Schrödinger semigroups - geometric estimates in terms of the occupation time

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Abstract

The difference of Schrödinger and Dirichlet semigroups is expressed in terms of the Laplace transform of the Brownian motion occupation time. This implies quantitative upper and lower bounds for the operator norms of the corresponding resolvent differences. One spectral theoretical consequence is an estimate for the eigenfunction for a Schrödinger operator in a ball where the potential is given as a cone indicator function.

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1 Introduction

The Feynman-Kac formula is a powerful tool in stochastic spectral analysis to study spectral properties of partial differential operators. In this article we shall consider the Laplace transform of the occupation timeof the Wiener process in certain regions Γ of $\mathbb{R}^d$. These considerations have several spectral theoretical consequences, in particular one can give quantitative upper and lower bounds for the resolvent differences of Schrödinger operators and Dirichlet operators in terms of the height of the positive part of the potential.

In some more detail, this means the following. Let H_0 be the self-adjoint realisation of $-\frac{1}{2}\Delta$ in $L^2(\mathbb{R}^d)$. Let $W = W_+ - W_-$ be a potential in Kato's class. The positive part of the potential W_+ is assumed to be high on a set $\Gamma \subset \mathbb{R}^d$. Let $U = W_+ 1_\Gamma$ and $V = W_+ 1_{\mathbb{R}^d - \Gamma} - W_-$. Schrödinger operators of the form

$$H = H_0 + V + U,$$

U positive and large on Γ arise naturally in several physical models, for instance, in $N-$body models, in solid state physics with periodic potentials and in atomic systems. The main feature in all these examples is that Γ is an unbounded region in $\mathbb{R}^d$ having in general non-smooth boundaries. Replacing the potential U by MU where M represents the height, we compare the physical system given by $H_M = H_0 + V + MU$ with an artificial system where the potential barrier MU is of infinite height ($M = \infty$). The corresponding Hamiltonian of this artificial system is the Dirichlet operator $H_\Sigma = (H_0+V)_\Sigma$, $\Sigma = \mathbb{R}^d - \Gamma$, the self-adjoint operator in $L^2(\Sigma)$ generated by the differential expression $-\frac{1}{2}\Delta + V$ with Dirichlet boundary conditions on $\partial\Gamma$. Because these operators act in different Hilbert spaces we introduce the restriction operator J where $Jf := f\mid_\Sigma$.

We now explain the connection between these Schrödinger or Dirichlet operators and Brownian motion. Let $X = (\Omega, \mathcal{F}, P_x, X_t)$ be Brownian motion in $\mathbb{R}^d$ with expectation denoted by E_x. The occupation time of Brownian motion in Γ is defined as

$$T_{t,\Gamma} := \mid \{s \in [0,t] : X_s \in \Gamma\} \mid \tag{1}$$

($\mid \cdot \mid$ is Lebesgue measure). The operator norm of the semi-group difference $Je^{-tH_M} - e^{-tH_\Sigma}J$ can be estimated by the Laplace transform of the occupation time *i.e.*

$$E_0 e^{-MT_{t,\Gamma}}$$

which will be one of the central objects of study in this article. Suppose Γ and Γ^c satisfy a uniform cone condition *i.e.* there exist angles α (resp. α') and heights h (resp. h') such that for any point $y \in \partial\Gamma$ we can find a cone C of angle α and height h (resp. a cone C' of angle α' and height h') contained in Γ with vertex y (resp. contained in Γ^c). Then the asymptotics of the Laplace transform for Γ is governed by the Laplace transform for the cone C:

$$E_0 e^{-MT_{t,C}} \tag{2}$$

We give upper and lower bounds for this quantity of the form

$$a(Mt)^{-A(C)} \leq E_0 e^{-MT_{t,C}} \leq b(\frac{\ln Mt}{Mt})^{A(C)} \tag{3}$$

for some $a, b > 0$ and $Mt > 2$. The cone constant $A(C)$ is defined in terms of the lowest eigenvalue of the Laplace-Beltrami operator on $S^{d-1} - (S^{d-1} \cap C)$ (S^{d-1} the unit sphere in $\mathbb{R}^d$) with Dirichlet boundary conditions. Because the cone condition is uniform on $\partial\Gamma$, $A(C)$ is always smaller or equal to $1/2$.

The estimate (3) has several spectral theoretical consequences, since one has quantitative error estimates in M, not just the bare convergence. The most important consequence is that for regular resolvent values

$$\| J(H_M + a)^{-1} - (H_\Sigma + a)^{-1} J \| \leq cM^{-(1/2-\epsilon)} \tag{4}$$

for some $\epsilon > 0$, $c = c(a, V, \epsilon, d)$. This in turn can be used to give quantitative estimates for the limiting absorption principle

$$\| \langle x\rangle^{-\alpha}(J(H_M + \lambda \pm i0)^{-1} - (H_\Sigma + \lambda \pm i0)^{-1}J)\langle x\rangle^{-\alpha} \| \tag{5}$$

with $\langle x\rangle = (1+ \mid x \mid^2)^{1/2}$, $\alpha > 1/2$, which provides error estimates for the spectral measures and scattering matrices. These and further applications will be described in a forthcoming article.

Here we mention another spectral application. Let C be a cone with vertex at the origin. Let $(H_0 + M1_C)_B$ be the Dirichlet operator in $L^2(B)$ where B is a ball in $\mathbb{R}^d$. Denote by ϕ_n the eigenfunction corresponding to the $n-$th eigenvalue of this operator. Then the value of the eigenfunction at the origin may be estimated via

$$\mid \phi_n(0) \mid \leq c(n) E_0 e^{-MT_{1,C}}$$

$$\leq c(n)(\frac{\ln M}{M})^{A(C)} \tag{6}$$

A typical feature in stochastic spectral analysis is the following: the assumptions and models come from physical situations and do not involve any stochastic element, as also the results, as in (4) and (6), but the method of proof relies heavily on the theory of stochastic processes. Here we studied in detail the Wiener trajectories in cones using known and new results in this theory.

The article is organised as follows. In section 2 the asymptotics of the occupation time in a cone is estimated from above and below. This entails upper (section 3) and lower (section 4) bounds for the operator norm of semigroup and resolvent differences. Our convergence results are restricted to a certain class of Lipschitz domains; we provide an example to show that for certain singularity regions Γ the convergence fails (section 3).

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2 Asymptotics of the occupation time in a cone

We shall examine here the precise asymptotics of the Laplace transform of the occupation time in a cone, which is determined by the cone constant A depending on the angle of the cone.

A cone C in $\mathbb{R}^d$, $d \geq 2$ is a set of the form

$$C = \{x \in \mathbb{R}^d : (x, e_1) \geq p \parallel x \parallel\}, \ -1 < p < 1 \tag{7}$$

Here $(\cdot,\cdot)$ is the standard inner-product in $\mathbb{R}^d$ and $e_1 := (1, 0, \ldots, 0)$. The angle of the cone is defined to be $\alpha := 2 \arccos p$. Let F be the closed subset of S^{d-1}, the unit sphere in $\mathbb{R}^d$ centred at the origin, given by $F := C \cap S^{d-1}$. Let (X_t, P_x) be Brownian motion in $\mathbb{R}^d$. Then the total occupation time of Brownian motion in $\Gamma \subset \mathbb{R}^d$ up to time $t > 0$ is defined by $T_{t,\Gamma} := | \{s \in [0,t] : X_s \in C\} |$. We shall denote by $| \cdot |$ the d-dimensional Lebesgue measure. Given any Borel set $B \subset \mathbb{R}^d$, the first hitting time of B is defined by $\sigma(B) := \inf\{t > 0 : X_t \in B\}$.

Definition 2.1 *The cone constant is defined via*

$$A = A(C) := \frac{\lambda_1(F^c)}{(\nu^2 + 2\lambda_1(F^c))^{1/2} + \nu} \tag{8}$$

where $\nu := d/2 - 1$ *and* $\lambda_1(F^c)$ *is the lowest eigenvalue of the Laplace-Beltrami operator* $-\frac{1}{2}\Delta$ *on* $F^c := S^{d-1} - F$ *with Dirichlet boundary conditions on* ∂F^c.

