EXPLICIT GROWTH ESTIMATES FOR SOLUTIONS OF P–ADIC DIFFERENTIAL SYSTEMS IN A REGULAR SINGULAR DISK

by

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ÜBUNGEN ZUR INFINITESIMALRECHNUNG II

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Hausaufgaben

1. Sei L die Menge aller Punkte $(x, y) \in \mathbf{R}^2$, für die das Produkt der Entfernungen von den Punkten $(1,0)$ und $(-1,0)$ gleich 1 ist. Finden Sie die Punkte $(x, y) \in L$, für die $|x|$ oder $|y|$ maximal ist.

2. Für $a, b \in \mathbf{R}$ sei $\delta(a, b) = \inf\left\{\sqrt{(x-a)^2 + (y-b)^2} \,|\, x, y \in \mathbf{R},\, xy = 1\right\}$. Bestimmen Sie die maximale offene Teilmenge $U \subset \mathbf{R}^2$, für die $\delta|_U$ glatt ist.

3. Zeigen Sie: Zu jedem $A \in \mathrm{SL}_n(\mathbf{R})$ existiert eine offene Teilmenge $V \in \mathbf{R}^{n\times n}$ mit $A \in V$ und eine glatte und bijektive Abbildung $f\colon U \to V \cap \mathrm{SL}_n(\mathbf{R})$, wo U eine offene Teilmenge des $\mathbf{R}^{n^2-1}$ ist. Hierbei bezeichnet $\mathbf{R}^{n\times n}$ den Vektorraum der $n \times n$–Matrizen mit Koeffizienten in $\mathbf{R}$ und $\mathrm{SL}_n(\mathbf{R})$ die Teilmenge aller Matrizen mit Determinante 1.

4. Sei $f \in C^1(\mathbf{R}^n)$ und $A = \{x \in \mathbf{R}^n \,|\, f(x) = 0\}$. Gegeben seien ein $a \in A$ und ein Vektor $x_0 \in \mathbf{R}^n$, sodaß $\operatorname{grad} f(a) \neq 0$ und $\operatorname{grad} f(a) \cdot x_0 = 0$. Zeigen Sie: Es existiert eine stetig differenzierbare Abbildung $w\colon I \to A$, wo I ein offenes, den Nullpunkt enthaltenes Intervall bezeichnet, sodaß $w(0) = a$ und $w'(0)(1) = x_0$ gilt.

Anwesenheitsaufgaben

1. Seien b_1, b_2 linear unabhängige Vektoren im $\mathbf{R}^3$; bezeichne $\delta(a)$ den Abstand des Punktes $a \in \mathbf{R}^3$ zur *Geraden* $g = \{x \in \mathbf{R}^3 \,|\, b_1 \cdot x = b_2 \cdot x = 0\}$, d.h. es sei

$$\delta(a) = \inf\left\{\sqrt{(x-a)^2} \,|\, x \in g\right\}.$$

Berechnen Sie $\delta(a)$.

2. Verallgemeinern Sie die vorstehende Aufgabe von 3 auf den Fall von n Dimensionen.

3. Finden Sie alle Abbildungen $w\colon I \to \mathbf{R}^2$ (I eine offenes, die 0 enthaltenes Intervall in $\mathbf{R}$), sodaß $w(0) = 0$ und $p(w(t), t) = 0$ für alle $t \in I$ gilt. Hierbei bezeichnet p das Polynom $p = (3x^2 - y^2)y - (x^2 + y^2)^2$.

4. Es seien a, b, c reelle Zahlen mit $abc \neq 0$, und es sei E die Menge der reellen Tripel (x, y, z) mit $ax^2 + by^2 + cz^2 = 1$. Bestimmen Sie die lokalen Extrema der Abbildung $f\colon E \to \mathbf{R}$, $\quad (x, y, z) \mapsto x^2 + y^2 + z^2$.

Explicit growth estimates for solutions
of p–adic differential systems in a regular
singular disk

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1 Introduction

1.1 Let k be an algebraically closed field of characteristic 0, complete under an absolute value $|\ |$ normalized by $|p| = p^{-1}$ at some prime p.

We consider the solution of the differential system

$$(*) \qquad x\,d/_{dx} Z = GZ$$

given in the form $Z = Y \times {}^{C}$, where

a) G and Y are $\mu \times \mu$ matrices whose entries are analytic functions on the unit disk (without boundary),

b) $Y(0)$ is the unit matrix,

c) C is a nilpotent matrix.

