Moduli spaces of local systems and higher Teichmüller theory

V.V. Fock, A. B. Goncharov

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1 Introduction

1 An algebraic-geometric approach to higher Teichmüller theory. Let S be a surface with n holes. The Teichmüller space $\mathcal{T}_S$ is the moduli space of complex structures on S modulo diffeomorphisms isotopic to the identity. We will usually assume that S is hyperbolic, i.e. $\chi(S) < 0$. Then the Poincaré uniformisation theorem identifies it with the space of all faithful representations $\pi_1(S) \to PSL_2(\mathbb{R})$ with discrete image, modulo conjugations by $PSL_2(\mathbb{R})$. Traditionally the Teichmüller theory is considered as a part of analysis and geometry.

The Teichmüller space is by no means a set of real points of an algebraic variety. Indeed, discreteness is not a condition of an algebraic nature. In this paper we show that, nevertheless, there is an algebraic-geometric approach to the Teichmüller theory. Moreover we show that it admits a generalization where PSL_2 is replaced by a split semi-simple algebraic group G with trivial center. Here are the main features of our story.

Let G be a split reductive algebraic group over $\mathbb{Q}$. Let $\widehat{S}$ be a pair consisting of a compact oriented surface S with n holes and a finite (possibly empty) collection of marked points on the boundary considered modulo isotopy. We define two new moduli spaces, denoted $\mathcal{X}_{G,\widehat{S}}$ and $\mathcal{A}_{G,\widehat{S}}$, closely related to the classical moduli space $\mathcal{L}_{G,S}$ of representations $\pi_1(S) \to G$ modulo conjugations. We will always assume that G has trivial center (resp. simply-connected) for the $\mathcal{X}$- (resp. $\mathcal{A}$-) moduli space. In particular when $\widehat{S}$ is a disk with marked points on the boundary we get moduli spaces of cyclically ordered configurations of points in the flag variety $\mathcal{B} := G/B$ and twisted cyclic configurations of points of the principal affine variety $\mathcal{A} := G/U$.

We show that if S does have holes, i.e. $n > 0$, then these moduli spaces carry interesting additional structures. In particular for each of them we define a distinguished collection of coordinate systems, equivariant under the action of the mapping class group of S. We prove that their transition functions are subtraction free, and thus provide a *positive* atlas on the corresponding moduli space.

We say that X is a *positive variety* if it admits a positive atlas, i.e. a distinguished collection of coordinate systems as above. If X is a positive variety, we can take points of X with values in any *semifield*, i.e. a set equipped with operations of addition, multiplication and division, e.g. $\mathbb{R}_{>0}$. We define *higher Teichmüller spaces* $\mathcal{X}^+_{G,\widehat{S}}$ and $\mathcal{A}^+_{G,\widehat{S}}$ as the sets of $\mathbb{R}_{>0}$-points of the corresponding positive spaces $\mathcal{X}_{G,\widehat{S}}$ and $\mathcal{A}_{G,\widehat{S}}$. Precisely, they consist of the real points of the corresponding moduli spaces whose coordinates in one, and hence in any, of the constructed coordinate systems are positive. Our approach for general G uses G. Lusztig's theory of positivity in semisimple Lie groups [L1]-[L2]. For $G = SL_m$ we have an elementary and self-contained approach developed in Sections 8-9.

We prove that the representations $\pi_1(S) \to G(\mathbb{R})$ underlying the points of $\mathcal{X}^+_{G,\widehat{S}}$ are discrete and hyperbolic. For $G = PSL_2(\mathbb{R})$ we recover the Fuchsian representations. If S is compact, a component in the moduli space $\mathcal{L}_{G,S}$ was defined and studied by N. Hitchin [H1] by a completely different method. A partial generalization to punctured S see in [BAG].

Here is how the classical Teichmüller spaces appear. If $n > 0$, the space $\mathcal{T}_S$ has a boundary. Let us consider its 2^n-sheeted cover $\mathcal{T}^+_S \to \mathcal{T}_S$, obtained by choosing orientations of each of the boundary components. It is ramified exactly at the boundary of $\mathcal{T}_S$. This space was studied in [F]. We show that in the absence of marked points, i.e. when $S = \widehat{S}$, the space $\mathcal{X}^+_{PGL_2,S}$ is identified with the Teichmüller space $\mathcal{T}^+_S$, and $\mathcal{A}^+_{SL_2,S}$ is identified with the decorated Teichmüller space $\mathcal{T}^d_S$ defined by R. Penner [P1]. The restrictions of our canonical coordiantes to the corresponding Teichmüller spaces are the coordinates defined in loc. cit. Our two moduli spaces, their positivity, and especially the discussed below motivic data seem to be new even in the classical setting.

It makes sense to consider points of our moduli spaces with values in an arbitrary semifield. An interesting situation appears when we consider exotic *tropical* semifields $\mathbb{A}^t$. Let us recall their definition. Let $\mathbb{A}$ be one of the three sets $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$. Replacing the operations of multiplication, division and addition in $\mathbb{A}$ by the operations of addition, subtraction and taking the maximum we obtain the tropical semifield $\mathbb{A}^t$. We show that the set of points of the moduli space $\mathcal{A}_{SL_2,S}$ with values in the tropical semifield $\mathbb{R}^t$ is identified with Thurston's transversal measured laminations on S [Th]. The $\mathbb{Q}^t$- and $\mathbb{Z}^t$-points of $\mathcal{A}_{SL_2,S}$ give us rational and integral laminations. Now taking the points of the other moduli spaces with values in tropical semifields we get generalizations of the lamination spaces. We show that for any positive variety X the projectivisation of the space $X(\mathbb{R}^t)$ serves as the Thurston type boundary for the space $X^+ := X(\mathbb{R}_{>0})$.

We suggest that there exists a remarkable duality between the $\mathcal{A}$-moduli space for the group G and the $\mathcal{X}$-moduli space for the Langlands dual group ${}^LG$. Here is one of (conjectural) manifestations of this duality. Let X be a positive variety. A rational function F on X is called *a good positive Laurent polynomial* if it is a Laurent polynomial with positive integral coefficients in every coordinate system of the positive atlas on X. We conjecture that the set of $\mathbb{Z}^t$-points the $\mathcal{A}/\mathcal{X}$-moduli space for G should parametrize vectors of a canonical basis in the space of all good positive Laurent polynomials on the $\mathcal{X}/\mathcal{A}$-moduli space for the Langlands dual group ${}^LG$, for the positive atlases defined in our paper. For $G = SL_2$ or PGL_2 we elaborate this theme in Section 11.

If S has no holes, the moduli space $\mathcal{L}_{G,S}$ has a natural symplectic structure. If $n > 0$, the moduli space $\mathcal{X}_{G,\widehat{S}}$ has a natural Poisson structure (it is provided by the construction in [FR]). We introduce a degenerate symplectic structure on the moduli space $\mathcal{A}_{G,\widehat{S}}$, an analog of the Weil-Petersson form. To define the Weil-Petersson form we construct its K_2-avatar. We show that it admits an even finer motivic avatar related to the motivic dilogarithm and the second motivic Chern class $c_2^{\mathcal{M}}$ of the universal G-bundle on BG. In fact the explicit construction of the class $c_2^{\mathcal{M}}$ for $G = SL_m$ given in [G3] delivers the canonical coordinates on the $\mathcal{A}_{SL_m,\widehat{S}}$-space. It was our starting point.

The Poisson structure and the Weil-Petersson form are compatible in a natural way: the quotient of the $\mathcal{A}$-space along the null foliation of the Weil-Petersson form is embedded into the $\mathcal{X}$-space as a symplectic leaf of the Poisson structure. The representations of $\pi_1(S)$ corresponding to its points are the ones with unipotent monodromies around each hole.

Recall that there is an embedding $SL_2 \hookrightarrow G$, as a principal SL_2–subgroup, well defined up to a conjugation. It leads to natural embeddings (G' denotes the adjoint group)

$$\mathcal{A}_{SL_2,S} \hookrightarrow \mathcal{A}_{G,S}, \quad \mathcal{X}_{PGL_2,S} \hookrightarrow \mathcal{X}_{G',S}$$

and their counterparts for the Teichmüller and lamination spaces.

Summarizing, we suggest that

The pair of moduli spaces $(\mathcal{A}_{G,\widehat{S}}, \mathcal{X}_{G',\widehat{S}})$ is the algebraic-geometric avatar of higher Teichmüller theory.

After the key properties of the $(\mathcal{X}, \mathcal{A})$-moduli spaces were discovered and worked out, we started to look for an adequate framework describing their structures. It seems that such a framework is provided by the object which we call *(orbi)-cluster ensemble* and develop in [FG2]. It generalizes the *cluster algebras*, discovered and studied in a remarkable series of papers by S. Fomin and A. Zelevinsky [FZI], [FZII], [FZ3]. A cluster ensemble is a pair of positive spaces $\mathcal{X}_{\mathcal{E}}$ and $\mathcal{A}_{\mathcal{E}}$ defined for any integral valued skew-symmetrizable function $\mathcal{E}$ on the square of a finite set. The cluster algebra is closely related to the positive space $\mathcal{A}_{\mathcal{E}}$. The orbi-cluster ensemble is even a more general structure.

For any G there exists a orbi-cluster ensemble $(\mathcal{A}_{\mathcal{E}(G,\widehat{S})}, \mathcal{X}_{\mathcal{E}(G,\widehat{S})})$ closely related to the pair of moduli spaces $(\mathcal{A}_{G,\widehat{S}}, \mathcal{X}_{G,\widehat{S}})$. For $G = SL_2$ the corresponding pairs of positive spaces coincide. For other G's the precise relationship between them is more complicated. We want to stress that usually, e.g. when $\widehat{S} = S$ and S has at least two punctures, the pair of positive spaces $(\mathcal{A}_{G,\widehat{S}}, \mathcal{X}_{G,\widehat{S}})$ corresponds not to a cluster, but rather to an orbi-cluster ensemble. In Section 9 we explain this for $G = SL_m$. A more technical general case is treated in the sequel to this paper. The fact that Penner's decorated Teichmuller space is related closely to a cluster algebra was independently observed in [GSV2].

We show in [FG2] that cluster ensembles have essentially all properties of the two moduli spaces described above, including the relation with the motivic dilogarithm. Our point is that

A big part of the higher Teichmüller theory can be generalized in the framework of (orbi)-cluster ensembles, providing a guide for investigation of the latter.

The relationship between the higher Teichmüller theory and cluster ensembles is very fruitful in both directions. Indeed, the higher Teichmüller theory suggests what kind of structures and results need to be developed in the theory of cluster ensembles. For example, we suggest in Section 4 loc. cit. that there exists a remarkable duality between the $\mathcal{X}_{\mathcal{E}}$ and $\mathcal{A}_{\mathcal{E}}$ spaces, which in the special case of our pair of moduli spaces includes the one discussed above. This duality should include the canonical bases of Lusztig and the Laurent phenomenon of Fomin and Zelevinsky [FZ3] as special cases.

We show in [FG2] that any cluster ensemble can be quantized. In the sequel to this paper we apply this to quantize the moduli space $\mathcal{X}_{G,\widehat{S}}$. The quantization is given by means of the quantum dilogarithm. Remarkably the quantization is governed by the motivic avatar of the Weil-Petersson form on the $\mathcal{A}_{G,\widehat{S}}$ moduli space.

The rest of the Introduction contains a more detailed discussion of main definitions and results.

2. The two moduli spaces. Let

$$S = \overline{S} - D_1 \cup ... \cup D_n$$

where $\overline{S}$ is a compact oriented two dimensional manifold without boundary of genus g, and $D_1, ..., D_n$ are non intersecting open disks. The boundary ∂S of S is a disjoint union of oriented circles $C_1, ..., C_n$.

Definition 1.1 *A marked surface $\widehat{S}$ is a pair $(S, \{x_1, ..., x_k\})$, where S is a compact oriented surface and $\{x_1, ..., x_k\}$ is a finite set of distinct boundary points, considered up to an isotopy. The set of boundary points can be empty.*

We have $\widehat{S} = S$ if there are no boundary points. Let us define the *punctured boundary* $\partial\widehat{S}$ of $\widehat{S}$ by

$$\partial\widehat{S} := \partial S - \{x_1, ..., x_k\}$$

It is a union of circles and (open) intervals. Let N be the number of all connected component of $\partial\widehat{S}$. We say that a marked surface $\widehat{S}$ is *hyperbolic* if $g > 1$, or $g = 1, N > 0$, or $g = 0, N \geq 3$. We will usually assume that $\widehat{S}$ is hyperbolic, and $N > 0$.

Recall that an H–local system is a principle H-bundle with a flat connection. There is a well known bijection between the homomorphisms of the fundamental group $\pi_1(X, x)$ to a group H modulo conjugation, and the isomorphism classes of H–local systems on a nice topological space X. It associates to a local system its monodromy representation.

Let G be a split reductive algebraic group over $\mathbb{Q}$. Then $\mathcal{L}_{G,S}$ is the moduli space of G-local systems on S. It is an algebraic stack over $\mathbb{Q}$, and its generic part is an algebraic variety over $\mathbb{Q}$. We introduce two other moduli spaces, denoted $\mathcal{A}_{G,\widehat{S}}$ and $\mathcal{X}_{G,\widehat{S}}$, closely related to $\mathcal{L}_{G,S}$. They are also algebraic stacks over $\mathbb{Q}$.

The flag variety $\mathcal{B}$ parametrizes Borel subgroups in G. Let B be a Borel subgroup. Then $\mathcal{B} = G/B$. Further, $U := [B, B]$ is a maximal unipotent subgroup in G.

Let $\mathcal{L}$ be a G-local system on S. We assume that G acts on $\mathcal{L}$ from the right. We define the associated *flag bundle* $\mathcal{L}_\mathcal{B}$ and *principal affine bundle* $\mathcal{L}_\mathcal{A}$ by setting

$$\mathcal{L}_\mathcal{B} := \mathcal{L} \times_G \mathcal{B} = \mathcal{L}/B; \qquad \mathcal{L}_\mathcal{A} := \mathcal{L}/U \tag{1}$$

Definition 1.2 *Let G be a split reductive group. A* framed *G-local system on $\widehat{S}$ is a pair $(\mathcal{L}, \beta)$, where $\mathcal{L}$ is a G-local system on S, and β a flat section of the restriction of $\mathcal{L}_\mathcal{B}$ to the punctured boundary $\partial\widehat{S}$. The space $\mathcal{X}_{G,\widehat{S}}$ is the moduli space of framed G–local systems on $\widehat{S}$.*

Let us assume now that G is simply-connected. The maximal length element w_0 of the Weyl group of G has a natural lift to G, denoted $\overline{w}_0$. Set $s_G := \overline{w}_0^2$. Then one shows that s_G is in the center of G and $s_G^2 = e$. In particular $s_G = e$ if the order of the center of G is odd, i.e. G is of type $A_{2k}, E_6, E_8, F_4, G_2$. If $s_G \neq e$, e.g. $G = SL_{2k}$, the definition of the moduli space $\mathcal{A}_{G,\widehat{S}}$ is more subtle. So in the Introduction we do not consider this case.

Definition 1.3 *Let G be is a simply-connected split semisimple algebraic group, and $s_G = e$. A* decorated *G-local system on $\widehat{S}$ is a pair $(\mathcal{L}, \alpha)$, where $\mathcal{L}$ is G-local system on S, and α a flat section of the restriction of $\mathcal{L}_\mathcal{A}$ to $\partial\widehat{S}$. The space $\mathcal{A}_{G,\widehat{S}}$ is the moduli space of decorated G-local systems on $\widehat{S}$.*

Let X be a G–set. The elements of $G\backslash X^n$ are called *configurations* of n elements in X. The coinvariants of the cyclic translation are called *cyclic configurations* of n elements in X.

Example. Let $\widehat{S}$ be a disk with $k \geq 3$ boundary points, so $\mathcal{L}$ is a trivial G-local system. Then $\mathcal{X}_{G,\widehat{S}}$ is identified with the cyclic configuration space of k flags in G, that is coinvariants of the cyclic shift on $G\backslash\mathcal{B}^k$. Indeed, choosing a trivialization of $\mathcal{L}$, we identify its fibers with G. Then the section β over the punctured boundary of the disk determines a cyclically ordered collection of flags. A change of the trivialization amounts to a left diagonal action of G. So a cyclic configuration of k flags is well defined. It determines the isomorphism class of the framed G-local system on the marked disk. Similarly, assuming $s_G = e$, the space $\mathcal{A}_{G,\widehat{S}}$ is identified with the cyclic configuration space of k points in the principal affine space $\mathcal{A} := G/U$.

Forgetting β we get canonical projection

$$\pi : \mathcal{X}_{G,\widehat{S}} \longrightarrow \mathcal{L}_{G,S}$$

If $\widehat{S} = S$, it is a finite map at the generic point of degree $|W|^n$, where W is the Weyl group of G.

We show that the moduli spaces $\mathcal{X}_{G,\widehat{S}}$ and $\mathcal{A}_{G,\widehat{S}}$ have some interesting additional structures. To explain this we need a few simple definitions.

3. Positive structure of the moduli spaces. Let H be a split algebraic torus over $\mathbb{Q}$, so $H \stackrel{\sim}{=} \mathbb{G}_m^N$. Following [BeK], a rational function f on H is called *positive* if it can be written as $f = f_1/f_2$ where f_1, f_2 are linear combinations of characters with positive integral coefficients. A *positive rational map* between two split algebraic tori H_1, H_2 is a rational map $f : H_1 \to H_2$ such that for any character χ of H_2 the composition $\chi \circ f$ is a positive rational function on H_1. A composition of positive rational functions is positive .

Example. The map $x = a + b, y = b$ is positive, but its inverse $b = y, a = x - y$ is not.

We define a *positive divisor* in H as the divisor of a positive rational function on H. Here is our working definition of a positive atlas.

Definition 1.4 *A positive atlas on an irreducible scheme/stack X over $\mathbb{Q}$ is a family of birational isomorphisms*

$$\psi_\alpha : H_\alpha \longrightarrow X, \quad \alpha \in \mathcal{C}_X, \tag{2}$$

called positive coordinate systems on X, parametrised by a non empty set $\mathcal{C}_X$, such that:

i) each ψ_α is regular at the complement to a positive divisor in H_α;

ii) for any $\alpha, \beta \in \mathcal{C}_X$ the transition map $\psi_\beta^{-1} \circ \psi_\alpha : H_\alpha \longrightarrow H_\beta$ is a positive rational map.

A *positive scheme* is a scheme equipped with a positive atlas. In particular a positive variety is rational. Given a positive scheme X, we associate to it an open domain $X(\mathbb{R}_{>0}) \hookrightarrow X(\mathbb{R})$ called the *positive part* of X. Namely, a point $p \in X(\mathbb{R})$ lies in $X(\mathbb{R}_{>0})$ if and only if its coordinates in one, and hence any positive coordinate system on X are positive. Composing ψ_α^{-1} with an isomorphism $H_\alpha \stackrel{\sim}{\to} \mathbb{G}_m^N$ we get coordinates $\{X_i^\alpha\}$ on X, providing an isomorphism of manifolds

$$\varphi_\alpha : X(\mathbb{R}_{>0}) \stackrel{\sim}{\longrightarrow} \mathbb{R}^{\dim X}, \quad p \in X(\mathbb{R}_{>0}) \longmapsto \{\log X_i^\alpha(p)\} \in \mathbb{R}^{\dim X} \tag{3}$$

Let Γ be a group of automorphisms of X. We say that a positive atlas (2) on X is Γ-*equivariant*, if Γ acts on the set $\mathcal{C}_X$, and for every $\gamma \in \Gamma$ there is an isomorphisms of algebraic tori $i_\gamma : H_\alpha \stackrel{\sim}{\longrightarrow} H_{\gamma(\alpha)}$ making the following diagram commutative:

$$\begin{array}{ccc} H_\alpha & \stackrel{\psi_\alpha}{\longrightarrow} & X \\ \downarrow i_\gamma & & \downarrow \gamma_* \\ H_{\gamma(\alpha)} & \stackrel{\psi_{\gamma(\alpha)}}{\longrightarrow} & X \end{array} \tag{4}$$

Here γ_* is the automorphism of X provided by γ. If a positive atlas on X is Γ-equivariant, then the group Γ acts on $X(\mathbb{R}_{>0})$.

Recall that the mapping class group Γ_S of S is the group of all outer automorphisms of $\pi_1(S)$ preserving the conjugacy classes provided by simple loops around each of the punctures. It acts naturally on the moduli spaces $\mathcal{X}_{G,\widehat{S}}$, $\mathcal{A}_{G,\widehat{S}}$, and $\mathcal{L}_{G,S}$.

Theorem 1.5 *Let G be a split semisimple simply-connected algebraic group, and G' the corresponding adjoint group. Assume that $n > 0$. Then each of the moduli spaces $\mathcal{X}_{G',\widehat{S}}$ and $\mathcal{A}_{G,\widehat{S}}$ has a positive Γ_S–equivariant atlas.*

Here is an outline of our construction of positive atlas on $\mathcal{X}_{G',\widehat{S}}$. Let us picture the boundary components of S without marked points as punctures. In the presence of marked points let us choose a single distinguished point on each connected component of the punctured boundary $\partial\widehat{S}$. We define an *ideal triangulation* T of $\widehat{S}$ as a triangulation of S with all vertices at the punctures and distinguished points. Each triangle of this triangulation can be considered as a disc with three marked points. These points are located on the edges of the triangle, one per each edge. The restriction of an element of $\mathcal{X}_{G',\widehat{S}}$ to a triangle t of an ideal triangulation provides a framed G-local system on the described above marked triangle $\widehat{t}$. It is an element of $\mathcal{X}_{G',\widehat{t}}$. So we get a projection $p_t : \mathcal{X}_{G',\widehat{S}} \to \mathcal{X}_{G',\widehat{t}}$. An edge of the triangulation T is called *internal* if it is not on the boundary of S. Given an oriented internal edge e of T we define a rational projection $p_e : \mathcal{X}_{G',\widehat{S}} \to H$. Let us orient all internal edges of an ideal triangulation. Then the collection of projections p_t and p_e provide us a rational map

$$\pi_{\mathbf{T}} : \mathcal{X}_{G',\widehat{S}} \longrightarrow \prod_{\text{triangles } t \text{ of } T} \mathcal{X}_{G',\widehat{t}} \times H_{G'}^{\{\text{internal edges of } T\}} \tag{5}$$

Theorem 1.6 (Decomposition Theorem). *For any ideal triangulation T of $\widehat{S}$, the map (5) is a birational isomorphism.*

Several different proves of this theorem are given in Section 6. A similar Decomposition Theorem for the space $\mathcal{A}_{G,\widehat{S}}$ is proved in Section 7, see (104).

We define a positive atlas on $\mathcal{X}_{G',\widehat{t}}$. Then the right hand side of (5) provides a positive atlas given by the product of the ones on $\mathcal{X}_{G',\widehat{t}}$ and H. We prove that the transition functions between the atlases provided by different triangulations are subtraction free. Therefore the collection of birational isomorphisms $\{\pi_T\}$ for all ideal triangulations of $\widehat{S}$ provides a (bigger) positive atlas on $\mathcal{X}_{G',\widehat{S}}$. It is Γ_S-equivariant by the construction.

For $G = SL_m$ we have a stronger result. A positive atlas is called *regular* if the maps ψ_α are regular open embeddings.

Theorem 1.7 *The moduli spaces $\mathcal{X}_{PGL_m,\widehat{S}}$ and $\mathcal{A}_{SL_m,\widehat{S}}$ have regular positive Γ_S–equivariant atlases.*

Theorem 1.5 is proved in Sections 6-7 using the results of Section 5. Theorem 1.7 is proved in Sections 8-9 using simpler constructions of positive atlases on the SL_m-moduli spaces given in Section 8. These constructions can be obtained as special cases of the ones given in Sections 5-7, but they have some extra nice properties special for SL_m, and do not rely on Sections 5-7. In particular our X- (resp. A-) coordinates on the configuration space of triples of flags (resp. affine flags) in an m-dimensional vector space are manifestly invariant under the cyclic (resp. twisted cyclic) shift. One can not expect this property for a general group G, and this creates some technical issues, making formulations and proofs heavier. In addition to that, the construction of positive coordinate systems given in Section 8 is simpler and can be formulated using the classical language of projective geometry. We continue discussion of the SL_m case in the Section 1.8.

4. Applications of positivity: higher Teichmüller spaces and laminations.

Definition 1.8 *The higher Teichmüller spaces $\mathcal{X}^+_{G',\widehat{S}}$ and $\mathcal{A}^+_{G',\widehat{S}}$ are the real positive parts of the moduli spaces $\mathcal{X}_{G,\widehat{S}}$ and $\mathcal{A}_{G,\widehat{S}}$:*

$$\mathcal{X}^+_{G',\widehat{S}} := \mathcal{X}_{G',\widehat{S}}(\mathbb{R}_{>0}) \hookrightarrow \mathcal{X}_{G,\widehat{S}}(\mathbb{R}), \qquad \mathcal{A}^+_{G,\widehat{S}} := \mathcal{A}_{G',\widehat{S}}(\mathbb{R}_{>0}) \hookrightarrow \mathcal{A}_{G,\widehat{S}}(\mathbb{R})$$

The mapping class group Γ_S acts on $\mathcal{X}^+_{G',\widehat{S}}$ and $\mathcal{A}^+_{G,\widehat{S}}$. Let

$$\mathcal{L}^+_{G',S} := \pi\left(\mathcal{X}^+_{G',S}\right) \subset \mathcal{L}_{G',S}(\mathbb{R})$$

We call the points of $\mathcal{L}^+_{G',S}$ *positive local systems* on S, and their monodromy representations *positive representations* of $\pi_1(S)$. There is a natural embedding

$$\mathcal{L}^+_{G',S} \hookrightarrow \mathcal{X}^+_{G',S}$$

splitting the canonical projection $\mathcal{X}^+_{G',S} \to \mathcal{L}^+_{G',S}$. The following result is proved in Section 6.8.

Theorem 1.9 *Let G be a split real reductive algebraic group with trivial center. Then the image of a positive representation of $\pi_1(S)$ is a discrete subgroup in $G(\mathbb{R})$.*

Since the isomorphisms ψ_α describing the positive atlas on $\mathcal{X}_{G,\widehat{S}}$ are defined explicitly, we obtain a parametrisation of a class of discrete subgroups in $G(\mathbb{R})$ which generalizes the Fuchsian subgroups of $PSL_2(\mathbb{R})$.

In Section 10 we show that our construction specialized to the case $G = SL_2$ for the $\mathcal{A}$-space and $G = PGL_2$ for the $\mathcal{X}$-space, gives the classical Teichmüller spaces $\mathcal{T}^d(S)$ and $\mathcal{T}^h(S)$:

Theorem 1.10 *Let S be a hyperbolic surface with $n > 0$ holes. Then*

a) The Teichmüller space $\mathcal{T}^+_S$ is isomorphic to $\mathcal{X}^+_{PGL_2,S}$. Further, $\mathcal{L}^+_{PGL_2,S}$ is the classical Teichmüller space $\mathcal{T}_S$.

b) The decorated Teichmüller space $\mathcal{T}^d_S$ is isomorphic to $\mathcal{A}^+_{SL_2,S}$.

Combining with isomorphisms $\mathcal{X}^+_{PGL_2,S} \stackrel{\sim}{=} \mathbb{R}^{6g-6+3n}$ and $\mathcal{A}^+_{SL_2,S} \stackrel{\sim}{=} \mathbb{R}^{6g-6+3n}$ we arrive at isomorphisms of manifolds

$$\mathcal{T}^+_S \stackrel{\sim}{=} \mathbb{R}^{6g-6+3n} \quad \text{and} \quad \mathcal{T}^d_S \stackrel{\sim}{=} \mathbb{R}^{6g-6+3n}$$

The first gives an elementary proof of the classical Teichmüller theorem for surfaces with holes.

When S is compact, a Teichmüller component of the space of representations $\pi_1(S) \to G(\mathbb{R})$ modulo conjugation was defined by N. Hitchin [H1]. His approach is analytic. It is based on the deep analytic results of N. Hitchin [H2], K. Corlette [C], C. Simpson [S] and S. Donaldson [D], and makes an essential use of a complex structure on S. It is very natural but rather non explicit. Our approach is different: it is combinatorial, explicit, and does not use any complex structure on S. The two approaches are completely independent. The precise relationship between the two versions of Teichmüller spaces is not clarified at the moment.

We introduce *integral, rational and real G–lamination spaces* as the tropical limits of the positive moduli spaces $\mathcal{X}_{G',\widehat{S}}$ and $\mathcal{A}_{G,\widehat{S}}$, i.e. the sets of points of these spaces in the semifields $\mathbb{Z}^t, \mathbb{Q}^t, \mathbb{R}^t$. The projectivisations of the real lamination spaces are the Thurston type boundaries of the corresponding Teichmüller spaces.

5. Positivity and hyperbolicity of the monodromy representations. Consider the universal G-local system on the moduli space $S \times \mathcal{X}_{G,S}$. Its fiber over $S \times p$ is the local system corresponding to the point p of the moduli space $\mathcal{X}_{G,S}$. Let $\mathbb{F}_{G,S}$ be the field of rational functions on the moduli space $\mathcal{X}_{G,S}$. The monodromy of the universal local system around a loop on S is a conjugacy class in $G(\mathbb{F}_{G,S})$.

Observe that given a positive variety X, there is a subset $\mathbb{Q}_+(X)$ of the field of rational functions of X consisting of the functions which are positive in one, and hence in any of the positive coordinate systems on X. Clearly $\mathbb{Q}_+(X)$ is a semifield, *positive semifield* of the positive variety X.

Theorem 1.5 delivers a positive atlas on the moduli space $\mathcal{X}_{G,S}$. Therefore we have the corresponding positive semifield $\mathbb{F}^+_{G,S}$ in the field $\mathbb{F}_{G,S}$.

According to Lusztig [L1] the group G has a natural positive atlas. It is defined using a pair (B^+, B^-) of opposite Borel subgroups in G. So we have a well defined subset $G(\mathbb{F}^+_{G,S})$.

Theorem 1.11 *The monodromy of the universal G-local system on $S \times \mathcal{X}_{G,S}$ around any homotopy non-trivial non-boundary loop on S is conjugated in $G(\mathbb{F}_{G,S})$ to an element of $G(\mathbb{F}^+_{G,S})$.*

The monodromy around a boundary component is conjugated to an element of $B^{\pm}(\mathbb{F}^+_{G,S})$, where $B^{\pm}$ means one of the Borel subgroups B^+ or B^-.

Observe that although this theorem is about monodromies of local systems on S, it can not be even stated without introducing the moduli space $\mathcal{X}_{G,S}$ and a positive atlas on it. Here is a corollary formulated entirely in classical terms.

An element of $G(\mathbb{R})$ is called *regular hyperbolic* if it is contained in a unique $\mathbb{R}$-split maximal torus of $G(\mathbb{R})$. The corresponding conjugacy class bears the same name. In particular it is regular and semisimple. Regular hyperbolic elements of $GL_n(\mathbb{R})$ are the elements with real distinct eigenvalues.

Theorem 1.12 *The monodromy of a positive local system on S is faithful. Moreover its monodromy around any homotopy nontrivial non-boundary loop is regular hyperbolic.*

It follows immediately from Theorem 1.11. Indeed, Definition 1.8 and Theorem 1.11 imply that the monodromy of a positive representation around a homotopy nontrivial non-boundary loop is conjugated to an element of $G(\mathbb{R}_{>0})$. Further, by the Gantmacher-Krein theorem [GKr] for $G = GL_n$, and Theorem 5.6 in [L1] in general, any element of $G(\mathbb{R}_{>0})$ is regular hyperbolic. To prove faithfulness observe that the identity element does not belong neither to $G(\mathbb{R}_{>0})$ nor to $B^{\pm}(\mathbb{R}_{>0})$.

6. The universal higher Teichmüller spaces and positive $G(\mathbb{R})$-opers. In the special case when $\widehat{S}$ is a disc with k boundary points, the higher Teichmüller space $\mathcal{X}^+_{G,\widehat{S}}$ parametrises *positive configurations* of k flags for $G(\mathbb{R})$. One of their key properties is invariance under the cyclic shift. Using this, we introduce *universal higher Teichmüller spaces* $\mathcal{X}^+_G$:

Definition 1.13 *The universal higher Teichmüller space $\mathcal{X}^+_G$ consists of all maps*

$$\beta : \mathbb{P}^1(\mathbb{Q}) \longrightarrow \mathcal{B}(\mathbb{R}) \tag{6}$$

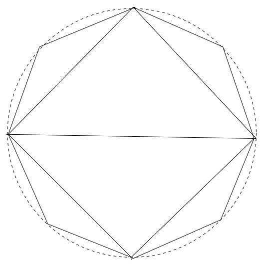
considered modulo the action of $G(\mathbb{R})$, which are positive *in the following sense: for any finite subset $x_1, ..., x_k$ of $\mathbb{P}^1(\mathbb{Q})$, whose cyclic order is induced by a natural one on $\mathbb{P}^1(\mathbb{Q})$, the configuration of flags $(\beta(x_1), ..., \beta(x_n))$ is positive.*

For $G = PGL_2$ we recover the universal Teichmüller space considered by Bers and Penner [P3].

The higher Teichmüller spaces $\mathcal{X}^+_{G,S}$ can be embedded in the universal one as follows. Let Δ be a torsion free subgroup of $PSL_2(\mathbb{Z})$ and $S_\Delta := \mathcal{H}/\Delta$, where $\mathcal{H}$ denotes the hyperbolic plane. Then S_Δ is equipped with a distinguished ideal triangulation, given by the image of the Farey triangulation of the hyperbolic plane. Conversely, the subgroup $\Delta \subset PSL_2(\mathbb{Z})$ is determined uniquely up to conjugation by an ideal triangulation of S. A framed $G(\mathbb{R})$-local system on S_Δ provides a representation $\rho : \Delta \to G(\mathbb{R})$, and naturally gives rise to a ρ-equivariant map (6). This property characterizes the higher Teichmüller space for S_Δ:

Theorem 1.14 *There is a natural bijection between the points of the space $\mathcal{X}^+_{G,S_\Delta}$ and pairs (a representations $\rho : \Delta \to G(\mathbb{R})$, a ρ-equivariant map (6)), considered modulo $G(\mathbb{R})$-conjugation.*

The universal higher Teichmüller space $\mathcal{X}^+_G$ is equipped with a natural positive coordinate atlas. Namely, recall the Farey triangulation of the hyperbolic plane, see Picture (7). The set of vertices of the Farey triangulation is identified with $\mathbb{P}^1(\mathbb{Q})$. Using the Farey triangulation just the same way as ideal triangulations of S, we obtain the following result, the Decomposition Theorem for the universal higher Teichmüller space $\mathcal{X}^+_G$, providing coordinate systems on $\mathcal{X}^+_G$.



(7)

Theorem 1.15 *There exists a canonical isomorphism*

$$\varphi_G : \mathcal{X}^+_G \longrightarrow {\mathcal{X}^+_{G,3}}^{\{\textit{Farey triangles}\}} \times H(\mathbb{R}_{>0})^{\{\textit{Farey diagonals}\}} \quad (8)$$

In Section 7.5 we define universal higher Teichmüller spaces $\mathcal{A}^+_G$ and prove similar results for them.

Positive continuous maps

$$\mathbb{P}^1(\mathbb{R}) \longrightarrow \mathcal{B}(\mathbb{R}) \quad (9)$$

are objects of independent interest. We prove in Section 6.9 that positive smooth maps (9) are integral curves of a canonical non-integrable distribution on the flag variety, provided by the simple positive roots, whose dimension equals to the rank of G. So they can be viewed as the $G(\mathbb{R})$-opers in the sense of Beilinson and Drinfeld on the circle. We call them *positive* $G(\mathbb{R})$*-opers*.

It is well known that $PGL_m(\mathbb{R})$-opers are nothing else but smooth curves in $\mathbb{RP}^{m-1}$: a smooth projective curve gives rise to its osculating curve in the flag variety. Recall (cf. [Sch]) that a curve $C \subset \mathbb{RP}^{m-1}$ is *convex* if any hyperplane intersects it in no more then $m-1$ points. We show in Section 8.12 that positive $PGL_m(\mathbb{R})$-opers correspond to smooth convex curves in $\mathbb{RP}^{m-1}$. More generally, positive continuous maps (9) give rise to C^1-smooth convex curves. Thus positive continuous maps (9) generalize projective convex curves to the case of an arbitrary reductive group G. It is interesting to note that convex curves appeared in some classical problems of analysis ([GKr]).

7. The Weil-Petersson form on the moduli space $\mathcal{A}_{G,\widehat{S}}$ and its K_2 and motivic avatars. We define the Weil-Petersson form $\Omega_{G,\widehat{S}}$ on the moduli space $\mathcal{A}_{G,\widehat{S}}$ by constructing a K_2-class on $\mathcal{A}_{G,\widehat{S}}$. We suggest that *many interesting symplectic structures can be upgraded to their* K_2*-avatars.* Let us explain how a K_2-class provides a symplectic form.

Recall that thanks to the Matsumoto theorem [Mi] the group $K_2(F)$ of a field F is the quotient of the abelian group Λ^2F^* by the subgroup generated by the elements $(1-x)\wedge x$, $x \in F^* - \{1\}$, called the Steinberg relations. If $F = \mathbb{Q}(Y)$ is the field of rational functions on a variety Y there is a homomorphism

$$d\log : K_2(\mathbb{Q}(Y)) \longrightarrow \Omega^2_{\log}(\mathrm{Spec}(\mathbb{Q}(Y))); \quad f_1 \wedge f_2 \longmapsto d\log f_1 \wedge d\log f_2$$

Applying it to a class $w \in K_2(\mathbb{Q}(Y))$ we get a rational 2-form $d\log(w)$ with logarithmic singularities. Let us formulate a condition on w which guarantees that the 2-form $d\log(w)$ is non singular, and hence defines a degenerate symplectic structure of Y. Let Y_1 be the set of all irreducible divisors in a variety Y. There is a *tame symbol* homomorphism:

$$\mathrm{Res} : K_2(\mathbb{Q}(Y)) \longrightarrow \oplus_{D \in Y_1}\mathbb{Q}(D)^*; \qquad \{f,g\} \longmapsto \mathrm{Rest}_D\Big(f^{v_D(g)}/g^{v_D(f)}\Big) \quad (10)$$

where Rest_D is restriction to the generic point of D, and $v_D(g)$ is the order of zero of g at the generic point of the divisor D. Set

$$H^2(Y, \mathbb{Z}_{\mathcal{M}}(2)) := \mathrm{Ker}(\mathrm{Res}) \hookrightarrow K_2(\mathbb{Q}(Y)), \qquad H^2(Y, \mathbb{Q}_{\mathcal{M}}(2)) := H^2(Y, \mathbb{Z}_{\mathcal{M}}(2)) \otimes \mathbb{Q} \quad (11)$$

The map $d\log$ transforms the tame symbol to the residue map. So its restriction to (11) provides a homomorphism

$$d\log : H^2(Y, \mathbb{Z}_{\mathcal{M}}(2)) \longrightarrow \Omega^2(Y_0)$$

where Y_0 is the nonsingular part of Y. We define a class

$$W_{G,\widehat{S}} \in H^2(\mathcal{A}_{G,\widehat{S}}, \mathbb{Q}_{\mathcal{M}}(2))^{\Gamma_S} \tag{12}$$

and set

$$\Omega_{G,\widehat{S}} := d\log(W_{G,\widehat{S}}) \in \Omega^2(\mathcal{A}_{G,\widehat{S}})^{\Gamma_S}$$

We also define the class $W_{G,S}$ for a compact oriented surface S. One can show that for $G = SL_2$ the form $\Omega_{SL_2,S}$ coincides with W. Goldman's symplectic structure [Gol1] on the space $\mathcal{L}_{SL_2,S}$. Its restriction to the Teichmüller component is the Weil-Petersson symplectic structure. The K_2-avatar of the Weil-Petersson form is a new object even in the classical case $G = SL_2$.

Motivic avatar of the Weil-Petersson form. We show that the K_2-class $W_{G,\widehat{S}}$ is a manifestation of a richer algebraic structure described by using the weight two motivic cohomology understood via the Bloch-Suslin complex, and the motivic dilogarithm. Namely, we construct a *motivic avatar*

$$\mathbb{W}_{G,\widehat{S}} \in H^2_{\Gamma_S}(\mathcal{A}_{G,\widehat{S}}, \mathbb{Q}_{\mathcal{M}}(2))$$

of the form $\Omega_{G,\widehat{S}}$. It lives in the Γ_S–equivariant weight two motivic cohomology of $\mathcal{A}_{G,\widehat{S}}$.

The 2-form $\Omega_{G,\widehat{S}}$ as well as its avatars are lifted from similar objects on the moduli space $\mathcal{U}_{G,\widehat{S}}$ of the unipotent framed G-local systems on $\widehat{S}$. The corresponding 2-form on the moduli space $\mathcal{U}_{G,\widehat{S}}$ is non degenerate, and thus provides a symplectic structure on this space.

Our construction uses as a building block the second motivic Chern class $c_2^{\mathcal{M}}$ of the universal G–bundle over the classifying space $BG_{\bullet}$. When $G = SL_m$, there is an explicit cocycle for the class $c_2^{\mathcal{M}}$ constructed in [G3]. Applying to it our general procedure we arrive at an explicit cocycle for the class $\mathbb{W}_{SL_m,\widehat{S}}$. Magically this construction delivers canonical coordinates on the space $\mathcal{A}_{SL_m,\widehat{S}}$. Their properties are outlined in the next subsection.

8. The (orbi)-cluster ensemble structure in the PGL_m/SL_m case. Let T be an ideal triangulation of $\widehat{S}$. Take the triangle

$$x + y + z = m, \quad x, y, z \geq 0$$

and consider its triangulation given by the lines $x = p$, $y = p$, $z = p$ where $0 \leq p \leq m$ is an integer. The *m–triangulation* of a triangle is a triangulation isotopic to this one. Let us m–triangulate each triangle of the triangulation T, so each side gets $m + 1$ vertices. We get a subtriangulation, called the *m–triangulation* of T. An edge of the m–triangulation of T is called an *internal edge* if it does not lie on a side of the original triangulation T. The orientation of S provides orientations of the internal edges. Indeed, take a triangle t of the triangulation T. The orientation of S provides an orientation of its boundary. An internal edge e sitting inside of t is parallel to a certain side of t, so the orientation of this side induces an orientation of e. This is illustrated on Figure 1. Here on the right there are two adjacent triangles of the triangulation T. On the left we show the 4–triangulation of these triangles. Their internal edges are oriented accordantly to the cyclic structure coming from the clockwise orientation of the plain.

Given an ideal triangulation T of $\widehat{S}$, we define a canonical coordinate system $\{\Delta_i\}$ on the space $\mathcal{A}_{SL_m,\widehat{S}}$, parametrized by the set

$$\mathrm{I}_m^T := \{\text{vertices of the } m\text{–triangulation of } T\} \text{ - } \{\text{vertices at the punctures of } S\} \tag{13}$$

Further, we define a canonical coordinate system $\{X_j\}$ on $\mathcal{X}_{PGL_m,\widehat{S}}$ parametrised by the set

$$\mathrm{J}_m^T := \mathrm{I}_m^T - \{ \text{ the vertices at the boundary of } S \ \}$$

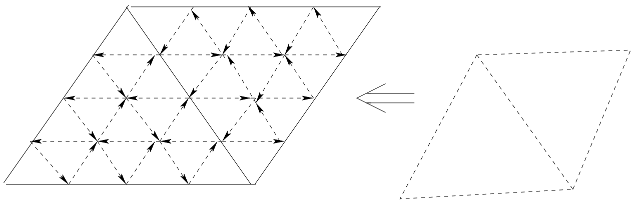


Figure 1: The 4-triangulation arising from a triangulation of a surface

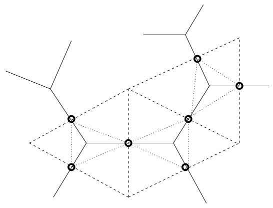


Figure 2: The set I_2^T is naturally identified with the set of all internal edges of T.

Definition 1.16 *A skew symmetric $\mathbb{Z}$–valued function ε_{pq} on the set of vertices of the m-triangulation of T is given by*

$$\varepsilon_{pq} := \#\{\textit{oriented edges from } p \textit{ to } q\} - \#\{\textit{oriented edges from } q \textit{ to } p\} \tag{14}$$

The Weil-Petersson form and the Poisson structure in these coordinates are given by

$$\Omega_{SL_m,\widehat{S}} = \sum_{i_1,i_2} \varepsilon_{i_1 i_2} d\log\Delta_{i_1} \wedge d\log\Delta_{i_2}, \qquad \{X_{j_1}, X_{j_2}\}_{PGL_m,\widehat{S}} = \sum_{j_1 j_2} \varepsilon_{j_1,j_2} X_{j_1} X_{j_2} \tag{15}$$

The K_2-avatar of the Weil-Petersson form is given by $W_{SL_m,\widehat{S}} = \sum_{i_1,i_2} \varepsilon_{i_1 i_2}\{\Delta_{i_1}, \Delta_{i_2}\}$

We call an ideal triangulation T of $\widehat{S}$ is *special* if it contains, as a part, one of the graphs shown on Figure. A marked surface $\widehat{S}$ is *special* if it admits a special triangulation, and *regular* otherwise.

Figure 3: Special graphs, called a virus and an eye.

Lemma 1.17 *A marked hyperbolic surfaces $\widehat{S}$ is special if and only if it contains at least two holes, and in addition one of the holes has no marked points on the boundary.*

The sets I_m^T and J_m^T can be defined for any, not necessarily finite, triangulation T. An especially interesting example is provided by the Farey triangulation. It is obviously regular. It has no boundary, so the I and J sets coincide, and denoted by I_m. Since the function ε_{pq} for the set I_m has a finite support for every given p, all structures of the cluster ensemble but the W-class make sense.

Theorem 1.18 *a) The universal pair of positive spaces $(\mathcal{X}_{PGL_m}, \mathcal{A}_{SL_m})$ is a part of the cluster ensemble for the function ε_{pq} on the set I_m.*

b) If $\widehat{S}$ is regular, the pair of positive spaces $(\mathcal{X}_{PGL_m,\widehat{S}}, \mathcal{A}_{SL_m,\widehat{S}})$ is a part of the cluster ensemble for the function ε_{pq} from Definition 1.16.