Proposition 2.2 *Let* C *be the cone (7). There exists a positive constant* c *such that for all* $t \geq e$

$$P_0(T_{t,C} \leq 1) \leq c(\frac{\ln t}{t})^A \tag{9}$$

Proof. Let $v \in S^{d-1} \cap int(C)$ be a vector from 0 to the interior of F and fix $a > 0$ for now. Denote the first hitting time of $av + C$ by σ_a. Then

$$P_0(T_{t,C} \leq 1) = P_0(T_{t,C} \leq 1, \sigma_{2a} \leq t) + P_0(T_{t,C} \leq 1, \sigma_{2a} > t) \tag{10}$$

We estimate the first term on the right-hand side. It is clear that there exists $\lambda > 0$ such that

$$\eta + Q_a \subset C - (2av + C) \text{ for all } \eta \in av + \partial C$$

where Q_a is the cube of side-length $a\lambda$ centred at the origin. We therefore have that

$$P_0(T_{t,C} \leq 1, \sigma_{2a} \leq t) = P_0(T_{\sigma_a,C} + T_{t-\sigma_a,C} \circ \theta_{\sigma_a} \leq 1, \sigma_a < \sigma_{2a} \leq t)$$

$$\leq P_0(T_{\sigma_{2a}-\sigma_a,C} \circ \theta_{\sigma_a} \leq 1) = E_0 P_{X_{\sigma_a}}(T_{\sigma_{2a},C} \leq 1) \leq P_0(\sigma(Q_a^c) \leq 1)$$

Let $P_0^{(1)}$ be the law of one-dimensional Brownian motion starting at 0 and τ the first exit time of $[-1, 1]$. We now use the classical estimate

$$\text{there exists } k > 0 \text{ such that for all } t > 0,\ P_0^{(1)}(\tau < t) \leq ke^{-1/2t} \tag{11}$$

After a time-change we obtain

$$P_0(T_{t,C} \leq 1, \sigma_{2a} \leq t) \leq ce^{-\lambda^2 a^2/2} \tag{12}$$

for a constant $c > 0$. The second term on the right-hand side of (10) may be estimated using [8] proposition 2.3 by

$$P_0(T_{t,C} \leq 1, \sigma_{2a} > t) \leq C(\frac{a^2}{t})^A \tag{13}$$

Put $a^2 = \frac{2A}{\lambda^2} \ln t$, then (12) and (13) yield

$$P_0(T_{t,C} \leq 1) \leq c(\frac{\ln t}{t})^A \ \Box$$

We now give an improved version of [2] Theorem 5.4.

Theorem 2.3 *Let C be the cone (7). There exists a constant $c > 0$ such that for all $t > 0$ and all $M > 0$ with $Mt > e$*

$$E_0 e^{-MT_{t,C}} \leq c(\frac{\ln Mt}{Mt})^A \quad A = A(C)$$

Proof. Let $\epsilon_n := e^{-n}$, $n = 0, 1, \ldots$ Note that $P_0(T_{t,C} \leq \epsilon_n t) = P_0(T_{\epsilon_n^{-1},C} \leq 1)$. Thus

$$\begin{aligned}
E_0 e^{-MT_{t,C}} &= \sum_{n=0}^{\infty} \int_{\epsilon_{n+1} t}^{\epsilon_n t} e^{-M\theta} P_0(T_{t,C} \in d\theta) \\
&\leq \sum_{n=0}^{\infty} e^{-Mte^{-(n+1)}} P_0(T_{t,C} \leq e^{-n} t) \\
&\leq c \sum_{n=0}^{\infty} e^{-Mte^{-(n+1)}} (\frac{n}{e^n})^A \\
&\leq c \int_1^{\infty} e^{-Mte^{-x}} (\frac{x}{e^x})^A dx \\
&= c \int_0^1 e^{-Mty} \mid y \ln y \mid^A y^{-1} dy \\
&= c \int_0^{Mt} e^{-z} \mid \frac{z}{Mt} \ln \frac{z}{Mt} \mid^A z^{-1} dz \\
&\leq c(\frac{\ln Mt}{Mt})^A \int_0^{\infty} e^{-z} (1 + \mid \ln z \mid)^A z^{A-1} dz
\end{aligned}$$

using $| \ln \frac{z}{Mt} | \leq | \ln Mt | (1+ | \ln z |)$ if $Mt > e$ and $z > 0$. □
We now demonstrate that the upper bound on the Laplace transform of the occupation time in a cone is almost optimal by providing an accompanying lower bound.

Lemma 2.4 *Let C be the cone (7). There exists a positive constant c such that*

$$P_0(T_{t,C} < 1) \geq ct^{-A}, \ t > 2$$

Proof. Denote by F_δ the δ−neighbourhood of F in S^{d-1} and by C_δ the cone generated by F_δ. We will take δ sufficiently small so that $S^{d-1} - F_\delta \neq \emptyset$. We then have by [8] 2.3

$$\begin{aligned} P_0(T_{t,C} < 1) &\geq P_0(X_1 \in B(0,1) \cap C_\delta^c; \sigma(C) \circ \theta_1 > t-1) \\ &= \int_{B(0,1)\cap C_\delta^c} p(1,0,\eta) P_\eta(\sigma(C) > t-1) d\eta \\ &\geq \int_{B(0,1)\cap C_\delta^c} p(1,0,\eta) c(\frac{t-1}{|\eta|^2})^{-A} d\eta \\ &\geq c \int_{B(0,1)\cap C_\delta^c} p(1,0,\eta) |\eta|^{2A} d\eta \, t^{-A} \ \square \end{aligned}$$

We can now obtain a lower bound on the Laplace transform of the occupation time in a cone as an easy corollary of the last lemma.

Proposition 2.5 *Let C be the cone (7). There exists a positive constant c such that for $Mt > 2$*

$$E_0 e^{-MT_{t,C}} \geq c(Mt)^{-A}$$

Proof. The result follows easily from the observation

$$E_0 e^{-MT_{t,C}} \geq e^{-1} P_0(T_{Mt,C} < 1) \geq c(Mt)^{-A} \ \square$$

Remark. Let $H := \{x \in \mathbb{R}^d : x_1 \geq 0\}$ denote the half-space in $\mathbb{R}^d$. By the arc-sine law [5] proposition 4.4.11 there exist positive constants c, c' such that $c(Mt)^{-1/2} \leq E_0 e^{-MT_{t,H}} \leq c'(Mt)^{-1/2}$ for large Mt. This together with our results in 2.3 and 2.5 allow us to deduce $A = A(H) = 1/2$ independently of the dimension d and hence that $\lambda_1(H)$, the bottom eigenvalue of the Laplace-Beltrami operator $-\frac{1}{2}\Delta$ on S^{d-1} with Dirichlet boundary conditions has the value $\frac{1}{2}(d-1)$. Let C_α be the cone of angle α, $0 < \alpha \leq \pi$ and suppose $d \geq 3$. Then $\lambda_1(F_\alpha^c) \downarrow 0$ as $\alpha \downarrow 0$. For, let $(\mathcal{F}, \mathcal{E})$ be the Dirichlet form on $L^2(S^{d-1}, \tau)$ (τ surface area measure on S^{d-1}) corresponding to the Laplace-Beltrami operator $-\frac{1}{2}\Delta$ on S^{d-1}, and Cap the associated capacity. Then because the Brownian motion does not hit points, $Cap(F_\alpha^c) \downarrow 0$, $\alpha \downarrow 0$ implying that $\bigcup \mathcal{F}_{F_\alpha}$ is $\mathcal{E}_1-$dense in $\mathcal{F}$ (see [7],[4]) where $\mathcal{F}_{F_\alpha} = \{u \in \mathcal{F} : u = 0 \ \tau - a.e. \text{ on } F_\alpha^c\}$. It follows from this and the min-max principle that $\lambda_1(F_\alpha^c) \downarrow 0$, $\alpha \downarrow 0$. Since the function $A(C)$ is monotone increasing in C, in the sense that $A(C') \leq A(C)$ whenever $C' \subset C$ we can deduce that $\lambda_1(F_\alpha^c) \leq cd$ for some constant $c > 0$ independent of d, and $0 < \alpha \leq \pi$, $d \geq 2$.

