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by

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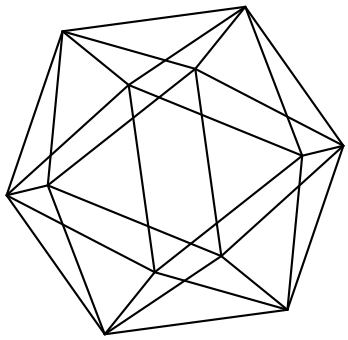

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SECTIONS OF QUADRICS OVER $\mathbb{A}^1_{\mathbb{F}_q}$

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Abstract. Given finitely many closed points in distinct fibers of a non-degenerate quadric over $\mathbb{A}^1_{\mathbb{F}_q}$, we ask for conditions under which there is a section passing through the closed points, possibly with higher order (nilpotence) conditions. This could be thought of as a quadratic version of Lagrange interpolation, and it is equivalent to proving strong approximation for non-degenerate quadrics over $\mathbb{F}_q[t]$. We show that under mild conditions on the quadratic form F over $\mathbb{F}_q[t]$ in d variables, $f, g \in \mathbb{F}_q[t]$, $\boldsymbol{\lambda} \in \mathbb{F}_q[t]^d$, if $d \geq 5$ then for $\deg f \geq (4+\varepsilon)\deg g + O(1)$ we have a solution $\mathbf{x} \in \mathbb{F}_q[t]^d$ to $F(\mathbf{x}) = f$ such that $\mathbf{x} \equiv \boldsymbol{\lambda} \bmod g$, where the the big-Oh notation does not depend on $f, g, \boldsymbol{\lambda}$. For $d = 4$, we show the same is true for $\deg f \geq (6+\varepsilon)\deg g + O(1)$. This gives us a new proof (independent of the Ramanujan conjecture over function fields proved by Drinfeld) that the diameter of any k-regular Morgenstern Ramanujan graphs G is at most $(2+\varepsilon)\log_{k-1}|G| + O_\varepsilon(1)$. In contrast to the $d = 4$ case, our result is optimal for $d \geq 5$. Along the way, we prove a stationary phase theorem over function fields that is of independent interest.

1. Introduction

1.1. **Motivation.** We begin by considering a natural geometric problem regarding quadratic forms over $\mathbb{F}_q[t]$. Suppose F is a quadratic form in d variables over $\mathbb{F}_q[t]$. Suppose f is a polynomial in $\mathbb{F}_q[t]$. We may then consider the affine variety X_f given by setting $F(\mathbf{x}) = f$, $\mathbf{x} \in \mathbb{A}^d_{\mathbb{F}_q[t]}$. We may view this as a family $\pi : X_f \to \mathbb{A}^1_{\mathbb{F}_q}$ over $\mathbb{A}^1_{\mathbb{F}_q}$. Suppose we have a collection of closed points $p_1, \dots, p_m$ in $\mathbb{A}^1_{\mathbb{F}_q}$. Choose, for each i, a point $\boldsymbol{\lambda}_i := (\lambda_1(p_i), \dots, \lambda_d(p_i))$ in the fiber $X_{p_i} := X_f \times_{\mathbb{F}_q[t]} \kappa(p_i)$ over p_i. Can we find a section $s : \mathbb{A}^1_{\mathbb{F}_q} \to X_f$ of the structure morphism $\pi : X_f \to \mathbb{A}^1_{\mathbb{F}_q}$ that maps each p_i to $\boldsymbol{\lambda}_i$ with some prescribed higher order (nilpotence) conditions of order m_i? This problem could be thought of as a quadratic version of the classical Lagrange interpolation. We show that if F is non-degenerate in $d \geq 5$ variables, then there is such a section provided that $\deg f \geq (4+\varepsilon)\sum_i m_i \deg p_i + O_{\varepsilon,F}(1)$, where the implied constant depends only on ε and the quadratic form F (in fact, we show a stronger result depending on anisotropic cones defined in definition 1.1). We also show that this condition is optimal. On the other hand, if $d = 4$, we show that this is true at least if $\deg f \geq (6+\varepsilon)\sum_i m_i \deg p_i + O_{\varepsilon,F}(1)$. That being said, we conjecture that $4+\varepsilon$ still suffices in the $d = 4$ case. In fact, as can be found in another paper by the two authors [TZ19], we have shown that the optimality of $4+\varepsilon$ when working with the class of quadratic forms in the construction of Morgenstern Ramanujan graphs follows from a twisted Linnik-Selberg conjecture over function fields. That paper relies heavily on the computations and techniques

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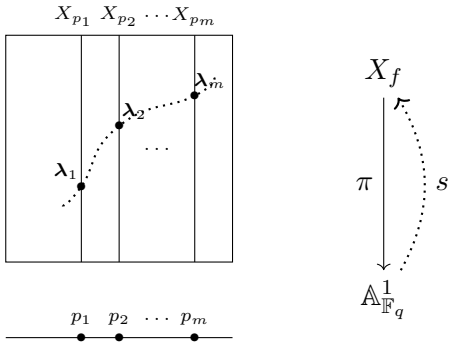
developed in this paper. The setup of the problem is pictorially represented by the following figure.



There is another more algebraic way of formulating the problem; in fact, this other formulation is more common. By packaging all the closed points p_i and all their multiplicities $m_i > 0$ (which can be viewed as irreducible polynomials $p_i(t)$ in $\mathbb{F}_q[t]$ raised to the power of m_i) into one polynomial $g(t) := \prod_i p_i(t)^{m_i}$, we can use the Chinese remainder theorem to reformulate the problem as an optimal strong approximation problem for quadratic forms over function fields. More precisely, we ask for the following. Suppose we have a quadratic form F in d variable over $\mathbb{F}_q[t]$, and polynomials $g, f \in \mathbb{F}_q[t]$. Additionally, we are given polynomials $\lambda_1, \dots, \lambda_d \in \mathbb{F}_q[t]$. We want to know when we have an integral solution $\mathbf{x} := (x_1, \dots, x_d) \in \mathbb{F}_q[t]^d$ to the system

$$\begin{cases} F(\mathbf{x}) = f, \\ \mathbf{x} \equiv \boldsymbol{\lambda} \bmod g, \end{cases} \tag{1}$$

where $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_d)$ and $\mathbf{x} \equiv \boldsymbol{\lambda} \bmod g$ means $x_i \equiv \lambda_i \bmod g$ for every $1 \le i \le d$. For a prime ideal ϖ of $\mathbb{F}_q[t]$, we write $\mathbb{F}_q[t]_\varpi$ for the completion of $\mathbb{F}_q[t]$ at ϖ. We say all local conditions for the system (1) are satisfied, if $X_f(K_\infty) := \{\mathbf{x} \in K_\infty^d : F(\mathbf{x}) = f\} \neq \emptyset$ and $F(\mathbf{x}) = f$ has a local solution $\mathbf{x}_\varpi \in \mathbb{F}_q[t]_\varpi^d$ for all prime ideals ϖ of $\mathbb{F}_q[t]$ such that $\mathbf{x}_\varpi \equiv \boldsymbol{\lambda} \bmod \varpi^{\mathrm{ord}_\varpi(g)}$. In the following $K_\infty := \mathbb{F}_q(\!(1/t)\!)$, $\widehat{(-)} := q^{(-)}$. K_∞^d is equipped with the norm $|\mathbf{x}| := \max_i |x_i|$ for any $\mathbf{x} = (x_1, \dots, x_d) \in K_\infty^d$. Consider the following definition.

Definition 1.1 (Anisotropic cone)**.** We say $\Omega \subset K_\infty^d$ is an anisotropic cone with respect to the quadratic form $F(\mathbf{x})$ if there exists fixed positive integers ω and ω' such that:

(1) If $\mathbf{x} \in \Omega$ then $f\mathbf{x} \in \Omega$ for every $f \in K_\infty$.
(2) If $\mathbf{x} \in \Omega$ and $\mathbf{y} \in K_\infty^d$ with $|\mathbf{y}| \le |\mathbf{x}|/\widehat{\omega}$, then $\mathbf{x} + \mathbf{y} \in \Omega$.
(3) $\widehat{\omega'}|F(\mathbf{x})| \ge |\mathbf{x}|^2$.

Remark 2. Whenever considering the equation $F(\mathbf{x}) = f$ along with an anisotropic cone Ω, we assume that $\Omega \cap X_f(K_\infty) \neq \emptyset$.

The main result of this paper is the following theorem.

Theorem 1.2. *Suppose q is a power of a fixed odd prime number, and let F be a non-degenerate quadratic form over $\mathbb{F}_q[t]$ in $d \ge 4$ variables and of discriminant Δ. Let $f, g \in \mathbb{F}_q[t]$ be nonzero polynomials such that $(f\Delta, g) = 1$, and let $\boldsymbol{\lambda} \in \mathbb{F}_q[t]^d$ be a d-tuple of polynomials at least one of whose coordinates is relatively prime to g. Finally, suppose that all local conditions for the system (1) are satisfied and $\Omega \cap X_f(K_\infty) \neq \emptyset$. If $d \ge 5$, then for any anisotropic cone Ω and for $\deg f \ge (4+\varepsilon)\deg g + O_{\varepsilon,F,\Omega}(1)$, there is a solution $\mathbf{x} \in \Omega \cap \mathbb{F}_q[t]^d$ to (1). If $d = 4$, this holds at least for $\deg f \ge (6+\varepsilon)\deg g + O_{\varepsilon,F,\Omega}(1)$.*

As a corollary, we obtain the following strong approximation result.

Corollary 1.3 (Strong approximation)**.** *With the notation as above, if $d \geq 5$ and all local conditions to the system* (1) *are satisfied, for* $\deg f \geq (4+\varepsilon)\deg g + O_{\varepsilon,F}(1)$*, there is a solution* $\mathbf{x} \in \mathbb{F}_q[t]^d$ *to* (1)*. If $d = 4$, this holds at least for* $\deg f \geq (6+\varepsilon)\deg g + O_{\varepsilon,F}(1)$.

In order to obtain this corollary, the main theorem 1.2 implies that it suffices to show that we can cover K_∞^d by finitely many anisotropic cones such that, for any given f, $X_f(K_\infty)$ intersects at least one of them. See Lemma 5.1 for a proof of this.

Before discussing the optimality our main theorem for $d \geq 5$, let us make some remarks regarding its proof. Though the proof uses the function field analogue of the circle method to prove the analogue of the theorem over the integers proved in the first author's paper [T. 19a], there are differences between the two papers. Though our theorem proves the stronger statement that a solution exists in an anisotropic cone, restricting to such an anisotropic cone is essential in our proof. Choosing a weight function centered at the origin will not suffice for isotropic quadratic forms (which are plentiful in positive characteristic); this would lead to suboptimal results even for $d \geq 5$. In order to obtain optimal results for $d \geq 5$, it is essential that we choose appropriate weighted sum of solutions within anisotropic cones. In order to deduce strong approximation for F, we show in Lemma 5.1 that for each f we can construct an anisotropic cone depending only on the class of f in $K_\infty^\times/K_\infty^{\times 2}$ such that $X_f(K_\infty)\cap\Omega \neq \emptyset$. This is one technicality that arises when working over positive characteristics as opposed to over $\mathbb{Q}$. Additionally, most of the proofs of the central results in the function field case are necessarily different than the ones over the integers. One of the main differences between the two papers is that in order to compute the oscillatory integrals, a stationary phase theorem over function fields had to be developed which is of independent interest. A feature of the function field case is that by using this stationary phase theorem, we can determine the oscillatory integrals in terms of (a complicated expression involving) Kloosterman sums at the infinite place. Another difference with the integer case is that we had to use a different method for the computation of the contribution of the main term. Moreover, due to a lack of literature on exponential sums over function fields with non-irreducible modulus, we had to work a little harder to obtain the necessary desired bounds regarding exponential sums. Along the way, we give proofs of the function-field analogues of the results of Heath-Brown (see [HB96a]) needed for the circle method in this setting. The tools developed in this paper are used in another paper of the authors in order to study the diameter of Morgenstern Ramanujan graphs [TZ19].

Remark 3. For $F(\mathbf{x}) = x_1^2 + \ldots + x_d^2$, we can take $\Omega = \{\mathbf{x} \in K_\infty^d : \forall i,\ \deg x_1 > \deg x_i\}$. Note that when $\deg f \leq 4\deg g - 3$, then the system need not have a solution in $\mathbb{F}_q[t]^d \cap \Omega$. For instance, when $\boldsymbol{\lambda} = (1, 0, \ldots, 0)$ and $f \equiv 1 + 2t^{\deg g - 1} g \bmod g^2$, then a solution implies the existence of $(t_1, \ldots, t_d) \in \mathbb{F}_q[t]^d$ such that

$$(1 + t_1 g)^2 + (t_2 g)^2 + \ldots + (t_d g)^2 \equiv 1 + 2t^{\deg g - 1} g \bmod g^2,$$

that is, $t_1 \equiv t^{\deg g - 1} \bmod g$. Since the solution is in Ω, the degree of f is equal to the degree of $(1 + t_1 g)^2$, and so $\deg f \geq 2(2\deg g - 1) = 4\deg g - 2$. This shows that the factor $4 + \varepsilon$ is optimal for $d \geq 5$, and is the best possible factor for $d = 4$. In fact, we conjecture that it is also optimal for $d = 4$.

Conjecture 1.4. *For $d = 4$ in Theorem 1.2, if* $\deg f \geq (4+\varepsilon)\deg g + O_{\varepsilon,F,\Omega}(1)$*, the same conclusion holds. In other words, the factor $4 + \varepsilon$ is optimal for all $d \geq 4$.*

Let us comment on why there is a difference between the $d = 4$ case and the $d \geq 5$ case, and argue why proving the optimal result for $d = 4$ is difficult, even in the case of function fields. In Proposition 7.1, we show that the error term of the counting function with respect to the main

contribution satisfies the following bound:

$$\left|\sum_{1\leq|r|\leq\widehat{Q}}\sum_{\mathbf{c}\neq 0}^{\mathrm{exc}}|gr|^{-d}S_{g,r}(\mathbf{c})I_{g,r}(\mathbf{c})\right|\leq\sum_{1\leq|r|\leq\widehat{Q}}\sum_{\mathbf{c}\neq 0}^{\mathrm{exc}}|gr|^{-d}|S_{g,r}(\mathbf{c})||I_{g,r}(\mathbf{c})|\ll_{\varepsilon}\widehat{Q}^{\frac{d+3}{2}+\varepsilon}|g|^{\frac{d-3}{2}+\varepsilon}(1+|g|^{-\frac{d-5}{2}+\varepsilon}),$$

where $\sum^{\mathrm{exc}}$ denotes summation over exceptional vectors (we do not give the definition of *exceptional* vectors here). When $d\geq 5$, we have $1+|g|^{-\frac{d-5}{2}+\varepsilon}=O(1)$, while when $d=4$ this is of order $|g|^{1/2}$ which forces upon us a weaker bound on the error and so a suboptimal result in this case. See the proof of the main theorem on the final page for the precise reason. We remark, however, that by using the triangle inequality in the above sum, we seem to be losing some extra cancellation that would lead to an improved version of the $d=4$ case; we are only using the Weil bound and not using a possible cancellation in the sums of Kloosterman sums themselves. For example, as can be found in the paper [TZ19] by the two authors, once we restrict to the Morgenstern quadratic forms (in 4 variables), we can reduce the optimality to a *twisted* version of the Linnik-Selberg conjecture (Conjecture 1.4 of *loc.cit*) which we suspect is true. Over function fields, the classical Linnik-Selberg conjecture is true and is equivalent to the Ramanujan conjecture proved by Drinfeld. See the work of Cogdell and Piatetski-Shapiro [CPS90] for a proof of this. Our twisted Linnik-Selberg conjecture is a generalization, and does not seem to easily follow from the usual Ramanujan conjecture. As pointed out in that paper, if we have a more complicated quadratic form in $d=4$ variables, then even the reduction to a natural cancellation similar to the Linnik-Selberg conjecture does not seem possible. This partially attests to the difficulty of obtaining the optimal result for $d=4$. That being said, since the Ramanujan conjecture over $\mathbb{F}_q(t)$ is, in contrast to that over $\mathbb{Q}$, proved, there is greater hope of proving such a result over function fields. This strong approximation for Morgenstern quadratic forms is intimately connected with the diameter of Morgenstern quadratic forms. We now discuss the connection of strong approximation for quadratic forms to Ramanujan graphs.

As mentioned above, another motivation for the consideration of this problem is related to the construction of Ramanujan graphs with optimal diameters. We begin by defining Ramanujan graphs. Fix an integer $k\geq 3$, and let G be a k-regular connected graph with the adjacency matrix A_G. It follows that k is an eigenvalue of A_G. Let λ_G be the maximum of the absolute value of all the other eigenvalues of A_G. By the Alon-Boppana Theorem [LPS88], $\lambda_G\geq 2\sqrt{k-1}+o(1)$, where $o(1)$ goes to zero as $|G|\to\infty$. We say that G is a Ramanujan graph if $\lambda_G\leq 2\sqrt{k-1}$.

The first explicit construction of Ramanujan graphs is due to Lubotzky-Phillips-Sarnak [LPS88], and independently by Margulis [Mar88]. It is a Cayley graph of $\mathrm{PGL}_2(\mathbb{Z}/q\mathbb{Z})$ or $\mathrm{PSL}_2(\mathbb{Z}/q\mathbb{Z})$ with $p+1$ explicit generators for every prime p and integer q. The optimal spectral gap on the LPS construction is a consequence of the Ramanujan bound on the Fourier coefficients of the weight 2 holomorphic modular forms, which justifies their naming. We refer the reader to [Sar90, Chapter 3], where a complete history of the construction of Ramanujan graphs and other extremal properties of them are recorded. In particular, Lubotzky-Phillips-Sarnak proved that the diameter of every k-regular Ramanujan graph G is bounded by $2\log_{k-1}|G|+O(1)$. This is still the best known upper bound on the diameter of a Ramanujan graph. It was conjectured that the diameter is bounded by $(1+\varepsilon)\log_{k-1}|G|$ as $|G|\to\infty$; see [Sar90, Chapter 3]. However, the first author proved that for some infinite families of LPS Ramanujan graphs the diameter is bigger than $4/3\log_{k-1}|G|+O(1)$; see [T. 18]. The first author has conjectured that the diameter of the LPS Ramanujan graphs is asymptotically $4/3\log_{k-1}|G|+o(\log_{k-1}|G|)$; the upper bound follows from an optimal strong approximation conjecture for integral quadratic forms in 4 variables; see [T. 19a, Conjecture 1.3]. The following theorem of Lubotzky-Phillips-Sarnak links the diameter of the LPS Ramanujan graphs to the strong approximation on the sphere.

Theorem 1.5 (Due to Lubotzky-Phillips-Sarnak [LPS88]). *Let* $v := \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \in G$, *where* G *is the LPS Ramanujan graph associated to* p *and* q. *There is a bijection between non-backtracking paths* $(v_0, \ldots, v_h)$ *of length* h *from* $v_0 = id$ *to* $v_h = v$ *in* G, *and the set of integral solutions to the following diophantine equation*

$$\begin{aligned} & x_1^2 + x_2^2 + x_3^2 + x_4^2 = N, \\ & \begin{bmatrix} x_1 + ix_2 & x_3 + ix_4 \\ -x_3 - ix_4 & x_1 - ix_2 \end{bmatrix} \equiv \lambda \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \ mod\ 2q \text{ for some } \lambda \in \mathbb{Z}/2q, \end{aligned} \tag{4}$$

where $N = p^h$. *In particular, the distance between* id *and* v *in* G *is the smallest exponent* h *such that* (4) *has an integral solution.*

We state a version of the optimal strong approximation conjecture for the sphere, which when combined with this theorem implies that the diameter of LPS Ramanujan graphs is at most $(\frac{4}{3} + \varepsilon) \log_{k-1} |G| + O_\varepsilon(1)$; see [RS17, T. 17] for further numerical evidence regarding this conjecture.

Conjecture 1.6. *Suppose that* N, m *and* $\lambda_1, \ldots, \lambda_4$ *are given integers such that*

$$N \equiv \sum_{i=1}^{4} \lambda_i^2 \mod m.$$

Assuming that $N \gg m^{4+\varepsilon}$, *there exists an integral solution* $(x_1, \ldots, x_4)$ *to the system*

$$\begin{aligned} & x_1^2 + x_2^2 + x_3^2 + x_4^2 = N, \\ & x_l \equiv \lambda_l \ mod\ m \text{ for } 1 \leq l \leq 4. \end{aligned}$$

This conjecture is inspired by the conjecture of Sarnak on the distribution of integral points on the sphere S^3. Indeed, given $R > 0$ such that $R^2 \in \mathbb{Z}$, we let $C(R)$ denote the maximum volume of any cap on the $(d-1)$-dimensional sphere $S^{d-1}(R)$ of radius R which contains no integral points. Sarnak defined [Sar15] the covering exponent of integral points on the sphere by:

$$K_d := \limsup_{R \to \infty} \frac{\log \left(\# S^{d-1}(R) \cap \mathbb{Z}^d \right)}{\log \left(\text{vol } S^{d-1}(R)/C(R) \right)}.$$

In his letter [Sar15] to Aaronson and Pollington, Sarnak showed that $4/3 \leq K_4 \leq 2$. To show that $K_4 \leq 2$, he appealed to the Ramanujan bound on the Fourier coefficients of weight k modular forms, while the lower bound $4/3 \leq K_4$ is a consequence of an elementary number theory argument. Furthermore, Sarnak states some open problems [Sar15, Page 24]. The first one is to show that $K_4 < 2$ or even that $K_4 = 4/3$.

It follows from Theorem 1.8 and Corollary 1.9 of [T. 19a] that $K_d = 2 - \frac{2}{d-1}$ for $d \geq 5$ and $4/3 \leq K_4 \leq 2$; see also [T. 19b] for bounds on the average covering exponent. Browning-Kumaraswamy-Steiner [BKS17] showed that $K_4 = 4/3$, subject to the validity of a twisted version of a conjecture of Linnik about cancellation in sums of Kloosterman sums; see also Remark 6.8 of [T. 19a]. We have shown, as will appear in a forthcoming paper, that a twisted version of the Linnik-Selberg conjecture proves the optimal bound for the diameter of Morgenstern Ramanujan graphs. Since the untwisted version of the Linnik-Selberg conjecture over function fields has already been proved using the Ramanujan conjecture over function fields (proved by Drinfeld), we are hopeful that we will be able to prove the desired twisted version of the conjecture. We will discuss this connection in a future paper.