The couple (Y,C) is uniquely determined by these conditions; in fact $C = G(0)$, and Y is invertible.

1.2. In this paper, we shall estimate the Taylor coefficients of $Y = \sum_{n\geq 0} Y_n x^n$, or what amounts to the same, the growth of $\|Y\|_0(r) = \operatorname*{Sup}_{|x|<r} \|Y(x)\|$ as r becomes close to 1.

Let us put, once for all, $\nu = \mu - 1 + \mathrm{ord}_p(\mu - 1)!$ Our main result is:

Theorem 1. Assume that $\|G\|_0(1)$ is finite, say $\|G\|_0(1) \le p^g$, $g \in \mathbb{N}$. Then for every $n \in \mathbb{N}$, one has the bound:

$$\|Y_n\| \le p^{g\nu} \cdot t(n)^\nu, \quad t(n) = \operatorname*{Max}_{1 \le m \le n} \frac{1}{|m|} \le n.$$

When $p \ge \mu$, this estimate is essentially best possible.

Corollary 1. For any $r \le s$ close enough to 1^-, one has the bound

$$(**) \qquad \mathrm{Max}(\|Y\|_0(r), \|Y^{-1}\|_0(r)) < p(\nu/e)^\nu \mathrm{Max}(1, \|G\|_0(s))^\nu (\log s/r)^{-\nu}.$$

In particular, if G is bounded and if $p \ge \mu$, then Y and Y^{-1} have logarithmic growth of rate $\mu - 1$ at the boundary.

In fact, it is possible to exhibit a bound similar to (**) (theorem 2) if, instead of assuming $G(0)$ to be nilpotent, we merely assume that its eigenvalues belong to $\mathbb{Q} \cap \mathbb{Z}_p$.

1.3. A logarithmic behaviour of rate $\mu - 1$ was first recognized by B. Dwork [6] in his early studies of Picard–Fuchs differential systems, as a consequence of the existence of a strong Frobenius structure, and was then expected in much more general situations. Without any Frobenius structure assumption, he and P. Robba succeeded in giving explicit growth estimates of the same kind in the case $C = 0$, using the very ingenious technique of "Frobenius factorization" [8]. Similar arguments allowed further A. Adolphson, Dwork and S. Sperber [1] to handle the extreme opposite case $C^{\mu-1} \neq 0$.

Since however this technique seemed to have reached its limits, the present author suggested to turn back to Dwork's earlier proof, using the existence of a weak Frobenius structure on every "reasonable" p–adic differential system, as established by C. Christol (cf. e.g. [5]); this idea provided a new proof in the case $C^{\mu-1} \neq 0$ [3]. The only difficulty in extending this argument was the lack of an effective version of the so–called cyclic vector lemma, i.e. the reduction of a differential system to a differential equation. This gap was fortunately filled in by Dwork [7], and a suitable refinement of his technique using again Frobenius transforms, (cf. proposition 2) allows to carry over the proof in the general case. Note however that in the special cases considered by Adolphson, Dwork, Robba and Sperber, their results are slightly more precise than ours.

1.4 One motivation in looking for explicit bounds lies in the global theory, i.e. when k is replaced by a number field K [4] [2]. Recall that for any Taylor series $M = \sum_{n\geq 0} M_n x^n$, where M_n are matrices with entries in K, one introduces the following basic invariants [2]:

$$\rho(M) = \sum_v \overline{\lim}\, \frac{1}{n} \log \mathrm{Max}(\|M_0\|_v,\ldots,\|M\|_v)$$

$$\sigma(M) = \overline{\lim} \sum_v \frac{1}{n} \log \mathrm{Max}(\|M_0\|_v,\ldots,\|M_n\|_v)$$

where the sum ranges over all places v of k, and $|\ |_v$ is normalized by $|x|_v = |x|_{\mathbb{Q}_p}^{-[K_v:\mathbb{Q}_p]/[K:\mathbb{Q}]}$ if $v|p$ (p prime or ∞), for every $x \in \mathbb{Q}$.

Thus $\exp(-\rho(M))$ is just the product of those of the radii of convergence of the v–adic Taylor series defined by M, which are smaller than one; while $\sigma(M) - \rho(M)$ measures the growth of a common denominator of $M_0,\ldots,M_n$, as $n \longrightarrow \infty$. In general there is no connection between $\rho(M)$ and $\sigma(M)$ (which may be infinite).

