# ALMOST UNRAMIFIED AUTOMORPHIC REPRESENTATIONS FOR SPLIT GROUPS OVER $\mathbf{F}_q(t)$

AMRITANSHU PRASAD

ABSTRACT. Let $G$ be a split reductive group over a finite field $\mathbf{F}_q$, $\mathbf{F} = \mathbf{F}_q(t)$ and $\mathbf{A}$ the adèles of $\mathbf{F}$. We describe the irreducible automorphic representations of $G(\mathbf{A})$ which have non-zero vectors invariant under Iwahori subgroups at two places and maximal compact subgroups at all other places in terms of the irreducible representations of Iwahori-Hecke algebras and the Satake isomorphism.

**Introduction.** Let $\mathbf{F}_q$ denote the finite field with $q$ elements and let $G$ be a split reductive group over $\mathbf{F}_q$. Fix a Borel subgroup $B$ of $G$ defined over $\mathbf{F}_q$ and a maximal $\mathbf{F}_q$-split torus $T$ contained in $B$. Consider the global field $\mathbf{F} = \mathbf{F}_q(t)$ with ring of adèles $\mathbf{A}$. For each valuation $v$ of $\mathbf{F}$ let $F_v$ denote the corresponding completion of $\mathbf{F}$. Let $O_v$ denote the ring of integers in $F_v$. Denote by $0$ and $\infty$ the valuations uniformized by $t$ and $t^{-1}$ respectively. Set

$$K_v = \begin{cases} \text{pre-image of } B(\mathbf{F}_q) \text{ under } G(O_v) \to G(\mathbf{F}_q) & \text{if } v = \infty, 0 \\ G(O_v) \text{ otherwise.} \end{cases}$$

Let $\mathbf{K} = \prod_v K_v$. Let

$$M = L^2(G(F)\backslash G(\mathbf{A})/\mathbf{K}).$$

Denote by $H_v$ the Hecke algebra of compactly supported measures on $G(F_v)$ that are left and right invariant under $K_v$. An irreducible representation $(\pi, V)$ of $\otimes_v H_v$ occurring discretely in $M$ has a decomposition

$$(\pi, V) = \otimes_v(\pi_v, V_v),$$

where $\pi_v$ is an irreducible representation of $H_v$. The representation $\pi_v$ is called the local constituent of $\pi$ at $v$.

The main theorem of this article describes the local constituents of the irreducible representations of $\otimes_v H_v$ that occur in the discrete part of $M$. For each $(\pi, V)$ occurring discretely in $M$ the representations $(\pi_v, V_v)$ are explicitly described in terms of the irreducible square-integrable representations of an Iwahori-Hecke algebra (for $v = \infty, 0$) or the Satake isomorphism (for the remaining $v$).

**Relation to Eisenstein series.** Let $X$ denote the space of all characters

$$T(\mathbf{F})\backslash T(\mathbf{A})/T(\mathbf{O}) \to \mathbf{C}^\times.$$

There is a natural isomorphism of $X$ with $\mathrm{Spec}(\mathbf{C}[\mathrm{Pic}(\mathbf{P}^1) \otimes X_*(T)])$, where $X_*(T)$ is the lattice $\mathrm{Hom}(\mathbf{G}_m, T)$ of rational cocharacters of $T$. The isomorphism $\deg :$ $\mathrm{Pic}(\mathbf{P}^1) \to \mathbf{Z}$ induces an isomorphism $\mathrm{Pic}(\mathbf{P}^1) \otimes X_*(T) \to X_*(T)$ and hence and isomorphism $\deg : \mathrm{Spec}(\mathbf{C}[\mathrm{Pic}(\mathbf{P}^1) \otimes X_*(T)]) \to \hat{T}$, where $\hat{T} = \mathrm{Spec}(\mathbf{C}[X_*(T)])$ is

*Date*: 28 August, 2002.

a maximal torus in the Langlands dual group of $G$. In particular $X$ is isomorphic to a complex torus.

