Flowers on Riemannian manifolds.

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Abstract

In this paper we will present two upper bounds for the length of a smallest "flower-shaped" geodesic net in terms of the volume and the diameter of a manifold. Minimal geodesic nets are critical points of the length functional on the space of graphs immersed into a Riemannian manifold. Let M^n be a closed Riemannian manifold of dimension n. We prove that there exists a minimal geodesic net that consists of one vertex and at most $2n-1$ geodesic loops based at that vertex of total length $\leq 2n!d$, where d is the diameter of M^n. We also show that there exists a minimal geodesic net that consists of one vertex and at most $3^{(n+1)^2}$ loops of total length $\leq 2(n+1)!^2 3^{(n+1)^3} FillRadM^n \leq 2(n+1)!^{\frac{5}{2}} 3^{(n+1)^3}(n+1)n^n vol(M^n)^{\frac{1}{n}}$, where $FillRadM^n$ denotes the filling radius and $vol(M^n)$ denotes the volume of M^n.

Introduction

Question 1. In 1983 M. Gromov asked whether there exists a constant $c(n)$ such that the length of a shortest closed geodesic, $l(M^n)$, on a closed Riemannian manifold M^n is bounded above by $c(n)vol(M^n)^{\frac{1}{n}}$, where $vol(M^n)$ is the volume of M^n, (see [G]).

A similar question can be asked about the relationship between $l(M^n)$ and the diameter d of the manifold. Namely on can ask the following
Question 2. Is there a constant $\tilde{c}(n)$, such that $l(M^n) \leq \tilde{c}(n)d$?

The answers to these questions are still unknown.

In this paper we prove an analog of these inequalities for geodesic flowers.

Definition 0.1 *A* **minimal geodesic flower** *is a bouquet of (finitely many) geodesic loops based at the same point which satisfies the following* *stationarity condition: the sum of unit vectors at the base point tangent to all geodesic arcs of the flower and directed from the base point equals to zero. A minimal geodesic flower that has less than or equal to m loops will also be called a minimal geodesic m-flower, (see fig. 1 (b)). Note that we do not require that all of these m loops are distinct. In other words, distinct loops can have positive integer multiplicities, but the sum of the multiplicities has to be at most m.*

Note that closed geodesics are minimal geodesic flowers and that geodesic loops are minimal geodesic flowers if and only if they are closed geodesics. The stationarity condition implies that minimal geodesic flowers are stationary 1-varifolds (from the perspective of the geometric measure theory). This means that if X is a smooth vector field on the manifold, Φ_t^X is the corresponding 1-parameter flow of diffeomorphisms and F is a minimal geodesic flower, then $t = 0$ is a critical point for the function $L(t)$ defined as the length of $\Phi_t^X(F)$.

Minimal geodesic flowers are comparatively rare. For a generic analytic closed Riemannian manifold the set of minimal geodesic flowers is countable.

In this paper we prove the following theorems:

Theorem 0.2 *Let M^n be a closed Riemannian manifold of dimension n and of diameter d. Let q be the smallest number such that $\pi_q(M^n) \neq \{0\}$. Then there exists a non-trivial geodesic flower F on M^n, consisting of at most $2q-1$ geodesic loops, at most q of which are distinct, such that the length of F is bounded above by $2q!d \leq 2n!d$.*

Theorem 0.3 *Let M^n be a closed Riemannian manifold of dimension n. Then there exists a non-trivial geodesic flower F that consists of at most $3^{(n+1)^2}$ geodesic loops, at most $\frac{n^2+3n}{2}$ of which are distinct, of total length $\leq 2(n+1)!^{\frac{5}{2}} 3^{(n+1)^3}(n+1)n^n vol(M^n)^{\frac{1}{n}}$, where $vol(M^n)$ is the volume of M^n.*

This paper is a continuation of our work on geodesic nets , [NR1], [NR2], [R]. In [NR1] we have obtained curvature-free upper bounds for the length of a smallest stationary 1-cycle in terms of the volume and in terms of the diameter of a manifold. Stationary 1-cycles are critical points of the length functional on the space of integral 1-cycles. They can be viewed as homological analogs of closed geodesics. In [NR2] and [R] we found similar estimates for the length of a smallest stationary geodesic net of a particular shape that was called a stationary m-cage. In fact, geodesic flowers form a subclass of a class of minimal cages. Below is a rigorous definition of minimal geodesic nets, stationary 1-cycles and minimal cages.

