient to prove the integral in (23) converges absolutely. We can refer to (6) (for $n < 0$), to (5) (for $n = 0,1$), and to (10), (5) (for $n \geq 2$) to find the upper bound for

$$\Big|Ai^{(n)}\Big(\frac{zu}{t^{1/3}}\Big)Ai^{(m)}\Big(\frac{zv}{(\tau-t)^{1/3}}\Big)\Big|,$$

which is sufficient to ensure the integral in (23) converges absolutely for the functions f, g under consideration.

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