That being said, our main Theorem 1.2 above can be used to a new proof, independent of the Ramanujan conjecture over function fields, that the diameter of k-regular Morgenstern Ramanujan

graphs G are bounded above by $(2+\varepsilon)\log_{k-1}|G| + O_\varepsilon(1)$. Let us first recall the construction of Ramanujan graphs due to Morgenstern.

Consider the quaternion algebra

$$\mathcal{A} := k\mathbf{1} + k\mathbf{i} + k\mathbf{j} + k\mathbf{ij},\ \mathbf{i}^2 = \nu,\ \mathbf{j}^2 = x-1,\ \mathbf{ij} = -\mathbf{ji},$$

where ν is not a square in $\mathbb{F}_q$, and $k := \mathbb{F}_q(t)$. Let us assume that q is odd. The quaternion algebra we should take for even q can be found in Section 5 of Morgenstern's paper [Mor94]. Let

$$\mathcal{S} := \mathbb{F}_q[t]\mathbf{1} + \mathbb{F}_q[t]\mathbf{i} + \mathbb{F}_q[t]\mathbf{j} + \mathbb{F}_q[t]\mathbf{ij}$$

be the integral part of $\mathcal{A}$. Given $\xi = a+b\mathbf{i}+c\mathbf{j}+d\mathbf{ij}$ in $\mathcal{A}$, its conjugate is defined as $\overline{\xi} := a-b\mathbf{i}-c\mathbf{j}-d\mathbf{ij}$. Furthermore, we have the norm

$$N(\xi) := \xi\overline{\xi} = a^2 - b^2\nu + (d^2\nu - c^2)(t-1).$$

As can be found in Lemmas 4.2 and 4.4 of Morgenstern's [Mor94], it is possible to construct elements $\xi_1, \dots, \xi_{q+1}$ of norm t (called elements of *basic norm* t) such that every element x of $\mathcal{S}$ such that $N(x) = t^n$ has the unique factorization

$$x = t^r u\theta_1 \dots \theta_m,$$

where $2r+m = n$, $N(u) = 1$, θ_i are basic norm t, and t does not divide $\theta_1 \dots \theta_m$. Theorem 5.5 of Morgenstern's [Mor94] states that such a $x = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{ij}$ in $\mathcal{S}$ of norm t^n is a multiple of basic norms t if and only if $a-1, b \equiv 0 \bmod t-1$. Define

$$\Lambda(t-1) := \left\{ x = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{ij} \in \mathcal{S} : \begin{array}{c} a-1, b \equiv 0 \bmod t-1, \\ N(x) \text{ is a power of } t, \\ t \text{ does not divide } x \end{array} \right\}.$$

From the above discussion, it follows that $\Lambda(t-1)$ is a free group generated by $\xi_1, \dots, \xi_{\frac{q+1}{2}}$ (if we reorder the basic norm t elements so that the rest are conjugates of the first half of them). The construction of the Ramanujan graphs given by Morgenstern is obtained by taking the Cayley graph of the quotient $\Gamma_g := \Lambda(t-1)/\Lambda(g)$ with respect to the $q+1$ basic norm t elements. Here, given $g \in \mathbb{F}_q[t]$ is an irreducible polynomial prime to $t(t-1)$, we have by definition

$$\Lambda(g) := \left\{ x = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{ij} \in \Lambda(t-1) : \begin{array}{c} b, c, d \equiv 0 \bmod g(t), \\ (a, g) = 1 \end{array} \right\}.$$

See Theorem 4.10 of [Mor94] for details. This Cayley graph is a Cayley graph of either $\mathrm{PGL}_2(\mathbb{F}_{q^d})$ or $\mathrm{PSL}_2(\mathbb{F}_{q^d})$, where d is the degree of the polynomial g. This is obtained by constructing a map $\mu : \Lambda(t-1) \to PGL_2(\mathbb{F}_{q^d})$. See Morgenstern's paper [Mor94] for a detailed discussion of this point. From the unique factorization of elements in $\Lambda(t-1)$ as products of basic norm t elements, we have the analogue of the above Theorem 1.5 of Lubotzky, Phillips, and Sarnak. Our main Theorem 1.2 applied to the (anisotropic) quadratic form

$$F(a,b,c,d) = a^2 - b^2\nu + (d^2\nu - c^2)(t-1)$$

gives us that the diameter of this k-regular Ramanujan graph $G := \Gamma_g$ ($k = q+1$ here) is at most $(6+\varepsilon)\log_q q^d + O_\varepsilon(1)$. Since $\mathrm{PGL}_2(\mathbb{F}_{q^d})$ and $\mathrm{PSL}_2(\mathbb{F}_{q^d})$ are of orders $q^{3d} - q^d$ and $\frac{q^{3d}-q^d}{2}$, respectively, this is $(2+\varepsilon)\log_{k-1}|G| + O_\varepsilon(1)$, as required. Similarly, we can deal with the case when q is even. We therefore have the following (known) corollary of our strong approximation result. However, our proof is independent of the Ramanujan conjecture over function fields (that is now a well-known deep theorem of Drinfeld).

Corollary 1.7. *The diameter of k-regular Morgenstern Ramanujan graphs G is at most*

$$(2+\varepsilon)\log_{k-1}|G| + O_\varepsilon(1).$$

Note that the proof that the diameter satisfies this bound is independent of the Ramanujan conjecture; however, the fact that the graphs G are indeed Ramanujan graphs still uses the Ramanujan conjecture. Since by Conjecture 1.4 we expect the optimal bound of $4+\varepsilon$ to hold at least for anisotropic quadratic forms in 4 variables as well, we expect the stronger upper bound $(\frac{4}{3}+\varepsilon)\log_{k-1}|G|+O_\varepsilon(1)$ to be true.

Our method is based on a version of the circle method that is developed in the work of Heath-Brown over the integers [HB96a], and modified by Browning and Vishe for function fields [BV15]. We improve the known upper bounds on some oscillatory integrals that come from the infinite place. In fact, we give an exact formula for these integrals in terms of the Kloosterman sums and our optimal upper bound are a consequence of Weil's bound on Kloosterman's sums.

2. The delta method for small target

In this section, we define a weighted sum $N(w,\boldsymbol{\lambda})$ counting the number of integral solutions of our problem. We then use the delta method to give an expression for it in terms of exponential sums and oscillatory integrals. This is done by giving an expansion of the delta function using the decomposition of $\mathbb{T}$ (that we shall define below) found in the paper [BV15] of Browning and Vishe. In this section, we also set up the basic notation that we shall use in this paper.

2.1. **Notation.** Let $K=\mathbb{F}_q(t)$ and let $\mathcal{O}=\mathbb{F}_q[t]$ be its ring of integers. The prime at infinity t^{-1}–which we denote by ∞–gives us the completion K_∞ of K with respect to the norm

$$|a/b|_\infty := q^{\deg a-\deg b}.$$

We often omit the ∞ from the notation $|.|_\infty$ and simply write $|.|$. For every d, we define the natural norm on K_∞^d by $|\mathbf{a}|:=\max_i|a_i|$. This endows K_∞^d and $\mathcal{O}_\infty^d$ with metric topologies. By considering the other places as well, we may construct the ring of adeles as $\mathbf{A}_K^d$. We do not discuss this construction here as it plays a minor role in this paper.

Note that we may identify K_∞ with the field

$$\mathbb{F}_q(\!(1/t)\!)=\left\{\sum_{i\le N}a_it^i : \text{for } a_i\in\mathbb{F}_q \text{ and some } N\in\mathbb{Z}\right\}$$

and put

$$\mathbb{T}=\{\alpha\in K_\infty : |\alpha|<1\}=\left\{\sum_{i\le -1}a_it^i : \text{for } a_i\in\mathbb{F}_q\right\}.$$

Let $\delta\in\mathbb{T}$. Then $\mathbb{T}/\delta\mathbb{T}$ is the set of cosets $\alpha+\delta\mathbb{T}$, of which there are $|\delta|$.

In the function field setting, smooth functions $f:F\to\mathbb{C}$ from a non-archimedian local field F are precisely the locally constant functions. The analogue here of Schwarz functions in real analysis is the notion of Schwarz-Bruhat functions which are the smooth (locally constant) functions $f:F\to\mathbb{C}$ with compact support. We denote the set of Schwarz-Bruhat functions on F by $S(F)$. We can then extend this notion to Schwarz-Bruhat functions on F^n by defining such a function to be one that is a Schwarz-Bruhat function in each coordinate. We could similarly define the space of Schwarz-Bruhat functions $S(\mathbf{A}_F^n)$ on the adeles $\mathbf{A}_F^n$. As mentioned above, adeles do not play an important role in this paper; our focus will be on the infinite place.

2.2. **Characters.** There is a non-trivial additive character $e_q : \mathbb{F}_q \to \mathbb{C}^*$ defined for each $a \in \mathbb{F}_q$ by taking $e_q(a) = \exp(2\pi i \mathrm{tr}(a)/p)$, where $\mathrm{tr} : \mathbb{F}_q \to \mathbb{F}_p$ denotes the trace map. This character induces a non-trivial (unitary) additive character $\psi : K_\infty \to \mathbb{C}^*$ by defining $\psi(\alpha) = e_q(a_{-1})$ for any $\alpha = \sum_{i \leq N} a_i t^i$ in K_∞. In particular it is clear that $\psi|_{\mathcal{O}}$ is trivial. More generally, given any $\gamma \in K_\infty$, the map $\alpha \mapsto \psi(\alpha\gamma)$ is an additive character on K_∞. We then have the following orthogonality property.

Lemma 2.1 (Kubota, Lemma 7 of [Kub74])**.**

$$\sum_{\substack{b \in \mathcal{O} \\ |b| < \widehat{N}}} \psi(\gamma b) = \begin{cases} \widehat{N}, & \text{if } |(\!(\gamma)\!)| < \widehat{N}^{-1}, \\ 0, & \text{otherwise,} \end{cases}$$

for any $\gamma \in K_\infty$ and any integer $N \geq 0$, where $(\!(\gamma)\!)$ is the part of γ with all degrees negative.

We also have the following

Lemma 2.2 (Kubota, Lemma 1(f) of [Kub74])**.** *Let $Y \in \mathbb{Z}$ and $\gamma \in K_\infty$. Then*

$$\int_{|\alpha| < \widehat{Y}} \psi(\alpha\gamma) d\alpha = \begin{cases} \widehat{Y}, & \text{if } |\gamma| < \widehat{Y}^{-1}, \\ 0, & \text{otherwise.} \end{cases}$$

In particular, if we set $Y = 0$, then we obtain the following expression for the delta function on $\mathcal{O}$:

$$\delta(x) = \int_{\mathbb{T}} \psi(\alpha x) d\alpha,$$

where

$$\delta(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{otherwise.} \end{cases}$$

2.3. **The delta function.** The idea now is to decompose $\mathbb{T}$ into a disjoint union of balls (with no minor arcs) which is the analogue of Kloosterman's version of the circle method in this function field setting. This is done via the following lemma of Browning and Vishe [BV15, Lemma 4.2].

Lemma 2.3. *For any $Q > 1$ we have a disjoint union*

$$\mathbb{T} = \bigsqcup_{\substack{r \in \mathcal{O} \\ |r| \leq \widehat{Q} \\ r \text{ monic}}} \bigsqcup_{\substack{a \in \mathcal{O} \\ |a| < |r| \\ (a,r)=1}} \left\{ \alpha \in \mathbb{T} : |r\alpha - a| < \widehat{Q}^{-1} \right\}.$$

The following follows from Lemma 2.3.

Lemma 2.4. *Let $Q \geq 1$ and $n \in \mathcal{O}$. We have*

$$\delta(n) = \frac{1}{\widehat{Q}^2} \sum_{\substack{r \in \mathcal{O} \\ |r| \leq \widehat{Q} \\ r \text{ monic}}} \sideset{}{^*}\sum_{|a| < |r|} \psi\left(\frac{an}{r}\right) h\left(\frac{r}{t^Q}, \frac{n}{t^{2Q}}\right) \tag{5}$$

where we henceforth put

$$\sideset{}{^*}\sum_{|a| < |r|} := \sum_{\substack{a \in \mathcal{O} \\ |a| < |r| \\ (a,r)=1}}.$$

and h is only defined for $x \neq 0$ as:

$$h(x, y) = \begin{cases} |x|^{-1} & \text{if } |y| < |x| \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We have

$$\delta(n) = \sum_{\substack{r\in\mathcal{O}\\ |r|\le\widehat{Q}\\ r \text{ monic}}} \sideset{}{^*}\sum_{|a|<|r|} \psi\left(\frac{an}{r}\right) \int_{|\alpha|<|r|^{-1}\widehat{Q}^{-1}} \psi(\alpha n)\, d\alpha.$$

It is easy to check that

$$\frac{1}{\widehat{Q}^2} h\left(\frac{r}{t^Q}, \frac{n}{t^{2Q}}\right) = \int_{|\alpha|<|r|^{-1}\widehat{Q}^{-1}} \psi(\alpha n)\, d\alpha.$$

The lemma follows by substituting the above formula. □

Proof. Indeed, using Lemma 2.3, we may rewrite the integral expression of the delta function as

$$\begin{aligned}
\delta(x) &= \int_{\mathbb{T}} \psi(\alpha x) d\alpha \\
&= \sum_{\substack{r\in\mathcal{O}\\ |r|\le\widehat{Q}\\ r \text{ monic}}} \sideset{}{^*}\sum_{|a|<|r|} \int_{|r\alpha-a|<\widehat{Q}^{-1}} \psi(\alpha x) d\alpha \\
&= \sum_{\substack{r\in\mathcal{O}\\ |r|\le\widehat{Q}\\ r \text{ monic}}} \sideset{}{^*}\sum_{|a|<|r|} \psi\left(\frac{ax}{r}\right) \int_{|\alpha|<|r|^{-1}\widehat{Q}^{-1}} \psi(\alpha x) d\alpha,
\end{aligned}$$

where the last equality follows from a linear change of variables. Note that if we define

$$h(x,y) := |x|^{-1} \int_{\mathbb{T}} \psi(yx^{-1}u) du,$$

then

$$\begin{aligned}
h\left(\frac{r}{t^Q}, \frac{x}{t^{2Q}}\right) &= \widehat{Q}|r|^{-1} \int_{\mathbb{T}} \psi\left(\frac{xu}{rt^Q}\right) du \\
&= \widehat{Q}^2 \int_{|\alpha|<|r|^{-1}\widehat{Q}^{-1}} \psi(\alpha x) d\alpha.
\end{aligned}$$

The last statement follows from Lemma 2.2. □

2.4. **Smooth sum $N(w, \boldsymbol{\lambda})$.** As previously stated, we want to take a weight function $w \in S(K_\infty^d)$ and use it to define a weighted sum over all the solutions whose existence we want to show. We will denote such a sum by $N(w, \boldsymbol{\lambda})$, and then we will use the circle method to give a lower bound for this quantity. A positive lower bound would prove existence of the desired solutions.

Let w be a compactly supported (Schwarz-Bruhat) weight function defined on K_∞^d. Assume that $\mathbf{x} \in \mathcal{O}^d$ satisfies the conditions $F(\mathbf{x}) = f$ and $\mathbf{x} \equiv \boldsymbol{\lambda} \bmod g$. We uniquely write $\mathbf{x} = g\mathbf{t} + \boldsymbol{\lambda}$, where $\mathbf{t} \in \mathcal{O}^d$ and $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_d)$ for λ_i of degree strictly less than that of g. Define

$$k := \frac{f - F(\boldsymbol{\lambda})}{g}. \tag{6}$$

If $F(\mathbf{x}) = f$, then $g^2 F(\mathbf{t}) + 2g\boldsymbol{\lambda}^T A\mathbf{t} = f - F(\boldsymbol{\lambda})$ which implies that $g | 2\boldsymbol{\lambda}^T A\mathbf{t} - k$. Then, $F(\mathbf{t}) + \frac{1}{g}(2\boldsymbol{\lambda}^T A\mathbf{t} - k) = 0$. We also define

$$G(\mathbf{t}) := \frac{F(g\mathbf{t} + \boldsymbol{\lambda}) - f}{g^2} = F(\mathbf{t}) + \frac{1}{g}(2\boldsymbol{\lambda}^T A\mathbf{t} - k).$$

Finally, we define

$$N(w,\boldsymbol{\lambda}) := \sum_{\mathbf{t}} w(g\mathbf{t}+\boldsymbol{\lambda})\delta(G(\mathbf{t})),$$

where $\mathbf{t} \in \mathcal{O}^d$. Note that $N(w,\boldsymbol{\lambda})$ is the weighted number of $\boldsymbol{x} \in \mathcal{O}^d$ satisfying the conditions the conditions $F(\mathbf{x}) = f$ and $\mathbf{x} \equiv \boldsymbol{\lambda} \bmod g$. We apply the delta expansion in (5) to $\delta(G(\mathbf{t}))$. Note that (2.4) holds only for values of $\mathcal{O}$. Moreover, $G(\mathbf{t}) \in \mathcal{O}$ if and only if $g|2\boldsymbol{\lambda}^T A\mathbf{t} - k$. Using Lemma 2.1, we have for $\gamma \in K_\infty$

$$\frac{1}{|g|}\sum_{\substack{\ell\in\mathcal{O}\\ |\ell|<|g|}} \psi(\gamma\ell) = \begin{cases} 1 & \text{if } |((\gamma))| < |g|^{-1} \\ 0 & \text{otherwise.}\end{cases}$$

In particular,

$$\frac{1}{|g|}\sum_{\substack{\ell\in\mathcal{O}\\ |\ell|<|g|}} \psi\left(\frac{(2\boldsymbol{\lambda}^T A\mathbf{t}-k)\ell}{g}\right) = \begin{cases} 1 & \text{if } |((\frac{2\boldsymbol{\lambda}^T A\mathbf{t}-k}{g}))| < |g|^{-1} \\ 0 & \text{otherwise.}\end{cases}$$

The condition

$$\left|((\frac{2\boldsymbol{\lambda}^T A\mathbf{t}-k}{g}))\right| < |g|^{-1}$$

is satisfied precisely when

$$((\frac{2\boldsymbol{\lambda}^T A\mathbf{t}-k}{g})) = 0,$$

that is, when $g|2\boldsymbol{\lambda}^T A\mathbf{t} - k$. Consequently, we may rewrite

$$N(w,\boldsymbol{\lambda}) = \frac{1}{|g|}\sum_{\substack{\ell\in\mathcal{O}\\ |\ell|<|g|}}\sum_{\mathbf{t}} \psi\left(\frac{(2\boldsymbol{\lambda}^T A\mathbf{t}-k)\ell}{g}\right) w(g\mathbf{t}+\boldsymbol{\lambda})\delta(G(\mathbf{t})).$$

Then, applying (5) and splitting the sum over $\mathbf{t}$ as a sum of sums over different congruence classes modulo gr, we obtain

$$\begin{aligned}
& N(w,\boldsymbol{\lambda}) \\
= \;& \frac{1}{|g|\widehat{Q}^2}\sum_{\substack{\ell\in\mathcal{O}\\ |\ell|<|g|}}\sum_{\mathbf{t}}\sum_{\substack{r\in\mathcal{O}\\ |r|\le\widehat{Q}\\ r \text{ monic}}}\sideset{}{^*}\sum_{|a|<|r|} \psi\left(\frac{(2\boldsymbol{\lambda}^T A\mathbf{t}-k)\ell}{g}+\frac{aG(\mathbf{t})}{r}\right) w(g\mathbf{t}+\boldsymbol{\lambda})h\left(\frac{r}{t^Q},\frac{G(\mathbf{t})}{t^{2Q}}\right) \\
= \;& \frac{1}{|g|\widehat{Q}^2}\sum_{\substack{\ell\in\mathcal{O}\\ |\ell|<|g|}}\sum_{\mathbf{t}}\sum_{\substack{r\in\mathcal{O}\\ |r|\le\widehat{Q}\\ r \text{ monic}}}\sideset{}{^*}\sum_{|a|<|r|} \psi\left(\frac{(a+r\ell)(2\boldsymbol{\lambda}^T A\mathbf{t}-k)+agF(\mathbf{t})}{gr}\right) w(g\mathbf{t}+\boldsymbol{\lambda})h\left(\frac{r}{t^Q},\frac{G(\mathbf{t})}{t^{2Q}}\right) \\
= \;& \frac{1}{|g|\widehat{Q}^2}\sum_{\substack{\ell\in\mathcal{O}\\ |\ell|<|g|}}\sum_{\substack{r\in\mathcal{O}\\ |r|\le\widehat{Q}\\ r \text{ monic}}}\sideset{}{^*}\sum_{|a|<|r|}\sum_{\mathbf{b}\in\mathcal{O}^d/(gr)}\sum_{\mathbf{s}\in\mathcal{O}^d} \psi\left(\frac{(a+r\ell)(2\boldsymbol{\lambda}^T A\mathbf{b}-k)+agF(\mathbf{b})}{gr}\right) w(g(\mathbf{b}+gr\mathbf{s})+\boldsymbol{\lambda}) \\
& \cdot h\left(\frac{r}{t^Q},\frac{G(\mathbf{b}+gr\mathbf{s})}{t^{2Q}}\right).
\end{aligned}$$

The Poisson summation formula for $f \in S(\mathbf{A}_K^d)$ states that

$$\sum_{x\in K^d} f(\mathbf{x}) = \sum_{x\in K^d} \widehat{f}(\mathbf{x}),$$

where

$$\widehat{f}(\mathbf{y}) := \int_{\mathbf{A}_K^d} f(\mathbf{x})\psi(\langle \mathbf{x}, \mathbf{y}\rangle)d\mathbf{x}.$$

From this, one deduces (see Lemma 2.1 of [BV15], for example) that for $v \in S(K_\infty^d)$,

$$\sum_{\mathbf{t}\in\mathcal{O}^d} v(\mathbf{t}) = \sum_{\mathbf{c}\in\mathcal{O}^d} \int_{K_\infty^d} \psi(\langle \mathbf{c}, \mathbf{t}\rangle)v(\mathbf{t})d\mathbf{t}.$$

Applying this to the $\mathbf{s}$ variable in the above expression of $N(w, \boldsymbol{\lambda})$, we obtain the expression

$$\begin{aligned}
& N(w, \boldsymbol{\lambda}) \\
= \; & \frac{1}{|g|\widehat{Q}^2} \sum_{\substack{\ell\in\mathcal{O}\\ |\ell|<|g|}} \sum_{\substack{r\in\mathcal{O}\\ |r|\leq\widehat{Q}\\ r \text{ monic}}} {\sum_{|a|<|r|}}^{*} \sum_{\mathbf{b}\in\mathcal{O}^d/(gr)} \sum_{\mathbf{c}\in\mathcal{O}^d} \psi\left(\frac{(a+r\ell)(2\boldsymbol{\lambda}^T A\mathbf{b} - k) + agF(\mathbf{b})}{gr}\right) \\
& \cdot \int_{K_\infty^d} \psi\left(\langle \mathbf{c}, \mathbf{t}\rangle\right) w(g(\mathbf{b}+gr\mathbf{t}) + \boldsymbol{\lambda}) h\left(\frac{r}{t^Q}, \frac{G(\mathbf{b}+gr\mathbf{t})}{t^{2Q}}\right) d\mathbf{t} \\
= \; & \frac{1}{|g|\widehat{Q}^2} \sum_{\substack{\ell\in\mathcal{O}\\ |\ell|<|g|}} \sum_{\substack{r\in\mathcal{O}\\ |r|\leq\widehat{Q}\\ r \text{ monic}}} {\sum_{|a|<|r|}}^{*} \sum_{\mathbf{c}\in\mathcal{O}^d} \sum_{\mathbf{b}\in\mathcal{O}^d/(gr)} |gr|^{-d}\psi\left(\frac{(a+r\ell)(2\boldsymbol{\lambda}^T A\mathbf{b} - k) + agF(\mathbf{b}) - \langle \mathbf{c}, \mathbf{b}\rangle}{gr}\right) \\
& \cdot \int_{K_\infty^d} \psi\left(\frac{\langle \mathbf{c}, \mathbf{t}\rangle}{gr}\right) w(g\mathbf{t} + \boldsymbol{\lambda}) h\left(\frac{r}{t^Q}, \frac{G(\mathbf{t})}{t^{2Q}}\right) d\mathbf{t}
\end{aligned}$$

We express this in the condensed form

$$N(w, \boldsymbol{\lambda}) = \frac{1}{|g|\widehat{Q}^2} \sum_{\substack{r\in\mathcal{O}\\ |r|\leq\widehat{Q}\\ r \text{ monic}}} \sum_{\mathbf{c}\in\mathcal{O}^d} |gr|^{-d} S_{g,r}(\mathbf{c}) I_{g,r}(\mathbf{c}), \tag{7}$$

where $I_{g,r}(\mathbf{c})$ and $S_{g,r}(\mathbf{c})$ are defined by

$$I_{g,r}(\mathbf{c}) := \int_{K_\infty^d} h\left(\frac{r}{t^Q}, \frac{G(\mathbf{t})}{t^{2Q}}\right) w(g\mathbf{t} + \boldsymbol{\lambda}) \psi\left(\frac{\langle \mathbf{c}, \mathbf{t}\rangle}{gr}\right) d\mathbf{t}, \tag{8}$$

and

$$S_{g,r}(\mathbf{c}) := \sum_{\substack{\ell\in\mathcal{O}\\ |\ell|<|g|}} {\sum_{|a|<|r|}}^{*} S_{g,r}(a, \ell, \mathbf{c}) \tag{9}$$

with

$$S_{g,r}(a, \ell, \mathbf{c}) := \sum_{\mathbf{b}\in\mathcal{O}^d/(gr)} \psi\left(\frac{(a+r\ell)(2\boldsymbol{\lambda}^T A\mathbf{b} - k) + agF(\mathbf{b}) - \langle \mathbf{c}, \mathbf{b}\rangle}{gr}\right). \tag{10}$$

In the next two sections, we bound from above $S_{g,r}$ and $I_{g,r}$.

