A NOTE ON THE #-UNKNOTTING OPERATION

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ABSTRACT. In this note we use a connection between local moves on knots and the singularity type of certain PL-spheres in 4-manifolds and use it to recover (stronger in principle versions of) known results as well as obtain new ones regarding certain #-unknotting numbers of knots.

1. **Introduction**

In [M] H. Murakami defines the #-local move below on a projection of a knot (notice how it differs from a *pass move* only by orientation):

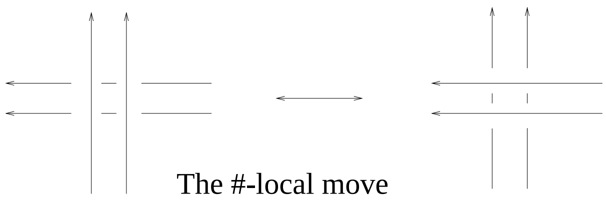

FIGURE 1

The move shown in the figure below will be refered to as a $\pm - \Xi$ move (plus for right to left in the figure, minus for reverse):

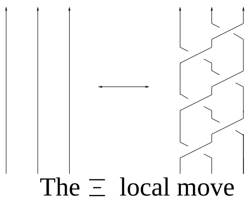

FIGURE 2

Y. Ohyama shows in [O] (Proposition 2.11) that a $\pm - \Xi$ move is locally equivalent to a #-local move. He shows infact that a $(+ : -)$, #-local move can be locally expressed in terms of three successive Ξ-operations (first two $+\Xi$ and then a $-\Xi$ operation) and that a $+\Xi$-local move is given locally by a single $(+ : -)$ #-local move.

H. Murakami proves in [M], by making use of non-orientable Seifert surfaces, that the #-local move is an unknotting operation (i.e. every knot has a projection with a finite number of local # crossings so that, when changed, a projection of the unknot is obtained). This is easily seen to define a metric $d_G^{\#}$ on the space of all knots, simply by defining $d_G^{\#}(K_1, K_2)$ to be the minimum number (over all projections and sequences of moves) of #-crossing changes necessary to pass from a projection of K_1 to one of K_2. The #-unknotting number $u^{\#}$ of a knot K is then nothing but $d_G^{\#}(K, O)$, where O is the unknot. As a consequence of Ohyama's aforementioned theorem the Ξ local move is also an unknotting operation and we can define similarly d_G^{Ξ} and u^{Ξ}. We wish to use a connection between local moves on knot diagrams and certain PL-embedded spheres in 4-manifolds and use it to recover (stronger in principle versions of) Murakami's Theorems 3.2, 3.5 and Corollary 3.3 as an application. This observation together with Ohyama's theorem allows us to get some new estimates on the #-unknotting number which involve $\sigma_{\frac{1}{3}}$. The aforementioned connection is inspired by a simple geometric idea of S. Suzuki in [S]. The results will be obtained by invoking results of Viro and Gilmer and Kervaire-Milnor, Freedman-Kirby.

We now introduce some language which will improve the write-up of the rest of this note. By $\mathcal{U}_K^{\#}(p, n)$ we will denote (in analogy with [CL] where $\mathcal{U}_K(p, n)$ are defined for the ordinary crossing change) the set of all knots which have a projection which can be turned to one for K by p $(+ : -)$ (left to right in Figure 1) and n $(- : +)$ (reverse direction of figure 1) moves. To say then that the #-local move is an unknotting operation can be translated into saying that every knot K belongs to $\mathcal{U}_O^{\#}(p, n)$ for some p, n non-negative integers. The same discussion holds for the Ξ-local move and we can define $\mathcal{U}_K^{\Xi}(p, n)$ etc. Ohyama's Proposition 2.11 of [O] can then be stated as saying that $\mathcal{U}_O^{\#}(p, n) \subset \mathcal{U}_O^{\Xi}(2p + n, p + 2n)$ and $\mathcal{U}_O^{\Xi}(p, n) \subset \mathcal{U}_O^{\#}(p, n)$. Notice that these inclusions imply $u^{\#} \leq u^{\Xi} \leq 3u^{\#}$. It should also be pointed out that:

$$u^{\#}(K) = \min \{p + n \colon K \in \mathcal{U}_O^{\#}(p, n)\},$$
$$u^{\Xi}(K) = \min \{p + n \colon K \in \mathcal{U}_O^{\Xi}(p, n)\}.$$

2. From local moves to 4-manifolds

The main lemma which establishes the claimed connection is essentially nothing but what is called *fusion* and *fision* (the two directions). We were motivated by Suzuki's idea in [S].