1.5. Let $Z = Y\,x^C$ be a solution of the differential system

(∗) $x d/_{dx}\, Z = GZ$,

and let us now simply assume that the entries of the $\mu \times \mu$ matrix Y (resp. G) are formal power series with coefficients in K (resp. in a subring of K finitely generated over $\mathbb{Z}$), that $\det Y(0) \neq 0$, and that C is a nilpotent constant matrix.

Corollary 2. With $M = (Y, Y^{-1})$, one has the inequality

$$\sigma(M) \leq \rho(M) + \mu - 1.$$

This result was already obtained via more global methods in [2] (erratum), cf. section 7 below.

Concerning the relation between ρ and σ, one may risk the following

Conjecture. Let $y \in K[[x]]$ be a G–function (i.e. $\sigma(y) < \infty$ and y satisfies a linear differential equation of order $\mu \geq 1$, $\mathcal{L}y = 0$ with coefficients in $K[x]$). Then

i) $\rho(y) \leq \sigma(y) \leq \rho(y) + \mu - 1$,

ii) $\sigma(y) - \rho(y) \in \mathbb{Q}$,

iii) if 0 is an ordinary point for $\mathcal{L}$, and if $\sigma(y) = \rho(y)$, then y is algebraic over $K(x)$.

1.6. I thank B. Dwork and G. Christol for their interest in the present work, which has benefitted very much from the influence of Christol's article [5] and Dwork's letter [7] (which contains a weaker form of lemma 4 below).

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2. Preliminary remarks

2.1 Until section 5, we shall relax condition c) into

d) the eigenvalues of C belong to $\mathbb{Z}_p$, and distinct eigenvalues remain distinct modulo $\mathbb{Z}$.

2.2. Since k is algebraically closed, one can find a sequence of elements $\alpha \in k$, $|\alpha| \longrightarrow 1^-$, and assume, after changing the variable $x \longmapsto \alpha x$, that the entries of G extend to analytic functions on the unit disk with boundary. Note also that $Y_G^{-1} = {}^tY_{-{}^tG}$, so that it is enough to deal with $Y = Y_G$. Let E_0 denote the completion of the ring of rational functions without pole in $D(0,1^-)$ with respect to the Gauss norm, and let E_0' be the quotient field of E_0. We assume henceforth that $G \in M_\mu(E_0)$. For any matrix F in $M_\mu(E_0')$, we shall set $\|F\| = \operatorname{Max}_{i,j} |F_{ij}|_{\text{Gauss}}$, so that $\|F\| = \|F\|_0(1)$ if $F \in M_\mu(E_0)$.

The map $x \longmapsto x^p$ extends to an isometry of $M_\mu(E_0')$ (not surjective), which we denote by ϕ. In order to avoid multiple superscripts, we put $\phi_q := \phi^\alpha$ if $q = p^\alpha$ (so that $\phi_1 = \text{id}$).

2.3. Because the eigenvalues of $G(0)$ belong to $\mathbb{Z}_p$, Dwork–Robba's theorem applies: the differential system $(*)$ admits an invertible analytic solution in the generic unit disk [8] (if $G \in M_\mu(k(x))$, a completely elementary proof of this fact is given in [2] V 6.1). If in addition $(*)$ is the differential system associated to a differential equation, the usual Dwork–Frobenius lemma (cf. e.g. [2] IV 2.1) implies that $\|G\| \leq 1$.

Lemma 1. If $G(0)$ is nilpotent and $\|G\| \leq 1$, then $\|Y_n\| \leq 1$ for $n < p$.

Proof: this follows immediately from the general explicit formula for Y_n:

$$Y_n = (\prod_{i,j<\mu} (\epsilon_i - \epsilon_j + n)^{-1}) . (\mathrm{Adj}\, U\, (G(0) + nI, G(0)))(\sum_{m<n} G_{n-m} Y_m) ,$$

where $U(D,C)$ denotes the linear endomorphism $B \longmapsto DB - BC$ of $M_\mu(k)$ (cf. [5] or [2] III (13)), and $\epsilon_0, \ldots, \epsilon_{\mu-1}$ denote the eigenvalues of $G(0)$.

2.4 The proof of our main results uses a cycle of transformations of the differential system (∗). Let us remind the formalism associated with such transformations (loc.cit.).
We write θ for the derivation xd/dx. Let $H \in GL_\mu(E_0')$, and define Z' by $Z = HZ'$. Then Z' satisfies the differential system

$$\theta Z' = H[G]Z' \quad \text{where} \quad H[G] := H^{-1}GH - H^{-1}\theta H.$$

Let us assume that $H[G]$ has no pole at 0, and that $H[G](0)$ satisfies condition d) (this is certainly the case if $H(0)$ is defined and invertible). Let also Y' denote the uniform part of Z', normalized by $Y'(0) = I$. Then $(HY')^{-1}Y \in GL_\mu(k[x,1/x])$.