We assign a coordinate system on the $\mathcal{X}$- and $\mathcal{A}$-space to every triangulation T of $\widehat{S}$. For every T the function ε_{pq} from Definition 1.16 describes the Poisson structure on $\mathcal{X}_{PGL_m,\widehat{S}}$ and the W-class on $\mathcal{X}_{SL_m,\widehat{S}}$ in the corresponding coordinate system via formulas (15). However the transition functions between the coordinate systems assigned to special triangulations are different then the ones prescribed by the cluster ensemble prescriptions. Therefore, although Definition 1.16 assigns a cluster ensemble structure for every $\widehat{S}$, if $\widehat{S}$ is special, the pair of positive spaces $(\mathcal{X}_{PGL_m,\widehat{S}}, \mathcal{A}_{SL_m,\widehat{S}})$ is no longer a part of this, or in fact any other cluster ensemble structure. We show that it is related to a more general orbi-cluster ensemble structure.

Theorem 1.19 *If $\widehat{S}$ is special, the pair of positive spaces $(\mathcal{X}_{PGL_m,\widehat{S}}, \mathcal{A}_{SL_m,\widehat{S}})$ is a part of the orbi-cluster ensemble for the function ε_{pq} from Definition 1.16.*

The precise meaning of this theorem is discussed in Section 9. In the sequel to this paper we will prove a similar result for an arbitrary G, quantize the higher Teichmüller spaces and as a result construct an infinite dimensional unitary projective representation of the group Γ_S.

9. Coda. We suggest that investigation of the pair of moduli spaces $(\mathcal{A}_{G,\widehat{S}}, \mathcal{X}_{G',\widehat{S}})$ can be viewed as a marriage of representation theory and the theory of surfaces. Indeed, when the surface $\widehat{S}$ is simple, i.e. is a disc with marked points on the boundary or an annulus, we recover many aspects of the representation theory for the group G, quite often from a new point of view: positivity in $G(\mathbb{R})$, invariants in the tensor products of finite-dimensional representations, W-algebras, quantum groups, canonical bases, Springer variety etc. On the other hand, when $G = SL_2$ but $\widehat{S}$ is general we get the theory of Riemann surfaces.

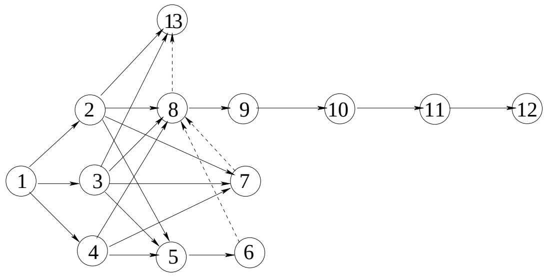


Figure 4: Leitfaden

Acknowledgments. N. Reshetikhin informed us about his work in progress with R. Kashaev, aimed at quantization of the moduli spaces of flat G-connections on S, containing a result similar to our decomposition (104). After the first version of this work was posted in the ArXive, we learned about the very interesting preprint of F. Labourie [Lab] who, in particular, proved discreteness and hyperbolicity of representations of $\pi_1(S) \to PSL_m(\mathbb{R})$, when S has no holes, lying in the Hitchin component, and proved that their limit sets are described by convex projective curves.

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2 The moduli spaces $\mathcal{A}_{G,\widehat{S}}$ and $\mathcal{X}_{G,\widehat{S}}$

Let us replace the holes on S by punctures, getting a surface S'. The local systems on S and S' are the same. Let us equip the surface S' with a complex structure. Then we get a complex algebraic curve $\mathbb{S}$. The moduli space of local systems on $\mathbb{S}$ has two different algebraic structures: the monodromic and the De Rham structures. Nevertheless the corresponding complex analytic spaces are isomorphic. The monodromic structure depends only on the fundamental group $\pi_1(\mathbb{S}(\mathbb{C}))$. Therefore it is determined by the topological surface S' underlying the curve $\mathbb{S}$. The corresponding moduli stack is denoted $\mathcal{L}_{G,S}$. To define the De Rham structure recall the Riemann-Hilbert correspondence ([D1]) between the isomorphism classes of G–local systems on $S' = \mathbb{S}(\mathbb{C})$ and algebraic G–connections with regular singularities on the algebraic curve $\mathbb{S}$. The moduli space $\mathcal{L}_{G,\mathbb{S}}$ of regular G-connections on $\mathbb{S}$ is an algebraic stack over $\mathbb{Q}$. Its algebraic structure depends on the algebraic structure of the curve $\mathbb{S}$. It is the De Rham algebraic structure. The Riemann-Hilbert correspondence provides an isomorphism of the complex analytic spaces $\mathcal{L}_{G,S}(\mathbb{C})$ and $\mathcal{L}_{G,\mathbb{S}}(\mathbb{C})$.

1. The moduli spaces $\mathcal{X}_{G,\widehat{S}}$. The flag variety $\mathcal{B}$ parametrises all Borel subgroups in G. Choosing a Borel subgroup B we identify it with G/B. Let $\mathcal{L}$ be a G-local system on S. Recall the associated *flag bundle* $\mathcal{L}_\mathcal{B} := \mathcal{L} \times_G \mathcal{B}$. We call the elements of a fiber of $\mathcal{L}_\mathcal{B}$ over x by *flags over x in* $\mathcal{L}$. A reduction of a G–local system $\mathcal{L}$ on a circle to B is just the same as a choice of a flag in a fiber of $\mathcal{L}$ invariant under the monodromy.

Examples. a) A flag in a vector space V is given by a filtration

$$0 = V_0 \subset V_1 \subset V_2 \subset ... \subset V_m = V, \quad \dim V_i = i \tag{16}$$

The flag variety for $G = GL_m$ parametrizes all flags in V.

b) Let $G = GL_m$. Recall the bijective correspondence between the GL_m–local systems and local systems of m-dimensional vector spaces. Namely, let V be a standard m-dimensional representation of GL_m. A GL_m–local system $\mathcal{L}$ provides a local system of vector spaces $\mathcal{L}' := \mathcal{L} \times_{\mathrm{GL_m}} V$. Conversely, given an m-dimensional local system $\mathcal{L}'$ we recover the corresponding GL_m–local system by taking all bases in fibers of $\mathcal{L}$. The flag bundle $\mathcal{L}_\mathcal{B}$ consists of all flags in the fibers of the local system $\mathcal{L}'$. If $G = SL_m$, $\mathcal{L}'$ is a local system with an invariant volume form.

Definition 2.1 *Let G be an arbitrary split reductive group. An enhanced G-local system on $\widehat{S}$ is a pair $(\mathcal{L}, \beta)$ where $\mathcal{L}$ is a G-local system on S and β is a flat section of the restriction of $\mathcal{L}_\mathcal{B}$ to the punctured boundary $\partial\widehat{S}$. $\mathcal{X}_{G,\widehat{S}}$ is the moduli space of enhanced G–local systems on $\widehat{S}$.*

One can rephrase this definition as follows:

i) If C_i is a boundary component without marked points, we choose a flag F_x over a certain point $x \in C_i$ invariant under the monodromy around C_i. Equivalently, we choose a reduction to the subgroup B of the restriction of the G-local system $\mathcal{L}$ to the boundary component C_i. The parallel transport of the flag F_x provides the restriction of the section β to C_i, and vice versa.

ii) If there is a non empty set $\{x_1, ..., x_p\}$ of the boundary points on the component C_i, the complement $C_i - \{x_1, ..., x_p\}$ is a union of p arcs. Let us pick a point y_j inside of each of these arcs, and choose a flag F_j over y_j. These flags may not be invariant under the monodromy. The flags F_j determine the restriction of the flat section β to the corresponding arc, and vice versa.

The Cartan group of G. A split reductive algebraic group G determines the Cartan group H. Namely, let B be a Borel subgroup in G. Then $H := B/U$ where U is the unipotent radical of B. If B' is another Borel subgroup then there is a unique isomorphism $B/U \longrightarrow B'/U'$. So the quotient B/U is canonically isomorphic to the Cartan subgroup H in G. Let

$$i_B : H \to B/U \tag{17}$$

be the canonical isomorphism. The Cartan group H is not considered as a subgroup of G.

The canonical projection $B \to H = B/U$ provides a natural projection

$$\kappa : \mathcal{X}_{G,S} \longrightarrow H^{\{\text{punctures of } S\}} \tag{18}$$

The moduli space $\mathcal{X}_{G,\widehat{S}}$ as a quotient stack. Let F be an arbitrary field. Choose a point $x \in S$. Let g be the genus of the surface $\overline{S}$. Let us choose generators $\alpha_1, ..., \alpha_g, \beta_1, ..., \beta_g, \gamma_1, ..., \gamma_n \in \pi_1(S, x)$

such that
$$\prod_{j=1}^{g}[\alpha_j,\beta_j]\prod_{i=1}^{n}\gamma_i = 1 \tag{19}$$
and where γ_i belongs to the conjugacy class in $\pi_1(S,x)$ determined by the i-th hole in S. It is well known that such a system of generators exists, and that (19) is the only relation between them in $\pi_1(S,x)$.

Definition 2.2 *An* enhanced representation $\pi_1(S,x)\to G(F)$ *is the following data:*
$$\Big(X_{\alpha_1},...,X_{\alpha_g},X_{\beta_1},...,X_{\beta_g},X_{\gamma_1},...,X_{\gamma_n},\{B_j\}\Big), \qquad X_? \in G(F), \quad j\in\pi_0(\partial\widehat{S}), \quad B_j\in\mathcal{B}(F) \tag{20}$$
where if j corresponds to a connected component C_j of the boundary of S without marked points on it, then $X_{\gamma_j}\in B_j(F)$.

Obviously the elements of the set (20) are the F-points of an affine algebraic variety denoted by $\widetilde{\mathcal{X}}_{G,\widehat{S}}$. The algebraic group G acts on this variety, and, by definition, the stack $\mathcal{X}_{G,\widehat{S}}$ is the quotient stack $\widetilde{\mathcal{X}}_{G,\widehat{S}}/G$.

Examples. 1. Let $\widehat{D}_n$ be a disc with n marked points on the boundary. Then $\widetilde{\mathcal{X}}_{G,\widehat{D}_n}=\mathcal{B}^n$, and $\mathcal{X}_{G,\widehat{D}_n}=G\backslash\mathcal{B}^n$ is the space of configurations of n flags.

2. Let S be an annulus. Then $\widetilde{\mathcal{X}}_{G,S}=\{(g,B_1,B_2)\}$ where B_i are Borel subgroups in G and $g\in B_1, g\in B_2$. The quotient $\mathcal{X}_{G,S}:=\widetilde{\mathcal{X}}_{G,S}/G$ is the Steinberg variety, see [CG].

2. Pinnings. Let us introduce some notations following [L1]. Let G be a split reductive simply-connected connected algebraic group over $\mathbb{Q}$. Let H be a split maximal torus of G and B^+,B^- a pair of opposed Borel subgroups containing H, with unipotent radicals U^+,U^-. Let U_i^+ $(i\in I)$ be the simple root subgroups of U^+ and let U_i^- be the corresponding root subgroups of U^-. Here I is a finite set indexing the simple roots. Denote by $\chi_i':H\longrightarrow\mathbb{G}_m$ the simple root corresponding to U_i^+. Let $\chi_i:\mathbb{G}_m\longrightarrow H$ be the simple coroot corresponding to χ_i'. We assume that for each $i\in I$ we are given isomorphisms $x_i:K\longrightarrow U_i^+$ and $y_i:K\longrightarrow U_i^-$ such that the maps
$$\begin{pmatrix}1&a\\0&1\end{pmatrix}\longmapsto x_i(a), \quad \begin{pmatrix}1&0\\b&1\end{pmatrix}\longmapsto y_i(b), \quad \begin{pmatrix}t&0\\0&t^{-1}\end{pmatrix}\longmapsto \chi_i(t)$$
provide a homomorphism $\varphi_i:SL_2(K)\longrightarrow G$. The datum $(H,B^+,B^-,x_i,y_i;i\in I)$ is called a *pinning* for G. The group G acts by conjugation on pinnings, making it into a principal homogeneous space for the group $G/\text{Center}(G)$. In particular any two pinnings for G are conjugated in G.

There is unique involutive antiautomorphism $\Psi:G\longrightarrow G$ such that for all $i\in I$ and $t\in H$ one has $\Psi(x_i(a))=y_i(a)$, $\Psi(y_i(a))=x_i(a)$ and $\Psi(t)=t$.

Example. If $G=SL_m$ then B^+,B^- are the subgroups of upper and lower triangular matrices, so H is the subgroup of diagonal matrices. Ψ is the transposition. If $G=PSL(V)$ where V is a vector space then a choice of pinning is the same thing as a choice of a projective basis in V_m.

Let $X^*(T)$ and $X_*(T)$ be the groups of characters $T\to\mathbb{G}_m$ and cocharacters $\mathbb{G}_m\to T$ for a split tori T. There is a pairing
$$<\cdot,\cdot>: X^*(T)\times X_*(T)\longrightarrow\mathbb{Z}, \quad \chi'\circ\chi:\mathbb{G}_m\longrightarrow\mathbb{G}_m, t\longmapsto t^{<\chi',\chi>}$$

Let $A=(a_{ij})$ be the Cartan matrix given by $a_{ij}:=<\chi_j',\chi_i>$. Recall the weight lattice $P\subset X^*(H)$. It is the subgroup of all $\gamma\in X^*(H)$ such that $<\gamma,\chi>\in\mathbb{Z}$ for all $i\in[1,...,r]$. Let $\{\omega_1,...,\omega_r\}$ be the $\mathbb{Z}$–basis of fundamental weights given by $<\omega_j,\chi_i>=\delta_{ij}$.

3. The element s_G. A choice of pinning for G provides an inclusion of the Cartan group H into G. Abusing notation we will denote the obtained subgroup by H and call it the Cartan subgroup corresponding to the pair (B^-, B^+) of opposite Borel subgroups. Then the element

$$\overline{s}_i := y_i(1)x_i(-1)y_i(1) = \varphi_i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

lifts to $\mathrm{Norm}_G(H)$ the generator $s_i \in W$. It is well known that the elements $\overline{s}_i$, $i \in I$, satisfy the braid relations. Therefore we can associate to each $w \in W$ its standard representative $\overline{w} \in \mathrm{Norm}_G(H)$ in such a way that for any reduced decomposition $w = s_{i_1}...s_{i_k}$ one has $\overline{w} = \overline{s}_{i_1}...\overline{s}_{i_k}$. Let w_0 be the maximal length element of W. We set $s_G := \overline{w}_0^2$. Then evidently $s_G^2 = e$.

Example. If $G = SL_m$ then $s_{SL_m} = (-1)^{m-1}e$.

Let R be the system of roots for the group G, and $R = R_+ \cup R_-$ its decomposition into a union of the sets of positive and negative roots. Then for any element $w \in W$ its length $l(w)$ equals to $|R_- \cap w(R_+)|$. For any root $\alpha = \sum n_i\alpha_i$ we set $\chi_\alpha(t) := \prod \chi_i(t)^{n_i}$.

Lemma 2.3 *Let* $w \in W$. *Then*

$$\overline{w}\overline{w^{-1}} = \prod_{\alpha \in R_- \cap w(R_+)} \chi_\alpha(-1)$$

Proof. By the induction on the length $l(w)$. If $l(w) = 1$ this is clear. Assume $w = w_iw'$ where $l(w') = l(w) - 1$. Then by the induction assumption

$$\overline{w}\overline{w^{-1}} = \overline{w}_i\overline{w'}\overline{w^{-1}}\overline{w}_i = w_i\Big(\prod_{\alpha \in R_- \cap w'(R_+)} \chi_\alpha(-1)\Big)w_i^{-1}\overline{w}_i\overline{w_i^{-1}} = \prod_{\alpha \in R_- \cap w'(R_+)} \chi_{w_i(\alpha)}(-1)\chi_{\alpha_i}(-1)$$

Since $w_i(R_- \cap w'(R_+)) \cup -\alpha_i = R_- \cap w(R_+)$, the lemma is proved.

Corollary 2.4 s_G *is a central element of* G *of order* 2.

Proof. Let 2ρ be the sum of all positive roots. Then (see Proposition 29, Chapter VI, paragraph 1 in [Bo]) we have $2\rho = \omega_1 + ... + \omega_r$. Denote by $\chi'_{2\rho}$ the character of H corresponding to 2ρ. Then for any coroot χ one has $< \chi'_{2\rho}, \chi > \in 2\mathbb{Z}$. Since $w_0^{-1} = w_0$, this combined with Lemma 2.3 implies that s_G commutes with the elements $x_i(t)$, and thus is a central element. The corollary is proved.

Recall the well defined up to a conjugation embedding $i : SL_2 \hookrightarrow G$ called the principal SL_2 subgroup of G.

Lemma 2.5 $s_G = i(s_{SL_2})$ *for the principal* SL_2 *subgroup of* G.

Proof. An easy exercise.

4. The moduli spaces $\mathcal{A}_{G,\widehat{S}}$. Let us choose a maximal unipotent subgroup U in G. The normalizer of U in G is a Borel subgroup B. Conversely, U is the commutant of B. The Cartan group H acts from the right on the principal affine variety $\mathcal{A} := G/U$: if $A = gU \in \mathcal{A}$ we set

$$A \cdot h = gU \cdot h := gi_B(h)U \tag{21}$$

Since H is commutative it, of course, can be considered as a left action. This action is free. The quotient $\mathcal{A}/H$ is identified with the flag variety $\mathcal{B} = G/B$. The canonical projection $\mathcal{A} \to \mathcal{B}$ is a principal H-bundle. More generally, let X be a right principal homogeneous space for G. Then X/U is an $\mathrm{Aut}X$–homogeneous space. Recall the *principal affine bundle* $\mathcal{L}_\mathcal{A} := \mathcal{L}/U$ associated to a G–local system $\mathcal{L}$ on S.

Let T'_S the punctured tangent bundle to the surface S, i.e. the tangent bundle with the zero section removed. Its fundamental group $\pi_1(T'S, x)$ is a central extension of $\pi_1(S, y)$ by $\mathbb{Z}$, where $y \in T_xS$:

$$0 \longrightarrow \mathbb{Z} \longrightarrow \pi_1(T'S, x) \longrightarrow \pi_1(S, y) \longrightarrow 0 \tag{22}$$

Let T'_yS be the punctured tangent space at a point y. The inclusion $T'_yS \hookrightarrow T'S$ induces an isomorphism of $\pi_1(T'_yS) \stackrel{\sim}{=} \mathbb{Z}$ with the central subgroup $\mathbb{Z}$ in (22). We denote by σ_S a generator of this central subgroup. It is well defined up to a sign.

Definition 2.6 *A twisted G-local system on S is a local system on $T'S$ with the monodromy s_G around σ_S.*

Observe that since s_G is of order two, this does not depend on the choice of the generator σ_S.

Let $\overline{\pi}_1(S, y)$ be the quotient of $\pi_1(S, y)$ by the central subgroup $2\mathbb{Z} \subset \mathbb{Z}$, so it is a central extension

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \overline{\pi}_1(T'S, x) \longrightarrow \pi_1(S, y) \longrightarrow 0$$

We denote by $\overline{\sigma}_S$ the order two central element in the subgroup $\mathbb{Z}/2\mathbb{Z}$. The twisted local systems on S are in bijective correspondence with the representations $\rho : \overline{\pi}_1(T'S, x) \to G$, considered modulo conjugation, such that $\rho(\overline{\sigma}_S) = s_G$.

Remark. Let us choose a complex structure on S. Then the uniformisation provides an embedding $i : \pi_1(S, y) \hookrightarrow PSL_2(\mathbb{R})$. It is easy to see that the group $\overline{\pi}_1(T'S, x)$ is isomorphic to the preimage of the subgroup $i(\pi_1(S, y))$ in $SL_2(\mathbb{R})$. In particular $\overline{\sigma}_S$ corresponds to the element $-e \in SL_2(\mathbb{R})$.

The group $\overline{\pi}_1(T'S, x)$ is isomorphic to the direct product $\mathbb{Z}/2\mathbb{Z} \times \pi_1(S, y)$, although this isomorphism is by no means canonical. The set of all isomorphisms

$$\overline{\pi}_1(T'S, x) \longrightarrow \mathbb{Z}/2\mathbb{Z} \times \pi_1(S, y)$$

which send the central subgroup $\mathbb{Z}/2\mathbb{Z}$ on the left to the left factor on the right, and reduce to the canonical isomorphism modulo the $\mathbb{Z}/2\mathbb{Z}$-subgroups, is a principal homogeneous space over $\mathrm{Hom}(\pi_1(S, y), \mathbb{Z}/2\mathbb{Z})$. Therefore choosing such an isomorphism we can identify the space of twisted G-local systems on S with the space of G-local systems on S. Observe that a choice of such an isomorphism is the same thing as a choice of a spin structure on S.

Let $\mathbf{C}_i$ be a little annulus containing a boundary component C_i of S as a component of its boundary. Observe that $\pi_1(T'\mathbf{C}_i)$ is canonically identified with $\mathbb{Z} \times \mathbb{Z}$: the first factor is $\pi_1(T'_x\mathbf{C}_i)$, and the second is generated by the tangent vectors to C_i. Let $\mathbf{L}$ be the isomorphism class of the G-local system on $\mathbf{C}_i$ whose monodromy along the element $(a, b) \in \mathbb{Z} \times \mathbb{Z} = \pi_1(T'\mathbf{C}_i)$ is s_G^a.

Let $x_1, ..., x_p$ be the marked points on the component C_i. Let us present the annulus $\mathbf{C}_i$ as a product $C_i \times [0, 1]$, and let $\mathbf{C}'_i := (C_i - \{x_1, ..., x_p\}) \times [0, 1]$. Then $\mathbf{C}'_i$ has k connected components.

Definition 2.7 *Let G be a split simply-connected reductive group. Let $\mathcal{L}$ be a G-local system on $T'S$ representing a twisted local system on S. A* decoration *on $\mathcal{L}$ is a choice of locally constant section α of the restriction of the principal affine bundle $\mathcal{L}_\mathcal{A}$ to $\cup_i \mathbf{C}'_i$.*

A decorated *twisted G–local system on a marked surface $\widehat{S}$ is a pair $(\mathcal{L}, \alpha)$, where $\mathcal{L}$ is a twisted G-local system on S, and α is a decoration on $\mathcal{L}$. The space $\mathcal{A}_{G,\widehat{S}}$ is the moduli space of decorated twisted G–local systems on $\widehat{S}$.*

When $s_G = e$ this definition reduces to the one given in the Definition 1.3. Just like in the $\mathcal{X}$-case, the moduli space $\mathcal{A}_{G,\widehat{S}}$ can be understood as the quotient moduli stack. We leave the obvious details to the reader.

Let $\mathrm{Conf}^*_n(\mathcal{A}) := G \backslash \mathcal{A}^n$ be the configuration space of n affine flags in G.

Definition 2.8 *Twisted cyclic shift map* $S : \mathrm{Conf}^*_n(\mathcal{A}) \longmapsto \mathrm{Conf}^*_n(\mathcal{A})$ *is given by*

$$S : (A_1, A_2, ..., A_n) \longmapsto (A_2, ..., A_n, s_G A_1)$$

Since s_G is a central element one has $A \cdot s_G = s_G A$. Therefore $S^n = \mathrm{Id}$ because

$$S^n(A_1, A_2, ..., A_n) = (s_G A_1, s_G A_2, ..., s_G A_n) \sim (A_1, A_2, ..., A_n)$$

The coinvariants of twisted cyclic map acting on the space $\mathrm{Conf}^*_n(\mathcal{A})$ are called *twisted cyclic configurations* of n affine flags in G. Let $\widetilde{\mathrm{Conf}}^*_n(\mathcal{A})$ be the moduli space of of twisted cyclic configurations of n affine flags in G.

Lemma 2.9 *Let* $\widehat{D}_k$ *be a disc* D *with* k *marked points at the boundary. Then* $\mathcal{A}_{G,\widehat{D}_k}$ *is identified with the twisted cyclic configuration space* $\widetilde{\mathrm{Conf}}^*_k(\mathcal{A})$.

Proof. The punctured tangent bundle to a disc is homotopy equivalent to a circle. So there is unique up to an isomorphism twisted G-local system on $\widehat{D}_k$. Consider a path in $T'D$ obtained by choosing a non zero tangent vector field on a simple loop near the boundary of D. The monodromy around this path is s_G. The marked points $x_1, ..., x_k$ on the boundary $C = \partial D$ divide C into a union of the arcs $C^*_1, ..., C^*_k$. We assume they follow the boundary clockwise. Let us choose a vector v_i tangent to D at a certain point of the arc C^*_i. We assume that the vectors v_i look inside of the disc D. For each vector v_i there is canonical homotopy class of path from v_i to v_1 in $T'D$ obtained by counterclockwise transport of v_i along C towards v_1. Restricting the decoration section α to $v_1, ..., v_k$ and then translating the obtained affine flags in the fibers over $v_2, ..., v_k$ to the fiber over v_1 we get a configuration of affine flags $(A_1, ..., A_k)$ over v_1. If we choose v_2 instead of v_1 as the initial vector, this configuration will be replaced by its twisted cyclic shift $(A_2, ..., A_n, s_G A_1)$. So we identified $\mathcal{A}_{G,\widehat{D}_k}$ with $\widetilde{\mathrm{Conf}}^*_k(\mathcal{A})$. The lemma is proved.

Recall the adjoint group G' corresponding to the group G. Since the canonical map $G \to G'$ kills the center, and s_G is in the center, a twisted G-local system $\mathcal{L}$ on $T'S$ provides a G'-local system $\mathcal{L}'$ on S. The canonical projection $p : \mathcal{L}_{\mathcal{A}} \longrightarrow \mathcal{L}'_{\mathcal{B}}$ provides a map

$$p : \mathcal{A}_{G,\widehat{S}} \longrightarrow \mathcal{X}_{G',\widehat{S}}, \qquad (\mathcal{L}, \alpha) \longmapsto (\mathcal{L}', \beta), \quad \beta := p(\alpha) \tag{23}$$

Lemma 2.10 *Let* k *be the number of boundary points. Then* $\mathrm{dim}\mathcal{X}_{G',\widehat{S}} + k \cdot \mathrm{dim}H = \mathrm{dim}\mathcal{A}_{G,\widehat{S}}$.

If $k = 0$, *one has* $\mathrm{dim}\mathcal{X}_{G',S} = \mathrm{dim}\mathcal{A}_{G,S} = \mathrm{dim}\mathcal{L}_{G,S} = (2g + n - 2)\mathrm{dim}G$.

Proof. The map (18) is surjective over the generic point. Consider a boundary component C_i without boundary points. The condition that a G–local system has a unipotent monodromy around C_i decreases the dimension by $\mathrm{rank}(G)$, while adding a decoration at s we increase the dimension by $\mathrm{dim}H = \mathrm{rank}(G)$. On the other hand over an arc the space of α's is G/U, while the space of β's is G/B. The lemma is proved.

5. Remarks, complements and examples. 1. If $\widehat{S} = S$, i.e. there is no boundary points, it may be convenient to assume that $S = \overline{S} - \{s_1, ..., s_n\}$ is a surface with n punctures. For each puncture there is a well defined isotopy class of a little oriented circle making a simple counterclockwise loop around s. Let $\mathcal{L}_s$ be the isomorphism class of the restriction of a G–local system $\mathcal{L}$ to it. An enhanced G–local system on S is then a G–local system $\mathcal{L}$ equipped with the following additional structure: for every puncture s we choose a reduction of the local system $\mathcal{L}_s$ to the Borel subgroup B.

One can always avoid talking about holes D_i in $\overline{S}$. Indeed, let S be a surface with n punctures. Denote by $\mathrm{Sp}_{s_i}\mathcal{L}$ the specialization of the local system $\mathcal{L}$ to the punctured tangent space $T_{s_i}\overline{S} - \{0\}$ at s_i. It is a local system on the punctured tangent space. The marked points are replaced by a collection of distinct rays in the punctured tangent bundle $T_{s_i}\overline{S} - \{0\}$. An enhanced structure on $\mathcal{L}$

is given by the following data at the punctures s_i: restrict the local system $\mathrm{Sp}_{s_i}\mathcal{L}$ to the complement of the chosen rays, and take its flat section β. This language is convenient when we work with an algebraic structure on S.

2. Let us look more closely at the projection (23) in the case when $\widehat{S} = S$. The monodromy of a G–local system on a circle is described by a conjugacy class in G. The monodromy provides a bijective correspondence between the conjugacy classes in G and the isomorphism classes of G–local system on a circle. A G–local system $\mathcal{L}$ on S is *unipotent* if for every boundary component C_i the monodromy of the restriction $\mathcal{L}_{|C_i}$ is unipotent.

Definition 2.11 *$\mathcal{U}_{G,S}$ is the moduli space of the enhanced unipotent G–local systems on S.*

Observe that a B-local system $\mathcal{L}_s$ on a circle has a unipotent monodromy if and only if it can be reduced to a U-local system. A decoration on $\mathcal{L}_s$ is the same as a reduction of the structure group to the subgroup U. So a decoration on a B–local system $\mathcal{L}_s$ on a circle exists if and only if $\mathcal{L}_s$ is unipotent. Thus a decorated G–local system on S is unipotent. The right action of H on $\mathcal{L}_s$ provides an action of H on the set of all decorations, making it into a principal homogeneous space over H.

Therefore forgetting the decoration we get a canonical projection

$$p' : \mathcal{A}_{G,S} \longrightarrow \mathcal{U}_{G,S} \tag{24}$$

It provides $\mathcal{A}_{G,S}$ with a structure of a principal $H_G^{\{\text{punctures}\}}$-bundle over $\mathcal{U}_{G,S}$, where H_G is the Cartan group of G. Observe that the map κ is the projection onto the group $H_{G'}^{\{\text{punctures}\}}$. There is a commutative diagram, where i is the natural closed embedding:

$$\begin{array}{ccccc} \mathcal{A}_{G,S} & & & & \\ p' \downarrow & \searrow p & & & \\ \mathcal{U}_{G',S} & \stackrel{i}{\longrightarrow} & \mathcal{X}_{G',S} & \stackrel{\kappa}{\longrightarrow} & H_{G'}^{\{\text{punctures}\}} \end{array} \tag{25}$$

One has $\mathcal{U}_{G',S} = p(\mathcal{A}_{G,S}) = \kappa^{-1}(e)$.

3. *Affine flags.* The points of $\mathcal{L}_{\mathcal{B}}$ parametrize the flags in the fibers of $\mathcal{L}$. In many cases, e.g. when the center of G is trivial, there is a similar interpretation of the principal affine bundle based on the notion of an affine flag.

Let A be a direct sum of one dimensional H-modules corresponding to all simple roots. Let $\mathcal{U}$ be the Lie algebra of U. The H-module $\mathcal{U}/[\mathcal{U},\mathcal{U}]$ is isomorphic to A.

Definition 2.12 *Suppose that the center of G is trivial. An affine flag is a pair*

$$\{B, i : A \stackrel{\sim}{\longrightarrow} \mathcal{U}/[\mathcal{U},\mathcal{U}]\} \tag{26}$$

where B is the Borel subgroup containing U, and i is an isomorphism of H-modules.

The group G acts by conjugation on affine flags. This action is transitive. Since the center of G is trivial, the stabilizer of the affine flag (26) is U. So the principal affine bundle parametrises the affine flags in the fibers of $\mathcal{L}$. The same conclusion can be maid for the classical groups with non trivial center. Here are some examples.

The classical affine flags. i). $G = GL_m$. Recall that an affine flag in a vector space V is given by a flag (16) in V plus a choice of non zero vectors v_i at each quotient V_i/V_{i-1} for all $i = 1, ..., m$. Equivalently, an affine flag in V can be defined by a sequence of non zero volume elements $\eta_i \in \det V_i$:

$$\eta_1 := v_1, \quad \eta_2 := v_1 \wedge v_2, \quad ..., \quad \eta_m := v_1 \wedge ... \wedge v_m;$$

Alternatively, consider the dual volume forms $\omega_i \in \det V_i^*$, $< \eta_i, \omega_i >= 1$. We may parametrise the affine flags by the sequence of the volume forms $(\omega_1, ..., \omega_m)$. The variety of all affine flags is identified with GL_m/U where U is the subgroup of upper triangular matrices in GL_m with units at the diagonal.

ii). *The SL_m case.* An affine flag in V determines a volume form in V. By an affine flag in a vector space with a given volume form ω we understand an affine flag whose volume form is the form ω. The variety all affine flags with a given volume form is identified with SL_m/U.

Example. An affine flag in a two dimensional vector space V is a pair {a non zero vector v_1 and a non zero 2–form ω in V}. The variety of all affine flags in a two dimensional symplectic vector space is identified with $\mathbb{A}^2 - \{0\}$.

3 Surfaces, graphs, mapping class groups and groupoids

This Section contains some background material, suitably adopted or generalized. We start from the ribbon graphs and their generalizations needed to describe two dimensional surfaces with marked points on the boundary. We use it throughout the paper. Then we introduce a convenient combinatorial model of the classifying orbi-space for mapping class groups which will be used in Section 12. In the end we recall some basic facts about the classical Teichmüller spaces which will be used in Section 10 and some other parts of the paper.

1. Ribbon graphs and hyperbolic surfaces. Recall that a *flag* of a graph Γ is a pair (v, E) where v is a vertex of Γ, E is an edge of Γ, and v is an an endpoint of E. A *ribbon graph* is a finite graph such that for each vertex a cyclic order of the set of all flags sharing this vertex is chosen. An example of a ribbon graph is provided by a graph drawn on an oriented surface S. The cyclic order of the flags sharing the same vertex is given by the counterclockwise ordering of the flags.

We say that an oriented path on a ribbon graph *turns left* at a vertex if we come to the vertex along the edge which is next (with respect to the cyclic order) to the edge used to get out of the vertex. A *face path* on a ribbon graph is a closed oriented path turning left at each vertex, which ends as soon as the path starts repeat itself. To visualize a face path imagine a ribbon graph glued of actual thin ribbons. Then a face path is a boundary component of such a ribbon graph. For example the punctured torus has one face path. A flag determines a face path. This face path can be visualized by a strip bounded by the face path. Conversely a face path plus a vertex (resp. edge) on this path determine uniquely the flag containing this vertex (resp. edge) situated on the face path.

Given a ribbon graph Γ we can get an oriented compact surface $S_{\Gamma,c}$ with a ribbon graph isomorphic to Γ lying on the surface. Namely, we take a disc for each of the face paths and glue its boundary to the graph along the face path. If we use punctured discs in the above construction we get an oriented surface S_Γ which can be obviously retracted onto Γ. Clearly S_Γ is obtained by making n punctures on $S_{\Gamma,c}$, where $n = F(\Gamma)$. Any hyperbolic surface can be obtained as S_Γ for certain ribbon graph Γ. Indeed, consider a triangulation of the surface with the vertices at the punctures, and take the dual graph.

2. Ideal triangulations of marked hyperbolic surfaces and marked trivalent graphs. Let $\widehat{S}$ be a marked hyperbolic surface. If its boundary component C_i has no boundary points on it, let us picture the corresponding hole on S as a puncture.

A *marked trivalent graphs on S of type $\widehat{S}$* is a graph Γ on S with the following properties:

i) The vertices of Γ are of valence three or one.

ii) The set of univalent vertices coincides with the set of marked points on the boundary ∂S.

iii) The surface S can be shrinked onto Γ.

We define an *ideal triangulation* of $\widehat{S}$ as a triangulation of S with the vertices either at the punctures or at the boundary arcs, such that each arc carries exactly one vertex of the triangulation, and each puncture serves as a vertex. The edges of an ideal triangulation either lie inside of S, or belong to the boundary of S.

The duality between triangulations and trivalent graphs provides a bijection between the isotopy classes of ideal triangulations of $\widehat{S}$ and isotopy classes of marked trivalent graphs on $\widehat{S}$. The ends of the dual graph are dual to the boundary edges of the ideal triangulation.

A *marked ribbon graph* is a graph with trivalent and univalent vertices equipped with a cyclic order of edges sharing each trivalent vertex. Given a marked ribbon graph one can construct the corresponding marked surface.

Consider a marked graph $(\overline{\Gamma}, v)$ which has just one 4-valent vertex in addition to trivalent and univalent vertices. Then there are two marked trivalent graphs Γ and Γ' related to $(\overline{\Gamma})$, as shown on the picture. We say that Γ' is obtained from Γ by a flip at the edge E shrinked to v. A similar operation on triangulations consists of changing an edge in a 4-gon of the triangulation. It is well known that any two isotopy classes of marked trivalent graphs on a surface are related by a sequence of flips.

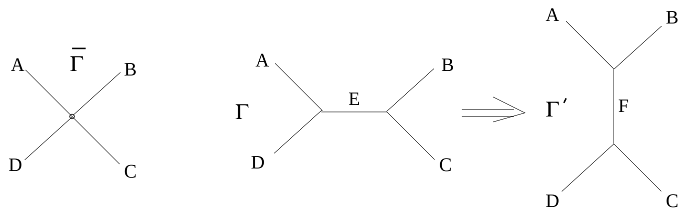


Figure 5: Flip at the edge E

3. The mapping class group of S. The mapping class group Γ_S is the quotient of the group of all orientation preserving diffeomorphisms of S modulo its connected component consisting of the diffeomorphisms isotopic to the identity. It has the following well known group theoretic description. Every hole D_i on S provides a conjugacy class in the fundamental group $\pi_1(S, x)$. The mapping class group Γ_S consists of all outer automorphisms of $\pi_1(S, x)$ preserving each of these conjugacy classes. The group Γ_S acts by automorphisms of the $\mathcal{A}$ and $\mathcal{X}$ moduli spaces. We will use a combinatorial description of the mapping class group as the automorphism group of objects of the *modular groupoid* of S defined below.

4. The graph complex $G_\bullet(S)$ **of a hyperbolic surface** S. It is a version of the complex used by Boardman and Kontsevich. To define it, recall the $\mathbb{Z}/2\mathbb{Z}$–torsor Or_Γ of orientations of a graph Γ, given by invertible elements in the determinant of free abelian group generated by the edges of Γ. Let $G_k(S)$ be the abelian group of $\mathbb{Z}$–valued functions $\varphi(\Gamma, \varepsilon_\Gamma)$ with finite support on the set of pairs $(\Gamma, \varepsilon_\Gamma)$, where Γ is an isotopy class of an embedded graph on S such that S can be retracted to it, $\varepsilon_\Gamma \in \mathrm{Or}_\Gamma$, one has $\varphi(\Gamma, -\varepsilon_\Gamma) = -\varphi(\Gamma, \varepsilon_\Gamma)$, and k is given by

$$k - 3 = \sum_{v \in V(\Gamma)} (\mathrm{val}(v) - 3) \tag{27}$$

Here $\mathrm{val}(v)$ is the valency of a vertex v of Γ. Observe that if all except one vertices of Γ are of valency three, then k is the valency of this one vertex. The differential $\partial : G_\bullet(S) \longrightarrow G_{\bullet-1}(S)$ is provided by contraction of the edges of Γ:

$$\partial\varphi(\Gamma, \varepsilon_\Gamma) := \sum_{E', \Gamma'} \varphi(\Gamma', \varepsilon_{\Gamma'}) \tag{28}$$

Here the sum is over all pairs (E', Γ'), where E' is an edge of Γ', such that the ribbon graph Γ'/E' obtained by shrinking of E' is isomorphic Γ, and $\varepsilon_{\Gamma'} = E' \wedge \varepsilon_\Gamma$. Then $\partial^2 = 0$. We place the trivalent graphs in the degree 2, and consider $G_\bullet(S)$ as a homological complex.

The graph complex of S is the chain complex of modular complex a finite dimensional contractible polyhedral complex $\mathbb{G}(S)$, called the modular complex of S, constructed in the next subsection.

5. The modular complex of S. Recall the n–dimensional Stasheff polytop K_n. It is a convex polytop whose combinatorial structure is described as follows. Consider plane trees with $n+3$ ends lying on an oriented circle. The ends are marked by the set $\{1, ..., n+3\}$, so that its natural order is compatible with the circle orientation. The codimension p faces of K_n are parametrized by the isotopy classes of such plane trees with $p+1$ internal vertices. Let F_T be the face corresponding to a tree T. Then F_T belongs to the boundary of $F_{\overline{T}}$ if and only if $\overline{T}$ is obtained from T by shrinking some internal edges. The Stasheff polytop K_n can be realized as a cell in the configuration space of $n+3$ cyclically ordered points on a circle, see [GM] for a survey.

Example. K_0 is a point, K_1 is a line segment, and K_2 is a pentagon.

Let us define a finite dimensional contractible polyhedral complex $\mathbb{G}(S)$, called the modular complex of S. The faces of the modular complex $\mathbb{G}(S)$ are parametrized by the isotopy classes of graphs Γ on S, homotopy equivalent to S. The polyhedron P_Γ corresponding to Γ is a product of the Stasheff polytops:

$$P_\Gamma := \prod_{v \in V(\Gamma)} K_{\mathrm{val}(v)-3} \tag{29}$$

Here $V(\Gamma)$ is the set of vertices of Γ, and $\mathrm{val}(v)$ is the valency of a vertex v. The face P_Γ is at the boundary of the face $P_{\overline{\Gamma}}$ if and only if $\overline{\Gamma}$ is obtained from Γ by shrinking of some edges. Precisely, the inclusion $P_\Gamma \hookrightarrow P_{\overline{\Gamma}}$ is provided by the similar data for the Stasheff polytops appearing in (29). The mapping class group $\mathrm{D}(S)$ acts on $\mathbb{G}(S)$ preserving the polyhedral structure.

Examples. 1. The vertices of $\mathbb{G}(S)$ correspond to the isotopy classes of the trivalent graphs on S. The edges – to the graphs with one 4–valent vertex. The two dimensional faces are either pentagons corresponding to graphs with unique 5–valent vertex, or squares corresponding to graphs with two 4–valent vertices.

2. Let S be a torus punctured at a single point. The mapping class group of S is isomorphic to $SL_2(\mathbb{Z})$. Then $\mathbb{G}(S)$ is the tree dual to the classical modular triangulation of the hyperbolic plane:

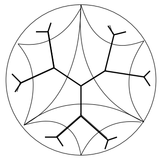


Figure 6: The tree dual to the modular triangulation

Proposition 3.1 *The graph complex $G_\bullet(S)$ is the chain complex of the modular complex $\mathbb{G}(S)$.*

Proof. The faces of the modular complex are glued according to the boundary map (28). An orientation ε_Γ of Γ provides an orientation of the face $P(\Gamma)$. The lemma now follows from the very definition.

Proposition 3.2 *The polyhedral complex $\mathbb{G}(S)$ is contractible. The stabilisers of the mapping class group acting on $\mathbb{G}(S)$ are finite.*

The proof is deduced from Strebel's theory of quadratic differentials on Riemann surfaces [S]. One can show that there exists a Γ_S-equivariant contraction of the Teichmüller space to the modular complex of S. So the modular complex is a "combinatorial version" of the Teichmüller space.

6. Modular groupoid and combinatorial description of the mapping class group. The quotient $\mathbb{G}(S)/\Gamma_S$ is a polyhedral orbifold. By proposition 3.2 its orbifold fundamental group is isomorphic to Γ_S. The *modular groupoid of* S is a combinatorial version of the Poincare groupoid of this orbifold. Precisely, the objects of the modular groupoid of S are Γ_S–orbits of the vertices of modular complex $\mathbb{G}(S)$, i.e. isomorphism classes of of the trivalent ribbon graphs of type S. The morphisms are represented by paths on the 1-skeleton composed with the graph automorphisms, i.e. the elements of Γ_S stabilizing the vertices. The two paths represent the same morphism if an only if they are homotopic in the 2–skeleton $\mathbb{G}_{\leq 2}(S)/\Gamma_S$.

Here is a description of the modular groupoid by generators and relations. Recall the flip determined by an oriented edge of the modular complex, see the picture (5). Every morphism in the modular groupoid is a composition of the elementary ones, given by flips and graph symmetries. There are two types of relations between the elementary morphisms provided by the two types of the two dimensional faces of $\mathbb{G}(S)$:

i) (*Square*). The flips performed at two disjoint edges commute.

ii) (*Pentagon*). The composition of the five flips corresponding to the (oriented) boundary of a pentagonal face of $\mathbb{G}(S)$ is zero.

They together with the graph symmetries generate all the relations between the morphisms in the modular groupoid.

7. Background on the Teichmüller spaces. The classical Teichmüller space $\mathcal{T}_S$ for a hyperbolic surface S can be defined as the moduli space of any of the following four different objects related to S:

1) Complex structures on S.

2) Faithful representations $\pi_1(S) \to PSL_2(\mathbb{R})$ with discrete image, modulo $PSL_2(\mathbb{R})$-conjugation.

3) Complete Riemannian metrics on S of curvature -1, called *complete hyperbolic metrics*.

4) Hyperbolic metrics on S with geodesic boundary ∂S.

Indeed, the uniformisation theorem implies 1) <=> 2). Taking the quotient of the hyperbolic plane $\mathcal{H}$ by the action of the image of $\pi_1(S)$ in $PSL_2(\mathbb{R})$ we go from 2) to 3). A Riemannian metric on S provides a complex structure on S, and hence the arrow 3) => 1). Taking the universal cover of S we see the equivalence 2) <=> 3). For each hole on a surface S with complete hyperbolic structure there is a unique geodesic, called *boundary geodesic*, homotopic to the boundary of the hole, except the case when the hole degenerates to a puncture, i.e. the corresponding end of S is a cusp. Cutting out the ends of the surface S along the boundary geodesics we get a surface with geodesic boundary. This leads to the equivalence 3) <=> 4).

The universal cover of a surface S equipped with a hyperbolic metric with geodesic boundary can be realized as the hyperbolic plane minus an infinite collection of disjoint discs bounding geodesics. These geodesics project onto the boundary geodesics on S. Precisely, let S' be a surface with complete hyperbolic metric corresponding to S, i.e. $S = S' - T_1 \cup ... \cup T_n$ where T_i is the tube cutted out from S' by the boundary geodesic corresponding to the i-th hole. The preimage of each boundary geodesic to the hyperbolic plane is a disjoint collection of geodesics on $\mathcal{H}$, with the monodromy group Γ acting transitively on it. The tube T_i lifts to a union of discs bounding these geodesics. Cutting out these discs we get a universal cover $\mathcal{H}_S$ of S sitting inside of $\mathcal{H}$. The cusps on the boundary of $\mathcal{H}_S$ form a Cantor set, which is the limit set of Γ.

If the number n of holes is positive, the Teichmüller space $\mathcal{T}_S$ is a manifold with corners. Choosing orientations of all boundary geodesics we get a version of the Teichmüller space denoted $\mathcal{T}_S^+$. Forgetting orientations of the geodesics we get a $2^n : 1$ covering $\mathcal{T}_S^+ \to \mathcal{T}_S$. On the other hand the orientation of S provides canonical orientations of the geodesics, and hence an embedding $\mathcal{T}_S \hookrightarrow \mathcal{T}_S^+$.