Lemma 2.1. *If $K \in \mathcal{U}_O^{\#}(p, n)$ then there is an embedded 2-sphere $A \subset M_{p,n} = \#_{p+n} S^2 \times S^2$, which is smooth everywhere except at one point where it is the cone on K and is such that on homology $[A] = 2(1, p - n) \oplus 0 \in H_2(M_{p,n}) = H_2(S^2 \times S^2 \#_{p+n-1} S^2 \times S^2) = H_2(S^2 \times S^2) \oplus_{p+n-1} H_2(S^2 \times S^2)$.*

Sketch of Proof Take the projection which becomes one of the unknot after applying the p $(+ : -)$ and n $(- : +)$, # local moves on it. Figure 3 below contains the idea of the proof. In it one can see how a $-$ to $+$ move results in expressing the resulting knot as the original one band-connect-summed (four bands) with the double of a Hopf link with negative linking number. The reverse direction is completely analogous.

We can then see that this leads to a "canonical" projection of the knot just as in [S] (the canonical form says essentially that every knot has a projection which is obtained from one of the unknot by "fusions" and "fisions" with the four component link obtained from the hopf link by doubling each of its two components). On the other hand the Kirby calculus picture of a punctured connected sum of k copies

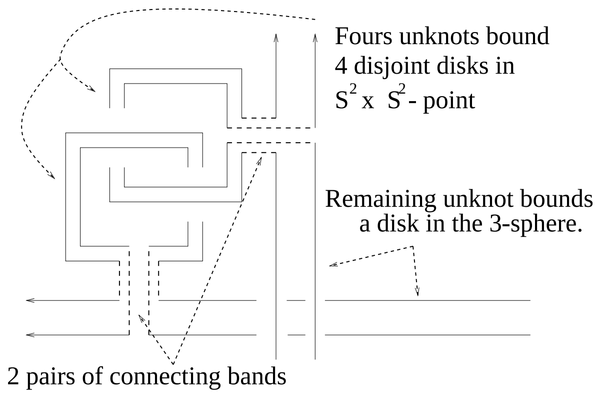

FIGURE 3

of $S^2 \times S^2$ is k copies of the Hopf link all of whose components are zero framed. This means that in a punctured $\#_{p+n} S^2 \times S^2$ our knot bounds a disk which on homology represents $\oplus_p (2,2) \oplus_n (2,-2)$. The disk consists of two copies of each of the core disks of the $2(p+n)$ 2-handles union with the bands (in the figure the bands are between the dotted lines) union with a disk that the remaining unknot bounds in the boundary 3-sphere. C. T. C. Wall's Theorems of [W1] AND [W2] on the transitivity of the automorphism group of the intersection form of such a manifold and its realizability by self-diffeomorphisms lead us to the desired class. Now close the manifold by adding a 4-ball and cone off the knot to get the singular sphere.

Remark 2.1. *1. Similarly as above we can see that if $K_1 \in \mathcal{U}^{\#}_{K_2}(p,n)$ then there is an embedded sphere in $M_{p,n}$ representing the same homology as in the lemma with two singular points where it is a cone on K_1 and the mirror image of K_2.*

2. Notice that $2u^{\#}(K)$ becomes the minimum second Betti number of $M_{p,n}$ as in the Lemma.

Similar considerations as in lemma 2.1 above imply then the analogous results in the case of a Ξ-local move. Because of this similarity and the geometric nature of the argument we ommit a sketch of proof in this case:

Lemma 2.2. *If $K \in \mathcal{U}^{\Xi}_{O}(p,n)$ then there is an embedded 2-sphere $A \subset N_{p,n} = \#_p CP^2 \#_n CP^2$, which is smooth everywhere except at one point where it is the cone on K and is such that on homology $[A] = 3(\oplus_p (1) \oplus_n (1)) \in H_2(N_{p,n}) = \oplus_p H_2(CP^2) \oplus_n H_2(CP^2)$.*

3. The estimates

Now we quote the 4-manifold theoretic results we need to achieve the promised estimates. The following theorem is proven in [G] by P. Gilmer (from where we copy it here) and as mentioned there a special case of it ($d = 2$ and one singular point) is also due to O. Ya. Viro. The version we quote here is only a special case of Gilmer's result (Theorem 4.1, Corollary 4.2) but it is more than what we need:

Theorem 3.1. *Suppose $A \subset M^4$ is an embedded sphere in a simply connected 4-manifold which is smooth everywhere except at q points where it is the cone on knots*

K_i, $i = 1, \dots, q$ and is such that on homology $[A] = \alpha = d\beta \in H_2(M^4)$, with d a power of a prime p. Then for every $0 < j < d$, $j \not\equiv 0 \bmod p$:

$$b_2(M^4) + q - 1 \geq |\sigma(M^4) - \frac{2}{d^2} j(d-j)\alpha^2 - \sum_i \sigma_{j/d}(K_i)|$$

For the following theorem see p. 66 of [K] but beware because it is written down incorrectly there! See also theorem 1.2 on p. 31 of [K]:

Theorem 3.2. *Suppose M^4 is closed, smooth,simply connected and $\omega \in H_2(M^4)$ is characteristic. If ω is represented by an embedded sphere which is smooth everywhere except at q singular points where it is a cone on knots K_i, $i = 1, \dots, q$ then it satisfies*

$$\frac{\sigma(M^4) - \omega \cdot \omega}{8} \equiv \sum_i Arf(K_i) \bmod 2$$

Such an ω is always representable by an embedded sphere which is smooth everywhere except at one singular point where it is a cone on a knot K.