Assume next that $H(0)$ is defined and invertible; then $Y = HY'H(0)^{-1}$. Furthermore H is invertible in $M_\mu(E_0)$ if and only if $H[G] \in M_\mu(E_0)$.

2.5. The proof will rely on the existence of a weak Frobenius structure, which we now recall.

Proposition 1 (Christol). In addition to a) b) d), assume that $||G|| \leq 1$ and that the eigenvalues of $G(0)$ belong to $p\mathbb{Z}_p$. Then there exists $A \in GL_\mu(E_0)$ such that

1) $A[G] = pF^\phi$, where $(F, Y_F, F(0))$ satisfies conditions a),b),d),
2) $Y_G = AY_F^\phi$
3) $||A|| = ||A^{-1}|| = 1$

cf. [5], or [2]V.

3. Cyclic vectors

3.1. Let V_q be a free $E_0^{\phi q}$–module with basis $e_0, \ldots, e_{\mu-1}$. We use this basis to identify the dual V_q^* of V_q with $\overset{\mu-1}{\Lambda} V_q$. This is an isometric identification if we put the obvious norms on V_q and V_q^*. There is a k–linear action of ∂ on V_q via coordinates.

Lemma 2. Let $\eta > 0$ and let $(w_i^*)_{i=0}^{\mu-1} \in V_q^*$. Assume that for every $v \in V$ such that $||v|| = 1$, one has $|| \sum (-1)^{\mu-1-i} w_i^* \partial^i v|| \leq \eta$. Then $||w_i^*|| \leq \eta q^i (\mu-1)!^{-1}$.

Proof: indeed, fix m and let $v = x^{q^\ell} e_m$ for $\ell = 0, \ldots, \mu-1$ successively. We get a linear system $\sum (-1)^{\mu-1-i} (w_i^* e_m) q^i \ell^i = \alpha_\ell$, $\ell = 0, \ldots, \mu-1$, with $\alpha_\ell \in E_0^{\phi q}$, $|\alpha_\ell| \leq \eta$.
By Cramer's rule, we find

$$q^i w_i^* e_m = \sum_\ell \pm \Big(\sum_{\substack{0 \leq m_j < \mu \\ m_j \neq \ell}} m_1 \ldots m_{\mu-1-i} \Big) \prod_{\substack{0 \leq k < j < \mu \\ k \neq \ell \\ j \neq \ell}} (j-k)\, \alpha_\ell\,,$$

whose norm is obviously bounded from above by

$$\eta \operatorname*{Max}_{\ell<\mu} |(\mu-1-\ell)!|^{-1}|\ell!|^{-1} \leq \eta((\mu-1)!)^{-1}.$$

Lemma 3. Let $F \in M_\mu(E_0^{\phi_q})$; we write $\Theta = \partial + {}^tF$ and, for any $w \in V_q$, let $m_i(w) = w \wedge \Theta w \wedge \ldots \wedge \Theta^{i-1}w \wedge \Theta^{i+1}w \wedge \ldots \wedge \Theta^{\mu-1}w$. Then for any $v \in V_q$, $\|v\| \leq 1$, one can write $\sum(-1)^{\mu-1-i}m_i^*(w)\Theta^i v = \sum(-1)^{\mu-1-i}\mu_i^*\partial^i v$, where

$$\mu_i^* = \sum_{i\geq j} A^{i,j}m_j^*,\ A^{i,j} \in M_\mu(E^{\phi_q}),\ A^{i,i} = I,\ A^{i,i-1} = -F,$$

and $\|A^{i,j}\| \leq \operatorname{Max}(1,\|F\|^{i-j})$.

Proof: by induction, it is clear that one may write $\Theta^i v$ in the form $\partial^i v + \sum_{j\leq i} (-1)^{i-j}({}^tA^{i,j})\partial^j v$, where the matrices $A^{i,j}$ enjoy all the properties being listed in the lemma. Next, one has $\sum_{i=1}^{\mu-1} (-1)^{\mu-i-1}m_i^*(w)\Theta^i v = \sum_{i=0}^{\mu-1}\sum_{j=0}^{i} (-1)^{\mu-j-1}(A^{i,j}m_j^*(w))\partial^j v$, which provides the desired expression for μ_i^*.