Definition 0.4 *(a) We define a minimal (or stationary) geodesic net* Γ *to be a graph immersed into a Riemannian manifold* M^n *satisfying the following two conditions:*
(1) each edge of Γ *is a geodesic segment;*
(2) the sum of unit vectors at each vertex tangent to the edges and directed from this vertex equals to zero.

(b) If, in addition, all the vertices of Γ *have even degrees then* Γ *is called a stationary* 1*-cycle.*

(c) If a minimal geodesic net Γ *has* 2 *vertices joined by at most* m *segments, (counted with multiplicities), or if* Γ *is a minimal geodesic flower, it is called a minimal geodesic* m*-cage, (or just a minimal geodesic cage), (see fig. 1).*

(d) A (not necessarily minimal or geodesic) immersion of a graph will be called a net. Also, nets that consist of a vertex together with at most m *(not necessarily geodesic) loops based at that point will be referred to simply as flowers and nets that are made of two vertices connected by at most* m *(not necessarily geodesic) segments or nets that are* m*-flowers will be referred to as* m*-cages, (or cages).*

Note that in our definition of a graph we allow it to have loops and multiple edges between vertices. This object is sometimes referred to as a multigraph.

It is easy to see that the conditions of Definition 0.4 ensure that a minimal geodesic net is a stationary point for the length functional on the space of immersed graphs, where the length of a graph is defined as a sum of lengths of the edges and the length of each edge is taken with the multiplicity corresponding to the multiplicity of the edge.

Let us now briefly describe the main idea of the proofs of Theorems 0.2 and 0.3. In [R] we obtained volume and diameter estimates for the length of a shortest minimal geodesic m-cage.

Note, that if one applies a length shortening process to a (non-degenerate) m-cage, it is possible for it to degenerate into a flower. That is, the length of one of its edges can become zero, and the two vertices will then coincide.

The new idea is that we can define a weighted length functional on the space of cages such that its gradient flow will “force” critical cages to

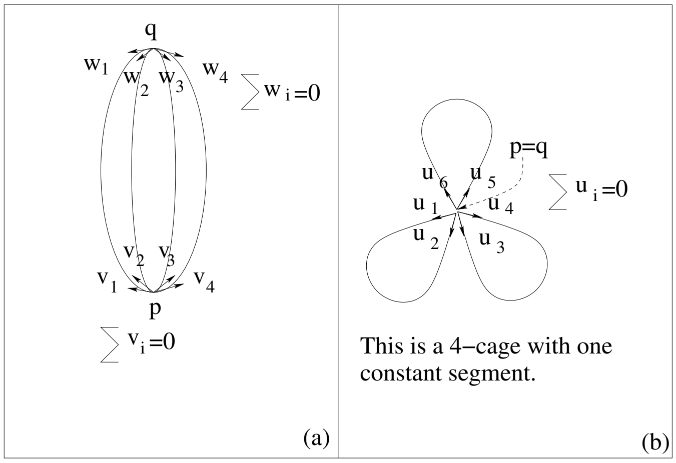

Figure 1: A non-degenerate stationary 4-cage and minimal geodesic 3-flower.

degenerate into geodesic flowers. In other words, it can be arranged so that a (non-degenerate) stationary m-cage is not a critical point of the new functional.

Example. Let us consider the space of 3-cages and let Γ be an element of this space. That is Γ is a graph with two vertices p and q and three edges e_1, e_2, e_3. Define $L(\Gamma) = length(e_1) + length(e_2) + 3length(e_3)$. Then a non-degenerate 3-cage can not be a critical point of L. Indeed, one of the conditions for it to be critical is that $v_1 + v_2 + 3v_3 = 0$, where v_1, v_2, v_3 are unit vectors tangent to e_1, e_2, e_3 respectively at p. Obviously, this condition cannot be satisfied. Therefore, if Γ is critical then one of the e_i's degenerates into a point and the cage degenerates into a flower.

We then combine this idea with the techniques of [R] that will be explained in the next section. Let us describe this more formally.

