ADIABATIC DECOMPOSITION OF THE ζ-DETERMINANT AND SCATTERING THEORY.

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ABSTRACT. We discuss the decompositon of the ζ-determinant of the square of the Dirac operator into contributions coming from different parts of the manifold. The "easy" case was worked out in paper [27]. Due to the assumptions made on the operators in [27], we were able to avoid the presence of the "small eigenvalues" which provide the large time contribution to the determinant. In the present work we analyze the general case. We offer a detailed analysis of the contribution made by the small eigenvalues. The idea is based on the method used by Werner Müller in his paper [23], where he studied the η-invariant on a manifold with cylindrical end. The analysis of the ζ-determinant, however, is more difficult. The difference is illustrated nicely in the proof of the Lesch-Wojciechowski formula for the ratio of the ζ-determinants of the Dirac operator subject to the two different boundary conditions provided in the final sections of the paper. The main results of the paper were announced in [28]

INTRODUCTION AND STATEMENT OF THE RESULTS

The study of the behaviour of the geometric invariants of the Dirac operators with respect to the decomposition of the underlying manifold is a fascinating subject with a long and interesting history. The history of the subject and the most important developements were described in many different sources. In particular, we refer to a beautiful review paper [22] for the account of the recent developements in this area of mathematics.

In this paper we concentrate on the decomposition of the ζ-determinants of the Dirac Laplacian. We now proceed with the precise definitions of the operators and invariants involved in our story.

Let $\mathcal{D} : C^\infty(M;S) \to C^\infty(M;S)$ be a compatible Dirac operator acting on sections of a bundle of Clifford modules S over a closed manifold M. The operator $\mathcal{D}$ is a self-adjoint operator, with discrete spectrum $\{\lambda_k\}_{k\in\mathbb{Z}}$. Assume that we have a decomposition of M as $M_1 \cup M_2$, where M_1 and

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M_2 are compact manifolds with boundaries such that

$$M = M_1 \cup M_2 \quad , \quad M_1 \cap M_2 = Y = \partial M_1 = \partial M_2 \, . \tag{0.1}$$

The ζ-determinant of the operator $\mathcal{D}$ is given by the formula

$$det_\zeta \mathcal{D} = e^{\frac{i\pi}{2}(\zeta_{\mathcal{D}^2}(0) - \eta_{\mathcal{D}}(0))} \cdot e^{-\frac{1}{2}\zeta'_{\mathcal{D}^2}(0)} \quad , \tag{0.2}$$

(see [32], see also the Introduction of [31]). The functions $\zeta_{\mathcal{D}^2}(s)$, and $\eta_{\mathcal{D}}(s)$ are defined as follows

$$\zeta_{\mathcal{D}^2}(s) = \sum_{\lambda_k \neq 0} (\lambda_k^2)^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} [Tr(e^{-t\mathcal{D}^2}) - dim\ ker\ \mathcal{D}]\ dt \tag{0.3}$$

and

$$\eta_{\mathcal{D}}(s) = \sum_{\lambda_k \neq 0} sign(\lambda_k) |\lambda_k|^{-s} = \frac{1}{\Gamma(\frac{s+1}{2})} \int_0^\infty t^{\frac{s-1}{2}} Tr\ \mathcal{D} e^{-t\mathcal{D}^2} dt \quad .$$

They are holomorphic functions of s for $Re(s)$ large and they both have meromorphic extensions to the complex plane with simple poles only. In both cases $s = 0$ is a regular point.

In this paper we are going to study the decomposition of the ζ-determinant of $\mathcal{D}$ into the contributions coming from the submanifolds M_1 and M_2 . More precisely, we concentrate on

$$e^{-\zeta'_{\mathcal{D}^2}(0)} \quad ,$$

the determinant of $\mathcal{D}^2$, which provides the modulus of the ζ-determinant of $\mathcal{D}$. The decomposition of the phase of the determinant is now a well-known story. We are going, however, to discuss this topic to a certain extent. The reason is that the adiabatic method we use here was originally developed in order to study the most important part of the phase, the η-invariant.

We make two further assumptions. First, we only discuss the odd-dimensional case

$$n = dim\ M = 2k + 1 \quad .$$

Second, we assume that M and the operator $\mathcal{D}$ have product structures in a neighborhood of the boundary Y . This means that there is a bicollar neighborhood $N \cong [-1, 1] \times Y$ of Y in M such that the Riemannian structure on M and the Hermitian structure on S are products when restricted to N . This implies that $\mathcal{D}$ has the following form when restricted to the submanifold N

$$\mathcal{D} = G(\partial_u + B) \quad . \tag{0.4}$$

Here u denotes the normal variable, $G : S|_Y \to S|_Y$ is a bundle automorphism, and B is a corresponding Dirac operator on Y. Moreover, G and B do not depend on u and they satisfy

$$G^* = -G \ , \ G^2 = -Id \ , \ B = B^* \ \text{ and } \ GB = -BG \ . \tag{0.5}$$

The first assumption is not of great importance from the technical point of view, and we discuss some technicalities and the minor differences between odd and even dimensional case, in the other paper. The second assumption has much more serious analytical consequences. The analysis of the non-product situation is much more complicated and we will not attempt to discuss it here.

Both the $\eta_{\mathcal{D}}(0)$ and $\zeta'_{\mathcal{D}^2}(0)$ are non-local invariants and their values depend on global geometric information. Therefore, at first sight, it seems rather naive to expect any decomposition formula for the invariants from which the ζ-determinant is constructed. However, there is a method for extracting the useful informtion. The second author got the idea after studying the short, but extremely influential paper by Singer (see [33]). In this beautiful research note Singer studied, among the other things, the boundary contribution to the the η-invariant of the boundary value problem. It follows from the formula (0.3) that the η-invariant can be expressed via the trace of the corresponding heat operator, or the integral over the manifold from the kernel of the corresponding operator. Singer obtained the aforementioned boundary contribution to the η-invariant by stretching the collar neighborhood of the boundary. The trace he studied was divided into the interior contribution, the cylinder contribution and the error term. The key here was the fact that the error term disappeared as the length of the collar approached infinity and that the cylinder contribution could be explicitly computed.

The second author got acquainted with [33] in the Fall of 1987 and decided to apply this method to the study of the decomposition of the η-invariant. In short, the essence of the what we call here the *Adiabatic Decomposition* of the invariants of the Dirac operator is the stretching of M in the normal direction to Y .

To formulate the decomposition formula for the ζ-determinant we have to describe the invariants on a manifold with boundary which enter the picture at this point. The operator B has a discrete spectrum with infinitely many positive and infinitely many negative eigenvalues. Let $\Pi_>$ denote the spectral projection onto the subspace spanned by the eigensections of B corresponding to the positive eigenvalues. Then $\Pi_>$ is an elliptic boundary condition for $\mathcal{D}_2 = \mathcal{D}|_{M_2}$ (see [1]; see [3] for an exposition of the theory of elliptic boundary problems for Dirac operators). In fact, any orthogonal projection satisfying

$$(0.6)\qquad -GPG = Id - P \quad \text{and} \quad P - \Pi_{>} \ \textit{is a smoothing operator}$$

is a self-adjoint elliptic boundary condition for the operator $\mathcal{D}_2$. This means that the associated operator

$$(\mathcal{D}_2)_P : dom\ (\mathcal{D}_2)_P \to L^2(M_2; S|_{M_2})$$

with $dom\ (\mathcal{D}_2)_P = \{s \in H^1(M_2; S|_{M_2}) \mid P(s|_Y) = 0\}$ is a self-adjoint Fredholm operator with $ker((\mathcal{D}_2)_P) \subset C^\infty(M_2; S|_{M_2})$ and discrete spectrum (see [37]). We denote by $\mathcal{G}r^*_\infty(\mathcal{D}_2)$ the space of P satisfying (0.6). The existence of the meromorphic extensions of the functions $\eta_{(\mathcal{D}_2)_P}(s)$, $\zeta_{(\mathcal{D}_2)^2_P}(s)$ to the whole complex plane and their nice behaviour in a neighborhood of $s = 0$ were established in [37]. It follows that the ζ-determinant of the operator $\mathcal{D}_P$, with $P \in \mathcal{G}r^*_\infty(\mathcal{D}_2)$ is well-defined. Let us observe that $Id - P \in \mathcal{G}r^*_\infty(\mathcal{D}_1)$, if P is an element of $\mathcal{G}r^*_\infty(\mathcal{D}_2)$. We denote by $\eta_{(D;P_1,P_2)}(s)$ the η-function of the operator $G(\partial_u + B) = \mathcal{D}|_{[0,1]\times Y}$, subject to the boundary condition P_2 at $u = 0$ and $Id - P_1$ at $u = 1$. We have the following pasting formula proved in [37]

$$(0.7)\qquad \eta_{\mathcal{D}}(0) = \eta_{(\mathcal{D}_1)_{Id-P_1}}(0) + \eta_{(\mathcal{D}_2)_{P_2}}(0) + \eta_{(D;P_1,P_2)}(0) \mod \mathbb{Z}\ .$$

The special case of this formula with the boundary condition $P_1 = P_2 = \Pi_>$, under the assumption $ker\ B = \{0\}$, (otherwise $(\mathcal{D}_2)_{\Pi_>}$ and $(\mathcal{D}_1)_{Id-\Pi_>}$ would not be self-adjoint) was discussed by Ron Douglas and the second author in the Spring of 1988 and most of this discussion was included in [13]. The result is

$$(0.8)\qquad \eta_{\mathcal{D}}(0) = \eta_{(\mathcal{D}_1)_{Id-\Pi_>}}(0) + \eta_{(\mathcal{D}_2)_{\Pi_>}}(0) \mod \mathbb{Z}\ .$$

The proof of (0.8) outlined by Douglas and the second author uses the fact that $\eta_{\mathcal{D}}(0)$, as well as the η-invariants of the boundary problems can be represented by the integral on the right side of the second equality in (0.2), where we put $s = 0$. This gives

$$(0.9)\qquad \begin{aligned} \eta_{\mathcal{D}}(0) &= \frac{1}{\sqrt{\pi}}\int_0^\infty \frac{1}{\sqrt{t}} Tr\ \mathcal{D}e^{-t\mathcal{D}^2} dt \\ &= \frac{1}{\sqrt{\pi}}\int_0^\infty \frac{dt}{\sqrt{t}} \int_M tr\ \mathcal{E}(t;x,x)dx\ , \end{aligned}$$

where $\mathcal{E}(t;x,x)$ denotes the kernel of the operator $\mathcal{D}e^{-t\mathcal{D}^2}$.

The first step in the proof is to replace the bicollar N by $N_R = [-R,R] \times Y$. We now use Duhamel's principle to split $\int_M tr\ \mathcal{E}(t;x,x)dx$ into contributions coming from each side, plus the cylinder contribution and the error term. Explicit representation of the operators $\mathcal{D}e^{-t\mathcal{D}^2}$, $(\mathcal{D}_1)_{Id-\Pi_>} e^{-t(\mathcal{D}_1)^2_{Id-\Pi_>}}$

and $(\mathcal{D}_2)_{\Pi_>}e^{-t(\mathcal{D}_2)^2_{\Pi_>}}$ on the corresponding cylinders show that in each case the cylinder contribution is trivial. It also easily follows that the error term disappears as $R \to \infty$.

Second, though $\eta_{\mathcal{D}}(0)$ is not local, its variation (for instance with respect to the parameter R) is local and therefore the value of the contributions does not vary with R. This is enough to make explicit calculations of the formula (0.8).

One has to stress that the invertibility of the tangential operator B implies significant analytical simplifications. It allows us to ignore the large time contribution to the η-invariant given by the integral

$$\int_{R^\varepsilon}^{\infty} t^{\frac{s-1}{2}} Tr\ \mathcal{D}e^{-t\mathcal{D}^2} dt \quad ,$$

for any $\varepsilon > 0$. This is due to the fact that the invertibility of B implies that all but finitely many eigenvalues of the operators involved stay away from 0 . There is finitely many small eigenvalues of the size e^{-cR} , and they produce only an integer contribution.

The situation is different in the case of the non-trivial kernel of the operator B . First of all we have to modify the spectral boundary conditions to obtain self-adjoint operators on manifolds M_1 and M_2 . Second, we have to deal with infinitely many eigenvalues approaching 0 as $R \to \infty$. Jeff Cheeger in his works on manifolds with conical singularities (see [9], [10]) introduced the appriopriate, finite-dimensional generalization of the Atiyah-Patodi-Singer conditions. He called them *"ideal boundary conditions"*. In this paper we use the *"ideal boundary conditions"* to define the operators on the manifolds with boundary. The Grassmannian $\mathcal{G}r^*_\infty(\mathcal{D}_2)$ of the self-adjoint boundary conditions of Atiyah-Patodi-Singer type discussed above was introduced by Bernhelm Boss-Bavnbek and second author (see [2], [13], [37]). The boundary correction term to the η-invariant, which appears in this situation was studied by Lesch and the second author in [20]. Lesch and Wojciechowski found a very nice expression for the variation of the η-invariant under the change of the *ideal* boundary condition. This was enough to formulate the decomposition formula (0.7) in the case P_1 and P_2 being *ideal* boundary conditions. Details of the proof appeared in [35] and [36]. The result was extended to the case of $P_1, P_2 \in \mathcal{G}r^*_\infty(\mathcal{D}_2)$ in [37].