8. Special graphs and flips. Recall that a trivalent graph on S is *special* if it contains, as a part, one of the two graphs shown on Figure 7 by punctured lines. We call them an eye and a virus. The triangulations dual to the eye and virus are shown on Figure 7 by solid lines. The sides shown by arrows are identified. The triangulations dual to the graphs containing a virus are characterised by the property that they contain a triangle with two sides identified. To obtain a diagram of the triangulation corresponding to an eye we cut one of the edges of the triangulation, the top one on the middle picture, getting the right diagram on Figure 7.

Figures 8 and 9 show that the special graphs can be obtained by flips from the usual graphs. Namely, on Figure 8 we obtain an eye by making a flip at the bottom horizontal edge of the graph

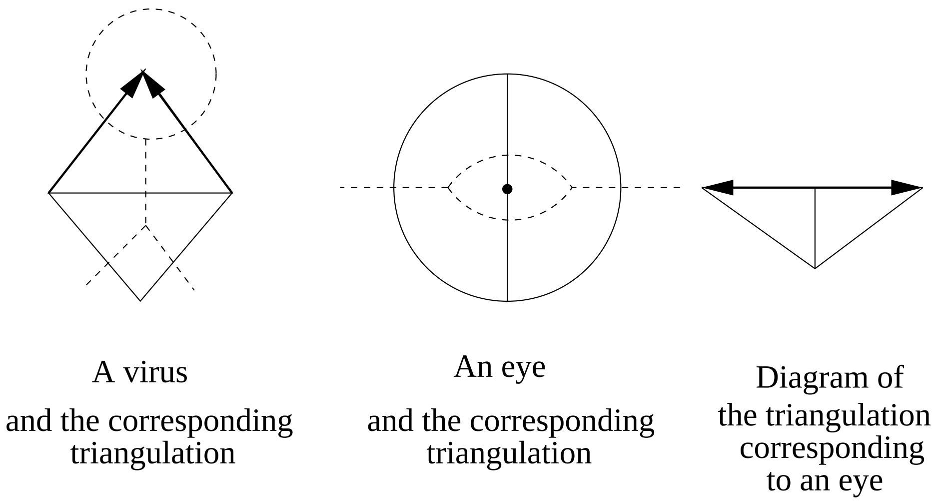


Figure 7: A virus and an eye, with the corresponding triangulations.

on the left. The bottom of Figure 9 shows that a flip at the leg of a virus produces an eye, and vice versa. A flip at the loop of a virus is not defined.

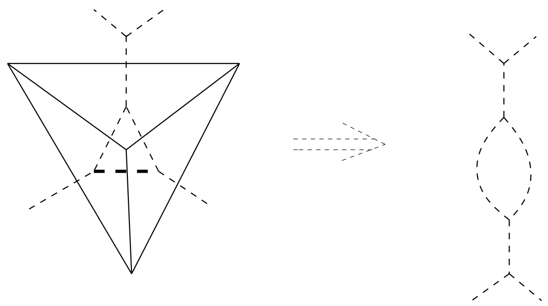


Figure 8: An eye is obtained by a flip at the bottom edge on the left.

Figure 9 shows that a flip from an eye to a virus, lifted to a $2:1$ cover of the surface ramified at the puncture at the center of the eye, turns out to be a composition of flips at the edges E_1 and E_2 of a regular graph. The edges E_1 and E_2 are the preimage of the edge E. The non-trivial automorphism of the cover is given by the central symmetry. Below we always treat a special triangulation of S by going to a cover of S where the triangulation becomes regular.

Proof of Lemma 1.17. Let S be a surface with a trivalent graph Γ on S, which is homotopy equivalent to S. Let us make a puncture of S, getting a new surface S'. Then adding a virus to Γ as shown on the picture we get a trivalent graph Γ' homotopy equivalent to S'. So S' is special. Since any marked surface with at least two holes, such that the boundary of one of the holes is without marked points, is special. Let us prove the converse claim. A virus separates the surface into two domains, each of them containing a whole. Moreover the one of them surrounded by the head of the virus has no marked points on its boundary. The lemma is proved.

4 Elements of positive algebraic geometry

1. Semifields and positive varieties. Following [BFZ96], we define a *semifield* as a set P endowed with the operations of addition and multiplication, which have the following properties:

i) Addition in P is commutative and associative.

ii) Multiplication makes P an abelian group.

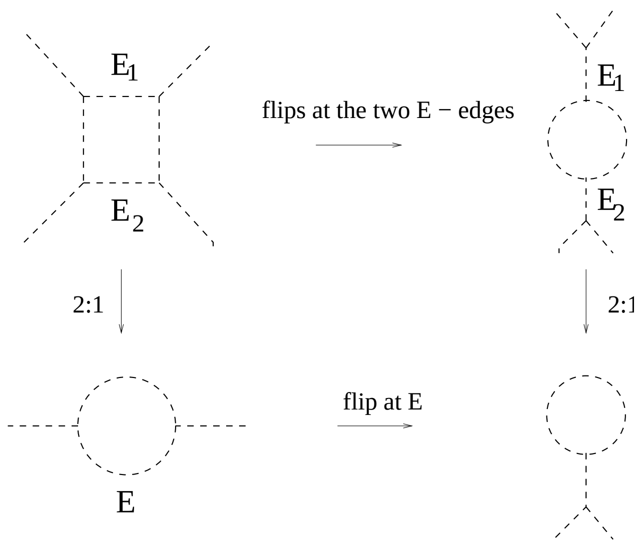


Figure 9: A flip from an eye to a virus is a folding of a composition of two flips on regular graphs

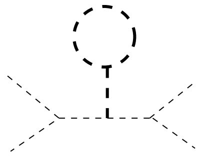


Figure 10: Adding a virus to a graph

iii) distributivity: $(a+b)c = ac + bc$ for $a, b, c \in P$.

Here are important examples of semifields:

a) $P = \mathbb{Q}_{>0} = \{x \in \mathbb{Q} | x > 0\}$,

b) $P = \mathbb{R}_{>0} = \{x \in \mathbb{R} | x > 0\}$,

c) Let K be a field and $P \subset K$ a semifield inside of K, with the addition and multiplication in P induced from K, like in the examples a) and b). Then there is a new semifield

$$K((\varepsilon))_P := \{f = a_s \varepsilon^s + a_{s+1}\varepsilon^{s+1} + ... | a_s \in P\} \subset K((\varepsilon))$$

consisting of all Laurent power series with the leading coefficient from P. For example starting from $\mathbb{Q}_{>0}$ we get a semifield denoted $\mathbb{Q}((\varepsilon))_{>0}$. Similarly one defines $\mathbb{R}((\varepsilon))_{>0}$ starting from $\mathbb{R}_{>0}$.

d) The tropical semifields: $P = \mathbb{Z}$, $P = \mathbb{Q}$ or $P = \mathbb{R}$ with the multiplication $\otimes$ and addition $\oplus$ given by

$$a \otimes b := a + b, \quad a \oplus b := \max(a, b) \tag{30}$$

Observe that in the examples a), b) and c) the semifield P is naturally embedded to a field, while in the example d) this is impossible since $a \oplus a = a$.

Lemma 4.1 *Assigning to $f \in K((\varepsilon))_P$ the minus degree of its leading coefficient we get a homomorphism of semifields*

$$-\deg : K((\varepsilon))_P \longrightarrow \mathbb{Z}^t$$

Proof. Clear.

There is a unique way to assign to a split tori H an abelian group $H(P)$ so that $\mathbb{G}_m(P) = P$ and $H \to H(P)$ is a functor from the category of split algebraic tori to the one of abelian groups. One has $H(\mathbb{Z}^t) = X_*(H)$ and $H(P) = X_*(H) \otimes_{\mathbb{Z}} P$, where P is considered here as an abelian group.

Let Pos be a category whose objects are split algebraic tori and morphisms are positive rational maps. A positive rational map $\varphi : H_1 \to H_2$ gives rise to a map of sets $\varphi_* : H_1(P) \to H_2(P)$, providing us a functor Pos $\longrightarrow$ Sets. In particular if φ is invertible in the category Pos then φ_* is an isomorphism. Therefore given a positive variety $(X, \{\psi_\alpha\})$ we get a collection of abelian groups $H_\alpha(P)$ and set isomorphisms $\psi_{\alpha,\beta} : H_\alpha(P) \to H_\beta(P)$ between them.

Recall the definition of a positive structure on a scheme/stack given in Section 1.3.

Definition 4.2 *Let P be a semifield and $(X, \{\psi_\alpha\})$ a positive scheme. Then its P–part $X(P)$ is a set whose elements are collections $\{p_\alpha \in H_\alpha(P)\}$ such that $\psi_{\alpha,\beta}(p_\alpha) = p_\beta$ for any $\alpha, \beta \in \mathcal{C}_X$.*

Decomposing H_α into a product of $\mathbb{G}_m$'s we get an isomorphism $\widetilde{\varphi}_\alpha : P^{\dim H_\alpha} \stackrel{\sim}{\longrightarrow} X(P)$.

The P-part $X(P)$ of X depends only on the positive birational class of X, and compatible positive structures give rise to the same P-part.

If $P = K_{>0}$ is a semifield in a field K we can realize the $K_{>0}$–part of X inside of the set of K-points of X. Indeed, thanks to the condition i) of definition 1.4 the map ψ_α is regular at $H_\alpha(K_{>0})$. Therefore there is a subset

$$X^+_{K_{>0}} := \psi_\alpha(H_\alpha(K_{>0})) \hookrightarrow X(K) \tag{31}$$

Thanks to the condition ii) of definition 1.4 it does not depend on α.

We say that a positive atlas $(\{\psi'_\alpha\}, \mathcal{C}')$ on X *dominates* another one $(\{\psi_\alpha\}, \mathcal{C})$ if there is an inclusion $i : \mathcal{C} \hookrightarrow \mathcal{C}'$ such that for any $\alpha \in \mathcal{C}$ the coordinate systems $\psi_\alpha : H_\alpha \to X$ and $\psi'_{i(\alpha)} : H'_{i(\alpha)} \to X$ are identified. So the positive atlases on X are partially ordered. If two positive atlases $\mathcal{C}'$ and $\mathcal{C}''$ dominate the same one $\mathcal{C}$, then there is a new positive atlas $\mathcal{C}' \cup \mathcal{C}''$ dominating all of them. Therefore for a given positive atlas on X there is the *maximal positive atlas* on X dominating it.

Let $(X, \{\psi_\alpha\}, \mathcal{C})$ be a positive scheme. Then $\psi_\alpha(H_\alpha)$ is a Zariski open subset of X. Their union over all $\alpha \in \mathcal{C}$ is a regular Zariski open subset of X. Applying this construction to the maximal positive atlas on X dominating the original one we get a regular Zariski open subset $X^{\rm reg} = X^{\rm reg}_{\mathcal{C}} \hookrightarrow X$ called the *regular part* of a positive variety X.

The P-part of X can be defined in a more general set up of positive spaces.

2. A category of positive spaces. Recall that a groupoid is a category where all morphisms are isomorphisms. Below we will assume that the set of isomorphisms between every two objects of a groupoid is non-empty, i.e. all groupoids are connected. Let $\mathcal{G}$ be a connected groupoid and α its object. The *fundamental group* $\Gamma_{\mathcal{G},\alpha}$ of the groupoid $\mathcal{G}$ based at α is the group of automorphisms of the object α. The fundamental groups based at different objects are isomorphic, but the isomorphism is well defined only up to an inner automorphism.

Definition 4.3 *A* positive space *$\mathcal{X}$ is a pair consisting of a groupoid $\mathcal{G}_\mathcal{X}$ and a functor*

$$\psi_\mathcal{X} : \mathcal{G}_\mathcal{X} \longrightarrow \mathrm{Pos} \tag{32}$$

The groupoid $\mathcal{G}_\mathcal{X}$ is called the coordinate groupoid of a positive space.

Thus for every object α of the coordinate groupoid $\mathcal{G}_\mathcal{X}$ there is an algebraic tori H_α, and for every morphism $f : \alpha \longrightarrow \beta$ in the groupoid there is a positive birational isomorphism $\psi_f : H_\alpha \longrightarrow H_\beta$.

Let $\mathcal{X}$ and $\mathcal{Y}$ be two positive spaces defined via the coordinate groupoids $\mathcal{G}_\mathcal{X}$ and $\mathcal{G}_\mathcal{Y}$ and functors $\psi_\mathcal{X}$ and $\psi_\mathcal{Y}$. A morphism $\mathcal{X} \to \mathcal{Y}$ is given by a functor $\mu_{\mathcal{X},\mathcal{Y}} : \mathcal{G}_\mathcal{X} \to \mathcal{G}_\mathcal{Y}$ and a natural transformation of functors $m : \psi_\mathcal{X} \to \psi_\mathcal{Y} \circ \mu_{\mathcal{X},\mathcal{Y}}$. A morphism is called a *strict morphism* if for every object $\alpha \in \mathcal{G}_\mathcal{X}$ the map $m_\alpha : \psi_\mathcal{X}(\alpha) \to \psi_\mathcal{Y} \circ \mu_{\mathcal{X},\mathcal{Y}}(\alpha)$ is a homomorphism of algebraic tori.

Given a semifield P and a positive space $\mathcal{X}$ defined via (32) we get a functor

$$\psi^P_\mathcal{X} : \mathcal{G}_\mathcal{X} \longrightarrow \mathrm{Sets}, \qquad \alpha \longmapsto H_\alpha(P), \quad H_\alpha := \psi_\mathcal{X}(\alpha)$$

It describes the set $\mathcal{X}(P)$ of P-points of $\mathcal{X}$. The fundamental group of the groupoid $\mathcal{G}_{\mathcal{X}}$ acts on $\mathcal{X}(P)$.

Example 1. A positive variety X provides a functor (32) as follows. The fundamental group of the coordinate groupoid $\mathcal{G}_{\mathcal{X}}$ is trivial, so it is just a set. Namely, it is the set parametrising positive coordinate charts of the atlas defining the positive structure of X. The functor $\psi_{\mathcal{X}}$ is given by $\psi_{\mathcal{X}}(\alpha) := H_\alpha$ and $\psi_{\mathcal{X}}(\alpha \to \beta) := \psi_{\alpha,\beta}$.

Example 2. Let X be a Γ-equivariant positive scheme. It provides a positive space $\mathcal{X}$ given by a functor (32) such that the fundamental group of the groupoid of $\mathcal{G}_{\mathcal{X}}$ is isomorphic to Γ.

3. The semiring of good positive Laurent polynomials for a positive space $\mathcal{X}$. Let $T_{i,\alpha}$ be the coordinates on the tori H_α. Then $\mathbb{Z}[H_\alpha]$ is the ring of Laurent polynomials in $T_{i,\alpha}$. Let us say that a Laurent polynomial F in the variables $T_{i,\alpha}$ is *good* it for every morphism $f : \beta \to \alpha$ in $\mathcal{G}_X$ the rational function f^*F is a Laurent polynomial in the coordinates $T_{i,\beta}$. Denote by $\mathbb{L}(\mathcal{X})$ the ring of all good Laurent polynomials for $\mathcal{X}$. We say that F is a *good positive Laurent polynomials* if for every morphism $f : \beta \to \alpha$ in $\mathcal{G}_X$ the function f^*F is a Laurent polynomial in the coordinates $T_{i,\beta}$ with positive integer coefficients. Denote by $\mathbb{L}_+(\mathcal{X})$ the semiring of good positive Laurent polynomials for $\mathcal{X}$.

Let $\mathbb{Q}_+(\mathcal{X})$ be the semifield of all positive rational functions for $\mathcal{X}$. Set $\widetilde{\mathbb{L}}_+(\mathcal{X}) = \mathbb{Q}_+(\mathcal{X}) \cap \mathbb{L}(\mathcal{X})$. It contains $\mathbb{L}_+(\mathcal{X})$, but may be bigger, as the following example shows: $x^2 - x + 1 = (x^3+1)/(x+1)$.

The semiring $\mathbb{L}_+(\mathcal{X})$ determines the set of *extremal elements* $\mathbf{E}(\mathcal{X})$, that is the elements which can not be decomposed into a sum of two non zero good positive Laurent polynomials. Any element of $\mathbb{L}_+(\mathcal{X})$ is a linear combination of the extremal elements with non negative integer coefficients.

Remark. If we replace a positive atlas on a scheme X by a one dominating it, the set of points of X with values in a semifield P does not change, while the (semi)ring of good (positive) Laurent polynomials can get smaller.

4. The schemes $\mathcal{X}^{**}$ and $\mathcal{X}^*$. Let $\mathcal{X} = (\psi_{\mathcal{X}}, \mathcal{G}_{\mathcal{X}})$ be a positive space. Let $\Gamma := \Gamma_{\mathcal{X},\alpha}$ be the fundamental group of the groupoid $\mathcal{G}_{\mathcal{X}}$. Below we define two Γ-equivariant positive schemes, $\mathcal{X}^* \hookrightarrow \mathcal{X}^{**}$, corresponding to $\mathcal{X}$ in the sense that $\mathcal{X}$ is the positive space assigned to the positive schemes $\mathcal{X}^*$ and $\mathcal{X}^{**}$. We start with the scheme $\mathcal{X}^{**}$.

Lemma 4.4 *Let $\mathcal{X}$ be a positive space. Then there exists unique positive regular Γ-equivariant scheme $\mathcal{X}^{**}$ corresponding to $\mathcal{X}$ which is "minimal" in the following sense. For any positive regular Γ-equivariant scheme $\mathcal{Y}$ corresponding to $\mathcal{X}$ there exists unique morphism $\mathcal{X}^{**} \hookrightarrow \mathcal{Y}$.*

Proof. Let us assume first that the fundamental group of the coordinate groupoid of $\mathcal{X}$ is trivial. Then the scheme $\mathcal{X}^{**}$ is obtained by gluing the coordinate tori H_α according to the birational isomorphisms $\psi_{\alpha,\beta}$. For any field K the set of its K-points is given by collections of elements $p_\alpha \in H_\alpha(K)$ such that $\psi_{\alpha,\beta} p_\alpha = p_\beta$. The general case is obtained by considering the universal covering groupoid for $\mathcal{G}_{\mathcal{X}}$. The lemma is proved.

The ring $\mathbb{L}[\mathcal{X}]$ is identified with the ring of global regular functions on $\mathcal{X}^{**}$.

Let $\{\psi_\alpha^{**}\}$ be the positive atlas of the scheme $\mathcal{X}^{**}$. The scheme $\mathcal{X}^*$ is defined as the subscheme of $\mathcal{X}^{**}$ given by the intersection of all coordinate charts of this atlas:

$$\mathcal{X}^* := \cap_{\alpha \in \mathcal{C}_{\mathcal{X}}} \mathrm{Im}\psi_\alpha^{**}(H_\alpha) \subset \mathcal{X}^{**}$$

It is a Γ-equivariant positive scheme: the positive atlas for $\mathcal{X}^{**}$ induces the one for $\mathcal{X}^*$. For any $\alpha \in \mathcal{C}_{\mathcal{X}}$ the scheme $\mathcal{X}^*$ is the complement to a possibly infinite collection of positive divisors in the torus H_α.

If the fundamental group of the groupoid $\mathcal{G}_{\mathcal{X}}$ is trivial, $\mathcal{X}^*$ is characterised by the following universality property. It is the maximal scheme such that for every α there exists a regular open embedding $i_\alpha : \mathcal{X}^* \hookrightarrow H_\alpha$ and $\psi_{\alpha,\beta} i_\alpha = i_\beta$. Reversing the arrows we arrive at the universality property for the scheme $\mathcal{X}^{**}$.

Let $(X, \{\psi_\alpha\}, \mathcal{C})$ be a positive regular scheme/stack. Then $X^{**} = \cup_{\alpha \in \mathcal{C}} \mathrm{Im}\psi_\alpha(H_\alpha)$ and one has $X^* \subset X^{**} \subset X$.

Lemma 4.5 *Let $(X, \{\psi_\alpha\}, \mathcal{C})$ be a positive regular scheme/stack, and X is noetherian. Then X^{**} can be covered by a finite collection of the open subsets $\psi_\alpha(H_\alpha)$.*

Proof. Since X is noetherian, any open subset of X is quasi-compact, i.e. any cover by open subsets admits a finite subcover. The subset X^{**} is open, and covered by the open subsets $\psi_\alpha(H_\alpha)$. Therefore there exists a finite subcover. The lemma is proved.

Therefore for a positive regular noetherian scheme X the ring $\mathbb{L}(X)$ as well as the semiring $\mathbb{L}_+(X)$ are determined by a finite number of conditions.

5. $\mathbb{Z}$–, $\mathbb{Q}$– and $\mathbb{R}$–tropicalizations and the Thurston boundary of a positive space. Let $\mathbb{A}$ be either $\mathbb{Z}$, or $\mathbb{Q}$, or $\mathbb{R}$. Denote by $\mathbb{A}^t$ the corresponding tropical semifield. Then a positive rational map $\varphi : H_1 \to H_2$ gives rise to a *piecewise-linear* map $\varphi_* : H_1(\mathbb{A}^t) \to H_2(\mathbb{A}^t)$.

Let $\mathcal{X}$ be a positive space. Then we have the manifold $\mathcal{X}(\mathbb{R}_{>0})$, and the $\mathbb{A}$-*tropicalization* $\mathcal{X}(\mathbb{A}^t)$ of $\mathcal{X}$. They are related as follows. First of all, one has $\mathcal{X}(\mathbb{Z}^t) \hookrightarrow \mathcal{X}(\mathbb{Q}^t) \hookrightarrow \mathcal{X}(\mathbb{R}^t)$. The multiplicative group $\mathbb{A}^*_{>0}$ of positive elements in $\mathbb{A}$ acts by automorphisms of the tropical semifield $\mathbb{A}^t$. Let us denote by $\{0\}$ the zero element in the abelian group $H(\mathbb{A}^t)$. Let $\mathbb{A}$ be either $\mathbb{Q}$ or $\mathbb{R}$. Set

$$\mathbb{P}H(\mathbb{A}^t) := \Big(H(\mathbb{A}^t) - \{0\}\Big)/\mathbb{A}^*_{>0}$$

Let $\varphi : H \to H'$ be a positive rational map. Then the piecewise-linear map $\varphi_* : H(\mathbb{A}^t) \to H'(\mathbb{A}^t)$ commutes with the multiplication by $\mathbb{A}^*_{>0}$, and $\varphi_*\{0\} = \{0\}$. Thus we get a morphism

$$\overline{\varphi}_* : \mathbb{P}H(\mathbb{A}^t) \longrightarrow \mathbb{P}H'(\mathbb{A}^t)$$

Definition 4.6 *Let $\mathcal{X}$ be a positive space. Let $\mathbb{A}$ be either $\mathbb{Q}$ or $\mathbb{R}$. The projectivisation $\mathbb{P}\mathcal{X}(\mathbb{A}^t)$ of the $\mathbb{A}$-tropicalisation of $\mathcal{X}$ is*

$$\mathbb{P}\mathcal{X}(\mathbb{A}^t) := \{\{p_\alpha \in \mathbb{P}H_\alpha(\mathbb{A}^t)\} \quad | \quad \overline{\psi}_{\alpha,\beta}(p_\alpha) = p_\beta \text{ for any } \alpha, \beta \in \mathcal{G}_X\}$$

Observe that $\mathbb{P}H_\alpha(\mathbb{R}^t)$ is a sphere, and the maps $\psi_{\alpha,\beta}$ are piecewise-linear maps of spheres. The set $\mathbb{P}\mathcal{X}(\mathbb{Q}^t)$ is an everywhere dense subset of $\mathbb{P}\mathcal{X}(\mathbb{R}^t)$.

The degree map from Lemma 4.1 provides a map

$$-\mathrm{deg} : \mathcal{X}(\mathbb{R}((\varepsilon))_{>0}) \longrightarrow \mathcal{X}(\mathbb{Z}^t)$$

An $\mathbb{R}((\varepsilon))_{>0}$-point $x(\varepsilon)$ of $\mathcal{X}$, considered modulo action of the group of positive reparametrizations of the formal ε-line, i.e. modulo substitutions $\varepsilon = \phi(\varepsilon)$, where $\phi(\varepsilon) \in \mathbb{R}((\varepsilon))_{>0}$, gives rise to a well defined element of $\mathbb{P}\mathcal{X}(\mathbb{Q}^t)$, provided that $\mathrm{deg}x(\varepsilon) \neq \{0\}$. Thus $\mathbb{P}\mathcal{X}(\mathbb{R}^t)$ serves as the Thurston type boundary of $\mathcal{X}(\mathbb{R}_{>0})$.

5 Positive configurations of flags

The first two subsections contain background material about positivity in algebraic semisimple groups, due to Lusztig, and Fomin and Zelevinsky. The rest of the Section is devoted to definition and investigation of positive structures on configurations of three and four generic flags in G.

1. A positive regular structure on U and G. We recall some basic results of G. Lusztig [L1], [L2]. The only difference is that we present them as results about existence and properties of regular positive structures on U and G. The proofs in loc.cit. give exactly what we need. However, as explained in Section 5.8, one needs to be careful when relating this point of view with the one used by Lusztig.

Let W be the Weyl group of G. Let $s_i \in W$ be the simple reflection corresponding to x_i, y_i. Let w_0 be the maximal length element in the Weyl group W, and $w_0 = s_{i_1}...s_{i_n}$ its reduced expression into a product of simple reflections. It is encoded by the sequence $\mathbf{i} := \{i_1, ..., i_n\}$ of simple reflections in the decomposition of w_0.

Proposition 5.1 *a)Each reduced expression of w_0, encoded by $\mathbf{i}$, provide an open regular embedding*

$$\psi_{\mathbf{i}} : \mathbb{G}_m^n \hookrightarrow U, \qquad (a_1, ..., a_n) \longmapsto x_{i_1}(a_1)...x_{i_n}(a_n) \tag{33}$$

b) The collection of embeddings $\psi_{\mathbf{i}}$, when $\mathbf{i}$ run through the set of all reduced expression of w_0, provides a positive structure on U.

c) Replacing x_{i_p} by y_{i_p} in (33) we get an open regular embedding $\psi_{\bar{\mathbf{i}}} : \mathbb{G}_m^n \hookrightarrow U^-$, providing a positive regular structure on U^-. The map $\Psi : U^- \to U^{\text{opp}}$ is a positive regular isomorphism.

Proof. Proposition 2.7 in [L1] and the proof of its part a) prove the proposition.

2. Positive regular structures on double Bruhat cells [FZ99]. The double Bruhat cell corresponding to an element $(u, v) \in W \times W$ is defined by

$$G^{u,v} := BuB \cap B^- vB^- \tag{34}$$

A *double reduced word* for the element $(u, v) \in W \times W$ is a reduced word for an element (u, v) of the Coxeter group $W \times W$. We will use the indices $\overline{1}, \overline{2}, ..., \overline{r}$ for the simple reflections in the first copy of W, and $1, 2, ..., r$ for the second copy. Then a double reduced word $\mathbf{j}$ for (u, v) is simply a shuffle of a reduced word for u written in the alphabet $[\overline{1}, \overline{2}, ..., \overline{r}]$, and a reduced word for v written in the alphabet $[1, 2, ..., r]$. Recall that $l(u)$ is the length of an element $u \in W$. We set $x_{\overline{i}}(t) := y_i(t)$. The following result is Theorem 1.2 in [FZ99].

Proposition 5.2 *a) Given a double reduced word $\mathbf{j} = (j_1, ..., j_{l(u)+l(v)})$ for (u, v) and $0 \leq k \leq l(u) + l(v)$ there is a regular open embedding*

$$x_{\mathbf{j},k} : H \times \mathbb{G}_m^{l(u)+l(v)} \hookrightarrow G^{u,v}$$

$$(h; t_1, ..., t_{l(u)+l(v)}) \longmapsto x_{i_1}(t_1)...x_{i_k}(t_{i_k})hx_{i_{k+1}}(t_{i_{k+1}})...x_{i_{l(u)+l(v)}}(t_{l(u)+l(v)})$$

b) The collection of all regular open embeddings $x_{\mathbf{j},k}$ provide a regular positive structure on the double Bruhat cell $G^{u,v}$.

Indeed, if $k = 0$ this is exactly what stated in loc. cit. Moving h we do not change the positive structure since the characters of H are positive by definition. It follows from Proposition 5.2 that $\dim G^{u,v} = r + l(u) + l(v)$.

The generalized minors and double Bruhat cells [FZ99]. Let $\mathbf{i} = (i_1, ..., i_n)$ be a reduced decomposition of w_0. Set

$$\overline{\overline{s}}_i := \varphi_i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \overline{\overline{w}}_0 := \overline{\overline{s}}_{i_1}...\overline{\overline{s}}_{i_n}$$

Then $\overline{\overline{w}}_0$ does not depend on choice of a reduced decomposition. Using both reduced decompositions $(i_1, ..., i_n)$ and $(i_n, ..., i_1)$ one immediately sees that

$$\overline{\overline{w}}_0 \overline{w}_0 = e, \quad \overline{\overline{w}}_0 = s_G \overline{w}_0, \quad \Psi(\overline{\overline{w}}_0) = \overline{w}_0, \quad \Psi(\overline{w}_0) = \overline{\overline{w}}_0$$

Let $G_0 := U^- HU$. Any element $x \in G_0$ admits unique decomposition $x = [x]_-[x]_0[x]_+$ with $[x]_- \in U_-$, $[x]_0 \in H$, $[x]_+ \in U_+$. Let $i \in [1, ..., r]$. Then according to [FZ99], the generalized minor $\Delta_{u\omega_i, v\omega_i}$ is the regular function on G whose restriction to the open subset $\overline{u} G_0 \overline{v}^{-1}$ is given by

$$\Delta_{u\omega_i, v\omega_i}(x) = \omega_i([\overline{u}^{-1} x \overline{v}]_0) \tag{35}$$

It was shown in loc. cit. that it depends only on the weights $u\omega_i, v\omega_j$. For example if $G = SL_{r+1}$ then for permutations $u, v \in S_{r+1}$ the function $\Delta_{u\omega_i, v\omega_i}$ is given by the $i \times i$ minor whose rows (columns) are labeled by the elements of the set $u([1, ..., i])$ (resp., $v([1, ..., i])$). Let $\mathbf{i}$ be a reduced decomposition of w_0 and $k \in \{[1, l(u) + l(v)]$. Following ([BFZ03], Section 2.3), we set

$$u_{\leq k} = u_{\leq k}(\mathbf{i}) = \prod_{l=1,...,k, \quad \varepsilon(i_l)=-1} s_{|i_l|} \tag{36}$$

$$v_{>k} = v_{>k}(\mathbf{i}) = \prod_{l=l(u)+l(v),...,k+1, \quad \varepsilon(i_l)=+1} s_{|i_l|} \tag{37}$$

where the index l is increasing (decreasing) in (36) (resp., (37)). For $k \in -[1, ..., r]$ we set $u_{\leq k} = e$ and $v_{>k} = v^{-1}$. Finally, for $k \in -[1, r] \cup [1, l(u) + l(v)]$, set

$$\Delta_{k;\mathbf{i}} := \Delta_{u_{\leq k}\omega_{|i_k|}, v_{>k}\omega_{|i_k|}}$$

Let us set

$$F(\mathbf{i}) = \{\Delta(k; \mathbf{i}) : k \in -[1, r] \cup [1, l(u) + l(v)]\}$$

Let

$$G^{u,v}(\mathbf{i}) := \{g \in G^{u,v} : \Delta(g) \neq 0 \text{ for all } \Delta \in F(\mathbf{i})\}$$

The lemma below is a part of Lemma 2.12 in [BFZ03], and follows directly from [FZ99].

Lemma 5.3 *The map $G^{u,v} \to \mathbb{A}^{r+l(u)+l(v)}$ given by $g \longmapsto (\Delta(g))_{\Delta \in F(\mathbf{i})}$ restricts to an isomorphism*

$$G^{u,v}(\mathbf{i}) \xrightarrow{\sim} \mathbb{G}_m^{r+l(u)+l(v)} \tag{38}$$

Let

$$\Delta_{\mathbf{i}} : \mathbb{G}_m^{r+l(u)+l(v)} \hookrightarrow G^{u,v} \tag{39}$$

be the regular open embedding provided by Lemma 5.3. The theorem below is a reformulation of the results in [FZ99].

Theorem 5.4 *The collection of embeddings $\{\Delta_{\mathbf{i}}\}$, when $\mathbf{i}$ runs through the set of all reduced decompositions of (u, v), provides a positive regular structure on the double Bruhat cell $G^{u,v}$. It is compatible with the positive regular structure defined in Proposition 5.2.*

3. A positive regular structure on the intersection of two opposite open Schubert cells in $\mathcal{B}$. Recall the two opposite Borel subgroups B^-, B^+. They determine the two open Schubert cells $\mathcal{B}_-$ and $\mathcal{B}_+$ in the flag variety $\mathcal{B}$, parametrising the Borel subgroups of G in generic position to B^- and B^+. One has

$$\mathcal{B}_- := \{u_- B^+ u_-^{-1}\}, \quad u_- \in U^-; \qquad \mathcal{B}_+ := \{u_+ B^- u_+^{-1}\}, \quad u_+ \in U^+ \tag{40}$$

The formula (40) provides isomorphisms $\alpha^- : U^- \xrightarrow{\sim} \mathcal{B}_-, \quad \alpha^+ : U^+ \xrightarrow{\sim} \mathcal{B}_+$. Let us set

$$\mathcal{B}_* := \mathcal{B}_- \cap \mathcal{B}_+, \quad U_*^+ := U^+ \cap B^- w_0 B^-, \quad U_*^- := U^- \cap B w_0 B$$

Lemma 5.5 *a) There are canonical isomorphisms*

$$\alpha_*^+ : U_*^+ \xrightarrow{\sim} \mathcal{B}_*, \quad u_+ \longmapsto u_+ B^- u_+^{-1}, \qquad \alpha_*^- : U_*^- \xrightarrow{\sim} \mathcal{B}_*, \quad u_- \longmapsto u_- B^+ u_-^{-1}$$

b) $\Psi : U_*^- \longrightarrow U_*^+$ is an isomorphism.

Proof. a) The map α_*^+ is evidently injective. Let us check the surjectivity. Given a pair (u_+, u_-) as in (40), such that $u_+B^-u_+^{-1} = u_-B^+u_-^{-1}$, let us show that $u_-w_0B^- \cap U^+$ is non empty, and hence is a point. Indeed, conjugating B^- by $u_-w_0B^-$ we get $u_-B^+u_-^{-1}$. By the assumption conjugation of B^- by u_+ gives the same result. Thus $u_+ \in u_-w_0B^-$. The part b) is obvious. The lemma is proved.

Lemma 5.5 implies that there is an isomorphism

$$\phi : U_*^- \stackrel{\sim}{\longrightarrow} U_*^+, \qquad \phi := (\alpha_*^+)^{-1}\alpha_*^-$$

Proposition 5.6 *a) For every reduced decomposition* $\mathbf{i}$ *of* w_0 *there are regular open embeddings*

$$\alpha_*^- \circ \psi_{\mathbf{i}} : (\mathbb{G}_m)^n \hookrightarrow \mathcal{B}_+; \qquad \alpha_*^+ \circ \psi_{\overline{\mathbf{i}}} : (\mathbb{G}_m)^n \hookrightarrow \mathcal{B}_- \tag{41}$$

b) The maps $\{\alpha_*^- \circ \psi_{\mathbf{i}}\}$ *provide a positive regular structure on* $\mathcal{B}_*$. *Similarly the maps* $\{\alpha_*^+ \circ \psi_{\mathbf{i}}\}$ *provide another positive regular structure on* $\mathcal{B}_*$.

c) The map ϕ *is positive. Therefore the two positive structures from the part b) are compatible, and thus provide a positive regular structure on* $\mathcal{B}_*$ *dominating them.*

Proof. a) According to Theorem 1.2 in [FZ99] there is a regular open embedding

$$x_{\mathbf{i}} : H \times \mathbb{G}_m^n \to G^{e,w_0} := B^+ \cap B^-w_0B^-, \qquad x_{\mathbf{i}}(h; t_1, ..., t_n) := hx_{i_1}(t_1) \cdot ... \cdot x_{i_n}(t_n)$$

Putting $h = e$ we land in $U^+ \cap B^-w_0B^-$.

b) Follows from Proposition 5.1.

c) In s. 3.2 of [L2] there is an algorithm for computation of a map $\phi' : U^- \longrightarrow U^+$, such that

$$uB^+u^{-1} = \phi'(u)B^-\phi'(u)^{-1}, \quad u \in U^- \tag{42}$$

Such a map is unique. It follows from loc. cit. that it is a positive rational map for the positive structures on U^+ and U^- given by Proposition 5.1. Its restriction to U_*^- is our ϕ. The proposition is proved.

4. A positive regular variety $\mathcal{V}$. Let $\mathcal{V}$ be the quotient of the affine variety U_* by the action of H:

$$\mathcal{V} := U_*/H := \mathrm{Spec}\mathbb{Q}[U_*]^H \tag{43}$$

Let us define a coordinate system on $\mathcal{V}$ corresponding to a reduced decomposition $\mathbf{i}$ of w_0. Let $\alpha_1, ..., \alpha_r$ be the set of all simple roots. Each $i_k \in \mathbf{i}$ is labeled by a simple root, providing a decomposition

$$\mathbf{i} = \mathbf{i}(\alpha_1) \cup ... \cup \mathbf{i}(\alpha_r)$$

Precisely, $j \in \mathbf{i}(\alpha_p)$ if and only if $i_j = p$. Let $l_k := |\mathbf{i}(\alpha_k)|$. So there is a positive regular open embedding

$$\psi_{\mathbf{i}} : \mathbb{G}_m^{\mathbf{i}(\alpha_1)} \times ... \times \mathbb{G}_m^{\mathbf{i}(\alpha_r)} \hookrightarrow U_*$$

The group H acts on the factor $\mathbb{G}_m^{\mathbf{i}(\alpha_k)}$ through its character χ_k'. Since the center of G is trivial, there is an isomorphism $(\chi_1', ..., \chi_r') : H \stackrel{\sim}{\longrightarrow} \mathbb{G}_m \times ... \times \mathbb{G}_m$. So H acts freely on the product. The map $\psi_{\mathbf{i}}$ transforms this action to the action of H on U_* by conjugation. Thus we get a regular open embedding

$$\overline{\psi}_{\mathbf{i}} : \mathbb{G}_m^{\mathbf{i}(\alpha_1)}/\mathbb{G}_m \times ... \times \mathbb{G}_m^{\mathbf{i}(\alpha_r)}/\mathbb{G}_m \hookrightarrow U_*/H$$

where the group $\mathbb{G}_m$ acts on $\mathbb{G}_m^{\mathbf{i}(\alpha_k)}$ diagonally. Let $a_{\alpha_k,1}, ..., a_{\alpha_k,l_k}$ be the natural coordinates on $\mathbb{G}_m^{\mathbf{i}(\alpha_k)}$. Then the natural coordinates on its quotient by the diagonal action of $\mathbb{G}_m$ are

$$t_{\alpha_k,j} := \frac{a_{\alpha_k,j+1}}{a_{\alpha_k,j}}, \quad 1 \leq j \leq l_k - 1 \tag{44}$$

Lemma 5.7 *The collection of regular open embeddings $\{\overline{\psi}_{\mathbf{i}}\}$, where $\mathbf{i}$ run through the set of all reduced expressions of w_0, provides a regular positive structure on $\mathcal{V}$.*

Proof. Follows immediately from the very definition and Proposition 5.1.

Similarly, replacing U by U^- we get a positive variety $\mathcal{V}^-$. It is handy to denote $\mathcal{V}$ by $\mathcal{V}^+$. The map Ψ evidently commutes with the action of H. It follows from the construction of the map ϕ given in s. 3.2 of [L2] that ϕ also commutes with the action of H. Thus they provide isomorphisms

$$\phi : \mathcal{V}^- \longrightarrow \mathcal{V}^+, \quad \Psi : \mathcal{V}^- \longrightarrow \mathcal{V}^+ \tag{45}$$

Abusing the notation we denote them the same way as the original maps.

Lemma 5.8 *The maps ϕ and Ψ in (45) are isomorphisms of positive regular varieties.*

Proof. For Ψ this is clear. For ϕ this follows from Proposition 5.6. The lemma is proved.

5. A positive structure on configurations of triples of flags. We say that a pair of flags is in *generic position* if they are conjugated to the pair (B^+, B^-), or, equivalently, project to the open G-orbit in $\mathcal{B} \times \mathcal{B}$. Recall that configurations of n–tuples of flags are by definition the G–orbits on $\mathcal{B}^n$. Denote by $\sim$ the equivalence relation provided by G–orbits. We say that a configuration of flags is *generic* if every two flags of the configuration are in generic position. The moduli space $\mathrm{Conf}_n(\mathcal{B})$ of generic configurations of $n > 2$ flags is a variety over $\mathbb{Q}$. Indeed, a generic configuration can be presented as

$$(B^-, B^+, u_1 B^+ u_1^{-1}, ..., u_k B^+ u_k^{-1}), \quad u_k \in U_*^-$$

Thus they are parametrised by a Zariski open part of $(U_*^- \times ... \times U_*^-)/H$.

Let us define a positive structure on the configuration space $\mathrm{Conf}_3(\mathcal{B})$. Let $\mathrm{pr} : U_*^\pm \to \mathcal{V}^\pm$ be the canonical projection. Consider a regular open embedding

$$\mathcal{V}^- \hookrightarrow \mathrm{Conf}_3(\mathcal{B}), \qquad v \longmapsto (B^-, B^+, vB^+v^{-1}) := (B^-, B^+, uB^+u^{-1}), \quad \mathrm{pr}(u) = v \tag{46}$$

Since H stabilizes the pair (B^-, B^+), this map does not depend on a choice of $u \in U_*^-$ lifting $v \in \mathcal{V}^-$.

Definition 5.9 *A positive structure on* $\mathrm{Conf}_3(\mathcal{B})$ *is defined by the map (46) and Lemma 5.7.*

According to Lemma 5.8 the positive structure is compatible with the one given by

$$\mathcal{V}^+ \hookrightarrow \mathrm{Conf}_3(\mathcal{B}), \qquad v \longmapsto (B^-, B^+, vB^-v^{-1}) := (B^-, B^+, uB^-u^{-1}), \quad \mathrm{pr}(u) = v \tag{47}$$

Yet another natural positive regular structure on $\mathrm{Conf}_3(\mathcal{B})$ is given by a regular open embedding

$$\mathcal{V}^+ \hookrightarrow \mathrm{Conf}_3(\mathcal{B}), \qquad v \longmapsto (B^-, v^{-1}B^-v, B^+) \tag{48}$$

Recall that the map Ψ acts on the flag variety $\mathcal{B}$ as well as on the configurations of flags.

Theorem 5.10 *Let G be a split reductive group with trivial center. Consider a positive atlas on* $\mathrm{Conf}_3(\mathcal{B})$ *provided by (46). Then one has:*

a) The cyclic shift $(B_1, B_2, B_3) \longmapsto (B_2, B_3, B_1)$ is a positive isomorphism on $\mathrm{Conf}_3(\mathcal{B})$. *Precisely:*

$$(B^-, B^+, uB^-u^{-1}) \sim (B^+, cB^-c^{-1}, B^-), \quad u \in \mathcal{U}_*^+, \quad c := (\phi\Psi)^2(u) \tag{49}$$

b) $\Psi : (B_1, B_2, B_3) \longmapsto (\Psi B_1, \Psi B_2, \Psi B_3)$ provides a positive isomorphism on $\mathrm{Conf}_3(\mathcal{B})$.

c) The two positive regular structures on $\mathrm{Conf}_3(\mathcal{B})$ *given by (46) and (48) are compatible.*

d) The reversing map $(B_1, B_2, B_3) \longmapsto (B_3, B_2, B_1)$ is a positive isomorphism on $\mathrm{Conf}_3(\mathcal{B})$.

So the positive structure on $\mathrm{Conf}_3(\mathcal{B})$ *is invariant under the action of the symmetric group S_3.*

Proof. a) Consider a configuration

$$(B^-, B^+, u_+B^-u_+^{-1}), \qquad \mathrm{pr}(u_+) \in \mathcal{V}_+ \tag{50}$$

Let us compute the composition $\Psi \circ \Psi$ applied to this configuration. Conjugating (50) by u_+^{-1} we get $(u_+^{-1}B^-u_+, B^+, B^-)$. It is, of course, equivalent to the initial configuration (50). Then applying Ψ we get

$$(u_-B^+u_-^{-1}, B^-, B^+), \qquad u_- = \Psi(u_+) \in U_*^-, \quad u_+ \in U_*^+ \tag{51}$$

According to (42) it can be written as

$$(v_+B^-v_+^{-1}, B^-, B^+) \quad \text{for some } v_+ \in U_*^+, \quad v_+ = \phi\Psi(u_-)$$

Conjugating by v_+^{-1} we get $(B^-, v_+^{-1}B^-v_+, B^+)$. Applying Ψ again we obtain

$$(B^+, v_-B^+v_-^{-1}, B^-), \qquad v_- := \Psi(v_+) \in U_*^-$$

We can write it as

$$(B^+, c_+B^-c_+^{-1}, B^-), \qquad c_+ := (\phi\Psi)^2(u_+) \in U_*^+$$

This configuration is the cyclic shift of a configuration (50). On the other hand, since $\Psi^2 = \mathrm{Id}$, it coincides with (50). It remains to use positivity of the maps ϕ (Proposition 5.6c)) and Ψ. The statement a) is proved.

b) The configuration (51) is a cyclic shift of the configuration (50). On the other hand, as the proof of a) shows, it is equivalent to $\Psi(B^-, B^+, u_+B^-u_+^{-1})$. The cyclic shift is positive by a). The statement b) is proved.

c) Conjugating (50) by u_+^{-1} we get $(u_+^{-1}B^-u_+, B^+, B^-)$. Applying the cyclic shift and using a) we get c).

d) By the parts c) and a) the following equivalences provide positive maps:

$$(B^-, B^+, u_+B^-u_+^{-1}) \sim (B^-, v_+^{-1}B^-v_+, B^+) \sim (v_+^{-1}B^-v_+, B^+, B^-)$$

Thus it remains to check that map $(B^+, B^-, v_+^{-1}B^-v_+) \longmapsto (B^+, B^-, v_+B^-v_+^{-1})$ induced by the inversion $v \longmapsto v^{-1}$ is a positive map on $\mathrm{Conf}_3(\mathcal{B})$. Although the inversion map is not positive on U, it induces a positive map on the quotient U_*/H. Indeed, inversion changes the order of factors in a reduced decomposition, which amounts to changing the reduced decomposition, and replaces each factor $x_i(t)$ by $x_i(-t)$. However this does not affect the ratios t_i/t_j used to define the positive atlas on U_*/H, and changing a reduced decomposition we get a positive map. The part d) is proved.

