Asymptotic Theory and Resurgent Functions

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Abstract

The aim of this paper[1] is to give a short and clear presentation of the new notion of a resurgent function based on the asymptotic approach.

Introduction

The resurgent function theory introduced in the beginning of eightees by Jean Écalle is at present used in many fields of mathematics, such as function theory, asymptotic theory of differential equations, dynamics systems theory and others. Up to the moment, there exits two quite different approaches to the construction of the resurgent functions theory which differ from each other, mainly, by the object of investigation. In the first approach, one deals with the *resummation problem for divergent series*, and, in the second, with the investigation of *asymptotic behavior of functions* (say, of exponential growth at infinity).

The first approach can be found in the classical works by J. Écalle [1] (see also [2], [3]), and the second was worked out by the authors (see [4] — [6]). The aim of the present paper is to give a clear and transparent presentation of the asymptotic

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[1]The text below is a presentation of the talk given by the authors on the seminar of Professor Werner Balser, Ulm University, in the summer of 1995. We use the opportunity to thank W. Balser for the possibility of providing a talk on his seminar.

approach to the resurgent function theory[2]. The detailed presentation of our approach the reader can find in the main part of the paper. Here we only emphasize one principal point lying in the basis of the asymptotic theory of resurgent functions.

From the viewpoint of the asymptotic approach, resurgent functions are, in essence, functions with discrete asymptotics. This means that we consider functions representable in the form of the (infinite) sum of "WKB-elements", that is, asymptotic expansions of the form

$$e^{S(x)} \sum_{j=0}^{\infty} a_j x^{-j}.$$

The equivalent formulation of this requirement is that the function $f(x)$ can be represented as

$$f(x) = \int_{\gamma} e^{-s} F(s,x)\, ds,$$

where the function $F(s,x)$ is a function with a discrete set of *simple singularities* in s for each fixed x, and γ is some contour in the complex plane $\mathbf{C}_s$. The latter means that the following representation

$$F(s,x) = \frac{a_0(x)}{s - S(x)} + \ln(s - S(x)) \sum_{k=0}^{\infty} \frac{(s - S(x))^k}{k!} a_{k+1}(x) + F_1(s,x)$$

with regular $F_1(s,x)$ takes place at any point $s = S(x)$ of singularity of $F(s,x)$.

1 Asymptotic expansion of functions of exponential growth

In this section, we consider the question what the asymototic expansions of functions of exponential growth can look like. Here we restrict ourselves by the case of functions of one complex variable having exponential growth of degree 1. Possible generalizations will be discussed below.

So, let $f(x)$ be a function of exponential growth of degree 1 (a function of exponential growth in the sequel) determined in the sector

$$S_\theta(R,\varepsilon) = \{x \in \mathbf{C} \mid \theta - \varepsilon < \arg x < \theta + \varepsilon,\ |x| > R\}$$

[2]We emphasize that, in this context, we mean the *construction* of the resurgent functions theory itself. The *application* of the "classical" Écalle's theory was done by many authors (see, for example, [2], [3], [7], and the bibliography therein).

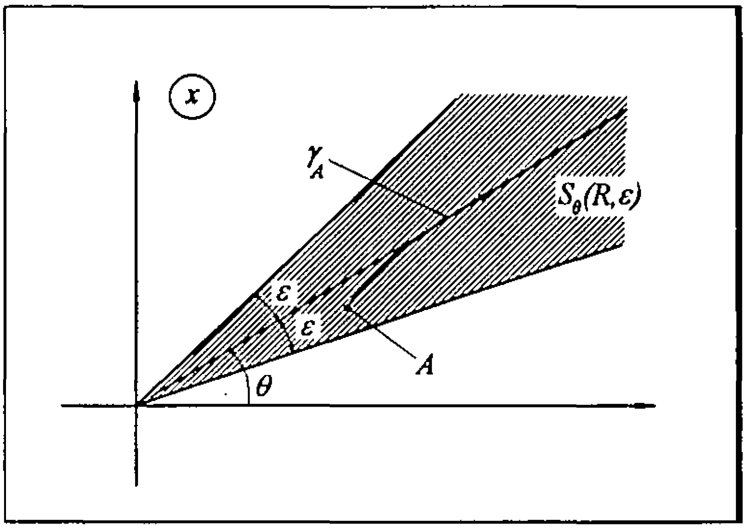

Figure 1.

of the complex plane **C** (for sufficiently large values of x; see Figure 1). This means that:

1) $f(x)$ is holomorphic in $S_\theta(R,\varepsilon)$;
2) $f(x)$ satisfies the inequality

$$|f(x)| \le Ce^{a|x|},$$

in each proper subsector $S_\theta(R',\varepsilon')$ of $S_\theta(R,\varepsilon)$ (here $R' > R$, $\varepsilon' < \varepsilon$; the constant C can depend on numbers R' and ε').

Now the question arises: how one can investigate the asymptotic behavior of a function $f(x)$ as $|x| \to \infty$ inside the sector $S_\theta(R,\varepsilon)$? We shall use here the well-known connection between the behavior of a function at infinity and the singulatiries of its Laplace image. Clearly, since functions under consideration are defined in the complex domain, one have to use the complex-analytic analogue of the Laplace transform — the so-called Borel-Laplace transform. For the reader's convenience we recall here the main definitions and statements concerning the theory of this transform. The detailed presentation the reader can find, for example, in the book [6].