3. Bounds on the exponential sums $S_{g,r}(\mathbf{c})$

In this section, we bound from above an averaged sum of the $S_{g,r}(\mathbf{c})$. Indeed, we prove the following.

Proposition 3.1. *We have the following upper bound*

$$\sum_{\substack{r\in\mathcal{O}\\ |r|<\widehat{X}}} |g|^{-d}|r|^{-\frac{d+1}{2}}|S_{g,r}(\mathbf{c})| \ll_\Delta |g|^\varepsilon \widehat{X}^{1+\varepsilon},$$

where $\widehat{X} = O(|f|^A)$ *for some fixed* A *and* $\Delta := \det A$.

Initially, a version of this result was proved by Heath-Brown (Lemma 28 of [HB96b]). This is a function field analogue of proposition 4.1 of the first author in [T. 19a]. We first prove a lemma indicating that most $S_{g,r}(a,\ell,\mathbf{c})$ vanish.

Lemma 3.2. *Unless* $\mathbf{c} \equiv 2(ar+\ell)A\boldsymbol{\lambda} \bmod g$, *we have* $S_{g,r}(a,\ell,\mathbf{c}) = 0$. *Consequently,* $S_{g,r}(\mathbf{c}) = 0$ *unless* $\mathbf{c} \equiv \alpha A\boldsymbol{\lambda} \bmod g$ *for some* $\alpha \in \mathcal{O}$.

Proof. Write $\mathbf{b} = r\mathbf{b}_1 + \mathbf{b}_2$, where $\mathbf{b}_1$ is a vector modulo g and $\mathbf{b}_2$ is a vector modulo r. We may then rewrite

$$S_{g,r}(a,\ell,\mathbf{c}) = \sum_{\mathbf{b}_2} \psi\left(\frac{(a+r\ell)(2\boldsymbol{\lambda}^T A\mathbf{b}_2 - k) + agF(\mathbf{b}_2) - \langle \mathbf{c}, \mathbf{b}_2\rangle}{gr}\right) \sum_{\mathbf{b}_1} \psi\left(\frac{2(a+r\ell)\boldsymbol{\lambda}^T A\mathbf{b}_1 - \langle \mathbf{c}, \mathbf{b}_1\rangle}{g}\right).$$

From Lemma 2.1, the second sum vanishes unless $\mathbf{c} \equiv 2(a+r\ell)A\lambda \bmod g$, which gives the first statement in the lemma. Since $S_{g,r}(\mathbf{c})$ is a sum of the $S_{g,r}(a,\ell,\mathbf{c})$, we obtain that it is zero unless possibly $\mathbf{c} \equiv \alpha A\boldsymbol{\lambda} \bmod g$ for some $\alpha \in \mathcal{O}$. □

By definition,

$$S_{g,r}(\mathbf{c}) = \sum_{\substack{\ell \in \mathcal{O} \\ |\ell|<|g|}} \sideset{}{^*}\sum_{|a|<|r|} \sum_{\mathbf{b} \in \mathcal{O}^d/(gr)} \psi\left(\frac{(a+r\ell)(2\boldsymbol{\lambda}^T A\mathbf{b} - k) + agF(\mathbf{b}) - \langle \mathbf{c}, \mathbf{b}\rangle}{gr}\right).$$

Since the sum over ℓ is zero unless $g | 2\boldsymbol{\lambda}^T A\mathbf{b} - k$, in which case it contributes a factor of $|g|$, we have

$$S_{g,r}(\mathbf{c}) = |g| \sideset{}{^*}\sum_{|a|<|r|} \sum_{\substack{\mathbf{b} \in \mathcal{O}^d/(gr) \\ g | 2\boldsymbol{\lambda}^T A\mathbf{b} - k}} \psi\left(\frac{a(2\boldsymbol{\lambda}^T A\mathbf{b} - k) + agF(\mathbf{b}) - \langle \mathbf{c}, \mathbf{b}\rangle}{gr}\right).$$

We will give a bound on each of the $S_{g,r}(\mathbf{c})$. We do so by first decomposing $S_{g,r}(\mathbf{c})$ into the product of two sums and then bounding each of the two sums separately.

Write $r = r_1 r_2$, where $r_i \in \mathcal{O}$ and $\gcd(r_1, 2\Delta g) = 1$ and such that the prime divisors of r_2 are among the prime divisors of $2\Delta g$. In particular, $\gcd(r_1, gr_2) = 1$, and so we may write

$$k = gr_2 k_1 + r_1 k_2$$

and

$$a = r_2 a_1 + r_1 a_2$$

for some $k_1, k_2 \in \mathcal{O}$ and unique $a_1 \in \mathcal{O}/(r_1)$, $a_2 \in \mathcal{O}/(r_2)$. Similarly, we may find vectors $\mathbf{b}_1 \in \mathcal{O}^d/(r_1)$ and $\mathbf{b}_2 \in \mathcal{O}^d/(gr_2)$ such that

$$\mathbf{b} = gr_2\mathbf{b}_1 + r_1\mathbf{b}_2.$$

If we set

$$S_1 := \sum_{a_1, \mathbf{b}_1} \psi\left(\frac{2r_2 a_1 \boldsymbol{\lambda}^T A\mathbf{b}_1 + a_1(gr_2)^2 F(\mathbf{b}_1) - \langle \mathbf{c}, \mathbf{b}_1\rangle - r_2 a_1 k_1}{r_1}\right), \tag{11}$$

and

$$S_2 := |g| \sideset{}{^*}\sum_{|a_2|<|r_2|} \sum_{\substack{\mathbf{b}_2 \in \mathcal{O}^d/(gr_2) \\ g | 2\boldsymbol{\lambda}^T A\mathbf{b}_2 - k\overline{r_1}}} \psi\left(\frac{2r_1 a_2 \boldsymbol{\lambda}^T A\mathbf{b}_2 + a_2 gr_1^2 F(\mathbf{b}_2) - \langle \mathbf{c}, \mathbf{b}_2\rangle - r_1 a_2 k_2}{gr_2}\right), \tag{12}$$

then we see from a simple substitution of the above that

$$S_{g,r}(\mathbf{c}) = S_1 S_2.$$

What we proceed to do is bound S_1 and S_2.

In order to bound S_1 from above, consider the following situation. Let $G(\mathbf{x}) := \mathbf{x}^T B\mathbf{x}$, where B is a symmetric matrix $B \in M_d(\mathcal{O})$ with $D := \det(B) \neq 0$. Furthermore, let $r \in \mathcal{O}$ be such that $\gcd(r, D) = 1$, and for each $t \in \mathcal{O}/(r)$, $\mathbf{c}, \mathbf{c}' \in \mathcal{O}^d/(r)$, define

$$S_r(G, \mathbf{c}, \mathbf{c}', t) := \sideset{}{^*}\sum_{|a|<|r|} \sum_{\mathbf{b}\in\mathcal{O}^d/(r)} \psi\left(\frac{a(G(\mathbf{b}) + \langle \mathbf{c}', \mathbf{b}\rangle + t) - \langle \mathbf{c}, \mathbf{b}\rangle}{r}\right).$$

We will prove the following lemma.

Lemma 3.3. *With the notation as above,*

$$S_r(G, \mathbf{c}, \mathbf{c}', t) = \left(\frac{D}{r}\right) \tau_r^d \mathrm{Kl}_r(G, \mathbf{c}, \mathbf{c}', t), \tag{13}$$

where $\tau_r := \sum_{|x|<|r|} \psi\left(\frac{x^2}{r}\right)$ *is the Gauss sum,* $\left(\frac{\cdot}{\cdot}\right)$ *is the Jacobi symbol, and* $\mathrm{Kl}_r(G, \mathbf{c}, \mathbf{c}', t)$ *is either a Kloosterman sum (for even* d*) or a Salié sum (for odd* d*). Furthermore, we have*

$$|S_1| \leq |r_1|^{\frac{d+1}{2}} \tau(r_1) |\mathrm{rad}_3'(r_1)|^{1/2} |\gcd(r_1, f)|^{1/2},$$

where $\tau(.)$ *is the divisor function, and* $\mathrm{rad}_3'(r_1)$ *is the product of the primes dividing* r_1 *with an odd power at least* 3*.*

In order to prove this lemma, we first reduce to the case where $r = \varpi^k$ for some irreducible $\varpi \in \mathcal{O}$. This is done via the following lemma.

Lemma 3.4 (Multiplicativity of $S_r(G, \mathbf{c}, \mathbf{c}', t)$)**.** *Suppose* $r = uv$ *for coprime* $u, v \in \mathcal{O}$*. Then*

$$S_r(G, \mathbf{c}, \mathbf{c}', t) = S_u(G, \bar{v}\mathbf{c}, \mathbf{c}', t) S_v(G, \bar{u}\mathbf{c}, \mathbf{c}', t).$$

Proof. Since u and v are coprime, as $\mathbf{b}_1$ ranges over $\mathcal{O}^d/(u)$ and $\mathbf{b}_2$ ranges over $\mathcal{O}^d/(v)$, the vector

$$\mathbf{b} = v\mathbf{b}_1 + u\mathbf{b}_2$$

ranges over a complete set of vectors modulo $uv = r$. Similarly, as a_1 ranges over $\mathcal{O}/(u)$ and a_2 ranges over $\mathcal{O}/(v)$,

$$a = va_1 + ua_2$$

ranges over a complete set of polynomials modulo $uv = r$. Making these substitutions, the summands in $S_r(G, \mathbf{c}, \mathbf{c}', t)$ become

$$\begin{aligned}
& \psi\left(\frac{a(G(\mathbf{b}) + \langle \mathbf{c}', \mathbf{b}\rangle + t) - \langle \mathbf{c}, \mathbf{b}\rangle}{r}\right) \\
= \; & \psi\left(\frac{(va_1 + ua_2)(G(v\mathbf{b}_1 + u\mathbf{b}_2) + \langle \mathbf{c}', v\mathbf{b}_1 + u\mathbf{b}_2\rangle + t) - \langle \mathbf{c}, v\mathbf{b}_1 + u\mathbf{b}_2\rangle}{uv}\right) \\
= \; & \psi\left(\frac{(va_1 + ua_2)(v^2 G(\mathbf{b}_1) + u^2 G(\mathbf{b}_2) + v\langle \mathbf{c}', \mathbf{b}_1\rangle + u\langle \mathbf{c}', \mathbf{b}_2\rangle + t) - v\langle \mathbf{c}, \mathbf{b}_1\rangle - u\langle \mathbf{c}, \mathbf{b}_2\rangle}{uv}\right) \\
= \; & \psi\left(\frac{a_1(v^2 G(\mathbf{b}_1) + \langle v\mathbf{c}', \mathbf{b}_1\rangle + t) - \langle \mathbf{c}, \mathbf{b}_1\rangle}{u}\right) \psi\left(\frac{a_2(u^2 G(\mathbf{b}_2) + \langle u\mathbf{c}', \mathbf{b}_2\rangle + t) - \langle \mathbf{c}, \mathbf{b}_2\rangle}{v}\right) \\
= \; & \psi\left(\frac{a_1(G(v\mathbf{b}_1) + \langle \mathbf{c}', v\mathbf{b}_1\rangle + t) - \langle \bar{v}\mathbf{c}, v\mathbf{b}_1\rangle}{u}\right) \psi\left(\frac{a_2(G(u\mathbf{b}_2) + \langle \mathbf{c}', u\mathbf{b}_2\rangle + t) - \langle \bar{u}\mathbf{c}, u\mathbf{b}_2\rangle}{v}\right).
\end{aligned}$$

Since u and v are coprime, $u\mathbf{b}_2$ and $v\mathbf{b}_1$ range over a complete set of residues modulo v and u, respectively. As a result,

$$S_r(G, \mathbf{c}, \mathbf{c}', t) = S_u(G, \bar{v}\mathbf{c}, \mathbf{c}', t) S_v(G, \bar{u}\mathbf{c}, \mathbf{c}', t),$$

as required. □

Since the characteristic of our base field is odd, we can diagonalize our quadratic form G modulo r, and write

$$G(\mathbf{x}) = \sum_{i=1}^{d} \alpha_i x_i^2.$$

Therefore,

$$S_r(G, \mathbf{c}, \mathbf{c}', t) = \sum_{|a|<|r|}^{*} \psi\left(\frac{at}{r}\right) \prod_{j=1}^{d} \sum_{b \in \mathcal{O}/(r)} \psi\left(\frac{a(\alpha_j b^2 + c_j' b) - c_j b}{r}\right).$$

We complete the square to obtain

$$\begin{aligned} S_r(G, \mathbf{c}, \mathbf{c}', t) &= \sum_{|a|<|r|}^{*} \psi\left(\frac{at}{r}\right) \prod_{j=1}^{d} \sum_{b \in \mathcal{O}/(r)} \psi\left(\frac{a\alpha_j \left(b + \overline{2a\alpha_j}(ac_j' - c_j)\right)^2 - \overline{4a\alpha_j}(ac_j' - c_j)^2)}{r}\right) \\ &= \sum_{|a|<|r|}^{*} \psi\left(\frac{at}{r}\right) \prod_{j=1}^{d} \psi\left(\frac{-\overline{4a\alpha_j}(ac_j' - c_j)^2)}{r}\right) \sum_{b \in \mathcal{O}/(r)} \psi\left(\frac{a\alpha_j \left(b + \overline{2a\alpha_j}(ac_j' - c_j)\right)^2}{r}\right) \end{aligned}$$

The internal sum is equal to $\left(\frac{a\alpha_j}{r}\right)\tau_r$, and so

$$S_r(G, \mathbf{c}, \mathbf{c}', t) = \tau_r^d \left(\frac{D}{r}\right) \psi\left(\frac{\sum_j \overline{2\alpha_j} c_j c_j'}{r}\right) \sum_{|a|<|r|}^{*} \left(\frac{a}{r}\right)^d \psi\left(\frac{a(t - \sum_j \overline{4\alpha_j} c_j'^2) - \bar{a} \sum_j \overline{4\alpha_j} c_j^2}{r}\right).$$

In light of Lemma 3.4, we proceed to bound $S_{\varpi^k}(G, \mathbf{c}, \mathbf{c}', t)$ for $k \geq 1$ and $\varpi \in \mathcal{O}$ irreducible. It suffices to bound the sums

$$\sum_{|a|<|\varpi^k|}^{*} \left(\frac{a}{\varpi^k}\right)^d \psi\left(\frac{a(t - \sum_j \overline{4\alpha_j} c_j'^2) - \bar{a} \sum_j \overline{4\alpha_j} c_j^2}{\varpi^k}\right).$$

We will be interested only in the case when $r = \varpi^k | r_1$, $G = (gr_2)^2 F$, $\mathbf{c}' = 2r_2 A\boldsymbol{\lambda}$, and $t = -r_2 k_1$. In this case,

$$t - \sum_j \overline{4\alpha_j} c_j'^2 \equiv -r_2 k_1 - F(\boldsymbol{\lambda})\bar{g}^2 \equiv (gr_1 k_2 - f)\bar{g}^2 \equiv -f\bar{g}^2 \bmod \varpi^k.$$

Similarly,

$$\sum_j \overline{4\alpha_j} c_j^2 \equiv \sum_j \overline{4g^2\eta_j} c_j^2 \bmod \varpi^k.$$

Making these substitutions and changing a to ag^2, we obtain

$$\sum_{|a|<|\varpi^k|}^{*} \left(\frac{a}{\varpi^k}\right)^d \psi\left(\frac{-af - \bar{a} \sum_j \overline{4\eta_j} c_j^2}{\varpi^k}\right).$$

Case $k = 1$: If $r = \varpi$, then the sum is a usual Salié (for d odd) or Kloosterman (for d even) sum over the finite field $\mathcal{O}/(\varpi)$. Consequently, we have the bound

$$\begin{aligned} |S_\varpi(G, \mathbf{c}, \mathbf{c}', t)| &\leq |\varpi|^{1/2} |\tau_\varpi|^d \left| \gcd\left(\varpi, t - \sum_j \overline{4\alpha_j} c_j'^2, \sum_j \overline{4\alpha_j} c_j^2\right)\right|^{1/2} \\ &= |\varpi|^{\frac{d+1}{2}} \left| \gcd\left(\varpi, t - \sum_j \overline{4\alpha_j} c_j'^2, \sum_j \overline{4\alpha_j} c_j^2\right)\right|^{1/2}, \end{aligned}$$

where $|\tau_\varpi| = |\varpi|^{1/2}$ follows from Proposition 2.4 of [Kow18]. In the study of S_1, we set $G = (gr_2)^2 F$, $t = -r_2 k_1$, $\mathbf{c}' = 2r_2 A\boldsymbol{\lambda}$, and $\varpi | r_1$. Recall that $F(\mathbf{x}) = \mathbf{x}^T A \mathbf{x}$, $\gcd(r_1, gr_2 D) = 1$, and $G = (gr_2)^2 F$ is diagonalizable with eigenvalues $\{\alpha_i\}$. If $r_1 = \varpi$, then

$$\sum_j \overline{4\alpha_j} c_j'^2 \equiv \overline{2gr_2}^2 c'^T A^{-1} c' \equiv \overline{g}^2 F(\boldsymbol{\lambda}) \bmod \varpi,$$

and so

$$\gcd\left(\varpi, t - \sum_j \overline{4\alpha_j} c_j'^2\right) = \gcd\left(r_1 = \varpi, gr_2 + \overline{g}^2 F(\boldsymbol{\lambda})\right) = \gcd(r_1, f).$$

Consequently, if $r_1 = \varpi$, we have

$$|S_1| \leq |\varpi|^{\frac{d+1}{2}} |\gcd(\varpi, f)|^{1/2}.$$

Case $k \geq 2$: By the computations, it suffices to show that

$$\left| \sum_{a|<|\varpi^k|}^{*} \left(\frac{a}{\varpi^k}\right)^d \psi\left(\frac{-af - \bar{a}\sum_j \overline{4\eta_j} c_j^2}{\varpi^k}\right) \right| \leq 2|\varpi|^{\lceil k/2 \rceil} |\gcd(\varpi^k, f)|^{1/2}.$$

Write $a = a_1 + a_2 \varpi^{\lceil k/2 \rceil}$, where a_1 is chosen modulo $\varpi^{\lceil k/2 \rceil}$ and is relatively prime to ϖ, and a_2 is chosen modulo $\varpi^{\lfloor k/2 \rfloor}$. Furthermore, note that

$$\overline{a_1 + a_2 \varpi^{\lceil k/2 \rceil}} \equiv \overline{a_1} - \overline{a_1}^2 a_2 \varpi^{\lceil k/2 \rceil} \bmod \varpi^{\lceil k/2 \rceil}.$$

Making these substitutions, we obtain

$$\begin{aligned}
\psi\left(\frac{-af - \bar{a}\sum_j \overline{4\eta_j} c_j^2}{\varpi^k}\right) &= \psi\left(\frac{-(a_1 + a_2\varpi^{\lceil k/2 \rceil})f - (\overline{a_1 + a_2\varpi^{\lceil k/2 \rceil}})\sum_j \overline{4\eta_j} c_j^2}{\varpi^k}\right) \\
&= \psi\left(\frac{-(a_1 + a_2\varpi^{\lceil k/2 \rceil})f - (\overline{a_1} - \overline{a_1}^2 a_2 \varpi^{\lceil k/2 \rceil})\sum_j \overline{4\eta_j} c_j^2}{\varpi^k}\right) \\
&= \psi\left(\frac{-a_1 f - \overline{a_1}\sum_j \overline{4\eta_j} c_j^2 + a_2 \varpi^{\lceil k/2 \rceil}\left(f + \overline{a_1}^2 \sum_j \overline{4\eta_j} c_j^2\right)}{\varpi^k}\right) \\
&= \psi\left(\frac{-a_1 f - \overline{a_1}\sum_j \overline{4\eta_j} c_j^2}{\varpi^k}\right) \psi\left(\frac{a_2\left(f + \overline{a_1}^2 \sum_j \overline{4\eta_j} c_j^2\right)}{\varpi^{\lfloor k/2 \rfloor}}\right).
\end{aligned}$$

Summation over $a_2 \bmod \varpi^{\lfloor k/2 \rfloor}$ gives us zero unless

$$f + \overline{a_1}^2 \sum_j \overline{4\eta_j} c_j^2 \equiv 0 \bmod \varpi^{\lfloor k/2 \rfloor},$$

in which case it contributes a factor of $|\varpi|^{\lfloor k/2 \rfloor}$. There are at most $2|\varpi|^{\lceil k/2 \rceil - \lfloor k/2 \rfloor}$ choices of $a_1 \bmod \varpi^{\lceil k/2 \rceil}$ such that the above congruence is true modulo $\varpi^{\lfloor k/2 \rfloor}$. Therefore,

$$\begin{aligned}
&\left| \sideset{}{^*}\sum_{|a|<|\varpi^k|} \left(\frac{a}{\varpi^k}\right)^d \psi\left(\frac{-af - \bar{a}\sum_j \overline{4\eta_j} c_j^2}{\varpi^k}\right)\right| \\
= \;& |\varpi^{\lfloor k/2 \rfloor}| \left| \sideset{}{^*}\sum_{\substack{|a_1|<|\varpi|^{\lceil k/2 \rceil} \\ \varpi^{\lfloor k/2 \rfloor} | f + \overline{a_1}^2 \sum_j \overline{4\eta_j} c_j^2}} \psi\left(\frac{-a_1 f - \overline{a_1}\sum_j \overline{4\eta_j} c_j^2}{\varpi^k}\right)\right| \\
\leq \;& 2|\varpi|^{\lceil k/2 \rceil} |\gcd(\varpi^k, f)|^{1/2}.
\end{aligned}$$

Hence,

$$|S_1| \leq 2|\varpi|^{\frac{dk}{2} + \lceil \frac{k}{2} \rceil} |\gcd(\varpi^k, f)|^{1/2}.$$

Combining these cases and using lemma 3.4, we obtain for every r_1

$$|S_1| \leq \tau(r_1) |\mathrm{rad}_3'(r_1)|^{1/2} |r_1|^{\frac{d+1}{2}} |\gcd(r_1, f)|^{1/2},$$

where $\mathrm{rad}_3'(r_1)$ is the product of the prime powers in the prime decomposition of r_1 dividing r_1 with *odd* powers at least 3. This concludes the proof of Lemma 13. From this, we obtain the desired bound on $|S_1|$ for each r_1.