The last assertion of the theorem is a corollary of a) and d). The theorem is proved.

6. A positive structure on $\mathrm{Conf}_4(\mathcal{B})$. Consider a triangulation of a convex 4-gon with the flags B_i assigned to its vertices, as on the picture. Choose in addition an orientation of the internal edge, from B_1 to B_3. Such a data, denoted by $\overline{T}$, is determined by an ordered configuration of 4 flags (B_1, B_2, B_3, B_4).

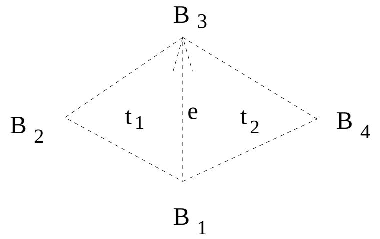


Figure 11: A triangulated quadrilateral with flags assigned to vertices

We will employ the following notation for the group action by conjugation on the flag variety:

$$u \cdot B := uBu^{-1}$$

Lemma 5.11 *The map*

$$(u_+, u_-) \longmapsto (B^-, u_+^{-1}\cdot B^-, B^+, u_-\cdot B^+) \tag{52}$$

provides a regular open embedding

$$i_{\overline{T}}: \mathrm{Conf}_4(\mathcal{B}) \hookrightarrow (U_*^+ \times U_*^-)/H \tag{53}$$

Proof. Let us identify (B_1, B_3) with the standard pair (B^-, B^+) and choose one of the pinnings related to (B^-, B^+). Then the element

$$(B_1, B_2, B_3, B_4) \in \mathrm{Conf}_4(\mathcal{B}) \tag{54}$$

can be written as the right hand side of (52). The action of the group H by conjugation on the pinnings with the given pair (B_1, B_3) is a simple transitive action. So the element (54) is described uniquely by an element of $(U_*^- \times U_*^+)/H$. The lemma is proved.

Just like in Section 5.2, there is a positive regular structure on $(U_*^+ \times U_*^-)/H$. Precisely, take a pair of reduced words $(\mathbf{i_1}, \mathbf{i_2})$ for w_0. Using $\mathbf{i_1}$ to define a positive coordinate system on U_*^+ and $\mathbf{i_2}$ for U_*^-, we get a regular open embedding

$$\overline{\psi}_{(\mathbf{i_1},\mathbf{i_2})}: \mathbb{G}_m^{\mathbf{i_1}(\alpha_1)+\mathbf{i_2}(\alpha_1)}/\mathbb{G}_m \times \ldots \times \mathbb{G}_m^{\mathbf{i_1}(\alpha_r)+\mathbf{i_2}(\alpha_r)}/\mathbb{G}_m \hookrightarrow (U_*^- \times U_*^+)/H$$

Similar to (44), we can introduce homogeneous coordinates on each of the factors $\mathbb{G}_m^{\mathbf{i_1}(\alpha_k)+\mathbf{i_2}(\alpha_k)}/\mathbb{G}_m$. Introducing the homogeneous coordinates one must take into account the fact that $\mathbb{G}_m$ acts on the factor $\mathbb{G}_m^{\mathbf{i_1}(\alpha_k)}$ through the character χ'_k, while on the factor $\mathbb{G}_m^{\mathbf{i_2}(\alpha_k)}$ via its inverse.

Theorem 5.12 *a) The rational map*

$$i_{\overline{T}}^{-1}: (U_*^+ \times U_*^-)/H \longrightarrow \mathrm{Conf}_4(\mathcal{B})$$

transforms the positive structure on $(U_*^+ \times U_*^-)/H$ to a positive structure on the variety $\mathrm{Conf}_4(\mathcal{B})$.

b) The cyclic shift on $\mathrm{Conf}_4(\mathcal{B})$ *is a positive rational map.*

c) The reversion $(B_1, B_2, B_3, B_4) \longmapsto (B_4, B_3, B_2, B_1)$ is a positive rational map on $\mathrm{Conf}_4(\mathcal{B})$. *So the positive structure on* $\mathrm{Conf}_4(\mathcal{B})$ *has a natural dihedral symmetry.*

Proof. a). The map i_T is not an isomorphism. Its image consists of the elements such that the corresponding pair of flags $(u_+^{-1}B^-u_+, u_-B^+u_-^{-1})$ is in generic position. So to prove a) it remains to show that the complement to the image of i_T is contained in a positive divisor.

Let $\overline{T}'$ be another triangulation of the 4-gon on Picture (11), obtained from $\overline{T}$ by changing the oriented edge $B_1 \to B_3$ to the one $B_2 \to B_4$. It provides another regular open embedding

$$i_{\overline{T}'}: \mathrm{Conf}_4(\mathcal{B}) \hookrightarrow (U_*^+ \times U_*^-)/H \tag{55}$$

We need the following proposition.

Proposition 5.13 *The rational map*

$$i_{\overline{T}} \circ i_{\overline{T}'}^{-1}: (U_*^+ \times U_*^-)/H \hookrightarrow (U_*^+ \times U_*^-)/H$$

as well as its inverse are positive rational maps.

Proof. Applying Proposition 5.2 to $\mathbf{j_1} := (\mathbf{i}, \bar{\mathbf{i}})$, $k = n$ and $\mathbf{j_2} := (\bar{\mathbf{i}}, \mathbf{i})$, $k = n$ we get

Lemma 5.14 *The map*

$$\eta: U_*^+ \times H \times U_*^- \longrightarrow U_*^- \times H \times U_*^+, \qquad \eta: (u_-, h, u_+) \longmapsto (a_+, t, a_-) \quad \text{if } u_-hu_+ = a_+ta_-$$

is a positive rational map.

Let (u_+, u_-) be as in (52). Then thanks to Corollary 5.14 there exists a positive rational map

$$(u_+, u_-) \longrightarrow (a_-, t, a_+), \quad a_\pm \in U_*^\pm, t \in H \quad \text{such that } u_+u_- = a_-a_+t$$

Set $g := a_-^{-1}u_+ = a_+t^{-1}u_-^{-1}$. Conjugating (52) by g we get

$$(a_+ \cdot B^-, B^-, a_-^{-1} \cdot B^+, B^+) = (\phi(a_+) \cdot B^+, B^-, \phi(a_-)^{-1} \cdot B^-, B^+)$$

Since ϕ is a positive isomorphism, this proves that the rational map $i_T \circ i_{T'}^{-1}$ is positive. Applying this argument three times we show that the rational map induced by the cube of the cyclic shift is positive. This is equivalent to positivity of the inverse of $i_T \circ i_{T'}^{-1}$. Proposition 5.13 is proved.

Let us return to the proof of Theorem 5.12. The map $i_{\overline{T}'} \circ i_{\overline{T}}^{-1}$ is certainly non regular at the subvariety of configurations of flags (B_1, B_2, B_3, B_4) determined by the condition that all pairs except (B_2, B_4) are in generic position. But according to Proposition 5.13 this subvariety is contained in a positive divisor. The part a) of the theorem is proved. The part b) follows immediately from Proposition 5.13. The part c) follows from the part d) of Theorem 5.10 and the following observation. The theorem is proved.

7. Another description of $\mathrm{Conf}_4(\mathcal{B})$. Recall the triangulation T of a convex 4-gon with the flags B_i assigned to its vertices as on the picture and the internal edge oriented from B_1 to B_3.

Proposition 5.15 *There exists a canonical birational isomorphism*

$$\varphi_T : \mathrm{Conf}_4(\mathcal{B}) \xrightarrow{\sim} \mathrm{Conf}_3(\mathcal{B}) \times H \times \mathrm{Conf}_3(\mathcal{B}) = \mathcal{V}^+ \times H \times \mathcal{V}^- \tag{56}$$

Proof. Let $(B_1, B_2, B_3, B_4) \in \mathrm{Conf}_4(\mathcal{B})$. Then there are projections

$$p_1 : \mathrm{Conf}_4(\mathcal{B}) \longrightarrow \mathrm{Conf}_3(\mathcal{B}), \quad (B_1, B_2, B_3, B_4) \longmapsto (B_1, B_2, B_3);$$

$$p_2 : \mathrm{Conf}_4(\mathcal{B}) \longrightarrow \mathrm{Conf}_3(\mathcal{B}), \quad (B_1, B_2, B_3, B_4) \longmapsto (B_1, B_3, B_4)$$

corresponding to the two triangles of the triangulation T.

We are going to define a rational projection

$$p_{\mathbf{e}} : \mathrm{Conf}_4(\mathcal{B}) \longrightarrow H \tag{57}$$

corresponding to the internal oriented edge $\mathbf{e}$ of the triangulation. Unlike the constructions in Section 5.4, it will not depend on the choice of a reduced decomposition $\mathbf{i}$ of w_0.

Recall the map (53). Recall that we assume that G has trivial center. Let $\mathcal{R}_U$ be the maximal abelian quotient of the unipotent radical U of a Borel subgroup B. The torus $H_U := B/U$ acts on $\mathcal{R}_U$. Let $\mathcal{R}_U^*$ be the unique dense H_U-orbit in $\mathcal{R}_U$. It is a principal homogeneous H_U–space. Let $\pi : U \to \mathcal{R}_U$ be the canonical projection. It provides a rational map

$$p_{\mathbf{e}} : (U_*^+ \times U_*^-)/H \longrightarrow (\mathcal{R}_{U^+}^* \times \mathcal{R}_{U^-}^*)/H = H, \quad (v_+, u_-) \longmapsto \pi(\Psi(v_+))/\pi(u_-)$$

Here the right hand side has the following meaning. For generic (v_+, u_-) both $\Psi(v_+)$ and u_- lie in the H-torsor $\mathcal{R}_{U^-}^*$, so their ratio is a well defined element of H.

Proposition 5.16 *The birational isomorphism (56) provides a positive atlas on the variety* $\mathrm{Conf}_4(\mathcal{B})$*. It is equivalent to the one defined in Section 5.4.*

Proof. It is an immediate corollary of the following lemma.

Lemma 5.17 *Let $(s_1, ..., s_m, t_1, ..., t_n)$ be the natural coordinates on $\mathbb{G}_m^{n+m}$. Set*

$$p_i = \frac{s_{i+1}}{s_i}, \quad i = 1, ..., m-1, \quad q_j = \frac{t_{j+1}}{t_j}, \quad j = 1, ..., n-1, \quad X = \frac{t_1}{s_m}, \quad Y = \frac{s_1 + ... + s_m}{t_1 + ... + t_n}$$

Then the two coordinate systems $(\{p_i\}, \{q_j\}, X)$ and $(\{p_i\}, \{q_j\}, Y)$ on the quotient of $\mathbb{G}_m^{n+m}$ by the diagonal action of $\mathbb{G}_m$ are related by positive transformations.

Proof. Follows immediately from the formula

$$\frac{1+p_1+p_1p_2+...+p_1...p_{m-1}}{1+q_1+q_1q_2+...+q_1...q_{n-1}} = p_1...p_{m-1}XY$$

Proposition 5.15 is proved.

8. A remark. According to Theorem 5.12, positive configurations of four flags have a dihedral symmetry. The following lemma shows that they do not have S_4-symmetry. Recall the H-invariant $p_{\mathbf{e}}(B_1, B_2, B_3, B_4)$ of a configuration of four flags defined by (57).

Lemma 5.18 *If* $p_{\mathbf{e}}(B_1, B_2, B_3, B_4) \in H(\mathbb{R}_{>0})$*, then* $p_{\mathbf{e}}(B_1, B_3, B_2, B_4) \in H(\mathbb{R}_{<0})$*.*

Proof. One has

$$(B_1, B_2, B_3, B_4) \sim (B^-, v_-^{-1}B^+v_-, B^+, u_+B^-u_+^{-1}) \sim \Big(B^-, B^+, v_-B^+v_-^{-1}, v_-u_+B^-(v_-u_+)^{-1}\Big)$$

where $v_- \in U^-(\mathbb{R}_{>0})$ and $u_+ \in U^+(\mathbb{R}_{>0})$. Writing $v_-u_+ = u'_+hv'_-$, where u'_+, h, v'_- are also positive, we get

$$(B_1, B_3, B_2, B_4) \sim \Big(B^-, v_-B^+v_-^{-1}, B^+, u'_+B^-(u'_+)^{-1}\Big)$$

Recall the canonical projection $\pi : U^- \to \mathcal{R}_{U^-} := U^-/[U^-, U^-]$. Elements of the abelian group $\mathcal{R}_{U^-}$ are represented uniquely as $\prod y_\alpha(t_\alpha)$, where α runs through all simple positive roots. If $v_- \in U^-(\mathbb{R}_{>0})$ then these t_α-coordinates of the element $\pi(v_-)$ are in $\mathbb{R}_{>0}$, while for $\pi(v_-^{-1})$ they are in $\mathbb{R}_{<0}$. Now the lemma follows immediately from the definition of the H-invariant. The lemma is proved.

9. A positive atlas on $\mathrm{Conf}_5(\mathcal{B})$ **and its dihedral symmetry**. Consider a convex pentagon with vertices labeled by a configuration $(B_1, ..., B_5)$. Let T be its triangulation. The two diagonals of the triangulation share a vertex. We will assume the diagonals are oriented out of the vertex. Then there is a birational isomorphism

$$\varphi_T : \mathrm{Conf}_5(\mathcal{B}) \xrightarrow{\sim} \mathrm{Conf}_3(\mathcal{B})^3 \times H^2 \tag{58}$$

where the factors $\mathrm{Conf}_3(\mathcal{B})$ correspond to triangles with ordered vertices of the triangulation T, and the factors H are assigned to the oriented diagonals of the triangulation. Precisely, assuming that the diagonals share the vertex B_1, the map is given by

$$(B_1, ..., B_5) \longmapsto (B_1, B_2, B_3) \times (B_1, B_3, B_4) \times (B_1, B_4, B_5) \times p_{\mathbf{e_{13}}}(B_1, B_2, B_3, B_4) \times p_{\mathbf{e_{14}}}(B_1, B_3, B_4, B_5)$$

The decomposition (58) provides a positive atlas of the space $\mathrm{Conf}_5(\mathcal{B})$. We are going to show that it is invariant under the natural action of the dihedral group D_5. So we have to show that it is invariant under the flips and reversing the order of the flags. The claim that a dihedral symmetry map followed by projection to a factor $\mathrm{Conf}_3(\mathcal{B})$ is positive immediately reduced to dihedral symmetry of positive atlases on the spaces $\mathrm{Conf}_n(\mathcal{B})$ for $n = 3, 4$. So it remains only to check similar claims for projections onto the H-factors. More specifically, the only statement not covered by the dihedral symmetry of positive atlases on the spaces $\mathrm{Conf}_n(\mathcal{B})$ for $n = 3, 4$ is this: a flip on the quadrilateral (B_1, B_2, B_3, B_4) of the triangulation on Figure 12, followed by projection to the H-invariant assigned to the diagonal B_1B_4, is a positive map.

It is convenient to use the following parametrisation of the space $\mathrm{Conf}_5(\mathcal{B})$. Consider the map

$$\frac{U^- \times U^+ \times U^+}{H} \longrightarrow \mathrm{Conf}_5(\mathcal{B}); \qquad (u_-, v_+, w_+) \longmapsto (B^-, v_+^{-1} \cdot B^-, v_+^{-1}w_+^{-1} \cdot B^-, B^+, u_- \cdot B^+) \tag{59}$$

This map is evidently a birational isomorphism. The positive atlases on $U^\pm$ provide in the standard way a positive atlas on the quotient on the left, and hence on $\mathrm{Conf}_5(\mathcal{B})$.

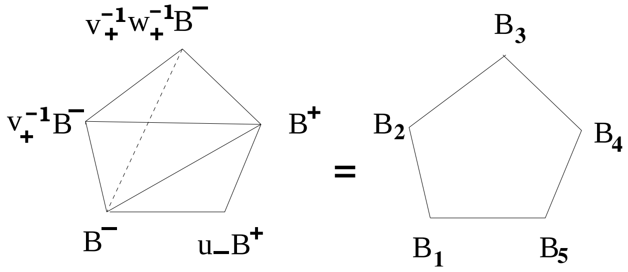


Figure 12: A triangulated pentagon with flags assigned to vertices

Lemma 5.19 *a) The positive atlas on* $\mathrm{Conf}_5(\mathcal{B})$ *given by the birational isomorphism (59) is equivalent to the standard one corresponding to the triangulation shown on Figure 12.*

b) The flip at the punctured diagonal on Figure 12 is a positive automorphism of $\mathrm{Conf}_5(\mathcal{B})$. *Thus the cyclic shift is a positive automorphism of* $\mathrm{Conf}_5(\mathcal{B})$.

c) The reversion $(B_1, ..., B_5) \longmapsto (B_5, ..., B_1)$ *is a positive automorphism of* $\mathrm{Conf}_5(\mathcal{B})$.

Proof. a) It is obvious that, taking into account the S_3-symmetry of the positive atlas on $\mathrm{Conf}_3(\mathcal{B})$, the configurations of triples of flags assigned to the three triangles of the triangulation shown on Figure 12 are parametrised by u_-, v_+, w_+, considered modulo the H-conjugation. So it remains to investigate the H-invariants assigned to the edges.

The two quadrilaterals of the triangulation provide two configurations of 4 flags. The first is

$$(B_1, B_2, B_3, B_4) \sim (v_+ \cdot B^-, B^-, w_+^{-1} \cdot B^-, B^+) = (\phi(v_+) \cdot B^+, B^-, w_+^{-1} \cdot B^-, B^+)$$

Applying the cyclic shift we recover the standard positive atlas on the configurations (B_1, B_2, B_3, B_4). Further, the configuration corresponding to the second quadrilateral is

$$(B_1, B_2, B_4, B_5) = (B^-, v_+^{-1} \cdot B^-, B^+, u_- \cdot B^-)$$

is already in the standard form.

b) After the flip at the diagonal B_1B_3 we get

$$(B_1, B_3, B_4, B_5) \sim (B^-, (w_+v_+)^{-1} \cdot B^-, B^+, u_- \cdot B^-)$$

c) Follows immediately from the properties of $\mathrm{Conf}_n(\mathcal{B})$ for $n = 3, 4$. The lemma is proved.

Remark. The configuration (B_1, B_3, B_4, B_5) is parametrised by the pair (w_+v_+, u_-) considered modulo the H-conjugation. So the group product in U^+ appears naturally here.

10. Positive configurations of K-flags: two points of view. Given a semifield P there are the P-parts $U_P^{\pm}$, H_P, G_P of the corresponding positive varieties. They are monoids. One has $G_P = U_P^+ H_P U_P^- = U_P^- H_P U_P^+$.

Let $K_{>0} \subset K$ be a semifield in a field K. Then there are two ways to define $K_{>0}$-positive configurations of flags. The first, preferable one, is this: we furnish the space $\mathrm{Conf}_n(\mathcal{B})$ with a positive atlas and take its $K_{>0}$-points. For $n = 3, 4, 5$ the positive atlas has been defined above, and the general case is deduced from this in Section 6. This method works for an arbitrary semifield. Below we elaborate another, more naive approach, where the fact that $K_{>0}$ is a semifield in a field K is important. We show that the two approaches lead to two *a priori* different notions, which nevertheless coincide if $K_{>0} = \mathbb{R}_{>0}$.

Recall $\mathrm{Conf}_n(\mathcal{B}(K)) := \mathcal{B}(K)^n/G(K)$.

Definition 5.20 *a) A configuration* $(B_1, B_2, B_3) \in \mathrm{Conf}_3(\mathcal{B}(K))$ *is* $K_{>0}$*-positive if*

$$(B_1, B_2, B_3) \sim (B^-, B^+, u \cdot B^+) \quad \textit{for some } u \in U^-_{K_{>0}}$$

b) A configuration $(B_1, B_2, B_3, B_4) \in \mathrm{Conf}_4(\mathcal{B}(K))$ is $K_{>0}$-positive if

$$(B_1, B_2, B_3, B_4) \sim (B^-, v^{-1} \cdot B^-, B^+, u \cdot B^+) \quad \text{for some } v \in U^+_{K_{>0}} \text{ and } u \in U^-_{K_{>0}}$$

The set of $K_{>0}$-positive configurations of flags for $n = 3, 4$ is denoted by $\mathrm{Conf}^+_n(\mathcal{B}(K))$. Repeating the proves of Theorems 5.10 and 5.12 we get the following:

Corollary 5.21 *Let $K_{>0}$ be a semifield in a field K. Let G be as in Theorem 5.10. Then the sets $\mathrm{Conf}^+_n(\mathcal{B}(K))$ for $n = 3, 4$ have a natural dihedral symmetry.*

Definition 5.22 *Choose a triangulation T of a convex n-gon whose vertices are labeled by flags $(B_1, ..., B_n)$. A configuration $(B_1, ..., B_n) \in \mathrm{Conf}_n(\mathcal{B}(K))$ is $K_{>0}$-positive if the four flags at the vertices of each quadrilateral of the triangulation T form a $K_{>0}$-positive configuration of flags.*

Corollary 5.21 shows that only dihedral order of vertices of each quadrilateral matters. Any two triangulations of a convex n-gon are related by a sequence of flips. Using Corollary 5.21 and the proof of Lemma 5.19 we see that the notion of $K_{>0}$-positivity does not change under a flip. So we get the following proposition, telling us that Definition 5.22 does not depend on choice of a triangulation:

Proposition 5.23 *If a configuration $(B_1, ..., B_n) \in \mathrm{Conf}_n(\mathcal{B}(K))$ is $K_{>0}$-positive with respect to one triangulation, it is positive with respect to any triangulation of the n-gon.*

Let us denote by $\mathrm{Conf}^+_n(\mathcal{B}(K))$ the set of all $K_{>0}$-positive configurations of K-flags.

Corollary 5.24 *The set $\mathrm{Conf}^+_n(\mathcal{B}(K))$ is invariant under the natural dihedral group action.*

Proof. Although the dihedral group action changes the triangulation used to define $K_{>0}$-positive configurations, by Proposition 5.23 it does not affect $K_{>0}$-positivity. The corollary is proved.

For an arbitrary semifield $K_{>0}$ one may have

$$\mathrm{Conf}^+_n(\mathcal{B}(K)) \neq \mathrm{Conf}_n(\mathcal{B})(K_{>0}) \tag{60}$$

Example. The configuration space of three distinct points on P^1 is defined as follows:

$$\mathrm{Conf}_3(P^1) := (P^1 \times P^1 \times P^1 - \text{diagonals})/SL_2$$

It is a single point. However for a field F the set of $SL_2(F)$-orbits on $P^1(F)^3 -$diagonals is $F^*/(F^*)^2$. Indeed, under the action of $SL_2(F)$ the first two points can be made 0 and ∞, and then the subgroup $\mathrm{diag}(t, t^{-1})$ of diagonal matrices in $SL_2(F)$ stabilizing these two points acts on $z \in F^*$ by $z \longmapsto t^2 z$. For example when $F = \mathbb{R}$ there are 2 orbits. Indeed, three ordered distinct points on $P^1(\mathbb{R})$ provide an orientation of $P^1(\mathbb{R})$. Moreover $\mathrm{Conf}^+_n(P^1(K)) = \mathrm{Conf}_n(P^1)(K_{>0})$ if and only if $(K_{>0})^2 = K_{>0}$, which may not be the case.

Here is a general set up. Assume that we have a map of F-varieties $f : X \to Z$, and an F-group G acts on X from the right. Assume that G acts on fibers of f, and this action is transitive on the fibers (over $\overline{F}$). Let $z \in Z(F)$. Set $Y := f^{-1}(z)$. It is a right homogeneous space for G. Let us assume that there exists a point $y \in Y(F)$. Then, setting $H := \mathrm{Stab}_G(y)$, we have $Y = H\backslash G$. So by ([Se], Cor 1 of Prop. 36 in I-5.4) there is canonical isomorphism

$$Y(F)/G(F) = \mathrm{Ker}\Big(H^1(F, H) \longrightarrow H^1(F, G)\Big)$$

The question whether there exists an F-point in Y is non trivial if Y is not a principal homogeneous space for G, and, as pointed out to us M. Borovoi, is treated for p-adic and number fields in the Subsection 7.7 of [Bor]. We will not need this in our story.

So if $Z := X/G$ is the quotient algebraic variety, there are two different sets: $Z(F)$ and $X(F)/G(F)$. The canonical map $X(F)/G(F) \longrightarrow Z(F)$ can be neither surjective nor injective.

However for positive configurations of real flags the situation is as nice as it could be:

Lemma 5.25 *For $n = 3, 4, 5$, one has $\mathrm{Conf}_n^+(\mathcal{B}(\mathbb{R})) = \mathrm{Conf}_n(\mathcal{B})(\mathbb{R}_{>0})$, and there is a canonical inclusion*

$$\mathrm{Conf}_n(\mathcal{B})(\mathbb{R}_{>0}) \hookrightarrow \mathrm{Conf}_n(\mathcal{B}(\mathbb{R}))$$

Proof. It boils down to the following two facts: $\mathcal{V}_*(\mathbb{R}_{>0}) = U_*(\mathbb{R}_{>0})/H(\mathbb{R}_{>0})$ and similarly $((U_*^- \times U_*^+)/H)(\mathbb{R}_{>0}) = U_*^-(\mathbb{R}_{>0}) \times U_*^+(\mathbb{R}_{>0})/H(\mathbb{R}_{>0})$. They follow from $(\mathbb{R}_+^*)^m = \mathbb{R}_+^*$. The lemma is proved.

A positive atlas on the space $\mathrm{Conf}_n(\mathcal{B})$ is defined in Section 6. Then it is straitforward to check that Lemma 5.25 remains valid for all $n > 2$. So the real positive configurations of flags in the sense of Definition 5.22 always coincide with the $\mathbb{R}_{>0}$-points of the positive space $\mathrm{Conf}_n(\mathcal{B})$, and are embedded into the configuration space of n real flags in generic position modulo the $G(\mathbb{R})$-action.

6 A positive structure on the moduli space $\mathcal{X}_{G,\widehat{S}}$

In this Section we give three different proofs, in Sections 6.1, 6.3 and 6.5, of the decomposition of the moduli space $\mathcal{X}_{G,\widehat{S}}$ according to a triangulation. This provides a positive atlas on $\mathcal{X}_{G,\widehat{S}}$.

0. Basic properties of positive atlas on $\mathrm{Conf}_n(\mathcal{B})$ **for** $n = 3, 4, 5$. Let us summarize the properties of the positive atlases on the spaces $\mathrm{Conf}_n(\mathcal{B})$, where $n = 3, 4, 5$, which are used below. The proves are given in Theorem 5.10, Theorem 5.12, and Lemma 5.19.

1) The space $\mathrm{Conf}_3(\mathcal{B})$ has a positive atlas invariant under the action of the group S_3.

2) There is a birational isomorphism

$$\varphi : \mathrm{Conf}_4(\mathcal{B}) \xrightarrow{\sim} \mathrm{Conf}_3(\mathcal{B}) \times H \times \mathrm{Conf}_3(\mathcal{B}) \tag{61}$$

where projections to the left and right factors $\mathrm{Conf}_3(\mathcal{B})$ are given by the maps

$$(B_1, B_2, B_3, B_4) \longmapsto (B_1, B_2, B_3) \quad \text{and} \quad (B_1, B_2, B_3, B_4) \longmapsto (B_1, B_3, B_4)$$

3) The birational isomorphism (61) provides a positive atlas on the space $\mathrm{Conf}_4(\mathcal{B})$. This positive atlas is invariant under the cyclic shift and reversion maps given by

$$(B_1, B_2, B_3, B_4) \longrightarrow (B_4, B_1, B_2, B_3), \qquad (B_1, B_2, B_3, B_4) \longrightarrow (B_4, B_3, B_2, B_1)$$

If we picture a configuration (B_1, B_2, B_3, B_4) at the vertices of a 4-gon as on Figure 11, then the cyclic invariance boils down to the invariance of the positive atlas on $\mathrm{Conf}_4(\mathcal{B})$ under a flip, and hence the invariance under the change of orientation of the diagonal e. The invariance under the reversion map shows the invariance of the positive atlas under the change of orientation of the plane where the 4-gon is located.

4) Consider the convex pentagon whose vertices are labeled by a configuration $(B_1, ..., B_5)$. Let T_1 be its triangulation by the diagonals B_1B_3 and B_1B_4. Then there is a birational isomorphism

$$\varphi_{T_1} : \mathrm{Conf}_5(\mathcal{B}) \xrightarrow{\sim} \Big(\mathrm{Conf}_3(\mathcal{B})\Big)^3 \times H^2 \tag{62}$$

where the factors $\mathrm{Conf}_3(\mathcal{B})$ correspond to the triangles of the triangulation T_1, and the factors H are assigned to the diagonals B_1B_3 and B_1B_4 of the triangulation.

The positive atlas of the space $\mathrm{Conf}_5(\mathcal{B})$ provided by (62) is invariant under the natural action of the dihedral group D_5.

1. A positive atlas on $\mathcal{X}_{G,\widehat{S}}$. Let us spell in detail the definition of the map π_T outlined in Section 1.2. Let $\widehat{S}$ be a marked hyperbolic surface. Let us choose a pair $\mathbf{T}$ given by

$$\mathbf{T} = \{\text{an ideal triangulation } T \text{ of } \widehat{S}, \text{ a choice of orientation of all internal edges of } T\} \tag{63}$$

Let assume first that the triangulation is regular in the sense of Section 4.8. Denote by $\mathrm{tr}(T)$ the set of the triangles of T, and by $ed_i(\mathbf{T})$ be the set of the internal edges of $\mathbf{T}$. Denote by $\mathcal{X}_{G,\widehat{t}}$ the moduli space of the framed G–local systems on a triangle t, considered as a disc with three marked points on the boundary given by certain three points inside of the edges, one point per each edge. Let $(\mathcal{L}, \beta) \in \mathcal{X}_{G,\widehat{S}}$. We can view a triangle t of the ideal triangulation as a disc with three marked points at the boundary, as we just explained. Restricting the framed local system $(\mathcal{L}, \beta) \in \mathcal{X}_{G,\widehat{S}}$ to such a triangle t we get an element of $\mathcal{X}_{G,\widehat{t}}$. Indeed, if a vertex s of the triangle t belongs to a boundary arc on $\widehat{S}$ then there is a flag at this point. If s is a puncture then we take the monodromy invariant flag at the intersection of the triangle with a little disc around the puncture. So we get a projection $p_t : \mathcal{X}_{G,\widehat{S}} \longrightarrow \mathcal{X}_{G,\widehat{t}}$. Further, each internal edge e of the triangulation T determines a pair of triangles t_1, t_2 sharing e. Their union $t_1 \cup t_2$ is a 4-gon with four marked points on the boundary, one per each side. Thus restricting $(\mathcal{L}, \beta)$ to $t_1 \cup t_2$ we get an element of $\mathcal{X}_{G,4}$. Its H–invariant corresponding to the oriented edge $\mathbf{e}$ is given by (57). It provides a rational projection $p_{\mathbf{e}} : \mathcal{X}_{G,\widehat{S}} \longrightarrow H$. The collection of projections $\{p_t\}, \{p_{\mathbf{e}}\}$ provides a rational map

$$\pi_{\mathbf{T}} : \mathcal{X}_{G,\widehat{S}} \longrightarrow \prod_{t \in \mathrm{tr}(T)} \mathcal{X}_{G,\widehat{t}} \times \prod_{\mathbf{e} \in ed_i(\mathbf{T})} H \tag{64}$$

If the triangulation T is special, the above construction works unless we consider the edge given by the loop of the virus. The construction of the H-invariant for this edge is obtained by going to the $2:1$ cover of S described in Section 4.8, and then using the above construction of the H-invariant. The same trick, described in detail in the Section 9.6 below, is used in the proof of the theorems in this Section in the part related to this edge.

Theorem 6.1 *Let G be a split semisimple algebraic group with trivial center and $\widehat{S}$ a marked hyperbolic surface. Then the map $\pi_{\mathbf{T}}$ is a birational isomorphism. The collection of rational maps $\{\pi_{\mathbf{T}}^{-1}\}$, where $\mathbf{T}$ run through all datas (63), provides a Γ_S–equivariant positive atlas on $\mathcal{X}_{G,\widehat{S}}$.*

Proof. Let us prove first the theorem in the special case when $\widehat{S} = \widehat{D}_n$ is a disc with n marked points on the boundary. We proceed by the induction on n. For $n = 3$ this is given the property 1) above. To make the inductive step from n to $n+1$ consider a convex $(n+1)$-gon P_{n+1} with the vertices $b_1, ..., b_{n+1}$ taken in the order compatible with the orientation of the disc. We assign a configuration of flags $(B_1, ..., B_{n+1})$ to the vertices of P_{n+1}, so that B_i is assigned to the vertex b_i. Let us triangulate this $(n+1)$-gon so that (b_1, b_2, b_3) is one of the triangles of the triangulations. Let P_n be the n-gon obtained by cutting out the triangle (b_1, b_2, b_3). By the induction assumption a generic data on the right hand side of (64) determines a configuration $(B_1, B_3, B_4, ..., B_{n+1}) \in \mathrm{Conf}_n(\mathcal{B})$ assigned to the vertices of the n-gon P_n. Adding the triangle (b_1, b_2, b_3) we add to this data the element $\beta_{123} \in \mathrm{Conf}_3(\mathcal{B})$ corresponding to this triangle, and an H-invariant $h_{13} \in H$ corresponding to the edge (b_1, b_3). Let (b_1, b_3, b_k) be the triangle of the triangulation adjacent to the edge (b_1, b_3), and $\beta_{13k} \in \mathrm{Conf}_3(\mathcal{B})$ is the invariant assigned to this triangle. According to the property 2) there is a unique configuration (B'_1, B'_2, B'_3, B'_k) of four flags with the invariant $(\beta_{123}, h_{13}, \beta_{13k})$. Observe that the configurations (B'_1, B'_3, B'_k) and (B'_1, B_3, B_k) are the same. Since G has trivial center, it acts without stable points on the space of generic triples of flags in G. Therefore there exists unique configuration $(B_1, B_2, ..., B_{n+1})$ such that $(B'_1, B'_2, B'_3, B'_k) = (B_1, B_2, B_3, B_k)$ corresponding to the data provided by the triangulation of the $(n+1)$-gon. So, given a triangulation T of the $(n+1)$-gon, the map (64) is a birational isomorphism for the disc $\widehat{D}_{n+1}$. Thus it provides a positive atlas on $\mathrm{Conf}_{n+1}(\mathcal{B})$. It follows from 3) that the positive atlas corresponding to the triangulation T' obtained from T by a flip is compatible with this positive atlas. The role of the mapping class group in this case is played by the dihedral group generated by the cyclic shift and reversion maps. This proves the theorem for the disc.

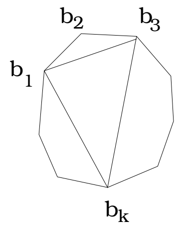


Figure 13: The quadrilateral corresponding to the triangle $b_1b_2b_3$

Now let us consider the general case. It is sufficient to show that, given an arbitrary field F, a generic point x of the right hand side of (64) provides a F-point y of the right hand side such that $\pi(x) = y$. Let us define a representation $\rho_x : \pi_1(S) \longrightarrow G(F)$ corresponding to x. Let T be an ideal triangulation of S. Lifting it to a universal cover $\widetilde{S}$ of S we obtain a triangulation $\widetilde{T}$. The triangulated surface $(\widetilde{S}, \widetilde{T})$ is isomorphic to the hyperbolic disc $\mathcal{H}$ equipped with the Farey triangulation. Let (a, b, c) be a triangle of $\widetilde{T}$. Let $\gamma \in G(F)$. Take a finite polygon P_γ of the triangulation $\widetilde{T}$ containing the triangles (a, b, c) and $(a', b', c') := \gamma(a, b, c)$. The polygon P_γ inherits a triangulation. So by the first part of the proof, the data x, restricted to the triangulated polygon, gives rise to a unique element $y(P_\gamma) \in \mathrm{Conf}_n(\mathcal{B}(F))$ corresponding to it. Let (B_a, B_b, B_c) be the corresponding configuration of flags attached to the vertices (a, b, c) of the polygon. Then there exists an element $g_\gamma \in G(F)$ such that

$$(B_{a'}, B_{b'}, B_{c'}) = g_\gamma(B_a, B_b, B_c)$$

Such an element is unique since G has trivial center. It is easy to see that $\gamma \to g_\gamma$ is a representation of $\pi_1(S, p)$. and it does not depend on the choice of the polygon. We leave construction of the framing as an easy exercise. The theorem is proved.

In the next subsection we work out an explicit construction of the framed local system corresponding to the generic data in (64).

2. The six canonical pinnings determined by a generic triple of flags in G. Let us choose two reduced words $\mathbf{i}^+$ and $\mathbf{i}^-$ of the element w_0. Set

$$U_*^-(\mathbf{i}^-) := \psi_{\mathbf{i}^-}(\mathbb{G}_m^n) \subset U_*^-, \qquad U_*^+(\mathbf{i}^+) := \psi_{\mathbf{i}^+}(\mathbb{G}_m^n) \subset U_*^+$$

The group H acts freely on each of these spaces. So the H-orbit of any element $u \in U_*^-(\mathbf{i}^-)$ intersects the subvariety

$$\overline{U_*^-}(\mathbf{i}^-) := \psi_{\mathbf{i}^-}\Big(e \times \mathbb{G}_m^{\mathbf{i}^-(\alpha_1)-1} \times e \times \mathbb{G}_m^{\mathbf{i}^-(\alpha_2)-1} \times ... \times e \times \mathbb{G}_m^{\mathbf{i}^-(\alpha_r)-1}\Big) \subset U_*^-(\mathbf{i}^-) \qquad (65)$$

in a single point, called the normalized representative $\overline{u}$. Similarly we define the normalized representative $\overline{v}$ of the H-orbit of an element $v \subset U_*^+(\mathbf{i}^+) := \psi_{\mathbf{i}^+}(\mathbb{G}_m^n)$ as the intersection point of the orbit with the subvariety

$$\overline{U_*^+}(\mathbf{i}^+) := \psi_{\mathbf{i}^+}\Big(\mathbb{G}_m^{\mathbf{i}^+(\alpha_1)-1} \times e \times \mathbb{G}_m^{\mathbf{i}^+(\alpha_2)-1} \times e \times ... \times \mathbb{G}_m^{\mathbf{i}^+(\alpha_r)-1} \times e\Big) \subset U_*^+(\mathbf{i}^+) \qquad (66)$$

Let (B_1, B_2, B_3) be a sufficiently generic triple of flags in G. Then one has

$$(B_1, B_2, B_3) = g_- \cdot (B^-, B^+, \overline{u}_- B^+ \overline{u}_-^{-1}), \quad u_- \in U_*^-(\mathbf{i}^-)$$

$$(B_1, B_2, B_3) = g_+ \cdot (B^-, \overline{v}_+^{-1} B^- \overline{v}_+, B^+), \quad v_+ \in U_*^+(\mathbf{i}^+)$$

for the uniquely defined elements $g_-, g_+ \in G$. ("Sufficiently generic", by definition, means that the above two formulas hold). Therefore such a triple (B_1, B_2, B_3) of flags in G determines the following two pinnings for G:

$$p^+_{(B_1,B_2,B_3)}(B_1, B_2) := g_- \cdot (B_-, B_+), \qquad p^-_{(B_1,B_2,B_3)}(B_1, B_3) := g_+ \cdot (B_-, B_+)$$

Using the cyclic shifts we define the rest of the pinnings. So we get the six pinnings for G, where the indices i are modulo 3:

$$p^{+}_{(B_1,B_2,B_3)}(B_i, B_{i+1}), \quad p^{-}_{(B_1,B_2,B_3)}(B_i, B_{i-1}), \tag{67}$$

These pinnings are determined by the cyclic configuration of the three flags (B_1, B_2, B_3) and the choice of two reduced words $\mathbf{i}^-, \mathbf{i}^+$ for w_0. Observe that $p^{\pm}$ corresponds to $g^{\mp}$. The + and − in (67) indicate whether the corresponding edge is oriented accordantly to the cyclic order of (B_1, B_2, B_3) or not. It is handy to picture the six pinning related to a triple $(B_1, B_2, B_3) \in \mathrm{Conf}^*_3(\mathcal{B})$ by arrows as on the picture below. Say that U *is a positive Zariski open subset of a positive variety* X if $X - U$

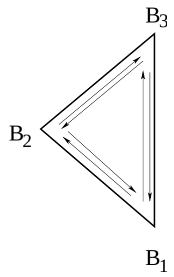


Figure 14: Arrows encode the six pinnings assigned to a configuration of three flags

is contained in a positive divisor of X. The discussion above proves the following lemma.

Lemma 6.2 *There exists a positive Zariski open subset* $\mathrm{Conf}^*_3(\mathcal{B}) \subset \mathrm{Conf}_3(\mathcal{B})$ *such that for any configuration* $(B_1, B_2, B_3) \in \mathrm{Conf}^*_3(\mathcal{B})$ *the construction above determines the six pinnings (67) for* G*. Changing a standard pinning for* G *amounts to a shift of each of our six pinnings by a common element in* G*.*

We leave a precise description of $\mathrm{Conf}^*_3(\mathcal{B})$ as the intersection of several explicitly given positive subvarieties of $\mathrm{Conf}_3(\mathcal{B})$ as a straitforward exercise.

The edge invariant. Given a generic configuration (B_1, B_2, B_3, B_4) of flags in G, such that both (B_1, B_2, B_3) and (B_1, B_3, B_4) are in $\mathrm{Conf}^*_3(\mathcal{B})$, there exists unique element

$$h_{(\mathbf{i}^-,\mathbf{i}^+)}(B_1, B_2, B_3, B_4) \in H$$

such that

$$p^{+}_{(B_1,B_3,B_4)}(B_1, B_3) = h_{(\mathbf{i}^-,\mathbf{i}^+)}(B_1, B_2, B_3, B_4) \cdot p^{-}_{(B_1,B_2,B_3)}(B_1, B_3) \tag{68}$$

The element $h_{(\mathbf{i}^-,\mathbf{i}^+)}(B_1, B_2, B_3, B_4) \in H$ in (68) is called the *edge invariant* of the configuration of flags (B_1, B_2, B_3, B_4) corresponding to the oriented edge (B_1, B_3).

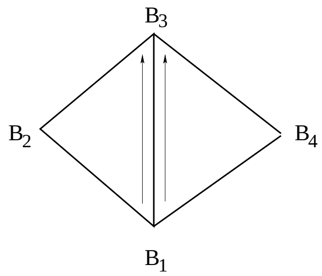


Figure 15: The edge invariant is obtained by comparison of the two pinnings shown by arrows

3. The positive structure of $\mathcal{X}_{G,\widehat{S}}$. To describe the rational inverse to this map we need a finer data assigned to a triangulation T of $\widehat{S}$.

Definition 6.3 *Denote by $\mathbb{T}$ the following data on $\widehat{S}$:*

i) An ideal triangulation T of a marked hyperbolic surface $\widehat{S}$.

ii) For each internal edge of the triangulation T a choice of orientation of this edge.

iii) For each triangle t of the triangulation T a choice of a vertex of this triangle.

iv) A pair of reduced decompositions $(\mathbf{i}^-, \mathbf{i}^+)$ of the maximal length element $w_0 \in W$.

We can visualize a choice described in iii) as a choice of an angle for each of the triangles t.

Theorem 6.4 *Let G be a split semisimple algebraic group with trivial center. Let $\widehat{S}$ be a marked hyperbolic surface and $\mathbb{T}$ a data assigned to an ideal triangulation T of $\widehat{S}$, as in Definition 6.3. Then*

a) There exists a regular open embedding

$$\psi_{\mathbb{T}} : \mathrm{Conf}_3^*(\mathcal{B})^{\{triangles\ of\ T\}} \times H^{\{edges\ of\ T\}} \hookrightarrow \mathcal{X}_{G,\widehat{S}} \tag{69}$$

b) The collection of regular open embeddings $\{\psi_{\mathbb{T}}\}$, when $\mathbb{T}$ runs through all possible datas from Definition 6.3 assigned to $\widehat{S}$, provides a Γ_S–equivariant positive structure on $\mathcal{X}_{G,\widehat{S}}$.

c) The positive structure defined in Theorem 6.1 is compatible with the one defined in b).

Proof. a) Recall the six pinning assigned according to Lemma 6.2 to an element of $C \in \mathrm{Conf}_3^*(\mathcal{B})$. Their dependence on the choice of a standard pinning (B^-, B^+) boils down to a shift by a common element in G. Further, let us suppose that we have two elements $C_1, C_2 \in \mathrm{Conf}_3^*(\mathcal{B})$ and an element $h \in H$. Then there exists unique configuration $(B_1, B_2, B_3, B_4) \in \mathrm{Conf}_4(\mathcal{B})$ such that

$$C_1 = (B_1, B_2, B_3), \quad C_2 = (B_1, B_3, B_4), \quad h_{(\mathbf{i}^-,\mathbf{i}^+)}(B_1, B_2, B_3, B_4) = h \tag{70}$$

Moreover if we choose one of the six pinnings corresponding to the configuration (B_1, B_2, B_3), then there is unique way to normalize the six pinnings corresponding to the configuration (B_1, B_3, B_4) so that formula (68) holds with $h_{(\mathbf{i}^-,\mathbf{i}^+)}(B_1, B_2, B_3, B_4) = h$. Therefore given (70) and one of the 12 pinnings shown by arrows on the picture we determine uniquely the rest of the pinnings. Let Γ be

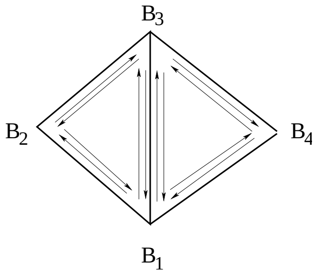


Figure 16: The twelve pinnings assigned to a quadrilateral

a graph on a marked hyperbolic surface $\widehat{S}$ dual to an ideal triangulation T of $\widehat{S}$. Consider a closed path $\mathbf{p}$ on the graph. We are going to attach an element $g_{\mathbf{p}} \in G$ to this path which will give rise to a homomorphism $\pi_1(S) \to G$. Let us consider the triangles of the triangulation corresponding to the vertices of the path $\mathbf{p}$, as on the picture. Let us choose one of these triangles and denote it t_1. The data $\mathbb{T}$ allows to assign to each triangle t a configuration of flags (B_1, B_2, B_3) sitting at the vertices of t, so that B_1 is assigned to the distinguished vertex (provided by the condition iii) in definition of $\mathbb{T}$), and the orientation of the triangle t given by the order of B_i's is the same as the one induced by the orientation of the surface S (clockwise on our pictures). It follows from the remarks made about pinnings above that given one of the six canonical pinnings associated to the flags sitting at the vertices of the triangle t_1, we determine uniquely the 6-tuples of pinnings corresponding to the triangles following t_1 along the path $\mathbf{p}$, till we return to the triangle t_1. Thus for a closed path $\mathbf{p}$

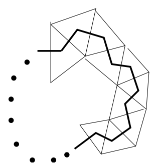


Figure 17: Constructing the local system

there are two 6-tuples of canonical pinnings assigned to the triangle t_1: the original one, and the one obtained after traveling along the loop $\mathbf{p}$. These two 6-tuples of pinnings differ by a shift in G. So we pick up an element $g_{\mathbf{p}} \in G$ transforming the original 6-tuple of pinnings to the final one. Clearly the map $\mathbf{p} \longmapsto g_{\mathbf{p}}$ is multiplicative: $\mathbf{p}_2\mathbf{p}_1 \longmapsto g_{\mathbf{p_2}}g_{\mathbf{p_1}}$. Moreover $\mathbf{p}^{-1} \longmapsto g_{\mathbf{p}}^{-1}$. Since $\mathbf{p}$ is a path on the graph, its homotopy class is determined uniquely by the path itself. So we get a group homomorphism $\pi_1(S, x) \to G$ where x is a base point on Γ inside of the triangle t_1.