For any function $\phi : B(\mathbf{O})\backslash G(\mathbf{O})/\mathbf{K} \to \mathbf{C}$ there exists a unique function $\phi_\chi : T(\mathbf{F})T(\mathbf{O})N(\mathbf{A})\backslash G(\mathbf{A})/\mathbf{K}$ whose restriction to $G(\mathbf{O})$ coincides with $\phi$ and such that

$$\phi_\chi(tg) = \delta^{\frac{1}{2}}(t)\chi(t)\phi_\chi(g)$$

where $\delta : B(\mathbf{A}) \to \mathbf{R}^+$ is the modulus character of $B(\mathbf{A})$. If $|\chi((t^{-1})^{\check{\alpha}})| < 1$ for each coroot $\check{\alpha}$ of $T$ in $G$ which is positive with respect to $B$, then the sum

$$E_\phi(\chi, g) = \sum_{\gamma \in B(\mathbf{F})\backslash G(\mathbf{F})} \phi_\chi(\gamma g)$$

converges absolutely and normally in $\chi$ and defines a function on $G(\mathbf{F})\backslash G(\mathbf{A})/\mathbf{K}$. For each $g$, $E_\phi(\chi, g)$ extends to a rational function on $X$.

**Theorem 1.** *$M_{\text{discrete}}$ is generated by functions of the form*

$$m(g) = f(\chi)E_\phi(\chi, g) \tag{2}$$

*where $f$ is a rational function on $X$ and $\chi \in X$.*

In view of Langlands's theory of Eisenstein series ([4], [5]), in order to prove Theorem 1 it suffices to verify that

**Lemma 3.** *Let $m$ be a complex-valued function on $G(\mathbf{F})\backslash G(\mathbf{A})/\mathbf{K}$. If $m_N \equiv 0$ then $m \equiv 0$. Here $m_N$ denotes the constant term of $m$ along the unipotent radical $N$ of $B$.*

*Proof.* Recall the following theorem [6]:

**Theorem 4.** *The map $\tilde{W} \to G(\mathbf{A})$ defined by*

$$\pi^\mu w \mapsto (t^{-1})^\mu \phi_\infty(w)$$

*induces a bijection of sets*

$$\tilde{W} \tilde{\to} G(\mathbf{F})\backslash G(\mathbf{A})/\mathbf{K}'.$$

Therefore, to prove Lemma 3 it suffices to verify that

$$\text{for any } w, w' \in \tilde{W},\ (t_w)_N(w') = 0 \text{ if and only if } w = w'. \tag{5}$$

Indeed,

$$\begin{aligned} (t_w)_N(w') &= \int_{N(\mathbf{F})\backslash N(\mathbf{A})} t_w(nw')dn \\ &= \text{measure } (N(\mathbf{F})\backslash(N(\mathbf{A}) \cap G(\mathbf{F})wKw'^{-1})) \end{aligned}$$

The latter is positive if and only if $w = w'$. □

**Hecke algebras.** For each valuation $v$ of $\mathbf{F}$ let $H_v$ denote the convolution algebra of compactly supported measures on $G(F_v)$ which are left and right invariant under $K_v$. Let $Z_v$ denote the center of $H_v$ for each $v$ and $R_v$ denote the lattice $T(F_v)/T(O_v)$.

**Proposition 6.** *For each $v$ there exists an isomorphism*

$$B_v : Z_v \to R_v^W$$

*such that $Z_v$ acts on $i_B^G\chi_v$ by $\chi_v \circ B_v$ for any character $\chi_v : T(F_v)/T(O_v) \to \mathbf{C}^\times$.*

*Proof.* For $v \neq \infty, 0$, $H_v$ is commutative so that $Z_v = H_v$ and $B_v$ is the Satake isomorphism. For $v = \infty$, or $0$ $B_v$ is the well known characterization of the center of an Iwahori-Hecke algebra attributed to Bernstein (see [8]). □

The map $X_*(T) \to R_v$ given by $\mu \mapsto \pi_v^\mu$, where $\pi_v$ is a uniformizing element for $F_v$ is independent of $\pi_v$ and is an isomorphism. This induces an isomorphism $\deg_v : X_v := \mathrm{Spec}(\mathbf{C}[R_v]) \to \hat{T}$. We have bijections

$$\mathrm{Spec} Z_v \xrightarrow{B_v^*} \mathrm{Spec} R_v^W \xrightarrow{\deg_v} \hat{T}/W \tag{7}$$

Let $\chi_s^v : Z_v \to \mathbf{C}$ denote the character corresponding to an element $s \in \hat{T}/W$ under the above identifications.