Definition 0.5 *(1) Let Γ be a (not necessarily geodesic) net with edges $e_1, ..., e_i, ..., e_k$. Then $L(\Gamma) = \Sigma_{i=1}^{k} m_i length(e_i)$, where $m_i \in \mathbf{Z}_+$ and $length(e_i)$ is the length of the edge e_i will be called a weighted length functional with weights $m_1, ..., m_k$. (Note that it corresponds to the regular length functional on the net, where each edge e_i is taken with a multiplicity m_i.)*
(2) A net N is critical with respect to a weighted length functional L with weights $m_i, i = 1, ..., k$ if for any one-parametric smooth flow of diffeomorphisms $\Phi_t, t = 0$ is a critical point of $\mu(t) = L(\Phi_t(N))$. It is equivalent to all edges being geodesic segments combined with the following stationarity

condition satisfied at every vertex of N: the weighted sum of unit vectors tangent to edges of N at that vertex and directed from it, equals to zero.

In this paper we will use certain specific generalized length functionals. We will establish the existence of their non-trivial critical points which are bouquets of geodesic loops of total length $\leq \tilde{c}(n)d$ and of total length $\leq c(n)vol(M^n)$, where d is the diameter and $vol(M^n)$ is the volume of M^n, using techniques of [R], modified Gromov's extension technique appearing in [G] and the idea illustrated by the above example. Although their critical points are not necessarily critical points of the length functional, one can make them into such by taking some of the loops with appropriate integer weights. Indeed, observe that if a geodesic flower F that consists of k distinct loops $e_1, ..., e_k$ is a critical point for the weighted length functional with weights $m_1, ..., m_k$ then a geodesic flower $\tilde{F}$ that consists of the geodesic loops e_i taken with multiplicities m_i, $i = 1, ..., k$ will be a minimal geodesic flower, that is a critical point for the regular length functional. This observation will be used throughout the paper.

1 The proof of Theorem 0.2.

Proof. The theorem will be proved by contradiction. Let M^n be a closed Riemannian manifold, such that $\pi_1(M^n) = ... = \pi_{q-1}(M^n) = \{0\}$ and $\pi_q(M^n) \neq \{0\}$. Let $f : S^q \longrightarrow M^n$ be a non-contractible map of a finely triangulated sphere to M^n. Assuming there are no "small" minimal geodesic flowers, we will extend this map to the disc D^{q+1} of dimension $q+1$, thus reaching a contradiction. To construct this extension, we will triangulate the disc as the cone over the chosen triangulation of the sphere. The procedure will then be inductive on skeleta of D^{q+1}. To begin with, the center of the disc will be mapped to an arbitrary point in M^n and the edges will be mapped to minimal geodesic segments that connect this point with corresponding vertices of the triangulation of the image sphere. The rest of the extension procedure reduces to "filling" m-cages by m-discs for all values of $m \leq q+1$, which is an inductive bootstrap procedure similar to the one used in [R]: Assuming that we have extended our map to the k-skeleton, we will explain how to extend it to the $(k+1)$-skeleton of D^{q+1}. In order to do that we will extend f to each $(k+1)$-dimensional simplex of D^{q+1} and in order to extend to a $(k+1)$-simplex it will be necessary to "fill" $(k+1)$-cages by $(k+1)$-dimensional discs. That is, consider the image of the boundary of the above simplex. It consists of $k+2$ k-dimensional simplices, one of which is so small that it can be treated as a point, (see remark on page 13 in [R], The idea is that we can contract this simplex to a point over itself, reducing our situation to the situation, where the simplex is treated as a point). Consider a $(k+1)$-cage Cg that corresponds to 1-skeleton of this simplex and apply a weighted length shortening process, where the weighted length functional is taken with weights $m_1 = ... = m_k = 1$, $m_{k+1} = k$.

Observe that in the absence of "small" minimal geodesic flowers, there will be no "small" critical points for the weighted length functional. As the result we can introduce a "weighted" version of the Birkhoff curve shortening process that will deform the space of all "small" flowers to the space of constant flowers, (see [C] for the detailed description of the Birkhoff curve shortening process). It can be formally defined as the usual Birkhoff-like 1-cycles shortening process described in a much more general situation in [NR1] and [R], but applied to the "weighted" flower, (i.e. the flower in which each loop is taken with the appropriate multiplicity).

Then the cage can be contracted to a point along a 1-parameter family of cages Cg_τ, $\tau \in [0,1]$ of smaller weighted length. Here we use the assumption that there are no "small" geodesic flowers. We can now construct a 1-parameter family of spheres S^k_τ of dimension k that starts with the image of the boundary of the given simplex and ends with a point, and thus generates a $(k+1)$-dimensional disc. Spheres are constructed by the procedure of "filling" cages Cg_τ at each τ described in [R]. That is given a $(k+1)$-cage Cg_τ, consider its $k+1$ "k-subcages", i.e. k-tuples $(e_1)_\tau, ..., (\hat{e}_j)_\tau, ..., (e_{k+1})_\tau$ obtained by ignoring one of the curves. By induction assumption, each of these subcages can be "filled" by discs of dimension k. (The base of induction is proved by contracting 2-cages, i.e. closed curves, by the usual Birkhoff curve shortening process. At this point we are using the assumption that there are no short periodic geodesics.)