Let us define an framed structure on this representation. Let s be a puncture on S. Consider all triangles of the triangulation sharing the vertex s. Let us choose a path $\mathbf{p_s}$ from the base point x to

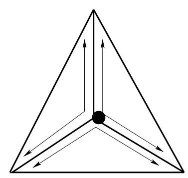


Figure 18: The link of a puncture s

a point inside of certain triangle t_s sharing the vertex s. Then given a pinning P_x in the triangle t_x containing x we get the six pinning assigned to the triangle T_s. Let P_s be one of them corresponding to an arrow going from the vertex s. Let (B, B') be the pair of Borel subgroups underlying this pinning, so that B is assigned to the vertex s. Going around s we define pinnings corresponding to the other triangles sharing s. Observe that all pinnings corresponding to the arrows starting at s have the same "initial" Borel subgroup, B. Therefore the monodromy of the local system corresponding to our representation around a small loop $\mathbf{l_s}$ going around s belongs to B. So if $P_s = gP_x$, then the element $\mathbf{p_s}^{-1}\mathbf{l_s}\mathbf{p_s}$ leaves the Borel subgroup $g^{-1}Bg$ invariant. This provides an framed structure of our representation at the puncture s.

We have defined the map $\psi_{\mathbb{T}}$. Let us show that it is injective. Let us define a left inverse map

$$\pi_{\mathbb{T}} : \mathcal{X}_{G,\widehat{S}} \longrightarrow \prod_{t \in \mathrm{tr}(T)} \mathcal{X}_{G,\widehat{t}} \times \prod_{\mathbf{e} \in \mathbf{ed_i}(\mathbb{T})} H$$

Here $\mathbf{ed_i}(\mathbb{T})$ is the set of internal edges of the triangulation T oriented as prescribed by the data $\mathbb{T}$. So the second product is over the set of all internal edges of T. The $\prod_{t \in \mathrm{tr}(T)} \mathcal{X}_{G,\widehat{t}}$-component of the map $\pi_{\mathbb{T}}$ is the same as for the map π_T. The H-component corresponding to an oriented internal edge $\mathbf{e}$ is the edge invariant from (68) assigned to $\mathbf{e}$. Notice that it is different from the one used in π_T. Then by the very construction we have $\pi_{\mathbb{T}} \circ \psi_{\mathbb{T}} = \mathrm{Id}$. Thus the map $\psi_{\mathbb{T}}$ is injective.

Lemma 6.5 *Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ be maps of irreducible algebraic varieties of the same dimension. Then $g \circ f = \mathrm{Id}_X$ implies that both f and g are birational isomorphisms.*

Proof. The conditions of the lemma imply that the maps f, g are dominant, so there are maps of fields $f^* : \mathbb{Q}(Y) \to \mathbb{Q}(X)$, $g^* : \mathbb{Q}(X) \to \mathbb{Q}(Y)$. Since $f^*g^* = \mathrm{Id}$, the lemma follows.

Lemma 6.6 *Let $\widehat{S}$ be a marked hyperbolic surface. Denote by $t(T)$ and $e_i(T)$ the number of triangles and internal edges of an ideal triangulation T of $\widehat{S}$. Then one has*

$$\dim \mathcal{X}_{G,\widehat{S}} = e_i(T)\dim H + t(T)(\dim U - \dim H) \tag{71}$$

This Lemma, of course, follows from the proof given in Section 6.1. However we will give an independent proof for completeness of the arguments.

Proof. Assume for a moment that the number k of marked boundary points on $\widehat{S}$ is equal to zero, i.e. $\widehat{S} = S$ and $e(T) = e_i(T)$ is the number of all edges. Then $\chi(S) = -e(T)+t(T), \quad 3e(T) = 2|t(T)|$. Therefore

$$e(T) = -3\chi(S), \quad t(T) = -2\chi(S)$$

It is well known that $\dim\mathcal{L}_{G,S} = \chi(S)\dim G$. The canonical map $\mathcal{X}_{G,S} \to \mathcal{L}_{G,S}$ is a finite map at the generic point. This follows from the well known fact that the monodromies of a generic G-local system on S around the pictures are regular conjugacy classes in G. This gives $\dim\mathcal{X}_{G,S} = -\chi(S)\dim G$. The equality (71) in the case $\widehat{S} = S$ follows immediately from these remarks. To check (71) in the general case observe that adding a marked point we increase the dimension by $\dim(G/B) = \dim U$. On the other hand adding a marked point we add by one the number of internal edges and triangles of a triangulation. So it remains to check the statement in the case when S is a disc or cylinder to have the base for the induction. The lemma is proved.

It follows from Lemma 6.6 both spaces in (69) are of the same dimension. Thus $\psi_{\mathbb{T}}$ is a birational isomorphism. The part a) of the theorem is proved.

b) It is sufficient to show that if T and T' are two triangulations of $\widehat{S}$ related by a flip, as on the picture below, and $\mathbb{T}$ and $\mathbb{T}'$ are two datas refining these triangulations, then the positive structures defined by $\psi_{\mathbb{T}}$ and $\psi_{\mathbb{T}'}$ are compatible. This follows from Theorem 5.12and Lemma 5.19. The part b) of the theorem is proved.

c) It follows from Proposition 5.16. The theorem is proved.

4. The higher Teichmüller $\mathcal{X}$-space and the $\mathcal{X}$-lamination space related to G.

Definition 6.7 *Let $\widehat{S}$ be a marked hyperbolic surface and G a split reductive algebraic group with trivial center.*

a) The Teichmüller space $\mathcal{X}^+_{G,\widehat{S}}$ is the $\mathbb{R}_{>0}$-positive part $\mathcal{X}_{G,\widehat{S}}(\mathbb{R}_{>0})$ of the positive space $\mathcal{X}_{G,\widehat{S}}$.

b) Let $\mathbb{A}^t$ be one of the tropical semifields $\mathbb{Z}^t, \mathbb{Q}^t, \mathbb{R}^t$. The set of $\mathcal{X}^{\mathbb{A}}$-laminations on $\widehat{S}$ corresponding to G is the set $\mathcal{X}_{G,\widehat{S}}(\mathbb{A}^t)$ of points of the positive space $\mathcal{X}_{G,\widehat{S}}$ with values in the tropical semifield $\mathbb{A}^t$.

According to Section 4.5 projectivisation of the set of real $\mathcal{X}$-laminations can be considered as the Thurston type boundary of the space $\mathcal{X}^+_{G,\widehat{S}}$.

Applying the definition of the regular part of a positive variety (Section 4.1) in our case we get the regular part $\mathcal{X}^{\rm reg}_{G,\widehat{S}} \hookrightarrow \mathcal{X}_{G,\widehat{S}}$. The higher Teichmüller space is a connected component in $\mathcal{X}^{\rm reg}_{G,\widehat{S}}(\mathbb{R})$.

Recall from Section 2.1 the set $\widetilde{\mathcal{X}}_{G,\widehat{S}}(\mathbb{R})/G(\mathbb{R})$ parametrising framed representations $\pi_1(S,x) \to G(\mathbb{R})$ modulo $G(\mathbb{R})$-conjugation. If $\widehat{D}_3$ is a disc with three marked points, then ordering these points we get an isomorphism $\widetilde{\mathcal{X}}_{G,\widehat{D}_3}(\mathbb{R}) = \mathcal{B}(\mathbb{R})^3$. Let T be an ideal triangulation of $\widehat{S}$. Then each triangle t of the triangulation provides a restriction map $\widetilde{\mathcal{X}}_{G,\widehat{S}}(\mathbb{R})/G(\mathbb{R}) \to \widetilde{\mathcal{X}}_{G,\widehat{t}}(\mathbb{R})/G(\mathbb{R})$. We employ quadrilaterals of the triangulation in a similar way.

An element of $\widetilde{\mathcal{X}}_{G,\widehat{S}}(\mathbb{R})/G(\mathbb{R})$ is called *positive* if its restriction to every quadrilateral of a given triangulation T belongs to ${\rm Conf}_4^+(\mathcal{B}(\mathbb{R}))$.

Corollary 6.8 *The set of positive elements in $\widetilde{\mathcal{X}}_{G,\widehat{S}}(\mathbb{R})/G(\mathbb{R})$ does not depend on the choice of triangulation T. It is isomorphic to $\mathcal{X}^+_{G,\widehat{S}}$. So there is an embedding $\mathcal{X}^+_{G,\widehat{S}} \hookrightarrow \widetilde{\mathcal{X}}_{G,\widehat{S}}(\mathbb{R})/G(\mathbb{R})$.*

Proof. Any two ideal triangulations of a hyperbolic marked surface are related by a sequence of flips. Corollary 5.21 plus Lemma 5.19 imply that a flip does not change the set of positive configurations. The second claim follows from Lemma 5.25 and the proof of Theorem 6.4.

5. A constructive proof of Theorem 6.1. Here is a convenient way to spell the definition of the framed local system given by the map $\psi_{\mathbb{T}}$, which we will use to study its monodromy. To simplify the exposition let us assume that $\widehat{S} = S$. To describe the answer we use the following picture. Starting from an ideal triangulation T of S, we construct a graph Γ'_T embedded into the surface by drawing small edges transversal to each side of the triangles and inside each triangle connect the ends of edges pairwise by three more edges, as shown on the picture. The edges of the little oriented triangles are called t-edges. The edges dual to the edges of the triangulation T are called e-edges.

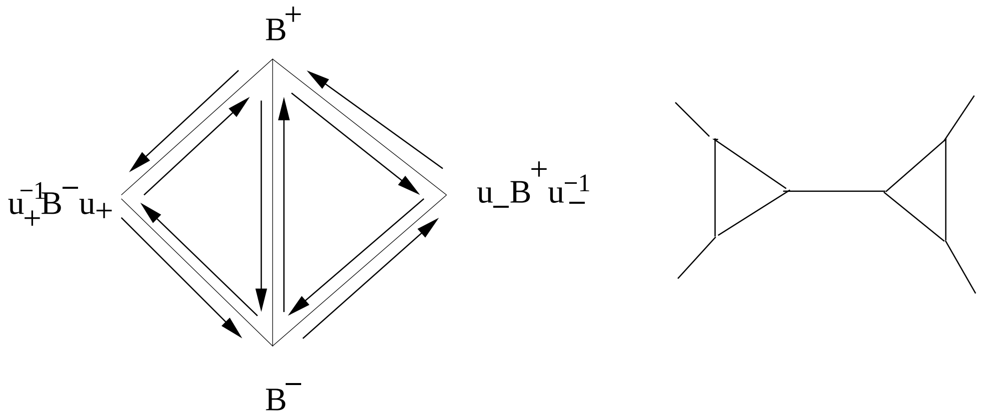


Figure 19: The graph Γ'_T

Given a left principal homogeneous G-space P, a point $p_0 \in P$, and a configuration (p_1, p_2) of two pints in P (that is, a pair of points in P considered up to the left diagonal action of G), there is an element $g_{p_0}(p_1, p_2) \in G$ defined as follows:

$$g_{p_0}(p_1, p_2) := g_1^{-1}g_2 \in G, \qquad \text{where } p_1 = g_1p_0, g_2 = g_2p_0 \tag{72}$$

We will apply this construction to the principal homogeneous space of pinnings for G. Recall that we choose a standard pinning $P_0 = (B^-, B^+)$ for G. Every edge of the graph Γ'_T provides a configuration of two pinnings in G, assigned naturally to the vertices of this edge. Indeed, the vertices of the graph on the right of the picture match the pinnings assigned to the triangulation on the left, as shown on the picture. We use only the pinnings compatible with the orientation of S. Therefore we can assign to every oriented edge $\mathbf{a}$ of the graph Γ'_T an element $M(\mathbf{a}) \in G$. Namely, it is the element assigned by (72) to the configuration of two pinnings attached to the oriented edge $\mathbf{a}$, and the standard pinning P_0. It follows from the very definitions that these elements enjoy the following properties:

1) $M(\mathbf{a})M(\overline{\mathbf{a}}) = \mathrm{Id}$, where $\overline{\mathbf{a}}$ denotes the edge as $\mathbf{a}$ taken with the opposite orientation.

2) $M(\mathbf{t_1})M(\mathbf{t_2})M(\mathbf{t_3}) = \mathrm{Id}$, where the oriented edges $\mathbf{t_1}, \mathbf{t_2}, \mathbf{t_3}$ make a small oriented triangle on the graph Γ'_T.

Thanks to 1) the elements $\{M(\mathbf{a})\}$ define a G-local system on the graph Γ'_T. Thanks to 2) it is descended to a G-local system on the graph Γ. Since Γ is homotopy equivalent to S, we got a G-local system on S.

6. Proof of Theorem 1.11. Given a loop on S, let us shrink it to a loop on Γ'_T. We may assume that this loop contains no consecutive t-edges: indeed, a composition of two t-edges is a t-edge. Thus we may choose an initial vertex on the loop so that the edges have the pattern *et...etet*. Let t and e be consecutive t- and e-edges of the graph Γ'_T. Then there is a unique way to orient them so that e goes after t, i.e. the oriented edge $\mathbf{e}$ starts at the end of the one $\mathbf{t}$. We say that a t-edge is oriented clockwise if it is compatible with the clockwise orientation of the corresponding triangle. Therefore the monodromy is computed as a product of elements of type $M(\mathbf{e})M(\mathbf{t})$, where $\mathbf{e}$ follows after $\mathbf{t}$.

This construction defines the universal framed G-local system on $\widehat{S}$. Recall the semifield $\mathbb{F}^+_{G,S}$ of positive rational functions on the moduli space $\mathcal{X}_{G,S}$ for the positive atlas defined above.

Proposition 6.9 *Let t and e be two consecutive t- and e-edges of the graph Γ'_T. Let us orient them so that* $\mathbf{e}$ *goes after* $\mathbf{t}$. *Then*

$$M(\mathbf{e})M(\mathbf{t}) \in \quad \begin{cases} B^+(\mathbb{F}^+_{G,S}) & \textit{if } \mathbf{t} \textit{ is oriented clockwise} \\ B^-(\mathbb{F}^+_{G,S}) & \textit{if } \mathbf{t} \textit{ is oriented counterclockwise} \end{cases}$$

Proof. We have to consider separately the cases when $\mathbf{t}$ is oriented clockwise, or counterclockwise, i.e. $\mathbf{t} = \mathbf{t_1}$ or $\mathbf{t} = \mathbf{t_2}$ where $\mathbf{t_1}$ and $\mathbf{t_2}$ are as on Figure 20.

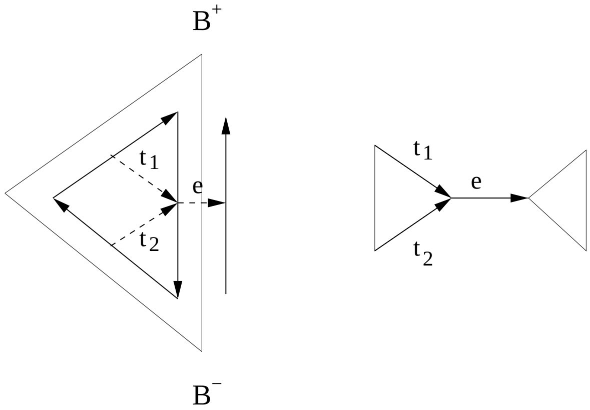


Figure 20: The oriented edges $\mathbf{t_1}$ and $\mathbf{t_2}$ on the graph Γ'_T

We denote by $[B_1, B_2, B_3]$ triples of flags, to distinguish from configurations of flags, and by $g \cdot [B_1, B_2, B_3]$ the result of the action of $g \in G$ on the triple $[B_1, B_2, B_3]$.

Case 1: $\mathbf{t} = \mathbf{t}_1$. Let $u_+ = h_+(\overline{u}_+)$ and $u_- = h_-(\overline{u}_-)$, where $h_\pm \in H$ acts on $U^\pm$ by conjugation. Then we have

$$\left[u_+^{-1}B^-u_+, B^+, B^-\right] = u_+^{-1} \cdot \left[B^-, B^+, u_+B^-u_+^{-1}\right] = u_+^{-1}h_+ \cdot \left[B^-, B^+, \overline{u}_+B^-\overline{u}_+^{-1}\right]$$
$$= u_+^{-1}h_+ \cdot \left[B^-, B^+, \phi(\overline{u}_+)B^+\phi(\overline{u}_+)^{-1}\right]$$

On the other hand

$$\left[B^-, B^+, u_-B^+u_-^{-1}\right] = h_- \cdot \left[B^-, B^+, \overline{u}_-B^+\overline{u}_-^{-1}\right] \tag{73}$$

Let $h(\phi(\overline{u}_+))(\overline{\phi(\overline{u}_+)}) = \phi(\overline{u}_+)$. Then

$$M(\mathbf{e})M(\mathbf{t}_1) = h(\phi(\overline{u}_+))^{-1}h_+^{-1}u_+h_- = h(\phi(\overline{u}_+))^{-1}\overline{u}_+h_+^{-1}h_- \in B^+(\mathbb{F}^+_{G,S})$$

Indeed, $h_+^{-1}h_-$ is the H-invariant, and $h(\varphi(\overline{u}_+))^{-1}$ is a positive rational function by positivity of ϕ.

Case 2: $\mathbf{t} = \mathbf{t}_2$. We have

$$\left[B^-, u_+^{-1}B^-u_+, B^+\right] = \left[B^-, \phi(u_+)^{-1}B^-\phi(u_+), B^+\right] = \phi(u_+)^{-1}\left[B^-, B^+, \phi(u_+)B^+\phi(u_+)^{-1}\right] =$$
$$\phi(u_+)^{-1}\left[B^-, B^+, u_+B^-u_+^{-1}\right] = \phi(u_+)^{-1}h_+\left[B^-, B^+, \overline{u}_+B^-\overline{u}_+^{-1}\right]$$

Comparing with (73), and using the fact that $u_+ = h_+\overline{u}_+h_+^{-1}$ implies $\phi(u_+) = h_+\phi(\overline{u}_+)h_+^{-1}$, we get

$$M(\mathbf{e})M(\mathbf{t}_2) = h_+^{-1}\phi(u_+)h_+h_+^{-1}h_- = \phi(\overline{u}_+)h_+^{-1}h_- \in B^-(\mathbb{F}^+_{G,S})$$

Indeed, $h_+^{-1}h_-$ is the H-invariant, and $\varphi(\overline{u}_+)$ is positive. The proposition is proved.

Let us return to the proof of the theorem. A loop on S contains elements of just one kind, B^+ or B^-, if and only if it is homotopic to the loop around a boundary component. Therefore the monodromy around any non-boundary loop is obtained as a product of elements both from $B^-(\mathbb{F}^+_{G,S})$ and $B^+(\mathbb{F}^+_{G,S})$. Thus it lies in $G(\mathbb{F}^+_{G,S})$. The theorem is proved.

Example. Let $G = PGL_2$, $u_+ = \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}$, $u_- = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}$. Then

$$(B^-, u_+^{-1}B^-u_+, B^+, u_-B^-u_-^{-1}) = (\infty, -y^{-1}, 0, x) = (\infty, -1, 0, xy)$$

The edge invariant equals xy, and coincides with the cross-ratio of the corresponding 4 points on $\mathbb{P}^1$.

7. The universal higher Teichmüller $\mathcal{X}$-space related to G.

Definition 6.10 *A set C is* cyclically ordered *if its any finite subset is cyclically ordered, and any inclusion of finite subsets preserves the cyclic order.*

To define a cyclic order on a set C it is sufficient to specify cyclic orders for all triples and quadruples of elements of C so that the following compatibility condition holds: forgetting any element of any cyclically ordered quadruple we get a triple whose induced cyclic order is the same as the prescribed one. Reversing a cyclic order on C we get a new one.

According to Theorems 5.10 and 5.12 the set of positive n-tuples of flags in $\mathcal{B}(\mathbb{R})$ is invariant under the operation of reversion of the cyclic order.

Definition 6.11 *i) A dihedral structure on C is a cyclic order considered up to reversion.*

ii) A map β from a set C with given dihedral structure to the flag variety $\mathcal{B}(\mathbb{R})$ is positive *if it maps every cyclically ordered quadruple (a, b, c, d) to a positive quadruple of flags $(\beta(a), \beta(b), \beta(c), \beta(d))$.*

Remarks. 1. Positive maps are necessarily injective. We say that C is a *positive subset* of the flag variety $\mathcal{B}(\mathbb{R})$, if there exists a cyclically ordered set C' and a positive map $\beta : C' \to \mathcal{B}(\mathbb{R})$ whose image is C.

2. There exist quadruples of flags in $\mathbb{RP}^2$ such that forgetting any of them we get a positive triple, while the quadruple itself is not positive. For example take four points (x_1, x_2, x_3, x_4) going clockwise on a circle, and consider the tangent flags to the points (x_1, x_3, x_2, x_4).

Consider the dihedral structure on $\mathbb{P}^1(\mathbb{Q})$ provided by the embedding $\mathbb{P}^1(\mathbb{Q}) \hookrightarrow \mathbb{P}^1(\mathbb{R})$.

Definition 6.12 *The universal Teichmüller space $\mathcal{X}^+_G$ is the quotient of the space of positive maps*

$$\beta : \mathbb{P}^1(\mathbb{Q}) \longrightarrow \mathcal{B}(\mathbb{R}) \tag{74}$$

by the natural action of the group $G(\mathbb{R})$ on it.

In other words, $\mathcal{X}^+_G$ is the space of configurations of real flags in $G(\mathbb{R})$ parametrised by $\mathbb{P}^1(\mathbb{Q})$.

Canonical decomposition of $\mathcal{X}^+_G$. To parametrise $\mathcal{X}^+_G$, let us choose three distinct points on $\mathbb{P}^1(\mathbb{Q})$, called $0, 1, \infty$. Recall the Farey triangulation, understood as a triangulation of the hyperbolic disc with a distinguished oriented edge (= flag). Then we have canonical identifications

$$\mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \infty = \{\text{vertices of the Farey triangulation}\} \tag{75}$$

The distinguished oriented edge connects 0 and ∞. So a point of $\mathcal{X}^+_G$ gives rise to a positive map

$$\{\text{vertices of the Farey triangulation}\} \longrightarrow \mathcal{B}(\mathbb{R})$$

considered up to the action of $G(\mathbb{R})$. Assigning to every Farey triangle the corresponding configuration of the three flags at the vertices, and to every Farey geodesic, oriented in a certain way, the H-invariant of the associated quadruple of flags, we get a canonical map

$$\varphi_G : \mathcal{X}^+_G \longrightarrow \mathcal{X}^+_{G,3}{}^{\{\text{Farey triangles}\}} \times H(\mathbb{R}_{>0})^{\{\text{Farey diagonals}\}} \tag{76}$$

Theorem 6.13 *The map (76) is an isomorphism.*

Proof. It is completely similar to the one of Theorem 6.1, and thus is omitted.

Example. When $G = PSL_2$ our universal Teichmüller space is essentially the one considered by L. Bers and R. Penner.

Recall the *Thompson group* $\mathbb{T}$ of all piece-wise linear automorphisms of $\mathbb{P}^1(\mathbb{Q})$. By definition, for every element $g \in \mathbb{T}$ there is a decomposition of $\mathbb{P}^1(\mathbb{Q})$ into a union of finite number of segments, which may overlap only at the ends, so that restriction of g to each segment is given by an element of $PGL_2(\mathbb{Q})$ (depending on the segment). It acts on $\mathcal{X}_G^+$ in an obvious way. Each element of the Thompson group can be presented as a composition of flips.

Let $\Gamma \in PSL_2(\mathbb{Z})$ be a torsion free finite index subgroup. Denote by $\mathcal{H}$ the hyperbolic disc equipped with the Farey triangulation. Set $S_\Gamma := \mathcal{H}/\Gamma$. The surface S_Γ inherits an ideal triangulation T_Γ. The higher Teichmüller space $\mathcal{X}^+_{G,S_\Gamma}$ is embedded into the universal one $\mathcal{X}^+_G$ as the subspace of Γ-invariants. Moreover, let

$$\varphi_{G,S_\Gamma} : \mathcal{X}^+_{G,S_\Gamma} \stackrel{\sim}{\longrightarrow} \mathcal{X}^{+\ \{\text{triangles of } T_\Gamma\}}_{G,3} \times H(\mathbb{R}_{>0})^{\{\text{diagonals of } T_\Gamma\}} \tag{77}$$

be the decomposition of the higher Teichmüller space according to a triangulation T_Γ of S, where we assume an orientation of the edges of T_Γ is chosen. This isomorphism is obtained by taking the $\mathbb{R}_{>0}$-points of the spaces in (64). Then the isomorphism (77) can be obtained by taking the Γ-invariants of the isomorphism (76).

When $G = PSL_m$, there is a more geometric way to describe points of the space $\mathcal{X}^+_G$, by (parametrised by $\mathbb{P}^1(\mathbb{Q})$) convex curves in $\mathbb{RP}^{m-1}$, see Section 8.10.

Remark. The universal Teichmüller space $\mathcal{X}^+_G$ admits a natural quantization, equivariant with respect to the action of the Thompson group. The quantum universal Teichmüller space provides a combinatorial approach to the W-algebra related to the group G. In the case $G = PSL_m$ the quantization is especially explicit, and follows from the results of Chapter 11 and [FG2]. The case $G = PSL_3$ is discussed in [FG3].

8. Discreteness of positive representations: Proof of theorem 1.9. Take a positive framed $G(\mathbb{R})$-local system on a hyperbolic surface S. Let T be an ideal triangulation of S. Consider a universal cover $\widetilde{S}$ and the corresponding triangulation $\widetilde{T}$. The triangulated surface $(\widetilde{S}, \widetilde{T})$ is isomorphic to the hyperbolic disc $\mathcal{H}$ equipped with the Farey triangulation. The vertices of the Farey triangulation are identified with $\mathbb{QP}^1$. Choose a point $\widetilde{p}$ on $\widetilde{S}$ and a trivialization of the local system at $\widetilde{p}$. Let $V(\widetilde{T})$ be the set of vertices of $\widetilde{T}$. Let us construct a map

$$\beta : V(\widetilde{T}) = \mathbb{QP}^1 \to \mathcal{B}(\mathbb{R})$$

Let $\widetilde{v}$ be a vertex of $\widetilde{T}$ and $\widetilde{\alpha}$ be a path from $\widetilde{p}$ to $\widetilde{v}$. Let p, v and α be the corresponding data on S. Let β be a small counterclockwise loop on S surrounding the point v and intersecting with α at one point r. Consider the flag at the fiber of the local system over r provided by the enhanced structure. The part of the path α between p and r maps it to the fiber over p, thus giving the desired map β.

The map β has the following properties.

1. Changing the base point $\widetilde{p}$ or trivialization at $\widetilde{p}$ results in a change $\beta \to h\beta$ for certain $h \in G(\mathbb{R})$.

2. The map β is $\pi_1(S)$-equivariant. Precisely, let $\gamma \in \pi_1(S, p)$ and $\rho : \pi_1(S, p) \to G(\mathbb{R})$ be the monodromy representation corresponding to our local system and the chosen trivialization over p. Then $\rho(\gamma)\beta(\widetilde{v}) = \beta(\gamma\widetilde{v})$, where γ acts on vertices of $\widetilde{T}$ by deck transformations.

3. β is a positive map.

The first two properties follow directly from the construction. The third one follows from the definition of a positive framed local system if the triple of vertices of $\widetilde{T}$ constitute vertices of a triangle or quadrilateral. If they do not, there exists a finite polygon of the triangulation $\widetilde{T}$ containing all these vertices. One can change the triangulation of this polygon by a finite number of flips so that the original triple/quadruple of points become vertices of one triangle/quadrilateral of the new triangulation. According to Theorem 5.21 if a local system is positive with respect to a triangulation, it is positive with respect to any other one, hence the property.

Let us fix a pair (B^-, B^+) of the opposite Borel subgroups in $G(\mathbb{R})$. Consider the following two subsets in the real flag variety:

$$\mathcal{B}^+ := \{u_- B^+ u_-^{-1}\}, \qquad \mathcal{B}^- := \{v_-^{-1} B^+ v_-\}, \quad u_-, v_- \in U^-(\mathbb{R}_{>0})$$

They are open in $\mathcal{B}(\mathbb{R})$. Let $\overline{\mathcal{B}}^+$ be the closure of $\mathcal{B}^+$ in $\mathcal{B}(\mathbb{R})$.

Lemma 6.14 *The sets $\overline{\mathcal{B}}^+$ and $\mathcal{B}^-$ are disjoint.*

Proof. One has $u_- B^+ u_-^{-1} = v_-^{-1} B^+ v_-$ if and only if $u_- = v_-^{-1}$, i.e. $u_- v_- = 1$. Let us show that this is impossible. Recall the homomorphism $\pi : U^- \to \mathcal{R}$ to the maximal abelian quotient. Then $\pi(u_-) = \prod_i x_i(t_i)$ and $\pi(v_-) = \prod_i x_i(s_i)$, where $t_i > 0, s_i \geq 0$, and the product is over all simple positive roots. Thus $0 = \pi(u_- v_-) = \prod_i x_i(t_i + s_i)$. But $t_i + s_i > 0$. This contradiction proves the lemma.

Now fix a triangle of $\widetilde{T}$ and let a, b, c be its vertices. The deck transformation along a loop γ sends the triangle (abc) to a triangle $(a'b'c')$ without common interior points with (abc). Without loss of generality we can assume that the vertices of the triangle $(a'b'c')$ are at the ark between a and c which does not contain b. However the point b' may coincide with a or c. Let us assume first this does not happen. Then, since β is positive, we have a positive quadruple of flags $(\beta(a), \beta(b), \beta(c), \beta(b'))$, and $\beta(b') = \rho(\gamma)\beta(b)$. We may assume that $\beta(a) = B^-, \beta(c) = B^+$. Then $\beta(b) \in \mathcal{B}^-$ and $\beta(b') \in \mathcal{B}^+$. Since $\mathcal{B}^-$ is open, there exists an open neighborhood O of the unit in $G(\mathbb{R})$ such that $g\beta(b) \in \mathcal{B}^-$ for any $g \in O$. Since $\mathcal{B}^+$ is disjoint with $\mathcal{B}^-$, $\rho(\gamma) \notin O$. In the case when b' coincides with a or c, we may assume without loss of generality that it coincide with c. Then the argument is the same except that in the very end we use the fact that $\mathcal{B}^-$ does not contain B^+, which is a special case of the lemma. The theorem is proved.

Proof of Theorem 1.14. It follows immediately from the above constructions.

9. Positive curves in $\mathcal{B}(\mathbb{R})$ are $G(\mathbb{R})$-opers on S^1. *A canonical distribution on $\mathcal{B}$.* The flag variety $\mathcal{B}$ is equipped with a G-invariant non-integrable distribution $\mathcal{D}$ whose dimension is equal to the rank of the group G. It is defined as follows. Let x be a point of $\mathcal{B}$. We may assume that the corresponding Borel subgroup is B^-. Then the tangent space $T_x\mathcal{B}$ is identified with the Lie algebra $\mathrm{Lie}(U^+)$. Consider the subspace of $\mathrm{Lie}(U^+)$ given by the direct sum of the one dimensional subspaces corresponding to the simple roots. Transforming it to the tangent space $T_x\mathcal{B}$, we get the fiber $\mathcal{D}_x$ of the distribution at the point x. It is easy to see that this definition is independent of the choices involved.

Proposition 6.15 *Let C be a positive differentiable curve of the flag variety $\mathcal{B}(\mathbb{R})$. Then C is an integral curve of the canonical distribution $\mathcal{D}$ on $\mathcal{B}(\mathbb{R})$.*

Proof. Let $c(s) \in \mathcal{B}(\mathbb{R})$ be a positive curve. We may assume that $c(0) = B^-, c(1) = B^+$. So $c(s) = x(s)B^-$ where $x(s) \in U^+(\mathbb{R}_{>0})$. Choose a reduced decomposition $w_0 = w_{i_1}...w_{i_n}$. Then the positivity of $x(s)$ implies that we can write it as $x(s) = \prod_{i_k} x_{i_k}(t_k(s))$, where $t_k(0) = 0$, $t_k(s) > 0$. Since $[x_i(s)x_j(t)] = O(st)$, linearizing $x(s)$ at $s = 0$ we get $\sum_k t'_{i_k}(0)E_{i_k} \in \mathrm{Lie}(U^+)$, where $x_i(t) = \exp(tE_i)$. The proposition is proved.

A smooth map from the circle S^1 to $\mathcal{B}(\mathbb{R})$ tangent to the canonical distribution is nothing else as a $G(\mathbb{R})$-*oper* on S^1 in the terminology of Beilinson and Drinfeld. So Proposition 6.15 asserts that positive differentiable loops in $\mathcal{B}(\mathbb{R})$ are $G(\mathbb{R})$-opers on S^1.

Describing $PGL_{n+1}(\mathbb{R})$-opers on real curves. A C^{n-1}-smooth (meaning $n-1$ times continuously differentiable) curve K in $\mathbb{RP}^n$ gives rise to *osculating curve* $\widetilde{K}$ in the real flag variety $\mathcal{B}_{n+1}(\mathbb{R})$ for

PGL_{n+1}. Namely, $\widetilde{K}$ is formed by the osculating flags to the curve at variable point p. It is an easy and well known fact that any differentiable $PGL_{n+1}(\mathbb{R})$-oper is the osculating curve for certain sufficiently smooth curve K in $\mathbb{RP}^n$. The curve K is recovered by applying the canonical projection π. The G-opers for other classical groups have a similar description.

We will show in Section 8.11 that positive $PGL_{n+1}(\mathbb{R})$-opers on S^1 come from convex curves in $\mathbb{RP}^n$. So positive $G(\mathbb{R})$-opers on real curves can be considered as a generalization of convex curves $\mathbb{RP}^n$ to the case of an arbitrary split semi-simple group G.

10. Semi-continuity of positive subsets of real flag varieties. Let $\varphi : C \to \mathcal{B}(\mathbb{R})$ be a positive map from a cyclically ordered set C. Changing the cyclic order of C to the opposite one we get a cyclically ordered set $\overline{C}$. By Theorem 5.12 the map φ, considered as a map from $\overline{C}$, provides a positive map $\overline{\varphi} : \overline{C} \to \mathcal{B}(\mathbb{R})$.

The group $G(\mathbb{R})$ acts on the set of maps from C to $\mathcal{B}(\mathbb{R})$. If a map $\varphi : C \to \mathcal{B}(\mathbb{R})$ is positive then for any $g \in G(\mathbb{R})$ the map $g\varphi$ is also positive. The $G(\mathbb{R})$-orbits on the set of positive maps are called *positive configurations of flags in* $\mathcal{B}(\mathbb{R})$.

Definition 6.16 *Let C be a cyclically ordered set. A map $\varphi : C \to \mathcal{B}(\mathbb{R})$ is* semi-continuous from the left *if for any sequence of distinct points $c_0, ..., c_n, ...$ in C, whose order is compatible with the cyclic order of C, the limit $\lim_{n\to\infty} \varphi(c_n)$ exists.*

A positive configuration in $\mathcal{B}(\mathbb{R})$ is semi-continuous from the left *if any map $\varphi : C \to \mathcal{B}(\mathbb{R})$ representing it is semi-continuous from the left.*

Changing the cyclic order to the opposite one we define the maps *semi-continuous from the right.*

Theorem 6.17 *Any positive map $\varphi : C \to \mathcal{B}(\mathbb{R})$ is semi-continuous from the left and from the right.*

Proof. Set $B_i := \varphi(c_i)$. The positivity of φ means that for every cyclically ordered quadruple $i < j < k < l < i$ the configuration of flags (B_i, B_j, B_k, B_l) is positive. Any positive configuration of 4 flags is equivalent to a configuration

$$(B^-, v \cdot B^-, vu \cdot B^-, B^+), \quad v, u \in U^-(\mathbb{R}_{>0}) \tag{78}$$

Lemma 6.18 *The 4-tuple of flags (78) representing a positive configuration is well defined up to the action of the positive tori $H(\mathbb{R}_{>0})$. A similar statement is valid for the triple $(B^-, u \cdot B^-, B^+)$.*

Proof. Suppose we have a 4-tuple of flags $(B^-, v' \cdot B^-, v'u' \cdot B^-, B^+)$ representing the same configuration, with $v', u' \in U(\mathbb{R}_{>0})$. Then there exists an element of $H(\mathbb{R})$ conjugating one to the other. The projection of u to the maximal abelian quotient $U^+/[U^+.U^+]$ has natural positive coordinates u_α parametrised by the simple roots α. The element $t \in H(\mathbb{R})$ acts on u_α by multiplication on $\chi_\alpha(t)$. Thus $\chi_\alpha(t) > 0$ for every simple root α. Since G has trivial center, this implies that $t \in H(\mathbb{R}_{>0})$. The lemma is proved.

It follows that the sequence of flags $(B_2, B_3, B_4, ..., B_0, B_1)$ can be written as

$$(B^-, v_3 \cdot B^-, v_4 \cdot B^-, ..., v_n \cdot B^-, ..., v_0 \cdot B^-, B^+); \quad v_k = u_3 u_4 ... u_k, \quad u_i \in U^+(\mathbb{R}_{>0}); \quad v_0 \in U^+(\mathbb{R}_{>0}) \tag{79}$$

Moreover we may assume that for every $n \geq 3$ one has $v_0 = v_n v_n'$ with $v_n' \in U^+(\mathbb{R}_{>0})$.

To proceed further we need the following fact. There exists a finite dimensional representation ρ of $G(\mathbb{R})$ and a basis in it such that the elements of $U(\mathbb{R}_{>0})$ act in this basis by unipotent upper triangular matrices with positive entries above the diagonal, and the map ρ provides an isomorphism of the flag variety $\mathcal{B}(\mathbb{R})$ onto its image. Indeed, take the adjoint representation. It follows from [L1] that the canonical basis in any finite dimensional representation has this property.

Then the values of each of the matrix elements in the sequence $\rho(v_i)$, $i = 3, 4, ...$, provide a monotonously increasing sequence of real numbers bounded by the value of the corresponding matrix element of $\rho(v_0)$. Thus there exists the limit $\lim_{i\to\infty}\rho(v_i)$. Since the map $\mathcal{B}(\mathbb{R}) \to \rho(\mathcal{B}(\mathbb{R}))$ is an isomorphism, and the image is closed, the sequence of Borel subgroups $\{B_i\}$ has a limit. The Theorem is proved.

11. Concluding remarks: quantum universal Teichmüller spaces and W-algebras. The space of $G(\mathbb{R})$-opers on the parametrised circle S^1 has a natural Poisson structure, the Gelfand-Dikii bracket, defined by Drinfeld and Sokolov in [DS]. On the other hand, since $P^1(\mathbb{Q})$ is a subset S^1, a positive $G(\mathbb{R})$-oper provides a point of the universal Teichmüller space $\mathcal{X}_G^+$. The universal Teichmüller space $\mathcal{X}_G^+$ has a natural Poisson structure. For instance if $G = PGL_m$ it is defined in the canonical coordinates by the function ε_{pq} from Definition 1.16 applied to the m-triangulation of the Farey triangulation. One can show that the Gelfand-Dikii bracket is compatible with the Poisson bracket on $\mathcal{X}_G^+$. The Poisson structure $\mathcal{X}_G^+$ admits a natural quantization equivariant under the action of the Thompson group (see [FG3] the exposition of the $G = PGL_3$ case). Therefore we suggest that *the quantum universal Teichmüller space can be considered as a combinatorial version of the W-algebra corresponding to G, divided by G.*

7 A positive structure on the moduli space $\mathcal{A}_{G,\widehat{S}}$

In this section we assume that G is a simply connected and connected.

We define a positive structure on the moduli space $\mathcal{A}_{G,\widehat{S}}$. In particular if $\widehat{S}$ is a disc with n marked points at the boundary we get a positive structure on the variety $\mathrm{Conf}_n(\mathcal{A})$ of configurations of n affine flags in generic position in G. We start with a definition of a positive structure on the varieties $\mathrm{Conf}_n(\mathcal{A})$ for $n = 2$ and $n = 3$. The definition in general is given by using an ideal triangulation of a hyperbolic marked surface $\widehat{S}$. We show that it does not depend on the choice of an ideal triangulation.

1. Generic configurations of pairs of affine flags. We say that a pair of affine flags is in generic position if the underlying pair of flags is in generic position. An n-tuple of affine flags is said to be in generic position if every two of them are in generic position. Denote by $\mathrm{Conf}_n(\mathcal{A})$ the variety of configurations of n affine flags in generic position in G. Similarly one defines a generic pair (A_1, B_2) where A_1 is an affine flag and B_2 is a flag. The generic pairs (A_1, B_2) form a principal homogeneous G–space.

Recall the element $\overline{w}_0 \in N(H)$. Recall that $[g]_0 := h$ if $g = u_- h u_+$, where $u_\pm \in U^\pm, h \in H$. Let $(A_1, A_2) = (g_1 U^-, g_2 \overline{w}_0 U^-) \in \mathrm{Conf}_2(\mathcal{A})$. We set

$$\alpha_2(g_1 U^-, g_2 \overline{w}_0 U^-) := [g_1^{-1} g_2]_0 \in H$$

We get a map $\alpha_2 : \mathrm{Conf}_2(\mathcal{A}) \to H$, $(A_1, A_2) \longmapsto \alpha_2(A_1, A_2)$.

Recall that the Cartan group H acts naturally from the right on the affine flag variety: $A \longmapsto A \cdot h$. Since H is commutative we can also consider it as a left action.

Proposition 7.1 *The map α_2 provides an isomorphism*

$$\alpha_2 : \mathrm{Conf}_2(\mathcal{A}) \xrightarrow{\sim} H, \quad (A_1, A_2) \longmapsto \alpha_2(A_1, A_2) \tag{80}$$

It has the following properties:

$$\alpha_2(A_1 \cdot h_1, A_2 \cdot h_2) = h_1^{-1} w_0(h_2) \cdot \alpha_2(A_1, A_2), \qquad \alpha_2(A_2, A_1) = s_G \cdot w_0\left(\alpha_2(A_1, A_2)^{-1}\right), \tag{81}$$

Proof. Bruhat decomposition shows that an arbitrary generic pair of affine flags (A_1, A_2) can be written uniquely as $(h \cdot U^-, \overline{w}_0 U^-)$. This provides the first claim of the proposition. To check the first of the properties (81) notice that $\alpha_2(h_1 U^-, \overline{w}_0 h_2 U^-) = h_1^{-1} w_0(h_2)$. The second of the properties (81) follows immediately from this one. The proposition is proved.

Observe that since $i_{B_1} = w_0 i_{B_2} w_0^{-1}$ one has $(A_1 \cdot h, A_2 \cdot w_0(h)) = h(A_1, A_2)$.

Recall the twisted cyclic shift map from the Definition 2.8.

Lemma 7.2 *The twisted cyclic shift map is a positive automorphism of* $\mathrm{Conf}_2(\mathcal{A})$.

Proof. Follows immediately from (81).

Remark. If $s_G \neq e$ then the cyclic shift is not a positive automorphism of $\mathrm{Conf}_2(\mathcal{A})$.

2. A positive structure on $\mathrm{Conf}_3(\mathcal{A})$. Let us define a map

$$\gamma : U_*^+ \longrightarrow H, \quad u^+ \longmapsto [u^+\overline{w}_0]_0 \in H \tag{82}$$

Lemma 7.3 *The map* γ *is a positive map*

Proof. Using the isomorphism $\mathbb{G}_m^I = H$ provided by $\omega_1, ..., \omega_r$ we have $\gamma(u_+) := \{\Delta_{\omega_i, w_0(\omega_i)}\}_{i\in I} \in \mathbb{G}_m^I$. It follows from Theorem 5.4 that it is a positive map. The lemma is proved.

Consider the regular open embeddings

$$\beta_1^- : B^- \hookrightarrow G/U^-, \quad b_- \longmapsto b_-\overline{\overline{w}}_0 U^-$$

$$\beta_2^- : B^+ \hookrightarrow G/U^-, \quad b_+ \longmapsto b_+ U^-$$

The Borel subgroups B^- and B^+ have the standard positive structures. They provide G/U^- with two positive structures. Similarly the regular open embeddings

$$\beta_1^+ : B^+ \hookrightarrow G/U^+, \quad b_+ \longmapsto b_+\overline{w}_0 U^+$$

$$\beta_2^+ : B^- \hookrightarrow G/U^+, \quad b_- \longmapsto b_- U^+$$

provide G/U^+ with two positive structures. Let $\Phi : B^- \to B^+$ be a rational map such that $\beta_2^+ = \beta_1^+\Phi$ on their common domain of the definition.

Lemma 7.4 *a) The two positive structures on* G/U^- *provided by the maps* β_1^-, β_2^- *are compatible.*

b) The two positive structures on G/U^+ *provided by the maps* β_1^+ *and* β_2^+ *are compatible.*

c) The maps Φ *and* Φ^{-1} *are positive rational maps.*

Proof. a) <=> b). Indeed, consider the involutive automorphism $\theta : G \longmapsto G$ which is uniquely determined by

$$\theta(h) := h^{-1}, \quad \theta(x_i(t)) := y_i(t), \quad \theta(y_i(t)) := x_i(t)$$

Observe that $\theta(\overline{\overline{w}}_0) = \overline{w}_0$ and $\theta(\overline{w}_0) = \overline{\overline{w}}_0$. Applying θ we immediately deduce b) from a) since $\theta(B^+) = B^-, \theta(B^-) = B^+, \theta(\overline{\overline{w}}_0) = \overline{w}_0$. Similarly b) implies a).