Suppose that the function $f(x)$ possesses properties 1) and 2) above. Then the integral

$$F(\xi) = \mathcal{B}[f(x)] = \frac{1}{2\pi i}\int\limits_{\gamma_A} e^{\xi x} f(x)\,dx, \tag{1}$$

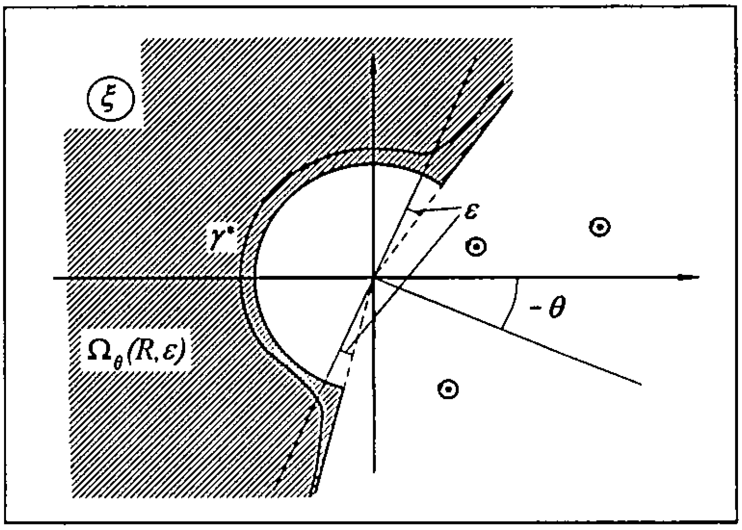

Figure 2.

is holomorphic in the region

$$\Omega_\theta(R,\varepsilon)=\left\{\xi\in\mathbb{C}\mid -\theta+\frac{\pi}{2}-\varepsilon<\arg\xi<-\theta+\frac{3\pi}{2}+\varepsilon,\ |\xi|>R\right\}.$$

This integral is taken over the contour γ_A coming from an arbitrary point A of the sector $S_\theta(R,\varepsilon)$ to infinity along this sector (one can suppose that this contour coincides with some ray for sufficiently large values of x; such a contour is shown on Figure 1).[3] The convergence of integral (1) in the above mentioned domain (this domain is shown on Figure 2) for sufficiently large R and sufficiently small ε can be proved with the help of standard estimates of integrals, and we leave this proof to the reader.

Definition 1 The function $F(\xi)$, given by (1), is called *the Borel transform* of the function $f(x)$.

Remark 1 The function $F(\xi)$, in general, depends on the choice of the point A in (1). However, two functions corresponding to different values of A differ from each other by an entire function of the variable ξ satisfying the inequality

$$|F(\xi)|\leq Ce^{a|\xi|}$$

[3]Here and below R and ε are some positive numbers which are not nesessarily the same in different formulas.

with some constants C and a. So, the Borel transform is, in essense, not a function but a *hyperfunction*, that is, a quotient class modulo entire functions of exponential type. (We shall not use this terminology here.)

The inversion of the Borel transform is given by

$$f(x) = \mathcal{L}\left[F(\xi)\right] = \int_{\gamma^*} e^{-\xi x} F(\xi)\, d\xi, \tag{2}$$

where the contour γ^* is shown on Figure 2.

More precisely, the following affirmation is valid.

Theorem 1 *Transforms* (1) *and* (2) *are inverses to each other.*

Definition 2 The function $f(x)$, given by (2), is called a *Laplace transform* of the function $F(\xi)$.

So, what is the connection between the behavior of a function $f(x)$ at infinity and the singularities of its Borel transform? We shall illustrate this connection on the following two examples.

Example 1 Suppose that a function $F(\xi)$ can be continued up to a *meromorphic* function on the entire plane $\mathbf{C}_\xi$ with the Laurent expansions at each its pole $\xi = \omega_j$, $j = 1, 2, \ldots$ given by

$$F(\xi) = \sum_{k=0}^{m_j} \frac{a_k^j}{(\xi - \omega_j)^{k+1}} + F_0(\xi),$$

where $F_0(\xi)$ is a holomorphic function near the point $\xi = \omega_j$. Then, moving the contour γ^* in the direction $\arg \xi = -\theta$ and using the residue theorem we obtain the asymptotic expansion of the function $f(x)$ as $x \to \infty$ inside the sector $S_\theta(R, \varepsilon)$:

$$f(x) \simeq \sum_j e^{\omega_j x} \sum_{k=0}^{m_j} \frac{(-1)^k a_k}{k!} x^k$$

up to functions with arbitrary exponential decrease at infinity.

Example 2 Suppose that the Borel transform $F(\xi)$ of the function $f(x)$ can be continued up to a *ramifying* function determined over the entire plane $\mathbf{C}_\xi$ with a discrete

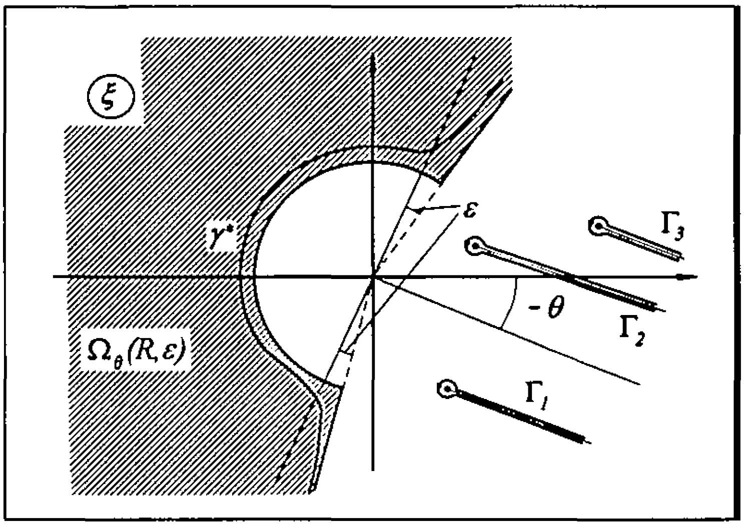

Figure 3.