Our Lemma 2.1 and Theorem 3.1 above easily imply:

Proposition 3.1. *If $K_1 \in \mathcal{U}^{\#}_{K_2}(p,n)$ then $-6p + 2n - \delta \leq \sigma(K_1) - \sigma(K_2) \leq 6n - 2p + \delta$, where δ is zero or one according as $K_2 = O$ or not.*

This easily implies Theorem 3.2 and Corollary 3.3 of [M] whereas Theorem 3.5 falls out of 3.2. In analogy we have the corresponding statement in the case of the Ξ move.

Proposition 3.2. *If $K \in \mathcal{U}^{\Xi}_{O}(p,n)$ then $-4p + 2n \leq \sigma_{1/3}(K) \leq -2p + 4n$ and $p + n \equiv Arf(K) \bmod 2$; in particular $u^{\Xi}(K) \equiv Arf(K) \bmod 2$.*

Using $\mathcal{U}^{\#}_{O}(p,n) \subset \mathcal{U}^{\Xi}_{O}(2p+n, p+2n)$ and $\mathcal{U}^{\Xi}_{O}(p,n) \subset \mathcal{U}^{\#}_{O}(p,n)$ one can get the following cross information about ($\#$, Ξ)-unknotting numbers which as far as we knot is new as we do not know of any estimates regarding $u_{\#}$ which involve $\sigma_{\frac{1}{3}}$:

Theorem 3.3. *If $K \in \mathcal{U}^{\#}_{O}(p,n)$ then $-6p \leq \sigma_{1/3}(K) \leq 6n$ and hence $|\sigma_{1/3}(K)| \leq 6u^{\#}(K)$. If $K \in \mathcal{U}^{\Xi}_{O}(p,n)$ then $-6p + 2n \leq \sigma(K) \leq 6n - 2p$.*

Remark 3.1. *Note that similar arguments can be applied for the classical unknotting number u. Changing p positive (n negative) crossings corresponds to p negative (n positive) full twists. Hence our knot is then the singularity type of a PL-sphere in $\#_p CP^2 \#_n \overline{CP}^2$ which represents the direct sum of twice the generator of each $\pm CP$. Applying Theorem 3.1 then shows $-2p \leq \sigma(K) \leq 2n$. This in particular is stronger than the well known $|\sigma(K)| \leq 2u(K)$, when $u(K) = p + n$. The general idea can be applied to any local move on knot diagrams. In fact we will do that for a new kind of unknotting operation in subsequent work. In that work we generalize the $\#$-local move and the observation here about its unknotting numbers is logically contained there. However the emphasis in that paper is the new local moves whereas here we want to bring out the possibility of estimating unknotting numbers via 4-manifold theory. The present note contains in addition Proposition 3.2 and Theorem 3.3. It should be noted that input from gauge theory for such unknotting operations are only possible for studying unknotting number one. The only possible input then for higher unknotting numbers is P. Gilmer's thesis. For carefully chosen knots higher unknotting numbers can be studied via branched covers as in* [CL].

References

[G] P. M. Gilmer, *Configurations of Surfaces in* 4*-manifolds,* Trans. of AMS, Vol 264, No. 2, 353-380, (1981)

[CL] T. D. Cochran, W. B. R. Lickorish, *Unknotting Information from* 4*-manifolds,* Trans. of the AMS, Vol. 297, No 1, 125-142, (1986)

[K] R. Kirby, *The topology of* 4*-manifolds,* LNM 1374 , Springer Verlag, (1981)

[M] H. Murakami, *Some Metrics on Classical Knots,* Math. Ann. 270, 35-45, (1985)

[O] Y. Ohyama, *Twisting and unknotting operations,* Revista Math. de la U.C. de Madrid, Vol. 7, No. 2 289-305, (1994)

[S] S. Suzuki, *Local Knots of* 2*-spheres in* 4*-manifolds* , Proc. Japan Acad., Vol. 45, No. 1 34-38, (1969)

[W1] C. T. C. Wall, *On Simply-connected 4-manifolds,* J . London Math. Soc. 39, 141-149, (1964)

[W2] C. T. C. Wall, *Diffeomorphism of* 4*-manifolds,* J . London Math. Soc. 39, 131-140, (1964)

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