3.2. For any $w \in V_q$, we denote by B_w the element of $M_\mu(E_0^{\phi_q})$ defined by $(B_w)_{ij} = e_{i-1}^*(\Theta^{j-1}w)$.

Lemma 4. For every $\epsilon > 0$, there exists $w \in V_q$ such that

i) $\|w\| \leq 1$

ii) $\det B_w \neq 0$

iii) $|(B_w)_{ij}| \leq \mathrm{Max}(1, \|F\|^{i-1})$

iv) $|(B_w^{-1})_{ij}| \leq (1+\epsilon)((\mu-1)!)^{-1} q^{j-1} \mathrm{Max}(1, \|F\|^{\mu-j})$.

Remark: the last norm is the norm of an element of $E_0'^{\,\phi_q}$.

Proof: let $\eta = \underset{\substack{|w| \leq 1 \\ w \in V_q}}{\mathrm{Sup}} |\det B_w|$. The existence of w satisfying i) ii) iii) is obvious; moreover $\eta > 0$, and one may choose w so that

v) $\eta \leq (1+\epsilon) |\det B_w|$.

For every v in the unit ball of V_q, the map $x \longmapsto \det B_{w+x^q v}$ is analytic on $D(0,1^-)$ and bounded by η; therefore

$\frac{1}{q!} \frac{d^q}{dx^q} (\det B_{w+x^q v})|_{x=0} = \sum (-1)^{\mu-1-i} m_i^*(w) \theta^i v = \sum (-1)^{\mu-1-i} \mu_i^* \vartheta^i v$ is also bounded by η.

By lemma 2 $|\mu_i^*| \leq \eta((\mu-1)!)^{-1}$ and by lemma 3 $m_i^*(w) = \mu_i^* - \sum_{j<i} A^{i,j} m_j^*(w)$, which gives

vi) $\|m_i^*(w)\| \leq \eta((\mu-1)!|^{-1} \mathrm{Max}(1, \|F\|^{\mu-i-1})$.

Because $|(B_w^{-1})_{ij}| = \dfrac{|m_{j-1}^*(w)\, e_{i-1}|}{|\det B_w|}$, iv) now follows from v) and vi).

Let $w \in V_1$ be as in the previous lemma, and let Z be a complete solution of $\vartheta Z = FZ$. It is readily checked that the rows of $B_w Z$ are obtained by successive applications of ϑ. In other words, the differential system $\vartheta Z' = B_w^{-1}[F]Z'$ is associated to a differential equation. On the other hand, this system may acquire apparent singularities, and the behaviour at 0 is no longer clear. But applying Christol's decomposition into singular factors to the matrix $B_w \in M_\mu(E_0)$, one finds that there exists $\Gamma \in GL_\mu(k(x))$ such that

i) $\Gamma[B_w^{-1}[F]] = (B_w^{-1}\Gamma)\,[F] \in M_\mu(E_0)$

ii) $B_w^{-1}\Gamma \in GL_\mu(E_0)$

iii) $\|\Gamma\| = \|\Gamma^{-1}\| = 1$. cf. [5] or [2]V.

In applications, the quantity of importance is $\|B_w\| \cdot \|B_w^{-1}\|$; direct application of lemma 4 would give the upper bound $(1+\epsilon)((\mu-1)!)^{-1}\,\mathrm{Max}(1,\|F\|^{2\mu-2})$. As we shall next show, this bound can be substantially improved on by using ϕ^q.

3.4. Let q be some power of p not less than $\mathrm{Max}(1,\|F\|)$, and let $F' = qF^{\phi_q}$, so that $\|F'\| \leq 1$. Let $w' \in V_q$ and $B_{w'}$ be as in the previous lemma, with respect to F'. Write $w' = w^{\phi_q}$ for some $w \in V_1$.