We now bound S_2 from above via the following lemma. The proof uses the Cauchy-Schwarz inequality.

Lemma 3.5. *For S_2 as above,*

$$|S_2| \ll_\Delta |g|^d |r_2|^{\frac{d}{2}+1}.$$

Proof. Recall that

$$S_2 := |g| \sideset{}{^*}\sum_{|a_2|<|r_2|} \sum_{\substack{\mathbf{b}_2 \in \mathcal{O}^d/(gr_2) \\ g | 2\boldsymbol{\lambda}^T A \mathbf{b}_2 - k\overline{r_1}}} \psi\left(\frac{2r_1 a_2 \boldsymbol{\lambda}^T A \mathbf{b}_2 + a_2 g r_1^2 F(\mathbf{b}_2) - \langle \mathbf{c}, \mathbf{b}_2 \rangle - r_1 a_2 k_2}{g r_2}\right).$$

Applying the Cauchy-Schwarz inequality to the a_2 variable, we obtain

$$\begin{aligned}
&|S_2|^2 \\
\leq \;& |g|^2 \varphi(r_2) \sideset{}{^*}\sum_{|a_2|<|r_2|} \left| \sum_{\substack{\mathbf{b}_2 \in \mathcal{O}^d/(gr_2) \\ g | 2\boldsymbol{\lambda}^T A \mathbf{b} - k\overline{r_1}}} \psi\left(\frac{2r_1 a_2 \boldsymbol{\lambda}^T A \mathbf{b}_2 + a_2 g r_1^2 F(\mathbf{b}_2) - \langle \mathbf{c}, \mathbf{b}_2 \rangle - r_1 a_2 k_2}{g r_2}\right)\right|^2 \\
= \;& |g|^2 \varphi(r_2) \sideset{}{^*}\sum_{|a_2|<|r_2|} \sum_{\substack{\mathbf{b}_2, \mathbf{b}_2' \in \mathcal{O}^d/(gr_2) \\ g | 2\boldsymbol{\lambda}^T A \mathbf{b}_2 - k\overline{r_1}, g | 2\boldsymbol{\lambda}^T A \mathbf{b}_2' - k\overline{r_1}}} \psi\left(\frac{2r_1 a_2 \boldsymbol{\lambda}^T A (\mathbf{b}_2 - \mathbf{b}_2') + a_2 g r_1^2 (F(\mathbf{b}_2) - F(\mathbf{b}_2')) - \langle \mathbf{c}, \mathbf{b}_2 - \mathbf{b}_2' \rangle}{g r_2}\right)
\end{aligned}$$

Making the substitution $\mathbf{u} = \mathbf{b}_2 - \mathbf{b}_2'$, we obtain

$$|S_2|^2 \leq |g|^2 \varphi(r_2) \sideset{}{^*}\sum_{|a_2|<|r_2|} \sum_{\substack{\mathbf{b}_2, \mathbf{u} \in \mathcal{O}^d/(gr_2) \\ g | 2\boldsymbol{\lambda}^T A \mathbf{b}_2 - k\overline{r_1}, g | 2\boldsymbol{\lambda}^T A \mathbf{u}}} \psi\left(\frac{2r_1 a_2 \boldsymbol{\lambda}^T A \mathbf{u} + a_2 g r_1^2 (2\mathbf{b}_2^T A \mathbf{u} + F(\mathbf{u})) - \langle \mathbf{c}, \mathbf{u} \rangle}{g r_2}\right).$$

The sum over $\mathbf{b}_2$ is zero unless $r_2|\Delta \gcd(\mathbf{u})$, which implies that the summation is non-zero only if

$$\mathbf{u} \in (r_2\mathcal{O}/(\gcd(\Delta, r_2)gr_2))^d \simeq (\mathcal{O}/(\gcd(\Delta, r_2)g))^d.$$

Hence,

$$\begin{aligned} |S_2|^2 &\leq |g|^2\varphi(r_2) \sideset{}{^*}\sum_{|a_2|<|r_2|} \sum_{\substack{\mathbf{b}_2\in\mathcal{O}^d/(gr_2)\\ g|2\boldsymbol{\lambda}^T A\mathbf{b}_2 - k\overline{r_1}}} \sum_{\substack{\mathbf{u}\in(\mathcal{O}/(\gcd(\Delta,r_2)g))^d\\ g|2\boldsymbol{\lambda}^T A\mathbf{u}}} 1 \\ &\ll_\Delta |g|^{2d}\varphi(r_2)^2|r_2|^d \\ &\ll_\Delta |g|^{2d}|r_2|^{d+2}. \end{aligned}$$

Taking square roots, we obtain

$$|S_2| \ll_\Delta |g|^d|r_2|^{\frac{d}{2}+1},$$

as required. □

We now put together the above results to prove Proposition 3.1.

Proof of Proposition 3.1. As before, write $r = r_1r_2$, where $\gcd(r_1, g\Delta) = 1$ and the prime divisors of r_2 are among those of $g\Delta$. By construction, we know that $|S_{g,r}(\mathbf{c})| = |S_1||S_2|$. Therefore, from Lemmas 3.3 and 3.5, we have

$$\begin{aligned} &\sum_{\substack{r\in\mathcal{O}\\ |r|<\widehat{X}}} |g|^{-d}|r|^{-\frac{d+1}{2}}|S_{g,r}(\mathbf{c})| \\ \ll_\Delta &\sum_{\substack{r\in\mathcal{O}\\ |r|<\widehat{X}}} \tau(r_1)|\mathrm{rad}_3'(r_1)|^{1/2}|r_2|^{1/2}|\gcd(r_1, f)|^{1/2} \\ \leq &\ \widehat{X}^\varepsilon \sum_{\substack{r\in\mathcal{O}\\ |r|<\widehat{X}}} |r_2|^{1/2}|\mathrm{rad}_3'(r_1)|^{1/2}|\gcd(r_1, f)|^{1/2} \\ = &\ \widehat{X}^\varepsilon \sum_{\substack{r_1\in\mathcal{O}\\ |r_1|<\widehat{X}}} |\mathrm{rad}_3'(r_1)|^{1/2}|\gcd(r_1, f)|^{1/2} \sum_{\substack{r_2\in\mathcal{O}\\ |r_2|<\widehat{X}/|r_1|}} |r_2|^{1/2}. \end{aligned}$$

The second (internal) sum can be bounded using

$$\sum_{\substack{r_2\in\mathcal{O}\\ |r_2|<\widehat{X}/|r_1|}} |r_2|^{1/2} \leq \sum_{d|(g\Delta)^\infty, |d|<\widehat{X}/|r_1|} |d|^{1/2}\frac{\widehat{X}}{|r_1d|} \leq \widehat{X}/|r_1| \sum_{d|(g\Delta)^\infty, |d|<\widehat{X}/|r_1|} 1 \ll \frac{\widehat{X}}{|r_1|}|g\Delta|^\varepsilon\widehat{X}^\varepsilon.$$

Hence,

$$\sum_{\substack{r_1\in\mathcal{O}\\ |r_1|<\widehat{X}}} |\mathrm{rad}_3'(r_1)|^{1/2}|\gcd(r_1, f)|^{1/2} \sum_{\substack{r_2\in\mathcal{O}\\ |r_2|<\widehat{X}/|r_1|}} |r_2|^{1/2} \ll \widehat{X}|g\Delta|^\varepsilon\widehat{X}^\varepsilon \sum_{\substack{r_1\in\mathcal{O}\\ |r_1|<\widehat{X}}} \frac{|\mathrm{rad}_3'(r_1)|^{1/2}|\gcd(r_1, f)|^{1/2}}{|r_1|},$$

from which the conclusion would follow if we show that $\sum_{\substack{r_1\in\mathcal{O}\\ |r_1|<\widehat{X}}} \frac{|\mathrm{rad}_3'(r_1)|^{1/2}|\gcd(r_1,f)|^{1/2}}{|r_1|} \ll \widehat{X}^\varepsilon$. First, note that for each T, we have

$$\sum_{\substack{r_1\in\mathcal{O}\\ |r_1|<\widehat{T}}} \frac{|\mathrm{rad}_3'(r_1)|^{1/2}}{|r_1|} \leq \prod_{\varpi:|\varpi|<\widehat{T}} \left(1 + \frac{1}{|\varpi|} + \frac{1}{|\varpi|^2} + \frac{1}{|\varpi|^{2.5}} + \frac{1}{|\varpi|^4} + \frac{1}{|\varpi|^{4.5}} + \dots\right).$$

On the other hand, a simple computation of geometric series gives us

$$1+\frac{1}{|\varpi|}+\frac{1}{|\varpi|^2}+\frac{1}{|\varpi|^{2.5}}+\frac{1}{|\varpi|^4}+\frac{1}{|\varpi|^{4.5}}+\ldots=1+\frac{1}{|\varpi|}+\frac{1}{|\varpi|^{1/2}(|\varpi|^{1/2}-1)(|\varpi|+1)}.$$

The product of such terms over all monic irreducible ϖ is less than

$$\exp\left(\sum_{|\varpi|<\widehat{T}}\left(\frac{1}{|\varpi|}+\frac{1}{|\varpi|^{1/2}(|\varpi|^{1/2}-1)(|\varpi|+1)}\right)\right)<\exp\left(2\sum_{|\varpi|<\widehat{T}}\frac{1}{|\varpi|}\right).$$

The number of monic irreducible polynomials of degree d over $\mathbb{F}_q$ is less than $\frac{2q^d}{d}$. Consequently,

$$\exp\left(2\sum_{|\varpi|<\widehat{T}}\frac{1}{|\varpi|}\right)\leq\exp\left(4\sum_{d=1}^{T}\frac{1}{d}\right)\sim\exp(4\gamma+4\log T)\ll\widehat{T}^{\varepsilon},$$

where γ is the Euler-Mascheroni constant. Therefore, we have

$$\sum_{\substack{r_1\in\mathcal{O}\\|r_1|<\widehat{T}}}\frac{|\mathrm{rad}_3'(r_1)|^{1/2}}{|r_1|}\ll\widehat{T}^{\varepsilon}.$$

Now, suppose $f=\varpi_1^{a_1}\ldots\varpi_s^{a_s}$ is the prime factorization of f. It then follows that

$$\sum_{\substack{r_1\in\mathcal{O}\\|r_1|<\widehat{X}}}\frac{|\mathrm{rad}_3'(r_1)|^{1/2}|\gcd(r_1,f)|^{1/2}}{|r_1|}\leq\sum_{\substack{0\leq i_j\leq a_j\\1\leq j\leq s}}\frac{1}{|\varpi_1^{i_1}\ldots\varpi_s^{i_s}|^{1/2}}\sum_{\substack{r_1\in\mathcal{O}\\|r_1|<\widehat{X}\\\varpi_j^{i_j}\nmid r_1\forall j}}\frac{|\mathrm{rad}_3'(r_1)|^{1/2}}{|r_1|}\ll\widehat{T}^{\varepsilon},$$

as required. □

4. Analytic functions on $\mathbb{T}^d$

In order to prove our main theorem, it turns out that we need to do analysis not just using polynomials over K_∞, but also using convergent Taylor series. We begin by defining a space of analytic functions defined on $\mathbb{T}^d$ that extends the space of polynomials. Let $\mathcal{O}_\infty:=\{x\in K_\infty:|\alpha|\leq 1\}$. Define

$$C^\omega(\mathbb{T}^d):=\left\{\sum_{(n_1,\ldots,n_d)\in\mathbb{N}_{\geq 0}^d}a_{(n_1,\ldots,n_d)}x_1^{n_1}\ldots x_d^{n_d}:a_{(n_1,\ldots,n_d)}\in\mathcal{O}_\infty\right\}.$$

It is easy to see that the above Taylor expansions are convergent for $(u_1,\ldots,u_d)\in\mathbb{T}^d$. When $d=1$, aside from polynomials in $\mathcal{O}_\infty[x]$, examples of analytic functions on $\mathbb{T}$ are

$$\frac{1}{1-x}:=\sum_{k=0}^{\infty}x^k,$$

and

$$(1+x)^{1/2}:=\sum_{k=0}^{\infty}\binom{1/2}{k}x^k.$$

This square root function is defined since the base characteristic is odd. We define the partial derivatives $\frac{\partial}{\partial x_i}$ for $1\leq i\leq d$ on $C^\omega(\mathbb{T}^d)$ to be the formal derivation operator which acts on the

monomials as: $\frac{\partial}{\partial x_i} x_1^{n_1} \dots x_d^{n_d} = n_i x_1^{n_1} \dots x_i^{n_i - 1} \dots x_d^{n_d}$ and extend them by linearity to power series. It is easy to check that it sends $C^\omega(\mathbb{T}^d)$ to itself. Let

$$C^\omega(\mathbb{T}^m, \mathbb{T}^n) := \{\Phi = (\phi_1, \dots, \phi_n) : \phi_j \in C^\omega(\mathbb{T}^m) \text{ and } \phi_j(0) \in \mathbb{T}\}.$$

For $\Phi \in C^\omega(\mathbb{T}^m, \mathbb{T}^n)$ define the Jacobi matrix $J\Phi := \left[\frac{\partial \phi_i}{\partial x_j}\right]$, where $1 \le i \le n$ and $1 \le j \le m$. For $m = n$ define the Jacobi determinant to be $\det(J\Phi)$. We also have the following change of variables formula, which readily follows from Igusa [Igu00, Lemma 7.4.2].

Lemma 4.1. *Let $\Gamma \subset K_\infty^n$ be a box defined by the inequalities $|x_i| < \widehat{R}_i$, for some real numbers $R_1, \dots, R_n$. Let $f : \Gamma \to \mathbb{C}$ be a continuous function. Then for any $M \in \mathrm{GL}_n(K_\infty)$ we have*

$$\int_\Gamma f(\mathbf{u}) d\mathbf{u} = |\det M| \int_{M\mathbf{v} \in \Gamma} f(M\mathbf{v}) d\mathbf{v}.$$

4.1. **The analytic automorphism of $\mathbb{T}^d$.** In this section, we define the group of the analytic automorphism of $\mathbb{T}^d$. We use this group in order to simplify and reduce the computations of our oscillatory integrals into Gaussian integrals. Recall that by Schwarz's Lemma the analytic automorphisms of the disk in the complex plane which fixes the origin are just rotations. Unlike the disk in the complex plane the analytic group of automorphisms of the disk $\mathbb{T}^d$ is enormous. Define

$$\mathcal{A}_\infty(\mathbb{T}^d) := \left\{\Phi \in C^\omega(\mathbb{T}^d, \mathbb{T}^d) : |\det(J\Phi(0))|_\infty = 1, \text{ and } \Phi(0) = 0\right\}.$$

Proposition 4.2. *$\mathcal{A}_\infty(\mathbb{T}^d)$ is a group under the composition of functions and it preserve the Haar measure on $\mathbb{T}^d$.*

First, we prove a lemma on diagonalizing symmetric matrices over K that we use in the proof of the preceding proposition. It is easy to see that $GL_d(\mathcal{O}_\infty) \subset C^\omega(\mathbb{T}^d, \mathbb{T}^d)$.

Lemma 4.3. *Suppose that $A \in M_{d\times d}(K_\infty)$ and $A^\intercal = A$. Then there exists $\gamma \in GL_d(\mathcal{O}_\infty)$ such that*

$$\gamma^\intercal A \gamma = D[\eta_1, \dots, \eta_d],$$

where $D[\eta_1, \dots, \eta_d]$ is the diagonal matrix with some $\eta_1, \dots, \eta_d \in K_\infty$ on its diagonal.

Proof. We proceed by induction on d. The lemma is trivial for $d = 1$. Without loss of generality, we assume that $A \in M_{d\times d}(\mathcal{O}_\infty)$ and $A \neq 0 \bmod t^{-1}$. Let $\bar{A}$ denote $A \bmod t^{-1}$ which is a matrix with $\mathbb{F}_q$ coefficients. Since $q \neq 2$, there exists a matrix $g \in GL_d(\mathbb{F}_q)$ which diagonalizes $\bar{A}$, and we have $g^\intercal \bar{A} g = D[\bar{\eta}_1, \dots, \bar{\eta}_d]$. Suppose that $\eta_1 \neq 0$. Let $A_1 := g^\intercal A g = \left[\mathbf{a}_1, \dots, \mathbf{a}_d\right] = \left[a_{i,j}\right]$, where $\mathbf{a}_i$ is the ith column vector of A_1, and $a_{i,j}$ is the ith and jth coordinate of A_1. Let

$$H := \begin{bmatrix} 1 & -\frac{a_{1,2}}{a_{1,1}} & \dots & -\frac{a_{1,d}}{a_{1,1}} \\ 0 & & & \\ \vdots & & I_{d-1\times d-1} & \\ 0 & & & \end{bmatrix}.$$

Note that $a_{1,1} \in \mathcal{O}_\infty^*$ is invertible. Hence $H \in GL_d(\mathcal{O}_\infty)$. Moreover, it is easy to check that

$$H^\intercal A_1 H = \begin{bmatrix} a_{11} & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & A_2 & \\ 0 & & & \end{bmatrix},$$

where $A_2^\intercal = A_2 \in M_{(d-1)\times(d-1)}(\mathcal{O}_\infty)$. The lemma follows from the induction hypothesis on A_2. $\square$

Proof of Proposition 4.2. By the product rule of the Jacobian it is easy to see that $\mathcal{A}_\infty(\mathbb{T}^d)$ is closed under the composition of functions. The identity function is the identity element of $\mathcal{A}_\infty(\mathbb{T}^d)$. It is enough to construct the inverse of $\Phi \in \mathcal{A}_\infty(\mathbb{T}^d)$. We prove the existence of the inverse by solving a recursive system of linear equations. First, we explain it when $d = 1$. We have $\Phi = \sum_{i=1}^\infty a_i x^i$, where $|a_1|_\infty = 1$. We wish to find $\Psi = \sum_{i=1}^\infty b_i x^i \in C^\omega(\mathbb{T}^d)$ such that $\Psi \circ \Phi(x) = x$. This implies that $b_1 = a_1^{-1}$ and the following system of equations hold for each $n \geq 2$

$$0 = b_n a_1^n + \sum_{i=1}^{n-1} b_i(\text{some polynomial in } a_1, \dots, a_{n-i+1}).$$

The above system of recursive linear equations have a unique solution where $b_n \in \mathcal{O}_\infty$. For general d, suppose that $\Phi := (\phi_1(x_1, \dots x_d), \dots, \phi_d(x_1, \dots, x_d)) \in \mathcal{A}_\infty(\mathbb{T}^d)$. By the definition of $\mathcal{A}_\infty(\mathbb{T}^d)$, we have $\det(J\Phi(0)) \in GL_d(\mathcal{O}_\infty)$. Let $\bar{\Psi} := J\Phi(0)^{-1} \in GL_d(\mathcal{O}_\infty)$. We note that $J(\bar{\Psi} \circ \Phi(0)) = I_{d\times d}$. Without loss of the generality, we assume that $J(\Phi(0)) = I_{d\times d}$. We wish to find $\Psi := (\psi_1(x_1, \dots x_d), \dots, \psi_d(x_1, \dots, x_d)) \in \mathcal{A}_\infty(\mathbb{T}^d)$ such that

$$\psi_i\big(\phi_1(x_1, \dots, x_d), \dots, \phi_d(x_1, \dots, x_d)\big) = x_i$$

for every $1 \leq i \leq d$. Suppose that

$$\phi_i := \sum_{(n_1,\dots,n_d)\in\mathbb{N}_{\geq 0}^d} a_{i,(n_1,\dots,n_d)} x_1^{n_1} \dots x_d^{n_d},$$
$$\psi_i := \sum_{(n_1,\dots,n_d)\in\mathbb{N}_{\geq 0}^d} b_{i,(n_1,\dots,n_d)} x_1^{n_1} \dots x_d^{n_d},$$

where $1 \leq i \leq d$. Let $|(n_1, \dots, n_d)| := \sum_{i=1}^d n_i$. For $(n_1, \dots, n_d) \in \mathbb{N}_{\geq 0}^d$, with $|(n_1, \dots, n_d)| \geq 2$, we have

$$0 = b_{i,(n_1,\dots,n_d)} + \sum_{(m_1,\dots,m_d)<(n_1,\dots,n_d)} b_{i,(m_1,\dots,m_d)}(\text{some polynomial in } a_{j,(k_1,\dots,k_d)}), \tag{14}$$

where $(k_1, \dots, k_d) \leq |(n_1, \dots, n_d)|$. Similarly, the above system of recursive linear equations have a unique solution where $b_{i,(n_1,\dots,n_d)} \in \mathcal{O}_\infty$. Finally, by the definition of $\mathcal{A}_\infty(\mathbb{T}^d)$, we have $|\det(J\Phi(0))|_\infty = 1$. This implies $|\det(J\Phi(\mathbf{x}))|_\infty = 1$ for every $\mathbf{x} \in \mathbb{T}^d$. This completes the proof of our lemma. □

Next, we prove a version of the Morse lemma for functions in $C^\omega(\mathbb{T}^d)$.