We then glue these $(k+1)$ discs as in the boundary of $(k+1)$-simplex to obtain S^k_τ. It is important to note that this process is continuous with respect to Cg_τ. This one-parameter family of spheres generates the desired $(k+1)$-dimensional disc that can be used to extend f.

The factorial dependence on q arises as follows. During the shortening of the weighted length of the $(k+1)$-cage it might happen that at $\tau = \tau_*$ the length of every edge in the cage but one becomes very small, while the length of that remaining edge becomes large. Although, it is large, it cannot be larger than $2kd$. This inequality holds because the weighted length is initially $\leq 2kd$ and is decreasing. Now recall that to construct a sphere "filling" this cage we need to consider $k+1$ k-subcages and construct discs that "fill" them. Without loss of generality, consider the subcage that is formed by the edges $e_1, ..., e_k$. To construct a disc we need to apply weighted length shortening with coefficients $1, 1, ..., 1, k-1$. The worst case scenario is when the length of e_k is almost $2kd$ and the lengths of $e_1, ..., e_{k-1}$ are very small. More formally, let l_i denote the length of $e_i, i = 1, ..., k+1$. Then $l_k = 2kd - l_1 - l_2 - ... - kl_{k+1}$ and the total weighted length of the above k-cage is at most $2(k-1)kd - (k-2)(l_1 + ... + l_{k-1}) - (k-1)kl_{k+1} \leq 2(k-1)kd$. If we continue with this reasoning until we get to the 2-cages, we will get the factorial bound $2q!d$. $\square$

Example. To illustrate the proof of Theorem 0.2 assume $q = 2$. Let us begin with a non-contractible map $f : S^2 \longrightarrow M^n$. The proof will be by contradiction. We will assume that there is no 2-flowers on M^n of length $\leq 4d$, where d is the diameter of M^n and will show that in this case we can, indeed, extend f to D^3 triangulated as a cone over S^2. The extension procedure will be inductive on skeleta of S^2.

Let us begin with the **0-skeleton** of D^3 that consists of the center of the disc $\tilde{p}$. It will be mapped to an arbitrary point $p \in M^n$.

Next, let us extend to the **1-skleton** of D^3. For that, consider an edge $[\tilde{p}, \tilde{v}_i]$. Let us map it to a shortest geodesic segment $[p, v_i = f(\tilde{v}_i)]$. Its length is, of course, $\leq d$.

To extend to the **2-skeleton** consider an arbitrary 2-simplex of the form $[\tilde{p}, \tilde{v}_{i_1}, \tilde{v}_{i_2}]$. Its boundary is mapped to a closed curve of length $\leq 2d + \delta$. Assuming there is no closed geodesics of length $\leq 2d + \delta$, this curve is contractible to a point by the Birkhoff curve shortening process. We will map the 2-simplex to the disc $[p, v_{i_1}, v_{i_2}]$ generated by the curve shortening homotopy.

Finally, let us extend to the **3-skeleton**. In order to do that, consider an arbitrary 3-simplex $[\tilde{p}, \tilde{v}_{i_1}, \tilde{v}_{i_2}, \tilde{v}_{i_3}]$. The image of its boundary is a 2-sphere glued from four 2-simplices, one of which is very small. It is, in fact, so small, that for the sake of the exposition we propose here to treat it as a point that we will denote by q. So, this sphere is formed by connecting two points p and q by geodesic segments e_1, e_2, e_3 and contracting each pair of closed curves to a point. Note also, that this construction provides us with a natural cell decomposition of this sphere into two 0-cells: namely points p, q, three 1-cells: e_1, e_2, e_3 and three 2-cells. Consider the 1-skeleton of this sphere under this decomposition. It is a net.