<u>Lemma 5. If</u> B_w <u>denotes the matrix constructed in</u> 3.2, <u>one has the relation</u>:

$$B_w = \begin{bmatrix} 1 & & & \\ & q^{-1} & & \\ & & \ddots & \\ & & & q^{1-\mu} \end{bmatrix} B', \quad B'^{\phi_q} = B_{w'} \; .$$

Proof: Write Z for a complete solution of the differential system $\partial Z = FZ$, so that $Z' = Z(x^q)$ is a complete solution of $\partial Z' = F'Z'$. The i^{th} row of $B_w Z$ is ∂^{i-1} applied to the first row, i.e. $\partial^{i-1}(\sum_{j=1}^{\mu} e^*_{j-1}(w)Z_j)$. Therefore the i^{th} row of

$(B_w Z)(x^q) = B_w^{\phi_q} Z'$ is $p^{1-i}\partial^{i-1}((\sum_{j=1}^{\mu} e^*_{j-1}(w)Z_j)^{\phi_q}) = p^{1-i}\partial^{i-1}(\sum_{j=1}^{\mu} e^*_{j-1}(w')Z'_j)$,

that is the i^{th} row of $B_{w'}Z'$. Using lemma 4 (with $\|F'\| \leq 1$) and lemma 5, one finds the bounds $\|B_w\| \leq q^{\mu-1}$, $\|B_w^{-1}\| \leq (1+\epsilon)((\mu-1)!)^{-1}$. Note that one may choose $q < p\,\mathrm{Max}(1,\|F\|)$.

Summarizing, and unwinding our definitions, one finds:

<u>Proposition 2. Let</u> $F \in M_\mu(E_0)$, and let $p^a \geq \mathrm{Max}(1,\|F\|)$. <u>For every</u> $\epsilon > 0$, <u>there exists</u> $B \in M_\mu(E_0)$ <u>such that</u>:

1) det $B \neq 0$
2) $\|B\|\,\|B^{-1}\| \leq (1+\epsilon)p^{a(\mu-1)+\mathrm{ord}_p(\mu-1)!}$
3) <u>the matrix</u> $B^{-1}[F]$ <u>has the form</u>

$$\begin{bmatrix} 0 & 1_1 & & \\ & & \ddots & \\ & & & 1 \\ a_1 & a_2 & \cdots & a_\mu \end{bmatrix}, \quad a_j \in E_0'\,.$$

4 <u>Iterated weak Frobenius structure</u>

The next result extends proposition 1 to higher powers of ϕ, and may be of independent interest. Recall that $\nu = \mu-1 + \mathrm{ord}_p(\mu-1)!$

<u>Proposition 3. In addition to conditions</u> a)b)d), <u>assume that</u> $\|G\| \leq 1$ <u>and that the eigenvalues of</u> $G(0)$ <u>belong to</u> $q\mathbb{Z}_p$. <u>Then for every</u> $\epsilon > 0$, <u>there exists</u> $A \in GL_\mu(E_0)$ <u>such that</u>

1) $A[G] = qF^{\phi^q}$, <u>where</u> $(F, Y_F, F(0))$ <u>satisfies conditions</u> a),b),d),
2) $Y_G = AY_F^{\phi^q}A(0)^{-1}$
3) $\|A\|\cdot\|A^{-1}\| \leq (1+\epsilon)q^\nu$
4) $\|F\| \leq 1$.

Proof: Applying proposition 1 to G, we get $A_1, F_1 \in M_\mu(E_0)$ with $A_1[G] = pF_1^\phi$, $\|F_1\| \le p$; to F_1 we apply proposition 2 (with $1+\epsilon$ replaced by $(1+\epsilon)^{a^{-1}}$); we next get rid of the (apparent) singularities of $B_1^{-1}[F_1]$ according to 3.3. The resulting transformation matrix is $\Delta := A_1 B_1^{-1\phi} \Gamma_1^\phi \in GL_\mu(E_0)$.

If we set $G_1 := B_1^{-1}\Gamma_1[F_1]$, we have

$$G_1 \in M_\mu(E_0)\,,$$
$$\|G_1\| \le 1 \text{ (by the Dwork–Frobenius lemma)},$$
$$G_1^\phi = \Delta[G],$$
$$Y_G = \Delta Y_{G_1}^\phi \, \Delta(0)^{-1}\,,$$
$$\|\Delta\| \cdot \|\Delta(0)^{-1}\| \le \|\Delta\| \, \|\Delta^{-1}\| \le (1+\epsilon)^{a^{-1}} p^\nu .$$

The proposition follows after a–fold iteration.