**Lemma 8.** *For each valuation $v$ of $\mathbf{F}$ and each $\omega_v \in Z_v$, we have*

$$E_\phi(\chi) \cdot \omega_v = \chi_v \circ B_v(\omega_v) E_\phi(\chi),$$

*where $\chi_v : T(F_v)/T(O_v) \to \mathbf{C}^\times$ is the restriction of $\chi$ to $T(F_v)$.*

*Proof.* We have

$$(E_\phi(\chi))_N(g) = \sum_{w \in W} \mathrm{meas}(N_w(\mathbf{F}) \backslash N_w(\mathbf{A})) I_w(\chi) \phi_\chi(g),$$

where $I_w(\chi) : i_{B(\mathbf{A})}^{G(\mathbf{A})} \chi \to i_{B(\mathbf{A})}^{G(\mathbf{A})w} \chi$ is the intertwining operator defined by

$$I_w(\chi)\phi_\chi = \int_{(N(\mathbf{A}) \cap w N(\mathbf{A}) w^{-1}) \backslash N(\mathbf{A})} \phi_\chi(w^{-1} n g) dn \tag{9}$$

for $\chi$ such that $|\chi(\pi^{\check{\alpha}})| < 1$ for each positive coroot $\check{\alpha}$ and then extended to all $\chi$ as a rational function on $X$. The function $\phi_\chi$ is of the form

$$\phi_\chi = \otimes_v \phi_{\chi,v}$$

where $\phi_{\chi,v}$ is function in $i_{B(F_v)}^{G(F_v)w} \chi_v^{K_v}$. The intertwining operator $I_w$ is a product of local intertwining operators:

$$I_w \phi_\chi = \otimes_v I_w^v \phi_{\chi,v}$$

where $I_w^v$ is defined as

$$I_w^v \phi_{\chi,v} = \int_{(N(F_v) \cap w N(F_v) w^{-1}) \backslash N(F_v)} \phi_\chi(w^{-1} n g) dn$$

for $\chi$ such that $|\chi(\pi_v^{\check{\alpha}_i})| < 1$ for each positive coroot $\check{\alpha}$ and extended to $X_v$ by a rational function.

Since $I_w^v \phi_{\chi,v} \in i_{B(F_v)}^{G(F_v)w} \chi$ and the image of $B_v$ is $W$-invariant, $Z_v$ also acts on $I_w^v \phi_{\chi,v}$ by $\chi_v \circ B_v$. Since the intertwining operators and constant term operations commute with the action of the Hecke algebras and since the constant term map is injective in our setting, Lemma 8 follows. □

**Local constituents at $\infty$ and $0$.** let $F$ denote the field $\mathbf{F}_q((\pi))$ of Laurent series in a variable $\pi$ with ring of integers $O$. Let $I$ be the Iwahori subgroup of $G(F)$ which is the preimage of $B(\mathbf{F}_q)$ under the map $G(O) \to G(\mathbf{F}_q)$ and let $H$ denote the corresponding Iwahori-Hecke algebra. The assignment $\pi \mapsto t$ (resp. $\pi \mapsto t^{-1}$) defines an isomorphism $F \to F_0$ (resp. $F \to F_\infty$) and hence an isomorphism $\theta_0 : H \to H_0$ (resp. $\theta_\infty : H \to H_\infty$). Let

$$N = L^2(\Delta G(F)\backslash(G(F)^2)/I^2).$$

Then $N$ is an $H \otimes H$ module whose discrete part is known by elementary and standard considerations (see e.g., [7]):

$$N_{\text{discrete}} = \bigoplus_{(\pi,V)\in\hat{H}} (\pi \otimes \tilde{\pi}, V \otimes \tilde{V}).$$

Here $\hat{H}$ denotes the set of isomorphism-classes of irreducible square-integrable representations of $H$. Moreover, by [7], there is an isometry of Hilbert spaces

$$I : N \to M$$

such that

$$I(n \cdot (h \otimes h')) = I(n) \cdot (\theta_\infty \circ \iota(h) \otimes \theta_0 \circ \iota \circ \overline{\kappa}(h')) \text{ for } h, h' \in H,\ n \in N.$$

where $\iota$ is the Iwahori-Matsumoto involution (introduced at the very end of [3]) and $\overline{\kappa}$ is an involution (defined in [7]) which when the derived group of $G$ is adjoint takes a representation to its contragredient. It follows that as an $H_\infty \otimes H_0$-module

$$M_{\text{discrete}} = \bigoplus_{(\pi,V)\in\hat{H}} (\pi \circ \iota \circ \theta_\infty \otimes \tilde{\pi} \circ \overline{\kappa} \circ \iota \circ \theta_0, V \otimes \tilde{V}). \tag{10}$$