3 Upper bound on semigroup and resolvent differences

In this section we are interested in perturbed operators H_M of the positive Laplacian $H_0 := -\frac{1}{2}\Delta$ in $L^2(\mathbb{R}^d)$, $d \geq 2$ of the form

$$H_M = H_0 + V + MU$$

where V is a potential uniformly bounded below and $U = 1_\Gamma$ for Γ a closed subset of $\mathbb{R}^d$ (the singularity region) and M is an arbitrary positive parameter. The mapping $J : L^2(\mathbb{R}^d, dx) \to L^2(\Sigma, dx)$, $\Sigma := \mathbb{R}^d - \Gamma$, is the obvious restriction. H_Σ will denote the operator H_0 on $L^2(\Sigma, dx)$ with Dirichlet boundary conditions on $\partial\Sigma$

We recall here the definition of uniform Lipschitz set from [2]. Given a closed set $\Gamma \subset \mathbb{R}^d$ define

$$\partial_l\Gamma := \{x \in \mathbb{R}^d : d(x, \partial\Gamma) \leq l\}, \ l > 0$$

Definition 3.1 *A closed set $\Gamma \subset \mathbb{R}^d$ is said to be a uniform Lipschitz set if there exist $0 < \delta < r < \infty$, $0 < L < \infty$, $0 < l < \infty$, $m \in \mathbb{N}$ and a countable collection B_k of open balls of radius r covering $\partial_l\Gamma$, such that*

> *for all k, $\partial\Gamma \cap B_k$ is locally the graph of a Lipschitz function with Lipschitz constant L;*

for all $x \in \partial_l\Gamma$, $d(x, B_k^c) \geq \delta > 0$ *for some* k;

each $x \in \partial_l\Gamma$ *belongs to at most* m *sets* B_k.

Remark. Given any uniform Lipschitz set as above there exists a cone $C = C_{int}$ (the subscript "*int*" stands for "interior") of angle $\alpha = 2\arctan L^{-1}$, $0 < \alpha \leq \pi$ and $r > 0$ such that

> for each $y \in \partial\Gamma$ there is a rigid transformation (translation, rotation) S_y of $\mathbb{R}^d$ such that $S_y C \cap B(y, r) \subset \Gamma$ and $S_y(0) = y$.

(see [10]). We quote the following from [2].

Proposition 3.2 *Let* Γ *be a uniform Lipschitz set in* $\mathbb{R}^d$. *Then there exists a positive constant* c *such that*

$$\sup_{x\in\Sigma} P_x(t - \epsilon < \sigma(\Gamma) < t) \leq c\epsilon^{1/2}(1 + t^{-1/2})$$

for any $0 < \epsilon < t$.

We now come to the main theorem of this section. We use the estimate, see [6] eq.III 2.8

$$\| Je^{-tH_M} - e^{-tH_\Sigma}J \| \leq \sup_{x\in\Sigma} E_x(e^{-MT_{t,\Gamma}} : \sigma(\Gamma) < t) \tag{14}$$

We note that the asymptotics in M and t of $E_.(e^{-MT_{t,\Gamma}} : \sigma(\Gamma) < t)$ are the same if Γ is a cone, but different in general. Since Γ is Lipschitz $\sigma(\Gamma) = \sigma(int(\Gamma))$ $P_. - a.s.$ so that $T_{t,\Gamma} > 0$ on $\{\sigma < t\}$; applying the monotone convergence theorem we see $\lim_{M\to\infty} E_.(e^{-MT_{t,\Gamma}} : \sigma < t) = 0$ for $t > 0$ fixed. On the other hand, for $x \in \Gamma$ $E_x e^{-MT_{t,\Gamma}} \geq e^{-ME_xT_{t,\Gamma}}$; if Γ is compact then $\lim_{t\to\infty} E_x T_{t,\Gamma} = l > 0$ and so $\lim_{t\to\infty} E_x e^{-MT_{t,\Gamma}} = e^{-Ml} > 0$ (see [2], [6]) for $M > 0$ fixed.

Theorem 3.3 *Suppose that* Γ *is a uniform Lipschitz set and that* $A = A(C) \leq 1/2$, $C = C_{int}$. *Then for each* $0 < \gamma < 1$, $0 < t < \infty$

$$\| Je^{-tH_M} - e^{-tH_\Sigma}J \| \leq u_\gamma(t)M^{-\gamma A}, \ M > 1$$

for some function $u_\gamma(t)$ *satisfying* $c_\gamma(a) := \int_0^\infty e^{-at}u_\gamma(t)dt < \infty$ *for all* $a > 0$, *and*

$$\| J(a + H_M)^{-1} - (a + H_\Sigma)^{-1}J \| \leq c_\gamma(a)M^{-\gamma A}, \ M > 1$$

Proof. For the sake of brevity we write $\sigma := \sigma(\Gamma)$, $\tau := \sigma(B(0,r)^c)$ where r is as in definition 3.1, $z := X_\sigma$. Then for $\epsilon > 0$

$$E_x(e^{-MT_{t,\Gamma}} : \sigma < t) = E_x(e^{-MT_{t,\Gamma}} : \sigma < t < \sigma + \tau \circ \theta_\sigma)$$
$$+E_x(e^{-MT_{t,\Gamma}} : \sigma < \sigma + \tau \circ \theta_\sigma < t, \tau \circ \theta_\sigma < \epsilon)$$
$$+E_x(e^{-MT_{t,\Gamma}} : \sigma < \sigma + \tau \circ \theta_\sigma < t, \tau \circ \theta_\sigma > \epsilon)$$
$$\leq E_x(E_z e^{-MT_{t-\sigma,S_z(C)}} : \sigma < t) + P_0(\tau < \epsilon) + E_0 e^{-MT_{\epsilon,C}} \tag{15}$$

Let $\tau_n := 1 - 2^{-n}$. Consider the expression $E_0 e^{-MT_{(1-\tau_{n+1})t,C}}$. If $M(1-\tau_{n+1})t > e$ we can use 2.3 and the estimate $(\frac{|\ln x|}{x})^A \leq c(\beta)x^{-\beta A}$ for some $c(\beta) > 0$ where $0 < \beta < 1$, $x > e$. If $M(1-\tau_{n+1})t < 1$ then $E_0 e^{-MT_{(1-\tau_{n+1})t,C}} \leq (M(1-\tau_{n+1})t)^{-\beta A}$. Note that $M(1-\tau_{n+1})$ belongs to the interval $(1, e)$ for at most two n's and for such n, $E_0 e^{-MT_{(1-\tau_{n+1})t,C}} \leq (M(1-\tau_{n+3})t)^{-\beta A}$. With these remarks and 3.2 we see

$$E_x(E_z e^{-MT_{t-\sigma,S_z(C)}} : \sigma < t) = \sum_{n=0}^{\infty} E_x(E_z e^{-MT_{t-\sigma,S_z(C)}} : \tau_n t < \sigma < \tau_{n+1} t)$$
$$\leq \sum_{n=0}^{\infty} E_0 e^{-MT_{(1-\tau_{n+1})t,C}} P_x(\tau_n t < \sigma < \tau_{n+1} t)$$
$$\leq c\sum_{n=0}^{\infty} (M(1-\tau_{n+1})t)^{-\beta A}((\tau_{n+1} - \tau_n)t)^{1/2}(1 + t^{-1/2})$$
$$\leq c(Mt)^{-\beta A}(1 + t^{-1/2}) \sum_{n=0}^{\infty} (\tau_{n+1} - \tau_n)^{1/2-\beta A} < \infty \tag{16}$$

because $\beta A < 1/2$. Returning to (15) we see that

$$E_x(e^{-MT_{t,\Gamma}} : \sigma < t) \leq cM^{-\beta A}t^{-\beta A}(1 + t^{-1/2}) + c'(M\epsilon)^{-\beta A} + c'' e^{-r^2/4\epsilon}$$

assuming $M\epsilon > e$. Now put $\epsilon = M^{-\alpha}$, $\alpha > 0$ to obtain

$$\leq cM^{-\beta A}t^{-\beta A}(1 + t^{-1/2}) + c'M^{-(1-\alpha)\beta A} + c'' M^{-\beta A}$$
$$\leq c(1 + t^{-\beta A}(1 + t^{-1/2}))M^{-(1-\alpha)\beta A}$$

for $M^{1-\alpha} > e$, independently of x. Note that $\int_0^\infty e^{-at}u_\gamma(t)dt < \infty$ for all $a > 0$ where $u_\gamma(t) = c(1 + t^{-\beta A}(1 + t^{-1/2}))$. The resolvent upper bound is obtained by integrating the result for the semi-group. □