We have to point out that, due to the fact that the variation of the η-invariant is local and that we have extra symmetries on the cylinder, there is no contribution to the gluing formula for the η-invariant made by *small eigenvalues* (or equivalently the large time contribution is trivial). On the contrary, the analysis of the contributions made by the *small eigenvalues* to the gluing formula for the ζ-determinant of the Dirac Laplacian is one of the most important part of the present exposition.

The decomposition of the η-invariant was also studied independently by many others mathematicians, who came up with several different variants of the gluing formula. Melrose and his collaborators used the process of blowing up the metric in the normal direction in the case where the initial structure was not product. This deformation and the necessary analytical apparatus was carefully described in a series of papers (see [15], [21] and especially [16]). They were able to analyze the integer contribution to the gluing formula, at least in the case of the signature operator. As a result they came up with the analytic description of the Wall's non-additivity formula for the signature of the manifold with corners (see [16]). The Wall's nonadditivity was also discussed by Bunke (see [6]), who also gave his proof of the gluing formula for the η-invariant. This formula was also discussed by Dai and Freed (see [12]), Brüning and Lesch (see [5]) and Werner Müller (see [23] and [24]).

Let us also mention the recent work of Kirk and Lesch [18]. They present the gluing formula for the η-invariant in which the integer contribution is expressed as a triple index involving the boundary condition and the Calderon projectors of $\mathcal{D}_1$ and $\mathcal{D}_2$. The key ingredient in the proof is the Scott-Wojciechowski Theorem for the determinant of the Dirac operator on a manifold with boundary (see [31]) and Brüning and Lesch approach to the *mod* $\mathbb{Z}$ gluing formula. To this day [18] contains the most complete and beautiful approach to the gluing of the η-invariant.

From the author's point of view the work of Werner Müller was especially stimulating and useful in their approach to the decomposition of the ζ-determinant. He considered the case of the non-invertible tangential operator B , and showed that the adiabatic limit of the η-invariant of the boundary problem was well-defined and equal to the corresponding invariant on a manifold with a cylindrical end. In order to prove this statement he had to make a systematic study of the behaviour of the small eigenvalues. He showed that their behaviour is ruled by *scattering matrix* on the manifold with cylindrical end attached. Though we can avoid the scattering theory in the study of the decomposition of the η-invariant, this is not so in the case of the ζ-determinant in the case of the non-invertible operator B .

The origin of present work may be traced to the paper [19]. The focus of the study by Klimek and the second author was on the η-invariants of the local boundary problems introduced by Singer in [33]. The analysis of the analytic torsion was obtained as a byproduct of the method used in the study of the η-functions, and was treated by the authors more as an exercise, than the beginning of a serious study. We also have to point out that the decomposition of the analytic torsion was studied by Hassell in [17]. This paper is close in spirit to our paper and the companion work [27], which deals with the easy case of the invertible operator B . Hassell studied the analytic torsion using the machinery developed by himself, Mazzeo and Melrose. Some further remarks on the subject can be found in the exposition [29].

To finish with the history of our subject we would also like to mention the work of Burghelea, Friedlander and Kappeler (see [7]). They were first to offer a decomposition formula for $det_\zeta \mathcal{D}^2$. Their formula involved the local elliptic boundary problems and the correction term, which, up to constant, was equal to the ζ-determinant of the elliptic operator on Y . Though the aforementioned operator lives on the hypersurface Y it depends on the global geometric information on the whole M . We discuss the relation of our work with the results of [7] in another place (see [30]).

Let us now start with a discussion of our approach to the decomposition of the ζ-determinant of the Dirac Laplacian. The logarithm of this determinant is equal to *"minus"* derivative of the ζ-function at $s = 0$ and we have the following formula

$$\frac{d}{ds}\zeta_{\mathcal{D}^2}(s)|_{s=0} = \int_0^\infty \frac{1}{t} Tr\ e^{-t\mathcal{D}^2} dt \ , \tag{0.10}$$

under the assumption that $\mathcal{D}$ is an invertible operator. The right side of (0.10) is "a priori" only a formal formula, but as explained, it makes sense in our context. We refer to a companion paper [27] for the discussion of this point.

We follow the original Douglas-Wojciechowski proof of the decomposition formula (0.8) for the η-invariant. However, we face two new problems, not present in the case of the η-invariant. First, $\frac{d}{ds}\zeta_{\mathcal{D}^2}(s)|_{s=0}$ is a much more *"non-local"* invariant, than the η-invariant. This results, for instance, in the fact that the value of $det_\zeta \mathcal{D}^2$ varies with the length of the cylinder. Second, the cylinder contribution to the determinant is now non-trivial.

We still follow the idea of paper [13] and we stretch our manifold M to separate M_1 and M_2 . Then, we split $\zeta'_{\mathcal{D}_R^2}(0)$ into contributions coming from different submanifolds plus cylinder contributions and the error terms. Here $\mathcal{D}_R$ denotes the operator $\mathcal{D}$ on the manifold M_R equal to the manifold M with N replaced by $N_R \cong [-R, R] \times Y$. We introduce a manifold with boundary

$$M_{1,R} = M_1 \cup [-R, 0] \times Y \ ,$$

where we identify the "old" collar neighborhood of the boundary Y on M_1 with $[-R-1,-R]\times Y$. Similarly we introduce the manifold $M_{2,R}$. The ζ-determinant is going to blow up as $R\to\infty$, hence we study the ratio

$$\frac{det_\zeta\mathcal{D}_R^2}{det_\zeta(\mathcal{D}_{1,R})^2_{P_1}\cdot det_\zeta(\mathcal{D}_{2,R})^2_{P_2}} \quad . \tag{0.11}$$

In (0.11) $\mathcal{D}_{i,R}$ denotes the operator $\mathcal{D}_R$ restricted to $M_{i,R}$ and P_i denotes the boundary condition, a finite dimensional modification of Atiyah-Patodi-Singer condition, defined below. We select

$$\sigma_1,\sigma_2 : ker\ B\to ker\ B \quad ,$$

two involutions on kernel of B which satisfy

$$G\sigma_i=-\sigma_i G \quad ,$$

and define $\pi_i=\frac{Id-\sigma_i}{2}$ the orthogonal projections onto -1 eigenspaces of σ_i . Then, we define

$$P_1=\Pi_< +\pi_1 \quad , \quad P_2=\Pi_> +\pi_2 \quad . \tag{0.12}$$

The projections P_i are the *"ideal boundary conditions"* mentioned before. They were introduced by Cheeger in [9] (see also [10]).

If we assume that all three operators in (0.11) are invertible, with eigenvalues bounded away from 0 (for any R large enough), then the quotient in (0.11) is bounded and we obtain a finite limit as $R\to\infty$. This situation was discussed in [27] (see also [26]). The assumptions made in [27] allow us to discard the large time contributions. In the general case, however, the presence of small eigenvalues makes the *"large time"* part extremely important. The analysis of the contributions made by the small eigenvalues of the operators $\mathcal{D}_R^2$, $(\mathcal{D}_{i,R})^2_{P_i}$ is the main achievement of the present paper.

The eigenvalues of $\mathcal{D}_R$ fall into three different categories. We have infinitely many large eigenvalues ($l-values$) bounded away from 0 . Secondly, there is an infinite family of small eigenvalues ($s-values$) of the size $O(\frac{1}{R})$. Last, we have a finite amount of eigenvalues which decay exponentially with R ($e-values$). There exists R_o such that, for any $R>R_o$ the number h_M of $e-values$ is independent of R . The following formula gives h_M

$$h_M=dim\ ker_{L^2}\mathcal{D}_{1,\infty}+dim\ ker_{L^2}\mathcal{D}_{2,\infty}+dim\ L_1\cap L_2 \quad , \tag{0.13}$$

where the operator $\mathcal{D}_{i,\infty}$ is the operator $\mathcal{D}_i$ extended in a natural way to the manifold $M_{i,\infty}$, which is equal to M_i with the infinite semicylinder $[0,\infty)\times Y$ (or $(-\infty,0]\times Y$) attached. More precisely, the operator $\mathcal{D}_i$

extends in a natural way to the manifold $M_{i,\infty}$. It has a unique closed self-adjoint extension in $L^2(M_{i,\infty}; S)$, which we denote by $\mathcal{D}_{i,\infty}$. The subspaces $L_i \subset ker\ B$ are the spaces of limiting values of extended L^2-solutions of $\mathcal{D}_{i,\infty}$ (see [1], see also [3]).

Remark 0.1. The decomposition of the eigenvalues of the operator $\mathcal{D}_R$ into different classes was discussed by Cappell, Lee and Miller (see [8]; see also [35]). The corresponding analysis for the operator $(\mathcal{D}_{i,R})_{P_i}$ was provided by Werner Müller (see [23]).

The spectrum of the operators $(\mathcal{D}_{i,R})_{P_i}$ splits in the same way as the spectrum of $\mathcal{D}_R$. The only difference is that the operators $(\mathcal{D}_{i,R})_{P_i}$ do not have non-trivial $e - values$ and the dimension of the space of the solutions of $(\mathcal{D}_{i,R})_{P_i}$ is equal to

$$h_i = dim\ ker\ (\mathcal{D}_{i,R})_{P_i} = dim\ ker_{L^2}\mathcal{D}_{i,\infty} + dim\ L_i \cap ker(\sigma_i - 1) \ . \tag{0.14}$$

It was already mentioned that, in the adiabatic limit, the different types of eigenvalues make their contribution at different time intervals. The contribution made by $l - values$ comes from the time interval $0 < t < R^{\varepsilon}$, where ε is a positive number. More precisely, it is not difficult to show that the $l - values$ contribution to the adiabatic limit of (0.11) from the time interval $[R^{\varepsilon}, \infty]$ disappears as $R \to \infty$ (see Section 1). The $s - values$ contribution comes from the interval $[R^{2-\varepsilon}, R^{2+\varepsilon}]$, for ε small enough.

The contribution made by $l - values$ was discussed in [27]. To be more precise in [27] (see also [26]) we discussed the case of the operator $\mathcal{D}$, such that $\mathcal{D}_{i,\infty}$ and the tangential operator B have trivial kernels. This condition implies the vanishing of the $e - values$ and $s - values$. This allows us to reduce the computation of the quotient in (0.11) to the corresponding quotient on the cylinder. The result is that the limit of (0.11) as $R \to \infty$ is equal to $2^{-\zeta_{B^2}(0)}$. Actually, even in the presence of $e-values$ and $s-values$, we are able to show that in the adiabatic limit the contribution from the time interval $[0, R^{2-\varepsilon}]$ comes from the cylinder only. This can be stated in the following way:

as $R \to \infty$ *the small time contribution forgets about the geometry involved and the operators in the quotient (0.11) can be replaced by their counterparts living on the cylinders* $((-\infty, 0] \cup [0, \infty)) \times Y$ *and* $(-\infty, +\infty) \times Y$ *.*

The precise meaning of the aforementioned statement can be found in Section 1. The method we use to prove this statement combines *Duhamel's Principle* and *finite propagation speed property of the wave operators.*

The computation of the $s-values$ contribution is the main achievement of the paper. We follow Werner Müller (see [23]) and use the $Scattering\, Theory$ to get a description of the $s-values$. The operators $\mathcal{D}_{i,\infty}$ on $M_{i,\infty}$ determine scattering matrices $C_i(\lambda)$. It appears that the matrix $C_{12} = C_1(0)C_2(0)$ determines the $s - values$ of the operator $\mathcal{D}_R$ for R large enough. More precisely we define a one dimensional Dirac operator $D(C_{12})$ equal to $-\frac{i}{2}\frac{d}{du}$ acting on sections of a flat vector bundle over S^1 determined by the unitary matrix $\overline{C}_{12}$ (complex conjugate of C_{12}). It turns out that, up to a negligible error, the $s-values$ of $\mathcal{D}_R$ are determined by the eigenvalues of the operator $D(C_{12})$. The exact correspondence is stated in Section 3. The operators $(\mathcal{D}_{i,R})_{P_i}$ define corresponding matrices S_{σ_i} . The finite-dimensional unitary operator S_{σ_i} is determined by the scattering matrix $C_i(0)$ and the involution σ_i . Those matrices provide us with the Dirac operators $D(S_{\sigma_i})$ on S^1 . The eigenvalues of $D(S_{\sigma_i})$, determine the $s - values$ of the operators $(\mathcal{D}_{i,R})_{P_i}$. Now, we can formulate the main result of the paper

Theorem 0.2. *The following formula holds:*

(0.15)
$$\lim_{R\to\infty} \frac{R^{-2h}\cdot det_\zeta \mathcal{D}_R^2}{det_\zeta(\mathcal{D}_{1,R})^2_{P_1}\cdot det_\zeta(\mathcal{D}_{2,R})^2_{P_2}} = 2^{-\zeta_{B^2}(0)}\cdot \frac{det_\zeta \frac{1}{4} D(C_{12})^2}{det_\zeta D(S_{\sigma_1})^2\cdot det_\zeta D(S_{\sigma_2})^2} \ ,$$

where $h = h_M - h_1 - h_2 = dim(L_1\cap L_2) - dim(L_1\cap ker(\sigma_1 - 1)) - dim(L_2\cap ker(\sigma_2 - 1))$.

Remark 0.3. Let us observe that we can choose involutions σ_i in such a way that $h = 0$.