**The Iwahori-Matsumoto involution.**

**Theorem 11.** *Let $(\pi, V) \in \hat{H}$. The central character of $(\pi, V)$ equals the central character of $(\pi \circ \iota, V)$.*

*Proof.* As in (7), the characters of the center of $H$ are parameterized by points in $\hat{T}/W$. Let $s \in \hat{T}/W$ be the central character of $H$ on $(\pi, V)$. Let $(r(\pi), U)$ denote the representation of $R$ that is obtained by taking the $A(O)$-invariants of the normalized Jacquet module (see [9, §2.3] for the correct normalization) of the irreducible representation of $G(\mathbf{F})$ whose $I$ invariants form the irreducible representation $(\pi, V)$ of $H$. By [2, Proposition 2.4] the canonical projection $V \to U$ is an isomorphism. Moreover, it follows easily from the definition of $B_v$ (see [8]), [2, Proposition 2.5] and [1, Lemma 1.5.1] that the restrictions to $R^W$ of the weights of $r(\pi)$ is also $s$. By [9, Proposition 13], weights of $r(\pi)$ and $r(\pi \circ \iota)$ on $U$ differ by conjugation by the longest element in $W$. Therefore, their restrictions to $R^W$ are the same. This means that the central character of $(\pi, V)$ equals that of $(\pi \circ \iota, V)$. $\square$

**The discrete spectrum.** For $(\pi, V) \in \hat{H}$, let $s(\pi) \in \hat{T}/W$ be the central character of $\pi$ on $V$. Let $\chi_s^v$ be as defined in the line following (7).

**Theorem 12.** *As an $\otimes_v H_v$-module,*

$$M_{\text{discrete}} = \bigoplus_{(\pi,V)\in\hat{H}} (\pi \circ \iota \circ \theta_\infty \otimes \tilde{\pi} \circ \overline{\kappa} \circ \iota \circ \theta_0 \otimes (\otimes_{v\neq 0,\infty} \chi^v_{s(\pi)^{\deg(v)}}), V \otimes \tilde{V}). \tag{13}$$

*Proof.* By Theorem 1 we may assume that $m$ is of the form (2) for some $\chi \in X$. Let $s$ be the image of $\chi$ in $\hat{T}$. By Lemma 8, the central character of $H_v$ on $M$ is $\chi_v \circ B_v$, where $\chi_v$ is the restriction of $\chi$ to $T(F_v)$. Obviously $\deg_v(\chi_v) = s^{\deg(v)}$ (see (7)). Taking $m$ in the direct summand corresponding to $\pi$ in (10) together with the fact that $s(\pi) = s(\pi \circ \iota)$ (Theorem 11) completes the proof of Theorem 12. $\square$

## References


[1] W. Casselman. Introduction to the theory of admissible representations of p-adic reductive groups. Unpublished notes, 1974-93.

[2] W. Casselman. The unramified principal series of $p$-adic groups. I. The spherical function. *Compositio Math.*, 40(3):387–406, 1980.

[3] N. Iwahori and H. Matsumoto. On some Bruhat decomposition and the structure of the Hecke rings of $p$-adic Chevalley groups. *Inst. Hautes Études Sci. Publ. Math. No.*, 25:5–48, 1965.

[4] R. Langlands. On the functional equations satisfied by Eisenstein series. Lecture Notes in Mathematics, Vol. 544. Springer-Verlag, Berlin-New York, 1976.

[5] C. Mœglin and J.-L. Waldspurger. Dcomposition spectrale et sries d'Eisenstein. (French) [Spectral decomposition and Eisenstein series] Une paraphrase de l'criture. [A paraphrase of Scripture] Progress in Mathematics, 113. Birkhuser Verlag, Basel, 1994. xxx+342 pp.

[6] A. Prasad. *Almost unramified discrete spectrum of split reductive groups over* $\mathbf{F}_q(t)$. PhD thesis, University of Chicago, 2001.

[7] A. Prasad. Almost unramified discrete spectrum for split groups over $\mathbf{F}_q(t)$. Duke Math. J. 113 (2002), no. 2, 237–257.

[8] A. Prasad. Iwahori-Hecke algebras (notes based on lectures by Kottwitz). http://aprasad.bravepages.com

[9] F. Rodier. Sur les représentations non ramifiées des groupes réductifs $p$-adiques; l'exemple de GSp(4). *Bull. Soc. Math. France*, 116(1):15–42, 1988.