Corollary 3.4 *Let Γ be a uniform Lipschitz set and assume in addition that*

> *for all $\pi - \epsilon < \alpha \leq \pi$ ($\epsilon > 0$) there exist $r(\alpha) > 0$ such that for all $z \in \partial\Gamma$*
> $$S_z C_\alpha \cap B(z, r(\alpha)) \subset \Gamma \tag{17}$$
> *where S_z is a rigid tranformation of $\mathbb{R}^d$ and C_α is a cone of angle α.*

Then for each $0 < \gamma < 1$, $0 < t < \infty$

$$\| Je^{-tH_M} - e^{-tH_\Sigma} J \| \leq u_\gamma(t) M^{-\gamma/2}, \text{ for all large } M > 0$$

for some function $u_\gamma(t)$ satisfying $c_\gamma(a) = \int_0^\infty e^{-at} u_\gamma(t) dt < \infty$ for all $a > 0$, and

$$\| J(a + H_M)^{-1} - (a + H_\Sigma)^{-1} \| \leq c_\gamma(a) M^{-\gamma/2}, \text{ for all large } M > 0$$

Proof. Pick $\pi - \epsilon < \alpha \leq \pi$, $0 < \gamma < 1$. Then apply 3.3 to obtain the estimate with exponent $\gamma A(C_\alpha)$. Since $A(C_\alpha) \to 1/2$ as $\alpha \to \pi$,and γ can be chosen arbitrarily close to 1 we have the result. □
Remark. We note that if Γ is compact with smooth boundary the last corollary applies.

We give an example to show that there are singularity regions Γ for which the semi-groups e^{-tH_M} do not converge to e^{-tH_Σ} in operator norm.

Define a singularity region Γ_{ab} by

$$\Gamma_{ab} := \{rx : x \in S^{d-1} - B_a, 1 \leq r \leq 1 + b\},\ a, b > 0 \tag{18}$$

where $e_1 := (1, 0, \dots, 0)$ and $B_a := S^{d-1} - S^{d-1} \cap B(e_1, a)$ and let $\Sigma_{ab} := \mathbb{R}^d - \Gamma_{ab}$.

Lemma 3.5 *There exist $\epsilon > 0$, $t > 0$, $r > 0$ such that given any $n \in \mathbb{N}$ we can find $a, b > 0$, $\phi \in L^2(\Sigma_{ab}; dx)$, $\| \phi \|_{L^2(\Sigma_{ab}, dx)} = 1$ such that for all $x \in B(0, 1/2)$*

$$E_x(e^{-nT_{\Gamma_{ab},t}} \phi(X_t) : \sigma(\Gamma_{ab}) < t < \sigma(B(0, r)^c)) \geq \epsilon \tag{19}$$

Proof. Let $p > 0$, then we can estimate

$$| E_x(e^{-nT_{\Gamma_{ab},t}}\phi(X_t) : \sigma(\Gamma_{ab}) < t < \sigma(B(0,r)^c)) - E_x\phi(X_t) |$$

$$\begin{aligned}
&\leq | E_x(e^{-nT_{\Gamma_{ab},t}}\phi(X_t) : \sigma(\Gamma_{ab}) < t < \sigma(B(0,r)^c)) - E_x(e^{-nT_{\Gamma_{ab},t}}\phi(X_t) : \sigma(\Gamma_{ab}) < t) | \\
&\quad + | E_x(e^{-nT_{\Gamma_{ab},t}}\phi(X_t) : \sigma(\Gamma_{ab}) < t) - E_x(e^{-nT_{\Gamma_{ab},t}}\phi(X_t)) | \\
&\quad + | E_x(e^{-nT_{\Gamma_{ab},t}}\phi(X_t)) - E_x\phi(X_t) | \\
&= | E_x(e^{-nT_{\Gamma_{ab},t}}\phi(X_t) : \sigma(\Gamma_{ab}) < t, \sigma(B(0,r)^c) < t) | \\
&\quad + | E_x(e^{-nT_{\Gamma_{ab},t}}\phi(X_t) : \sigma(\Gamma_{ab}) > t\} | \\
&\quad + | E_x(1 - e^{-nT_{\Gamma_{ab},t}})\phi(X_t) | \\
&\leq \| \phi \|_\infty P_x(\sigma(B(0,r)^c) < t) + \| \phi \|_\infty P_x(\sigma(\Gamma_{ab}) > t) + \| \phi \|_\infty E_x(1 - e^{-nT_{\Gamma_{ab},t}})
\end{aligned}$$

We thus have that

$$E_x(e^{-nT_{\Gamma_{ab},t}}\phi(X_t) : \sigma(\Gamma_{ab}) < t < \sigma(B(0,r)^c))$$

$$\geq E_x\phi(X_t) - P_x(\sigma(B(0,r)^c) < t) - P_x(\sigma(\Gamma_{ab}) > t) - E_x(1 - e^{-nT_{\Gamma_{ab},t}}) \quad (20)$$

assuming $\| \phi \|_\infty = 1$. Note that the right-hand side of (20) only depends on n through the last term. We now estimate each of the terms in (20). Let $\phi := 1_{A_{bc}}$ where $A_{bc} := \{sx : x \in S^{d-1}, 1 + b \leq s \leq c\}$ and c is chosen so that $m(A_{bc}) = 1$. Then

$$\begin{aligned}
E_x\phi(X_t) &= \int p(t,x,y)\phi(y)dy \\
&\geq c' t^{-d/2} e^{-c^2/2t}(c^d - (1+b)^d) \quad (21)
\end{aligned}$$

Let $P_0^{(1)}$ denote the law of 1−dimensional Brownian motion started at 0. It is clear there exists $l(r) > 0$ such that

$$x + [-l(r), l(r)]^d \subset B(0,r), \; x \in B(0,1/2) \quad (22)$$

Thus

$$\begin{aligned}
&P_x(\sigma(B(0,r)^c) < t) \leq P_x(\sigma((x + [-l(r), l(r)]^d)^c) < t) \\
&\leq P^{(1)}(\sigma([-l(r), l(r)]^c) < t) = P^{(1)}(\sigma([-1,1]^c) < l(r)^{-2}t) \\
&\leq k e^{-1/2l(r)^{-2}t} = k e^{-l(r)^2/2t} \quad (23)
\end{aligned}$$

for all $x \in B(0,1/2)$ by (11).
Now

$$P_x(\sigma(\Gamma_{ab}) > t) = P_x(\sigma(\Gamma_{ab}) > t, \sigma(S^{d-1}) < t) + P_x(\sigma(\Gamma_{ab}) > t, \sigma(S^{d-1}) > t)$$