Proposition 4.4 (Morse lemma over K_∞)**.** *Assume that $\phi(\mathbf{u})$ is an analytic function on $\mathbb{T}^d$ with a single critical point at 0 and the Hessian H_ϕ, where $|\det(H_\phi(0))|_\infty = 1$. Then there exists $\Psi \in \mathcal{A}_\infty(\mathbb{T}^d)$ with $J\Psi(0) = I_{d\times d}$ such that*

$$\phi(\Psi) = \phi(0) + \Psi^\intercal H_\phi(0)\Psi.$$

Proof. By Lemma 4.3 there exists a matrix $g \in GL_d(\mathcal{O}_\infty)$ such that $g^\intercal H_\phi(0) g = D[\lambda_1, \dots, \lambda_d]$. Since $H_\phi(0) \in GL_d(\mathcal{O}_\infty)$ then $\lambda_i \in \mathcal{O}_\infty$ and $|\lambda_i|_\infty = 1$. By changing the variables with g, we assume that $H_\phi(0)$ is a diagonal matrix. First, we explain it for $d = 1$. We have $\phi(x) = \phi(0) + \lambda x^2 + x^3 \sum_{n=0}^\infty a_n x^n$, where $|\lambda_i|_\infty = 1$. Let

$$\psi(x) := x\big(1 + x\sum_{n=0}^\infty \lambda^{-1} a_n x^n\big)^{1/2} = x\Big(\sum_{k=0}^\infty \binom{1/2}{k}\big(x\sum_{n=0}^\infty \lambda^{-1} a_n x^n\big)^k\Big) \in \mathcal{A}_\infty(\mathbb{T}),$$

where we used the taylor expansion $(1 + x)^{1/2} := \sum_{k=0}^\infty \binom{1/2}{k} x^k$. It is easy to check that $\phi = \phi(0) + \lambda\psi^2$. This completes the proof of the lemma for $d = 1$. For general d, we proceed by

induction on d. We explain our induction hypothesis next. Assume that

$$\phi(x_1,\dots,x_d)=\phi(0)+\sum_{i,j\geq 2}x_ix_j(\delta_{i,j}\lambda_i+h_{i,j}(x_1,x_2,\dots,x_d)),$$

for some $h_{i,j}(x_1,\dots,x_d)\in C^\omega(\mathbb{T}^d)$ and $\lambda_i\in\mathcal{O}_\infty$, where $h_{i,j}(0)=0$ and $|\lambda_i|_\infty=1$. Then

$$\phi=\phi(0)+\sum_{j\geq 1}\lambda_j\psi_j^2,$$

where $\psi_j=x_j+h_j(x_1,\dots,x_d)$ such that $h_j(x_1,\dots,x_d)$ has a critical point at 0. The induction hypothesis holds for $d=1$. We assume that it holds for $d-1$, and we prove it for d. We write

$$\phi(x_1,\dots,x_d)=\phi(0)+x_1^2(\lambda_1+h_{1,1}(x_1,\dots,x_d))+\sum_{j\geq 2}2x_1x_jh_{1,j}(x_1,x_2,\dots,x_d)\\+\sum_{i,j\geq 2}x_ix_j(\delta_{i,j}\lambda_i+h_{i,j}(x_1,x_2,\dots,x_d)),$$

for some $h_{i,j}(x_1,\dots,x_d)\in C^\omega(\mathbb{T}^d)$, where $h_{i,j}(0)=0$. Define

$$\psi_1:=x_1\big(1+\lambda_1^{-1}h_{1,1}\big)^{1/2}+\big(\lambda_1^{-1}\sum_{j\geq 2}x_jh_{1,j}(x_1,x_2,\dots,x_d)\big)\big(1+\lambda_1^{-1}\phi_1\big)^{-1/2}.$$

We have

$$\phi=\phi(0)+\lambda_1\psi_1^2+\sum_{i,j\geq 2}x_ix_j(\delta_{i,j}\lambda_i+h'_{i,j}(x_1,x_2,\dots,x_d)),\tag{15}$$

for some $h'_{i,j}(x_1,\dots,x_d)\in C^\omega(\mathbb{T}^d)$, where $h_{i,j}(0)=0$. By the induction induction hypothesis for $d-1$, we have

$$\phi=\phi(0)+\lambda_1\psi_1^2+\sum_{j\geq 2}\lambda_j\psi_j^2,$$

where $\psi_j=x_j+h_j(x_1,\dots,x_d)$ such that $h_j(x_1,\dots,x_d)$ has a critical point at 0. This concludes our lemma. □

4.2. **Stationary phase theorem over function fields.** In this section, we prove a version of the stationary phase theorem in the function fields setting that we use for computing the oscillatory integrals $I_{g,r}(\mathbf{c})$.
Let $f\in K_\infty$ and define

$$\mathcal{G}(f):=\begin{cases}\min(|f|_\infty^{-1/2},1) & \text{if } \mathrm{ord}(f) \text{ is } \textit{even},\\ |f|_\infty^{-1/2}\varepsilon_f & \text{if } \mathrm{ord}(f)\geq 1 \text{ and is } \textit{odd},\\ 1 & \text{otherwise},\end{cases}\tag{16}$$

where $\varepsilon_f:=\frac{G(f)}{|G(f)|}$ and $G(f):=\sum_{x\in\mathbb{F}_q}e_q(a_fx^2)$ is the gauss sum associated to a_f the top degree coefficient of f. Suppose that $\phi\in C^\omega(\mathbb{T}^d)$ has a single critical point at 0 with the Hessian H_ϕ, where $|\det(H_\phi(0))|_\infty=1$.

Proposition 4.5. *Suppose the above assumptions on ϕ and f. We have*

$$\int_{\mathbb{T}^d}\psi(f\phi(\mathbf{u}))d\mathbf{u}=\psi(f\phi(\mathbf{0}))\prod_{i=1}^d\mathcal{G}(f\lambda_i),$$

where $\lambda_i\in\mathcal{O}_\infty$ for $1\leq i\leq d$ are diagonal element of $g^\intercal H_\phi(0)g$ for some $g\in GL_d(\mathcal{O}_\infty)$ obtained in Lemma 4.3.

We begin the proof of the above proposition by proving some spacial cases of the proposition for the quadratic polynomials.

4.2.1. *Gaussian integrals over function field.* We define the analogue of the Gaussian integrals over the function field K and give an explicit formula for them.

Lemma 4.6. *For every $f \in K_\infty$, we have*

$$\int_{\mathbb{T}} \psi(fu^2)du = \mathcal{G}(f).$$

Proof. First, suppose that $\mathrm{ord}(f) = 2k$, where $k \geq 0$. We partition $\mathbb{T}$ into the cosets of $t^{-k}\mathbb{T}$. Let $\alpha + t^{-k}\mathbb{T} \subset \mathbb{T}$. We show that

$$\int_{\alpha+t^{-k}\mathbb{T}} \psi(fu^2)du = 0$$

for $\alpha \notin t^{-k}\mathbb{T}$. We have

$$\begin{aligned}\int_{\alpha+t^{-k}\mathbb{T}} \psi(fu^2)du = \int_{t^{-k}\mathbb{T}} \psi(f(\alpha+v)^2)dv &= \psi(f\alpha^2)\int_{t^{-k}\mathbb{T}} \psi(f(2\alpha v+v^2))dv \\ &= \psi(f\alpha^2)\int_{t^{-k}\mathbb{T}} \psi(f2\alpha v)dv = 0,\end{aligned}$$

where we used Lemma 2.2, $\mathrm{ord}(fv^2) \leq -2$ and $\mathrm{ord}(\alpha f) \geq k$. Therefore,

$$\int_{\mathbb{T}} \psi(fu^2)du = \int_{t^{-k}\mathbb{T}} \psi(fu^2)du = \int_{t^{-k}\mathbb{T}} du = |f|_\infty^{-1/2} = \mathcal{G}(f).$$

On the other hand, if $\mathrm{ord}(f) = 2k-1$, where $k \geq 1$. Similarly, for $\alpha \notin t^{-k+1}\mathbb{T}$

$$\begin{aligned}\int_{\alpha+t^{-k}\mathbb{T}} \psi(fu^2)du = \int_{v\in t^{-k}\mathbb{T}} \psi(f(\alpha+v)^2)du &= \psi(f\alpha^2)\int_{t^{-k}\mathbb{T}} \psi(f(2\alpha v+v^2))dv \\ &= \psi(f\alpha^2)\int_{t^{-k}\mathbb{T}} \psi(f2\alpha v)dv = 0,\end{aligned}$$

where we used Lemma 2.2, $\mathrm{ord}(fv^2) \leq -3$ and $\mathrm{ord}(\alpha f) \geq k$. Hence

$$\int_{\mathbb{T}} \psi(fu^2)du = \int_{t^{-k+1}\mathbb{T}} \psi(fu^2)du = q^{-k}G(f) = \mathcal{G}(f).$$

The last equality follows from the following. Indeed, by the definition of the integral, we have

$$\begin{aligned}\int_{t^{-k+1}\mathbb{T}} \psi(fu^2)du &= \lim_{m\to+\infty} q^{-m-k+1} \sum_{a_{-m}t^{-m-k+1}+\ldots+a_{-1}t^{-k}:a_i\in\mathbb{F}_q} \psi((a_{-m}t^{-m-k+1}+\ldots+a_{-1}t^{-k})^2 f) \\ &= \lim_{m\to+\infty} q^{-m-k+1} \sum_{a_{-m},\ldots,a_{-1}\in\mathbb{F}_q} e_q(a_f a_{-1}^2) \\ &= q^{-k}\sum_{x\in\mathbb{F}_q} e_q(a_f x^2).\end{aligned}$$

It is well-known, that $G(f) = q^{1/2}\varepsilon_f$. Consequently, $q^{-k}G(f) = |f|_\infty^{-1/2}\varepsilon_f$. We have therefore proved the result for $\mathrm{ord}(f) = 2k-1$, $k \geq 1$.

Finally, if $\mathrm{ord}(f) \leq -1$, then $\mathrm{ord}(fu^2) < -1$ for $u \in \mathbb{T}$. Consequently,

$$\int_{\mathbb{T}} \psi(fu^2)du = \int_{\mathbb{T}} du = 1.$$

This concludes the proof. □

Next, we give a formula for the Gaussian integral associate to any symmetric matrix $A \in M_{d\times d}(K_\infty)$. Define

$$\mathcal{G}(A) := \int_{\mathbb{T}^d} \psi(\mathbf{u}^\intercal A\mathbf{u}).$$

Lemma 4.7. *We have*

$$\mathcal{G}(A) = \prod_{i=1}^{d} \mathcal{G}(\lambda_i),$$

where $\lambda_i \in K_\infty$ for $1 \le i \le d$ are diagonal element of $g^\intercal Ag$ for some $g \in GL_d(\mathcal{O}_\infty)$ obtained in Lemma 4.3.

Proof. By Lemma 4.3, there exists $g \in GL_d(\mathcal{O}_\infty)$ such that $g^\intercal Ag = D[\lambda_1, \dots, \lambda_d]$. By the change of the variable formula in Lemma 4.1, we have

$$\begin{aligned}\mathcal{G}(A) = \int_{\mathbb{T}^d} \psi(\mathbf{u}^\intercal A\mathbf{u})d\mathbf{u} &= \int_{\mathbb{T}^d} \psi\left((g^{-1}\mathbf{u})^\intercal g^\intercal Ag(g^{-1}\mathbf{u})\right) d\mathbf{u} \\ &= \int_{\mathbb{T}^d} \psi\left(\sum_{i=1}^{d} \lambda_i v_i^2\right) d\mathbf{v} = \prod_{i=1}^{d} \mathcal{G}(\lambda_i),\end{aligned}$$

where $\begin{bmatrix} v_1 \dots v_d \end{bmatrix} = \mathbf{v} = g^{-1}\mathbf{u}$. This completes the proof of the lemma. □

Finally, we give a proof of the Proposition 4.5.

Proof of Proposition 4.5 . By Proposition 4.4, there exists $\Psi \in \mathcal{A}_\infty(\mathbb{T}^d)$ such that $\phi(\Psi) = \phi(0) + \Psi^\intercal H_\phi(0)\Psi$. By Proposition 4.2, Ψ is a measure preserving automorphism of $\mathbb{T}^d$. Hence,

$$\int_{\mathbb{T}^d} \psi(f\phi(\mathbf{u}))d\mathbf{u} = \int_{\mathbb{T}^d} \psi(f(\phi(0) + \Psi^\intercal H_\phi(0)\Psi))d\Psi.$$

By Lemma 4.7,

$$\int_{\mathbb{T}^d} \psi(f(\phi(0) + \Psi^\intercal H_\phi(0)\Psi))d\Psi = \psi(f\phi(\mathbf{0})) \prod_{i=1}^{d} \mathcal{G}(f\lambda_i),$$

where $\lambda_i \in \mathcal{O}_\infty$ for $1 \le i \le d$ are diagonal element of $g^\intercal H_\phi(0)g$ for some $g \in GL_d(\mathcal{O}_\infty)$ obtained in Lemma 4.3. This concludes the proof of our proposition. □

5. Bounds on the oscillatory integrals $I_{g,r}(\mathbf{c})$

In this section, we give explicit formulas for the oscillatory integrals $I_{g,r}(\mathbf{c})$ in terms of the Kloosterman sums (Salié sums). By Lemma 4.3, we suppose that $F(\gamma\mathbf{u}) = \sum_{\eta_i} \eta_i u_i^2$, where $\gamma \in GL_d(\mathcal{O}_\infty)$. Recall the additive character $\psi : K_\infty \to \mathbb{C}^*$ from §2.2, and

$$h(x,y) = \begin{cases} |x|^{-1} & \text{if } |y| < |x| \\ 0 & \text{otherwise.} \end{cases}$$

5.1. **Test function.** In this section, we define the test function w that we use for estimating the oscillatory integrals $I_{g,r}(\mathbf{c})$ at the end of this section. Recall the definition 1.1 of an anisotropic cone.

Lemma 5.1. *Let $F(\mathbf{x})$ be a non-degenerate quadratic form in $d \ge 4$ variables. We may then cover K_∞^d with four anisotropic cones such that for any given f, $X_f(K_\infty)$ intersects at least one of them.*

Proof. We show that each class in $K_\infty^\times/K_\infty^{\times 2}$, which consists of representative $1, \nu, t, \nu t$, where $\nu \in \mathbb{F}_q^\times$ is a quadratic non-residue, gives us an anisotropic cone, at least one of which intersects $X_f(K_\infty)$ for any given f. Indeed, since being an anisotropic cone is preserved by linear change of coordinates, we may assume without loss of generality that F is diagonal and the coefficients of F are also among these representatives. Furthermore, we may assume without loss of generality that f is one of the representatives $1, \nu, t, \nu t$ by uniformly scaling the coordinates (note that, by definition, anisotropic cones are invariant under scaling). After these reduction, by taking the set of $\mathbf{x} \in K_\infty^d$ such that $|F(\mathbf{x})| \geq \frac{1}{q^2}|\mathbf{x}|^2$, we obtain an anisotropic cone. Showing that the class of f in $K_\infty^\times/K_\infty^{\times 2}$ is represented by an element of this anisotropic cone follows from a simple case-by-case analysis. Suppose the class of f is νt. If one of the coefficients of F is νt, then we have a solution in the anisotropic cone. Otherwise, the coefficients are among $1, \nu, t$ and at least two of the coefficients are equal since $d \geq 4$. If -1 is a square, then every element of K_∞ can be written as the sum of two squares. Since at least one coefficient repeats, this implies that we can represent any element. On the other hand, -1 may be a quadratic non-residue, in which case we may assume $\nu = -1$. If both 1 and -1 show up as coefficients, we may represent any element of K_∞. Therefore, let us assume otherwise. We are reduced to showing that there is a solution in the anisotropic cone to the equations $t(x_1^2 + x_2^2 + x_3^2 + 1) = \pm x_4^2$, $t(x_1^2 + x_2^2 + 1) = \pm(x_3^2 + x_4^2)$, $t(x_1^2+1) = \pm(x_2^2+x_3^2+x_4^2)$, and $x_1^2+\ldots+x_4^2 = \pm t$ for any choice of signs. $x_1^2+x_2^2+1 = 0$ is solvable modulo any odd prime, and so the first and second equations have a solution in the anisotropic cone. Take $a, b \in \mathbb{F}_q^\times$ such that $a^2 + b^2 = -1$ (since -1 is a quadratic non-residue, $ab \neq 0$). For the third equation, let $(x_1, x_2, x_3, x_4) = \left(1, at\left(1 \mp \frac{1}{t}\right)^{1/2}, bt\left(1 \mp \frac{1}{t}\right)^{1/2}, t\right)$. Note that such squareroots exist in K_∞ because q is odd (see the beginning of Section 4 for the formula). For the final equation $\pm t = x_1^2 + \ldots + x_4^2$ let $(x_1, x_2, x_3, x_4) = \left(at\left(1 \pm \frac{1}{2t}\right)^{1/2}, bt\left(1 \pm \frac{1}{2t}\right)^{1/2}, t, 0\right)$. The other classes can be dealt with similarly; at the beginning, you can multiply the quadratic form by ν or t and scale the coordinates to reduce it to the above case that f has class νt. Note that the construction of the anisotropic cone associated to f depends only on the class of f in $K_\infty^\times/K_\infty^{\times 2}$. □

Remark 17. This lemma shows that given any f, we can find an anisotropic cone intersecting $X_f(K_\infty)$. This fact combined with our main theorem implies strong approximation for F (Corollary 1.3).

Fix an anisotropic cone Ω with respect to $F(\mathbf{x})$ (such that $\Omega \cap X_f(K_\infty) \neq \emptyset$).

Lemma 5.2. *Suppose that* $\mathbf{x} \in \Omega$ *and* $\mathbf{y} \notin \Omega$. *Then*

$$|\mathbf{x} \pm \mathbf{y}| \geq \max(|\mathbf{x}|, |\mathbf{y}|)/\widehat{\omega}.$$

Proof. It follows from property (2). □

For non-degenerate quadratic form $F(\mathbf{x}) = \mathbf{x}^\intercal A\mathbf{x}$, we say $F^*(\mathbf{x}) = \mathbf{x}^\intercal A^{-1}\mathbf{x}$ is the dual of $F(\mathbf{x})$. Note that $F(\mathbf{x}) = F^*(A\mathbf{x})$. Let $\Omega^* := A\Omega$.

Lemma 5.3. Ω^* *is an anisotropic cone with respect to* F^*.

Proof. It follows from the definition of Ω^*, F^* and anisotropic cones. □

Let w be the characteristic function of a ball centered at $\mathbf{x}_0 \in V_f \cap \Omega$:

$$w(\mathbf{x}) = \begin{cases} 1 \text{ if } |\mathbf{x} - \mathbf{x}_0| < |t^{-\alpha_0} f|^{1/2}, \\ 0 \text{ otherwise,} \end{cases}$$

where $\alpha_0 > \max \deg(\eta_i) + \omega$ is any large enough fixed integer such that

$$\left\{\mathbf{y} \in K_\infty^d : |\mathbf{y} - A\mathbf{x}_0| < |t^{-\alpha_0} f|^{1/2}\right\} \subset \Omega^*.$$

Note that if $w(\mathbf{x}) \neq 0$, then $\mathbf{x} \in \Omega$. Moreover,

$$w(g\mathbf{t}+\boldsymbol{\lambda}) = \begin{cases} 1 \text{ if } |\mathbf{t}-\mathbf{t}_0| < \widehat{R}, \\ 0 \text{ otherwise,} \end{cases}$$

where $\mathbf{x}_0 = g\mathbf{t}_0 + \boldsymbol{\lambda}$, and $R := \lceil \deg(f)/2 - deg(g) - \alpha_0/2 \rceil$.

5.2. **Bounding $I_{g,r}(\mathbf{c})$.** Recall that

$$G(\mathbf{t}) := \frac{F(g\mathbf{t}+\boldsymbol{\lambda}) - f}{g^2} = F(\mathbf{t}) + \frac{1}{g}(2\boldsymbol{\lambda}^T A\mathbf{t} - k),$$

where $k = \frac{f-F(\lambda)}{g}$. In this section, we assume that $Q := \lceil \deg(f)/2 - deg(g) \rceil + \max_i(\deg(\eta_i)) + \omega'$. We have

$$I_{g,r}(\mathbf{c}) = \int_{K_\infty^d} h\left(\frac{r}{t^Q}, \frac{G(\mathbf{t})}{t^{2Q}}\right) w(g\mathbf{t}+\boldsymbol{\lambda}) \psi\left(\frac{\langle \mathbf{c}, \mathbf{t}\rangle}{gr}\right) d\mathbf{t} = \int_{\substack{|\mathbf{t}-\mathbf{t}_0|<\widehat{R} \\ |G(\mathbf{t})|<\widehat{Q}|r|}} \frac{\widehat{Q}}{|r|} \psi\left(\frac{\langle \mathbf{c}, \mathbf{t}\rangle}{gr}\right) d\mathbf{t}. \tag{18}$$

Let $\kappa := \max_i |\frac{c_i}{g}|$.