(on its Riemannian surface) set of singularities $\{\xi = \omega_j,\ j = 1, 2, \ldots\}$, and that at any point of singularity $\xi = \omega_j$ of this function the representation

$$F(\xi) = \frac{a_0^j}{\xi - \omega_j} + \ln(\xi - \omega_j) \sum_{k=0}^{\infty} \frac{(\xi - \omega_j)^k}{k!} a_{k+1}^j + F_0(\xi) \tag{3}$$

takes place modulo holomorphic function $F_0(\xi)$ (the series on the right in the latter formula is supposed to be convergent in some neighborhood of the point $\xi = \omega_j$). For such a function, the above described deformation of the integration contour will lead to the following decomposition of the function $f(x)$ in the sum of integrals

$$f(x) \simeq \sum_j \int_{\Gamma_j} e^{-\xi x} F(\xi)\, d\xi,$$

taken over the contours Γ_j, drawn on Figure 3. With the help of the theory of classical Laplace transform, it is easy to show that each of these integrals admits the asymptotic expansion of the type

$$\int_{\Gamma_j} e^{-\xi x} F(\xi)\, d\xi \simeq e^{-\omega_j x} \sum_{k=0}^{\infty} \frac{a_k^i}{x^k} \tag{4}$$

as $k \to \infty$. The elements

$$e^{\omega_j x} \sum_{k=0}^{\infty} \frac{a_k^i}{x^k},$$

which have arised in the latter asymptotic expansion have the form of the well-known WKB-expansions (after the variable change $x = 1/(ih)$). They will be called *WKB-elements*. In these elements the functions ω^i play the role of a phase function (action), and the coefficients a_k^i — the role of an amplitude function of WKB-expansion.

So, one can expect that the asymptotic expansion of the function $f(x)$ has the form

$$f(x) \simeq e^{-\omega_1 x} \sum_{k=0}^{\infty} \frac{a_k^1}{x^k} + e^{-\omega_2 x} \sum_{k=0}^{\infty} \frac{a_k^2}{x^k} + \dots. \tag{5}$$

However, the interpretation of the obtained formula meats certain difficulties. The matter is that the definition of expansion (5) itself needs a refinement. Actually, let us suppose that the numbers ω_j are enumerated in accordance with increasing of their real parts (to be definite, we carry out our considerations for real values of x):

$$\mathrm{Re}\,\omega_1 < \mathrm{Re}\,\omega_2 < \dots.$$

Then expansion (5) is graded in accordance with the order of its terms as $x \to \infty$. Using the classical step-by step procedure of defining the asymptotic expansion, we write down the following relations:

$$f(x) - e^{-\omega_1 x} \sum_{k=0}^{N} \frac{a_k^1}{x^k} = O\left(x^{-(N+1)} e^{-\omega_1 x}\right),$$

which encounter orders included into the first sum on the right in (5). Later on, similar relations encountering orders included in the second sum must have the form

$$f(x) - e^{-\omega_1 x} \sum_{k=0}^{\infty} \frac{a_k^1}{x^k} - e^{-\omega_2 x} \sum_{k=0}^{N} \frac{a_k^2}{x^k} = O\left(x^{-(N+1)} e^{-\omega_2 x}\right). \tag{6}$$

The latter relation, however, makes sense only in the case when the series on the left converges. Unfortunately, this fact, as a rule, is not valid for an asymptotic series, and we arrive at the nesessity of *resummation of divergent power series.* Actually, if we have some resummation procedure which assigns a function

$$\sigma \left[e^{-\omega_1 x} \sum_{k=0}^{\infty} \frac{a_k^1}{x^k} \right]$$

to a divergent series

$$e^{-\omega_1 x}\sum_{k=0}^{\infty}\frac{a_k^1}{x^k}$$

such that this series occurs to be an asymptotic expansion of the above function (such a function will be called a *presum* of the series), relation (6) can be rewritten in the form

$$f(x)-\sigma\left[e^{-\omega_1 x}\sum_{k=0}^{\infty}\frac{a_k^1}{x^k}\right]-e^{-\omega_2 x}\sum_{k=0}^{N}\frac{a_k^2}{x^k}=O\left(x^{-(N+1)}e^{-\omega_2 x}\right)$$

which makes sense.

Using the above scheme of definition of an asymptotic expansion one has to take into account that the values of coefficients a_k^2 involved in the second *(recessive)* part of the asymptotic expansion (and even the existence of this recessive part) *depends on the choice of the concrete resummation method* σ. We remark that the integral on the right in (4) determines a presum of the series on the right in this relation since the integrand on the left in (4) is uniquely determined by the series. The resummation procedure based on relation (4) is called *Borel resummation procedure.* This procedure possesses a number of "good" properties, for example, the correspondence between formal series and their presums occurs to be an algebra homomorphism which commutes with the differentiation. This is exactly the procedure which will be used in the sequel for the definition of asymptotic expansions of the form (5). Below we denote this procedure by σ.

So, one can see that *if the Borel transform of the function* $f(x)$ *is an endlessly continuable function of the variable* ξ *(this means that it has not more than a discrete set of singularities on its Riemannian surface), and all singularities of this transform have the form* (3), *then this function has the asymptotic expansion of the form* (5). The requrement of endless continuability of the Borel image of a function is exactly the simplest definition of *resurgent function with simple singularities.* We remark also that the actions for resurgent functions with simple singularities are uniquely determined by Borel images of these functions as points of singularity of these images. However, as we shall see in the next section, this definition requires a serious generalization.

2 Counterexamples

This section contains examples of functions which appear in the real problems but are not included formally to the class of simplest resurgent functions defined above.