The first term can be estimated by

$$\begin{aligned} P_x(\sigma(\Gamma_{ab}) > t, \sigma(S^{d-1}) < t) &\leq P_x(X_{\sigma(S^{d-1})} \in B_a) \\ &\leq 2^d \tau(B_a) \end{aligned} \tag{24}$$

for $x \in B(0,1/2)$ by [9] theorem 3.1, where τ is surface measure on S^{d-1}. Note that there exists a box of side-length $2R$ such that

$$B(0,1) \subset x + [-R,R]^d, \; x \in B(0,1/2) \tag{25}$$

Thus for all $x \in B(0,1/2)$

$$\begin{aligned} P_x(\sigma(\Gamma_{ab}) > t, \sigma(S^{d-1}) > t) &\leq P_x(\sigma(S^{d-1}) < t) \\ \leq P_x(\sigma((x + [-R,R]^d)^c) > t) &\leq P_0^{(1)}(\sigma([-R,R]^c) > t)^d \\ = P^{(1)}(\sigma([-1,1]^c) > R^{-2}t)^d &\leq K e^{-\lambda R^{-2} t d} \end{aligned} \tag{26}$$

for some $\lambda > 0$ by [8] p.120
The domain Γ_{ab} is a subset of the annulus $A_b = \{rx : 1 \leq r < 1 + b\}$ thus $T_{\Gamma_{ab},t} \leq T_{A_b,t}$. So

$$E_x(1 - e^{-nT_{\Gamma_{ab},t}}) \leq E_x(1 - e^{-nT_{A_b,t}})$$

Moreover by symmetry $E_x(1 - e^{-nT_{A_b,t}})$ is constant on $\partial B(0,1/2)$ and

$$E_y(1 - e^{-nT_{A_b,t}}) \leq E_x(1 - e^{-nT_{A_b,t}})$$

for $y \in B(0,1/2)$, $x \in \partial B(0,1/2)$. We know that by letting b get sufficiently small $E_x(1 - e^{-nT_{A_b,t}})$ can be made arbitrarily small thus it is possible to choose $a = a(n), b = b(n)$ so that

$$E_x(1 - e^{-nT_{\Gamma_{ab},t}}) \leq \eta, \; x \in B(0,1/2) \tag{27}$$

for any $\eta > 0$. Substituting (21), (23),(24), (26),(27) into (20) we see that for $x \in B(0,1/2)$

$$E_x(e^{-nT_{\Gamma_{ab},t}}\phi(X_t) : \sigma(\Gamma_{ab}) < t < \sigma(B(0,r)^c)) \geq c' t^{-d/2} e^{-c^2/2t}(c^d - (1+b)^d)$$

$$- ke^{-l(r)^2/2t} - 2^d\tau(S^{d-1} - \Gamma_{ab}) - Ke^{-tR^{-2}td} - \eta \tag{28}$$

By choosing r, $l(r)$, t large and $\tau(B_a)$ small, the right-hand side of (28) can be uniformly bounded below by a positive constant $\epsilon > 0$ for all $x \in B(0, 1/2)$. □

Corollary 3.6 *For ϵ, t, r as above choose a_n, b_n such that (19) is true. Define*

$$\Gamma := \cup_{n=1}^{\infty}\{3rne_1 + \Gamma_{a_n b_n}\}, \ \Sigma := I\!R^d - \Gamma \tag{29}$$

Let $H_n := H_0 + n1_\Gamma$ where H_0 is the Laplace operator, H_Σ the Laplace operator with Dirichlet boundary conditions on Γ. Then the semi-group differences $Je^{-tH_n} - e^{-tH_\Sigma}J$ fail to converge to 0 in $L^2(I\!R^d)$ as $n \to \infty$.

Proof. Let ϕ be as in the lemma and define $\phi_n := \phi \circ \psi_n^{-1}$ where ψ_n is translation in the e_1-direction by $3rn$. Then

$$\| e^{-tH_n} - e^{-tH_\Sigma} \|^2 \geq \| (e^{-tH_n} - e^{-tH_\Sigma})\phi \|_2^2$$

$$= \int_\Sigma | E_x(e^{-nT_{\Gamma,t}}\phi_n(X_t) : \sigma < t) |^2 \, dx$$

$$\geq \int_{\psi_n B(0,1/2)} | E_x(e^{-nT_{t,\Gamma}}\phi_n(X_t) : \sigma(\Gamma_{a_n b_n}) < t < \sigma(B(\psi_n(0), r)^c)) |^2 \, dx$$

$$\geq \epsilon^2 \mid B(0, 1/2) \mid$$

so fails to converge to zero. □

4 Lower bound on semigroup and resolvent differences

In this section we shall consider perturbations of the positive Laplacian H_0 in $L^2(I\!R^d)$, $d \geq 2$ of the form

$$H_M := H_0 + V + MU$$

where V is a potential uniformly bounded above by a constant b, U is the indicator function of a singularity region Γ satisfying there exists $\Gamma_0 \subset \partial\Gamma$ with positive surface measure and a cone $C = C_{cov}$ such that for each $z \in \Gamma_0$ there is a rigid transformation S_z of $\mathbb{R}^d$ such that

$$S_z C^c - \{z\} \subset \mathbb{R}^d - \Gamma; \tag{30}$$

M is an arbitrary positive parameter. The subscript "cov" stands for "covering" to emphasize that $C_{int} \subset C_{cov}$ where C_{int} is the cone appearing in Theorem 3.3. and these cones are not the same in general; C_{int} is always contained in Γ, at least locally, while C_{cov} contains Γ.
We want to get lower bounds on $\| Je^{-tH_M} - e^{-tH_\Sigma} J \|$ to counterpart the result of the last section. Since we are working with the $L^2(\mathbb{R}^d)$−operator norm we automatically have the lower bound $\geq \| e^{-tH_M}\phi - J^* e^{-tH_\Sigma} J\phi \|$ for $\phi \in L^2(\mathbb{R}^d)$, $\| \phi \|_{L^2(\mathbb{R}^d)} = 1$. By the Feynman-Kac formula this translates into

$$= \int_\Sigma | E_x\{e^{-MT_{t,\Gamma} - \int_0^t V(X_s)ds}\phi(X_t) : \sigma(\Gamma) < t\} |^2 \, dx$$

Theorem 4.1 *Assume that assumption* (30) *is in force. Fix* $t \geq 1$. *Given* $p > 1$ *there exists a constant* $k(t)$ *such that*

$$\| Je^{-tH_M} - e^{-tH_\Sigma} J \| \geq k(t) M^{-pA}$$

for all sufficiently large $M > 0$.

Proof. We shall take $V \equiv 0$ to simplify the proof; it is only a slight modification to deal with V as in (30). We suppose also that Γ_0 has finite surface measure. Let $\phi \in L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, $\phi \geq 0$, $\| \phi \|_{L^2(\mathbb{R}^d)} \leq 1$. Let us once more adopt the notation of the proof of 3.3. Then for $0 < \theta < 1$

$$\{ \int_\Sigma | E_x(e^{-MT_{t,\Gamma}}\phi(X_t) : \sigma < t) |^2 \, dx\}^{1/2}$$

$$\geq \{ \int_\Sigma | E_x(e^{-MT_{t,\Gamma}}\phi(X_t) : \theta t < \sigma < t, z \in \Gamma_0) |^2 \, dx\}^{1/2}$$

$$\geq \{ \int_\Sigma | E_x(E_z(e^{-MT_{(1-\theta)t,\Gamma}}\phi(X_{t-\sigma})) : \theta t < \sigma < t, z \in \Gamma_0) |^2 \, dx\}^{1/2}$$

Note that for any $y \in \mathbb{R}^d$

$$\phi = \inf_{B(y,1)} \phi + \{\phi - \inf_{B(y,1)} \phi\} \geq \inf_{B(y,1)} \phi - \{\inf_{B(y,1)} \phi - \phi\} 1_{B(y,1)^c}$$

This and the triangle inequality give $\geq I_1 - I_2$ where

$$I_1 := \{\int_\Sigma \mid E_x(E_z(e^{-MT_{(1-\theta)t,\Gamma}} \inf_{B(z,1)} \phi) : \theta t < \sigma < t, z \in \Gamma_0) \mid^2 dx\}^{1/2} \quad (31)$$

$$I_2 := \{\int_\Sigma \mid E_x(E_z(e^{-MT_{(1-\theta)t,\Gamma}}(\inf_{B(z,1)} \phi - \phi(X_{t-\sigma}))1_{B(z,1)^c} \circ X_{t-\sigma} : \theta t < \sigma < t, z \in \Gamma_0) \mid^2 dx\}^{1/2} \quad (32)$$

Define $g_t := P_\cdot(z \in \Gamma_0; \theta t < \sigma < t)$; then $g_t \leq P_\cdot(\sigma(\Gamma_{0,\epsilon}) < t) \leq e^t E_\cdot e^{-\sigma(\Gamma_{0,\epsilon})}$ where $\Gamma_{0,\epsilon}$ is the ϵ-neighbourhood of Γ_0. Let $(\mathcal{F}, \mathcal{E})$ be the Dirichlet form associated to $d-$dimensional Brownian motion. Since $\Gamma_{0,\epsilon}$ is relatively compact, $\Gamma_{0,\epsilon}$ has finite $\mathcal{E}_1-$capacity and moreover $p^1_{\Gamma_{0,\epsilon}} = E_\cdot e^{-\sigma(\Gamma_{0,\epsilon})}$ is a version of the equilibrium $1-$potential of $\Gamma_{0,\epsilon}$ (see [4]); in particular $p^1_{\Gamma_{0,\epsilon}} \in L^2(\mathbb{R}^d)$ so that $g_t \in L^2(\Sigma)$. Let $b_t := \| g_t \|_{L^2(\Sigma)}$. Since Γ is a uniform Lipschitz set, $b_t > 0$.