Now we will apply a weighted length shortening process to this net. That is, we will shorten the weighted length $l(e_1) + l(e_2) + 2l(e_3)$. It is the same as shortening the length of a net, in which e_1 and e_2 are taken with multiplicity 1 and e_3 with the multiplicity 2. Critical points of this length shortening process are of the following shape: (a) stationary figure 8; (b) a closed geodesic. It is easy to see that a geodesic net consisting of two vertices p and q, three **distinct** non-trivial geodesic segments connecting them and a stationary condition at the vertices $v_1 + v_2 + 2v_3 = 0$ cannot be a critical point, since it can only be satisfied if e_1 and e_2 coincide. Thus, if we assume that there is no stationary figure 8 or closed geodesic of length $\leq 4d$ then the 1-skeleton can be contracted to a point, (see fig. 2 (a)). In the process we obtain a 1-parameter family of nets that we will denote Cg_τ, which can be extended to 1-parameter family of spheres S^2_τ, that begins with the original sphere and ends with a point, (see fig. 2 (b)). S^2_τ is constructed as follows: at each time τ we consider three pairs of curves and contract them to a point without the length increase. It can and will happen at some point that the length of one or two segments will decrease to zero and segments themselves will degenerate to points, (see fig. 2). In that case, we can still consider three pairs of curves, where one of the curves in two pairs will be a constant curve. We then fill each of these three pairs of curves by discs as we did above when we were extending to the 2-skeleton, i. e. using the curve shortening process. These 3 2-discs glued together form S^2_τ. Thus, if there is no geodesic flowers of length $\leq 4d$ then we can extend our map f to the 3-skeleton of D^3, reaching a contradiction.

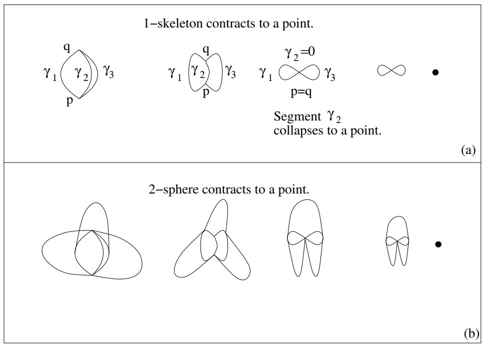

Figure 2: Deforming a 2-sphere to a point

2 The proof of Theorem 0.3.

Note that by similar methods one can also prove Theorem 0.3. Let us begin by stating the definition of the Filling Radius defined by M. Gromov in [G].

Definition 2.1 **[G]** *Let M^n be an abstract manifold and let $X = L^\infty(M^n)$ be the Banach space of bounded Borel functions f on M^n. Let M^n be isometrically imbedded in X, where the imbedding of M^n into X is the map that assigns to each point p of M^n the distance function $p \longrightarrow f_p = d(p,q)$. Then the filling radius $FillRadM^n$ is the infimum of $\varepsilon > 0$, such that M^n bounds in the ε-neighborhood $N_\varepsilon(M^n)$, i.e. homomorphism $H_n(M^n) \longrightarrow H_n(N_\varepsilon(M^n))$ vanishes, where $H_n(M^n)$ denotes the singular homology group of dimension n with coefficients in $\mathbf{Z}$, when M is orientable, and with coefficients in $\mathbf{Z}_2$, when M is not orientable.*

Alternatively, one can give a different definition of the filling radius of M^n by defining first $FillRad(M^n \subset X)$, the filling radius of M^n isometrically imbedded into some metric space X, as the smallest ε, such that M^n bounds in the ε-neighborhood of M^n and then taking the infimum over all of the isometric imbeddings. It was shown by M. Katz that $FillRadM^n \leq \frac{d}{3}$, where d is the diameter of M^n, (see [K]).

In [G] M. Gromov had found an estimate for the filling radius of a closed Riemannian manifold in terms of the volume of this manifold.

Theorem 2.2 **[G]** *Let M^n be a closed connected Riemannian manifold. Then $FillRadM^n \leq gc(n)(vol(M^n))^{\frac{1}{n}}$, where $gc(n) = (n+1)n^n(n+1)!^{\frac{1}{2}}$ and $vol(M^n)$ denotes the volume of M^n.*

In this section we will prove the following

Theorem 2.3 *Let M^n be a closed Riemannian manifold. Then there exists a minimal geodesic flower that consists of at most $3^{(n+1)^2}$ geodesic loops of total length $2(n+1)!3^{(n+1)^3}FillRadM^n$.*

Theorem 2.3 combined with Theorem 2.2 leads to the volume bound for the length of a smallest minimal geodesic flower as it was stated in Theorem 0.3.