1. Consider the function

$$f(x) = e^{\sqrt{x}}. \tag{7}$$

It is not hard to see that the Borel image of this function is an endlessly-continuable function with the only singularity at $\xi = 0$. So, the Borel transform theory prescribes the value of action for this function to be equal to zero. However, the computation of the amplitude function shows that the singularity at the origin has more complicated structure than (3) and, hence, the function in question is not a resurgent function with simple singularities (it does not determine a WKB-element with zero action). From the other hand, function (7) is evidently a WKB-element with the action $\sqrt{x}$. Thus, one can see that the classical Borel transform is accustomed to the consideration of WKB-elements with actions linear in x.

This difficulty can be overcome by using the so-called *k-Borel transform* [8]. It is defined as a composition

$$\mathcal{B}_k = \alpha_{1/k} \circ \mathcal{B} \circ \alpha_k,$$

where $\mathcal{B}$ is the classical Borel transform, and α_k is the substitution $x \mapsto x^k$. This transform takes into account functions (WKB-elements) with the action proportional to x^k.

2. As the next example we consider the function

$$f(x) = e^x + e^{\sqrt{x}}. \tag{8}$$

It is easy to see that the Borel transform of the first summand on the right is an endlessly-continuable function with simple singularities (corresponding to the phase function x and the amplitude function 1). On the opposite, the Borel transform of the second summand, though endlessly-continuable with the only singularity at $\xi = 0$, has at this point a *complicated* (non-simple) singularity which does not correspond to a WKB-expansion. As it was already mentioned above, to investigate this summand one has to use the k-Borel transform (for $k = 1/2$). So, there arises a nesessity of generalization of the resurgent functions theory to the case when *different components of one and the same function have different orders of exponential growth.* We remark that such functions arise while considering differential equations. For example, the differential equation

$$y'' + xy' + xy = 0$$

possesses solutions involving components with orders 2 and 1.

3. As the third example consider the function

$$f(x) = e^{x+\sqrt{x}}.$$

This function is a WKB-element with the action $x + \sqrt{x}$. Clearly, the k-Borel transform is *not applicable* for investigation of such functions for any choice of k since the application of such a transform gives one WKB-elements with the action homogeneous in x. This is the main difference between this example and the previous one, where one has, at least in principle, the possibility of decomposing the function in question into two components and applying to these components the k-Borel transforms with different k.

4. Last, it is nesessary to work out the theory of asymptotic expansions for the investigation of functions of *several variables* as these variables increase simultaneously. The first guess is a possibility of applying the multiple Borel transform[4]. However, this way leads to the representation of functions in question in the form of integrals over a multidimensional homology class in the multidimensional complex space. The investigation of such integrals can hardly be considered as more easy problem than the initial problem of investigation of asymptotic expansions.

Later on, while working out the multidimensional resurgent functions theory one should take into account (apart from the one-dimentional effects listed above) the possibility of appearance of different orders of exponential growth in different directions. The simplest example of a function with such a behavior is

$$f(x, y) = e^{x^2+y}.$$

In the next section, we shall show how one can construct the multidimensional resurgent functions theory encountering all the above listed effects.

3 Resurgent representation and resurgent functions

In this section, we deal with the definition of a resurgent function based on the resurgent representation introduced by the authors. To motivate the mentioned definition as well as to clarify the presentation, we shall use the inductive method of exposition. Namely, we begin with the generalization of the Borel-Laplace theory to functions of several variables and, using this generalization as a starting point, construct the general resurgent representation.

[4]Clearly, the notion of multiple Borel transform requires rigorous description

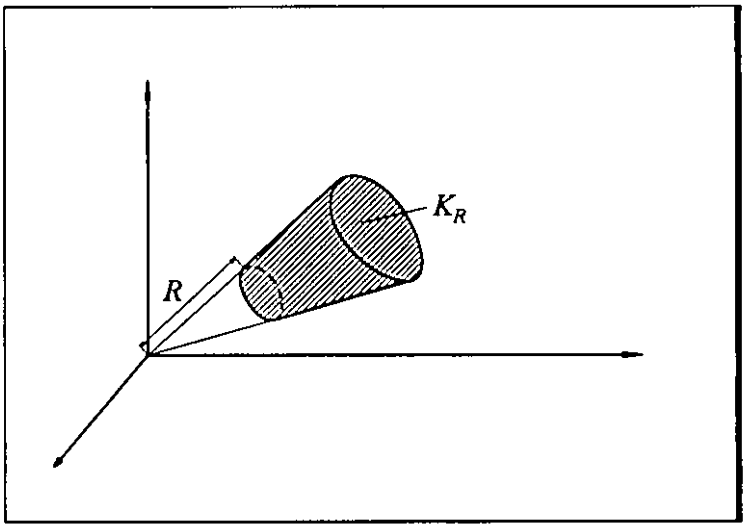

Figure 4.

3.1 Multidimensional Borel-Laplace transform

1. Let us consider the function $f(x)$, determined on the set

$$K_R = \{x \in K \mid |x| > R\} \subset \mathbf{C}^n,$$

where $x = (x^1, \ldots, x^n)$ are coordinates in $\mathbf{C}^n$, and K is some conical (that is, $\mathbf{R}_+$-invariant) set in $\mathbf{C}^n$ (see Figure 4). Suppose that:

1) $f(x)$ is holomorphic in the domain K_R;

2) $f(x)$ satisfies the inequality

$$|f(x)| \leq C e^{a|x|} \tag{9}$$

in each proper subset $K'_{R'}$, $K' \subset\subset K$, $R' > R$. We say that $f(x)$ is a function of *exponential growth of order* 1 in K_R.