Define $\psi = \psi(M,t) := t^{-\gamma} M^{-Aq} \beta^{-1}$ where γ, q, β are undetermined positive constants, and $\phi := \psi 1_{B(x_0,R)}$ where x_0 is some point in Γ_0 and $R = R(M,t)$ is chosen so that $\| \phi \|_{L^2(\mathbb{R}^d)} = 1$ *i.e.* $R = c\psi^{-1/d}$ where $c > 0$ constant. For now we assume that

$$B(x,1) \subset B(x_0,R) \text{ for all } x \in \Gamma_0 \quad (33)$$

$$\bigcup_{x \in \Gamma_0} B(x,(1-\eta)R) \subset B(x_0,R) \text{ for some } 0 < \eta < 1 \quad (34)$$

We can estimate I_1 using proposition 2.5 and (33) by

$$I_1 \geq c b_t \psi (M(1-\theta)t)^{-A} \quad (35)$$

if $M(1-\theta)t > 2$. We now bound I_2 from above using (34) and [8] (3g) via

$$I_2 \leq \psi\{\int_\Sigma E_x(P_z(X_{t-\sigma} \notin B(x_0,R)) : \theta t < \sigma < t, z \in \Gamma_0)^2 dx\}^2$$

$$\leq \psi\{\int_\Sigma E_x(P_z(\sigma(B(z,(1-\eta)R)^c) < (1-\theta)t) : \theta t < \sigma < t, z \in \Gamma_0)^2 dx\}^{1/2}$$

$$\leq c b_t \psi e^{-(1-\eta)^2 R^2 / 2(1-\theta)t} \quad (36)$$

$$\leq c(\alpha+1)!b_t\psi\{\frac{(1-\eta)^2R^2}{(1-\theta)t}\}^{-\alpha} \tag{37}$$

From (35) and (37) we obtain

$$\| Je^{-tH_M}-e^{-tH_\Sigma}J \| \geq cb_t\psi(M(1-\theta)t)^{-A}-c'(\alpha+1)!b_t\psi(1-\eta)^{-2\alpha}(1-\theta)^{\alpha}R^{-2\alpha}t^{\alpha}$$

$$= cb_t t^{-\gamma}M^{-Aq}\beta^{-1}(M(1-\theta)t)^{-A}$$

$$-c'(\alpha+1)!b_t t^{-\gamma}M^{-Aq}\beta^{-1}(1-\eta)^{-2\alpha}(1-\theta)^{\alpha}(t^{-\gamma}M^{-Aq}\beta^{-1})^{2\alpha/d}t^{\alpha}$$

$$= cb_t\beta^{-1}(1-\theta)^{-A}t^{-\gamma-A}M^{-(1+q)A}$$

$$-c'(\alpha+1)!b_t\beta^{-1-2\alpha/d}(1-\eta)^{-2\alpha}(1-\theta)^{\alpha}t^{-\gamma+\alpha-2\alpha\gamma/d}M^{-Aq(1+2\alpha/d)}$$

For $q>0$, $0<\theta,\eta<1$ fixed, choose α, β, γ, and M large enough so that

$$(1+2\alpha/d)q = 1+q \tag{38}$$

$$\gamma + A < \gamma + 2\alpha\gamma/d - \alpha \tag{39}$$

$$\frac{1}{2}c\beta^{-1}(1-\theta)^{-A} = c'(\alpha+1)!\beta^{-1-2\alpha/d}(1-\eta)^{-2\alpha}(1-\theta)^{\alpha} \tag{40}$$

$$1+\delta(\Gamma_0) < R = c''\beta^{1/d}t^{\gamma/d}M^{Aq/d} \tag{41}$$

$$(1-\eta)R+\delta(\Gamma_0) < R \text{ i.e. } \eta^{-1}\delta(\Gamma_0) < R \tag{42}$$

Here (41) and (42) guarantee (33) and (34) respectively; $\delta(\Gamma_0)$ is the diameter of $\Gamma_0 \subset \partial\Gamma$. We thus have

$$\| Je^{-tH_M}-e^{-tH_\Sigma}J \| \geq \frac{1}{2}cb_t\beta^{-1}(1-\theta)^{-A}t^{-\gamma-A}M^{-(1+q)A} \tag{43}$$

$$= k(t)M^{-pA}$$

for $p=1+q>1$ and all large M. □

Theorem 4.2 *Assume (30). Let $p>1$. Then for each $0<a<\infty$ there exists a positive constant $c(a)$ such that for M sufficiently large*

$$\| J(a+H_M)^{-1}-(a+H_\Sigma)^{-1}J \| \geq c(a)M^{-pA}$$

Proof. Fix $t_0 \geq 1, h > 0$ and let $\psi := t_0^{-\gamma} M^{-Aq} \beta^{-1}$ where γ, q, β are undetermined positive constants and let $\phi = \psi 1_{B(x_0,R)}$ where x_0 is some point in Γ_0 and R is chosen so that $\| \phi \|_{L^2(\mathbf{R}^d)} = 1$. Let $\tau := \sigma(B(0,(1-\eta)R)^c)$ and suppose that

$$\bigcup_{z \in \Gamma_0} B(z,(1-\eta)R) \subset B(x_0, R) \tag{44}$$

for some $0 < \eta < 1$. We then have for $0 < \theta < 1$

$$\| J(a+H_M)^{-1} - (a+H_\Sigma)^{-1}J \| \geq \| (J(a+H_M)^{-1} - (a+H_\Sigma)^{-1}J)\phi \|_{L^2(\Sigma)}$$

$$= \{ \int_\Sigma (\int_0^\infty e^{-at} E_x(e^{-MT_{t,\Gamma}} \phi \circ X_t : \sigma < t) dt)^2 dx \}^{1/2}$$

$$\geq \{ \int_\Sigma (\int_{t_0}^{t_0+h} e^{-at} E_x(e^{-MT_{t_0+h,\Gamma}} \phi \circ X_t : \sigma < t_0) dt)^2 dx \}^{1/2}$$

$$\geq \{ \int_\Sigma (\int_{t_0}^{t_0+h} e^{-at} E_x(e^{-MT_{t_0+h,\Gamma}} \phi \circ X_t : \theta t_0 < \sigma < t_0, z \in \Gamma_0) dt)^2 dx \}^{1/2}$$

$$\geq \psi \{ \int_\Sigma (E_x(e^{-MT_{t_0+h,\Gamma}} (\int_{t_0}^{t_0+h} e^{-at} 1_{B(x_0,R)} \circ X_t dt) : \theta t_0 < \sigma < t_0, z \in \Gamma_0))^2 dx \}^{1/2}$$

$$\geq \psi \int_{t_0}^{t_0+h} e^{-at} dt \{ \int_\Sigma (E_x(e^{-MT_{t_0+h,\Gamma}} : \tau \circ \theta_\sigma > h + (1-\theta)t_0,$$

$$\theta t_0 < \sigma < t_0, z \in \Gamma_0))^2 dx \}^{1/2}$$

$$\geq c(a)\psi \{ \int_\Sigma (E_x(E_z e^{-MT_{(1-\theta)t_0+h,\Gamma}} : \tau > h+(1-\theta)t_0) : \theta t_0 < \sigma < t_0, z \in \Gamma_0))^2 dx \}^{1/2}$$

$$\geq c(a)\psi \{ \int_\Sigma (E_x(E_z e^{-MT_{(1-\theta)t_0+h,\Gamma}} : \theta t_0 < \sigma < t_0, z \in \Gamma_0))^2 dx \}^{1/2}$$

$$-c(a)\psi \{ \int_\Sigma (E_x(E_z e^{-MT_{(1-\theta)t_0+h,\Gamma}} : \tau < h+(1-\theta)t_0) : \theta t_0 < \sigma < t_0, z \in \Gamma_0))^2 dx \}^{1/2}$$

$$\geq c(a)\psi \{ \int_\Sigma (E_x(E_z e^{-MT_{(1-\theta)t_0+h,\Gamma}}) : \theta t_0 < \sigma < t_0, z \in \Gamma_0))^2 dx \}^{1/2}$$

$$-c(a)\psi \{ \int_\Sigma E_x(P_z(\sigma(B(z,(1-\eta)R)^c) < (1-\theta)t_0+h) : \theta t_0 < \sigma < t_0, z \in \Gamma_0)^2 dx \}^{1/2}$$