The proof of Theorem 2.3 is based on the combination of the ideas from the proof of Theorem 0.2 and an adaptation of the trick by M. Gromov from [G] involving filling M^n by a polyhedron W^{n+1} in $L^\infty(M^n)$, attempting to extend the identity map on M^n to W^{n+1} and obtaining a geodesic flower as an obstruction to this extension.

The details of the proof of Theorem 2.3 are very similar to that of Theorem 0.2, except that instead of contracting m-cages, we will be contracting 1-skeletons of simplices. The spheres and discs are then built out of those 1-skeletons in a similar fashion. Also, for each n the weighted length functional applied to 1-skeletal net will be $\Sigma_{i=1}^{(n+1)(n+2)/2} 3^{i-1} length(e_i)$, where e_i's are the edges of 1-skeleton of an $(n+1)$-dimensional simplex, satisfying the condition that the edge e_1 is coming out of the vertex w_0, edges e_2, e_3 are coming out of the vertex w_1, edges e_4, e_5, e_6 are coming out of the vertex w_3, etc. Finally, the last $(n+1)$ edges are coming out of the same vertex w_{n+1}. This is the functional that will force a net to degenerate into a flower, which will be proven in the Merging Lemma below.

Proof of Theorem 2.3. Let us begin by assuming that the length of a shortest minimal geodesic flower is "large".

By the definition of the Filling Radius of M^n, M^n bounds in the $(FillRadM^n + \delta)$-neighborhood of M^n in $L^\infty(M^n)$. We can assume that it bounds a polyhedron W^{n+1}, (see [G]). That is, $\partial W^{n+1} = M^n$, when M^n is orientable and $\partial W^{n+1} = \partial M^n$ mod 2, when M^n is not orientable and W^{n+1} lies in the $(FillRadM^n + \delta)$-neighborhood of M^n.

Let W^{n+1} and M^n be triangulated in such a way that the diameter of any simplex in this triangulation is smaller than some small $\delta > 0$.

Assuming there is no "small" minimal geodesic flowers one can construct an extension of the identity map $id : M^n \longrightarrow M^n$ to W^{n+1}, thus reaching a contradiction.

This extension is constructed by induction on the dimension of skeleta of W^{n+1}.

Let us begin with the **0-skeleton**. Each vertex $\tilde{w}_i \in W^{n+1}$ will be mapped to a vertex $w_i \in M^n$, that is closest to $\tilde{w}_i$. Thus, $d(\tilde{w}_i, w_i) \leq FillRadM^n + \delta$.

Next, we will extend to the **1-skeleton**. Consider an edge of the form $[\tilde{w}_i, \tilde{w}_j] \subset W^{n+1} \backslash M^n$. It will be mapped to a minimal geodesic segment $[w_i, w_j]$ that connects w_i and w_j of length $\leq 2FillRadM^n + 3\delta$.

Now, let us go to the **2-skeleton**. Let $\tilde{\sigma}^2_{i_0,i_1,i_2} = [\tilde{w}_{i_0}, \tilde{w}_{i_1}, \tilde{w}_{i_2}]$ be an arbitrary 2-simplex. Its boundary is mapped to a closed curve of length $\leq 6FillRadM^n + 9\delta$. The assumption about all flowers being small implies, in particular that there are no closed geodesics of smaller length, therefore, we can contract this curve to a point along the curves of smaller length.

Moreover, the absence of "short" periodic geodesics implies that this curve shortening homotopy can be arranged to depend continuously on a curve. We will map $\tilde{\sigma}^2_{i_0,i_1,i_2}$ to a surface that is generated by the above homotopy, denoted as $\sigma^2_{i_0,i_1,i_2}$.

Next let us go to the **3-skeleton**. Consider an arbitrary 3-simplex $\tilde{\sigma}^3_{i_0,i_1,i_2,i_3} = [\tilde{w}_{i_0}, ..., \tilde{w}_{i_3}]$. By the previous step of the induction, its boundary is mapped to the following chain: $\Sigma^3_{j=0}(-1)^j \sigma^2_{i_i,...,\hat{i}_j,...,i_3}$. Consider its 1-skeleton. It will be a net, that we will denote by K_i. Let us apply weighted length shortening process for nets to continuously deform it to a point. The weighted length of K_i is defined as $\Sigma^6_{j=1} 3^{j-1} length(e_j)$, where e_j is an edge of the 1-skeleton. (We will not explicitly describe this length shortening process, but it can be found in [NR1] and it is very similar to the length shortening process for m-cages in [R]).