We study the relative ζ-function and its derivative to prove Theorem 0.2. We define

(0.16)
$$\zeta_{rel,R}(s) := \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}[Tr(e^{-t\mathcal{D}_R^2} - e^{-t(\mathcal{D}_{1,R})^2_{P_1}} - e^{-t(\mathcal{D}_{2,R})^2_{P_2}}) - h]\ dt \ .$$

We decompose $\zeta_{rel,R}(s)$ into two parts

$$\zeta_s^R(s) = \frac{1}{\Gamma(s)}\int_0^{R^{2-\varepsilon}} (\cdot)\ dt \ \ , \ \ \zeta_l^R(s) = \frac{1}{\Gamma(s)}\int_{R^{2-\varepsilon}}^\infty (\cdot)\ dt \ .$$

The derivatives of $\zeta_s^R(s), \zeta_l^R(s)$ at $s = 0$ give the small and large time contribution.

In Section 1 we deal with the small time contribution and prove the following proposition

Proposition 0.4.

$$\lim_{R\to\infty}\Big(\frac{d}{ds}\zeta_s^R(s)|_{s=0} - h(\gamma - (2-\varepsilon)\cdot ln\ R)\Big) = ln\ 2\cdot\zeta_{B^2}(0)\ . \tag{0.17}$$

This gives the *"small time contribution"* and, as a result, the first factor in the right side of (0.15).

In Section 2 we obtain the description of the small eigenvalues. We follow [23] and use scattering theory in order to get a description of the $s-values$ of the $\mathcal{D}_R$ and the boundary problems $(\mathcal{D}_{i,R})_{P_i}$, which allows us to make a comparison of the eigenvalues of those operators with the eigenvalues of the corresponding operators over S^1 . This is the central part of this paper.

In Section 3 we use the results of Section 2 to show that, in the adiabatic limit, the large time contribution to the quotient (0.11) is equal to

$$\frac{det_\zeta\frac{1}{4}D(C_{12})^2}{det_\zeta D(S_{\sigma_1})^2\cdot det_\zeta D(S_{\sigma_2})^2}\ .$$

This is the second factor on the right side of (0.15).

The $e-values$ make their presence via the factor R^{-2h} on the left side of (0.15).

The last two Sections deal with the Lesch-Wojciechowski formulas. In Section 4 we review the formula for the η-invariant and offer a new proof based on the method used in order to prove Theorem 0.2. This proof is much more complicated than the original presented in [20]. However, it is a nice illustration of the differences we encounter when we deal with ζ-determinant instead of the η-invariant. In the last Section we prove the corresponding formula for the ζ-determinant of the Dirac Laplacian. The nice feature of the formula is that the ratio of the ζ-determinants is expressed as the ratio of the determinants of the finite-dimensional matrices. We refer to Section 5 for details.

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1. Small Time Contribution

In this Section we compute

$$\lim_{R\to\infty} \frac{d}{ds}\zeta_s^R(s)|_{s=0} \quad .$$

This is done in two steps. First, we use *Duhamel's Principle* and *Finite Propagation Speed Property of the Wave Operator* to show that we can reduce the problem to computations on the cylinder. Then, we perform the explicit calculations on the cylinder. Both parts are fairly standard. The cylinder contribution have been recently computed in [27]. Therefore, we only discuss the details of the reduction scheme and refer to [27] for the explicit computation on the cylinder.

Let $\mathcal{E}_R(t;x,y)$ denote the kernel of the operator $e^{-t\mathcal{D}_R^2}$. We introduce the specific parametrix for $\mathcal{E}_R(t;x,y)$, which fits our main purpose to "*localize*" the contribution coming from the cylinder $[-R,R]\times Y$ and the interior of M_R . In fact, the interesting point here is that we use $\mathcal{E}_R(t;x,y)$ to construct this parametrix. Let $\mathcal{E}_{cyl}(t;x,y)$ denote the kernel of the operator $e^{-t(-\partial_u^2+B^2)}$ on the infinite cylinder $(-\infty,+\infty)\times Y$. We introduce a smooth, increasing function $\rho(a,b):[0,\infty)\to[0,1]$ equal to 0 for $0\le u\le a$ and equal to 1 for $b\le u$. We use $\rho(a,b)(u)$ to define

$$\phi_1 = 1-\rho(\frac{5}{7}R,\frac{6}{7}R) \quad , \quad \psi_1 = 1-\psi_2$$

and

$$\phi_2 = \rho(\frac{1}{7}R,\frac{2}{7}R) \quad , \quad \psi_2 = \rho(\frac{3}{7}R,\frac{4}{7}R).$$

We extend these functions to symmetric functions on the whole real line. These functions are constant outside the interval $[-R,R]$ and we use them to define the corresponding functions on a manifold M_R . Now, we define $Q_R(t;x,y)$ a *"parametrix"* for the kernel $\mathcal{E}_R(t;x,y)$ using the following formula

$$Q_R(t;x,y) = \phi_1(x)\mathcal{E}_{cyl}(t;x,y)\psi_1(y) + \phi_2(x)\mathcal{E}_R(t;x,y)\psi_2(y) \quad . \tag{1.1}$$

It follows from Duhamel's Principle that

$$\mathcal{E}_R(t;x,y) = Q_R(t;x,y) + (\mathcal{E}_R * \mathcal{C}_R)(t;x,y) \ , \tag{1.2}$$

where $\mathcal{E}_R * \mathcal{C}_R$ is a convolution given by

$$(\mathcal{E}_R * \mathcal{C}_R)(t;x,y) = \int_0^t ds \int_{M_R} dz \{\mathcal{E}_R(s;x,z)\mathcal{C}_R(t-s;z,y)\} \ ,$$

and the correction term $\mathcal{C}_R(t;x,y)$ is given by the formula

$$\mathcal{C}_R(t;x,y) = -\frac{\partial^2\phi_1}{\partial u^2}(x)\mathcal{E}_{cyl}(t;x,y)\psi_1(y) - \frac{\partial\phi_1}{\partial u}(x)\frac{\partial\mathcal{E}_{cyl}}{\partial u}(t;x,y)\psi_1(y)$$

$$-\frac{\partial^2\phi_2}{\partial u^2}(x)\mathcal{E}_R(t;x,y)\psi_2(y) - \frac{\partial\phi_2}{\partial u}(x)\frac{\partial\mathcal{E}_R}{\partial u}(t;x,y)\psi_2(y).$$

The following elementary Lemma follows from the construction of $Q_R(t;x,y)$.

Lemma 1.1. *The error term $\mathcal{C}_R(t;x,y)$ is equal to 0 outside of the cylinder $[-\frac{6}{7}R, \frac{6}{7}R] \times Y$. Moreover it is equal to 0 if the distance between x and y is smaller than $R/7$.*

Now, following Cheeger, Gromov and Taylor (see [11]; see also Section 3 of [10]), we use the *Finite Propagation Speed Property for the Wave Operator.* The technique introduced in [11] allows us to compare the heat kernels determined by the operators $\mathcal{D}_R^2$, $(\mathcal{D}_1)^2_{P_1}$ and $(\mathcal{D}_2)^2_{P_2}$ with the heat kernel of the operator $-\partial_u^2 + B^2$ on the cylinder $(-\infty,+\infty) \times Y$. We describe the case we need in our work. Let N_1 and N_2 be Riemannian manifolds of dimension n and S_i be spinors bundle with Dirac operators $\mathcal{D}_i$. Assume that there exists a decomposition $N_i = K_i \cup U_i$, where U_i is an open subset of N_i . Moreover, we assume that there exists an isometry $h : U_1 \to U_2$ covered by the unitary bundle isomorphism $\Phi_h : S_1|_{U_1} \to S_2|_{U_2}$, which intertwines Dirac operators $\mathcal{D}_1|_{U_1}$ and $\mathcal{D}_2|_{U_2}$. We identify

$$U \cong U_1 \cong U_2 \ ,$$

so that M_1 and M_2 have a common open subset U . Let $\mathcal{E}_i(t;x,y)$ denote the kernel of the operator $e^{-t\mathcal{D}_i^2}$. Then we have the following estimate on the difference of the heat kernels on U

Theorem 1.2. *(see [11].) Let $x, y \in U$ and let $r = min(d(x,K_1), d(y,K_1), d(x,K_2), d(y,K_2))$. Then there exist positive constants c_1, c_2 , such that*

$$\|\mathcal{E}_1(t;x,y) - \mathcal{E}_2(t;x,y)\| \le c_1 e^{-c_2 \frac{r^2}{t}} \quad . \tag{1.3}$$

Remark 1.3. The same technique also gives estimate (1.3) for the derivatives of the kernels in the normal direction.

In our situation $N_1 = M_R$ (or $M_{i,R}$) and N_2 is the cylinder $(-\infty, +\infty) \times Y$. Note that $\mathcal{E}_{cyl}(t;x,y)$, the kernel of the heat operator defined by operator $\mathcal{D}^2_{N_2} = -\partial_u^2 + B^2$, satisfies the standard estimate. More precisely, for small time we have

$$\|\mathcal{E}_{cyl}(t;(u,w),(v,z))\| \le c_1 t^{-\frac{n}{2}} e^{-c_3 \frac{(u-v)^2}{t}} \quad , \tag{1.4}$$

where $u, v \in \mathbf{R}$ and $w, z \in Y$. For large t , the operator e^{-tB^2} approaches $\pi_{ker\ B}$, the orthogonal projection onto $ker\ B$. It follows from the Sobolev imbedding theorem that $\pi_{ker\ B}(x,y)$, the kernel of this projection, is uniformly bounded on Y (see for instance (1.5) in [13]). Hence, for large time, the kernel of e^{-tB^2} is bounded by a constant, not by the factor $c_1 \cdot e^{-c_2 \frac{d^2(w,z)}{t}}$.

We are going to use (1.3) and (1.4) to estimate the error term $(\mathcal{E}_R * \mathcal{C})(t;x,y)$. Lemma 1.1 and Theorem 1.2 gives the following estimate

Proposition 1.4. *There exist positive constants c_1 , c_2 such that the following estimate holds for any $((u,w),(v,z)) \in supp\ \mathcal{C}_R(t;\cdot,\cdot)$*

(1.5)
$$\|\mathcal{C}_R(t;(u,w),(v,z))\| \le c_1 e^{-c_2 \frac{R^2}{t}} \quad , \quad \|\mathcal{E}_R(t;(u,w),(v,z))\| \le c_1 e^{-c_2 \frac{R^2}{t}} \quad .$$

Proof. We have

$$\|\mathcal{E}_R(t;(u,w),(v,z))\| \le \|\mathcal{E}_{cyl}(t;(u,w),(v,z))\|$$
$$+ \|\mathcal{E}_R(t;(u,w),(v,z)) - \mathcal{E}_{cyl}(t;(u,w),(v,z))\| \quad .$$

It is obvious that in the case $(u,w),(v,z) \in supp\, \mathcal{C}_R(t;\cdot,\cdot)$, both summands on the right side satisfy estimate (1.5) for some positive constants c_1 and c_2 . There is no problem with getting the same estimate for the normal derivatives of the kernels in the normal direction (see (1.3) and Remark 1.3), which takes care of the estimates for the kernel $\mathcal{C}_R(t;(u,w),(v,z))$. ☐

Now, we are ready to prove the main technical result of this section

Theorem 1.5.

$$\lim_{R\to\infty}\int_0^{R^{2-\varepsilon}} \frac{dt}{t}\int_{M_R} tr\ (\mathcal{E}_R * \mathcal{C}_R)(t;x,x)dx = 0 \ . \tag{1.6}$$

Proof. The Theorem follows from estimate (1.5), which implies

$$|tr\ (\mathcal{E}_R * \mathcal{C}_R)(t;x,x)| \leq \|(\mathcal{E}_R * \mathcal{C}_R)(t;x,x)\| \leq$$

$$\int_0^t ds \int_{M_R} \|\mathcal{E}_R(s;x,z)\mathcal{C}_R(t-s;z,x)\|dz =$$

$$\int_0^t ds \int_{[-\frac{6}{7}R,\frac{6}{7}R]\times Y} \|\mathcal{E}_R(s;x,z)\mathcal{C}_R(t-s;z,x)\|dz =$$

$$c_1^2 \cdot \int_0^t ds \int_{[-\frac{6}{7}R,\frac{6}{7}R]\times Y} e^{-c_2\frac{R^2}{s}} e^{-c_2\frac{R^2}{t-s}} dz \leq$$

$$c_3 R\cdot \int_0^t e^{-c_2\frac{tR^2}{s(t-s)}} ds \leq c_3 R\cdot \int_0^{\frac{t}{2}} e^{-2c_2\frac{R^2}{s}} ds \leq c_3 R\cdot \frac{t}{2} e^{-4c_2\frac{R^2}{t}} \ ,$$

where the last estimate is a consequence of the elementary inequality

$$\int_0^t e^{-\frac{c}{s}} ds \leq t e^{-\frac{c}{t}} \ .$$

We have proved

$$|tr\ (\mathcal{E}_R * \mathcal{C}_R)(t;x,x)| \leq c_4 t \cdot e^{-c_5\frac{R^2}{t}} \ , \tag{1.7}$$

which allows us to estimate as follows

$$|\int_0^{R^{2-\varepsilon}} \frac{dt}{t}\int_{M_R} tr\ (\mathcal{E}_R * \mathcal{C}_R)(t;x,x)dx| \leq$$

$$c_6 R \cdot \int_0^{R^{2-\varepsilon}} e^{-c_5 \frac{R^2}{t}} \leq c_6 R^{3-\varepsilon} e^{-c_5 R^{\varepsilon}} \quad ,$$

and (1.6) follows. □

The proof of (1.6) for the operator $(\mathcal{D}_{i,R})_{P_i}$ on a manifold $M_{i,R}$ is carried out in exactly the same manner and we obtain the main result of this Section

Theorem 1.6. *We have the following "adiabatic limit" formula*

(1.8)
$$\lim_{R\to\infty} \int_0^{R^{2-\varepsilon}} \frac{1}{t} \cdot (Tr\ e^{-t\mathcal{D}_R^2} - Tr\ e^{-t(\mathcal{D}_{1,R})_{P_1}^2} - Tr\ e^{-t(\mathcal{D}_{2,R})_{P_2}^2}) dt = ln\ 2 \cdot \zeta_{B^2}(0) \quad .$$

Proof. Estimate (1.6) gives

$$\lim_{R\to\infty} \int_0^{R^{2-\varepsilon}} \frac{1}{t} \cdot (Tr\ e^{-t\mathcal{D}_R^2} - Tr\ e^{-t(\mathcal{D}_{1,R})_{P_1}^2} - Tr\ e^{-t(\mathcal{D}_{2,R})_{P_2}^2}) dt =$$

$$\lim_{R\to\infty} \int_0^{R^{2-\varepsilon}} \frac{dt}{t} \cdot [\int_{M_{1,R}} (tr\ Q_R(t;x,x) - tr\ Q_{1,R}(t;x,x)) dx +$$
$$\int_{M_{2,R}} (tr\ Q_R(t;x,x) - tr\ Q_{2,R}(t;x,x)) dx] \quad .$$

Now, the interior contributions to the different parametrices, all determined by the kernel $\mathcal{E}_R(t;x,y)$, cancel out and we are left with the cylinder contribution only. An explicit computation of this contribution was presented in [27]. □

We need the next result to finish the analysis of the *"small time contribution"*.