Let us introduce a function

$$f_x(\lambda) = f\left(\lambda x^1, \ldots, \lambda x^n\right).$$

Clearly, the function $f_x(\lambda)$ considered as a function in variable $\lambda \in \mathbf{C}$ is determined in some sector $S_0(R, \varepsilon)$, $R > 0$, $\varepsilon > 0$ for each fixed x as a function of exponential growth of order 1. Hence, one can consider its Borel image

$$F\left(s, x^1, \ldots, x^n\right) = \mathcal{B}_{\lambda \to s}\left[f_x(\lambda)\right].$$

It is easy to see that $F(s,x)$ is a homogeneous function in (s,x) of degree 1:

$$F\left(\lambda s, \lambda x^1, \dots, \lambda x^n\right) = \lambda F\left(s, x^1, \dots, x^n\right).$$

The inverse (Laplace) transform of the function $F\left(s, x^1, \dots, x^n\right)$ can be written down in the form

$$f_x(\lambda) = f\left(\lambda x^1, \dots, \lambda x^n\right) = \int\limits_\gamma e^{\lambda s} F\left(s, x^1, \dots, x^n\right) ds,$$

and, hence,

$$f\left(x^1, \dots, x^n\right) = \mathcal{L}\left[F\left(s, x^1, \dots, x^n\right)\right] = \int\limits_\gamma e^{s} F\left(s, x^1, \dots, x^n\right) ds. \tag{10}$$

The function $F\left(s, x^1, \dots, x^n\right)$ will be called a *(multidimensional) Borel transform of the function* $f(x)$. Formula (10) is an inversion formula for the transform $\mathcal{B}$.

2. Suppose now that the function $f(x)$, holomorphic in the domain K_R satisfies the inequality

$$|f(x)| \leq Ce^{a|x|^k}$$

for some $k > 0$ (instead of (9)). Then the above considerations can be modified in the following way: the function

$$f_x(\lambda) = f\left(\lambda^{1/k} x^1, \dots, \lambda^{1/k} x^n\right)$$

is again a function of exponential growth of order 1, and we define its Borel image by the formula

$$F\left(s, x^1, \dots, x^n\right) = \mathcal{B}_k\left[f(x)\right] = \mathcal{B}_{\lambda \to s}\left[f_x(\lambda)\right],$$

where $\mathcal{B}$ is the classical Borel transform. In doing so, the function

$$F\left(s, x^1, \dots, x^n\right)$$

occurs to be a weighted homogeneous function in variables $\left(s, x^1, \dots, x^n\right)$:

$$F\left(\lambda s, \lambda^{1/k} x^1, \dots, \lambda^{1/k} x^n\right) = \lambda F\left(s, x^1, \dots, x^n\right),$$

and the inversion formula reads

$$f\left(x^1, \dots, x^n\right) = \mathcal{L}\left[F\left(s, x^1, \dots, x^n\right)\right] = \int\limits_\gamma e^{s} F\left(s, x^1, \dots, x^n\right) ds. \tag{11}$$

3. At last, let us consider the function $f(x)$ in K_R satisfying the inequality

$$|f(x)| \leq Ce^{a_1|x|^{k_1}+\ldots+a_n|x|^{k_n}}.$$

Then we put

$$f_x(\lambda) = f\left(\lambda^{1/k_1}x^1, \ldots, \lambda^{1/k_n}x^n\right),$$

and the corresponding formula for the k-Borel transform becomes

$$F\left(s, x^1, \ldots, x^n\right) = \mathcal{B}_k\left[f(x)\right] = \mathcal{B}_{\lambda\to s}\left[f_x(\lambda)\right]$$

(here $k = (k_1, \ldots, k_n)$), and the inversion formula reads

$$f\left(x^1, \ldots, x^n\right) = \mathcal{L}\left[F\left(s, x^1, \ldots, x^n\right)\right] = \int_\gamma e^s F\left(s, x^1, \ldots, x^n\right) ds. \tag{12}$$

One can see that the function $F(s, x^1, \ldots, x^n)$ possesses the following homogeneity properties:

$$F\left(\lambda s, \lambda^{1/k_1}x^1, \ldots, \lambda^{1/k_n}x^n\right) = \lambda F\left(s, x^1, \ldots, x^n\right).$$

The three situations considered above lead us to the conclusion that the *inversion formulas* (10), (11), and (12) *in all three cases are identical, and the growth properties of the function* $f(x)$ *are determined just by homogeneity properties of the integrand* $F(s, x^1, \ldots, x^n)$. This observation is exactly the starting point for the resurgent representation introduced in the next section.

3.2 Resurgent representation

Let us introduce the following definition:

Definition 3 The function $f(x)$, determined in the domain $K_R \subset \mathbf{C}^n$ is called to be *resurgent* if it can be represented in the form

$$\boxed{f(x) = \mathcal{L}\left[F(s, x)\right] = \int_\gamma e^s F(s, x)\, ds} \tag{13}$$

with an endlessly-continuous function $F(s, x^1, \ldots, x^n)$.

Remark 2 Clearly, each concrete problem requires exact determination of the class of functions $F(s, x)$. At the same time, we do not require that the function F is homogeneous in any reasonable sense.

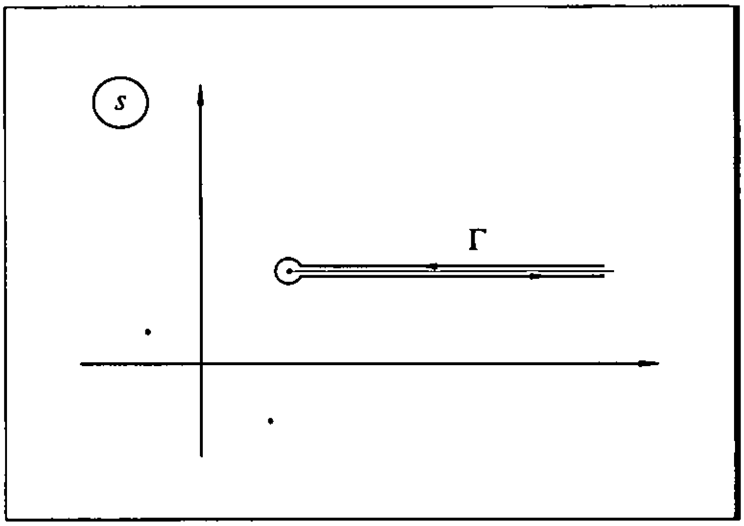

Figure 5.