Denote the first and second terms of the last expression by I_1 and I_2 respectively and note the similarity to (31) and (36). Argueing as in the proof of 4.1 we obtain

$$I_1 \geq c(a)b_{t_0}\psi(M((1-\theta)t_0+h))^{-A}$$

and

$$I_2 \leq c(a)(\alpha+1)!b_{t_0}\psi\{\frac{(1-\eta)^2R^2}{(1-\theta)t_0+h}\}^{-\alpha}$$

if (44). Continuing as before

$$\begin{aligned}\| J(a+H_M)^{-1}-(a+H_\Sigma)^{-1}J \| &\geq c(a)b_{t_0}\psi(M((1-\theta)t_0+h))^{-A} \\ &-c(a)(\alpha+1)!b_{t_0}\psi\{\frac{(1-\eta)^2R^2}{(1-\theta)t_0+h}\}^{-\alpha} \\ &= c(a)b_{t_0}t_0^{-\gamma}M^{-Aq}\beta^{-1}(M((1-\theta)t_0+h))^{-A} \\ &-c'(a)(\alpha+1)!b_{t_0}t_0^{-\gamma}M^{-Aq}\beta^{-1}(1-\eta)^{-2\alpha}((1-\theta)t_0+h)^{\alpha}(t_0^{-\gamma}M^{-Aq}\beta^{-1})^{4\alpha/d} \\ &\geq c(a)b_{t_0}\beta^{-1}(1-\theta)^{-A}t_0^{-\gamma-A}M^{-(1+q)A} \\ &-c'(a)(\alpha+1)!b_{t_0}\beta^{-1-4\alpha/d}(1-\eta)^{-2\alpha}(1-\theta)^{\alpha}t_0^{-\gamma+\alpha-4\alpha\gamma/d}M^{-Aq(1+4\alpha/d)}\end{aligned}$$

For $q>0$, $0<\theta,\eta<1$ fixed, choose α,β,γ, and M large enough so that

$$(1+4\alpha/d)q = 1+q$$

$$\gamma + A < \gamma + 4\alpha\gamma/d - \alpha$$

$$\frac{1}{2}c(a)\beta^{-1}(1-\theta)^{-A} = c'(a)(\alpha+1)!\beta^{-1-4\alpha/d}(1-\eta)^{-2\alpha}(1-\theta)^{\alpha}$$

$$1+\delta(\Gamma_0) < R = c''\beta^{2/d}t_0^{2\gamma/d}M^{2Aq/d}$$

$$(1-\eta)R+\delta(\Gamma_0) < R \; i.e. \; \eta^{-1}\delta(\Gamma_0) < R$$

We thus have

$$\begin{aligned}&\| J(a+H_M)^{-1}-(a+H_\Sigma)^{-1}J \| \\ &\geq \frac{1}{2}c(a)b_{t_0}\beta^{-1}(1-\theta)^{-A}t_0^{-\gamma-A}M^{-(1+q)A} \\ &= k(a)M^{-pA}\end{aligned}$$

where $p = 1+q > 1$ and M is large. □

The above results are by no means optimal if the singularity region Γ is taken to be a cone of angle $\alpha > \pi$. The following results give an improvement in such cases.

Let Γ be a closed subset of $I\!R^d$ and suppose surface area measure τ is defined. Let us assume in addition that

> there exists a cone C, closed sets $\Gamma_0^n \subset \partial\Gamma$, $r_n > 0$, $n \in I\!N$, $r_n \to \infty$ with $\tau(\Gamma_0^n) = 1$, $\delta(\Gamma_0^n) = 1$ such that for each n and $z \in \Gamma_0^n$ there exists a rigid transformation S_z of $I\!R^d$ such that
>
> $$(S_z C^c - \{z\}) \cap B(z, r_n) \subset \Sigma; \tag{45}$$
>
> moreover, the function $g_t^n := P_\cdot(z \in \Gamma_0^n, \theta t < \sigma < t)$, $0 < t < \infty$, $0 < \theta < 1$ satisfies $\inf_n b_t^n > 0$ where $b_t^n := \| g_t^n \|_{L^2(\Sigma)}$.

Theorem 4.3 *Let Γ be a singularity region satisfying (45). Fix $t \geq 1$. Given $p > 1$ there exists a constant $k(t)$ such that*

$$\| Je^{-tH_M} - e^{-tH_\Sigma} J \| \geq k(t) M^{-pA} \ (A = A(C))$$

for all sufficiently large $M > 0$.

Proof. Define $\tau_n := \sigma(B(0, r_n)^c)$. For $\phi \geq 0$, $\phi \in L^2(I\!R^d)$, $\| \phi \|_{L^2(I\!R^d)} = 1$ we have

$$\| Je^{-tH_M} - e^{-tH_\Sigma} J \| \geq$$

$$\{ \int_\Sigma E_x(E_z(e^{-MT_{(1-\theta)t,\Gamma}} \phi(X_{t-\sigma})) : \theta t < \sigma < t, z \in \Gamma_0^n)^2 dx \}^{1/2}$$

$$\geq \{ \int_\Sigma E_x(E_z(e^{-MT_{(1-\theta)t,\Gamma}} \phi(X_{t-\sigma}) : \tau_n > (1-\theta)t) : \theta t < \sigma < t, z \in \Gamma_0^n)^2 dx \}^{1/2}$$

$$\geq \{ \int_\Sigma E_x(E_z(e^{-MT_{(1-\theta)t,S_z(C)}} \phi(X_{t-\sigma}) : \tau_n > (1-\theta)t) : \theta t < \sigma < t, z \in \Gamma_0^n)^2 dx \}^{1/2}$$

$$\geq \{ \int_\Sigma E_x(E_z(e^{-MT_{(1-\theta)t,S_z(C)}} \phi(X_{t-\sigma})) : \theta t < \sigma < t, z \in \Gamma_0^n)^2 dx \}^{1/2}$$

$$- \{ \int_\Sigma E_x(E_z(e^{-MT_{(1-\theta)t,S_z(C)}} \phi(X_{t-\sigma}) : \tau_n < (1-\theta)t) : \theta t < \sigma < t, z \in \Gamma_0^n)^2 dx \}^{1/2}$$

$$\geq \{ \int_\Sigma (E_z(e^{-MT_{(1-\theta)t,S_z(C)}} \phi(X_{t-\sigma})) : \theta t < \sigma < t, z \in \Gamma_0^n)^2 dx \}^{1/2}$$

$$- ce^{-r_n^2/(1-\theta)t} b_t^n \tag{46}$$

Denoting the first term in the last expression by I we have as in 4.1 that $I^n \geq I_1^n - I_2^n$ where

$$I_1^n := \{ \int_\Sigma E_x(E_z(e^{-MT_{(1-\theta)t,S_z(C)}} \inf_{B(z,1)} \phi) : \theta t < \sigma < t, z \in \Gamma_0^n)^2 dx\}^{1/2}$$

$$I_2^n := \{ \int_\Sigma E_x(E_z(e^{-MT_{(1-\theta)t,S_z(C)}}(\inf_{B(z,1)} \phi - \phi(X_{t-\sigma}))1_{B(z,1)^c} \circ X_{t-\sigma} :$$

$$\theta t < \sigma < t, z \in \Gamma_0^n)^2 dx\}^{1/2} - ce^{-r(n)^2/(1-\eta)t} \| g_t^n \|_{L^2(\Sigma)}$$