Proposition 1.7. *We have the following equality*

$$(\zeta_s^R)'(0) = \int_0^{R^{2-\varepsilon}} t^{-1} Tr(e^{-t\mathcal{D}_R^2} - e^{-t(\mathcal{D}_{1,R})_{P_1}^2} - e^{-t(\mathcal{D}_{2,R})_{P_2}^2}) dt + h(\gamma - (2-\varepsilon) ln\ R) \quad .$$

Proof. First, we separate $\zeta_s^R(0)$ into two parts

$$\frac{1}{\Gamma(s)}\int_0^{R^{2-\varepsilon}} t^{s-1}Tr(e^{-t\mathcal{D}_R^2}-e^{-t(\mathcal{D}_{1,R})_{P_1}^2}-e^{-t(\mathcal{D}_{2,R})_{P_2}^2})\,dt-\frac{h}{\Gamma(s)}\int_0^{R^{2-\varepsilon}} t^{s-1}dt \ .$$

We have the elementary identity

$$\frac{d}{ds}\{\frac{h}{\Gamma(s)}\int_0^{R^{2-\varepsilon}} t^{s-1}dt\}\Big|_{s=0}=-h\gamma+h(2-\varepsilon)\log R \ .$$

The derivative

$$\frac{d}{ds}\{\frac{1}{\Gamma(s)}\int_0^{R^{2-\varepsilon}} t^{s-1}Tr(e^{-t\mathcal{D}_R^2}-e^{-t(\mathcal{D}_{1,R})_{P_1}^2}-e^{-t(\mathcal{D}_{2,R})_{P_2}^2})dt\}\Big|_{s=0}$$

is obtained in the standard way following computation of the $\zeta'_{\mathcal{D}^2}(0)$ (see [32], or the introduction to [27]). There is no problem with the pole at $s=0$, because the constant term in the asymptotic expansion of the

$$Tr(e^{-t\mathcal{D}_R^2}-e^{-t(\mathcal{D}_{1,R})_{P_1}^2}-e^{-t(\mathcal{D}_{2,R})_{P_2}^2})$$

disappears. Indeed, it is a linear combination of the corresponding terms in the expansion of $Tr\ e^{-t\mathcal{D}_R^2}$, $Tr\ e^{-t(\mathcal{D}_{1,R})_{P_1}^2}$ and $Tr\ e^{-t(\mathcal{D}_{2,R})_{P_2}^2}$, but all of them are equal to 0 . The vanishing of this coefficient for the Dirac operator on a closed odd-dimensional manifold is a classical result (see for instance [14]). The result for the boundary problems considered in this paper has been recently established in the Appendix to the paper [27]. This ends the proof of the Proposition. $\square$

The proof of Proposition 0.4 is complete.

2. SMALL EIGENVALUES AND SCATTERING MATRICES

In this section we investigate the relation between the $s-values$ of the operators $\mathcal{D}_R$, $(\mathcal{D}_{1,R})_{P_1}$ and $(\mathcal{D}_{2,R})_{P_2}$ and the scattering matrices $C_1(\lambda)$, $C_2(\lambda)$ determined by the operators $\mathcal{D}_{i,\infty}$ on $M_{i,\infty}$. We refer to [23] (section 4 and Section 8) for a more detailed exposition of the elements of *Scattering Theory* that we use in this paper.

Let μ_1 denote the lowest positive eigenvalue of the tangential operator B . For any $f\in ker(B)$ and $\lambda\in\mathbb{C}-\{(-\infty,-\mu_1]\cup[\mu_1,+\infty)\}$ we define

$E(f,\lambda)$ a generalized eigensection of $\mathcal{D}_{1,\infty}$ over $M_{1,\infty}$ determined by the couple $\{f,\lambda\}$ (see [23] Section 4). It is a smooth section of the bundle S over $M_{1,\infty}$, which satisfies

$$\mathcal{D}E(f,\lambda) = \lambda E(f,\lambda) \quad .$$

The section $E(f,\lambda)$ has the following form over $[0,\infty)\times Y$,

$$E(f,\lambda) = e^{-i\lambda u}(f - iGf) + e^{i\lambda u}C(\lambda)(f - iGf) + \theta(f,\lambda)$$

where θ is a square integrable section such that, for each u , $\theta(f,\lambda,(u,\cdot))$ is orthogonal to $ker(B)$. The operator $C(\lambda) : ker\ B \to ker\ B$ is regular and unitary and equals the "*Scattering Matrix*" for $|\lambda| < \mu_1$. Moreover, we have

$$C(\lambda)C(-\lambda) = Id \quad , \quad C(\lambda)G = -GC(\lambda) \quad .$$

In particular

$$C(0)^2 = Id \quad , \quad C(0)G = -GC(0) \quad .$$

Therefore $C(0)$ gives a distinguished unitary involution of $ker(B)$. In fact, the space $L_1 \subset ker\ B$, which appeared in the Introduction is equal to the $+1$ eigenspace of $C(0)$

$$L_1 = ker(C(0) - 1) \quad .$$

Now we study the $s-values$ of the operator $\mathcal{D}_R$ over M_R . We introduce a manifold

$$\bar{M}_R = M_{1,R} \sqcup M_{2,R} \quad .$$

The boundary of $\bar{M}_R$ is equal to two copies of Y . We consider $\mathcal{D}_R$ the operator on a closed manifold M_R , as the Dirac operator on $\bar{M}_R$, which satisfies the transmission boundary condition. In particular if $\lambda = \lambda(R)$ is an $s - value$ of $\mathcal{D}_R$ and ϕ denotes the corresponding eigensection, then ϕ satisfies the *"transmission boundary condition"*

$$\phi|_{\partial M_{1,R}} = \phi|_{\partial M_{2,R}} \quad .$$

We refer to [5] for a more detailed discussion of the transmission problem (see also [26]). We want to warn the reader that, although we discuss transmission boundary condition so it would be natural to consider $M_{1,R}$ as $M_{1,R} = M_1 \cup [-R, 0] \times Y$, we still parametrize the cylindrical parts as

$$M_{1,R} = M_1 \cup [0, R] \times Y \quad , \quad M_{2,R} = M_2 \cup [-R, 0] \times Y \quad .$$

In the following we assume that

$$|\lambda| < R^{-\kappa} \quad , \quad \|\phi\| = 1 \quad , \tag{2.1}$$

where $0 < \kappa < 1$. In particular we have $\lambda \leq \frac{\mu_1}{2}$. The section ϕ can be represented in the following way on $[0, R] \times Y \subset M_{1,R}$

$$\phi = e^{-i\lambda u}\psi_1 + e^{i\lambda u}\psi_2 + \phi_1$$

where $\psi_1 \in ker(G - i), \psi_2 \in ker(G + i)$ and ϕ_1 has the following form

$$\phi_1(u, y) = \sum_k a_k e^{-\sqrt{\mu_k^2 - \lambda^2}u}\{(\mu_k + \sqrt{\mu_k^2 - \lambda^2})\phi_k(y) + \lambda G\phi_k(y)\}$$

$$+ \sum_k b_k e^{\sqrt{\mu_k^2 - \lambda^2}u}\{(\lambda\phi_k(y) + (\mu_k + \sqrt{\mu_k^2 - \lambda^2})G\phi_k(y)\}$$

where $\{\mu_k\}_{k\in\mathbb{N}}$ is the set of positive eigenavlues of B and ϕ_k denotes the eigensection of B corresponding to μ_k . We will use the following estimate on the L^2-norm of ϕ_1 in the y-direction on the cylinder (see Lemma 2.1 in [35])

Lemma 2.1. *There exist positive constants c_1, c_2 such that*

$$\|\phi_1|_{\{R\}\times Y}\| \leq c_1 e^{-c_2 R} \quad .$$

We also have the following result (see [23] Proposition 8.14, see also Theorem 2.2 in [35]).

Proposition 2.2. *If the eigenvalue $\lambda(R)$ is not bounded away from 0 , then the zero$-$eigenmode, $e^{-i\lambda u}\psi_1 + e^{i\lambda u}\psi_2$, of ϕ is a non-trivial section of $S|Y$.*

Assume that ψ_1 is non-zero. Choose $\psi \in L_1$ such that $\psi_1 = \psi - iG\psi$. Then the generalized eigensection $E(\psi,\lambda)$ associated to ψ over $M_{1,\infty}$ has the following expression

$$E(\psi,\lambda) = e^{-i\lambda u}(\psi - iG\psi) + e^{i\lambda u}C_1(\lambda)(\psi - iG\phi) + \theta(\psi,\lambda)$$

over $[0,\infty) \times Y$. Following [23], we introduce $F = \phi|_{M_{1,R}} - E(\psi,\lambda)|_{M_{1,R}}$. Green's Theorem gives

$$0 = \langle DF, F\rangle_{M_{1,R}} - \langle F, DF\rangle_{M_{1,R}} = \int_Y \langle GF, F\rangle dy.$$

On the other hand, Lemma 2.1 shows that

$$\int_Y \langle GF, F\rangle dy = -i\| \ C_1(\lambda)\psi_1 - \psi_2\|^2 + O(e^{-c_3 R})$$

for some positive constant c_3. This produces the estimate

$$\| \ C_1(\lambda)\psi_1 - \psi_2\|^2 < e^{-c_3 R} \tag{2.2}$$

(compare [23]). It follows from (2.2) that ψ_2 is non-trivial section if and only if ψ_1 is non-trivial.

We repeat the argument for the eigensection ϕ of $\mathcal{D}_R$ over $[-R,0]\times Y \subset M_{2,R}$. We have

$$\phi = e^{i\lambda u}\psi_3 + e^{-i\lambda u}\psi_4 + \phi_2$$

where $\psi_3 \in ker(G+i), \psi_4 \in ker(G-i)$ and ϕ_2 is orthogonal to $ker(B)$. Choose $\psi \in L_2$ such that $\psi_3 = \psi + iG\psi$. Then the generalized eigensection $E(\phi,\lambda)$ associated to ϕ over $M_{2,\infty}$ can be written as

$$E(\psi,\lambda) = e^{i\lambda u}(\psi + iG\psi) + e^{-i\lambda u}C_2(\lambda)(\psi + iG\psi) + \theta(\psi,\lambda) \quad .$$

We obtain the expected estimate

$$\| \ C_2(\lambda)\psi_3 - \psi_4\|^2 < e^{-c_3 R} \quad . \tag{2.3}$$

Transmission boundary condition over $Y \sqcup Y = \partial M_{1,R} \sqcup \partial M_{2,R}$ implies the equality

$$e^{-i\lambda R}\psi_1 + e^{i\lambda R}\psi_2 = e^{-i\lambda R}\psi_3 + e^{i\lambda R}\psi_4$$

so that

$$\psi_1 = e^{2i\lambda R}\psi_4, \quad \psi_3 = e^{2i\lambda R}\psi_2 \ . \tag{2.4}$$

By (2.2), (2.3), (2.4), we have

$$\|e^{2i\lambda R}C_1(\lambda)\psi_4 - \psi_2\|^2 < e^{-c_3R} \ , \ \|e^{2i\lambda R}C_2(\lambda)\psi_2 - \psi_4\|^2 < e^{-c_3R} \ .$$

We combine those inequalities and obtain

$$\|e^{4i\lambda R}C_1(\lambda) \circ C_2(\lambda)\psi_2 - \psi_2\|^2 < e^{-c_4R} \ , \tag{2.5}$$

$$\|e^{4i\lambda R}C_2(\lambda) \circ C_1(\lambda)\psi_4 - \psi_4\|^2 < e^{-c_4R} \tag{2.6}$$

for some positive constant c_4.