Let us prove b). Recall that $b_+\overline{w}_0U^+ = b_-U^+$. We have to show that the maps $b_+ \to b_-$ and $b_- \to b_+$ are positive. Consider the first map. Write $b_+ = h_+u_+, b_- = h_-u_-$. We may assume without loss of generality that $h_+ = 1$. Then $h_- = \gamma(u_+)$, so it is a positive map by Lemma 7.3. The map $u_+ \longmapsto u_-$ is the map ϕ^{-1}, and hence it is positive. The claim is proved.

c) Follows from b). The lemma is proved.

3. Positive configurations of affine flags. Let (A_1, A_2, A_3) be a generic triple of affine flags in G. Denote by B_i the flag corresponding to A_i. So every two of the flags B_1, B_2, B_3 are in generic position. Let $\mathrm{Conf}(A_1, B_2, B_3)$ be the configuration space of generic triples (A_1, B_2, B_3). Then there is an isomorphism

$$\mathrm{Conf}(A_1, B_2, B_3) \stackrel{\sim}{\longrightarrow} U_*^+ \tag{83}$$

Indeed, let (U^-, w_0B^-) be a standard pair. Then there exists unique $u \in U_*^+$ such that

$$(A_1, B_2, B_3) \sim (U^-, w_0B^-, u_+B^-); \qquad u_+ \in U_*^+ \tag{84}$$

Lemma 7.5 *There is a canonical isomorphism*

$$\alpha_3 : \mathrm{Conf}_3(\mathcal{A}) \stackrel{\sim}{\longrightarrow} \mathrm{Conf}_2(\mathcal{A}) \times \mathrm{Conf}_2(\mathcal{A}) \times \mathrm{Conf}(A_1, B_2, B_3) \tag{85}$$

$$(A_1, A_2, A_3) \longmapsto (A_1, A_2) \times (A_2, A_3) \times (A_1, B_2, B_3)$$

Proof. The affine flags A_2 and A_3 are determined by the elements $\alpha_2(A_1, A_2)$ and $\alpha_2(A_1, A_3)$ in H. The lemma is proved.

The isomorphism α_3 combined with the isomorphisms (80) and (83) leads to an isomorphism

$$\alpha_3 : \mathrm{Conf}_3(\mathcal{A}) \stackrel{\sim}{\longrightarrow} H \times H \times U_*^+, \tag{86}$$

Lemma 7.6 *The inverse to the isomorphism α_3 is defined as follows:*

$$\alpha_3^{'} : H \times H \times U_*^+ \longrightarrow \mathrm{Conf}_3(\mathcal{A}), \quad (h_2, h_3, u_+) \longmapsto (U^-, h_2\overline{w}_0U^-, h_2w_0(h_3)s_Gu_+U^-)$$

Proof. Consider the natural projection

$$\mathbf{e}_{ij} : \mathrm{Conf}_3(\mathcal{A}) \longrightarrow H; \qquad (A_1, A_2, A_3) \longmapsto (A_i, A_j), \quad 1 \leq i < j \leq 3$$

It follows from the definition of the map α_2 that $\mathbf{e}_{12}\alpha_3^{'}$ is the projection on the first H-factor. Let us check that $\mathbf{e}_{23}\alpha_3^{'}$ is the projection on the second H-factor. One has

$$(h_2\overline{w}_0U^-, h_2w_0(h_3)s_Gu_+U^-) \sim (U^-, h_3s_G\overline{w}_0^{-1}u_+\overline{w}_0\overline{w}_0^{-1}U^-) = (U^-, h_3\overline{w}_0^{-1}(u_+)\overline{w}_0U^-)$$

Since $\overline{w}_0^{-1}(u_+) := \overline{w}_0^{-1}u_+\overline{w} \in U^-$, the statement follows. The similar claim about the U_*^+-factor is obvious. The lemma is proved.

Definition 7.7 *A positive regular structure on the variety* $\mathrm{Conf}_3(\mathcal{A})$ *is given by the isomorphism (86) and the standard positive regular structures on U_*^+ and H.*

Lemma 7.8 *The map $\mathbf{e}_{13} : \mathrm{Conf}_3(\mathcal{A}) \longrightarrow H$ is a positive rational map.*

Proof. Recall the map γ, see (82). Using the isomorphism (86) we have

$$\mathbf{e}_{13}\alpha_3'(h_2, h_3, u_+) = (U^-, h_2w_0(h_3)s_Gu_+U^-) = (U^-, h_2w_0(h_3)u_+\overline{w}_0\overline{w}_0U^-)$$

Comparing this with (35) we observe that this is equal to $h_2w_0(h_3)\gamma(u_+)$. The lemma is proved.

Proposition 7.9 *The cyclic shift is a positive rational map on the variety* $\mathrm{Conf}_3(\mathcal{A})$.

Proof. To simplify the calculation we may assume without loss of generality that $h_2 = h_3 = e$. So we start from the configuration $(U^-,\ \overline{w}_0U^-,\ u_+s_GU^-)$. Multiplying from the left by u_+^{-1} and then using $\overline{w}_0^{-1}u_+^{-1}\overline{w}_0 \in U^-$ we get

$$(U^-,\ \overline{w}_0U^-,\ u_+s_GU^-) \sim (u_+^{-1}U^-,\ \overline{w}_0U^-,\ s_GU^-)$$

Applying the inversion map $g \longmapsto g^{-1}$ we get

$$(U^-u_+,\ U^-\ \overline{w}_0^{-1},\ U^-s_G)$$

Applying the antiautomorphism Ψ we get

$$(u_-U^+,\ \Psi(\overline{w}_0^{-1})U^+,\ s_GU^+), \quad u_- := \Psi(u_+)$$

Since $\Psi(\overline{w}_0^{-1})^{-1}u_-\Psi(\overline{w}_0^{-1}) \in U^+$, this configuration can be written as

$$(u_-U^+,\ \Psi(\overline{w}_0^{-1})U^+,\ s_GU^+)$$

According to Lemma 7.4b) one has $u_-U^+ = v_+h\overline{w}_0U^+$. Here $v_+h = \Phi(u_-)$ and $v_+ = \phi(u_-)$. The map Φ is a positive map. Thus, since $v_+^{-1}U^+ = U^+$, we can write our configuration as

$$(v_+h\overline{w}_0U^+,\ \Psi(\overline{w}_0^{-1})U^+,\ s_GU^+) \sim (h\overline{w}_0U^+,\ v_+^{-1}\Psi(\overline{w}_0^{-1})U^+,\ s_GU^+)$$

Applying the inversion map we write it as

$$(U^+\overline{w}_0^{-1}h^{-1},\ U^+\Psi(\overline{w}_0)v_+,\ U^+s_G)$$

Then applying Ψ we get

$$(h^{-1}\Psi(\overline{w}_0^{-1})U^-,\ \Psi(v_+)\overline{w}_0U^-,\ s_GU^-)$$

Applying Lemma7.4a) we can write $\Psi(v_+)\overline{\overline{w}}_0U^- = h'\phi\Psi(v_+)U^-$, where the map $\Psi(v_+) \longmapsto h'$ is positive. So we get

$$(h^{-1}\overline{w}_0U^-,\ s_Gh'(\phi\Psi)^2(u_+)U^-,\ s_GU^-)$$

We conclude that

$$(U^-,\ \overline{w}_0U^-,\ u_+s_GU^-) \sim (h^{-1}\overline{w}_0U^-,\ s_Gh'(\phi\Psi)^2(u_+)U^-,\ s_GU^-) \tag{87}$$

Since the maps $\phi\Psi$, $u_+ \longmapsto h^{-1}$ and $u_+ \longmapsto h'$ are positive, formula (87) implies that the twisted cyclic shift is a positive map on $\mathrm{Conf}_3(\mathcal{A})$. The proposition is proved.

Let us consider the following a regular open embedding map

$$H^3 \times U_*^+ \times U_*^- \hookrightarrow \mathrm{Conf}_4'(\mathcal{A})$$

$$(h_2, h_3, h_4, u_+, v_-) \longmapsto (U^-,\ u_+^{-1}h_2U^-,\ h_3\overline{w}_0U^-,\ v_-h_4\overline{w}_0U^-) \tag{88}$$

Here $\mathrm{Conf}_4'(\mathcal{A})$ is the moduli space of configurations of four flags $(B_1, ..., B_4)$ such that all flags except the pair (B_2, B_4) are in generic position.

We are going to define a positive structure on the variety $\mathrm{Conf}_4(\mathcal{A})$ by using the birational map

$$H^3 \times U_*^+ \times U_*^- \longrightarrow \mathrm{Conf}_4(\mathcal{A}) \tag{89}$$

and the standard positive structures on H, U_*^- and U_*^-. We have to check however that this map is defined on a complement to a positive rational divisor.

Proposition 7.10 *a) The map (89) is defined on a complement to a positive rational divisor.*
b) The twisted cyclic shift provides a positive automorphism of the variety $\mathrm{Conf}_4(\mathcal{A})$.

Proof. Let us define a rational map

$$U^+ \times U^- \times H \times H \longrightarrow U^- \times U^+ \times H, \quad (u_+, v_-, h_2, h_4) \longmapsto (a_-, a_+, t) \tag{90}$$

by solving the equation $h_2^{-1}u_+h_4v_- = a_-a_+t$. It is a positive rational map. Let us set

$$g := a_-^{-1}h_2^{-1}u_+ = a_+th_4^{-1}v_-^{-1}$$

Then multiplying the configuration (88) by g from the left we get

$$(a_+th_4^{-1}U^-,\ U^-,\ a_-^{-1}h_2^{-1}h_3\overline{w}_0U^-,\ t\overline{w}_0U^-)$$

Applying to it the twisted cyclic shift we get

$$(U^{-},\ a_{-}^{-1}h_{2}^{-1}h_{3}\overline{w}_{0}U^{-},\ t\overline{w}_{0}U^{-},\ a_{+}th_{4}^{-1}s_{G}U^{-}) \tag{91}$$

We can write the configuration (91) in the form of (88):

$$(91) = (U^{-},\ \widetilde{u}_{+}^{-1}\widetilde{h}_{2}U^{-},\ \widetilde{h}_{3}\overline{w}_{0}U^{-},\ \widetilde{v}_{-}\widetilde{h}_{4}\overline{w}_{0}U^{-})$$

In other words, one must have

$$i)\ \ a_{-}^{-1}h_{2}^{-1}h_{3}\overline{w}_{0}U^{-} = \widetilde{u}_{+}^{-1}\widetilde{h}_{2}U^{-}, \quad ii)\ \ t = \widetilde{h}_{3}, \quad iii)\ \ a_{+}th_{4}^{-1}s_{G}U^{-} = \widetilde{v}_{-}\widetilde{h}_{4}\overline{w}_{0}U^{-} \tag{92}$$

Lemma 7.11 *The rational map $(h_2, h_3, h_4, u_+, v_-) \longmapsto (\widetilde{h}_2, \widetilde{h}_3, \widetilde{h}_4, \widetilde{u}_+, \widetilde{v}_-)$ provided by equations (92) is a positive rational map.*

Proof. For $\widetilde{h}_3$ and the equation ii) this follows from the positivity of the map (90). For the equation iii) and $\widetilde{v}_-, \widetilde{h}_4$ this follows immediately from Lemma 7.4a) since $s_G\overline{w}_0 = \overline{\overline{w}}_0$. To handle i) means to show that the rational map $B^- \to B^+$, $x_- \longmapsto x_+$ determined from the equation $x_-^{-1}\overline{w}_0U^- = x_+^{-1}U^-$ is a positive rational map. This equation is equivalent to $x_+ = \Psi\Phi\Psi(x_-)$. Indeed, applying the inversion antiautomorphism to this equation we get $U^-\overline{\overline{w}}_0x_- = U^-x_+$. Then applying Ψ we get $\Psi(x_-)\overline{w}_0U^+ = \Psi(x_+)U^+$. So $\Psi(x_+) = \Phi^{-1}\Psi(x_-)$, and hence $x_+ = \Psi\Phi^{-1}\Psi(x_-)$. By Lemma 7.4 the rational map $\Psi\Phi^{-1}\Psi : B^+ \to B^-$ is positive. So the lemma follows. Both statements of the proposition follows immediately from this lemma. The proposition is proved.

Let us consider the canonical projection (the edge projection)

$$\mathbf{e} = (\mathbf{e}_{12}, \mathbf{e}_{23}, \mathbf{e}_{13}) : \mathrm{Conf}_3(\mathcal{A}) \longrightarrow H \times H \times H \tag{93}$$

It follows from Lemma 7.6 and proof of Lemma 7.8 that one has

$$\mathbf{e}\alpha_3'(h_2, h_3, u_+) = (h_2, h_3, h_2w_0(h_3)\gamma(u_+)) \tag{94}$$

Therefore the map $\mathbf{e}$ is surjective. Let us define the variety $\mathcal{V}_G$ as its fiber over (e, e, e):

$$\mathcal{V}_G := \mathbf{e}^{-1}(e, e, e) \subset \mathrm{Conf}_3(\mathcal{A}) \tag{95}$$

Proposition 7.12 *a) The variety $\mathcal{V}_G$ has a natural positive structure.*

b) The positive structure on the variety $\mathcal{V}_G$ is invariant under the cyclic shift map.

Proof. a) We deduce this from Theorem 5.4 applied to the double Bruhat cell G^{e,w_0}. Observe that G^{e,w_0} can be identified with the configurations space $\mathrm{Conf}(A_1, B_2, A_3)$. Given a reduced decomposition $\mathbf{i}$ of w_0, the collection of the generalized minors $F(\mathbf{i}) = \{\Delta(k; \mathbf{i}), k \in [1, r] \cup [1, l(w_0)]\}$ provides a positive coordinate system on G^{e,w_0}. The minors $\Delta(k; \mathbf{i}), k \in [1, r]$ are the ones $\Delta_{\omega_i, w_0(\omega_i)}$, so they give the map γ. On the other hand among these minors there are $\Delta_{\omega_i,\omega_i}$ for every $i \in I = \{1, ..., r\}$. To see this write the sequence $\mathbf{i} = (i_1, ..., i_{l(w_0)})$. Then for each $p \in \{1, ..., r\}$ take the very right among all i_a's such that $i_a = p$. Denote it by $i_{a(p)}$. Then $\Delta(i_{a(p)}; \mathbf{i}) = \Delta_{\omega_p,\omega_p}$. The rest of the generalized minors, that is the minors

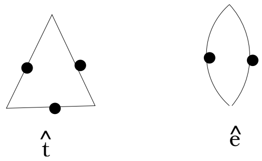
$$F'(\mathbf{i}) = \{\Delta(k; \mathbf{i}), k \in [1, l(w_0)] - \{a(1), ..., a(r)\}\}$$

give a positive coordinate system on $\mathcal{V}_G$. The part a) is proved.

b) Follows easily from a) and Proposition 7.9. The proposition is proved.

4. A positive structure on the moduli space $\mathcal{A}_{G,\widehat{S}}$. Let $\widehat{S}$ be a marked hyperbolic surface. Choose an ideal triangulation T of $\widehat{S}$. Denote by $\mathcal{A}_{G,\widehat{t}}$ the moduli space of the twisted decorated

G-local systems on the triangle t, considered as a disc with three marked points on the boundary. These marked points are located on the sides of the triangle, one point per each side. The punctured tangent space to a triangle t of the triangulation T sits in the punctured tangent space $T'S$. So restricting the local system $\mathcal{L}$ on $T'S$ representing an element of $\mathcal{A}_{G,\widehat{S}}$ to $T't$ we get an element of $\mathcal{A}_{G,\widehat{t}}$. So we get a projection $q'_t : \mathcal{A}_{G,\widehat{S}} \longrightarrow \mathcal{A}_{G,\widehat{t}}$. Similarly, each edge e of the triangulation T can be thickened a bit to became a disc with two distinguished points on its boundary provided by the vertices of e. They cut the boundary of the disc into two arcs. Let $\widehat{e}$ be this disc with two marked points on the boundary, one on each of these arcs.



(96)

The restriction provides a map $q_e : \mathcal{A}_{G,\widehat{S}} \longrightarrow \mathcal{A}_{G,\widehat{e}}$. All together, they provide a map

$$\varphi_T : \mathcal{A}_{G,\widehat{S}} \longrightarrow \prod_{t \in \mathrm{tr}(T)} \mathcal{A}_{G,\widehat{t}} \times \prod_{e \in ed(T)} \mathcal{A}_{G,\widehat{e}} \qquad (97)$$

However the target space of this map is too big. Indeed, let e be an edge of a triangle t. Then there is a natural projection $q_{\widehat{t},\widehat{e}} : \mathcal{A}_{G,\widehat{t}} \to \mathcal{A}_{G,\widehat{s}}$, and $q'_{\widehat{t}} q_{\widehat{t},\widehat{e}} = q_{\widehat{e}}$. The subvariety in the right hand side of (97) defined by these conditions is the right target space. It contains the following open subset.

Definition 7.13 *Let T be an ideal triangulation of $\widehat{S}$. The subvariety*

$$\mathcal{A}_{G,T} \subset \prod_{t \in \mathrm{tr}(T)} \widetilde{\mathrm{Conf}_3 \mathcal{A}} \times \prod_{e \in ed(T)} \widetilde{\mathrm{Conf}_2 \mathcal{A}}$$

is defined by the conditions $q'_{\widehat{t}} q_{\widehat{t},\widehat{e}} = q_{\widehat{e}}$ for every pair (e,t) where e is an edge of the triangle t of T.

Theorem 7.14 *Let G be a split simply-connected semisimple algebraic group. Let $\widehat{S}$ be a marked hyperbolic surface and T is an ideal triangulation of $\widehat{S}$. Then there exists a regular open embedding*

$$\nu_T : \mathcal{A}_{G,T} \hookrightarrow \mathcal{A}_{G,\widehat{S}} \qquad (98)$$

such that $\varphi_T \nu_T$ is the identity.

Proof. Let us introduce a convenient way to think of local systems on the punctured tangent bundle $T'S$ having the monodromy s_G around the circle T'_xS. Let Γ be a graph on S dual to an ideal triangulation T. Let us define a collection of points in $T'S$ as follows. For every edge e of the graph Γ let us choose an internal point $x(e)$ on this edge, and a pair of non zero tangent vectors $v_1(e)$ and $v_2(e)$ at this point looking in the opposite directions from e, as on the right Picture (99).

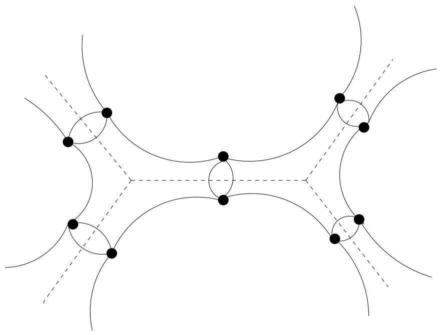

(99)

We are going to define a groupoid $\mathcal{G}_\Gamma$ with the objects given by the points $\{v_i(e)\}$, where e runs through all the edges of Γ and $i \in \{1,2\}$, which is equivalent to the fundamental groupoid of $T'S$. Therefore local $G(F)$-systems on $T'S$ can be understood as functors $\mathcal{G}_\Gamma \to G(F)$. Here is a system of generators of the groupoid $\mathcal{G}_\Gamma$.

1). In the torsor of paths in $T'_{x(e)}S$ connecting the vectors $v_1(e)$ and $v_2(e)$ there are two distinguished elements, p_+ and p_-, where p_+ moves the vector $v_1(e)$ clockwise, and p_- counterclockwise towards the vector $v_2(e)$. By definition $p_\pm \in \mathrm{Mor}_{\mathcal{G}_\Gamma}(v_1(e), v_2(e))$.

2). Let v be a vertex of Γ. Choose a sufficiently small neighborhood $\mathcal{D}_v$ of v containing the three edges sharing v. Then the space of nonzero tangent vectors at the points of $\Gamma \cap \mathcal{D}_v$ which are not tangent to Γ is a union of three connected simply connected components. Therefore if we denote by e_1, e_2, e_3 the three edges sharing v, each connected component has exactly two of the six vectors $v_i(e_j)$. By definition the unique path connecting each pair belongs to the morphisms in $\mathcal{G}_\Gamma$.

The arcs on the left Picture (99) illustrate the six generators of the groupoid near a vertex of Γ. They form a hexagon H_v. The arcs illustrate how the tangent vectors move. So for instance the left ark on Picture (99) intersecting Γ moves the tangent vector looking up clockwise to the one looking down. Putting the hexagons corresponding to all vertices of Γ together we get the set of the generators of the groupoid $\mathcal{G}_\Gamma$. On Picture (100) the objects of the groupoid $\mathcal{G}_\Gamma$ are pictured by fat points, and the generators by arcs.



(100)

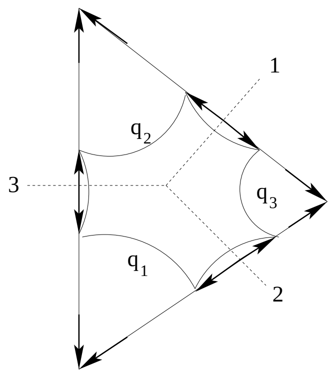
Here are two simple but crucial observations about this picture.

Lemma 7.15 *a) The composition of the six paths forming the hexagon H_v on the left picture in (99) amounts to rotation of a tangent vector by 4π.*

b) The composition p_+p_+ as well as p_-p_- amounts to rotation of a tangent vector by 2π.

Proof. A simple exercise.

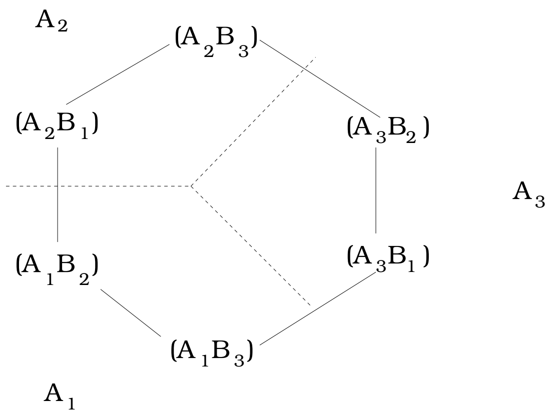
Let us consider a triangle t of the triangulation T dual to a vertex v of Γ. We may assume it intersects the edges e_j of Γ sharing the vertex v at the chosen points $x(e_j)$. Then there is a canonical homotopy class of path in $T'S$ going from the tangent vectors $v_i(e_j)$ along the corresponding side of the triangle t to proximity of the corresponding vertex of the triangle t, as on Picture (101).



(101)

Using this path and the decoration at this vertex of t we picking up an affine flag over the tangent vector $v_i(e_j)$. Let q be a side of the hexagon H_v which which does not intersect Γ on Picture 100 (i.e. sits in one of the connected components discussed in 2)). Then the affine flags sitting at the endpoints of q are obtained one from the other by the parallel transport along q.

Lemma 7.15a) implies that the monodromy around the hexagon H_v near a vertex v is the identity. So the restriction of the local system $\mathcal{L}$ to the hexagon H_v is trivial. Therefore we can transform all the affine flags into a fiber over one point, getting a configuration of three affine flags (A_1, A_2, A_3) in G corresponding to the vertex v, as on Picture (102). The affine flags are parametrised by the vertices of the triangle t. Denote by B_1, B_2, B_3 the corresponding flags. Then we can attach the pairs (A_i, B_j) to the vertices of the hexagon H_v as on Picture (102).



(102)

For each oriented edge $\mathbf{a}$ of the hexagon H_v corresponds an element $g_{\mathbf{a}}$ of G defined as follows. Consider a triple of flags (A_1, A_2, A_3) representing our configuration. Let (A_i, B_j) and $(A_{i'}, B_{j'})$ be the pairs assigned to the first and second vertices of the oriented edge $\mathbf{a}$. There exists unique element $g \in G$ transforming (A_i, B_j) to the standard pair (U^-, B^+). Set $(A'_1, A'_2, A'_3) := (gA_1, gA_2, gA_3)$. We define $g_{\mathbf{a}}$ as the unique element of G such that transforming the pair $(A'_{i'}, B'_{j'}) = g_{\mathbf{a}}(U^-, B^+)$.

There are two types of edges of the graph on Picture (100): the ones intersecting the graph Γ on Picture (100), and the ones which do not intersect it. Each edge of the first type determines an edge of the triangulation T, and each edges of the second type determines a vertex of T. If $\mathbf{p}$ (resp. $\mathbf{q}$) is an edge of the first (resp. second) type, then

$$g_{\mathbf{p}} \in \overline{w}_0 H, \quad g_{\mathbf{q}} \in U^- \tag{103}$$

Each edge of the graph on Picture (100) is assigned to a unique hexagon H_v. Applying the above construction to the oriented edges $\mathbf{a}$ of these hexagons we get a collection of elements $\{g_{\mathbf{a}}\}$ of G. It has the following properties:

i) Reversing orientation of an edge amounts to replacing the corresponding element by its inverse.

ii) The product of the elements corresponding to a path around the hexagon is the identity.

iii) The product of the elements corresponding to a simple path around a two-gon is s_G.

iv) Let $\mathbf{q}$ (resp. $\mathbf{p}$) be an oriented edge corresponding to a vertex (resp. edge) of the triangulation T. Then one has (103)

By Lemma 7.15b) the monodromy around each of the two-gons on Picture (100) must be s_G. This agrees with iii).

We can invert this construction.

Lemma 7.16 *Let $\{g_{\mathbf{a}}\}$ be an arbitrary collection of elements of $G(F)$ assigned to oriented edges of the graph on Picture (100), which satisfy the conditions i) - iv). Then there exists a twisted decorated $G(F)$-local system on $\widehat{S}$ such that the collection of elements $g_{\mathbf{a}}$ constructed for it coincide with the original collection.*

Proof. Thanks to the properties i) - iii) there exists unique functor $\mathcal{G}_\Gamma \to G(F)$ determined by the condition $\mathbf{a} \longmapsto g_{\mathbf{a}}$. Here $\mathbf{a}_1\mathbf{a}_2$ means that the path $\mathbf{a}_1$ follows $\mathbf{a}_2$. It provides a twisted $G(F)$-local system on $\widehat{S}$. The condition iv) is used to define a decoration. The lemma is proved.

Recall that a point of the variety $\mathcal{A}_{G,T}$ is given by a collection of twisted cyclic configurations (A_1^t, A_2^t, A_3^t) of affine flags in generic position, one for each triangle t of T, compatible over the internal edges of T. Given such a configuration (A_1^t, A_2^t, A_3^t) attached to a vertex v of Γ we assign to the oriented edges of the hexagon H_v the elements $g_{\mathbf{a}}$ as above. They satisfy the conditions i), ii), iv). Finally, the compatibility over the edges implies the condition iii). It remains to apply the construction from Lemma 7.16. The theorem is proved.

It remains to define a positive structure on the variety $\mathcal{A}_{G,T}$ and prove that the positive structures corresponding to different ideal triangulations T are compatible.

Consider $\widetilde{\mathrm{Conf}_3\mathcal{A}}$ as a subvariety of $\mathcal{A}_{G,\widehat{t}}$. Then choosing a vertex of the triangle t, we have

$$\widetilde{\mathrm{Conf}_3\mathcal{A}} = \mathcal{V}_G \times (\widetilde{\mathrm{Conf}_2\mathcal{A}})^3$$

Recall that a positive structure on $\mathcal{V}_G$ is given by the Proposition 7.12. The positive coordinate systems are parametrized by reduced decompositions $\mathbf{i}$ of the element w_0. A positive structure on $\widetilde{\mathrm{Conf}_2\mathcal{A}}$ is given by Proposition 7.1 and Lemma 7.2. This suggests the following definition.

Definition 7.17 *Denote by* $\mathbf{T}$ *the following data on* $\widehat{S}$*:*

i) An ideal triangulation T *of a marked hyperbolic surface* $\widehat{S}$*.*

ii) For each triangle t *of the triangulation* T *a choice of a vertex of this triangle.*

iii) A reduced decomposition $\mathbf{i}$ *of the maximal length element* $w_0 \in W$*.*

So given such a data $\mathbf{T}$ and using Theorem 7.14 we have an open embedding

$$\nu_{\mathbf{T}} : +\mathcal{V}_G^{\{\text{triangles of } T\}} \times H^{\{\text{edges of } T\}} \hookrightarrow \mathcal{A}_{G,\widehat{S}} \qquad (104)$$

Theorem 7.18 *Let* G *be a split simply-connected semisimple algebraic group. Let* $\widehat{S}$ *be a marked hyperbolic surface. Then the collection of regular open embeddings* $\{\nu_{\mathbf{T}}\}$*, when* $\mathbf{T}$ *runs through all the datas from the Definition 7.17 assigned to* $\widehat{S}$*, provides a* Γ_S*–equivariant positive structure on* $\mathcal{A}_{G,\widehat{S}}$*.*

Proof. We have to check that we get a compatible positive structure by changing any of the three components in the data $\mathbf{T}$ from the definition 7.17. For iii) this follows from Lusztig's theory [L1]. For ii) this follows from Proposition 7.9. To prove the statement for for i) we need to check that a flip as on Figure 21 provides a positive transformation. This follows from the Proposition 7.10. The theorem is proved.

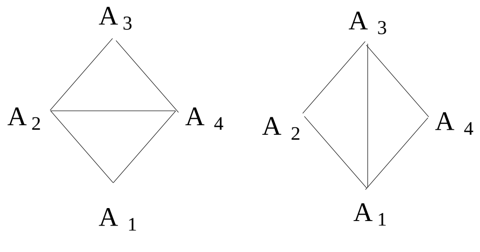


Figure 21: Flip

Definition 7.19 *Let* $\widehat{S}$ *be a hyperbolic marked surface and* G *a split simply-connected semisimple algebraic group.*

a) The decorated Teichmüller space $\mathcal{A}^+_{G,\widehat{S}}$ *is the set of all* $\mathbb{R}_{>0}$*-points of the positive space* $\mathcal{A}_{G,\widehat{S}}$*.*

b) Let $\mathbb{A}^t$ be one of the tropical semifields $\mathbb{Z}^t, \mathbb{Q}^t, \mathbb{R}^t$. The set $\mathcal{A}_{G,\widehat{S}}(\mathbb{A}^t)$ of points of the positive variety $\mathcal{A}_{G,\widehat{S}}$ with the values in the tropical semifield $\mathbb{A}^t$ is called the set of $\mathcal{A}^{\mathbb{A}}$-laminations on $\widehat{S}$ corresponding to G.

The space of real $\mathcal{A}$-laminations serves as the Thurston boundary of the Teichmüller space $\mathcal{A}^+_{G,\widehat{S}}$.

The decorated Teichmüller space $\mathcal{A}^+_{G,\widehat{D}_n}$ is identified with the space of positive twisted cyclic configurations of n affine flags in $G(\mathbb{R})$ on a disc $\widehat{D}_n$ with n marked points on the boundary.

Corollary 7.20 *The space $\mathcal{A}^+_{G,\widehat{D}_n}$ is invariant under the twisted cyclic shift.*

5. The universal Teichmüller $\mathcal{A}$-space for G. Let $\widetilde{\mathbb{P}}^1(\mathbb{R})$ be a two-sheeted covering of $\mathbb{P}^1(\mathbb{R})$, and $\widetilde{\mathbb{P}}^1(\mathbb{Q})$ the subset of its rational points. Let s be the nontrivial automorphism of the covering.

Let $\{p_1, ..., p_n\}$ be a cyclically ordered subset of $\mathbb{P}^1(\mathbb{Q})$, i.e. its order is compatible with one of the orientations of $\mathbb{P}^1(\mathbb{R})$. Then a choice of an initial point p_1 plus its lift $\widetilde{p}_1$ determines a lift $\{\widetilde{p}_1, ..., \widetilde{p}_n\}$ of the set $\{p_1, ..., p_n\}$ to $\widetilde{\mathbb{P}}^1(\mathbb{Q})$: we lift an oriented loop started at p_1, whose orientation agrees with the cyclic order of the points p_i, to a path starting at $\widetilde{p}_1$, and lift the points $p_1, ..., p_n$ using this path. We say $\{\widetilde{p}_1, ..., \widetilde{p}_n\}$ is a *coherent lift* of the cyclically ordered set $\{p_1, ..., p_n\}$.

Definition 7.21 *Let G be a split semisimple simply-connected algebraic group. A map*

$$\alpha : \widetilde{\mathbb{P}}^1(\mathbb{Q}) \longrightarrow \mathcal{A}(\mathbb{R}) \qquad \text{such that } \alpha(sp) = s_G\alpha(p) \tag{105}$$

is positive *if for any cyclically ordered n-tuple of points $\{p_1, ..., p_n\}$ on $\mathbb{P}^1(\mathbb{Q})$, and a coherent lift $\{\widetilde{p}_1, ..., \widetilde{p}_n\}$ of these points, the configuration of affine flags $(\alpha(\widetilde{p}_1), ..., \alpha(\widetilde{p}_n))$ is positive.*

The universal decorated Teichmüller space $\mathcal{A}^+_G$ *is the quotient of the space of positive maps (105) by the action of the group $G(\mathbb{R})$.*

Observe that if the configuration of affine flags $(\alpha(\widetilde{p}_1), ..., \alpha(\widetilde{p}_n))$ is positive for a certain coherent lift of the cyclically ordered set $\{p_1, ..., p_n\}$, it is positive for any such a coherent lift. So the positivity is a property of $\{p_1, ..., p_n\}$. Further, it follows from the main results of this Section that is sufficient to check the positivity for $n = 2, 3$ only.

Recall the positive space $\mathcal{V}_G$, see (95). Let $\mathcal{V}^+_G$ be its positive part. Identifying $\mathbb{P}^1(\mathbb{Q})$ with vertices of the Farey triangulation, we have a decomposition theorem for the universal Teichmüller space $\mathcal{A}^+_G$:

Theorem 7.22 *There exists a canonical isomorphism*

$$\mathcal{A}^+_G = \mathcal{V}_G^{+\{\textit{Farey triangles}\}} \times H(\mathbb{R}_{>0})^{\{\textit{Farey diagonals}\}}$$

To fix an isomorphism in this theorem, we have to choose a vertex for each of the Farey triangles.

The higher Teichmüller spaces $\mathcal{A}^+_{G,S}$ are embedded into the universal one just in the same fashion as for the $\mathcal{X}$-space.

8 Special coordinates on the $\mathcal{A}$ and $\mathcal{X}$ spaces when G is of type A_m

Let Γ be a marked trivalent ribbon graph. Then there are varieties $\mathcal{X}_{G,\Gamma}$ and $\mathcal{A}_{G,\Gamma}$ parametrising framed and decorated unipotent G-local systems on Γ. If Γ is of type $\widehat{S}$, they are isomorphic to $\mathcal{X}_{G,\widehat{S}}$ and $\mathcal{A}_{G,\widehat{S}}$. The corresponding isomorphisms depend on a choice of an isotopy class of embedding of Γ to $\widehat{S}$. They form a principal homogeneous space over the mapping class group.

In this chapter we introduce some natural functions $\{\Delta_i\}$ and $\{X_j\}$ on the varieties $\mathcal{A}_{\mathrm{SL_m},\Gamma}$ and $\mathcal{X}_{\mathrm{PGL_m},\Gamma}$. The combinatorial data parametrising these functions has been described already in the Section 1.5, and will be briefly recalled below.

1. The sets I_m^Γ and J_m^Γ parametrising the canonical coordinates. Let Γ be a marked trivalent graph on S of type $\widehat{S}$. Then there is an ideal triangulation T_Γ of $\widehat{S}$ dual to Γ. It is determined up to an isomorphism by the ribbon structure of Γ. Precisely, let Γ_1, Γ_2 be two graphs as above such that the corresponding marked ribbon graphs $\overline{\Gamma}_1$ and $\overline{\Gamma}_2$ are isomorphic. Then an isomorphism $\overline{\Gamma}_1 \to \overline{\Gamma}_2$ determines uniquely an element g of the mapping class group of S such that $g\Gamma_1$ is isotopic to Γ_2, and hence gT_{Γ_1} is isotopic to T_{Γ_2}.

Summarizing, a marked ribbon trivalent graph Γ of type $\widehat{S}$ determines an ideal triangulation $T = T_\Gamma$ of a marked surface $\widehat{S}_\Gamma$ isomorphic to $\widehat{S}$. The isotopy class of an isomorphism $\widehat{S}_\Gamma \to \widehat{S}$ is determined by an isotopy class of an embedded graph Γ. In this chapter we work with the marked ribbon trivalent graph Γ, the corresponding triangulation $T = T_\Gamma$, and surface $\widehat{S} = \widehat{S}_\Gamma$. Recall the m-triangulation of an ideal triangulation T_Γ of the surface $\widehat{S}_\Gamma$ defined in the Section 1.5. It provides the oriented graph described there and denoted T_m. Recall the two sets, I_m^T and J_m^T, attached to the triangulation. Since the triangulation T and the corresponding graph Γ determine each other, we can use the upper script Γ in the notations. So

$$\mathrm{I}_m^\Gamma := \{\text{vertices of the } m\text{–triangulation of } T_\Gamma\} \text{ - } \{\text{vertices at the punctures of } S\} \qquad (106)$$

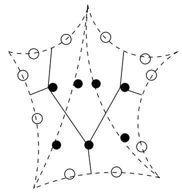
$$\mathrm{J}_m^\Gamma := \mathrm{I}_m^\Gamma - \{ \text{ the vertices at the boundary of } S \ \}$$

Remark. If $\Gamma \longmapsto \Gamma'$ is a flip then there is canonical bijection $\mathrm{I}_m^\Gamma \longrightarrow \mathrm{I}_m^{\Gamma'}$. However the oriented graphs T_m and T'_m are different.

Examples. 1. The set J_2^Γ is identified with the set of all internal (= not ends) edges of Γ.

2. $\widehat{S} = S$ if and only if $\mathrm{I}_m^\Gamma = \mathrm{J}_m^\Gamma$ is the same set.

3. The picture below illustrates the 3-triangulation of a disk with 5 marked points on the boundary. The disk is given by a pentagon. The ideal triangulation of the disk is given by triangulation of the pentagon. The boldface points illustrate the elements of the set I_3^Γ. Adding to them the points visualized by little circles we get the set J_3^Γ. So $|\mathrm{I}_3^\Gamma| = 7$ and $|\mathrm{J}_3^\Gamma| = 17$.



A parametrisation of the sets I_m^Γ and J_m^Γ. Choose a vertex v of Γ. It is dual to a triangle of the triangulation T. The inner vertices of the m–triangulation of this triangle are described by triples of nonnegative integers with sum m

$$(a, b, c), \qquad a + b + c = m, \quad a, b, c \in \mathbb{Z}_{\geq 0} \qquad (107)$$

such that at least two of the numbers a, b, c are different from zero.

We say that a triple (a, b, c) is of *vertex type* if $a, b, c > 0$, and picture the integers a, b, c at the pieces of faces of Γ sharing the vertex v:

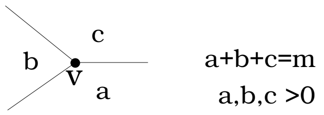


Figure 22: A triple (a, b, c) of vertex type

If one (and hence the only one) of the integers a, b, c is zero, the other two determine an edge e containing v. In this case we say that the triple (a, b, c) is of *edge type*:

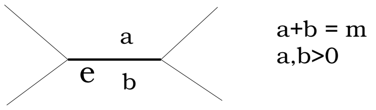


Figure 23: A pair (a, b) of the edge type

The vertex/edge type elements correspond to the vertices of the m–triangulation located inside of the triangles/edges of the dual triangulation $\widehat{\Gamma}$.

2. The regular functions Δ_i on the space $\mathcal{A}_{\mathrm{SL_m},\Gamma}$. They are parametrised by $i \in \mathrm{I}_m^\Gamma$.

We use the following notation. Let V be a vector space of dimension n and $\omega \in \det V^*$ a volume form in V. Let $v_1, ..., v_n$ be vectors of V. Set

$$\Delta(v_1, ..., v_n) := \Delta_\omega(v_1, ..., v_n) :=< v_1 \wedge ... \wedge v_n, \omega >$$

Let $i \in \mathrm{I}_m^\Gamma$. Then there is a triangle t_v of the triangulation T containing i, dual to a vertex v of Γ. This triangle is unique if i is of vertex type. Otherwise there are two such triangles.

The vertex functions. Let L_v be the fiber of the local system L at a vertex v. Then, by definition, it carries a monodromy invariant volume form ω. There are three monodromy operators $M_{v,\alpha}$, $M_{v,\beta}$ and $M_{v,\Gamma}$ acting on L_v. They correspond to the three face paths α, β, Γ starting at v. Given a decorated SL_m–local system L on Γ we get three affine flags

$$X = (x_1, ..., x_m); \quad Y = (y_1, ..., y_m); \quad Z = (z_1, ..., z_m)$$

at L_v. Each of the flags X, Y and Z is invariant under the corresponding monodromy operator:

$$M_{v,\alpha}(X) = X, \quad M_{v,\beta}(Y) = Y, \quad M_{v,\Gamma}(Z) = Z$$

Recall that i is parametrised by a solution (a, b, c) of (107), and that the numbers (a, b, c) match the flags X, Y, Z. Put $x_{(a)} := x_1 \wedge ... \wedge x_a$. The vertex function $\Delta_i = \Delta^v_{a,b,c}$ is defined by

$$\Delta_i(\mathcal{L}) := \Delta^v_{a,b,c}(X, Y, Z) := \Delta_\omega(x_{(a)} \wedge y_{(b)} \wedge z_{(c)}) = \Delta_\omega(x_1 \wedge ... \wedge x_a \wedge y_1 \wedge ... \wedge y_b \wedge z_1 \wedge ... \wedge z_c)$$

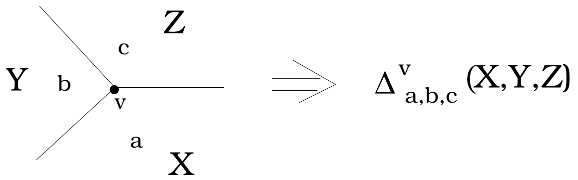


Figure 24: The data giving rise to a vertex function

The edge functions. Let L_e be the fiber of the local system L at an edge e. The SL_m–structure provides a volume form ω in L_e. There are two affine flags

$$X = (x_1, ..., x_m); \quad Y = (y_1, ..., y_m)$$

at L_e invariant under the monodromies around the two face paths α and β sharing the edge e: $M_{v,\alpha}(X) = X, M_{v,\beta}(Y) = Y$. We picture each of the flags located at the corresponding face. We define the edge functions by

$$\Delta_i(\mathcal{L}) = \Delta^e_{a,b}(X, Y) := \Delta_\omega(x_{(a)} \wedge y_{(b)}) = \Delta_\omega(x_1 \wedge ... \wedge x_a \wedge y_1 \wedge ... \wedge y_b)$$

3. The rational functions X_j on the space $\mathcal{X}_{PGL_m,\Gamma}$. Let $i \in \mathrm{J}_m^\Gamma$. Take all the vertices j connected with i by an edge in the graph T_m. Then there is one (if i of vertex type) or two (if i is of

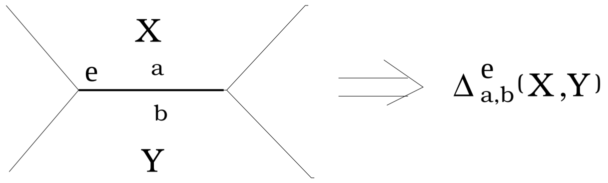


Figure 25: The data giving rise to an edge function

edge type) triangles of T containing all these j's. Consider the invariant flags attached to the vertices of these triangles. There are three such flags if i is vertex type, and four if it is of the edge type, see the pictures below. We can arrange these flags to sit in the same vector space L_v (respectively L_e). For each of these flags choose an affine flag dominating it. We denote by $\widetilde{F}$ the affine flag dominating flag F. Then for each j as above we have the Δ_j–coordinate calculated for these affine flags. We denote it $\widetilde{\Delta}_j$. We set

$$X_i = \prod_{j \in \mathrm{I}^{\Gamma}_m} (\widetilde{\Delta}_j)^{\varepsilon_{ij}} \tag{108}$$

The right hand side evidently does not depend on the choice of dominating affine flags. Below we work out this definition for the vertex and edge type i's.

The vertex functions $X^v_{a,b,c}$. They are given by

$$X^v_{a,b,c}(X,Y,Z) := -\frac{\Delta_{a-1,b+1,c}(\widetilde{X},\widetilde{Y},\widetilde{Z})\Delta_{a,b-1,c+1}(\widetilde{X},\widetilde{Y},\widetilde{Z})\Delta_{a+1,b,c-1}(\widetilde{X},\widetilde{Y},\widetilde{Z})}{\Delta_{a+1,b-1,c}(\widetilde{X},\widetilde{Y},\widetilde{Z})\Delta_{a,b+1,c-1}(\widetilde{X},\widetilde{Y},\widetilde{Z})\Delta_{a-1,b,c+1}(\widetilde{X},\widetilde{Y},\widetilde{Z})}$$

The structure of this formula is illustrated on the picture:

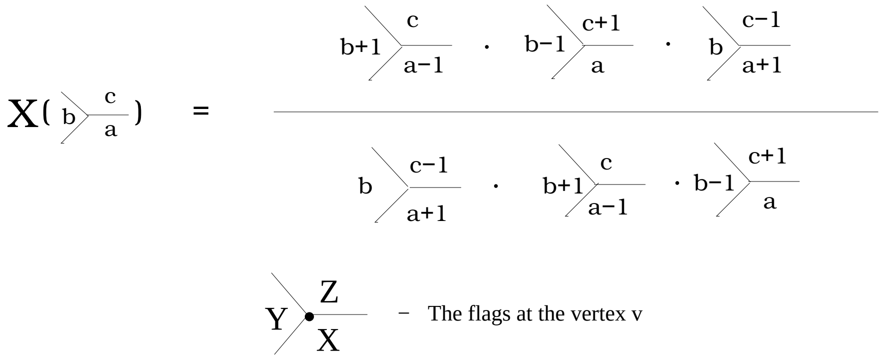


Figure 26: The vertex function

The edge functions $X^e_{a,b}$. They are given by

$$X^e_{a,b}(\widetilde{X},\widetilde{Y},\widetilde{Z},\widetilde{T}) := -\frac{\Delta_{a,b-1,1}(\widetilde{X},\widetilde{Z},\widetilde{T})\Delta_{a-1,1,b}(\widetilde{X},\widetilde{Y},\widetilde{Z})}{\Delta_{a-1,b,1}(\widetilde{X},\widetilde{Z},\widetilde{T})\Delta_{a,1,b-1}(\widetilde{X},\widetilde{Y},\widetilde{Z})}$$

The structure of this formula is illustrated on the picture:

Recall the canonical projection

$$p : \mathcal{A}_{SL_n,\widehat{S}} \longrightarrow \mathcal{X}_{PGL_n,\widehat{S}} \tag{109}$$

Proposition 8.1 *One has*

$$p^* X^{\Gamma}_i = \prod_{j \in \mathrm{I}^{\Gamma}_m} (\Delta^{\Gamma}_j)^{\varepsilon^{\Gamma}_{ij}} \tag{110}$$

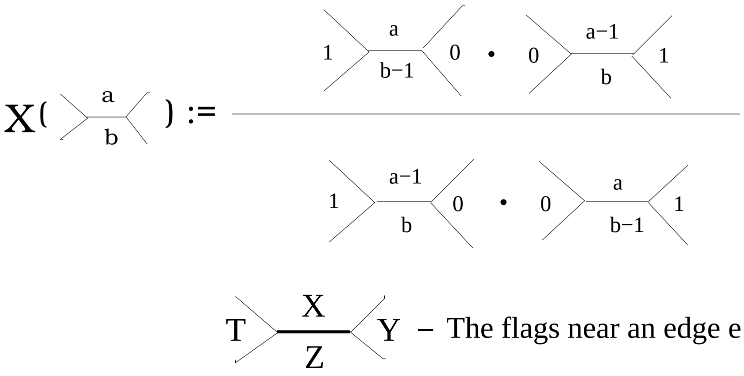


Figure 27: The edge function

Proof. This is a reformulation of the formula (108). The proposition is proved.