Using exactly the same argument as in 4.1 we obtain

$$I \geq \frac{1}{2} c b_t^n \beta^{-1}(1-\theta)^{-A} t^{-\gamma-A} M^{-(1+q)A}$$

The constants $c, \beta, \theta, \gamma, q$ can be chosen independently of n. By hypothesis b_t^n is uniformly bounded below. Thus the above inequality is valid for all M bigger than some number not depending on n. So $I \geq k(t)M^{-pA}$ where $p = 1 + q > 1$. Finally

$$\| Je^{-tH_M} - e^{-tH_\Sigma} J \| \geq k(t)M^{-pA} - ce^{-r_n^2/(1-\theta)t} b_t^n$$

valid for all n. Since b_t^n is uniformly bounded in n, we let $n \to \infty$ to get the result. □

Theorem 4.4 *Let Γ satisfy (45). Then for each $0 < a < \infty$ there exists a positive constant $c(a)$ such that for M sufficiently large*

$$\| J(a + H_M)^{-1} - (a + H_\Sigma)^{-1} J \| \geq c(a)M^{-pA} \ (A = A(C))$$

Proof. As in 4.2 we have

$$\| J(a + H_M)^{-1} - (a + H_\Sigma)^{-1} J \| \geq$$

$$c(a)\{ \int_\Sigma (E_x(E_z e^{-MT_{(1-\theta)t_0+h,\Gamma_0^n}} \inf_{B(z,1)} \phi) : \theta t_0 < \sigma < t_0, z \in \Gamma_0^n))^2 dx\}^{1/2}$$

$$-c(a)\psi\{ \int_\Sigma E_x(P_z(\sigma(B(z,(1-\eta)R)^c) < (1-\theta)t_0+h) : \theta t_0 < \sigma < t_0, z \in \Gamma_0^n)^2 dx\}^{1/2}$$

Denoting as in 4.3 $\tau_n = \sigma(B(0, r_n)^c)$ the first right-hand term may be estimated

$$\geq c(a)\{\int_\Sigma (E_x(E_z e^{-MT_{(1-\theta)t_0+h,\Gamma_0^n}} \inf_{B(z,1)} : \tau_n > (1-\theta)t_0 + h) :$$

$$\theta t_0 < \sigma < t_0, z \in \Gamma_0^n)^2 dx\}^{1/2}$$

$$\geq c(a)\{\int_\Sigma (E_x(E_z e^{-MT_{(1-\theta)t_0+h,S_z(C)}} \inf_{B(z,1)} : \tau_n > (1-\theta)t_0 + h) :$$

$$\theta t_0 < \sigma < t_0, z \in \Gamma_0^n)^2 dx\}^{1/2}$$

$$\geq c(a)\{\int_\Sigma (E_x(E_z e^{-MT_{(1-\theta)t_0+h,\Gamma_0^n}} \inf_{B(z,1)}) : \theta t_0 < \sigma < t_0, z \in \Gamma_0^n)^2 dx\}^{1/2}$$

$$-c(a)\{\int_\Sigma (E_x(E_z e^{-MT_{(1-\theta)t_0+h,\Gamma_0^n}} \inf_{B(z,1)} : \tau_n \leq (1-\theta)t_0 + h) :$$

$$\theta t_0 < \sigma < t_0, z \in \Gamma_0^n)^2 dx\}^{1/2}$$

$$\geq c(a)\{\int_\Sigma (E_x(E_z e^{-MT_{(1-\theta)t_0+h,\Gamma_0^n}} \inf_{B(z,1)}) : \theta t_0 < \sigma < t_0, z \in \Gamma_0^n)^2 dx\}^{1/2}$$

$$-c(a)e^{-r_n^2/((1-\theta)t_0+h)}$$

We thus have

$$\| J(a + H_M)^{-1} - (a + H_\Sigma)^{-1} J \| \geq$$

$$c(a)\{\int_\Sigma (E_x(E_z e^{-MT_{(1-\theta)t_0+h,\Gamma_0^n}} \inf_{B(z,1)}) : \theta t_0 < \sigma < t_0, z \in \Gamma_0^n)^2 dx\}^{1/2}$$

$$-c(a)\psi\{\int_\Sigma E_x(P_z(\sigma(B(z,(1-\eta)R)^c) < (1-\theta)t_0 + h) :$$

$$\theta t_0 < \sigma < t_0, z \in \Gamma_0^n)^2 dx\}^{1/2}$$

$$-c(a)e^{-r_n^2/((1-\theta)t_0+h)}$$

Now argue as in 2.3 to obtain the conclusion. □
Note that any cone Γ satisfies (45) for $C = H$, the half-plane so it follows immediately

Corollary 4.5 *Let Γ be a cone of angle α, $0 < \alpha < 2\pi$. Fix $t \geq 1$. Given $p > 1$ there exists a constant $k(t)$ such that*

$$\| Je^{-tH_M} - e^{-tH_\Sigma}J \| \geq k(t)M^{-p/2}$$

for all sufficiently large M. For each $0 < a < \infty$ there exists $c(a) > 0$ such ahat for all large M

$$\| J(a + H_M)^{-1} - (a + H_\Sigma)^{-1}J \| \geq c(a)M^{-p/2}$$

Conjecture. P.Duclos [3] has shown

$$\| J(a + H_M)^{-1} - (a + H_\Sigma)^{-1}J \| \leq c(a)M^{-1/2}$$

where Γ is any cone. Together with the last result the expected rate of decay is $O(M^{-1/2})$. In the light of 3.4 one would expect this also for a large class of Lipschitz domains.

5 Application to decay of eigenfunctions

Let B be an open ball in $\mathbb{R}^d$, $d \geq 2$ centred at the origin, and C be a cone. Let H_M be the operator $H_M := -\frac{1}{2}\Delta + M1_C$, $M > 0$ on $L^2(B)$ with Dirichlet boundary conditions on ∂B. The operator H_M has eigenvalues $0 \leq \lambda_M^n \leq \lambda_M^{n+1}$, $n = 1, 2, \ldots$ and $\lambda_M^n \leq \lambda^n$ where λ^n is the n-th eigenvalue of $H := -\frac{1}{2}\Delta$ on $L^2(B)$ with Dirichlet boundary conditions on ∂B. Let ϕ_M^n be a normalised L^2–eigenvector with eigenvalue λ_M^n. Denote by T_t^M the L^2–heat semi-group for H_M. By [1], T_t^M has an integral kernel $p^M(t, x, y)$ which is jointly continuous on $(0, \infty) \times B \times B$ and $T_t^M\phi_M^n$ is continuous so that $\phi_M^n = e^{\lambda_M^n t}T_t^M\phi_M^n$ is continuous.

Proposition 5.1 *Fix n. There exists a positive constant $c(n)$ such that*

$$| \phi_M^n(0) | \leq c(n)(\frac{\ln M}{M})^A, M > e$$

where $A = A(C)$ is the cone constant.

Proof. We first estimate $| \phi_M^n |$ as follows.

$$\phi_M^n(x) = e^{\lambda_M^n} \int_B {,}^M (1, x, y)\phi_M^n(y)dy \tag{47}$$

$$\leq e^{\lambda_M^n}\{\int_B p^M(1,x,y)^2 dy\}^{1/2}$$

$$\leq e^{\lambda_M^n}\{\int_B p(1,x,y)^2 dy\}^{1/2}$$

where $p(t,x,y)$ is the integral kernel of the heat semi-group associated to H the last line following by the Feynman-Kac representation of T_t^M. Thus $\mid \phi_M^n(x) \mid\leq c(n)$. It then follows from (47) that

$$\mid \phi_M^n(0) \mid\leq c(n)\int_B p^M(1,0,y)dy$$

$$= c(n)E_0(e^{-MT_{1,C}} : \sigma > 1)$$

$$\leq c(n)E_0 e^{-MT_{1,C}}$$

$$\leq c(n)(\frac{\ln M}{M})^A$$

Here σ is the first hitting time of B^c. □

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