There exist a positive number δ such that the unitary operators $C_1(\lambda)$, $C_2(\lambda)$ are the analytic functions of λ for $\lambda \in (-\delta, \delta)$. We define operator $C_{12}(\lambda)$ as

$$C_{12}(\lambda) = C_1(\lambda) \circ C_2(\lambda) : ker(G+i) \to ker(G+i) \ .$$

The operator $C_{12}(\lambda)$ is a unitary operator and it is an analytic function of λ for $\lambda \in (-\delta, \delta)$. Therefore, there exist real analytic functions $\alpha_j(\lambda)$ of $\lambda \in (-\delta, \delta)$ such that $\exp(i\alpha_j(\lambda))$ are the corresponding eigenvalues of $C_{12}(\lambda)$ for $\lambda \in (-\delta, \delta)$. We fix the constant δ_1, $0 < \delta_1 < \delta$, and define

$$r_j = max_{\lambda \in (-\delta_1, \delta_1)} |\alpha_j'(\lambda)| \ .$$

The function $f(\lambda) = 4R\lambda + \alpha_j(\lambda)$ is strictly increasing on $(-\delta_1, \delta_1)$, if we assume that $R \geq r_j$. Choose $R_0 \geq max(r_j, \delta_1^{-\frac{1}{\kappa}})$. For $R \geq R_0$ and $k \in \mathbb{Z}$, there exists at most one solution of the equation

$$4R\lambda + \alpha_j(\lambda) = 2\pi k, \quad |\lambda| < R^{-\kappa} \ . \tag{2.7}$$

Let $k_{j,max} = k_{j,max}(R)$ be the maximal k for which (2.7) has a solution. Then

$$|k_{j,max}| \leq R^{1-\kappa} \tag{2.8}$$

for $R \geq R_0$. We now use (2.5) to show that $\lambda(R)$ satisfies

$$4R\lambda(R) + \alpha_j(\lambda(R)) = 2\pi k + O(e^{-cR}) \quad , \qquad |\lambda(R)| < R^{-\kappa} \ . \tag{2.9}$$

We have proved the following result

Proposition 2.3. *There exists R_0 such that for $R > R_0$ the $s-value$ $\lambda(R)$ of $\mathcal{D}_R$ satisfies*

$$4R\lambda(R) + \alpha_j(\lambda(R)) = 2\pi k + O(e^{-cR}) \tag{2.10}$$

for an integer k with $|k| < R^{1-\kappa}$, where $\exp(i\alpha_j(\lambda))$ is an eigenvalue of the unitary operator $C_1(\lambda) \circ C_2(\lambda) : ker(G+i) \to ker(G+i)$.

The map $C_{12}(0) : ker(G+i) \to ker(G+i)$ is not the identity map. However, we have the following proposition

Proposition 2.4. *The map $C_{12}(0)$ is equal to the identity when restricted to the subspace $I_+(L_1 \cap L_2)$ where $I_+ = G - i : ker(B) \to ker(G+i)$.*

Proof. The subspace L_i is the $+1$-eigenspace of the operator $C_i(0)$. Hence it follows from the definition, that the restriction of the map $C_1(0) \circ C_2(0)$ to $L_1 \cap L_2$ is the identity map. The claim follows easily from the following diagram

$$\begin{array}{ccc} L_1 \cap L_2 & \xrightarrow{I_+} & ker(G+i) \\ {\scriptstyle C_1(0)\circ C_2(0)}\downarrow & & \downarrow{\scriptstyle C_1(0)\circ C_2(0)} \\ L_1 \cap L_2 & \xrightarrow{I_+} & ker(G+i) \ . \end{array}$$

□

It follows from Proposition 2.4 that the multiplicity of the eigenvalue $+1$ of the operator $C_{12}(0)$ is equal to $dim(L_1 \cap L_2)$.

Moreover, (2.10) implies that the corresponding eigenvalues of the operator $\mathcal{D}_R$ are, in fact, $e-values$. Here we mean eigenvalues $\lambda(R)$, which corresponds to $\alpha_j(0) = 0$ and $k = 0$ in the formula (2.10) in the Proposition 2.3. This provides another proof of the result, originally shown in [8]

Theorem 2.5. *The dimension of the space of eigensections corresponding to $e-values$ which are not determined by $ker_{L^2}(\mathcal{D}_{i,\infty})$ for $i=1,2$ is equal to the dimension of the space $L_1 \cap L_2$.*

The corresponding analysis for the operators $(\mathcal{D}_{i,R})_{P_i}$ for $i=1,2$ follows in the same manner. The case of the boundary condition determined by $C_1(0)$ was studied in [23], but it is not difficult to show, that everything works well for any involution σ on $ker(B)$. Let ϕ be an eigensection of $(\mathcal{D}_{1,R})_{P_1}$, which corresponds to the $s-value$ $\lambda = \lambda(R)$. That is,

$$\mathcal{D}_{1,R}\phi = \lambda\phi \quad \text{with} \quad |\lambda| < R^{-\kappa}$$

for some fixed κ with $0<\kappa<1$ and

$$P_1(\phi|_{\{R\}\times Y}) = 0. \tag{2.11}$$

The section ϕ can be represented in the following way on $[0,R]\times Y \subset M_{1,R}$

$$\phi = e^{-i\lambda u}\psi_1 + e^{i\lambda u}\psi_2 + \phi_1$$

where $\psi_1 \in ker(G-i), \psi_2 \in ker(G+i)$ and ϕ_1 is smooth L^2 section orthogonal to $ker(B)$. By (2.11), we have

$$e^{-2i\lambda R}P_1(\psi_1) = -P_1(\psi_2) \ . \tag{2.12}$$

The next result corresponds to Proposition 2.2.

Proposition 2.6. *If the eigenvalue $\lambda(R)$ is not bounded away from 0 , then the $zero-eigenmode$, $e^{-i\lambda u}\psi_1 + e^{i\lambda u}\psi_2$, of ϕ is a non-trivial section on the cylinder. Moreover, the following formula holds*

$$\|C_1(\lambda)\psi_1 - \psi_2\|^2 < e^{-c_3 R} \ . \tag{2.13}$$

We define

$$I_- = G + i : ker(B) \to ker(G-i)$$

$$I_{\sigma_1} = I_-|_{ker(\sigma_1+1)} : ker(\sigma_1+1) \to ker(G-i) \ ,$$

$$P_{\sigma_1} = \frac{1}{2}(\sigma_1 - 1) : ker(B) \to ker(\sigma_1+1)$$

and

$$S_{\sigma_1}(\lambda) = -P_{\sigma_1} \circ C_1(\lambda) \circ I_{\sigma_1} : ker(\sigma_1 + 1) \to ker(\sigma_1 + 1) \quad .$$

We know that there exists a unique $\phi \in ker(\sigma_1+1)$ such that $\psi_1 = \phi - iG\phi$. Using the equation (2.12) and (2.13), we have

$$\|e^{2i\lambda R} S_{\sigma_1}(\lambda)\phi - \phi\|^2 < e^{-c_3 R}$$

for $\phi \in ker(\sigma_1+1)$. Similarly we define S_{σ_2} using $C_2(0)$, I_+ instead of $C_1(0)$, I_- by

$$S_{\sigma_2}(\lambda) := -P_{\sigma_2} \circ C_2(\lambda) \circ I_{\sigma_2} : ker(\sigma_2 + 1) \to ker(\sigma_2 + 1) \quad ,$$

where

$$I_{\sigma_2} := I_+|_{ker(\sigma_2+1)} : ker(\sigma_2 + 1) \to ker(G + i) \quad .$$

We also have

$$\|e^{2i\lambda R} S_{\sigma_2}(\lambda)\phi - \phi\|^2 < e^{-c_3 R}$$

for $\phi \in ker(\sigma_2 + 1)$. We repeat the argument we used to prove the Proposition 2.3 and obtain

Proposition 2.7. *There exists R_0 such that for $R > R_0$ the $s-value$ $\lambda = \lambda(R)$ of $(\mathcal{D}_{i,R})_{P_i}$ satisfies*

$$2R\lambda(R) + \beta_j(\lambda(R)) = 2\pi k + O(e^{-cR}) \tag{2.14}$$

for an integer k with $|k| < R^{1-\kappa}$, where $\exp(i\beta_j(\lambda))$ denotes an eigenvalue of the unitary operator $S_{\sigma_i}(\lambda) = -P_{\sigma_i} \circ C_i(\lambda) \circ I_{\sigma_i} : ker(\sigma_i+1) \to ker(\sigma_i+1)$.

The restriction of $S_{\sigma_i}(0)$ to $ker(\sigma_i + 1) \cap ker(C_i(0) + 1)$ is equal to Id , so that the number of $\beta_j(0)$'s with $\beta_j(0) = 0$ is the same as

$$dim(ker(\sigma_i + 1) \cap ker(C_i(0) + 1)) = dim(ker(\sigma_i - 1) \cap ker(C_i(0) - 1)) \quad .$$

We have the following proposition

Proposition 2.8. *There is a natural isomorphism*

$$ker((\mathcal{D}_{i,R})_{P_i}) \cong ker_{L^2}(\mathcal{D}_{i,\infty}) \oplus ker(\sigma_i - 1) \cap L_i$$

for $i = 1, 2$.

Proof. Let $\phi \in ker(\mathcal{D}_{1,R})_{P_1}$. Section ϕ satisfies $G(\frac{d}{du} + B)\phi = 0$ on the cylinder $[0, R] \times Y$, and it has the following representation when restricted to this cylinder

$$\phi = \phi_0 + \sum_{\mu_j > 0} c_j e^{-\mu_j u} \phi_j$$

where $(\sigma_i - 1)(\phi_0) = 0$. We follow [23] (see also section 6 of [13]) and use this expansion to extend ϕ to a smooth section $\tilde{\phi}$ on $M_{1,\infty}$ satisfying $D_{1,\infty}\tilde{\phi} = 0$. This means that $\tilde{\phi}$ belongs to the extended L^2-kernel of $D_{1,\infty}$. Hence ϕ_0 is an element of L_1 . Let $E(\phi_0, \lambda)$ be the generalized eigensection attached to ϕ_0. Then $\tilde{\phi} - \frac{1}{2}E(\phi_0, 0)$ is square integrable and $D_{1,\infty}(\tilde{\phi} - \frac{1}{2}E(\phi_0, 0)) = 0$, and the map

$$\phi \longrightarrow (\tilde{\phi} - \frac{1}{2}E(\phi_0, 0), \phi_0)$$

gives the expected isomorphism. □

It follows from Proposition 2.8 that the number of elements in the set $\{\beta_j(0) \,:\, \text{eigenvalue of } S_{\sigma_i}(0) \mid \beta_j(0) = 0 \}$ is equal to the dimension of the subspace of $ker((\mathcal{D}_{i,R})_{P_i})$, which is not determined by $ker_{L^2}(\mathcal{D}_{i,\infty})$ for $i = 1, 2$.

Now we define our model operator. Let $C : W \to W$ denote a unitary operator acting on d-dimensional vector space W with the eigenvalues $e^{i\alpha_j}$ for $j = 1, \cdots, d$. We introduce the operator $D(C)$,

$$D(C) := -i\frac{1}{2}\frac{d}{du} : C^\infty(S^1, E_C) \to C^\infty(S^1, E_C)$$

where E_C is the flat vector bundle over $S^1 = \mathbb{R}/\mathbb{Z}$ defined by the holonomy $\overline{C}$, the complex conjugate of C . The spectrum of $D(C)$ is equal to

$$\{ \pi k - \frac{1}{2}\alpha_j \mid k \in \mathbb{Z}, j = 1, \cdots, d \} \ . \tag{2.15}$$

The operators $C_{12} = C_1(0) \circ C_2(0)$, $S_{\sigma_1} := S_{\sigma_1}(0)$ and $S_{\sigma_2} := S_{\sigma_2}(0)$ are the unitary operators acting on the finite dimensional vector spaces. Hence we can define self-adjoint, elliptic operators $D(C_{12}), D(S_{\sigma_1}), D(S_{\sigma_2})$ on S^1.

Proposition 2.9. *There exists R_0 such that for $R > R_0$, the $s-value$ $\lambda(R)$ of $\mathcal{D}_R$ (or $(\mathcal{D}_{i,R})_{P_i}$) satisfies*

$$2R\lambda(R) = \lambda_k + O(\frac{1}{R^\kappa}) \quad (or \quad R\lambda(R) = \lambda_k + O(\frac{1}{R^\kappa})) \ , \tag{2.16}$$

where λ_k is an eigenvalue with $|\lambda_k| \leq R^{1-\kappa}$ of $D(C_{12})$ (or $D(S_{\sigma_i})$).

Proof. We know that $\alpha_j(\lambda)$ is an analytic function of λ for $\lambda \in (-\delta, \delta)$, which has the following expansion

$$\alpha_j(\lambda) = \alpha_{j0} + \alpha_{j1}\lambda + \alpha_{j2}\lambda^2 + \cdots \ .$$

It follows from (2.15) that the eigenvalues of $D(C_{12})$ are of the form $\pi k - \frac{1}{2}\alpha_{j0}$. Equation (2.10) implies

$$2R\lambda(R) = (\pi k - \frac{1}{2}\alpha_{j0}) - \frac{1}{2}\lambda(R)(\alpha_{j1} + \alpha_{j2}\lambda(R) + \cdots) + O(e^{-cR})$$

which gives the (2.16) since $|\lambda(R)| \leq R^{-\kappa}$. ☐

3. Large time contribution

In this section, we prove the following proposition

Proposition 3.1.

$$\lim_{R\to\infty} \{ \int_{R^{2-\varepsilon}}^\infty t^{-1}[Tr(e^{-t\mathcal{D}_R^2} - e^{-t(\mathcal{D}_{1,R})_{P_1}^2} - e^{-t(\mathcal{D}_{2,R})_{P_2}^2}) - h] \ dt + h(\gamma + \varepsilon \cdot ln \ R) \tag{3.1}$$

$$- \frac{d}{ds} \{ \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1}[Tr(e^{-t\frac{1}{4}D(C_{12})^2} - e^{-tD(S_{\sigma_1})^2} - e^{-tD(S_{\sigma_2})^2}) - h] dt \} \Big|_{s=0} \} = 0$$

where $h = dim(L_1 \cap L_2) - dim(L_1 \cap ker(\sigma_1 - 1)) - dim(L_2 \cap ker(\sigma_2 - 1))$.