Clearly, integral (13) can be represented as the sum of integrals of the same type over standard contours Γ_j (see Figure 5). Each integral over a standard contour we shall call a *component* of the function $f(x)$.

Suppose now that the function $F(s,x)$ has simple singularities, that is, it is representable in the form

$$F(s,x) = \frac{a_0(x)}{s - S(x)} + \ln(s - S(x)) \sum_{j=0}^{\infty} \frac{(s - S(x))^j}{j!} a_{j+1}(x)$$

near each its point of singularity. Then the corresponding function $f(x)$ has the asymptotic expansion

$$f(x) \simeq e^{S(x)} \sum_{j=0}^{\infty} a_j(x).$$

We present here the resurgent representation of functions of one variable included into the above considered (counter)examples:

$$e^x + e^{\sqrt{x}} = \int_\gamma e^{-s} \left\{ \frac{1}{2\pi i} \left[\frac{1}{s+x} + \frac{1}{s+\sqrt{x}} \right] \right\} ds,$$

$$e^{x+\sqrt{x}} = \int_\gamma e^{-s} \left\{ \frac{1}{2\pi i (s + x + \sqrt{x})} \right\} ds.$$

We remark that the above defined functions

$$\begin{aligned} F_1(s,x) &= \frac{1}{2\pi i}\left[\frac{1}{s+x}+\frac{1}{s+\sqrt{x}}\right], \\ F_2(x) &= \frac{1}{2\pi i(s+x+\sqrt{x})} \end{aligned}$$

are not homogeneous (and even weighted-homogeneous) ones in (s,x).

Let us formulate here one more important affirmation on the introduced representation used in the applications to investigation of differential equations by the resurgent analysis method. Namely, the following theorem describes the commutation between the introduced representation and the differentiation operation.

Theorem 2 *The following commutation formulas*

$$\boxed{\frac{\partial}{\partial x^i}\circ\mathcal{L}=\mathcal{L}\circ\left[\left(\frac{\partial}{\partial s}\right)^{-1}\frac{\partial}{\partial x^i}\right]}$$

are valid.

As we shall see below, the commutation formulas of the above form transform differential equations for functions of x into equations *quantized with respect to smoothness* for functions of (s,x). This fact is of extreme importance since in the representation of functions of (s,x) asymptotics in variables x become asymptotics in smoothness.

4 Applications

Here we shall consider the two applications of the above introduced notion of resurgent function. The first application deals with the investigation of fundamental systems of solutions to ordinary differential equations, and the second with classification of functions of exponential growth by the type of their asymptotic expansion at infinity. Clearly, a lot of applications of this theory are out of the frasmework of this paper (such as the investigation of behavior at infinity of solutions to partial differential equations, see [9]). The reader can find these applications (as well as the more detailed presentation of the theory) in the book [6] by the authors.

4.1 Investigation of ordinary differential equations

Consider the differential equation

$$y^{(n)} + P_{n-1}(x)\, y^{(n-1)} + \ldots + P_1(x)\, y' + P_0(x)\, y = 0 \tag{14}$$

with the polynomial coefficients $P_j(x)$, and denote by

$$H\left(x, \frac{d}{dx}\right) = \frac{d^n}{dx^n} + P_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}} + \ldots + P_1(x)\frac{d}{dx} + P_0(x)$$

the corresponding differential operator.

The following affirmation takes place.

Theorem 3 *Equation* (14) *has a full system of resurgent solutions with simple singularities.*

We remark that since any solution of equation (14) is a linear combination of solutions from the fundamental system, it occurs that all solutions to this equation are resurgent functions with simple singularities.

The idea of the proof[5] of Theorem 3 is as follows. Using the resurgent representation

$$y(x) = \mathcal{L}[Y(s,x)],$$

we transform equation (14) for y into the equation

$$H\left(x, \left(\frac{\partial}{\partial s}\right)^{-1}\frac{\partial}{\partial x}\right) Y(s,x) = 0$$

for Y which is quantized with respect to smoothness. In doing so, one can not only prove the existence of resurgent solutions to the initial equation but also investigate the asymptotic behavior of solutions as $x \to \infty$ (see the book [6]).

Let us illustrate this scheme on the example of the above mentioned equation

$$y'' - xy' + xy = 0.$$

The corresponding equation for the "resurgent image" $Y(s,x)$ reads

$$\left(\frac{\partial}{\partial s}\right)^{-2}\frac{\partial^2 Y}{\partial x^2} - x\left(\frac{\partial}{\partial s}\right)^{-1}\frac{\partial Y}{\partial x} + xY = 0$$

[5] The proof of this theorem was carried out in collaboration with E. Delabaere, H. Dillinger, and F. Pham.

or

$$\frac{\partial^2 Y}{\partial x^2} - x\frac{\partial^2 Y}{\partial s \partial x} + x\frac{\partial^2 Y}{\partial s^2} = 0.$$

The singularities of solutions to this equation can be found from the corresponding Hamilton-Jacobi equation[6]

$$(S')^2 - xS' + x = 0.$$

We obtain

$$p(x) = S'(x) = \frac{x}{2} \pm \sqrt{\frac{x^2}{4} - x}.$$

Choosing the branch of the square root in accordance to the formula

$$\sqrt{\frac{x^2}{4} - x} = \frac{x}{2}\sqrt{1 - \frac{4}{x}}, \quad \sqrt{1} = 1$$

for large values of x, we obtain

$$p_+(x) = \frac{x}{2} + \frac{x}{2}\sqrt{1 - \frac{4}{x}} = x - 1 - \frac{1}{4x} + O\left(\frac{1}{x^2}\right),$$

and

$$p_-(x) = \frac{x}{2} - \frac{x}{2}\sqrt{1 - \frac{4}{x}} = 1 + \frac{1}{4x} + O\left(\frac{1}{x^2}\right).$$