It is easy to see using lemma 6.6 that the total number of the X–functions equals to the dimension of the space of all PGL_m-local systems on S.

4. The cross-ratio and the triple ratio. The space of orbits of the group PGL_2 acting on the 4-tuples of distinct points on P^1 is one dimensional. There is a classical coordinate on the orbit space, called the *cross-ratio* of four points $x_1, ..., x_4$. To define it we present P^1 as the projectivisation of a two dimensional vector space V, and choose four vectors $v_1, ..., v_4$ in V projecting to the given points $x_1, ..., x_4$ in $P(V)$. Then

$$r^+(x_1, ..., x_4) := \frac{\Delta(v_1, v_2)\Delta(v_3, v_4)}{\Delta(v_1, v_4)\Delta(v_2, v_3)} \tag{111}$$

One has

$$r^+(x_1, x_2, x_3, x_4) = r^+(x_2, x_3, x_4, x_1)^{-1}; \quad r^+(x_1, x_2, x_3, x_4) = -1 - r^+(x_1, x_3, x_2, x_4)$$

To check the second equality we use the Plücker relation

$$\Delta(v_1, v_2)\Delta(v_3, v_4) - \Delta(v_1, v_3)\Delta(v_2, v_4) + \Delta(v_1, v_4)\Delta(v_2, v_3)$$

We also need a *triple ratio* of three flags in P^2 as defined in s. 3.5 of [G2]. The space of the orbits of the group PGL_3 acting on the triples of flags F_1, F_2, F_3 in P^2 is one dimensional. The triple ratio $r_3^+(F_1, F_2, F_3)$ is a natural coordinate on this space. To define it, let us present P^2 as a projectivisation of a three dimensional vector space V. Choose three affine flags A_1, A_2, A_3 dominating the flags F_1, F_2, F_3 in $P(V)$. We represent the flags A_i as the pairs (v_i, f_i) where $v_i \in V, f_i \in V^*$ and $< v_i, f_i >= 0$. Then set

$$r_3^+(F_1, F_2, F_3) := \frac{< f_1, v_2 >< f_2, v_3 >< f_3, v_1 >}{< f_1, v_3 >< f_2, v_1 >< f_3, v_2 >}$$

This definition obviously does not depend on the choice of the pair (v, f) representing a flag F. It is useful to rewrite this definition using the presentation of the affine flags given by

$$A_1 = (x_1, x_1 \wedge x_2), \quad A_2 = (y_1, y_1 \wedge y_2), \quad A_3 = (z_1, z_1 \wedge z_2)$$

Then

$$r_3^+(F_1, F_2, F_3) := \frac{\Delta(x_1, x_2, y_1)\Delta(y_1, y_2, z_1)\Delta(z_1, z_2, x_1)}{\Delta(x_1, x_2, z_1)\Delta(y_1, y_2, x_1)\Delta(z_1, z_2, y_1)} \tag{112}$$

The cross and triple ratios provide isomorphisms

$$r^+ : \{4\text{–tuples of distinct points of } \mathbb{P}^1\}/PGL_2 \xrightarrow{\sim} \mathbb{P}^1 - \{0, -1, \infty\}$$

$$r_3^+ : \{\text{generic triples of flags of } \mathbb{P}^2\}/PGL_3 \stackrel{\sim}{\longrightarrow} \mathbb{P}^1 - \{0, -1, \infty\}$$

Let $F_i = (x_i, L_i)$ be a flag in P^2, so x_i is a point on a line L_i, and $i = 1, 2, 3$. Consider the triangle formed by the lines L_1, L_2, L_3. Denote by y_1, y_2, y_3 its vertices, so y_i is opposite to the line L_i. Let $\widehat{x}$ be the point of intersection of the line L_3 and the line through the points x_1 and x_2. Then one has

$$r_3^+(F_1, F_2, F_3) = r^+(y_1, y_2, \widehat{x}, x_3)$$

Remark. A yet different definition of the triple ratio one can find in Section 4.2 of [G5].

Lemma 8.2 *A triple (F_1, F_2, F_3) of real flags in $\mathbb{R}P^2$ is positive if and only if the points x_i are at the boundary of a convex domain bounded by the lines of the flags, as on Figure 28.*

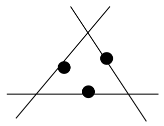


Figure 28: A positive triple of flags

Proof. Any triple of flags satisfying the condition of the lemma can be presented as follows: $v_1 = (0, 1, 1); v_2 = (1, 0, 1); v_3 = (1, t, 0); f_1 = x_1, f_2 = x_2, f_3 = x_3$, where $t > 0$. Its triple ratio is t.

Remark. It is often useful to change the sign in the definition of the cross- and triple ratios: $r := -r^+$ and $r_3 := -r_3^+$.

5. Expressing the X-functions via the cross- and triple ratios. *The vertex functions.* Recall that F_k is the k-dimensional subspace of a flag F. For generic configuration of flags (X, Y, Z) the quotient

$$\frac{L_v}{X_{a-1} \oplus Y_{b-1} \oplus Z_{c-1}} \tag{113}$$

is a three dimensional vector space. It inherits a configuration of three flags $(\overline{X}, \overline{Y}, \overline{Z})$. Namely, the flag $\overline{X}$ is given by the projection of the flag $(X_a/X_{a-1} \subset X_{a+1}/X_{a-1} \subset ...)$ onto (113), and so on. The vertex X–function $X_{a,b,c}$ equals to their triple ratio:

$$X_{a,b,c}(X, Y, Z) = r_3^+(\overline{X}, \overline{Y}, \overline{Z})$$

The edge X–functions. For a generic configuration of flags (X, Y, Z, T) in L_e then the quotient

$$\frac{L_e}{X_{a-1} \oplus Y_{b-1}}$$

is a two dimensional vector space. It inherits a configuration $(\overline{X}_a, \overline{Y}_1, \overline{Z}_b, \overline{T}_1)$ of four distinct one dimensional subspaces: $\overline{X}_a := X_a/X_{a-1}$ and so on. Their cross–ratio is the edge X-function $X_{a,b}$:

$$X_{a,b}(X, Y, Z, T) = r^+(\overline{X}_a, \overline{Y}_1, \overline{Z}_b, \overline{T}_1)$$

6. The main result. Let us choose an isotopy class Γ of a marked trivalent graph on $\widehat{S}$ isomorphic to a marked ribbon graph Γ. It provides an isomorphism of the moduli spaces $f_\Gamma : \mathcal{X}_{PGL_m,\Gamma} \longrightarrow \mathcal{X}_{PGL_m,\widehat{S}}$. Thus we get the rational functions $X_j^\Gamma := f_\Gamma^* X_j$ on the moduli spaces $\mathcal{X}_{PGL_m,\widehat{S}}$. Similarly there are regular functions Δ_i^Γ

Theorem 8.3 *a) The collections of rational functions $\{X_j^\Gamma\}$, $j \in \mathrm{J}_m^\Gamma$, where Γ runs through the set of all isotopy classes of marked trivalent graphs on $\widehat{S}$ of type $\widehat{S}$, provide a positive regular atlas on $\mathcal{X}_{PGL_m,\widehat{S}}$.*

b) The collections of regular functions $\{\Delta_i^\Gamma\}$, $i \in \mathrm{I}_m^\Gamma$, provide a positive regular atlas on $\mathcal{A}_{SL_m,\widehat{S}}$.

Plan of the proof. In Sections 8.7 and 8.8 we give a yet another definition of the X–coordinates on the configuration space of triples and quadruples of flags in V_m. We show in Lemmas 8.4, 8.6, and 8.8 that it gives the same X-coordinates as in Section 8.2. The results of Section 8.8 make obvious that they are related to the general definition for a standard reduced decomposition of w_0, see Lemma 8.8. Similarly our Δ-coordinates on the space $\mathcal{A}_{SL_m,\Gamma}$ are evidently related to the A-coordinates from Section 7 for a standard reduced decomposition of w_0. However our special coordinates have some remarkable features which are absent in the general case even for SL_m and nonstandard reduced decompositions of w_0. For example the defined in Section 8.2-8.3 coordinates on the configuration space of triples of (affine) flags in V_m are manifestly invariant under the cyclic (twisted cyclic) shift.

So the positivity statements in Theorem 8.3 are special cases of the ones in Theorem 1.5. On the other hand these statements follow immediately from the results of Section 9. This way we get simpler proofs of the positivity results, which are independent from the proof of the Theorem 1.5.

We show that our X-functions provide a regular atlas in Section 8.9. A similar statement about the Δ-coordinates is reduced to the one about the X-coordinates using the fact that the projection $\mathcal{A}_{SL_m,\widehat{S}} \longrightarrow \mathcal{X}_{PGL_m,\widehat{S}}$ is obtained by factorization along the action of a power of the Cartan group.

Let us start the implementation of this plan.

7. Projective bases determined by three flags. A *projective basis* in a vector space V is a basis in V up to a multiplication by a common scalar. Projective bases in V form a principal homogeneous space for $PGL(V)$. Let A, B, C be three flags at generic position in an m-dimensional vector space V. Below we introduce several different projective bases in V related to these flags, and compute the elements of PGL_m transforming one of these bases to the other.

Consider the m–triangulation of a triangle shown on the picture. Each side of the triangle carries m vertices of the m–triangulation. There are two types of the triangles, the triangles looking down, and the triangles looking up. We picture them as the white and grey triangles. Let us assign the flags A, B, C to the vertices of the big triangle. The side across the A-vertex is called the A–side, etc.. The vertices of the m–triangulation are parametrized by non negative integer solutions of the equation $a + b + c = m - 1$. The (a, b, c)–coordinates of a vertex show the distance from the vertex to the A, B and C sides of the triangle. A vertex A of the big triangle provides two arrows which start at A and look along one of the two sides sharing A. A *snake* is a path in the one-skeleton of the m–triangulation from one of the vertices of the big triangle to the opposite side, which each time goes in the direction of one of the two arrows assigned to the vertex.

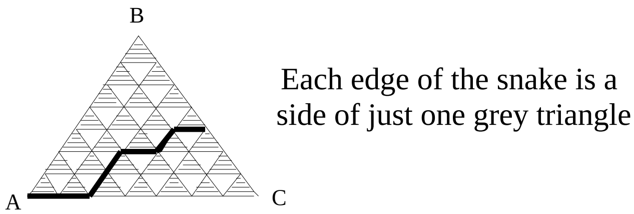


Figure 29: A snake

A snake provides a projective basis in V. Indeed, let $F^a := F_{m-a}$ be the codimension a subspace of a flag F. We attach to each vertex (a, b, c) a one–dimensional subspace $V^{a,b,c} := A^a \cap B^b \cap C^c$. The one–dimensional subspaces V_1, V_2, V_3 attached to the vertices of any grey triangle t span a two–dimensional subspace $V(t)$. The subspaces V_1, V_2, V_3 provide six projective bases in $V(t)$, assigned to the oriented sides of the triangle t. Namely, we assume that the triangle t is counterclockwise oriented. Then if the side V_1V_2 is oriented according to the orientation of t, we choose vectors $v_1 \in V_1$ and $v_2 \in V_2$ so that $v_1 + v_2 \in V_3$. If not, we choose vectors $v_1 \in V_1$ and $v_2 \in V_2$ so that $v_1 - v_2 \in V_3$, as shown on the picture.

Remark. This definition of the six projective bases agrees with the one given in Section 5.2.

Each edge of the triangulation is a boundary of just one grey triangle. Using this observation,

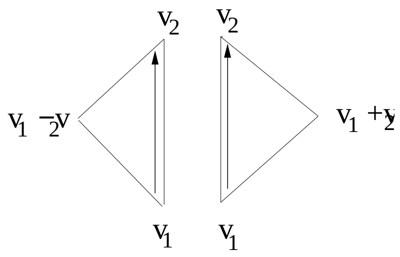


Figure 30: Defining two projective bases in a two dimensional space

let us construct a projective basis corresponding to a snake. Let $e_1, ..., e_{m-1}$ be the edges of the snake in their natural order, so the A–vertex belongs to e_1. Denote by $t_1, ..., t_{m-1}$ the grey triangles containing these edges. Each edge e_i determines a projective basis in the subspace $V(t_i)$. A nonzero vector $v_1 \in V^{m-1,0,0}$ determines a natural basis $v_1, ..., v_m$ parametrized by vertices of the snake. Indeed, v_1 plus a projective basis in V_{t_1} corresponding to e_1 determines a vector v_2 in $V(t_1)$. The vector v_2 plus a projective basis in $V(t_2)$ corresponding to e_2 provides v_3, and so on. All together they provide a projective basis.

8. Transformations between different projective bases related to three flags. Let us compute the element of PGL_m transforming the projective basis corresponding to a snake to the one corresponding to another snake. We assume that the group PGL_m acts on projective bases from the left. First, let us do it for the snakes sharing the same vertex of the big triangle, say the vertex A. An elementary move of a snake is one of the following two local transformations on Figure 31:

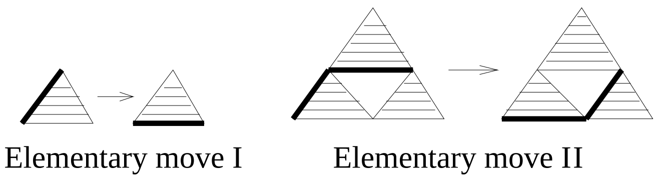


Figure 31: Elementary moves of the snake

We can transform one snake to another using a sequence of elementary moves. An example for $m = 3$ see on Figure 32. In general such a sequence of elementary moves is not unique.

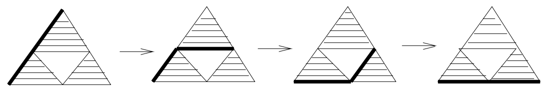


Figure 32: Moving the snake in the case $m = 3$

Let us calculate the elements of PGL_2 and PGL_3 corresponding to the elementary moves I and II. For the elementary move I we get

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \longmapsto \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_1 + v_2 \end{pmatrix}$$

Here the vectors $v_1, v_2, v_1 + v_2$ in $V(t)$ correspond to the vertices of a grey triangle t, so (v_1, v_2) is the projective basis related to the left edge of t, and $(v_1, v_1 + v_2)$ is the one related to the bottom edge of t.

The vertex X–coordinates are assigned to the white triangles. Namely, let α, β, γ be the distances from a white triangle to the sides of the triangle, $\alpha + \beta + \gamma = m - 3$. Then the flags A, B, C induce the flags $\overline{A}, \overline{B}, \overline{C}$ in the three dimensional quotient

$$\frac{V}{A_\alpha \oplus B_\beta \oplus C_\gamma} \tag{114}$$

The triple ratio of these three flags is the X–coordinate corresponding to the white triangle.

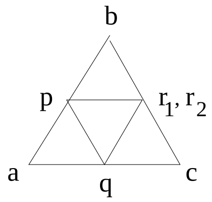
Now consider an elementary move of type II corresponding to a white triangle. Let us calculate the corresponding element of PGL_m, and show that it depends only on the X-coordinate assigned to this white triangle. Denote by a a vector spanning the one dimensional subspace $\overline{A}_1$ of the flag $\overline{A}$. There are two choices for the vectors spanning $\overline{B}_2 \cap \overline{C}_2$: the vector r_1 is defined via the first snake, and the vector r_2 defined via the second one on the picture describing the elementary move II. Then

$$r_2 = X r_1 \tag{115}$$

Lemma 8.4 *The number X in (115) is the X–coordinate of the flags $\overline{A}, \overline{B}, \overline{C}$.*

Proof. A choice of vector a provides the vectors b, c, p, q spanning the subspaces $\overline{B}_1$, $\overline{C}_1$, $\overline{A}_2 \cap \overline{B}_2$, $\overline{A}_2 \cap \overline{C}_2$ respectively, and defined using the left (for b and p) and the right (for c and q) snakes. The three grey triangles provide the relations

$$r_1 = p + b, \quad q = a + p, \quad c = q + r_2 \tag{116}$$



Therefore, as easily follows from (115) and (116), one has

$$X = \frac{\Delta(p, q, r_2)}{\Delta(p, q, r_1)} = -\frac{\Delta(p, q, c)}{\Delta(p, q, b)}$$

It is easy to check that the second fraction equals to the X-coordinate of the flags $\overline{A}, \overline{B}, \overline{C}$. The lemma is proved.

Example. Let us compute the transformation between the projective bases corresponding to the two snakes on Figure 32. We present it as a composition: $(a, p, b) \longmapsto (a, p, r_1) \longmapsto (a, q, r_1) \longmapsto (a, q, r_2) \longmapsto (a, q, c)$. So thanks to (116) and Lemma 8.4 the corresponding matrix is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & X \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1+X & X \end{pmatrix} \in PGL_3; \tag{117}$$

Let us generalize this computation. Let $\varphi_i : SL_2 \hookrightarrow GL_m$ be the canonical embedding corresponding to the i-th root $\lambda_i - \lambda_{i+1}$. Set

$$F_i = \varphi_i \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad H_i(x) := \mathrm{diag}(\underbrace{1, ..., 1}_{i}, x, ..., x)$$

So $H_i : \mathbb{G}_m \longrightarrow H$ is the coroot corresponding to the root $\lambda_i - \lambda_{i+1}$. So $H_i(x)$ commutes with F_j and E_j if $i \neq j$. So the left hand side of (117) can be written as $F_2 H_2(X) F_1 F_2 = F_2 F_1 H_2(X) F_2$.

Let $X_{a,b,c}$, where $a + b + c = m - 3$, $a, b, c \geq 0$, be the vertex coordinates of the configuration of flags (A, B, C). Let $M_{AB \to AC}$ be the element transforming the projective basis corresponding to the snake AB to the one for the snake AC.

Proposition 8.5 *a) One has*

$$M_{AB\to AC} = \prod_{j=1}^{m-1}\left(\Big(\prod_{i=j}^{m-2} H_{i+1}(X_{m-i-2,i-j,j-1})F_i\Big)F_{m-1}\right) \tag{118}$$

b) The element $M_{AB\to BA}$ is the transformation $S: e_i \longmapsto (-1)^i e_{m-i}$.
c) One has

$$M_{CA\to BA} = SM^{-1}_{AB\to AC}S^{-1} \tag{119}$$

For example for $m=2$ we get $M_{AB\to AC} = F_1$, and for $m=3$ and $m=4$ we get

$$M_{AB\to AC} = F_2H_2(X_{000})F_1F_2, \qquad M_{AB\to AC} = F_3H_3(X_{001})F_3F_3H_2(X_{100})F_1H_3(X_{010})F_2F_3$$

Remark. In the formula (118) between every two elements F_i, $i>1$, which have no F_i's in between, there is unique element $H_i(X)$ between them. Its precise location between the two F_i's is not essential since $H_i(X)$ commutes with all the F's located between the two F_i's.

Proof. a) Let us define a sequence of elementary moves transforming the snake AB to the snake AC. The crucial intermediate steps of this sequence are shown on the picture below. We show the snakes which go i steps towards C, and then $m-i-1$ steps in the direction AB, for $i=1,...,m-2$. In between we use the obvious elementary moves transforming the i-th snake to the the $(i+1)$-st.

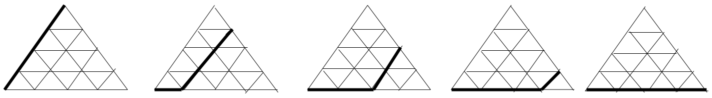


Figure 33: Moving the snake

Now, just like in the above example, our formulas and Lemma 8.4 give the formula (118).

b) Let (A,B,C) be a cyclically ordered triple of distinct one dimensional subspaces in a two dimensional vector space. Let $[v_1,v_2]$ (respectively $[u_1,u_2]$) be the projective basis determined by the snake AB (respectively BA). This means that $v_1+v_2\in C$ (respectively $u_1-u_2\in C$). It follows that $u_1=v_2, u_2=-v_1$. This proves the claim in the two dimensional case. The general case immediately reduces to this.

c) This is obvious. The proposition is proved.

Remark. Since $SF_i^{-1}S^{-1} = E_{m-i}$, there is a formula for $M_{CA\to BA}$ similar to (118). In this formula we use the other standard reduced decomposition of w_0 encoding the sequence of E_i's then the one encoding the sequence of F_i's in (118).

Consider a configuration of four flags (A,B,C,D). We assign them to vertices of a quadrilateral triangulated by the diagonal AC. There are two projective bases related to the snake AC: the one, $P_-(AC)$, comes from the triangle ABC, and the other, $P_+(AC)$, comes from the triangle ACD.

Lemma 8.6 *Let $x_a = X_{a,m-2-a}$ be the coordinates attached to the edge AC. Then*

$$M_{P_-(AC)\to P_+(AC)} = \mathrm{diag}(1, x_0, x_0x_1, ..., x_0x_1...x_{m-2}) \tag{120}$$

Proof. Follows from the SL_2 case, which was considered in the Example in the end of Section 6.7.

9. Comparison with the definitions of Sections 4 **and** 5. Recall the subsets $\overline{U}^-_*(\mathbf{i})\subset U^-$ and $\overline{U}^+_*(\mathbf{j})\subset U^+$ defined by reduced decompositions $\mathbf{i}$ and $\mathbf{j}$ of w_0, see (65) and (66). Elements $\overline{u}_+\in\overline{U}^+_*(\mathbf{i})$ and $\overline{u}_-\in\in\overline{U}^+_*(\mathbf{i})$ have canonical decompositions:

$$\overline{u}_+ = \prod_{j\in\mathbf{j}} x_j(s_j), \quad \overline{u}_- = \prod_{i\in\mathbf{i}} y_i(t_i), \qquad s_j,t_i\in F^* \tag{121}$$

The elements of the sets $\mathbf{i}$ and $\mathbf{j}$ are labeled by the simple roots $\alpha_1, ..., \alpha_r$, providing the decompositions

$$\mathbf{i} = \mathbf{i}(\alpha_1) \cup ... \cup \mathbf{i}(\alpha_r) \qquad \mathbf{j} = \mathbf{j}(\alpha_1) \cup ... \cup \mathbf{j}(\alpha_r)$$

Our convention was the following: let α be a simple root, j^+_α is the right element of $\mathbf{j}(\alpha)$, i^-_α is the left element of $\mathbf{i}(\alpha)$. Then for each simple root α one has

$$x_{j^+_\alpha}(s_{j^+_\alpha}) = 1, \qquad y_{i^-_\alpha}(t_{i^-_\alpha}) = 1, \tag{122}$$

Consider the following two reduced decomposition of the maximal length element in S_m:

$$\mathbf{i} = (m, m-1, m, ..., 2, ..., m, 1, 2, ..., m), \qquad \mathbf{j} = (m, ..., 1, m, ..., 2, ..., m, m-1, m)$$

Consider the m-triangulation of the triangle ABC, as on Figure 29. Its grey triangles are presented as a union of m strips going parallel to the side BC. We enumerate the strips so that the k-th strip contains k the grey triangles. These numbers match the simple roots: k corresponds to $\lambda_k - \lambda_{k+1}$. Then the string $\mathbf{i}$ provides an enumeration of the grey triangles from the top to the bottom, going in each layer parallel to the line AC from A to C. Let t be a white triangle and t' the grey triangle sharing a side with t, obtained from t by going towards the vertex C. Recall that our special X-coordinates are naturally attached to the white triangles. We assign the X-coordinate attached to the white triangle t, to the corresponding grey triangle t'. Finally, we assign 1 to the grey triangles neighboring the side AB.

Lemma 8.7 *One has*

$$M_{AB\to AC} = \overline{u}^-_{AB\to AC} h^-_{AB\to AC}, \qquad \overline{u}^-_{AB\to AC} \in U^-_*(\mathbf{i}), \quad h^-_{AB\to AC} \in H$$

The canonical coordinates (121) of the element $\overline{u}^-_{AB\to AC}$ are given by the X-coordinates assigned as above to the grey triangles, provided the grey triangles are enumerated by the elements of the set $\mathbf{i}$ *as was explained above.*

The element $h^-_{AB\to AC}$ is the product of the elements $H_{i_t}(X_t)$ assigned to the grey triangles t.

Proof. Follows from the very definition and formulas (121), (118) and (122).

A projective basis in V_m is the same thing as a pinning for PGL_m.

Lemma 8.8 *The six projective bases corresponding to the six oriented edges of the triangle ABC coincide with the six canonical pinnings attached to the triple of flags (A, B, C) in V_m in Section 5.2.*

Proof. Lemma 8.7 implies that the projective basis assigned to the snake AB coincides with the one assigned to the side AB of the triangle whose vertices are labeled by the configuration of flags

$$(A, B, C) = (B^-, B^+, u_- B^+ u_-^{-1})$$

The case of the snakes BC and CA is similar. To treat the case of the snake BA we use the remark in Section 8.8. The lemma is proved.

10. Constructing a framed local system with arbitrary given non-zero coordinates. Let Γ be a trivalent graph on $\widehat{S}$, of the type of $\widehat{S}$. In Section 8.2 we defined the X-functions on the moduli space $\mathcal{X}_{PGL_m,\widehat{S}}$ corresponding to Γ. Recall that they are parametrized by the set J^Γ_m. Below we are going to show how to invert this construction. Precisely, let F be a field. Then given $X_i \in F^*$, $i \in \mathrm{J}^\Gamma_m$ we will construct a framed PGL_m-local system on $\widehat{S}$ with the given X-coordinates $\{X_i\}$.

Consider a decomposition of $\widehat{S}$ into hexagons and rectangles obtained as follows. Replace the edges of Γ by rectangles and the vertices by hexagons, as on the left picture. Let Δ be the 1–skeleton of this decomposition.

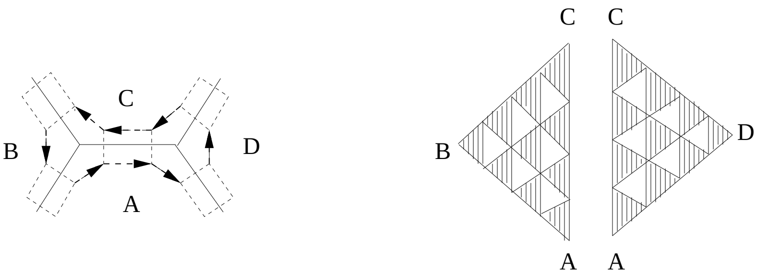


Figure 34: The graph Δ

Let G a split reductive algebraic group. Then, given a framed G-local system on $\widehat{S}$, every vertex v of Γ provides a configuration of three flags in the vicinity of v. Therefore there are the six canonical pinning assigned to v. These pinnings match the vertices of the hexagon of Δ corresponding to v.

To construct a framed $G(F)$-local system on $\widehat{S}$ we assign to every oriented edge $\mathbf{e}$ of Δ an element $M(\mathbf{e}) \in G(F)$. Let $\mathbf{p} := \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3...$ be a path on Δ, where $\mathbf{ef}$ means $\mathbf{e}$ follows $\mathbf{f}$. Then $\mathbf{p}$ gives rise to $M(\mathbf{p}) := M(\mathbf{e}_1)M(\mathbf{e}_2)M(\mathbf{e}_3)....$

Lemma 8.9 *Suppose that elements $M(\mathbf{e}) \in G(F)$ satisfy the following conditions:*
i) If $\mathbf{e}$ does not intersect the graph Γ then $M(\mathbf{e}) \in B(F)$.
ii) Let $\overline{\mathbf{e}}$ be the edge $\mathbf{e}$ equipped with the opposite orientation. Then $M(\mathbf{e})M(\overline{\mathbf{e}}) = \mathrm{Id}$.
iii) The element assigned to a path around a rectangle or hexagon is the identity.
Then the elements $M(\mathbf{p})$ depend only on the homotopy class of the path $\mathbf{p}$, providing a $G(F)$-local system on S. Moreover it has a natural framing on $\widehat{S}$.

Proof. The properties ii) and iii) guaranty that we get a $G(F)$-local system on S, and i) provides a natural framing. The lemma is proved.

Let us construct such elements $M(\mathbf{e})$. The graph Δ has three types of edges: the edges intersecting Γ, the edges of the rectangles which do not intersect Γ, and the edges of the hexagons which do not intersect Γ. We denote them, respectively, by $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. We will assume that the $\mathbf{c}$-edges of the hexagons are oriented counterclockwise, i.e. in the opposite way to the cyclic order of the edges and hence the flags in the vicinity of v. The following definition is suggested by Proposition 8.5 and Lemma 8.6.

a) Set $M(\mathbf{a}) := S$.

b) Let $\mathbf{b}$ be an edge of a rectangle which goes along an oriented edge $\mathbf{e}$ of Γ. Let $x_a = X^{\mathbf{e}}_{a,m-2-a}$ be the coordinates attached to the edge $\mathbf{e}$. Then $M(\mathbf{b})$ is given by the right hand side of (120).

c) Let $\{X_{a,b,c}\}$, where $a+b+c = m-3$ and $a, b, c \geq 0$, be the coordinates assigned to a vertex v of Γ. Then integers a, b, c match the edges of the hexagon surrounding v which do not intersect Γ. Let $\mathbf{c}$ be such an edge assigned to a, counterclockwise oriented. Then $M(\mathbf{c})$ is given by the right hand side of (118).

Theorem 8.10 *Let F be a field, and we are given $X_i \in F^*$, where $i \in \mathrm{J}^{\Gamma}_m$. Then*
a) The elements $M(\mathbf{a}), M(\mathbf{b}), M(\mathbf{c}) \in PGL_m(F)$ satisfy the conditions of Lemma 8.9. So they give rise to a framed local system on $\widehat{S}$.
b) Let $\mathcal{L}$ be a framed $PGL_m(F)$–local system on $\widehat{S}$. Then the framed $PGL_m(F)$–local system $\mathcal{L}(X)$ provided by the X–coordinates of $\mathcal{L}$ is isomorphic to $\mathcal{L}$.

Proof. Follows immediately from Proposition 8.5 and Lemma 8.6. The theorem is proved.

Let H_m (respectively $\mathcal{V}_m$) be the algebraic torus parametrising the internal edge (respectively vertex) X–coordinates. Then H_m is isomorphic to the Cartan group of PGL_m, and $\mathcal{V}_m = \mathbb{G}_m^{(m-2)(m-1)/2}$.

The part a) of Theorem 8.10 provides a regular map

$$\psi_\Gamma : H_m^{\{\text{internal edges of } \Gamma\}} \times \mathcal{V}_m^{\{\text{vertices of } \Gamma\}} = \mathbb{G}_m^{\mathrm{J}_m^\Gamma} \longrightarrow \mathcal{X}_{PGL_m,\widehat{S}}$$

Corollary 8.11 *The map ψ_Γ is a birational isomorphism, and is an injective regular map.*

Proof. The part b) of Theorem 8.10 plus the dimension count imply that all conditions of Lemma 6.5 are valid for the map ψ_Γ. So it is a birational isomorphism. Since ψ_Γ is regular, it is injective. The corollary is proved.

Definition 8.12 *The atlas given by the collection of embeddings $\{\psi_\Gamma\}$, when Γ run through all isotopy classes of marked trivalent graphs on $\widehat{S}$ of type $\widehat{S}$, is called* **special atlas** *on $\mathcal{X}_{PGL_m,\widehat{S}}$.*

By Lemma 8.7 the special atlas is a subatlas of the atlas defined in Section 5. This implies that the special atlas is a positive atlas. An elementary proof of this claim is contained in Section 9. This positivity claim plus Theorem 8.10 gives a complete proof of Theorem 8.3.

In the next subsection we investigate the monodromy properties of the universal PGL_m-local system on S with respect to the special positive atlas on $\mathcal{X}_{PGL_m,\widehat{S}}$.

11. Positive Laurent property of the monodromy of universal PGL_m-local system. A *special good positive Laurent polynomial on $\mathcal{X}_{PGL_m,S}$* is a rational function which in every coordinate system $\{T_i\}$ from the special positive atlas on $\mathcal{X}_{PGL_m,S}$ is a Laurent polynomial with positive integral coefficients. We say that a matrix with entries in the field $\mathbb{F}_{PGL_m,S}$ is a *special good totally positive (integral) Laurent matrix* if in every special positive coordinate system, every minor of this matrix is a good positive integral Laurent polynomial. There is a similar definition for the upper/lower triangular matrices.

Theorem 8.13 *The monodromy of the universal PGL_m-local system $\mathcal{L}_m$ around a non-boundary loop on S is conjugated to a special good totally positive Laurent matrix. The monodromy around a boundary loop is conjugated to an upper/lower triangular special good totally positive Laurent matrix.*

Proof. It is similar to the proof of Theorem 1.11. Starting from the connection on the graph Δ given by the elements $M(\mathbf{a}), M(\mathbf{b}), M(\mathbf{c})$ we define a connection on the corresponding graph Γ'_T, shown on Figure 20, as follows. Observe that the oriented $\mathbf{t}$-edges of Γ'_T match the oriented $\mathbf{c}$-edges of Δ. Further, the $\mathbf{b}$-edges of Δ have canonical orientations: the ones compatible with the clockwise orientation of the holes. So oriented $\mathbf{e}$-edges of Γ'_T match the $\mathbf{b}$-edges of Δ. So we set

$$M(\mathbf{t}) := SM(\mathbf{c}) = M(\mathbf{a})M(\mathbf{c}); \qquad M(\mathbf{e}) := M(\mathbf{b})S = M(\mathbf{b})M(\mathbf{a})$$

assuming that the $\mathbf{t}$ and $\mathbf{e}$-edges of Γ'_T match the corresponding edges of Δ. Then $M(\mathbf{e})M(\mathbf{t})$ is a lower triangular special totally positive integral Laurent matrix. Indeed, it is obtained by a product of the matrices F_j, $H_i(x)$ and $M(\mathbf{b})$. Similalry, if $\mathbf{e}$ follows $\overline{\mathbf{t}}$, then $M(\mathbf{e})M(\overline{\mathbf{t}}) = M(\mathbf{e})M(\mathbf{t})^{-1} = M(\mathbf{b})SM(\mathbf{c})S^{-1}$ is an upper triangular special totally positive integral Laurent matrix: see Remark after Proposition 8.5. The rest is as in the proof of Theorem 1.11. The theorem is proved.

Let $\rho : PGL_m \to \mathrm{Aut}(V)$ be a finite dimensional representation. Consider the associated local system $\mathcal{L}_\rho := \mathcal{L}_m \times_{PGL_m} V$ on $S \times \mathcal{X}_{PGL_m}$.

Corollary 8.14 *a) Let M_α be the monodromy of the universal local system around a loop α. Then, for any integer $n > 0$, $\mathrm{tr}(M_\alpha^n)$ is a special good positive Laurent polynomial on $\mathcal{X}_{PGL_m,S}$.*

b) For any finite dimensional representation ρ of PGL_m the monodromy of the universal local system $\mathcal{L}_\rho$ around any loop on S is conjugated to a matrix whose entries are special good positive rational Laurent polynomials on $\mathcal{X}_{PGL_m,\widehat{S}}$.

c) The trace of the monodromy of $\mathcal{L}_\rho$ around any loop is a special good positive rational Laurent polynomial on $\mathcal{X}_{PGL_m,\widehat{S}}$.

Proof. a) Follows immediately from Theorem 8.13.

b) Take the canonical basis in V. Then the elements E_i and F_i of PGL_m act by matrices with non-negative ratinal coefficients, and the elements $H_i(X)$ act by diagonal matrices whose coefficients are monomials in $X^{\pm 1}$. c) follows from b). The corollary is proved.

Conjecture 8.15 $\mathrm{tr}(M^n_\alpha)$ *can not be presented as a sum of two non-zero elements of* $\mathbb{L}_+(\mathcal{X}_{PGL_m,S})$*:*

$$\mathrm{tr}(M^n_\alpha) \in \mathbf{E}(\mathcal{X}_{PGL_m,S})$$

12. Convex curves in $\mathbb{RP}^n$ and positive curves in the flag variety for $PGL_{n+1}(\mathbb{R})$. A curve K in $\mathbb{RP}^n$ is *convex* if any hyperplane contains no more then n points of K. This definition goes back to Schoenberg [Sch]. It is interesting that every convex algebraic curve is projectively equivalent to the Veronese curve given by $(x_0 : x_1) \longmapsto (x_0^n : x_0^{n-1}x_1 : ... : x_1^n)$.

We start from a useful inductive criteria of positivity of configurations of flags in $\mathbb{RP}^n$.

Lemma 8.16 *A configurations of m flags in $\mathbb{RP}^2$ is positive if and only if the lines of the flags form a convex m-gon, and the points of the flags are at the boundary of this m-gon, but not at the vertices, see Figure 35.*

Proof. For $m = 3$ this is Lemma 8.2. The general case is deduced from it, see Section 1 of [FG3].

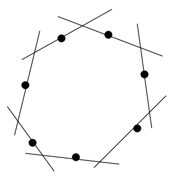


Figure 35: A positive configuration of six flags

Denote by $\mathcal{B}_{n+1}(\mathbb{R})$ the flag variety for $PGL_{n+1}(\mathbb{R})$. There are two natural projections

$$\pi : \mathcal{B}_{n+1}(\mathbb{R}) \longrightarrow \mathbb{RP}^n, \quad \widehat{\pi} : \mathcal{B}_{n+1}(\mathbb{R}) \longrightarrow \widehat{\mathbb{RP}}^n$$

Let $(F_1, ..., F_m)$ be a generic configuration of flags in $\mathbb{RP}^n$. Set $x_k := \pi(F_k)$. We define a configuration of flags $(x_k|F_1, ..., F_m)$ in $\mathbb{RP}^{n-1}$ by projection with the center at the point x_k. Namely, let F be a flag in $\mathbb{RP}^n$ whose $(n-2)$-dimensional subspace does not contain x_k. Then F provides a flag in the projective space of all lines passing through x_k: its p-dimensional subspace consists of the lines through x_k passing through the similar subspace of F. This way the flags F_i, $i \neq k$, give rise to the flags $\overline{F}_i$. Finally, the p-dimensional subspace of the the flag F_k provides $(p-1)$-dimensional subspace of the flag $\overline{F}_k$. Now we set $(x_k|F_1, ..., F_m) := (\overline{F}_1, ..., \overline{F}_m)$.

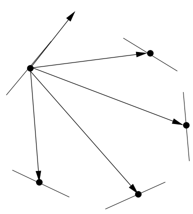


Figure 36: Projecting a configuration of five flags

Lemma 8.17 *Let $n \geq 3$. A configuration of flags $(F_1, ..., F_m)$ in $\mathbb{RP}^n$ is positive if and only if all configurations $(x_1|F_1, ..., F_m), ..., (x_m|F_1, ..., F_m)$ are positive.*

Proof. Let us prove the implication <=. Take a convex m-gon P_m with vertices labelled by the flags $F_1, ..., F_m$. Observe that, assuming $n \geq 3$, given a triangulation of the polygon P_m, the corresponding canonical cooordinates of the configuration $(F_1, ..., F_m)$ are defined as follows: we take a triple or quadruple of flags assigned to a triangle or quadrilateral of the triangulation, project them to the flags in $\mathbb{RP}^2$, and then take the canonical coordinates of the obtained configuration. It follows from this that positivity of the configurations $(x_1|F_1, ..., F_m), ..., (x_m|F_1, ..., F_m)$ implies positivity of all the coordinates of the original configuration $(F_1, ..., F_m)$ with respect to given triangulation.

Let us prove that if the configuration $(F_1, ..., F_m)$ is positive, then the one $(x_m|F_1, ..., F_m)$ is also positive. Since the property of positivity of configurations of flags is cyclically invariant, this would imply the implication =>. Consider the triangulation of the polygon P_m by the diagonals from the vertex F_m. Consider the triple (F_k, F_{k+1}, F_m) assigned to the vertices of one of the triangles. Then, assuming $n \geq 3$, the set of the coordinates corresponding to the triple $(x_m|F_k, F_{k+1}, F_m)$ is a subset of the set of coordinates of the triple (F_k, F_{k+1}, F_m). A similar statement is true for the coordinates assigned to a diagonal (F_k, F_m) of the polygon. The claim follows immediately from this. The lemma is proved.

Remark. The implication <= of Lemma 8.17 is false for $n = 2$.

Let K be an oriented continuous convex curve in $\mathbb{RP}^n$. Let x_1, x_2 be two points of K. We say that $x_1 < x_2$ if x_2 follows x_1 according to the orientation of the curve K. Given a collection of points $X = x_1 < x_2 < ... < x_n$ on K, we associate to it a flag

$$F_X := (x_1, [x_1x_2], ..., [x_1...x_n])$$

where $[x_1...x_k]$ denotes the subspace spanned by the points $x_1, ..., x_k$. Convexity of K implies that $\dim[x_1...x_k] = k - 1$, so F_X is indeed a flag. More generally, given n-tuples of points

$$X(1) := \{x_1(1) < ... < x_n(1)\}, \quad ... \quad , X(m) := \{x_1(m) < ... < x_n(m)\} \tag{123}$$

we get the flags $F_{X(1)}, ..., F_{X(m)}$. We say that $X(a) < X(b)$ if $x_i(a) < x_j(b)$ for all $1 \leq i, j \leq n$.

Proposition 8.18 *Let K be an oriented continuous convex curve in $\mathbb{RP}^n$. Assume that $X(1) < X(2) < ... < X(m)$ in (123). Then the configuration of flags $(F_{X(1)}, ..., F_{X(m)})$ is positive.*

Proof. Lemma 8.16 implies Proposition 8.18 for $n = 2$. So we may assume that $n > 2$. The projection of a convex curve from a point on this curve is a convex curve. So Lemma 8.17 implies Proposition 8.18 by descending induction on n. The proposition is proved.

The osculating curves. Let K be an oriented continuous convex curve in $\mathbb{RP}^n$. Choose a point $x \in K$, and consider an n-tuple of points $x < x_2 < ... < x_n$ on K. It provides a flag. We claim that when the points x_i approach the point x keeping their order, the corresponding family of flags has a limit. We prove this claim below. The limit is denoted $F_r(x)$ and called the *right osculating flag* at x. The map $x \longmapsto F_r(x)$ provides a map $\beta_r : K \to \mathcal{B}_{n+1}(\mathbb{R})$, which, as one easily sees using the proof of Proposition 8.18, is positive. The curve $\widetilde{K}_r := \beta_r(K)$ is called the *right osculating curve* to K. Similarly starting from the points $x > x_2 > ... > x_n$ we define the *left osculating curve* $\widetilde{K}_l$ to K.

To prove the claim, consider a sequence of $(n-1)$-tuples of points $X(k) := x_2(k) < ... < x_n(k)$, $k = 1, 2, ...$, such that $x < x_i(k)$ and $x_i(k)$ converges to x as $k \to \infty$. Let us assume that $X(a) < X(b)$. Projecting K from x we get a convex curve in $\mathbb{RP}^{n-1}$, and a sequence of flags $F_{X(a)}$ on it. This sequence is positive by Proposition 8.18. Thus by Theorem 6.17 it has a limit $\overline{F}_r(x)$. The limit evidently does not depend on the choice of sequence $X(k)$. Then $F_r(x)$ is the unique flag containing x and projecting to the flag $\overline{F}_r(x)$.

Theorem 8.19 *a) Let C be a positive curve in the flag variety $\mathcal{B}_{n+1}(\mathbb{R})$. Then $\pi(C)$ and $\widehat{\pi}(C)$ are convex curve of $\mathbb{RP}^n$ and $\widehat{\mathbb{RP}}^n$.*

b) Let K be a continuous convex curve in $\mathbb{RP}^n$. Then the osculating curves $\widetilde{K}_l$ and $\widetilde{K}_r$ are positive curves in $\mathcal{B}_{n+1}(\mathbb{R})$. So if K is a C^{n-1}-smooth, then $\widetilde{K}_l = \widetilde{K}_r = \widetilde{K}$ is a positive curve in $\mathcal{B}_{n+1}(\mathbb{R})$.

c) The rule $K \longmapsto \widetilde{K}$ gives rise to a bijective correspondence

$$C^n\text{-smooth convex curves in } \mathbb{RP}^n < - > C^1\text{-smooth positive curves in } \mathcal{B}_{n+1}(\mathbb{R}) \qquad (124)$$

Proof. a) It follows immediately from the following crucial proposition:

Proposition 8.20 *Let $(F_1, ..., F_{n+1})$ be a positive configuration of flags in $\mathbb{RP}^n$. Let $(x_1, ..., x_{n+1})$ be the corresponding configuration of points in $\mathbb{RP}^n$, i.e. $\pi(F_i) = x_i$. Then these points are not contained in a hyperplane.*

Proof. Observe that a positive configuration of flags consists of the flags in generic position. Lemma 8.17 implies that $(x_{n+1}|F_1, ..., F_n)$ is a positive configuration of flags. Further, the claim of Proposition 8.20 for the configuration of flags $(x_{n+1}|F_1, ..., F_n)$ in $\mathbb{RP}^{n-1}$ evidently implies the one for the initial flags $F_1, ..., F_{n+1}$. So arguing by the induction on $n \geq 2$ we reduce the Proposition to the case $n = 2$. Let us prove the Proposition when $n = 2$.

Let V_3 be a three dimensional vector space. Choose a volume form ω in V_3. For any two vectors a, b we define the cross-product $a \times b \in V_3^*$ by $< a \times b, c >:= \Delta(a, b, c)$. The volume form ω defines the dual volume form in V_3^*, so we can define $\Delta(x, y, z)$ for any three vectors in V_3^*. The following fact is Lemma 5.1 from [G4].