Lemma 5.4. *Suppose that $\kappa < \frac{|r|}{\widehat{R}}$, then $I_{g,r}(\mathbf{c}) = \psi\left(\frac{\langle \mathbf{c},\mathbf{t}_0\rangle}{gr}\right) I_{g,r}(0)$.*

Proof. Since $\max_i(|c_i|) < \frac{|gr|}{\widehat{R}}$ and $|\mathbf{t}-\mathbf{t}_0| < \widehat{R}$, $\psi\left(\frac{\langle \mathbf{c},\mathbf{t}\rangle}{gr}\right) = \psi\left(\frac{\langle \mathbf{c},\mathbf{t}_0\rangle}{gr}\right)$. Hence, we have

$$I_{g,r}(\mathbf{c}) = \psi\left(\frac{\langle \mathbf{c}, \mathbf{t}_0\rangle}{gr}\right) \int_{\substack{|\mathbf{t}-\mathbf{t}_0|<\widehat{R} \\ |G(\mathbf{t})|<\widehat{Q}|r|}} \frac{\widehat{Q}}{|r|} d\mathbf{t} = \psi\left(\frac{\langle \mathbf{c}, \mathbf{t}_0\rangle}{gr}\right) I_{g,r}(0).$$

This completes the proof of our lemma. □

Lemma 5.5. *Let Q, R and $\mathbf{t}_0$ be as above, and suppose that $|\mathbf{t}-\mathbf{t}_0| < \widehat{R}$. Then $|G(\mathbf{t})| < \widehat{Q}|r|$ is equivalent to $|F(\mathbf{t}) - k/g| < \widehat{Q}|r|$. Moreover, if $|G(\mathbf{t})| < \widehat{Q}|r|$, then $|G(\mathbf{t}+\boldsymbol{\zeta})| < \widehat{Q}|r|$ for every $\boldsymbol{\zeta} \in K_\infty^d$, where $|\boldsymbol{\zeta}| \leq \min(|r|, \widehat{R})$.*

Proof. Since $\mathbf{t}_0 \in \Omega$, by property (3) in Lemma 5.1, $|\mathbf{t}_0| \leq |f|^{1/2}\widehat{\omega'}^{1/2}/|g|$. Recall that $Q = \lceil \deg(f)/2 - deg(g) \rceil + \max_i(\deg(\eta_i)) + \omega'$. Since $\frac{|\boldsymbol{\lambda}|}{|g|} < 1$, and $|\mathbf{t}_0| < |f|^{1/2}\widehat{\omega'}^{1/2}/|g|$ then $|\frac{1}{g}(2\boldsymbol{\lambda}^T A\mathbf{t}_0)| < \widehat{Q}$. Hence, for $|\mathbf{t}-\mathbf{t}_0| < \widehat{R}$, $|G(\mathbf{t})| < \widehat{Q}|r|$ is equivalent to $|F(\mathbf{t}) - k/g| < \widehat{Q}|r|$. Moreover, suppose that $|\boldsymbol{\zeta}| \leq \min(|r|, \widehat{R})$, and $|\mathbf{t}-\mathbf{t}_0| < \widehat{R}$, then

$$|G(\mathbf{t}+\boldsymbol{\zeta}) - G(\mathbf{t})| \leq \max\left(|F(\boldsymbol{\zeta})|, |\boldsymbol{\zeta}^\intercal A(\mathbf{t}+\boldsymbol{\lambda}/g)|\right) \leq \max(|\boldsymbol{\zeta}^\intercal A\boldsymbol{\zeta}|, \widehat{Q}|\boldsymbol{\zeta}|) \leq \widehat{Q}|r|,$$

where we used $\frac{|\boldsymbol{\lambda}|}{|g|} < 1$, $|A| = \max_i\widehat{(\deg}(\eta_i))$. Hence, if $|G(\mathbf{t})| < \widehat{Q}|r|$, then

$$|G(\mathbf{t}+\boldsymbol{\zeta})| \leq \max(|G(\mathbf{t})|, |G(\mathbf{t}+\boldsymbol{\zeta}) - G(\mathbf{t})|) < \widehat{Q}|r|. \tag{19}$$

This concludes the proof of our lemma. □

We say $\mathbf{c}$ is an ordinary vector if

$$\kappa \geq \widehat{Q}/\widehat{R}. \tag{20}$$

Lemma 5.6. *Suppose that $\mathbf{c}$ is an ordinary vector and $|r| \leq \widehat{Q}$. Then,*

$$I_{g,r}(\mathbf{c}) = 0. \tag{21}$$

Proof. By (18) and (19), we have

$$I_{g,r}(\mathbf{c}) = \int_{\substack{|\mathbf{t}|<\widehat{R}\\ |G(\mathbf{t})|<\widehat{Q}|r|}} \frac{\widehat{Q}}{|r|}\psi\left(\frac{\langle \mathbf{c},\mathbf{t}\rangle}{gr}\right)d\mathbf{t} = \frac{\widehat{Q}}{|r|}\int_{\substack{|\mathbf{t}|<\widehat{R}\\ |G(\mathbf{t})|<\widehat{Q}|r|}} \frac{1}{\min(|r|,\widehat{R})^d}\int_{|\boldsymbol{\zeta}|<\min(|r|,\widehat{R})}\psi\left(\frac{\langle \mathbf{c},\mathbf{t}+\boldsymbol{\zeta}\rangle}{gr}\right)d\boldsymbol{\zeta}d\mathbf{t}.$$

Since $|\mathbf{c}| \geq |g|\widehat{Q}/\widehat{R}$, and $|r| < \widehat{Q}$ then $\int_{|\boldsymbol{\zeta}|<\min(|r|,\widehat{R})}\psi\left(\frac{\langle \mathbf{c},\boldsymbol{\zeta}\rangle}{gr}\right)d\boldsymbol{\zeta} = 0$. This concludes the lemma. □

We say $\mathbf{c} \neq 0$ is an *exceptional* vector if $\kappa < \widehat{Q}/\widehat{R}$. For the exceptional vectors c, we represent $I_{g,r}(\mathbf{c})$ in terms of the Kloosterman sums (Salié sums) at ∞. For $\alpha \in K_\infty$ with $|\alpha|_\infty = \widehat{l}^2$, define

$$\mathrm{Kl}_\infty(\alpha,\psi) := \int_{|x|_\infty=\widehat{l}}\psi\left(\frac{\alpha}{x}+x\right)dx,$$

and

$$\mathrm{Sa}_\infty(\alpha,\psi) := \int_{|x|_\infty=\widehat{l}}\varepsilon_x\psi\left(\frac{\alpha}{x}+x\right)dx,$$

where ε_x were defined in (16). By Weil's estimate on the Kloosterman sums and the Salié sums, we show that $\mathrm{Kl}_\infty(\psi,\alpha) \ll |\alpha|^{1/4}$, and $\mathrm{Sa}_\infty(\psi,\alpha) \ll |\alpha|^{1/4}$.

Proposition 5.7. *Suppose that* $\mathbf{c}$ *is an exceptional vector and* $\kappa \geq \eta\frac{|r|}{\widehat{R}}$ *and* $d \geq 4$*, where* $\eta > \widehat{\omega}$ *is a fixed large enough constant integer. For* $\mathbf{c} \in \Omega^*$*, we have*

$$|I_{g,r}(\mathbf{c})| \ll \widehat{Q}^d\Big(\frac{|\mathbf{c}|\widehat{Q}}{|gr|}\Big)^{-\frac{d-1}{2}} \tag{22}$$

where the implied constant in $\ll$ *only depends on* F *and* Ω *. Otherwise,* $\mathbf{c} \notin \Omega^*$ *and* $I_{g,r}(\mathbf{c}) = 0$.

We give the proof of the above proposition after proving some auxiliary lemmas. For $\alpha \in K$ and $l \in \mathbb{Z}$, define

$$B_\infty(\psi,l,\alpha) := \int_{|x|_\infty=\widehat{l}}\psi(\frac{\alpha}{x}+x)dx,$$

$$\widetilde{B}_\infty(\psi,l,\alpha) := \int_{|x|_\infty=\widehat{l}}\varepsilon_x\psi(\frac{\alpha}{x}+x)dx.$$

We write $\alpha = t^{2l+k}\alpha'(1+\tilde{\alpha})$ and $x = t^l x'(1+\tilde{x})$ for unique $\tilde{\alpha},\tilde{x} \in \mathbb{T}$ and $\alpha', x' \in \mathbb{F}_q$. Note that for $k = 0$, we have $B_\infty(\psi,l,\alpha) = \mathrm{Kl}_\infty(\psi,\alpha)$ and $\widetilde{B}_\infty(\psi,l,\alpha) = \mathrm{Sa}_\infty(\psi,\alpha)$. In the following lemma, we give an explicit formula for $B_\infty(\psi,l,\alpha)$ in terms of the Kloosterman sums; see [CPS90, Lemma 3.4] for a similar calculation.

Lemma 5.8. *We have*

$$B_\infty(\psi,l,\alpha) := \begin{cases} (q-1)\widehat{l} & \text{if } \max(l+k,l) < -1, \text{ and } k \neq 0,\\ -\widehat{l} & \text{if } \max(l+k,l) = -1, \text{ and } k \neq 0,\\ 0 & \text{if } \max(l+k,l) > -1, \text{ and } k \neq 0.\end{cases}$$

$$\mathrm{Kl}_\infty(\psi,\alpha) := \begin{cases} (q-1)\widehat{l} & \text{if } l < -1,\\ \widehat{l}\mathrm{Kl}(\alpha',\mathbb{F}_q) & \text{if } l = -1,\\ \widehat{l}\sum_{x'^2=\alpha'}\psi\Big(2t^l x'(1+\tilde{\alpha})^{1/2}\Big)\mathcal{G}(2x't^l) & \text{if } \alpha' \text{ is a quadratic residue,}\\ 0 & \text{if } \alpha' \text{ is not a quadratic residue.}\end{cases}$$

Similarly,

$$\widetilde{B}_\infty(\psi,l,\alpha):=\begin{cases}(q-1)\widehat{l} & \text{if } \max(l+k,l)<-1, \text{ and } k\neq 0,\\ -\widehat{l} & \text{if } l+k=-1, \text{ and } k>0,\\ \widehat{l}\tau_\psi(\varepsilon) & \text{if } l=-1, \text{ and } k<0,\\ 0 & \text{if } \max(l+k,l)>-1, \text{ and } k\neq 0.\end{cases}$$

where $\tau_\psi:=\sum_{a\in\mathbb{F}_q}e_q(a)\chi(a)$, *where* χ *is the quadratic character in* $\mathbb{F}_q$. *Finally,*

$$\mathrm{Sa}_\infty(\psi,l,\alpha):=\begin{cases}(q-1)\widehat{l} & \text{if } l<-1,\\ \widehat{l}\mathrm{Sa}(\alpha',\mathbb{F}_q) & \text{if } l=-1,\\ \widehat{l}\sum_{x'^2=\alpha'}\psi\Big(2t^lx'(1+\tilde{\alpha})^{1/2}\Big)\mathcal{G}(2x't^l) & \text{if } \alpha' \text{ is a quadratic residue,}\\ 0 & \text{if } \alpha' \text{ is not a quadratic residue.}\end{cases}$$

Proof. Suppose that $k>0$. We have

$$B_\infty(\psi,l,\alpha)=\int_{|x|_\infty=\widehat{l}}\psi(\frac{\alpha}{x}+x)dx=\widehat{l}\sum_{x'\in\mathbb{F}_q^*}\int_{\mathbb{T}}\psi\Big(\frac{t^{l+k}\alpha'(1+\tilde{\alpha})}{x'(1+\tilde{x})}+t^lx'(1+\tilde{x})\Big)d\tilde{x}.$$

Fix $\tilde{\alpha}\in\mathbb{T}$ and $\alpha',x'\in\mathbb{F}_q$, and define the analytic function $u(\tilde{x})$ as

$$u(\tilde{x}):=\frac{\alpha'(1+\tilde{\alpha})}{x'(1+\tilde{x})}+t^{-k}x'(1+\tilde{x})-\big[\frac{\alpha'(1+\tilde{\alpha})}{x'}+t^{-k}x'\big],$$

where $\tilde{x}\in\mathbb{T}$. We note that $u(0)=0$, and $|\frac{\partial u}{\partial\tilde{x}}(0)|=|-\frac{\alpha'(1+\tilde{\alpha})}{(1+\tilde{x})^2x'}+t^{-k}x'|=1$. Hence $u\in\mathcal{A}_\infty(\mathbb{T})$. By changing the variable to $u(\tilde{x})$, we have

$$B_\infty(\psi,l,\alpha)=\widehat{l}\sum_{x'\in\mathbb{F}_q^*}\psi\Big(\frac{\alpha'(1+\tilde{\alpha})t^{l+k}}{x'}+x't^l\Big)\int_{\mathbb{T}}\psi(t^{l+k}u)du=\begin{cases}(q-1)\widehat{l} & \text{if } l+k<-1,\\ -\widehat{l} & \text{if } l+k=-1,\\ 0 & \text{otherwise.}\end{cases}$$

On the other hand, suppose that $k<0$. Fix $\tilde{\alpha}\in\mathbb{T}$ and $\alpha',x'\in\mathbb{F}_q$, and define the analytic function $v(\tilde{x})$ as

$$v(\tilde{x}):=t^k\frac{\alpha'(1+\tilde{\alpha})}{x'(1+\tilde{x})}+x'(1+\tilde{x})-\big[t^k\frac{\alpha'(1+\tilde{\alpha})}{x'}+x'\big],$$

where $\tilde{x}\in\mathbb{T}$. We note that $|\frac{\partial v}{\partial\tilde{x}}(0)|=|-\frac{t^k\alpha'(1+\tilde{\alpha})}{(1+\tilde{x})^2x'}+x'|=1$. Hence $v\in\mathcal{A}_\infty(\mathbb{T})$. By changing the variable to $v(\tilde{x})$, we have

$$B_\infty(\psi,l,\alpha)=\widehat{l}\sum_{x'\in\mathbb{F}_q^*}\psi\Big(\frac{\alpha'(1+\tilde{\alpha})t^{l+k}}{x'}+x't^l\Big)\int_{\mathbb{T}}\psi(t^lv)dv=\begin{cases}(q-1)\widehat{l} & \text{if } l<-1,\\ -\widehat{l} & \text{if } l=-1,\\ 0 & \text{otherwise.}\end{cases}$$

Finally suppose that $k=0$. Fix $\tilde{\alpha}\in\mathbb{T}$ and $\alpha',x'\in\mathbb{F}_q$. Suppose that $x'^2\neq\alpha'$ in $\mathbb{F}_q$, and define the analytic function $w(\tilde{x})$ as

$$w(\tilde{x}):=\frac{\alpha'(1+\tilde{\alpha})}{x'(1+\tilde{x})}+x'(1+\tilde{x})-\big[\frac{\alpha'(1+\tilde{\alpha})}{x'}+x'\big],$$

where $\tilde{x}\in\mathbb{T}$. We note that $|\frac{\partial w}{\partial\tilde{x}}(0)|=|-\frac{\alpha'(1+\tilde{\alpha})}{(1+\tilde{x})^2x'}+x'|=|-\frac{\alpha'(1+\tilde{\alpha})-(1+\tilde{x})^2x'^2}{(1+\tilde{x})^2x'}|=1$ and $w\in\mathcal{A}_\infty(\mathbb{T})$. Otherwise $x'^2=\alpha'$ in $\mathbb{F}_q$. Define $x_0:=(1+\tilde{\alpha})^{1/2}-1\in\mathbb{T}$ and

$$h(\tilde{x}):=\frac{\alpha'(1+\tilde{\alpha})}{x'(1+\tilde{x})}+x'(1+\tilde{x})-\big[2x'(1+\tilde{\alpha})^{1/2}\big].$$

It is easy to see that $h(x_0)=0$, $\frac{\partial h}{\partial \tilde{x}}(x_0)=0$ and $\frac{\partial^2 h}{\partial^2 \tilde{x}}(x_0)=\frac{2x'}{(1+\tilde{\alpha})^{1/2}}$. Hence x_0 is a critical point with $|\frac{\partial^2 h}{\partial^2 \tilde{x}}(x_0)|=1$. By the stationary phase theorem, we have

$$B_\infty(\psi,l,\alpha)=\widehat{l}\sum_{x'^2\neq\alpha'}\psi\Big(\frac{\alpha'(1+\tilde{\alpha})t^l}{x'}+x't^l\Big)\int_{\mathbb{T}}\psi(t^lw)dw+\widehat{l}\sum_{x'^2=\alpha'}\psi\Big(2t^lx'(1+\tilde{\alpha})^{1/2}\Big)\mathcal{G}(2x't^l)$$

Suppose that α' is a quadratic non-residue in $\mathbb{F}_q$. Then, from above it follows that

$$B_\infty(\psi,l,\alpha)=\begin{cases}(q-1)\widehat{l} & \text{if } l<-1,\\ \widehat{l}\mathrm{Kl}(\alpha',\mathbb{F}_q) & \text{if } l=-1,\\ 0 & \text{otherwise.}\end{cases}$$

Finally, assume that α' is a quadratic residue in $\mathbb{F}_q$. We have

$$B_\infty(\psi,l,\alpha)=\begin{cases}(q-1)\widehat{l} & \text{if } l<-1,\\ \widehat{l}\mathrm{Kl}(\alpha',\mathbb{F}_q) & \text{if } l=-1,\\ \widehat{l}\sum_{x'^2=\alpha'}\psi\Big(2t^lx'(1+\tilde{\alpha})^{1/2}\Big)\mathcal{G}(2x't^l) & \text{otherwise.}\end{cases}$$

This concludes the proof of the first part of the lemma. The argument for $\widetilde{B}_\infty(\psi,l,\alpha)$ is similar. Recall that $\varepsilon_x=1$ unless l is odd, which is the quadratic character evaluated at the top coefficient of t^2x. The second part of the lemma follows from the same lines, and we skip the details. □

Proof of Proposition 5.7. By Lemma 5.5, $|G(\mathbf{t})|<\widehat{Q}|r|$ is equivalent to $|F(\mathbf{t})-k/g|<\widehat{Q}|r|$ for $|\mathbf{t}-\mathbf{t}_0|<\widehat{R}$. By Lemma 2.2, we have

$$\int_{\mathbb{T}}\psi\Big(\frac{\alpha}{rt^Q}(F(\mathbf{t})-k/g)\Big)d\alpha=\begin{cases}1, & \text{if } |F(\mathbf{t})-k/g|<\widehat{Q}|r|,\\ 0, & \text{otherwise.}\end{cases}$$

We replace the above integral for detecting $|F(\mathbf{t})-k/g|<\widehat{Q}|r|$. Hence, by (18)

$$I_{g,r}(\mathbf{c})=\frac{\widehat{Q}}{|r|}\int_{\mathbb{T}}\int_{|\mathbf{t}-\mathbf{t}_0|<\widehat{R}}\psi\left(\frac{\langle\mathbf{c},\mathbf{t}\rangle}{gr}+\frac{\alpha}{rt^Q}(F(\mathbf{t})-k/g)\right)d\mathbf{t}d\alpha.$$

Recall that $F(\gamma\mathbf{y})=\sum_{\eta_i}\eta_iy_i^2$ for some $\gamma\in GL_d(\mathcal{O}_\infty)$. We change variables to $\mathbf{y}=\begin{bmatrix}y_1\\ \vdots\\ y_d\end{bmatrix}=\gamma^{-1}\mathbf{t}$, and obtain

$$\frac{\langle\mathbf{c},\mathbf{t}\rangle}{gr}+\frac{\alpha}{rt^Q}(F(\mathbf{t})-k/g)=\frac{-\alpha k}{rgt^Q}+\frac{1}{r}\Big(\sum_i\frac{c_i'y_i}{g}+\frac{\alpha\eta_iy_i^2}{t^Q}\Big),$$

where $\begin{bmatrix}c_1'\\ \vdots\\ c_d'\end{bmatrix}=\gamma^\intercal\mathbf{c}$. Let $\mathbf{y}_0:=\gamma^{-1}\mathbf{t}_0$. Then γ is a bijection between $\Big\{\mathbf{t}\in K_\infty^d:\ |\mathbf{t}-\mathbf{t}_0|<\widehat{R}\Big\}$ and $\Big\{\mathbf{y}\in K_\infty^d:\ |\mathbf{y}-\mathbf{y}_0|<\widehat{R}\Big\}$. Hence, $I_{g,r}(\mathbf{c})=\frac{\widehat{Q}}{|r|}\int_{\mathbb{T}}\psi(\frac{-\alpha k}{rgt^Q})I_{g,r}(\alpha,\mathbf{c})d\alpha$, where

$$I_{g,r}(\alpha,\mathbf{c}):=\prod_{i=1}^d\int_{|y_i-y_{i0}|<\widehat{R}}\psi\left(\frac{1}{r}\left(\frac{c_i'y_i}{g}+\frac{\alpha\eta_iy_i^2}{t^Q}\right)\right)dy_i,$$

where $\begin{bmatrix} y_{10} \\ \vdots \\ y_{d0} \end{bmatrix} = \mathbf{y}_0$. We write $z_i := y_i - y_{i0}$. We have

$$I_{g,r}(\alpha, \mathbf{c}) := \prod_{i=1}^{d} \int_{|z_i|<\widehat{R}} \psi\left(\frac{1}{r}\left(\frac{c_i'(z_i+y_{i0})}{g} + \frac{\alpha\eta_i(z_i+y_{i0})^2}{t^Q}\right)\right) dz_i,$$

The phase function has a critical point at $\frac{-c_i' t^Q}{2g\eta_i\alpha} - y_{i0}$. This critical point is inside the domain of the integral, if $|\kappa_i| < \widehat{R}$, where $\kappa_i := \frac{c_i' t^Q}{g\eta_i\alpha} + 2y_{i0}$. Note that κ_i is a function of α. Given $\alpha \in \mathbb{T}$, we partition the indices into:

$$CR := \left\{1 \le i \le d : |\kappa_i| < \widehat{R}\right\},$$
$$NCR := \left\{1 \le i \le d : |\kappa_i| \ge \widehat{R}\right\}.$$

For $i \in NCR$, we change the variables to $v_i = z_i + \kappa_i^{-1} z_i^2$. It is easy to check that this change of variables belongs to $\mathcal{A}_\infty(\mathbf{t} < R)$. For $i \in CR$, we change the variables to $w_i = z_i + \kappa_i/2$. Hence,

$$\begin{aligned} I_{g,r}(\alpha, \mathbf{c}) = &\prod_{i\in NCR} \psi\left(\frac{1}{r}\left(\frac{c_i' y_{i0}}{g} + \frac{\alpha\eta_i y_{i0}^2}{t^Q}\right)\right) \int_{|v_i|<\widehat{R}} \psi\left(\frac{\alpha\eta_i}{rt^Q}\kappa_i v_i\right) dv_i \\ &\times \prod_{i\in CR} \psi(-\frac{t^Q {c_i'}^2}{4rg^2\eta_i\alpha}) \int_{|w_i|<\widehat{R}} \psi\left(\frac{\alpha\eta_i}{rt^Q} w_i^2\right) dw_i. \end{aligned} \tag{23}$$

By Lemma 2.2 and Lemma 4.6, we have

$$\int_{|v_i|<\widehat{R}} \psi\left(\frac{\alpha\eta_i}{rt^Q}\kappa_i v_i\right) dv_i = \begin{cases} \widehat{R}, & \text{if } |\frac{\alpha\eta_i}{rt^Q}\kappa_i| < 1/\widehat{R}\ , \\ 0, & \text{otherwise,} \end{cases} \tag{24}$$

$$\int_{|w_i|<\widehat{R}} \psi\left(\frac{\alpha\eta_i}{rt^Q} w_i^2\right) dw_i = \widehat{R}\mathcal{G}\left(\frac{\alpha\eta_i t^{2R}}{rt^Q}\right).$$

Suppose that $\mathbf{c}' \notin \Omega^*$. By Lemma 5.2, $\max_{1\le i\le d} |\kappa_i| \ge |\mathbf{y}_0|/\widehat{\omega} \ge \widehat{R}$. On the other hand, recall that $\kappa := \max_i |\frac{c_i}{g}|$. Since $\begin{bmatrix} c_1' \\ \vdots \\ c_d' \end{bmatrix} = \gamma^\intercal \mathbf{c}$ and $\gamma \in GL_d(\mathcal{O}_\infty)$, $\kappa = \max_i |\frac{c_i}{g}| = \max_i |\frac{c_i'}{g}|$. By Lemma 5.2,

$$\max_{1\le i\le d} \left|\kappa_i \frac{\alpha\eta_i}{t^Q}\right| = \max_{1\le i\le d} \left(\frac{c_i'}{g} + \frac{2\alpha\eta_i y_{i0}}{t^Q}\right) \ge \kappa/\widehat{\omega}$$

By our assumption, $\kappa \ge \eta\frac{|r|}{\widehat{R}}$. Since $\eta > \widehat{\omega}$, $\max_{1\le i\le d} |\frac{\alpha\eta_i}{rt^Q}\kappa_i| \ge 1/\widehat{R}$. By equations (23) and (24), we have $I_{g,r}(\mathbf{c}) = 0$ for $\mathbf{c}' \notin \Omega^*$.