5. Growth estimates (nilpotent case)

5.1. We now prove theorem 1.

Remind that $G(0)$ is assumed to be nilpotent, and that we have made the reduction 2.2. We first apply proposition 2 to G, $a = g$, and then decompose the resulting matrix B into singular factors (3.3). We obtain $Y = B^{-1}\Gamma \, {}_{B^{-1}\Gamma[G]}(B^{-1}\Gamma)(0)^{-1}$, $\|B^{-1}\Gamma[G]\| \le 1$ (Dwork-Frobenius) and $\|B^{-1}\Gamma\| \cdot \|(B^{-1}\Gamma)^{-1}(0)\| \le \|B^{-1}\Gamma\| \cdot \|(B^{-1}\Gamma)^{-1}\| \le (1+\epsilon)p^{g\nu}$. A fortiori $\|Y_n\| \le (1+\epsilon)p^{g\nu} \max_{m \le n} \|(Y_{B^{-1}\Gamma[G]})_m\|$, and by letting $\epsilon \longrightarrow 0$ and replacing G by $B^{-1}\Gamma[G]$ we are reduced to the case $\|G\| \le 1$.

The conditions of proposition 3 are then fullfilled for any choice of $q = p^a$. We fix $n \geq 0$, and choose $a = \mathrm{ord}_p n$, so that $q = t(n)$ with the notation of theorem 1.

Let us write, according to the latter proposition:

$$Y = AY_F(x^q)A(0)^{-1},$$

with $\|F\| \leq 1$, $\|A\| \, \|A(0)^{-1}\| \leq (1+\epsilon)t(n)^{\nu}$.

Equating the n^{th} Taylor coefficient in both sides, one sees immediately that only the $p-1$ first Taylor coefficients of Y_F are involved.

Using lemma 1, one deduces that $\|Y_n\| \leq (1+\epsilon)t(n)^{\nu}$, and it only remains to let $\epsilon \longrightarrow 0$.

5.2. Let us deduce corollary 1 from this. At a first step, assume that G is bounded. Then the inequality similar to (**) but with $\|G\|_0(1)$ instead of $\|G\|_0(\nu)$ in the right hand side is a consequence of the following elementary lemma (whose proof is left to the reader):

<u>Lemma 6</u>. <u>For any matrix</u> U <u>with entries analytic on</u> $D(0,1^-)$, <u>and any positive constant</u> κ <u>the following are equivalent</u>:

a) <u>for every</u> $r \in [e^{-\kappa/p}, 1[$, $\|U\|_0(r) \leq \kappa^{\nu}(\log 1/r)^{-\nu}$
b) for every $n \geq \nu p \kappa$, $\|U_n\| \leq (\frac{e\kappa}{\nu})^{\nu} n^{\nu}$.

The general case follows after a change of variable $x \longmapsto \beta x$, $|\beta| \longrightarrow s$.

6. Growth estimates (case of rational exponents)

6.1. We now assume that $(G,Y,G(0))$ satisfies conditions a) b) d), and that the eigenvalues of $G(0)$ are rational numbers.

We assume moreover that $G(0)$ is in Jordan normal form and that $\|G\|_0(1) \leq 1$.

6.2 By the same reduction as in 2.2., one may assume moreover that $G \in M_\mu(E_0)$. In order to apply proposition 3, one has to use shearing transforms to reduce to the case where the diagonal elements of $G(0)$ are divisible by q.

Explicitly, let D be a diagonal matrix with integer entries $0 \leq \Delta_{ii} < q$, such that D acts as a homothety on the Jordan blocks of $G(0)$, and such that
$\eta_i = (D + G(0))_{ii} \in q\mathbb{Z}_p \cap [0,q[$ for $i = 1,\ldots,\mu$, and distinct η_i's remain distinct mod $\mathbb{Z}$. Because D and $G(0)$ commute, $D + G(0) = H[G](0)$, where $H = x^{-D}$. Moreover $Y = x^{-D}Y_{H[G]}x^D$, $H[G] \in M_\mu(E_0)$, $\|H[G]\| \leq 1$. However reasoning coefficientwise as before is not allowed because the letter formula involves negative powers of x; but note that

i) $\|Y\|_0(r) < r^{-q}\|Y_{H[G]}\|_0(r)$ for $0 < r < 1$.

6.3. One may now apply proposition 3 to $H[G]$:

$$Y_{H[G]} = AY_F^{\phi_q}A(0)^{-1}, \quad \|F\| \leq 1, \quad F(0) = \frac{1}{q}\,A(0)^{-1}H[G](0)A(0)\,,$$

ii) $\|Y_{H[G]}\|_0(r) \leq (1+\epsilon)q^\nu\|Y_F\|_0(r^q)$.