Proposition 3.1 shows that the large time contribution to the adiabatic decomposition formula for the ζ-determinant is equal to

$$\frac{det_\zeta \frac{1}{4} D(C_{12})^2}{det_\zeta D(S_{\sigma_1})^2 \cdot det_\zeta D(S_{\sigma_2})^2} \quad .$$

We start with the following result which corresponds to the Proposition 1.7 in our study of the *"small time contribution"*

Proposition 3.2. *We have the following formula for the small time contribution defined by the* 1*-dimensional operators*

$$\lim_{R\to\infty} \Big(\frac{d}{ds} \{ \frac{1}{\Gamma(s)} \int_0^{R^{-\varepsilon}} t^{s-1} [Tr(e^{-t\frac{1}{4}D(C_{12})^2} - e^{-tD(S_{\sigma_1})^2} - e^{-tD(S_{\sigma_2})^2}) - h] dt \} \Big|_{s=0}$$
$$- h(\gamma + \varepsilon \cdot ln\ R) \Big) = 0 \quad .$$

Proof. First, note that there exists a constant c_1 such that

$$|Tr(e^{-t\frac{1}{4}D(C_{12})^2} - e^{-tD(S_{\sigma_1})^2} - e^{-tD(S_{\sigma_2})^2})| < c_1\sqrt{t} \quad . \tag{3.2}$$

This is due to the fact that $D(C_{12})^2$, $D(S_{\sigma_1})^2$ and $D(S_{\sigma_1})^2$ are Dirac Laplacians on S^1 , so the expansion of the trace of the heat operator for each of those Laplacians is of the form

$$\sqrt{\frac{\pi}{t}} \cdot \frac{dim\ ker\ B}{2} + O(\sqrt{t}) \quad .$$

Now a factor of $\frac{1}{4}$ in front of $D(C_{12})^2$ provides a factor of 2 in front of $\sqrt{\frac{\pi}{t}}$ in the expansion for the operator $\frac{1}{4}D(C_{12})^2$. This allows us to estimate

$$|\int_0^{R^{-\varepsilon}} t^{-1} Tr(e^{-t\frac{1}{4}D(C_{12})^2} - e^{-tD(S_{\sigma_1})^2} - e^{-tD(S_{\sigma_2})^2})\ dt| \le$$
$$\le c_2 \cdot \int_0^{R^{-\varepsilon}} \frac{dt}{\sqrt{t}} = 2c_2 \cdot R^{-\frac{\varepsilon}{2}} \quad ,$$

We also have

$$\frac{d}{ds} \{ \frac{h}{\Gamma(s)} \int_0^{R^{-\varepsilon}} t^{s-1} dt \} \Big|_{s=0} = -h(\gamma + \varepsilon \cdot ln\ R) \quad .$$

This completes the proof. $\square$

It follows from the Proposition 3.2 that equality (3.1) is equivalent to the following equation

$$\lim_{R\to\infty}\int_{R^{2-\varepsilon}}^{\infty} t^{-1}[Tr(e^{-t\mathcal{D}_R^2} - e^{-t(\mathcal{D}_{1,R})_{P_1}^2} - e^{-t(\mathcal{D}_{2,R})_{P_2}^2}) - h]\ dt$$

$$= \lim_{R\to\infty}\frac{d}{ds}\{\frac{1}{\Gamma(s)}\int_{R^{-\varepsilon}}^{\infty} t^{s-1}[Tr(e^{-t\frac{1}{4}D(C_{12})^2} - e^{-tD(S_{\sigma_1})^2} - e^{-tD(S_{\sigma_2})^2}) - h]\ dt\}\Big|_{s=0}$$

$$= \lim_{R\to\infty}\int_{R^{-\varepsilon}}^{\infty} t^{-1}[Tr(e^{-t\frac{1}{4}D(C_{12})^2} - e^{-tD(S_{\sigma_1})^2} - e^{-tD(S_{\sigma_2})^2}) - h]\ dt \quad .$$

We change variables and obtain

$$\int_{R^{2-\varepsilon}}^{\infty} t^{-1}[Tr(e^{-t\mathcal{D}_R^2} - e^{-t(\mathcal{D}_{1,R})_{P_1}^2} - e^{-t(\mathcal{D}_{2,R})_{P_2}^2}) - h]dt$$

$$= \int_{R^{-\varepsilon}}^{\infty} t^{-1}[Tr(e^{-tR^2\mathcal{D}_R^2} - e^{-tR^2(\mathcal{D}_{1,R})_{P_1}^2} - e^{-tR^2(\mathcal{D}_{2,R})_{P_2}^2}) - h]\ dt \quad .$$

Now (3.1) is equivalent to

$$\lim_{R\to\infty}\int_{R^{-\varepsilon}}^{\infty} t^{-1}[Tr(e^{-tR^2\mathcal{D}_R^2} - e^{-tR^2(\mathcal{D}_{1,R})_{P_1}^2} - e^{-tR^2(\mathcal{D}_{2,R})_{P_2}^2}) - h]dt \tag{3.3}$$

$$= \lim_{R\to\infty}\int_{R^{-\varepsilon}}^{\infty} t^{-1}[Tr(e^{-t\frac{1}{4}D(C_{12})^2} - e^{-tD(S_{\sigma_1})^2} - e^{-tD(S_{\sigma_2})^2}) - h]\ dt \quad .$$

We split

$$Tr(e^{-tR^2\mathcal{D}_R^2} - e^{-tR^2(\mathcal{D}_{1,R})_{P_1}^2} - e^{-tR^2(\mathcal{D}_{2,R})_{P_2}^2})$$

into two parts

$$Tr_{I,R}(e^{-tR^2\mathcal{D}_R^2} - e^{-tR^2(\mathcal{D}_{1,R})_{P_1}^2} - e^{-tR^2(\mathcal{D}_{2,R})_{P_2}^2}) \quad ,$$

$$Tr_{II,R}(e^{-tR^2\mathcal{D}_R^2} - e^{-tR^2(\mathcal{D}_{1,R})_{P_1}^2} - e^{-tR^2(\mathcal{D}_{2,R})_{P_2}^2})$$

where $Tr_{I,R}(\cdot)$ ($Tr_{II,R}(\cdot)$) is taken over the eigenvalues of $R^2\mathcal{D}_R^2$, $R^2(\mathcal{D}_{1,R})_{P_1}^2$, $R^2(\mathcal{D}_{2,R})_{P_2}^2$ which are larger (smaller or equal to) than $R^{\frac{1}{2}}$. The next Proposition shows that $Tr_{I,R}(\cdot)$ can be neglected as $R\to\infty$

Proposition 3.3. *We have the following estimate*

$$\int_{R^{-\varepsilon}}^{\infty} t^{-1} Tr_{I,R}(e^{-tR^2\mathcal{D}_R^2} - e^{-tR^2(\mathcal{D}_{1,R})^2_{P_1}} - e^{-tR^2(\mathcal{D}_{2,R})^2_{P_2}})dt \le b_3 e^{-b_4 R^{\frac{1}{2}-\varepsilon}}$$

for some positive constants b_3, b_4.

Proof. Let $\lambda_{k_0}(R)$ denote the smallest "*large*" eigenvalue of $\mathcal{D}_R^2$, that is, we have

$$\lambda_{k_0}(R) > \frac{1}{R\sqrt{R}} \quad .$$

We estimate $Tr_{I,R}e^{-tR^2\mathcal{D}_R^2}$, which is the sum of the "*large*" eigenvalues

$$Tr_{I,R}e^{-tR^2\mathcal{D}_R^2} = \sum_{\lambda_k \ge \frac{1}{R\sqrt{R}}} e^{-tR^2\lambda_k} = \sum_{\lambda_k \ge \frac{1}{R\sqrt{R}}} e^{-(tR^2-1)\lambda_k} e^{-\lambda_k} <$$

$$e^{-(tR^2-1)\lambda_{k_0}} \sum_{\lambda_k \ge \frac{1}{R\sqrt{R}}} e^{-\lambda_k} < e^{-(tR^2-1)\lambda_{k_0}} Tr\ e^{-\mathcal{D}_R^2} <$$

$$b_1 R e^{-(tR^2-1)\frac{1}{R\sqrt{R}}} < b_1 R e^{-b_2 t\sqrt{R}} \quad ,$$

for some positive constants b_1, b_2 . Now we have

$$\int_{R^{-\varepsilon}}^{\infty} \frac{1}{t} Tr_{I,R} e^{-tR^2\mathcal{D}_R^2} dt < \int_{R^{-\varepsilon}}^{\infty} \frac{1}{t} b_1 R e^{-b_2 t\sqrt{R}} dt <$$

$$\frac{b_1}{b_2} R^{\frac{1}{2}+\varepsilon} \int_{b_2 R^{\frac{1}{2}-\varepsilon}}^{\infty} e^{-v} dv < b_3 e^{-b_4 R^{\frac{1}{2}-\varepsilon}} \quad .$$

The trace $Tr_{I,R}e^{-tR^2(\mathcal{D}_{i,R})^2_{P_i}}$ for $i = 1, 2$ is estimated in the same way. □

We also split $Tr(e^{-t\frac{1}{4}D(C_{12})^2} - e^{-tD(S_{\sigma_1})^2} - e^{-tD(S_{\sigma_2})^2})$ into two parts

$$Tr_{I,R}(e^{-t\frac{1}{4}D(C_{12})^2} - e^{-tD(S_{\sigma_1})^2} - e^{-tD(S_{\sigma_2})^2}) \quad ,$$

$$Tr_{II,R}(e^{-t\frac{1}{4}D(C_{12})^2} - e^{-tD(S_{\sigma_1})^2} - e^{-tD(S_{\sigma_2})^2}) \quad .$$

where $Tr_{I,R}(\cdot)$ $(Tr_{II,R}(\cdot))$ is taken over the eigenvalues of $\frac{1}{4}D(C_{12})^2$, $D(S_{\sigma_1})^2$, $D(S_{\sigma_2})^2$ which are larger (smaller or equal to) than $R^{\frac{1}{2}}$. The following

Proposition corresponds to the Proposition 3.3 and its proof is essentially the same as the proof of Proposition 3.3

Proposition 3.4. *There exist positive constants c_1, c_2 such that*

$$\int_{R^{-\varepsilon}}^{\infty} t^{-1} Tr_{I,R}(e^{-t\frac{1}{4}D(C_{12})^2} - e^{-tD(S_{\sigma_1})^2} - e^{-tD(S_{\sigma_2})^2}) \leq c_1 e^{-c_2 R^{\frac{1}{2}-\varepsilon}} \quad .$$

Now we can see that the equality (3.3) is equivalent to

$$(3.4) \quad \lim_{R\to\infty} (\int_{R^{-\varepsilon}}^{\infty} t^{-1}[Tr_{II,R}(e^{-tR^2\mathcal{D}_R^2} - e^{-tR^2(\mathcal{D}_{1,R})^2_{P_1}} - e^{-tR^2(\mathcal{D}_{2,R})^2_{P_2}}) - h]dt$$

$$- \int_{R^{-\varepsilon}}^{\infty} t^{-1}[Tr_{II,R}(e^{-t\frac{1}{4}D(C_{12})^2} - e^{-tD(S_{\sigma_1})^2} - e^{-tD(S_{\sigma_2})^2}) - h] \; dt) = 0 \quad .$$

Equation (3.4) is a consequence of the next result

Proposition 3.5. *For sufficiently large R , there exist positive constants c_6, c_7 independent of R and t , such that*

$$|Tr_{II,R}(e^{-tR^2\mathcal{D}_R^2}) - Tr_{II,R}(e^{-t\frac{1}{4}D(C_{12})^2})| < c_6 \frac{t}{R^{\frac{1}{4}}} e^{-c_7 t} \quad ,$$

$$|Tr_{II,R}(e^{-tR^2(\mathcal{D}_{i,R})^2_{P_i}}) - Tr_{II,R}(e^{-tD(S_{\sigma_i})^2})| < c_6 \frac{t}{R^{\frac{1}{4}}} e^{-c_7 t}$$

for any positive t.