Next, we suppose that $\mathbf{c}' \in \Omega^*$ and prove inequality (22). By equations (23) and (24), $I_{g,r}(\mathbf{c}) = 0$ unless $|\alpha| = \widehat{l}$, where $\widehat{l} := \kappa\frac{\widehat{Q}}{|A\mathbf{t}_0|}$. Note that $|\alpha| = \widehat{l} \gg \kappa$. By equations (23) and (24), we have

$$I_{g,r}(\alpha, \mathbf{c}) = \widehat{R}^d \prod_{i=1}^{d} \left(\delta_{\widehat{R}\le\kappa_i<\frac{|r|\widehat{Q}}{\widehat{R}|\alpha\eta_i|}} \psi\left(\frac{1}{r}\left(\frac{c_i' y_{i0}}{g} + \frac{\alpha\eta_i y_{i0}^2}{t^Q}\right)\right) + \delta_{\kappa_i<\widehat{R}} \psi(-\frac{t^Q {c_i'}^2}{4rg^2\eta_i\alpha})\mathcal{G}\left(\frac{\alpha\eta_i t^{2R}}{rt^Q}\right)\right). \tag{25}$$

The contribution of the first term on the right hand side is zero unless $\widehat{R} \leq \frac{|r|\widehat{Q}}{\widehat{R}|\alpha\eta_i|}$, which implies

$$|\alpha| \leq \frac{|r|}{\widehat{R}}\left(\frac{\widehat{Q}}{\widehat{R}|\eta_i|}\right) \ll \frac{|r|}{\widehat{R}}.$$

By comparing the preceding inequality with $\alpha \gg \kappa$, we have $\kappa \ll \frac{|r|}{\widehat{R}}$. By choosing η large enough, this contradicts with our assumption $\kappa \geq \eta\frac{|r|}{\widehat{R}}$. Therefore, for large enough η

$$I_{g,r}(\mathbf{c}) = \frac{\widehat{Q}\widehat{R}^d}{|r|}\int_{|\alpha|=\widehat{l}} \psi(\frac{-\alpha k}{rgt^Q}) \prod_{\kappa_i<|\alpha|} \psi(-\frac{t^Q {c_i'}^2}{4rg^2\eta_i\alpha})\mathcal{G}\left(\frac{\alpha\eta_i t^{2R}}{rt^Q}\right) d\alpha.$$

By (16), we have

$$\prod_i \mathcal{G}\left(\frac{\alpha\eta_i t^{2R}}{rt^Q}\right) = \pm\varepsilon_\alpha^v \prod_i \min\left(1, \left(\frac{\widehat{l}\widehat{R}^2|\eta_i|}{|r|\widehat{Q}}\right)^{-1/2}\right),$$

where $v = 0, 1$ depending on parity of the degrees of η_i and α and quadratic residue of their top coefficients. Hence,

$$I_{g,r}(\mathbf{c}) = \frac{\widehat{Q}\widehat{R}^d}{|r|}\sum_{\kappa<\widehat{l}<1} \pm \prod_i \min\left(1, \left(\frac{\widehat{l}\widehat{R}^2|\eta_i|}{|r|\widehat{Q}}\right)^{-1/2}\right)\int_{|\alpha|=\widehat{l}} \psi(\frac{-\alpha k}{rgt^Q})\psi(-\frac{t^Q F^*(\mathbf{c})}{4rg^2\alpha})\varepsilon_\alpha^v d\alpha,$$

where $F^*(\mathbf{c}) = \sum_i \frac{{c_i'}^2}{\eta_i}$. By Lemma 5.8, we have

$$\begin{aligned}\int_{|\alpha|=\widehat{l}} \psi(\frac{-\alpha k}{rgt^Q})\psi(-\frac{t^Q F^*(\mathbf{c})}{4rg^2\alpha})\varepsilon_\alpha^v d\alpha &= \begin{cases} |\frac{rgt^Q}{k}|B_\infty(\psi, l+\deg(\frac{k}{rgt^Q}), \frac{kF^*(\mathbf{c})}{4r^2g^3}) & \text{for } v=0,\\ |\frac{rgt^Q}{k}|\tilde{B}_\infty(\psi, l+\deg(\frac{k}{rgt^Q}), \frac{kF^*(\mathbf{c})}{4r^2g^3}) & \text{for } v=1,\end{cases}\\ &= \begin{cases} |\frac{rgt^Q}{k}|\mathrm{Kl}_\infty(\psi, \frac{kF^*(\mathbf{c})}{4r^2g^3}) & \text{if } 2l=\deg(\frac{t^{2Q}F^*(\mathbf{c})}{kg}), \text{ and } v=0\\ |\frac{rgt^Q}{k}|\mathrm{Sa}_\infty(\psi, \frac{kF^*(\mathbf{c})}{4r^2g^3}) & \text{if } 2l=\deg(\frac{t^{2Q}F^*(\mathbf{c})}{kg}), \text{ and } v=1\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

Therefore, by using the Weil bound on the Kloosterman sums (Salie sums), we have

$$|I_{g,r}(\mathbf{c})| \ll \frac{\widehat{Q}\widehat{R}^d}{|r|}\left(\frac{|F^*(\mathbf{c})|^{1/2}\widehat{R}^2}{|f|^{1/2}|r|}\right)^{-d/2}\left|\frac{rgt^Q}{k}\right|\left|\frac{fF^*(\mathbf{c})}{r^2g^4}\right|^{1/4} \ll \widehat{Q}^d\Big(\frac{|F^*(\mathbf{c})|^{1/2}\widehat{Q}}{|gr|}\Big)^{-\frac{d-1}{2}},$$

where we used $|f|^{1/2} \gg \widehat{Q}|g|$. Since $|\mathbf{c}| \ll |F^*(\mathbf{c})|^{1/2}$ for $\mathbf{c} \in \Omega^*$, this concludes Proposition 5.7. □

6. Main contribution to counting function

In this section, we study the main contribution to the counting function $N(w, \boldsymbol{\lambda})$. We first begin by estimating the contribution in $N(w, \boldsymbol{\lambda})$ from the terms where $\mathbf{c} = 0$. In order to do so, we first prove the following lemma which gives an estimate on the the norm of $I_{g,r}(\mathbf{0})$ for $|r|$ not too large. We then show that the contribution from the other terms is small. Finally, we show that contribution from 0 can be written in terms of local densities.

Lemma 6.1. *Suppose $\varepsilon > 0$. With the notation as before and for $1 \leq |r| \leq \widehat{Q}^{1-\varepsilon}$, we have*

$$|I_{g,r}(\mathbf{0})| = C_F\widehat{Q}^d(1 + O(\widehat{Q}^{-\varepsilon}))$$

for some constant C_F as $\varepsilon \to 0^+$.

Proof. It follows from equation 18 that

$$I_{g,r}(\mathbf{0}) = \frac{\widehat{Q}}{|r|}\int_{\substack{|\mathbf{t}-\mathbf{t}_0|<\widehat{R}\\ |G(\mathbf{t})|<\widehat{Q}|r|}} d\mathbf{t} = \frac{\widehat{Q}}{|r|}\int_{\substack{|g\mathbf{t}+\boldsymbol{\lambda}-\mathbf{x}_0|\leq|t^{-\alpha_0}f|^{1/2}\\ |F(g\mathbf{t}+\boldsymbol{\lambda})-f|<\widehat{Q}|r||g|^2}} d\mathbf{t}.$$

Making the substitution $\mathbf{x} = g\mathbf{t}+\boldsymbol{\lambda}$ gives us the equality

$$I_{g,r}(\mathbf{0}) = \frac{\widehat{Q}}{|r||g|^d}\int_{|\mathbf{x}-\mathbf{x}_0|\leq|t^{-\alpha_0}f|^{1/2}:|F(\mathbf{x})-f|<\widehat{Q}|r||g|^2} d\mathbf{x}.$$

Furthermore, by Lemma 2.2 and Fubini, we may rewrite this as

$$\begin{aligned} I_{g,r}(\mathbf{0}) &= \frac{\widehat{Q}}{|r||g|^d}\int_{|\mathbf{x}-\mathbf{x}_0|\leq|t^{-\alpha_0}f|^{1/2}}\int_{\mathbb{T}}\psi\left(\frac{F(\mathbf{x})-f}{rg^2t^Q}\alpha\right)d\alpha d\mathbf{x}\\ &= \frac{\widehat{Q}}{|r||g|^d}\int_{\mathbb{T}}\int_{|\mathbf{x}|<\widehat{D}}\psi\left(\frac{F(\mathbf{x}+\mathbf{x}_0)-f}{rg^2t^Q}\alpha\right)d\mathbf{x}d\alpha\\ &= \frac{\widehat{Q}\widehat{D}^d}{|r||g|^d}\int_{\mathbb{T}}\int_{\mathbb{T}^d}\psi\left(\frac{F(\mathbf{x}+t^{-D}\mathbf{x}_0)-f/t^{2D}}{rg^2t^{Q-2D}}\alpha\right)d\mathbf{x}d\alpha \end{aligned}$$

where $D := \lceil\frac{1}{2}(-\alpha_0+\deg f+1)\rceil$ and the last equality follows from scaling the $\mathbf{x}$ coordinate by a factor of $\widehat{D}$. Making the substitution $\beta = \frac{\alpha}{rg^2t^{Q-2D}}$, we obtain the equality

$$I_{g,r}(\mathbf{0}) = \frac{\widehat{Q}^2\widehat{D}^d}{|g|^{d-2}\widehat{2D}}\int_{|\beta|<\frac{\widehat{2D}}{\widehat{Q}|r||g|^2}}\int_{\mathbb{T}^d}\psi\left((F(\mathbf{x}+t^{-D}\mathbf{x}_0)-f/t^{2D})\beta\right)d\mathbf{x}d\beta.$$

Note that the integral is equal to

$$\frac{\widehat{2D}}{\widehat{Q}|r||g|^2}\mathrm{vol}\left(\left\{\mathbf{x}\in\mathbb{T}^d : |F(\mathbf{x}+t^{-D}\mathbf{x}_0)-f/t^{2D}|\leq\frac{\widehat{Q}|r||g|^2}{\widehat{2D}}\right\}\right)\geq 0.$$

Consequently, the first integral is a non-negative real number and can be viewed as a density. Note that $\mathbf{x}=0$ is a zero of $F(\mathbf{x}+t^{-D}\mathbf{x}_0)-f/t^{2D}$. Consequently, by Lemma 6.2 proved next, we can choose $\widehat{Q}$ large enough (depending on ε and the F) such that the integral corresponds to taking integrals for $|\beta|$ larger than the threshhold after which it is positive (see next lemma). The conclusion follows.

□

We prove the following lemma that was used in the proof of the previous lemma.

Lemma 6.2. *Let L be an integer, and let Q be a polynomial over K_∞ such that $Q(\mathbf{0}) = f/t^{2D}$, and consider*

$$\int_{\mathbb{T}^d}\int_{|\beta|\leq\widehat{L}}\psi\left((Q(\mathbf{x})-f/t^{2D})\beta\right)d\beta d\mathbf{x}.$$

The limit as $L\to\infty$ exists and is a strictly positive number $\sigma_\infty > 0$.

Proof. As in the computation in the proof of the previous lemma, we have the equality

$$\int_{\mathbb{T}^d}\int_{|\beta|\leq\widehat{L}}\psi\left((Q(\mathbf{x})-f/t^{2D})\beta\right)d\beta d\mathbf{x} = \widehat{L}\mathrm{vol}\left(\left\{\mathbf{x}\in\mathbb{T}^d : |Q(\mathbf{x})-f/t^{2D}|\leq\widehat{L}^{-1}\right\}\right).$$

Note that $\mathrm{vol}(t^{-L}\mathbb{T}) = \widehat{L}^{-1}$. Each $\mathbf{x}\in\mathbb{T}$ such that $|Q(\mathbf{x})-f/t^{2D}|\leq\widehat{L}^{-1}$ gives us a coset $\mathbf{x}+t^{-L}\mathbb{T}^d$ of solutions in $Q^{-1}(t^{-L}\mathbb{T})$. Hence, using $\mathrm{vol}(t^{-L}\mathbb{T}^d) = \widehat{L}^{-d}$, we have

$$\mathrm{vol}(Q^{-1}(t^{-L}\mathbb{T})) = \widehat{L}^{-d}|\{\mathbf{x}+t^{-L}\mathbb{T}^d\in\mathbb{T}^d/t^{-L}\mathbb{T}^d : |Q(\mathbf{x})-f/t^{2D}|\leq\widehat{L}^{-1}\}|.$$

Therefore,

$$\begin{aligned} & \widehat{L}\mathrm{vol}\left(\left\{\mathbf{x}\in\mathbb{T}^d : |Q(\mathbf{x})-f/t^{2D}|\leq \widehat{L}^{-1}\right\}\right) \\ = \ & \widehat{L}^{-d+1}|\{\mathbf{x}+t^{-L}\mathbb{T}^d\in\mathbb{T}^d/t^{-L}\mathbb{T}^d : |Q(\mathbf{x})-f/t^{2D}|\leq \widehat{L}^{-1}\}| \\ = \ & \frac{|\{\mathbf{x}+t^{-L}\mathbb{T}^d\in\mathbb{T}^d/t^{-L}\mathbb{T}^d : |Q(\mathbf{x})-f/t^{2D}|\leq \widehat{L}^{-1}\}|}{\widehat{L}^{d-1}}. \end{aligned}$$

By Hensel's Lemma, for large enough L, this latter quantity stabilizes. Since there is a solution in $\mathbb{T}$ to the equation $Q(\mathbf{x})=f/t^{2D}$, namely $\mathbf{0}$, the above quantity is strictly positive as well. The conclusion follows. □

We now show that when $\widehat{Q}^{1-\varepsilon}\leq|r|\leq\widehat{Q}$, then the contribution of the terms in $N(w,\boldsymbol{\lambda})$ when $\mathbf{c}=\mathbf{0}$ and corresponding to such r is small. This follows from the following more general statement for all $\mathbf{c}$.

Lemma 6.3.

$$\sum_{\widehat{Q}^{1-\varepsilon}\leq|r|\leq\widehat{Q}}|gr|^{-d}|S_{g,r}(\mathbf{c})||I_{g,r}(\mathbf{c})|\ll_{\varepsilon,\Delta}|g|^{\varepsilon}\widehat{Q}^{\frac{d+3}{2}+\varepsilon}$$

Proof. Suppose $\widehat{Q}^{1-\varepsilon}\leq|r|\leq\widehat{Q}$. It is easy to see from the definition of $I_{g,r}(\mathbf{c})$ that for such r,

$$|I_{g,r}(\mathbf{c})|\ll\widehat{Q}^{d+\varepsilon}.$$

Using this, we obtain

$$\begin{aligned} \sum_{\widehat{Q}^{1-\varepsilon}\leq|r|\leq\widehat{Q}}|gr|^{-d}|S_{g,r}(\mathbf{c})||I_{g,r}(\mathbf{c})| &= \sum_{\widehat{Q}^{1-\varepsilon}\leq|r|\leq\widehat{Q}}|r|^{-\frac{d-1}{2}}|g|^{-d}|r|^{-\frac{d+1}{2}}|S_{g,r}(\mathbf{c})||I_{g,r}(\mathbf{c})| \\ &\leq \widehat{Q}^{d+\varepsilon}\sum_{(1-\varepsilon)Q<k\leq Q}\left(q^k\right)^{-\frac{d-1}{2}}\sum_{|r|=q^k}|g|^{-d}|r|^{-\frac{d+1}{2}}|S_{g,r}(\mathbf{c})| \end{aligned}$$

By Proposition 3.1,

$$\sum_{|r|=q^k}|g|^{-d}|r|^{-\frac{d+1}{2}}|S_{g,r}(\mathbf{c})|\ll|g|^{\varepsilon}\left(q^k\right)^{1+\varepsilon}.$$

Therefore,

$$\widehat{Q}^{d+\varepsilon}\sum_{(1-\varepsilon)Q<k\leq Q}\left(q^k\right)^{-\frac{d-1}{2}}\sum_{|r|=q^k}|g|^{-d}|r|^{-\frac{d+1}{2}}|S_{g,r}(\mathbf{0})|\ll|g|^{\varepsilon}\widehat{Q}^{d+\varepsilon}\sum_{(1-\varepsilon)Q<k\leq Q}\left(q^k\right)^{-\frac{d-3}{2}+\varepsilon}\ll|g|^{\varepsilon}\widehat{Q}^{\frac{d+3}{2}+\varepsilon},$$

as required. □

In order to put this lemma into greater perspective, we use the next two lemmas to estimate

$$\sum_{r:1\leq|r|\leq\widehat{T}}|gr|^{-d}S_{g,r}(0).$$

Lemma 6.4. *For $d\geq4$ and every $\mathbf{c}$, the sum*

$$\sum_{r}|r|^{-d}S_{g,r}(\mathbf{c})$$

is absolutely convergent.

Proof. Using Lemmas 3.3 and 3.5, we obtain

$$\begin{aligned}|r|^{-d}|S_{g,r}(\mathbf{c})| &\ll_\Delta \tau(r_1)|r|^{-d}|r_1|^{\frac{d+1}{2}}|\mathrm{rad}_3'(r_1)|^{1/2}|r_2|^{d/2+1}|\gcd(r_1,f)|^{1/2}\\ &= \tau(r_1)|r|^{-d/2+1}|\mathrm{rad}_3'(r_1)|^{1/2}\frac{|\gcd(r_1,f)|^{1/2}}{|r_1|^{1/2}}\\ &\le |r|^{-d/2+1+\epsilon}|f|^{1/2}\frac{|\mathrm{rad}_3'(r_1)|^{1/2}}{|r_1|^{1/2}}.\end{aligned}$$

Hence,

$$\sum_{|r|\le\widehat{X}}|r|^{-d}|S_{g,r}(\mathbf{c})| \ll_{\Delta,g,f} \sum_{1\le N\le X}\widehat{N}^{-d/2+1+\epsilon}\sum_{|r_1|\le\widehat{N}}\frac{1}{|r_1|^{1/2}} \ll_{\Delta,g,f}\sum_{1\le N\le X}\widehat{N}^{-d/2+3/2+2\epsilon}.$$

The last summation is a partial sum of a geometric series, and so the associated infinite sum is convergent since $d\ge 4$. □

Lemma 6.5. *For any $\varepsilon>0$, we have*

$$\sum_{r:1\le|r|\le\widehat{T}}|r|^{-d}S_{g,r}(0)=\sum_r|r|^{-d}S_{g,r}(0)+O_{\varepsilon,\Delta}(\widehat{T}^{3/2-\frac{d}{2}+\varepsilon}).$$

Proof. Write

$$\sum_r|r|^{-d}S_{g,r}(0)=\sum_{r:1\le|r|\le\widehat{T}}|r|^{-d}S_{g,r}(0)+\sum_{|r|>\widehat{T}}|r|^{-d}S_{g,r}(0).$$

The triangle inequality gives us

$$\left|\sum_{|r|>\widehat{T}}|r|^{-d}S_{g,r}(0)\right|\le\sum_{\widehat{N}=\widehat{T}}^{\infty}\widehat{N}^{-d}\sum_{|r|=\widehat{N}}|S_{g,r}(0)|.$$

From $S_{g,r}(0)=S_1S_2$ and Lemmas 3.3 and 3.5, we have

$$|S_{g,r}(0)|\ll_{\Delta,g}\tau(r_1)|r|^{d/2}|r_1|^{1/2}|\mathrm{rad}_3'(r_1)|^{1/2}|r_2||\gcd(r_1,f)|^{1/2},$$

using which we obtain

$$\begin{aligned}\sum_{N=T}^{\infty}\widehat{N}^{-d}\sum_{|r|=\widehat{N}}|S_{g,r}(0)| &\ll_\Delta \sum_{N=T}^{\infty}\widehat{N}^{-d/2}\sum_{|r|=\widehat{N}}\tau(r_1)|r_1|^{1/2}|\mathrm{rad}_3'(r_1)|^{1/2}|r_2||\gcd(r_1,f)|^{1/2}\\ &\le \sum_{N=T}^{\infty}\widehat{N}^{1-\frac{d}{2}}\sum_{|r_1|\le\widehat{N}}\tau(r_1)|r_1|^{-1/2}|\mathrm{rad}_3'(r_1)|^{1/2}|\gcd(r_1,f)|^{1/2}\\ &\le \sum_{N=T}^{\infty}\widehat{N}^{3/2-\frac{d}{2}+\varepsilon}\sum_{|r_1|\le\widehat{N}}\frac{|\gcd(r_1,f)|^{1/2}|\mathrm{rad}_3'(r_1)|^{1/2}}{|r_1|}\\ &= \sum_{N=T}^{\infty}\widehat{N}^{3/2-\frac{d}{2}+2\varepsilon}=O_{\varepsilon,\Delta}(\widehat{T}^{3/2-\frac{d}{2}+\varepsilon}),\end{aligned}$$

where we have used that $d\ge4$. Using this, we obtain that

$$\sum_{1\le|r|\le\widehat{T}}|r|^{-d}S_{g,r}(0)=\sum_r|r|^{-d}S_{g,r}(0)+O_{\varepsilon,\Delta}(\widehat{T}^{3/2-\frac{d}{2}+\varepsilon}).$$

From Lemma 6.4, the infinite sum is absolutely convergent. The conclusion follows. □

We now want to show that the infinite sum

$$\sum_r |r|^{-d} S_{g,r}(0)$$

can be entirely written in terms of number theoretic information.