We **claim** that only minimal geodesic flowers can be minimal geodesic nets with such a functional. This follows from the Merging Lemma below, but in order to better illustrate what can happen during the considered process we will present here a more transparent proof that works in this simple case.

Proof of the claim.

Case 1. (See fig. 3 (a)). First we will show that a non-degenerate 1-skeleton cannot be a minimal geodesic net with respect to the above functional. Let v_{ij} denote the unit vector tangent to e_i at w_j. Stationarity conditions at vertices $w_0, ..., w_4$ imply

(1) $v_{10} + 3^2 v_{30} + 3^3 v_{40} = 0$;

(2) $v_{11} + 3v_{21} + 3^4 v_{51} = 0$;

(3) $3v_{22} + 3^2 v_{32} + 3^5 v_{62} = 0$;

(4) $3^3 v_{43} + 3^4 v_{53} + 3^5 v_{63} = 0$.

These conditions cannot be satisfied unless each vertex merges with some other vertex, which leads to configurations on fig. 3 (b), (c), (d).

Case 2. Let us consider the configuration depicted on fig. 3 (b). There vertex w_0 merges with w_1 and w_2 merges with w_3. The stationarity condition at $w_0 = w_1$ implies that

$v_{10} + v_{11} + 3v_{21} + 3^2 v_{30} + 3^3 v_{40} + 3^4 v_{51} = 0$.

This condition cannot be satisfied unless this vertex merges with the remaining vertex $v_2 = v_3$, but in that case we will have a flower. Note that it is possible, although it is not shown on the figure, that either or both of the loops e_1 and e_6 disappear. Althought, these cases need to be

considered separately, a similar consideration will tell us that this graph must degenerate into a flower as well.
Cases 3 and 4, depicted on fig. 3 (c) and (d), are done in a similar fashion, which completes the proof of the claim. □

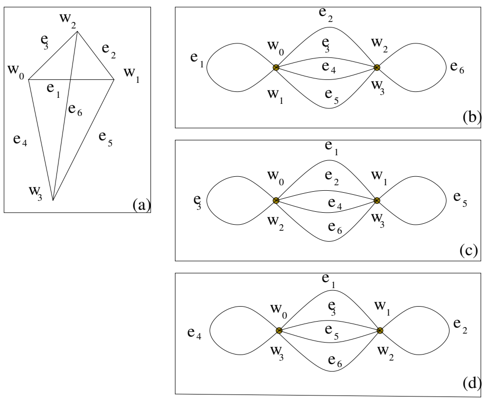

Figure 3: The graphs of the above shape cannot minimize the weighted length functional $l(\Gamma) = \Sigma_{j=1}^{6} 3^{j-1} e_j$.

Now note that at each time t during this weighted length shortening deformation, we can use the net $(K_i)_t$ to construct a 2-dimensional sphere S_t^2 in a way that is analogous to the similar construction in the proof of Theorem 0.2. This 1-parameter family of 2-spheres can be regarded as a 3-disc that we will denote as $\sigma^3_{i_0,\ldots,i_3}$. We will assign it to $\tilde{\sigma}^3_{i_0,\ldots,i_3}$.

We can continue in a similar fashion until we reach the $(n+1)$-skeleton of W, thus constructing a singular chain on M^n, that has the fundamental class $[M^n]$ as its boundary, and therefore, arriving at a contradiction. The only somewhat non-trivial fact that we need in order to finish the induction step is the following Merging Lemma.

Lemma 2.4 (Merging Lemma) *Let K be a net corresponding to a 1-skeleton of $m+1$ dimensional simplex in a Riemannian manifold M^n. Let $e_i, i = 1, \ldots, \frac{(m+1)(m+2)}{2}$ be the edges, enumerated as follows: e_1 connects the vertices w_0 and w_1, e_2, e_3 share the vertex w_2, e_4, e_5, e_6 share the vertex w_3, $e_7, \ldots., e_{10}$ share the vertex w_4, etc. Consider the following weighted length functional on K: $L(K) = \Sigma_{j=1}^{\frac{(m+1)(m+2)}{2}} 3^{j-1} length(e_j)$. Then the only non-trivial critical points of this functional are minimal geodesic flowers.*