On the other hand, the explicit formula for Y_{F_m} being stated in the proof of lemma 1, together with the formula $\prod_{n\leq m}(\xi+n)^{-1} = \frac{1}{m!}\sum_{n\leq m}\binom{m}{n}\frac{(-1)^j}{\xi+n}$, shows that

iii) $\|Y_F\|_0(r^q) \leq \operatorname{Sup}_{m\geq 0}\big(\operatorname{Max}_{n\leq m,\ i,j\leq\mu}\prod|\epsilon_i-\epsilon_j+n|^{-1}\big)(r^q p^{\mu^2/p-1})^m$, where ϵ_i, $i = 0,\ldots,\mu-1$ denote the eigenvalues of $F(0)$.

Note that $\epsilon_i = \eta_i/q$, and let N denote a <u>positive common denominator</u> for the η_i. Because of our choice of D, one has the bound:

iv) $\operatorname{Max}_{n\leq m}\prod_{i,j<\mu}|\epsilon_i - \epsilon_j + n|^{-1} \leq N^{\mu^2}(1+m)^{\mu^2}$.

6.4. Let us fix $\eta \in]0,1[$; assuming that $r \geq \eta^{\mu^2}$, let q be the minimal (integral) power of p such that $r^q \leq \eta^{\mu^2}\cdot p^{-\mu^2/p-1}$. We set $\tau = \tau(\eta) = \operatorname{Sup}_{m\geq 0}(1+m)\,\eta^{m-1}$.

Combining iii and iv, we find

v) $\|Y_F\|_0(r^q) \leq (N\eta\tau)^{\mu^2}$,

and putting i) ii) v) together, and letting $\epsilon \longrightarrow 0$:

vi) $\|Y\|_0(r) < r^{-q}q^{\nu}\cdot(N\eta\tau)^{\mu^2} \leq (p^{p/p-1}\cdot\tau N)^{\mu^2}q^{\nu}$.

At least, notice that $q < (\mu^2/p)(\log 1/\eta + \frac{\log p}{p-1})(\log 1/r)^{-1}$. Therefore, we have proved the following result.

<u>Theorem 2. Let</u> G,Y <u>be as in</u> 6.1, <u>and let</u> N <u>be as before a positive common denominator for the exponents</u>. <u>We fix</u> $\eta \in]0,1[$. <u>Then for every</u> $r \in]\eta^{\mu^2},1[$, <u>one has the bound</u>:

$$(***) \qquad \mathrm{Max}(\|Y\|_0(r), \|Y^{-1}\|_0(r)) < \kappa(\log 1/r)^{\nu},$$
$$\kappa = (\mu^2 p^{-1}(\log 1/\eta + (p-1)^{-1}\log p))^{\nu}(p^{p/p-1}\cdot\tau(\eta)\cdot N)^{\mu^2}.$$

Remark: a simple application of corollary 1, after the preliminary change of variable $x \longmapsto x^N$, would give a logarithmic growth $N\nu$ instead of ν.

7 Remarks on the global case.

We indicate how to deduce corollary 2 from theorem 1. First of all, we have the inequality of convergence radii: $R_v(G) \leq R_v(Y,Y^{-1})$. On the other hand, because the entries of G are "globally bounded", one has $R_v(G) = 1$, $\|G\|_0(1) \leq 1$, for almost every place v. For each of these places, one may apply theorem 1 after some change of variable $x \longmapsto \alpha x$, $|\alpha| \to R_v(Y,Y^{-1})^{-}$.

One gets $\sigma(Y,Y^{-1}) \leq \rho(Y,Y^{-1}) + \overline{\lim_n} \sum_{\substack{v \\ p(v)\leq n}} \frac{1}{n}\log t(n)^{v_v} = \rho(Y,Y^{-1}) + \mu-1$.

(among the places for which $\|G\|_0(1) \leq 1$, only those whose residual characteristic is less than $n+1$ can provide a positive summand $\log |Y_n|_v$).

Note that this corollary was already obtained in [2]V (erratum). In fact, the term $(\mu-1)(\mu^2+1)$ which the main text gives in place of $\mu-1$ can be reduced to $\mu-1$, just by replacing the lines 8–22 on p.108 by: "${}^t(H_v^{\mathrm{inv}*})^{-1}H^{\mathrm{cores}\phi} \in GL_\mu(E_0,v)$; it follows that (6.4.3.) : $h_n(Y_{F_v}) \leq \frac{\mu-1}{n}\log|p|_v^{-1} + \frac{1}{p}h_{[n/p]}(Y_{H_v}{}^{\mathrm{cores}}{}_{[E_v]})$."

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