Proof. We know that the dimension of the kernel of $\frac{1}{4}D(C_{12})^2$ is h_M , and it follows that the contributions of the $e-values$ and zero eigenvalues cancel out in the difference of the traces

$$Tr_{II,R}(e^{-tR^2\mathcal{D}_R^2}) - Tr_{II,R}(e^{-t\frac{1}{4}D(C_{12})^2}) \quad .$$

Hence, in the following we neglect those eigenvalues. Now, we use the analysis of $s-values$ developed in Section 2. We fix $\kappa = \frac{3}{4}$. It follows from Proposition 2.9 that, for any eigenvalue $\lambda(R)$ of $\mathcal{D}_R$ with $|\lambda(R)| < R^{-\frac{3}{4}}$ there exists an analytic function $\alpha(\lambda)$ such that

$$R\lambda(R) = \lambda_j + \frac{1}{4}\lambda(R)\alpha(\lambda(R)) + O(e^{-cR})$$

where λ_j is an eigenvalue of $\frac{1}{2}D(C_{12})$ with $|\lambda_j| < R^{\frac{1}{4}}$. Therefore, there exists a function $c_3(R)$ such that

$$R^2\lambda(R)^2 = \lambda_j^2 + c_3(R)\frac{\lambda_j}{R^{\frac{3}{4}}} + \frac{c_3(R)^2}{R^{\frac{3}{2}}} \quad ,$$

$$|c_3(R)| \le c_4 \quad ,$$

for any sufficiently large R . We use the elementary inequality $|e^{-\lambda} - 1| \le |\lambda|e^{|\lambda|}$ to get

$$|e^{-tR^2\lambda(R)^2} - e^{-t\lambda_j^2}| = |e^{-t\lambda_j^2}(e^{-t[R^2\lambda(R)^2-\lambda_j^2]} - 1)|$$

$$\le (\frac{c_3(R)|\lambda_j|}{R^{\frac{3}{4}}} + \frac{c_3(R)^2}{R^{\frac{3}{2}}})te^{-(\lambda_j^2 - \frac{c_3(R)|\lambda_j|}{R^{\frac{3}{4}}} - \frac{c_3(R)^2}{R^{\frac{3}{2}}})t} \le \frac{max(c_4, c_4^2)t}{R^{\frac{1}{2}}}e^{-\frac{1}{2}\lambda_j^2 t} \quad .$$

In the last inequality we used the fact that $|\lambda_j| < R^{\frac{1}{4}}$. Let us fix a sufficiently large R and a positive t. We take the sum over finitely many eigenvalues λ_j^2 of $\frac{1}{4}D(C_{12})^2$ with $\lambda_j^2 < R^{\frac{1}{2}}$ for fixed R, and obtain

$$|Tr_{r,II,R}(e^{-tR^2\mathcal{D}_R^2}) - Tr_{II,R}(e^{-t\frac{1}{4}D(C_{12})^2})| \le c_5\frac{t}{R^{\frac{1}{2}}}\sum_{\lambda_j^2<R^{\frac{1}{2}}} e^{-\frac{1}{2}\lambda_j^2 t} \quad .$$

for some constant c_5 . The operator $D(C_{12})^2$ is a Laplace type operator over S^1 , hence the number of eigenvalues λ_j^2 with $\lambda_j^2 < R^{\frac{1}{2}}$ can be estimated by $R^{\frac{1}{4}}$. Therefore, we have

$$c_5\frac{t}{R^{\frac{1}{2}}}\sum_{\lambda_j^2<R^{\frac{1}{2}}} e^{-\frac{1}{2}\lambda_j^2 t} \le c_6\frac{t}{R^{\frac{1}{2}}}R^{\frac{1}{4}}e^{-\lambda_1^2 t}$$

where λ_1 denotes the first non-zero eigenvalue of $\frac{1}{4}D(C_{12})^2$. Note that c_6 and λ_1 are independent of R and t. This proves the first claim. The proof of the second claim goes the same way. $\square$

The proof of Proposition 3.1 is now complete. We have shown that

$$(3.5) \quad \lim_{R\to\infty}\Big(\zeta_s^{R'}(0) - h(\gamma - (2-\varepsilon)\cdot ln\ R) + \zeta_l^{R'}(0) + h(\gamma + \varepsilon\cdot ln\ R)\Big) =$$

$$ln\, 2\cdot\zeta_{B^2}(0)+\frac{d}{ds}\{\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}[Tr(e^{-t\frac{1}{4}D(C_{12})^2}-e^{-tD(S_{\sigma_1})^2}-e^{-tD(S_{\sigma_2})^2})-h]dt\}\Big|_{s=0}\}\ .$$

which is equal to the formula (0.15) from Theorem 0.2.

4. Relations with the decomposition of the η-invariant

In this Section we discuss the relation of our results to the decomposition formulas for the η-invariant. The formula (0.7) has been proved by several authors and the proof we discuss in this Section is not the simplest one. Still we believe that it is worthy to present the *"scattering approach"* to the decomposition of the η-invariant. The key in our proof is to show that the *scattering data* provide you with the Lesch-Wojciechowski formula which gives the cylinder contribution in the decomposition formula for the η-invariant (see [20]).

Let us remind the reader that the η-function of a Dirac operator on a closed manifold, introduced in [1], is defined as

$$\eta_{\mathcal{D}}(s) = \sum_{\lambda\neq 0} sign(\lambda)|\lambda|^{-s}\quad ,$$

where the sum is taken over all non-zero eigenvalues of $\mathcal{D}$. The η-function is well-defined for $Re(s)$ large and it has a meromorphic extension to the whole complex plane and $s=0$ is a regular point, hence $\eta_{\mathcal{D}}(0)$ is well-defined. If we consider $\{\mathcal{D}_r\}$, a smooth family of the Dirac operators, then the function $r\to\eta_{\mathcal{D}_r}(0)$ is a well-defined smooth function modulo integer contribution. The reason is that every time the eigenvalue changes sign the value of the $\eta_{\mathcal{D}_r}(0)$ jumps. Following [1] we introduce the η-invariant of $\mathcal{D}$ as

$$\eta(\mathcal{D}) = \frac{1}{2}\cdot(\eta_{\mathcal{D}}(0) + dim\ ker\ \mathcal{D})\ . \tag{4.1}$$

The η-invariant is a smooth function of the Dirac operator modulo integer contribution. The η-function of the boundary problem, defined by the boundary condtion of the type discussed earlier in this paper is well-defined and it has the same properties as the η-function of the Dirac operator on a closed manifold. Now, let us assume that we have a given Dirac operator $\mathcal{D}$ on a closed odd-dimensional manifold M , which decomposes in the way described in the Introduction. The following result was proved by several authors and we refer to [36] and [37] for the proof and bibliographical remarks.

Theorem 4.1. *We have the following formula*

$$\eta(\mathcal{D}_M) = \eta((\mathcal{D}_1)_{P_1}) + \eta((\mathcal{D}_2)_{P_2}) + \eta(\mathcal{D};\sigma_1,\sigma_2) \quad mod\ \mathbb{Z} \ , \tag{4.2}$$

where $\eta(\mathcal{D};\sigma_1,\sigma_2)$ denotes the η-invariant of the operator $\mathcal{D}$ restricted to the collar $N \cong [-1,1]\times Y$ (hence equal to $G(\partial_u + B)$) , subject to the boundary condition P_2 at $u=-1$ and P_1 at $u=1$ (see [36]).

Though the formula holds for any generalized self-adjoint boundary conditions of Atiyah-Patodi-Singer type (see [37]) we restrict our attention here to the *"ideal boundary conditions"* defined in the Introduction, hence

$$P_1 = \Pi_< + \pi_1 \quad , \quad P_2 = \Pi_> + \pi_2 \quad ,$$

where π_i is an orthogonal projection onto the -1-eigenspace of an involution $\sigma_i : ker\ B \to ker\ B$, which anticommutes with G . Let us observe that

$$U = \sigma_1\sigma_2 : ker\ B \to ker\ B$$

is the unitary operator, such that

$$UG = GU \quad , \quad det\ U = 1 \quad , \quad U^* = \sigma_1 U \sigma_1 \ . \tag{4.3}$$

It follows that the spectrum of U is invariant under complex conjugation. Moreover, let $\mathcal{G}_\pm$ denote the $\pm i$ eigenspace of the operator $G : ker\ B \to ker\ B$, then the maps $U_\pm = U|_{\mathcal{G}_\pm} : \mathcal{G}_\pm \to \mathcal{G}_\pm$ are well-defined.

Let $T(U_\pm) : \mathcal{G}_\pm \to \mathcal{G}_\pm$ denote a self-adjoint operator such that

$$U_\pm = e^{iT(U_\pm)} \ . \tag{4.4}$$

The following result proved in Section 2 of [20] was the key ingredient in the proof of Theorem 4.1.

Theorem 4.2. *(see [20]). We have the following formulas*

$$\eta_{(\mathcal{D};\sigma_1,\sigma_2)}(0) = -\frac{Tr\ T(U_+)}{\pi} = \frac{Tr\ T(U_-)}{\pi} \quad mod\ \mathbb{Z} \ , \tag{4.5}$$

and

$$\eta(\mathcal{D};\sigma_1,\sigma_2) = \frac{Tr\ T(U_-)}{2\pi} \quad mod\ \mathbb{Z} \ .$$

Remark 4.3. Let us also notice that the formula for the η-invariant of the operator $D(V)$ on S^1 defined by $V \in U(N)$

$$\eta(D(V)) = -\frac{Tr\ T(V)}{2\pi} \quad mod\ \mathbb{Z}\ . \tag{4.6}$$

(see for instance [4]).

One way to prove the decomposition formula (4.2) is to use the *adiabatic analysis* we developed in the proof of Theorem 0.2. This analysis easily gives us the following Theorem

Theorem 4.4. *The following decomposition formula holds for the η-invariant*

$$\eta(\mathcal{D}) - \eta(\mathcal{D}_{\sigma_1}) - \eta(\mathcal{D}_{\sigma_2}) \tag{4.7}$$

$$= \eta(D(C_{12})) - \eta(D(S_{\sigma_1})) - \eta(D(S_{\sigma_2})) \quad mod \quad \mathbb{Z}$$

Proof. In fact the analysis for the η-invariant is simpler than for the ζ-determinant of the Dirac Laplacian. It provides us with the expected formula

$$\lim_{R\to\infty} \{\eta(\mathcal{D}_R) - \eta((\mathcal{D}_{1,R})_{\sigma_1}) - \eta((\mathcal{D}_{2,R})_{\sigma_2})\}$$

$$= \eta(D(C_{12})) - \eta(D(S_{\sigma_1})) - \eta(D(S_{\sigma_2})) \quad \mathrm{mod} \quad \mathbb{Z}$$

Now, we use the fact that the η-invariant on the manifolds M_R, $M_{1,R}$ and $M_{2,R}$ is independent of R (modulo integer contributions). This was carefully explained in [23] (see Proposition 2.16). □

It seems that we have a problem here. The cylinder contribution in formula (4.2) in Theorem 4.1, contains only boundary data, and

$$\eta(D(C_{12})) - \eta(D(S_{\sigma_1})) - \eta(D(S_{\sigma_2}))$$

the cylinder contribution in the formula (4.7) in Theorem 4.4 seems to contain global information from M encoded into *scattering matrices*. However, there is a cancellation mechanism which leads to the equality of the $\eta(\mathcal{D}; \sigma_1, \sigma_2)$ and the right side of (4.7).

The scattering matrix $C_i = C_i(0)$ can be represented in the following way, with respect to the decomposition $ker\ B = \mathcal{G}_+ \oplus \mathcal{G}_-$

$$C_i = \begin{pmatrix} 0 & C(i)_- \\ C(i)_+ & 0 \end{pmatrix} \quad where \quad C(i)_\pm C(i)_\mp = Id \quad .$$

Similar formulas hold for the involutions σ_i . We see that

$$C_{12} = C(1)_+ C(2)_- : \mathcal{G}_- \to \mathcal{G}_- \quad .$$

We also need the corresponding formula for the operator S_{σ_1}

$$S_{\sigma_1} = -P_{\sigma_1} \circ C_1 \circ I_{\sigma_1} = -\frac{1}{2} \begin{pmatrix} -Id & \sigma(1)_- \\ \sigma(1)_+ & -Id \end{pmatrix} \begin{pmatrix} 0 & C(1)_- \\ C(1)_+ & 0 \end{pmatrix} \begin{pmatrix} 2i & 0 \\ 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} -i\sigma(1)_- C(1)_+ & 0 \\ iC(1)_+ & 0 \end{pmatrix} \quad .$$

The subspace $ker(\sigma_1 + Id)$ is the graph of the operator

$$\begin{pmatrix} Id & 0 \\ 0 & -\sigma(1)_+ \end{pmatrix} \quad ,$$

which means that each element of $\mathcal{G}_i$ is represented in the form $\begin{pmatrix} f \\ -\sigma(1)_+ f \end{pmatrix}$, for some $f \in \mathcal{G}_+$. This allows us to represent the map $S_{\sigma_1} = -P_{\sigma_1} \circ C_1 \circ I_{\sigma_1} : ker(\sigma_1 + Id) \to ker(\sigma_1 + Id)$ as

$$(4.8) \qquad S_{\sigma_1} \begin{pmatrix} f \\ -\sigma(1)_+ f \end{pmatrix} = \begin{pmatrix} -i\sigma(1)_- C(1)_+ & 0 \\ iC(1)_+ & 0 \end{pmatrix} \begin{pmatrix} f \\ -\sigma(1)_+ f \end{pmatrix}$$
$$= \begin{pmatrix} -i\sigma(1)_- C(1)_+ f \\ \sigma(1)_+ (i\sigma(1)_- C(1)_+ f) \end{pmatrix} \quad .$$

It turns out that, from the spectral point of view, the operator S_{σ_1} is equal to the operator

$$-i\sigma(1)_- C(1)_+ : \mathcal{G}_+ \to \mathcal{G}_+ \quad ,$$

or equivalently to the operator

$$-iC(1)_+ \sigma(1)_- : \mathcal{G}_- \to \mathcal{G}_- \quad .$$

The corresponding analysis for the operator S_{σ_2} implies that S_{σ_2} is equivalent to

$$iC(2)_- \sigma(2)_+ : \mathcal{G}_+ \to \mathcal{G}_+ \quad or \quad i\sigma(2)_+ C(2)_- : \mathcal{G}_- \to \mathcal{G}_- \quad .$$