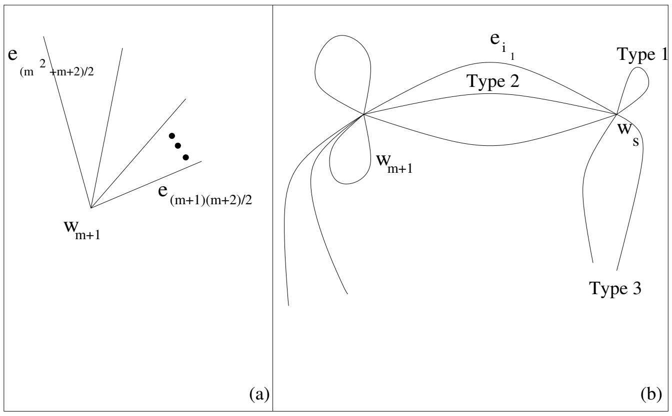

Figure 4: Illustration of the Merging Lemma

Proof.

First note that a non-degenerate K cannot be a critical point for the weighted length functional defined above, since the stationarity condition will not be satisfied at any of the vertices of K. Now suppose the vertices of K have merged with each other in some ways. We will show that K is not a critical point unless all of them have merged. In order to see that, consider a vertex w_{m+1} and some vertex w_s with which it did not merge, (see fig. 4 (b)). Note that according to the hypothesis of the lemma, the last m edges are all coming out of the vertex w_{m+1}, (see fig. 4 (a)). Moreover, they are connecting it with the remaining vertices $w_0, ..., w_m$. Let us look at all of the edges that are coming out of w_s. They will be of three types: (1) loops, (2) edges connecting w_s and w_{m+1} and (3) edges connecting w_s with some other vertex. Among all of these edges, the one with the highest index will be an edge of type (2). Let us denote this edge as e_{i_1} The stationarity condition at w_s is $\Sigma_{j=1}^{k} 3^{i_j - 1} v_{i_j} = 0$, where $k \leq (m+1)(m+2)$, v_{i_j} is a unit vector tangent to one of the edges at w_s and $i_1 > i_j$ for all $j > 1$. (Note that each edge of type (1) corresponds to two different vectors v_{i_1}, v_{i_2}.) Obviously, this condition cannot be satisfied. Therefore, this configuration must degenerate unless all the vertices coincide with w_{m+1} $\square$

Finally, note that the estimate (that can definitely be improved) arises as follows. Suppose we have extended the identity map $id : M^n \longrightarrow M^n$ to the k-skeleton of W^{n+1} and now want to extend it to the $(k+1)$-skeleton. Then consider an arbitrary $(k+1)$-simplex of W^{n+1}, its 1-skeleton and its image that we will denote as N. To extend the map to this simplex, we will construct a $(k+1)$-dimensional disc that fills N. N consists of at most $k+2$ vertices and at most $\frac{(k+2)(k+1)}{2}$ edges. Each edge has a weight assigned to it as in the Merging Lemma above. Assuming there is no "small" geodesic flowers, N can be deformed to a point along the 1-parameter family of nets $N_\tau, \tau \in [0,1]$, where the weighted length of N_τ decreases with τ. Next for each τ we construct a sphere S^k_τ that fills N_τ. It is consructed by constructing k-dimensional discs and gluing them as in the boundary of a $(k+1)$-dimensional simplex. In order to construct those discs, we consider subnets that are obtained by ignoring a vertex and all the edges that are coming out of this vertex.

Let us estimate the maximal length of each edge in a subnet. Definitely it is $\leq 2FillRadM^n \frac{(k+2)(k+1)}{2} 3^{\frac{(k+2)(k+1)-2}{2}}$, i.e. the maximal number of edges in N_τ times the maximal weighted length of the edges in N_τ. Therefore, the total weighted length of the whole subnet is $\leq 2FillRadM^n \frac{(k+1)k(k+2)(k+1)}{2^2} 3^{\frac{(k+1)k-2}{2}} 3^{\frac{(k+2)(k+1)-2}{2}}$. If we continue in such a manner starting from the $(n+1)$-skeleton of W^{n+1} we would obtain a bound of $2FillRadM^n \frac{(n+2)!(n+1)!}{2^n} 3^{\Sigma_{k=0}^{n} \frac{(k+2)(k+1)-2}{2}} \leq 2FillRadM^n (n+1)!^2 3^{(n+1)^3}$.

Note also that the maximal number of geodesic loops in the geodesic flower can be estimated by the number of edges in N, but taken with multiplicities that correspond to weights, thus it is bounded by geometric sum $\Sigma_{j=1}^{\frac{(n+2)(n+1)}{2}} 3^{j-1} \leq 3^{(n+1)^2}$. □

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