The values of action can be now found simply by integrating:

$$\begin{aligned} S_+(x) &= \tfrac{x^2}{2} - x - \tfrac{1}{4}\ln x + O\left(\tfrac{1}{x}\right), \\ S_-(x) &= x + \tfrac{1}{4}\ln x + O\left(\tfrac{1}{x}\right). \end{aligned}$$

Hence, the asymptotic expansion of solutions to the initial problem must have the form

$$\begin{aligned} y(x) &= e^{S_+(x)}\sum_j x^{-j}a_j^+ + e^{S_-(x)}\sum_j x^{-j}a_j^- \\ &= \exp\left\{\frac{x^2}{2} - x - \frac{1}{4}\ln x + O\left(\frac{1}{x}\right)\right\}\sum_j x^{-j}a_j^+ \\ &+ \exp\left\{x + \frac{1}{4}\ln x + O\left(\frac{1}{x}\right)\right\}\sum_j x^{-j}a_j^-. \end{aligned}$$

[6]With respect to the WKB-representation

$$Y(s,x) = e^{S(x)\partial/\partial s}a(s,x).$$

(see [10])

Remark 3 There is another method of investigating equations of the type (14) based on the so-called *multisummability* notion (see [11], [12], [13], [14], [15], [16], [8], and others). Namely, one constructs a formal solution of the differential equation in question (that is, a solution in formal power series), and then resummate the obtained solution. In doing so, due to the presence of different components with different order of exponential growth, the simplest resummation procedure with some fixed value of k occurs to be inapplicable. The reason for this phenomenon is that, if one tries to resummate a formal series

$$\sum_{j=0}^{\infty} x^{-j} a_j$$

which splits into the sum of the two series

$$\sum_{j=0}^{\infty} x^{-j} a_j = \sum_{j=0}^{\infty} x^{-j} a_j^1 + \sum_{j=0}^{\infty} x^{-j} a_j^2 \tag{15}$$

being asymptotic expansions of two functions $f_1(x)$ and $f_2(x)$ of different orders of exponential growth (k_1 and k_2, $k_1 > k_2$, respectively), then:

1) If we apply the formal k-Borel transform with largest value of $k = k_1$, to the right-hand part of (15), then the second sum will be transformed to the series with zero radius of convergence, and, hence, the formal k-Borel transform of the series in question *does not determine an analytic function even in a very small neighborhood of the origin.* So, only the formal k-Borel transform with the minimal value of $k = k_2$ is applicable.

2) Later on, the result of the application the formal k_2-Borel transform to series (15) is a sum of the two functions $F_1(\xi)$ and $F_2(\xi)$. It occurs that *the first of these two functions is a function of exponential growth with order more than k_2, so that the k_2-Laplace transform occurs to be inapplicable to this function.*

Thus, the resummation procedure fails both for $k = k_1$ and $k = k_2$ (as well as for any other value of k). Hence, on the first step one has to apply the formal k_2-Borel transform for minimal order $k = k_2$ obtaining, as a result a convergent series in the dual variable ξ. Then, on the second step, one has to use *the acceleration operator* taking Borel images of functions of exponential order k_2 to functions of exponential order k_1 (this operator is none more than the operator of a power variable change written in the dual representation; it occurs to be an integral operator over some contour in the complex plane). As a result one obtains a function for which the Laplace transform is well-defined. The final step is the application of the corresponding Laplace transform to the function obtained with the help of the acceleration procedure. On this step one obtains the solution to equation in question.

Clearly, this procedure (which allows to prove the existence of resurgent solutions) cannot give any information about the asymptotic behavior of these solutions at infinity, since the result of the application of the acceleration procedure is not cleqarly a function with simple singularities.

4.2 Classification of functions of exponential growth

In this concluding section, we present a classification of functions of exponential growth in several variables based on the asymptotic expansions of these functions as $x \to \infty$. Due to the discussion above, it is clear that this classification must be based on the

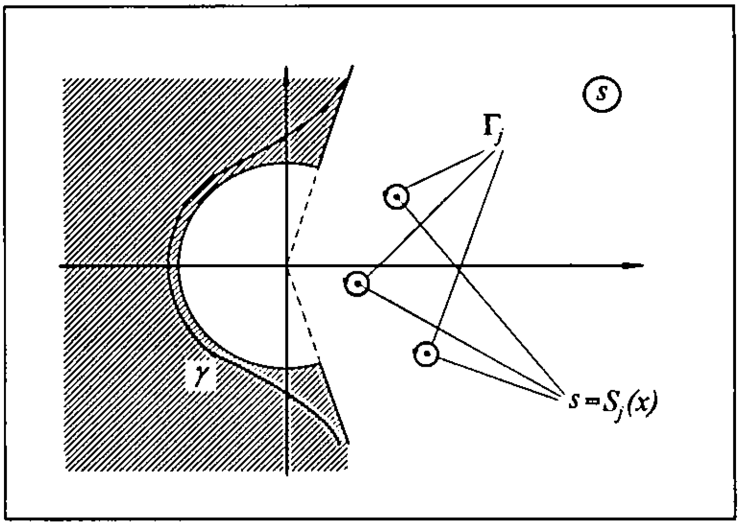

Figure 6.

description of singularities of the corresponding functions in the space of variables (s, x).

1. *Polar singularities* (see Figure 6).