As a result, when we study the quantity $\frac{det(C_{12})}{det(S_{\sigma_1}) det(S_{\sigma_2})}$, we obtain

$$\frac{det(C_{12})}{det(S_{\sigma_1})det(S_{\sigma_2})} = det\left(C_{12}S_{\sigma_2}^{-1}S_{\sigma_1}^{-1}\right)$$

$$= det\ C(1)_+\sigma(2)_-\sigma(1)_+C(1)_- = det\ \sigma(2)_-\sigma(1)_+ \quad .$$

In the same manner we analyze the trace of the logarithm of the corresponding operator which, of course, leads us to the η-invariant. Let us remind (see (4.4)) that we introduced a self-adjoint matrix $T(U_-)$, such that $U_- = exp(iT(U_-))$, where

$$U_- = \sigma(1)_+\sigma(2)_- \quad .$$

If we combine our computations with formula (4.6), we have

$$\eta(D(C_{12})) - \eta(D(S_{\sigma_1})) - \eta(D(S_{\sigma_2})) = -\frac{Tr\ T(\overline{\sigma(2)_-\sigma(1)_+})}{2\pi}$$

$$\frac{Tr\ T(\sigma(2)_-\sigma(1)_+)}{2\pi} = \frac{Tr\ T(\sigma(1)_+\sigma(2)_-)}{2\pi} = \frac{Tr\ T(U_-)}{2\pi} \quad mod\ \mathbb{Z} \quad .$$

This ends the proof of the following Theorem

Theorem 4.5. *The cylinder contributions in the formulas (4.2) and (4.7) are equal to each other*

$$\eta(\mathcal{D};\sigma_1,\sigma_2) = \eta(D(C_{12})) - \eta(D(S_{\sigma_1})) - \eta(D(S_{\sigma_2})) \quad mod\ \mathbb{Z} \quad . \tag{4.9}$$

5. Lesch-Wojciechowski formula for the ζ-determinant of the Dirac Laplacians

Let us now assume that we have a given Dirac operator $\mathcal{D}$ on a compact manifold N with boundary Y and that we specified two different boundary conditions determined by involutions σ_1 , σ_2 of the kernel of the tangential operator B . The formula for the difference $\eta(\mathcal{D}_{P_2}) - \eta(\mathcal{D}_{P_1})$ can be obtained by studying the variation of the η-invariant. This is how the original proof in [20] went. We obtained the equality

$$\eta(\mathcal{D}_{P_2}) - \eta(\mathcal{D}_{P_1}) = \eta(\mathcal{D};\sigma_1,\sigma_2) \quad mod\ \mathbb{Z} \quad . \tag{5.1}$$

and we used this equality to study the decomposition formula for the η-invariant (see [35], [36], [37]).

You can actually reverse the argument. We just proved the decomposition formula for the η-invariant using the *adiabatic argument* and *scattering matrices* in the previous Section. Now, the formula (5.1) follows if we apply the decomposition formula in the case of the manifold $M = \tilde{N} = (-N) \cup N$ where $M_1 = (-N)$ is the manifold N with the reverse orientation. The corresponding Dirac operator is constructed via the well-known doubling construction introduced by the second author in [34]. We define a bundle of Clifford modules on $\tilde{N}$ as $S^G = S \cup_G S$. The set of smooth sections of S^G can be identified with the set of couples (s_1, s_2) , where

$$s_1 \in C^\infty((-N) \cup [0,1) \times Y; S) \;\; \textit{and} \;\; s_2 \in C^\infty((-1,0] \times Y \cup N; S) \;\; ,$$

and we identify

$$s_2(u,y) \cong G(y) s_1(u,y)$$

over $(-1,1) \times Y$. The operator $\tilde{\mathcal{D}}$ on $C^\infty(M; S^G)$ is defined via the formula

$$\tilde{\mathcal{D}}(s_1, s_2) = (-\mathcal{D}s_1, \mathcal{D}s_2) \;\; . \tag{5.2}$$

This definition works due to the fact that $-\mathcal{D}$ on $(-1,1) \times Y$ is represented as

$$-\mathcal{D}|_{(-1,1)\times Y} = -G(-\partial_u + B)$$

while $\mathcal{D}|_{(-1,1)\times Y} = G(\partial_u + B)$. We can now check

$$G((-\mathcal{D})s_1)|_{(-1,1)\times Y}) = (-\partial_u + B)s_1 = -(-\partial_u + B)G^2 s_1$$

$$= -(-G\partial_u - GB)Gs_1 = \mathcal{D}s_2|_{(-1,1)\times Y} \;\; .$$

A similar computation shows that the scattering matrix C_1 is equal to $-C_2$, where $C_2 = C_2(0)$ is the scattering matrix determined by the operator $\mathcal{D}_{2,\infty}$ on $N_{2,\infty} \cong ((-\infty, 0] \times Y) \cup N$. It follows from the formula (4.6) that $\eta(D(C_{12})) = \eta(D(-Id)) = 0$ and now formula (4.9) from Theorem 4.5 gives (5.1), which can be also reformulated as

Proposition 5.1. *The following equality holds for the η-invariants of the operators $\mathcal{D}_{P_1}$ and $\mathcal{D}_{P_2}$*

$$\eta(\mathcal{D}_{P_2}) - \eta(\mathcal{D}_{P_1}) = -\eta(D(S_{\sigma_1})) - \eta(D(S_{\sigma_2})) \;\; mod \; \mathbb{Z} \;\; . \tag{5.3}$$

The interesting feature of formula (5.3) is that, according to (4.6), the number on the right side of (5.3) is the trace of a finite dimensional matrix. In this final Section of our paper we prove the corresponding result for the ζ-determinant of the Dirac Laplacian.

We consider only the case of invertible operators $\mathcal{D}_{P_1}$ and $\mathcal{D}_{P_2}$. It follows that $D(S_{\sigma_i})$ are invertible operators. We also have $ker_{L^2}\mathcal{D}_\infty$ equal to $\{0\}$, and $C_{12} = -Id$, hence the operator $D(C_{12})$ is invertible as well. Theorem 0.2 provides the formulas

$$\lim_{R\to\infty} \frac{det_\zeta \mathcal{D}_R^2}{det_\zeta(\mathcal{D}_{1,R})^2_{\tilde{P}_1} \cdot det_\zeta(\mathcal{D}_{2,R})^2_{P_2}} = \frac{2^{-\zeta_{B^2}(0)} \cdot det_\zeta \frac{1}{4} D(C_{12})^2}{det_\zeta D(S_{\sigma_1})^2 \cdot det_\zeta D(S_{\sigma_2})^2} \ ,$$

and

$$\lim_{R\to\infty} \frac{det_\zeta \mathcal{D}_R^2}{det_\zeta(\mathcal{D}_{1,R})^2_{\tilde{P}_1} \cdot det_\zeta(\mathcal{D}_{2,R})^2_{P_1}} = \frac{2^{-\zeta_{B^2}(0)} \cdot det_\zeta \frac{1}{4} D(C_{12})^2}{det_\zeta D(S_{\sigma_1})^2 \cdot det_\zeta D(S_{\sigma_1})^2} \ ,$$

where $\tilde{P}_1 = \Pi_< + \pi_1$, as the original P_1 projection is equal to $\Pi_> + \pi_1$. The result is the following equality

$$\lim_{R\to\infty} \frac{det_\zeta(\mathcal{D}_{2,R})^2_{P_2}}{det_\zeta(\mathcal{D}_{2,R})^2_{P_1}} = \frac{det_\zeta D(S_{\sigma_2})^2}{det_\zeta D(S_{\sigma_1})^2} \ . \tag{5.4}$$

Theorem 0.2 gives us only the adiabatic equality in (5.4). However, we observed in [29] that the quotient on the left side of (5.4) is independent of R . The proof relies on the results of the Scott-Wojciechowski work on the determinants of the boundary problems (see [31]) and on the Nicolaescu study of the dynamics of the Calderon projections (see [25]). We have the folowing Proposition

Proposition 5.2. *Under the assumptions we have made in this section the quotient*

$$\frac{det_\zeta(\mathcal{D}_{2,R})^2_{P_2}}{det_\zeta(\mathcal{D}_{2,R})^2_{P_1}}$$

does not depend on R.

Proof. The proof of this proposition is an application of the following Scott–Wojciechowski formula, Proposition 4.1 in [31],

$$\frac{\det_\zeta(\mathcal{D}_R)^2_{P_1}}{\det_\zeta(\mathcal{D}_R)^2_{P_2}} = |\det_{Fr} U_{P_2}(U_{P_1})^{-1} S_R(P_1) S_R(P_2)^{-1}|^2 \tag{5.5}$$

where $\det_{Fr}$ is the Fredholm determinant, and

$$U_{P_2}(U_{P_1})^{-1} : \mathrm{Range}(P_1) \to \mathrm{Range}(P_2)$$

is a unitary map which depends only on P_1, P_2. The operator $S_R(P_i)$ is defined by

$$S_R(P_i) := P_i P(\mathcal{D}_R) : \mathcal{H}(\mathcal{D}_R) \to \text{Ran } P_i \quad ,$$

where $P(\mathcal{D}_R)$ is the Calderon projection defined by $\mathcal{D}_R$ and $\mathcal{H}(\mathcal{D}_R)$ is the Cauchy data space (see for instance [3]) . The definitions of the boundary condition P_i implies the following decomposition of $S_R(P_i)$

$$\Pi_> S_R(P_i) = \Pi_> P(\mathcal{D}_R) \quad and \quad \sigma_i S_R(P_i) = \sigma_i P(\mathcal{D}_R) \quad .$$

We also have the corresponding decomposition of $S_R(P_i)^{-1}$ into its restrictions to the images of $\Pi_>$ and σ_i which implies the obvious identities

$$\sigma_i P(\mathcal{D}_R) S_R(P_i)^{-1} = \sigma_i \quad and \quad \Pi_> P(\mathcal{D}_R) S_R(P_i)^{-1} = \Pi_> \quad .$$

Hence, the operator $S_R(P_1)S_R(P_2)^{-1}$ has the following form

(5.6)

$$\begin{aligned} S_R(P_1)S_R(P_2)^{-1} &= \begin{pmatrix} \Pi_> S_R(P_1)S_R(P_2)^{-1}\Pi_> & \Pi_> S_R(P_1)S_R(P_2)^{-1}\sigma_2 \\ \sigma_1 S_R(P_1)S_R(P_2)^{-1}\Pi_> & \sigma_1 S_R(P_1)S_R(P_2)^{-1}\sigma_2 \end{pmatrix} \\ &= \begin{pmatrix} \Pi_> & 0 \\ 0 & \sigma_1 S_R(P_1)S_R(P_2)^{-1}\sigma_2 \end{pmatrix} \quad . \end{aligned}$$

Nicolaescu proved that the dynamics of the Cauchy data is determined by the operator e^{-RB} on $L^2(Y, S|_Y)$ (see [25]). However, this operator is equal to the identity when restricted to the subspace $ker(B)$. It follows that $\sigma_1 S_R(P_1)S_R(P_2)^{-1}\sigma_2$ is independent of R. Now the Proposition follows from (5.5) and (5.6). $\square$

We combine (5.4), Proposition 5.2, and the formula for the one dimensional det_ζ to express the quotient

$$\frac{det_\zeta(\mathcal{D}_{2,R})^2_{P_2}}{det_\zeta(\mathcal{D}_{2,R})^2_{P_1}}$$

as a finite-dimensional determinant. Let us remind the reader of the formula for the ζ-determinant of the operator $D(U)^2$. Let us start with the operator $D_a = -i\frac{d}{dx} + a$ acting on functions on $S^1 = \mathbb{R}\ mod\ 2\pi\mathbb{Z}$. In this case there is the well-known formula (see for instance [4])

$$det_\zeta D_a^2 = 4 \cdot \sin^2 \pi a \quad . \tag{5.7}$$

Now, if we have an operator $-i\frac{d}{dx}$ on $S^1 = \mathbb{R}\ mod\ \mathbb{Z}$ acting on sections of the bundle determined by the holonomy $U \in U(N)$, this translates into a direct sum of the operators $-i\frac{d}{dx} + \frac{a_j}{2\pi}$, where e^{ia_j} is an eigenvalue of U . Hence the corresponding formula is

$$det_\zeta D(U)^2 = 4^N \prod_{j=1}^{N} \sin^2 \frac{a_j}{2} \ . \tag{5.8}$$

We can use (5.8) to rewrite the right side of (5.4) in terms of the suitable $sin - matrices$. Another possibility is to use equality

$$det_\zeta D(U) = 4^N det\ \frac{2 - U - U^{-1}}{4} \ , \tag{5.9}$$

which is equivalent to (5.8). This leads to the Lesch-Wojciechowski formula for the ζ-determinant of the Dirac Laplacian

Theorem 5.3. *The following formula holds for the quotient of the ζ-determinants of the square of $\mathcal{D}$ subject to the boundary conditions defined by involutions σ_1 and σ_2*

$$\frac{det_\zeta \mathcal{D}^2_{P_2}}{det_\zeta \mathcal{D}^2_{P_1}} = det\Big(\Big(Id - \frac{S_{\sigma_2} + S^{-1}_{\sigma_2}}{2}\Big)\Big(Id - \frac{S_{\sigma_1} + S^{-1}_{\sigma_1}}{2}\Big)^{-1}\Big) \ . \tag{5.10}$$

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