Lemma 6.6.

$$\sum_r |r|^{-d} S_{g,r}(0) = |g|^d \prod_{\varpi} \sigma_{\varpi},$$

where ϖ ranges over the monic irreducible polynomials in $\mathbb{F}_q[t]$, and

$$\sigma_{\varpi} := \lim_{k\to\infty} \frac{|\left\{\mathbf{x} \bmod \varpi^{k+\nu_{\varpi}(g)} : F(\mathbf{x}) \equiv f \bmod \varpi^{k+\nu_{\varpi}(g)},\ \mathbf{x} \equiv \boldsymbol{\lambda} \bmod \varpi^{\nu_{\varpi}(g)}\right\}|}{|\varpi|^{(d-1)k}},$$

and is strictly positive.

Proof. We know that the infinite sum is absolutely convergent. Define for each $N \geq 0$ the analogue of the factorial

$$(N)! := \prod_{\substack{|f|\leq \widehat{N} \\ f \text{ monic}}} f.$$

Write

$$\begin{aligned}
&\sum_{r|(N)!} |r|^{-d} S_{g,r}(0) \\
&= \sum_{r|(N)!} |r|^{-d} \sum_{\substack{a \bmod gr \\ (a,r)=1}} \sum_{\mathbf{b}\in\mathcal{O}^d/(gr)} \psi\left(\frac{a(2\boldsymbol{\lambda}^T A\mathbf{b} - k + gF(\mathbf{b}))}{gr}\right) \\
&= \frac{1}{|(N)!|^d} \sum_{r|(N)!} \sum_{\substack{a \bmod gr \\ (a,r)=1}} \left|\frac{(N)!}{r}\right|^d \sum_{\mathbf{b}\in\mathcal{O}^d/(gr)} \psi\left(\frac{a(2\boldsymbol{\lambda}^T A\mathbf{b} - k + gF(\mathbf{b}))}{gr}\right) \\
&= \frac{1}{|(N)!|^d} \sum_{r|(N)!} \sum_{\substack{a \bmod gr \\ (a,r)=1}} \sum_{\mathbf{b}\in\mathcal{O}^d/(g(N)!)} \psi\left(\frac{a(2\boldsymbol{\lambda}^T A\mathbf{b} - k + gF(\mathbf{b}))}{gr}\right) \\
&= \frac{1}{|(N)!|^d} \sum_{\mathbf{b}\in\mathcal{O}^d/(g(N)!)} \sum_{r|(N)!} \sum_{\substack{a \bmod gr \\ (a,r)=1}} \psi\left(\frac{a(2\boldsymbol{\lambda}^T A\mathbf{b} - k + gF(\mathbf{b}))}{gr}\right).
\end{aligned}$$

Since

$$\sum_{r|(N)!} \sum_{\substack{a \bmod gr \\ (a,r)=1}} = \sum_{a \bmod g(N)!},$$

$$\sum_{r|(N)!} |r|^{-d} S_{g,r}(0) = \frac{1}{|(N)!|^d} \sum_{a \bmod g(N)!} \sum_{\mathbf{b}\in\mathcal{O}^d/(g(N)!)} \psi\left(\frac{a(2\boldsymbol{\lambda}^T A\mathbf{b} - k + gF(\mathbf{b}))}{g(N)!}\right).$$

Furthermore, this latter quantity is equal to

$$|g| \frac{\left|\{\mathbf{b} \in \mathcal{O}^d/(g(N)!) : 2\boldsymbol{\lambda}^T A\mathbf{b} - k + gF(\mathbf{b}) \equiv 0 \bmod g(N)!\}\right|}{|(N)!|^{d-1}}.$$

Let us write $(N)! = \varpi_1^{a_1} \dots \varpi_\ell^{a_\ell}$. Then

$$2\boldsymbol{\lambda}^T A\mathbf{b} - k + gF(\mathbf{b}) \bmod g(N)!$$

is the same as having

$$F(g\mathbf{b}+\lambda)-f\equiv \mathrm{mod}\varpi_i^{a_i+2\nu_{\varpi_i}(g)}$$

for each $i=1,\dots,\ell$. We conclude that

$$\begin{aligned}
&\sum_{r|(N)!}|r|^{-d}S_{g,r}(0)\\
=\;&|g|^d\prod_{\varpi|(N)!}\frac{\left|\{\mathbf{b}\in\mathcal{O}^d/(\varpi^{\nu_\varpi((N)!)+\nu_\varpi(g)}):F(\varpi^{\nu_\varpi(g)}\mathbf{b}+\boldsymbol{\lambda})\equiv f \bmod \varpi^{\nu_\varpi((N)!)+2\nu_\varpi(g)}\}\right|}{\left|\varpi^{\nu_\varpi((N)!)+\nu_\varpi(g)}\right|^{d-1}}\\
=\;&|g|^d\prod_{\varpi|(N)!}\frac{\left|\{\mathbf{x}\in\mathcal{O}^d/(\varpi^{\nu_\varpi((N)!)+\nu_\varpi(g)}):F(\mathbf{x})\equiv f \bmod \varpi^{\nu_\varpi((N)!)+\nu_\varpi(g)},\ \mathbf{x}\equiv\boldsymbol{\lambda} \bmod \varpi^{\nu_\varpi(g)}\}\right|}{\left|\varpi^{\nu_\varpi((N)!)+\nu_\varpi(g)}\right|^{d-1}}.
\end{aligned}$$

Letting $N\to\infty$ gives us

$$\sum_r|r|^{-d}S_{g,r}(0)=|g|^d\prod_\varpi\sigma_\varpi,$$

where σ_ϖ are as in the statement of the lemma. By Hensel's lemma and the fact that our system satisfies all local conditions, the local densities are strictly positive.

□

7. Proof of the main theorem

In this section, we prove our main theorem. Though we obtain a theorem for $d\geq 4$, it is only optimal when $d\geq 5$. We assume that we have a non-degenerate quadratic form over $\mathbb{F}_q[t]$ in $d\geq 4$ variables. We would like to show that under good conditions, we have strong approximation. Though the conclusion will be optimal in $d\geq 5$ variables, it will not be so for $d=4$ variables. We first give a bound on the contributions of the nonzero exceptional vectors to our counting function.

Proposition 7.1. *For any non-degenerate quadratic form F over $\mathbb{F}_q[t]$ in $d\geq 4$ variables, and for any $\varepsilon>0$, we have*

$$\sum_{1\leq|r|\leq\widehat{Q}}\sum_{\mathbf{c}\neq 0}^{\mathrm{exc}}|gr|^{-d}|S_{g,r}(\mathbf{c})||I_{g,r}(\mathbf{c})|\ll_\varepsilon \widehat{Q}^{\frac{d+3}{2}+\varepsilon}|g|^{\frac{d-3}{2}+\varepsilon}(1+|g|^{-\frac{d-5}{2}+\varepsilon}),$$

where $\sum^{\mathrm{exc}}$ denotes summation over exceptional vectors.

We prove this proposition by rewriting

$$\sum_{1\leq|r|\leq\widehat{Q}}\sum_{\mathbf{c}\neq 0}^{\mathrm{exc}}|gr|^{-d}S_{g,r}(\mathbf{c})I_{g,r}(\mathbf{c})=E_1+E_2,$$

where

$$E_1:=\sum_{\mathbf{c}\neq 0}^{\mathrm{exc}}\sum_{1\leq|r|\leq\frac{\widehat{R}|\mathbf{c}|}{\eta|g|}}|gr|^{-d}S_{g,r}(\mathbf{c})I_{g,r}(\mathbf{c})$$

and

$$E_2:=\sum_{\mathbf{c}\neq 0}^{\mathrm{exc}}\sum_{\frac{\widehat{R}|\mathbf{c}|}{\eta|g|}<|r|\leq\widehat{Q}}|gr|^{-d}S_{g,r}(\mathbf{c})I_{g,r}(\mathbf{c}).$$

This division of the sum into two parts is suggested by Proposition 5.7.

Lemma 7.2.

$$|E_1| \ll \widehat{Q}^{\frac{d+3}{2}+\varepsilon}|g|^{\frac{d-3}{2}+\varepsilon}(1+|g|^{-\frac{d-5}{2}+\varepsilon}),$$

where the constant depends only on ε, the quadratic form, and Ω, and then showing that E_1 and E_2 satisfy the above bound.

Proof. By Proposition 5.7, we know that for $|r| \leq \frac{\widehat{R}|\mathbf{c}|}{\eta|g|}$

$$|I_{g,r}(\mathbf{c})| \ll \widehat{Q}^d \left(\frac{\widehat{Q}|\mathbf{c}|}{|gr|}\right)^{-\frac{d-1}{2}},$$

where the implied constant depends only on the quadratic form and Ω. Using this, we obtain

$$\begin{aligned} |E_1| &\ll \widehat{Q}^d \sideset{}{^{\mathrm{exc}}}\sum_{\mathbf{c}\neq 0} \sum_{1\leq|r|\leq\frac{\widehat{R}|\mathbf{c}|}{\eta|g|}} |gr|^{-d}|S_{g,r}(\mathbf{c})| \left(\frac{\widehat{Q}|\mathbf{c}|}{|gr|}\right)^{-\frac{d-1}{2}} \\ &= \widehat{Q}^{\frac{d+1}{2}} \sideset{}{^{\mathrm{exc}}}\sum_{\mathbf{c}\neq 0} \left(\frac{|\mathbf{c}|}{|g|}\right)^{-\frac{d-1}{2}} \sum_{1\leq|r|\leq\frac{\widehat{R}|\mathbf{c}|}{\eta|g|}} |g|^{-d}|r|^{-\frac{d+1}{2}}|S_{g,r}(\mathbf{c})|. \end{aligned}$$

By Proposition 3.1,

$$\sum_{1\leq|r|\leq\frac{\widehat{R}|\mathbf{c}|}{\eta|g|}} |g|^{-d}|r|^{-\frac{d+1}{2}}|S_{g,r}(\mathbf{c})| \ll |g|^{\varepsilon}\left(\frac{\widehat{R}|\mathbf{c}|}{\eta|g|}\right)^{1+\varepsilon} \ll |g|^{\varepsilon}\left(\frac{\widehat{Q}|\mathbf{c}|}{|g|}\right)^{1+\varepsilon},$$

where the implied constant depends only on the quadratic form, ε, and Ω. Here, we are also using the fact that R and Q are of the same order up to a constant depending on the quadratic form and Ω. Consequently,

$$|E_1| \ll \widehat{Q}^{\frac{d+1}{2}} \sideset{}{^{\mathrm{exc}}}\sum_{\mathbf{c}\neq 0} \left(\frac{|\mathbf{c}|}{|g|}\right)^{-\frac{d-1}{2}} |g|^{\varepsilon}\left(\frac{\widehat{Q}|\mathbf{c}|}{|g|}\right)^{1+\varepsilon} = \widehat{Q}^{\frac{d+3}{2}+\varepsilon}|g|^{\frac{d-3}{2}+\varepsilon} \sideset{}{^{\mathrm{exc}}}\sum_{\mathbf{c}\neq 0} |\mathbf{c}|^{-\frac{d-3}{2}}.$$

Note that the exceptional vectors $\mathbf{c}$ are all congruent to $\alpha A\boldsymbol{\lambda}$ modulo g for some varying polynomial α. By assumption, at least one coordinate of $\boldsymbol{\lambda}$ is relatively prime to g, say the first one. Since every exceptional $\mathbf{c}$ is congruent to $\alpha A\boldsymbol{\lambda} \bmod g$ for some α depending on $\mathbf{c}$, the first coordinate varies through all polynomials modulo g as $\mathbf{c}$ and so as α varies. Consequently,

$$\sideset{}{^{\mathrm{exc}}}\sum_{\mathbf{c}\neq 0} |\mathbf{c}|^{-\frac{d-3}{2}+\varepsilon} \ll \sum_{0\neq|\alpha|<|g|} |\alpha|^{-\frac{d-3}{2}+\varepsilon} \ll 1+|g|^{-\frac{d-5}{2}+\varepsilon},$$

from which we obtain

$$|E_1| \ll \widehat{Q}^{\frac{d+3}{2}+\varepsilon}|g|^{\frac{d-3}{2}+\varepsilon}(1+|g|^{-\frac{d-5}{2}+\varepsilon}),$$

where the constant depends only on ε, the quadratic form F, and Ω. □

Similarly, we have the same bound on E_2.

Lemma 7.3.

$$|E_2| \ll \widehat{Q}^{\frac{d+3}{2}+\varepsilon}|g|^{\frac{d-3}{2}+\varepsilon}(1+|g|^{-\frac{d-5}{2}+\varepsilon}),$$

where the constant depends only on ε, the quadratic form, and Ω.

Proof. In this case, $|r| > \frac{\widehat{R}|\mathbf{c}|}{\eta|g|}$ for which we have the trivial bound

$$|I_{g,r}(\mathbf{c})| \ll \widehat{Q}^{d+\varepsilon},$$

where the implied constant depends only on ε, the quadratic form, and Ω. Using this, we obtain

$$\begin{aligned}
|E_2| &\ll \widehat{Q}^{d+\varepsilon} \sideset{}{^{\mathrm{exc}}}\sum_{\mathbf{c}\neq 0} \sum_{\frac{\widehat{R}|\mathbf{c}|}{\eta|g|}<|r|\leq \widehat{Q}} |gr|^{-d} |S_{g,r}(\mathbf{c})| \\
&= \widehat{Q}^{d+\varepsilon} \sideset{}{^{\mathrm{exc}}}\sum_{\mathbf{c}\neq 0} \sum_{\frac{\widehat{R}|\mathbf{c}|}{\eta|g|}<|r|\leq \widehat{Q}} |r|^{-\frac{d-1}{2}} |g|^{-d} |r|^{-\frac{d+1}{2}} |S_{g,r}(\mathbf{c})| \\
&= \widehat{Q}^{d+\varepsilon} \sideset{}{^{\mathrm{exc}}}\sum_{\mathbf{c}\neq 0} \sum_{k=1+\log_q \frac{\widehat{R}|\mathbf{c}|}{\eta|g|}}^{Q} \left(q^k\right)^{-\frac{d-1}{2}} \sum_{|r|=q^k} |g|^{-d} |r|^{-\frac{d+1}{2}} |S_{g,r}(\mathbf{c})|.
\end{aligned}$$

By Proposition 3.1, for each k,

$$\sum_{|r|=q^k} |g|^{-d} |r|^{-\frac{d+1}{2}} |S_{g,r}(\mathbf{c})| \ll |g|^{\varepsilon} (q^k)^{1+\varepsilon}.$$

Hence

$$\begin{aligned}
|E_2| &\ll \widehat{Q}^{d+\varepsilon} |g|^{\varepsilon} \sideset{}{^{\mathrm{exc}}}\sum_{\mathbf{c}\neq 0} \sum_{k=1+\log_q \frac{\widehat{R}|\mathbf{c}|}{\eta|g|}}^{Q} \left(q^k\right)^{-\frac{d-3}{2}+\varepsilon} \\
&\ll \widehat{Q}^{d+\varepsilon} |g|^{\varepsilon} \sideset{}{^{\mathrm{exc}}}\sum_{\mathbf{c}\neq 0} \left(\frac{\widehat{Q}|\mathbf{c}|}{|g|}\right)^{-\frac{d-3}{2}+\varepsilon} \\
&= \widehat{Q}^{\frac{d+3}{2}+\varepsilon} |g|^{\frac{d-3}{2}+\varepsilon} \sideset{}{^{\mathrm{exc}}}\sum_{\mathbf{c}\neq 0} |\mathbf{c}|^{-\frac{d-3}{2}+\varepsilon}.
\end{aligned}$$

As before,

$$\sideset{}{^{\mathrm{exc}}}\sum_{\mathbf{c}\neq 0} |\mathbf{c}|^{-\frac{d-3}{2}+\varepsilon} \ll \sum_{0\neq|\alpha|<|g|} |\alpha|^{-\frac{d-3}{2}+\varepsilon} \ll 1 + |g|^{-\frac{d-5}{2}+\varepsilon},$$

from which the conclusion follows. □

We are now ready to prove our main theorem. Note that from remark 3 this is optimal for $d \geq 5$.

Proof of the main theorem 1.2. Recall that

$$N(w, \boldsymbol{\lambda}) = \frac{1}{|g|\widehat{Q}^2} \sum_{\substack{r\in\mathcal{O} \\ |r|\leq \widehat{Q} \\ r \text{ monic}}} \sum_{\mathbf{c}\in\mathcal{O}^d} |gr|^{-d} S_{g,r}(\mathbf{c}) I_{g,r}(\mathbf{c}). \tag{26}$$

By Lemma 5.6, Lemma 6.3, and Proposition 7.1, we have

$$\begin{aligned} N(w,\boldsymbol{\lambda}) &= \frac{1}{|g|\widehat{Q}^2}\sum_{\substack{r\in\mathcal{O}\\ |r|\leq\widehat{Q}^{1-\varepsilon}\\ r \text{ monic}}} |gr|^{-d}S_{g,r}(\mathbf{0})I_{g,r}(\mathbf{0}) + O_{\varepsilon,F,\Omega}\left(\frac{\widehat{Q}^{\frac{d+3}{2}+\varepsilon}|g|^{\frac{d-3}{2}+\varepsilon}(1+|g|^{-\frac{d-5}{2}+\varepsilon})}{|g|\widehat{Q}^2}\right)\\ &= \frac{1}{|g|\widehat{Q}^2}\sum_{\substack{r\in\mathcal{O}\\ |r|\leq\widehat{Q}^{1-\varepsilon}\\ r \text{ monic}}} |gr|^{-d}S_{g,r}(\mathbf{0})I_{g,r}(\mathbf{0}) + O_{\varepsilon,F,\Omega}\left(\widehat{Q}^{\frac{d-1}{2}+\varepsilon}|g|^{\frac{d-5}{2}+\varepsilon}(1+|g|^{-\frac{d-5}{2}+\varepsilon})\right). \end{aligned}$$

By Lemma 6.1, $I_{g,r}(\mathbf{0}) = C\widehat{Q}^d(1+O_\varepsilon(\widehat{Q}^{-\varepsilon}))$ for some constant $C>0$. Hence,

$$\frac{1}{|g|\widehat{Q}^2}\sum_{\substack{r\in\mathcal{O}\\ |r|\leq\widehat{Q}^{1-\varepsilon}\\ r \text{ monic}}} |gr|^{-d}S_{g,r}(\mathbf{0})I_{g,r}(\mathbf{0}) = \frac{C\widehat{Q}^{d-2}(1+O_\varepsilon(\widehat{Q}^{-\varepsilon}))}{|g|}\sum_{\substack{r\in\mathcal{O}\\ |r|\leq\widehat{Q}^{1-\varepsilon}\\ r \text{ monic}}} |gr|^{-d}S_{g,r}(\mathbf{0}).$$

On the other hand, by Lemma 6.5 and Lemma 6.6,

$$\sum_{\substack{r\in\mathcal{O}\\ |r|\leq\widehat{Q}^{1-\varepsilon}\\ r \text{ monic}}} |gr|^{-d}S_{g,r}(\mathbf{0}) = \prod_{\varpi}\sigma_\varpi + O\left(\frac{\widehat{Q}^{-\frac{d-3}{2}+\varepsilon}}{|g|^d}\right).$$

As a result, we finally obtain

$$\begin{aligned} N(w,\boldsymbol{\lambda}) &= \frac{C\widehat{Q}^{d-2}(1+O_\varepsilon(\widehat{Q}^{-\varepsilon}))}{|g|}\left(\prod_{\varpi}\sigma_\varpi + O\left(\frac{\widehat{Q}^{-\frac{d-3}{2}+\varepsilon}}{|g|^d}\right)\right) + O_{\varepsilon,F,\Omega}\left(\widehat{Q}^{\frac{d-1}{2}+\varepsilon}|g|^{\frac{d-5}{2}+\varepsilon}(1+|g|^{-\frac{d-5}{2}+\varepsilon})\right)\\ &= \frac{C\widehat{Q}^{d-2}}{|g|}\prod_{\varpi}\sigma_\varpi + O_{\varepsilon,F,\Omega}\left(\widehat{Q}^{\frac{d-1}{2}+\varepsilon}|g|^{\frac{d-5}{2}+\varepsilon}(1+|g|^{-\frac{d-5}{2}+\varepsilon})\right)\\ &= \frac{C\widehat{Q}^{d-2}}{|g|}\prod_{\varpi}\sigma_\varpi\left(1+O_{\varepsilon,F,\Omega}\left(\frac{\widehat{Q}^{\frac{d-1}{2}+\varepsilon}|g|^{\frac{d-3}{2}+\varepsilon}(1+|g|^{-\frac{d-5}{2}+\varepsilon})}{\widehat{Q}^{d-2}}\right)\right)\\ &= \frac{C\widehat{Q}^{d-2}}{|g|}\prod_{\varpi}\sigma_\varpi\left(1+O_{\varepsilon,F,\Omega}\left(\frac{|g|^{d-3+\varepsilon}(1+|g|^{-\frac{d-5}{2}+\varepsilon})}{|f|^{\frac{d-3}{4}-\varepsilon}}\right)\right)\\ &= \frac{C\widehat{Q}^{d-2}}{|g|}\prod_{\varpi}\sigma_\varpi\left(1+O_{\varepsilon,F,\Omega}\left((1+|g|^{-\frac{d-5}{2}+\varepsilon})\left(\frac{|g|^{4+\varepsilon}}{|f|}\right)^{\frac{d-3}{4}}\right)\right). \end{aligned}$$

Therefore, if $d\geq 5$, we can take $|f|\gg|g|^{4+\varepsilon}$, while if $d=4$, we can take $|f|\gg|g|^{6+\varepsilon}$. □

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