In this case, the expansion of the function $F(s, x)$ at its points of singularity $s = S_j(x)$ has the form

$$F\left(s, x^1, \ldots, x^n\right) = F(s, x) = \sum_{k=1}^{m} \frac{a_k^{(j)}(x)}{\left[s - S_j(x)\right]^k} + F_0(s, x),$$

where the function $F_0(s, x)$ is regular at the point $s = S_j(x)$. Accordingly, the asymptotics of the corresponding function $f(x)$ is

$$f(x) = \sum_j \int_{\Gamma_j} e^{-s} F(s, x)\, ds \simeq \sum_j e^{-S_j(x)} \sum_{k=1}^{m} \frac{(-1)^{k-1}}{(k-1)!} a_k^{(j)}(x). \tag{16}$$

2. *Ramifying discrete singularities* (see Figure 7).

In the case of simple singularities (this case is the most interesting one for applications) the asymptotic expansion of the function $F(s, x)$ at points of its singularity

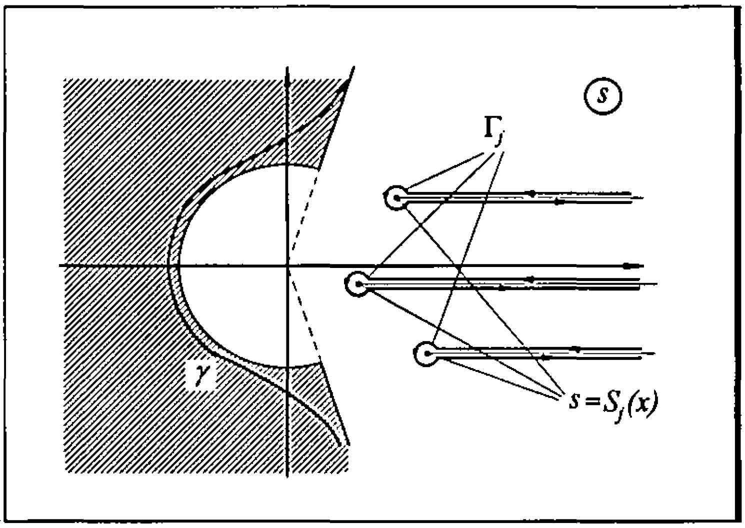

Figure 7.

$s = S_j(x)$ has the form

$$F(s,x) = \frac{a_0^{(j)}(x)}{s - S_j(x)} + \ln(s - S_j(x)) \sum_{k=0}^{\infty} \frac{[s - S_j(x)]^k}{k!} a_{k+1}^{(j)}(x),$$

and, correspondingly, the asymptotic expansion of the initial function $f(x)$ as $x \to \infty$ is

$$f(x) = \sum_j \int_{\Gamma_j} e^{-s} F(s,x)\, ds \simeq \sum_j e^{-S_j(x)} \sum_{k=0}^{\infty} a_k^{(j)}(x). \tag{17}$$

The two types of functions of exponential growth considered above correspond to the case of *functions with discrete asymptotics*. This terminology is inspired by the fact that asymptotic expansions (16) and (17) include exponentials $e^{-S_j(x)}$, corresponding to a discrete set of values of action $s = S_j(x)$ (this is a direct consequence of the discrete type of singularities of the function $F(s,x)$). However, it is also possible that the singularities of the function $F(s,x)$ form domains of non-zero measure (the so-called *continuous asymptotics*). In this latter case one can also distinguish two cases.

3. *Compact singularities of a nonzero measure* (see Figure 8).

In this case the singularities of the function $F(s,x)$ are contained in some compact set K of the complex plane $\mathbf{C}_s$. In the case when $F(s,x)$ has a univalent character

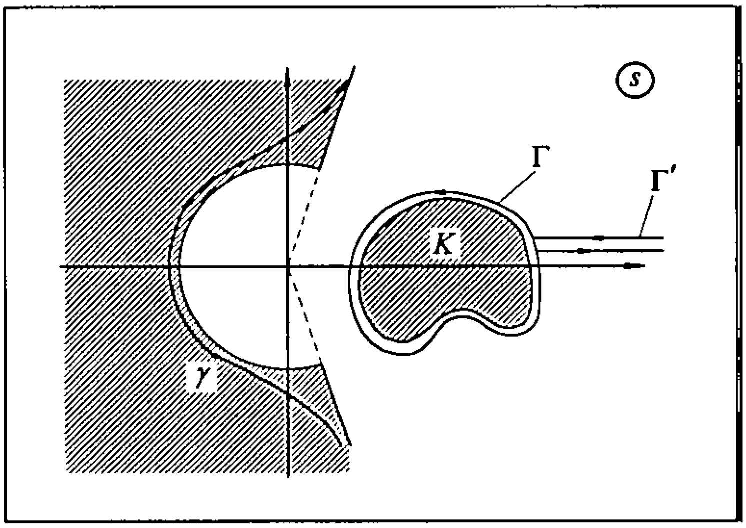

Figure 8.

while encircling the set K, the asymptotics of the initial function $f(x)$ can be written down in the form

$$f(x) \simeq \int_{\Gamma} e^{-s} F(s,x)\, ds$$

of application of an analytic functional determined by the function $F(s,x)$, to the exponential e^{-s}.

In the general case when the function $F(s,x)$ ramifies around the compact set K this asymptotics can be represented as the application of the hyperfunction determined by $F(s,x)$ to the same exponential e^{-s}:

$$f(x) \simeq \int_{\Gamma'} e^{-s} F(s,x)\, ds.$$

Both last cases are characterized by the fact that the asymptotic expansion of the function $f(x)$ involves continually many exponents of the form e^{-s}.

4. *General case.*

In this case no additional restrictions are posed on the set of singularities of the function $F(s,x)$. However, one can show that in this case the function $f(x)$ can be represented as an (infinite) sum of results of application of different analytic functionals to the exponential e^{-s} (see also [17], where the case of functions of power growth is considered).

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