Lemma 8.21 *For any 6 vectors in generic position $a_1, a_2, a_3, b_1, b_2, b_3$ in V_3 one has*

$$\Delta(a_1, a_2, b_2) \cdot \Delta(a_2, a_3, b_3) \cdot \Delta(a_3, a_1, b_1) - \Delta(a_1, a_2, b_1) \cdot \Delta(a_2, a_3, b_2) \cdot \Delta(a_3, a_1, b_3) =$$

$$\Delta(a_1, a_2, a_3) \cdot \Delta(a_1 \times b_1, a_2 \times b_2, a_3 \times b_3)$$

Proof. Any configuration of six generic vectors in V_3 is equivalent to the following one:

$$\begin{array}{cccccc} a_1 & a_2 & a_3 & b_1 & b_2 & b_3 \\ - & - & - & - & - & - \\ 1 & 0 & 0 & x_1 & y_1 & z_1 \\ 0 & 1 & 0 & x_2 & y_2 & z_2 \\ 0 & 0 & 1 & x_3 & y_3 & z_3 \end{array}$$

Then both the left and right hand sides are equal to $y_3z_1x_2 - x_3y_1z_2$. The lemma is proved.

Let (A, B, C) be a positive configuration of three flags in $\mathbb{RP}^2$. The flags A, B, C can be defined by pairs of vectors $(a_1, a_2), (b_1, b_2), (c_1, c_2)$ in V_3, so that the flag A is determined by the affine flag $(a_1, a_1 \wedge a_2)$, and so on. Recall the triple ratio (112). Using Lemma 8.21 we have

$$1 + r_3^+(A, B, C) = 1 + \frac{\Delta(a_1, a_2, b_1)\Delta(b_1, b_2, c_1)\Delta(c_1, c_2, a_1)}{\Delta(a_1, a_2, c_1)\Delta(b_1, b_2, a_1)\Delta(c_1, c_2, b_1)} =$$

$$\frac{\Delta(a_1, b_1, c_1)\Delta(a_1 \times a_2, b_1 \times b_2, c_1 \times c_2)}{\Delta(a_1, a_2, c_1)\Delta(b_1, b_2, a_1)\Delta(c_1, c_2, b_1)}$$

Therefore $\Delta(a_1, b_1, c_1) \neq 0$ if $r_3^+(A, B, C) > 0$. Proposition 8.20, and hence the part a) of Theorem 8.19 are proved.

b) Observe that projection with the center at a point $x \in K$ maps the osculating flags on K to the ones on the projected curve. So the part b) follows immediately from Lemma 8.17.

c) By Proposition 6.15 a differentiable positive curve C in $\mathcal{B}_{n+1}(\mathbb{R})$ is tangent to the canonical distribution, and hence is the osculating curve for the n times differentiable curve $\pi(C)$ in $\mathbb{RP}^n$. This curve is convex by the part a). The converse statement is the part b) of the theorem. The theorem is proved.

9 The pair $(\mathcal{X}_{PGL_m,\widehat{S}}, \mathcal{A}_{SL_m,\widehat{S}})$ corresponds to an orbi-cluster ensemble

In this Section we formulate precisely and prove Theorems 1.18 and 1.19.

1. A dictionary. We assume that the reader is familiar with the notion of cluster ensemble as defined in [FG2]. A cluster ensemble is determined by the same data as the cluster algebras of Fomin-Zelevinsky [FZI]: it consists of a finite set I and a skew-symmetrizable function $\mathcal{E} = (\varepsilon_{ij})$ on $\mathrm{I}\times\mathrm{I}$. There is an I-marked graph Tr_I whose vertices v parametrize the $\mathcal{A}$- and $\mathcal{X}$- coordinate systems, and whose edges adjacent to a given vertex are in bijective correspondence with the set I. For a given vertex v we have the $\mathcal{A}$-coordinates $\{A_i^v\}$ parametrized by the set I. Further, there is a subset $\mathrm{J} \subset \mathrm{I}$. The $\mathcal{A}$-coordinates parametrized by the complement $\mathrm{I} - \mathrm{J}$ are called tropical $\mathcal{A}$-coordinates. The set J parametrizes the $\mathcal{X}$-coordinates $\{X_j^v\}$.

Here is a dictionary relating the ingredients of the cluster ensemble structure to the data assigned to the $(\mathcal{A}_{SL_n,\widehat{S}}, \mathcal{X}_{PGL_n,\widehat{S}})$ moduli spaces. Let us choose a marked trivalent graph Γ on S, of type $\widehat{S}$, and take the corresponding m-triangulation $T_m = T_m(\Gamma)$ of S. We understood T_m as an oriented graph. Recall that an oriented graph G determines a skew-symmetric function $\varepsilon_{pq}(G)$ on the set of pairs of its vertices, see (14).

i) The set I of the cluster ensemble is the set I_m^Γ of vertices of the graph $T_m(\Gamma)$ minus the punctures of S, and the subset J is given by J_m^Γ. The sets I_m^Γ as well as J_m^Γ are canonically identified for different Γ's.

ii) The cluster function $\mathcal{E} = \mathcal{E}_{SL_m,\widehat{S}}$ is the function $\varepsilon_{pq} = \varepsilon_{pq}(T_m)$.

iii) The oriented graph $T_m(\Gamma)$ corresponds to a vertex v of the cluster complex $C_\mathcal{E}$.

iv) The coordinates Δ_i^Γ are the cluster ensemble coordinates A_i^v.

v) The coordinates X_j^Γ are the cluster ensemble coordinates X_j^v.

vi) Formula (110) shows that the function X_j^Γ and Δ_i^Γ are related under the projection p the same way as the cluster ensemble coordinates X_j^v and A_i^v, see [FG2].

However an interpretation of a flip in the cluster language is a non trivial problem, which we address now. We are going to prove that a flip can be presented as a composition of mutations.

Describing a flip and its action on the cluster function. Let Γ' be a graph obtained from Γ by a flip at an edge E. Denote by F the new edge of Γ'. Let $\{A,B,C,D\}$ (resp. $\{A',B',C',D'\}$) be the set of edges of the Γ (resp. Γ') adjacent to E (resp. F). Then there is canonical bijection $\{A,B,C,D\} \longrightarrow \{A',B',C',D'\}$ such that $X \longmapsto X'$. Let us number by $1,2,3,4$ the pieces of faces in the vicinity of the edge E. They match the ones in the vicinity of F. We assume that their order is compatible with the cyclic order provided by the ribbon structure of Γ, see Figure (37). Denote by T'_m the m-triangulation corresponding to Γ'.

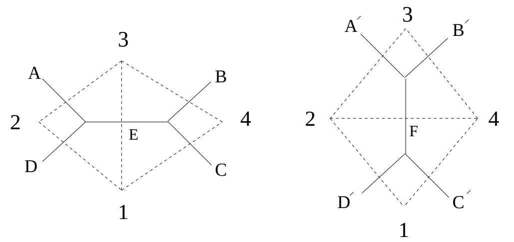


Figure 37: Describing a flip

The claim that a flip can be obtained as a composition of mutations means the following:

i) The composition of mutations transforms the function $\varepsilon_{pq}(T_m)$ to $\varepsilon_{pq}(T'_m)$

ii) The transformation rules for the $\mathcal{A}$- and $\mathcal{X}$-coordinates under a flip coincide with the ones provided by the sequence of mutations.

2. A sequence of mutations representing a flip at an edge E. We present it as a composition of $m-1$ steps. Consider the diamond with the vertices $1,2,3,4$ on Figure (37). One can inscribe into the m-triangulation of this diamond rectangles of the size $i\times(m-i)$, where $i = 1, ..., m-1$, centered at the center of the diamond. Here i (resp. $(m-i)$) is the length in direction of the edge F (resp. E). We subdivide this rectangle into $i(m-i)$ equal squares. Figure (39) illustrates this.

Step i. Do mutations in the direction of the elements of the set $\mathrm{I} := \mathrm{I}_m$ corresponding to the centers of little squares of the $i \times (m-i)$ rectangle.

For example at the step 1 (resp. $m-1$) we do mutations corresponding to the $m-1$ vertices located inside of the vertical (resp. horizontal) diagonal of the diamond.

Proposition 9.1 *The described sequence of mutations transforms the cluster function $\varepsilon_{pq}(T_m)$ to the one $\varepsilon_{pq}(T'_m)$.*

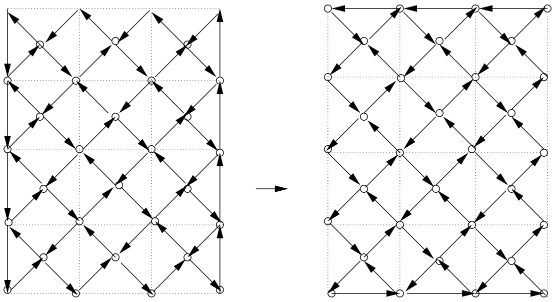


Figure 38:

Proof. Consider an $m \times n$ rectangle where $m, n \geq 1$ are integers. Subdivide it into mn equal squares, mark the vertices and centers of the squares. Connect the marked points by arrows as on Figure (38). Denote by $R_{m,n}$ the obtained oriented graph. Observe that the right picture on Figure (38) is obtained from the left one by inverting directions of the arrows at each square center, and replacing the arrows at the sides of the rectangle by the ones at the top and bottom, keeping the counterclockwise orientation.

Lemma 9.2 *Consider the cluster function ε_{pq} corresponding to the oriented graph on the left of Figure (38). Let us perform a sequence of flips at all centers of the mn squares, in any order. Then the obtained cluster function ε'_{pq} is described by the picture on the right of Figure (38).*

Proof. Follows immediately from the definitions.

Example. The two pictures (39) and (40) illustrate the case $m = 6$. We show a part of the mutated graph after the steps 1 and 2. On each picture we show the distinguished rectangles corresponding to this and previous steps.

Contemplation will convince the reader that after $m-1$ steps we get the graph of the function $\varepsilon_{pq}(T'_m)$. The proposition is proved.

3. Hypersimplices and a decomposition of a simplex. Recall [GGL] that hypersimplex $\Delta^{k,l}$, where $k + l = n - 1$, is a section of the n-dimensional cube $0 \leq x_i \leq 1$ by the hyperplane $\sum x_i = n - 1 - k$. The hypersimplex $\Delta^{k,l}$ can be obtained as the convex hull of the centers of k-dimensional faces of an $(n-1)$–dimensional simplex.

Example. $\Delta^{1,1}$ is an octahedron. It is the convex hull of the centers of the edges of a tetrahedron.

Let m be a positive integer. Consider the simplex

$$\Delta^n_{(m)} := \{(x_0, ..., x_n) \in \mathbb{R}^n \quad | \quad \sum_{i=0}^{n} x_i = m, \quad x_i \geq 0\} \tag{125}$$

Lemma 9.3 *The hyperplanes $x_i = k$ where k is an integer cut the simplex (125) into a union of hypersimplices.*

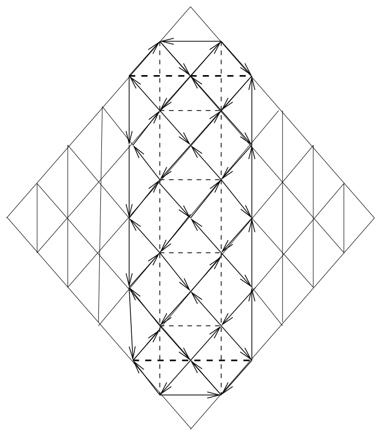


Figure 39:

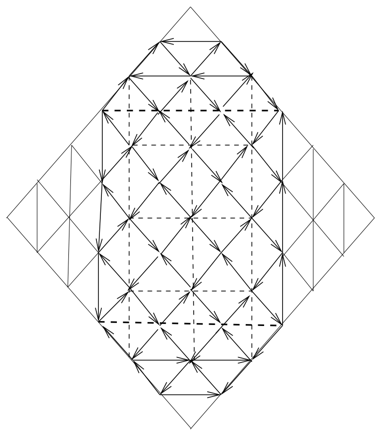


Figure 40:

Proof. Consider decomposition of $\mathbb{R}^n$ into unit cubes with the faces $x_i = k$ where k is an integer. The hyperplane $\sum a_i = m$ intersect each of these cubes either by an empty set, or by a hypersimplex. The lemma is proved.

We call it the hypersimplicial decomposition of the simplex $\Delta^n_{(m)}$.

Example. When $n = 2$ we get an m–triangulation of a triangle defined in Section 1.6.

Consider the three dimensional simplex $\Delta^3_{(m)}(E)$ given by

$$x_1 + x_2 + x_3 + x_4 = m, \quad x_i \geq 0, \tag{126}$$

whose vertices are labeled by $1, 2, 3, 4$, as on Figure (41). Its edges are dual to the A, B, C, D, E, F edges, and get the same labels. The faces match the vertices of the edges E and F. The two faces matching the vertices of E are called the e–faces, and the other two are called the f–faces.

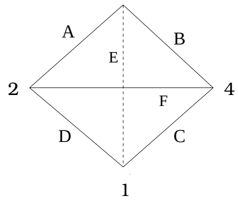


Figure 41:

The hypersimplicial decomposition of $\Delta^3_{(m)}(E)$ induces the m–triangulation of its faces. The faces of $\Delta^3_{(m)}(E)$ are identified with the corresponding triangles of T or T', so that the m–triangulations of the faces match the ones of the triangles. Thus to make a flip we glue the simplex $\Delta^3_{(m)}(E)$ so that its e–faces match the corresponding two triangles of T. Then the f–faces of $\Delta^3_{(m)}(E)$ provide the two triangles of T'.

4. The action of a flip on the A-coordinates. Let us attach to the integral points inside of the simplex (126) functions on the configurations space of four affine flags in an m-dimensional vector space V_m with a volume form ω. Given an integral solution (a, b, c, d) of (126) and 4-tuple of affine flags (X, Y, Z, T) we set

$$\Delta_{a,b,c,d}(X, Y, Z, T) := \Delta_\omega(x_{(a)} \wedge y_{(b)} \wedge z_{(c)} \wedge t_{(d)})$$

where $x_{(a_1)} = x_1 \wedge ... \wedge x_{a_1}$ and so on. If one of the numbers a, b, c, d is zero we get the functions defined in Section 8.2. The Plücker relations allow to derive some identities between these functions which we are going to describe now.

The hypersimplicial decomposition of the three dimensional simplex (126) is a decomposition into tetrahedrons of type $\Delta^{2,0}$, octahedrons $\Delta^{1,1}$, and tetrahedrons of type $\Delta^{0,2}$. The octahedrons play an important role in our story. They are parametrized by the solutions (a_1, a_2, a_3, a_4) of the equation

$$a_1 + a_2 + a_3 + a_4 = m - 2, \quad a_i \in \mathbb{Z}, a_i \geq 0 \tag{127}$$

The collection of vertices of these octahedrons coincide with the set of the non negative integral solutions of the equation (126). The vertices of the (a_1, a_2, a_3, a_4)–octahedron have the coordinates

$$(a_1, a_2, a_3, a_4) + (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4), \quad \varepsilon_i \in \{0, 1\}, \quad \sum_{i=1}^{4} \varepsilon_i = 2 \tag{128}$$

We will use a shorthand $\Delta_{b_1,b_2,b_3,b_4} := \Delta_{b_1,b_2,b_3,b_4}(X, Y, Z, T)$.

Lemma 9.4 *For any solution $\overline{a} = (a_1, a_2, a_3, a_4)$ of (127) one has*

$$\Delta_{\overline{a}+(1,1,0,0)}\Delta_{\overline{a}+(0,0,1,1)} + \Delta_{\overline{a}+(1,0,0,1)}\Delta_{\overline{a}+(0,1,1,0)} = \Delta_{\overline{a}+(1,0,1,0)}\Delta_{\overline{a}+(0,1,0,1)} \tag{129}$$

Proof. Let $X = (x_1, x_1 \wedge x_2, x_1 \wedge x_2 \wedge x_3, ...)$ be an affine flag in V_m. Let $X_1 \subset X_2 \subset ... \subset X_m = V_m$ be the corresponding flag. The affine flag X induces an affine flag in each of the quotients V_m/X_i. The four affine flags (X, Y, Z, T) in generic position in V_m induce four affine flags in the quotient

$$\frac{V_m}{X_{a_1} \oplus Y_{a_2} \oplus Z_{a_3} \oplus T_{a_4}} \tag{130}$$

Since $a_1 + a_2 + a_3 + a_4 = m - 2$, this quotient is two dimensional, and the induced affine flags provide a configuration of vectors $(\overline{x}_{a_1+1}, ..., \overline{t}_{a_4+1})$. We define a volume form ω' in the space (130) by

$$\omega'(v_1 \wedge v_2) := \Delta(x_{(a_1)} \wedge y_{(a_2)} \wedge z_{(a_3)} \wedge t_{(a_4)} \wedge v_1 \wedge v_2) \tag{131}$$

Set $(v_1, v_2, v_3, v_4) := (\overline{x}_{a_1+1}, ..., \overline{t}_{a_4+1})$ in the two dimensional space (130). One easily checks that multiplying (129) by $(-1)^{a_2+a_4}$ we get the Plücker relation

$$\omega'(v_1 \wedge v_2)\omega'(v_3 \wedge v_4) + \omega'(v_1 \wedge v_4)\omega'(v_2 \wedge v_3) = \omega'(v_1 \wedge v_3)\omega'(v_2 \wedge v_4)$$

The lemma is proved.

One can interpret it as follows. The six vectors $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$ from (128) match the edges of the graphs Γ and Γ' as shown on Figure 42. For instance $(1, 1, 0, 0)$ matches the edge A on the Picture

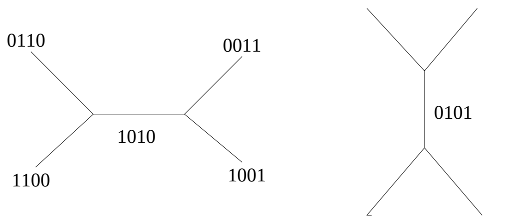


Figure 42:

(37) and so on. Set $\Delta_{\overline{a},A} := \Delta_{\overline{a}+(1,1,0,0)}$ and so one. Then we have

$$\Delta_{\overline{a},A}\Delta_{\overline{a},C} + \Delta_{\overline{a},B}\Delta_{\overline{a},D} = \Delta_{\overline{a},E}\Delta_{\overline{a},F} \tag{132}$$

The defined in Section 9.2 sequence of mutations can be described geometrically as follows. The octahedrons of the hypersimplicial decomposition of $\Delta^3_{(m)}(E)$ form $m-1$ horizontal layers, if we assume that the edges E and F parallel to a horizontal plane, and on the i-th step we add the octahedrons on the i-th level.

Recall that a mutation in the direction of an edge $e_k : v \to v'$ changes just one A-coordinate, namely A^v_k to $A^{v'}_k$, and the new coordinate is defined from the exchange relation

$$A^v_k A^{v'}_k = \prod_{\varepsilon_{ki}>0} (A^v_i)^{\varepsilon_{ki}} + \prod_{\varepsilon_{ki}<0} (A^v_i)^{-\varepsilon_{ki}}$$

In our situation the mutation corresponding to an octahedron replaces $\Delta_{\overline{a},E}$ by $\Delta_{\overline{a},F}$, so the exchange relation is given by (132).

5. The action of a flip on the X-coordinates. Recall the canonical projection $p : \mathcal{A}_{SL_n,\widehat{S}} \longrightarrow \mathcal{X}_{PGL_n,\widehat{S}}$. Observe that in the special case when $\widehat{S}$ is a disc with n marked points the canonical projection $p : \mathrm{Conf}_n(\mathcal{A}) \to \mathrm{Conf}_n(\mathcal{B})$ is a map onto. Thus to prove any identity between the functions X^Γ_i on $\mathrm{Conf}_n(\mathcal{B})$ it is sufficient to prove them for the pull backs $p^*X^\Gamma_i$. Since we already proved the that the A-coordinates and the cluster functions transform under a flip the same way they transform under the composition of mutations corresponding to a flip, and since we have the cluster type formula (110) for $p^*X^\Gamma_i$, the X-coordinates also transform under a flip the same way as under the corresponding composition of mutations.

The functions corresponding to the octahedrons of hypersimplicial decomposition. The cross–ratio of the vectors $\overline{x}_{a_1+1}, ..., \overline{t}_{a_4+1}$ provides a function on configurations of 4–tuples of flags in V_m:

$$R_{a_1,a_2,a_3,a_4}(X,Y,Z,T) := r^+(\overline{x}_{a_1+1}, \overline{y}_{a_2+1}, \overline{z}_{a_3+1}, \overline{t}_{a_4+1}) \tag{133}$$

These functions correspond to the octahedrons of the hypersimplicial decomposition of the simplex $\Delta^3_{(m)}(E)$. They are determined by the part of the graph Γ in the vicinity of E. A flip $\Gamma \to \Gamma'$ at the edge E changes these functions to their inverses.

Proof of Theorem 1.18. Since the Farey triangulation is not special in the sense of Section 3.8, the results obtained above imply the proof of Theorem 1.18i). To get Theorem 1.18ii) we use in addition Lemma 1.17.

To formulate precisely and prove Theorem 1.19 it remains to consider the special triangulations. We do it below.

6. Treatment of the special graphs. Recall the special trivalent ribbon graphs defined in Section 3.8. By definition, they contain a part given by either an eye or a virus. There are well defined $\mathcal{X}$- and $\mathcal{A}$-coordinates assigned to edges and triangles of an eye. For the $\mathcal{X}$-coordinates assigned to an edge E of an eye this deserves a comment. Namely, let Q_E be the rectangle having E as an edge, see the middle graph on Figure 7. The two vertices opposing the edge E are glued. However the four flags assigned to the vertices of Q_E are generic: the flags assigned to the two glued vertices differ by a monodromy transformation. Thus the corresponding $\mathcal{X}$-coordinates on the edge E are well defined. Contrary to this, to define the $\mathcal{X}$-coordinates corresponding to the loop of a virus we have to go to the $2:1$ covering of S pictured on Figure 9 and described below. So it is no surprise that we face problems comparing the effects at this loop of cluster and geometric transformations, and that these problems are resolved by going to the cover.

Let us consider first the case $G = PGL_2$. Figure 43 shows how we obtain a virus by making a flip on an eye at the edge E, and tells us how the coordinates on the moduli space $\mathcal{X}_{PGL_2,\widehat{S}}$ are transformed by this flip. Figure 44 tells us the effect of the cluster transformation, mutation, for the cluster ensemble corresponding to the same moduli space, under the same flip. Observe that the only difference is for the $\mathcal{X}$-coordinate A assigned to the edge obtained by gluing the two edges shown by arrows on the Figure. In the geometric case the transformation is $A \longmapsto AX$, while in the cluster case the coordinate A does not change. Since $\mathcal{X}_{PGL_2,\widehat{S}}$ is embedded into $\mathcal{X}_{PGL_m,\widehat{S}}$, the cluster rule for the transformation of the $\mathcal{X}$-coordinates for $G = PGL_2$ also does not agree with the geometric one. There is a similar problem for the $\mathcal{A}$-coordinates. Below we are going to show how to resolve this problem.

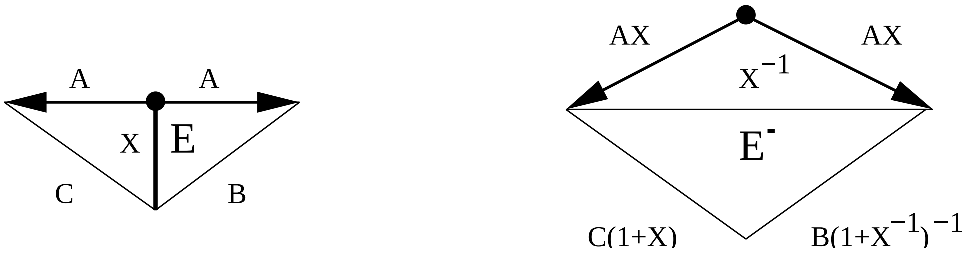


Figure 43: A virus is obtained from an eye by the flip at the edge E.

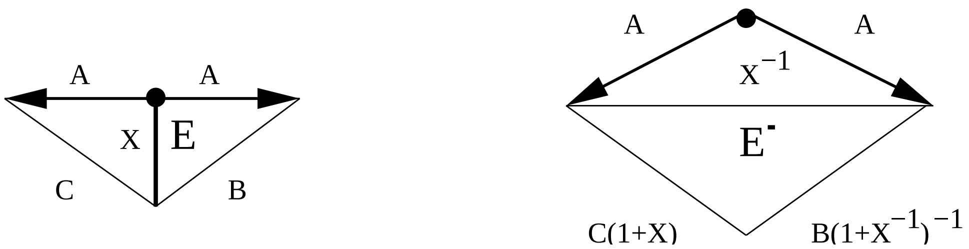


Figure 44: Cluster transformation of coordinates for the flip at E making a virus from an eye.

Let v be the vertex of triangulation shown by the fat point on Figure 43. Let us construct a $2:1$ cover $\widetilde{S} \to S$ ramified at the single point v of the compactification of S. Then there is an embedding $i : \mathcal{X}_{G,S} \hookrightarrow \mathcal{X}_{G,\widetilde{S}}$. Its image is identified with the set of the stable points of the involution σ of the covering.

The coordinate system on either $\mathcal{A}$- or $\mathcal{X}$-space corresponding to a trivalent graph Γ on S is described by a torus $T^*_{S,\Gamma}$, where $*$ stays either for $\mathcal{A}$ or for $\mathcal{X}$. A flip $\Gamma \to \Gamma'$ at an edge E provides a birational automorphism $\varphi^*_E : T^*_{S,\Gamma} \to T^*_{S,\Gamma'}$, The coordinate system on $\widetilde{S}$ corresponding to the lifted graph $\widetilde{\Gamma}$ on $\widetilde{S}$ is described by the torus $T^*_{\widetilde{S},\widetilde{\Gamma}}$. One has

$$T^*_{S,\Gamma} = (T^*_{\widetilde{S},\widetilde{\Gamma}})^{\sigma=\mathrm{Id}} \tag{134}$$

Lifting the trivalent graph on S to the cover $\widetilde{S}$, we get the graph on the left of Figure 45. The involution σ interchanges the edges marked by the same letter, and acts as the central symmetry with respect to the 4-valent vertex. We will make the flip at the edge E_1, followed by the flip at the edge E_2. These two flips are illustrated on Figure 45.

We claim that, using the identification (134), the geometric coordinate transformation corresponding to the flip at the edge E on Figure 43 equals to the composition of two flips on the cover $\widetilde{S}$. Each of these flips is a product of mutations for the cluster ensemble $(\mathcal{X}_{PSL_m,\widetilde{S}}, \mathcal{A}_{PSL_m,\widetilde{S}})$.

Proposition 9.5 *i) The composition of the flips at the edges E_1 and E_2, restricted to the subtorus (134), is equivalent to the flip at the edge E on S, i.e. the following diagram is commutative:*

$$\begin{array}{ccc} T^*_{S,\Gamma} & \stackrel{\varphi^*_E}{\longrightarrow} & T^*_{S,\Gamma'} \\ \downarrow = & & \downarrow = \\ (T^*_{\widetilde{S},\widetilde{\Gamma}})^{\sigma=\mathrm{Id}} & \stackrel{\varphi^*_{E_2}\circ\varphi^*_{E_1}}{\longrightarrow} & (T^*_{\widetilde{S},\widetilde{\Gamma}'})^{\sigma=\mathrm{Id}} \end{array}$$

The flips at E_1 and E_2 commute.

ii) Each of the flips at the edges E_1 and E_2 is a product of mutations in the cluster ensemble defined using the trivalent graph on $\widetilde{S}$ (partly shown on Figure 45) and the function ε_{ij} related to it.

This proposition allows to deduce many properties of the coordinate transformations of the flips from the an eye to a virus and back from the similar properties of mutations on the cover $\widetilde{S}$.

Proof. i) This boils down to the very definitions.

ii) It is instructive to consider first the PGL_2 case. Then the coordinate transformations for the $\mathcal{X}$-coordinates are as follows:

$$B_1 = B(1+X^{-1})^{-1}, \quad C_1 = C(1+X), \quad \overline{A}_1 = A(1+X), \quad A_1 = A(1+X^{-1})^{-1}$$

$$\overline{A}_2 = A(1+X)(1+X^{-1})^{-1} = AX, \quad A_2 = A(1+X^{-1})^{-1}(1+X) = AX$$

Thus the composition of two mutations on Figure 45 commutes with the automorphism σ of the cov-

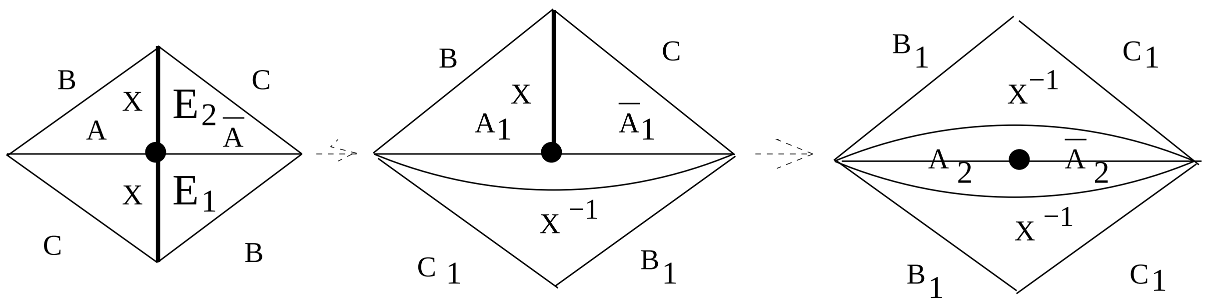


Figure 45: A flip on an eye is a composition of two mutations on a 2 : 1 cover.

ering, and thus corresponds to a transformation on S. The obtained transformation is the geometric transformation of coordinates corresponding to the flip at the edge E, see Figure 43.

Now let us address the general case, when $G = PGL_m/SL_m$. We say that an edge E is *generic* if for the quadrilateral Q_E of the triangulation having E as the diagonal both maps $\mathcal{A}_{G,\widehat{S}} \to \mathcal{A}_{G,Q_E}$ and $\mathcal{X}_{G,\widehat{S}} \to \mathcal{X}_{G,Q_E}$ provided by the restriction to Q_E are surjective at the generic point. For the $\mathcal{A}$-space a flip at a generic edge is a composition of mutations. Indeed, the $\mathcal{A}$-coordinates on a quadrilateral are defined using the four affine flags assigned to the vertices of the quadrilateral. The quadrilaterals corresponding to the edges E_1 on the left picture and E_2 on the middle picture of Figure 45 are generic. This gives the claim ii) for the $\mathcal{A}$-coordinates.

The $\mathcal{X}$-coordinates in the internal part of the quadrilateral Q_E also depend only on the four flags assigned to the vertices of Q_E. However the $\mathcal{X}$-coordinates at an external edge F depend also on the fifth flag, and so we have to take into account the pentagon $P_{E,F}$ containing as the internal diagonals the edge E and the external edge F. The natural projection $\mathcal{X}_{G,\widehat{S}} \to \mathcal{X}_{G,P_{E,F}}$ provided by the restriction to the pentagon $P_{E,F}$ is surjective at the generic point. This is clear for the edge E_1. To check this for the edge E_2 observe that although the pentagon $P_{E,F}$ has two vertices identified (at the vertex w on Figure 46), the flags at these vertices differ by the monodromy around the puncture v, and thus they are generic. The claim ii) for the $\mathcal{X}$-coordinates follows from this. The proposition, and hence Theorem 1.19 are proved.

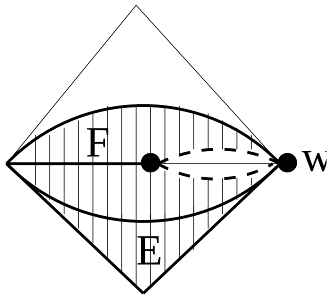


Figure 46: The two vertices of the shaded pentagon $P_{E,F}$ are identified at the vertex w

7. Integral points of higher Teichmüller spaces. The following proposition is due to Grothendieck. We will prove it in a slightly different way.

Proposition 9.6 *There is canonical bijection between the ribbon graphs and finite index torsion free subgroups of $SL_2(\mathbb{Z})$ modulo conjugation.*

Construction. Recall that ribbon graphs encodes pairs (S,T), where S is a surface with punctures, and T an ideal triangulation of S, considered modulo diffeomorphisms of surfaces. A torsion free finite index subgroup $\Delta \subset SL_2(\mathbb{Z})$ provides a surface $S_\Delta := \mathcal{H}/\Delta$ with triangulation given by the image of the Farey triangulation of $\mathcal{H}$. Vice versa, given a pair (S,T) we consider the point of the Teichmüller space whose coordinates with respect to the triangulation T are equal to 1 at every edge of T. According to, say, Section 8.10, the monodromy representation of such a local system provides a subgroup $\Delta_{(S,T)}$ of $PSL_2(\mathbb{Z})$, well defined up to a conjugation. One checks that the triangulated surface corresponding to $\Delta_{(S,T)}$ is isomorphic to the pair (S,T). Indeed, to compute the coordinate assigned to an edge E of the triangulation T, we take the two triangles sharing E to the universal cover, which is the hyperbolic plane with the Farey triangulation. There we get the two triangles $(\infty,-1,0)$ and $(\infty,0,1)$. So the coordinate of E is $r^+(\infty,-1,0,1)=1$. The proposition is proved.

Definition 9.7 *A point of the Teichmüller space $\mathcal{X}^+_{G,S}$ is integral if the corresponding monodromy representation can be represented by a homomorphism $\pi_1(S) \hookrightarrow G(\mathbb{Z})$.*

So the pairs (S,T) considered modulo $\mathrm{Diff}_0(S)$ describe the integral points of the Teichmüller space $\mathcal{X}^+_{PGL_2,S}$. Here is a hypothetical generalization.

Consider a cluster coordinate system on $\mathcal{X}_{PGL_m,S}$, and take a faithful representation $\pi_1(S) \hookrightarrow PGL_m(\mathbb{Q})$ corresponding to the point with all cluster $\mathcal{X}$-coordinates equal to 1.

Question 9.8 *Is it true that the above construction provides an inclusion $\pi_1(S) \hookrightarrow PGL_m(\mathbb{Z})$, and every integral point of the Teichmüller space $\mathcal{X}^+_{PGL_m,S}$ appears this way?*

Example. The cluster coordinate system assigned to an ideal triangulation of S provides a map given as a composition $\pi_1(S) \hookrightarrow PSL_2(\mathbb{Z}) \hookrightarrow PSL_m(\mathbb{Z})$, where the first map comes from Proposition 9.6, and the second is given by the irreducible m-dimensional representation of SL_2.

10 The classical Teichmüller spaces

1. Proof of Theorem 1.10a). Let p be an point of $\mathcal{X}^{+}_{PGL_2,S}$. Let us assign to p the corresponding representation $\rho_p : \pi_1(S) \to PSL_2(\mathbb{R})$. By Theorem 1.9 we get a discrete faithful representation of $\pi_1(S)$, and hence a point of the classical Teichmüller space. A framed structure for ρ_p is equivalent to choice of orientations of all holes in S. Indeed, a framed structure near a hole is given by choice of a point on $\mathbb{R}P^1$ invariant under the monodromy $M_{\rm hole}$ around the hole. To see this, we identify $\mathbb{R}P^1$ with the absolute of $\mathcal{H}$. Then the $M_{\rm hole}$-invariant points are the ends of the unique $M_{\rm hole}$-invariant geodesic covering the boundary geodesic on the surface $\mathcal{H}/\rho_p(\pi_1(S))$. (An element $M_{\rm hole} \in PSL_2(\mathbb{R})$ is defined up to a conjugation by the image of $\pi_1(S)$). So choosing one of them we orient the geodesic towards the chosen point. So we get an inclusion

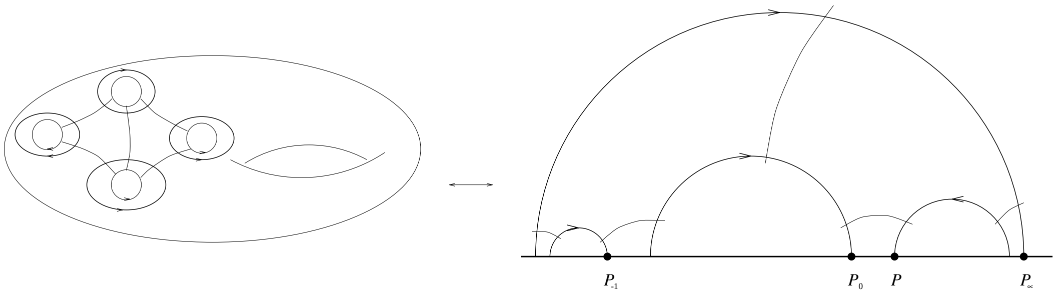
$$A : \mathcal{X}^{+}_{PGL_2,S} \hookrightarrow \mathcal{T}^{+}_{S}$$

Similarly there is a tautological inclusion B left inverse to A:

$$B : \mathcal{T}^{+}_{S} \hookrightarrow \mathcal{X}_{PGL_2,S}(\mathbb{R}), \qquad B \circ A = {\rm Id}$$

It remains to show that A is surjective or, equivalently, the image of the map B lies in $\mathcal{X}_{PGL_2,S}(\mathbb{R}_{>0})$. So we have to show that our coordinates on $\mathcal{X}_{PGL_2,S}(\mathbb{R})$ take positive values on the image of B.

Let S be a surface equipped with a hyperbolic metric with geodesic boundary. Choose a trivalent graph Γ on S and cut S along the sides of the dual graph $\Gamma^{\vee}$ into hexagons, as on Picture (135). The two hexagons sharing an edge form an octagon. It has four geodesic sides surrounding holes. Let us lift the octagon to the hyperbolic plane and continue the geodesic sides till the absolute $\mathbb{R}P^1$.



(135)

The orientations of the holes induce the orientations of the geodesics on $\mathcal{H}$, and hence the end-points of these four geodesics. Therefore we get four points $p_1, ..., p_4$ on the absolute $\mathbb{R}P^1$. The orientation of S induces a cyclic order of geodesic sides of the octagon on S, and hence a cyclic order of the points $p_1, ..., p_4$. The key fact is that this cyclic order is compatible with the orientation of the absolute. Therefore the cross-ratio $r^+(p_1, ..., p_4)$ is positive. Finally, if we assume that p_1 is the point incident to the diagonal E of the octagon, then $r^+(p_1, ..., p_4)$ is the restriction of our coordinate function X^{Γ}_{E} to $\mathcal{X}^{+}_{S}$. The part a) of the theorem is proved.

Before we start the proof of the part b) of Theorem 1.10, it is instructive to consider the following toy example.

2. Describing the decorated Teichmüller space for disc with n marked points. Let V_2 be a real two dimensional vector space with a symplectic form ω. There is canonical map $s : V_2 \longrightarrow S^2V_2, \quad v \longmapsto v \cdot v$. A symplectic structure in V_2 determines a quadratic form $(\cdot, \cdot)$ in S^2V_2:

$$(v_1 \cdot v_2, v_3 \cdot v_4) := \frac{1}{2}\Big(\omega(v_1, v_3)\omega(v_2, v_4) + \omega(v_1, v_4)\omega(v_2, v_3)\Big)$$

In particular

$$(s(v_1), s(v_2)) = \omega(v_1, v_2)^2 \tag{136}$$

It is of signature $(2,1)$. The vectors $s(v)$ are isotropic and lie on one half of the isotropic cone in S^2V_2, denoted Q_+. We identify Q_+ with the quotient of $V_2 - 0$ by the reflection $v \longmapsto -v$.

Observe that the projectivisation PQ_+ of the cone Q_+ is identified with the projective line $PV_2 = \mathbb{R}P^1$. So an orientation of the vector space V_2 induces an orientation of PQ_+. A point q on the cone Q_+ projects to the point $\overline{q} \in PQ_+$. A configuration $(q_1, ..., q_n)$ of n points on the cone Q_+ is *positive* if the corresponding projective configuration of points $(\overline{q}_1, ..., \overline{q}_n)$ in $PQ_+ = PV_2$ is positive, i.e. its cyclic order coincides with the one induced by the orientation of the projective line PV_2.

Denote by $\mathrm{Conf}_n^+(Q_+)$ the space positive configurations of n points $(q_1, ..., q_n)$ on the cone Q_+, and by $\mathrm{CConf}_n^+(Q_+)$ the corresponding space of cyclic positive configurations.

Recall that the space $\widetilde{\mathrm{Conf}_n}(V_2)$ of twisted cyclic configurations of n vectors in the vector space V_2. Let $\widetilde{\mathrm{Conf}}_n^+(V_2)$ be the set of $\mathbb{R}_{>0}$-points for the positive atlas defined in Section 7.

Proposition 10.1 *There is canonical isomorphism*

$$\mathrm{CConf}_n^+(Q_+) \stackrel{\sim}{\longrightarrow} \widetilde{\mathrm{Conf}}_n^+(V_2) \tag{137}$$

Proof. Let us interpret the space $\mathrm{CConf}_n^+(Q_+)$ in terms of the real vector space V_2. Since $Q_+ = (V_2 - \{0\})/\pm 1$, a configuration of n points in Q_+ is the same thing as a $PSL_2(\mathbb{R})$-configuration $(\pm v_1, ..., \pm v_n)$ of n non-zero vectors in V_2, each considered up to a sign. Now starting from such a cyclic $PSL_2(\mathbb{R})$-configuration $(\pm v_1, ..., \pm v_n)$ we make a twisted cyclic $SL_2(\mathbb{R})$-configuration of n vectors in V_2 as follows. Consider a one dimensional subspace L in V_2 which does not contain any of the vectors $\pm v_i$. Let L' be one of the two connected components of $V_2 - L$. Let v_i' be the one of the two vectors $\pm v_i$ located in L'. We may assume that $(v_1', ..., v_n')$ is the order of the vectors v_i' in L': otherwise we change cyclically the numerations of the vectors v_i. Then the correspondence

$$(\pm v_1, ..., \pm v_n) \longrightarrow (v_1', ..., v_n')$$

provides map (137). To check that it is well defined notice that if we move the line L so that it crosses the line spanned by the vector v_1, but not v_2, then we will get a configuration $(v_2', ..., v_n', -v_1')$ which is twisted cyclic equivalent to $(v_1', ..., v_n')$. The proposition is proved.

The isomorphism (137) transforms the real coordinates $|\Delta(\pm v_i, \pm v_j)|$, $i < j$, on the left to the positive coordinates $\Delta(v_i', v_j')$, $i < j$, on the right.

3. Proof of Theorem 1.10b). Let us realize the hyperbolic plane as the sheet of the hyperboloid

$$-x_1^2 - x_2^2 + x_3^2 = 1, \quad x_3 > 0$$

Denote by $< x, x' >$ the symmetric bilinear form corresponding to the quadratic form $-x_1^2 - x_2^2 + x_3^2$. The horocycles on the hyperbolic plane are described by vectors x on the cone Q_+, defined by the equation $< x, x >= 0$ in the upper half space $x_3 > 0$. Namely, such a vector x provides a horocycle

$$h_x := \{\xi \quad | \quad < \xi, x >= 1, \quad < \xi, \xi >= 1, \quad \xi_3 > 0\}$$

The distance $\rho(h_x, h_{x'})$ between the two horocycles h_x and $h_{x'}$ is defined as follows. Take the unique geodesic perpendicular to both horocycles. The horocycles cut out on it a geodesic segment of finite length. Then $\rho(h_x, h_{x'})$ is the length of this segment taken with the sign plus if the segment is outside of the discs inscribed into the horocycles, and with the sign minus otherwise, see the picture.

Observe that $< x, x' >$ is positive unless x is proportional to x'. The distance between the two horocycles can be computed by the formula

$$\rho(h_x, h_{x'}) = \frac{1}{2} \log < x, x' > \tag{138}$$

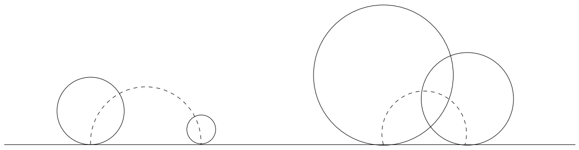


Figure 47: The distance between the horocycles can be any real number

Now take a hyperbolic topological surface S with n punctures equipped with a complete hyperbolic metric g. Its universal cover is the hyperbolic plane. The punctures are described by $\pi_1(S)$–orbits of the arrows on the cone $< x, x >= 0$, $x_3 > 0$.

According to Penner [P1] a decoration of (S, g) is given by choice of a horocycle h_i for each of the punctures s_i. Let $\mathcal{A}_S^+$ be the moduli space of decorated complete hyperbolic surfaces of given topological type S. Penner's coordinates a_E^Γ on it are defined using a trivalent graph Γ on S and parametrised by the set $\{E\}$ of edges of Γ. Namely, consider the ideal triangulation of S dual to the graph Γ. For each side of the triangulation there is a unique geodesic between the punctures isotopic to the side. The coordinate a_E^Γ is the length of the geodesic γ_E of the triangulation dual to E.

We can interpret this definition as follows. Choose a lift $\widetilde{\gamma}_E$ of the geodesic γ_E on the hyperbolic plane. It is defined up to the action of $\pi_1(S)$. Let l_1 and l_2 be the arrows on the cone $< x, x >= 0$ corresponding to the boundary of $\widetilde{\gamma}_E$. So the plane spanned by l_1 and l_2 intersects the hyperboloid by a geodesic $\widetilde{\gamma}_E$. The arrows l_1, l_2 project to the punctures s_1, s_2 bounding γ_E. A choice of horocycle(s) at the puncture s_i, $i = 1, 2$, is equivalent to the choice of vector(s) x_i on the arrow l_i.

We need to show that the coordinates a_i^Γ coincide with our Δ–coordinates. We can spell the definition of the coordinates a_i^Γ as follows. Choose an edge E of the ribbon graph Γ. Let F_1 and F_2 be the two face paths sharing the edge E. Then the decoration on a local $PSL_2(\mathbb{R})$–system $\mathcal{L}$ on S provides vectors $\pm v_1$ and $\pm v_2$, each defined only up to a sign, in the fiber $\mathcal{L}_E$ of the local system $\mathcal{L}$ over the edge E. The vectors $\pm v_i$ are invariant up to a sign under the monodromy along the face path F_i. Then we assign to the edge E the coordinate

$$\Delta_E^\Gamma := |\Delta(v_1, v_2)|$$

Observe that (136) implies $\Delta(v_1, v_2)^2 = (s(v_1), s(v_2))$. So thanks to (138)

$$a_E^\Gamma = \log|\Delta(v_1, v_2)| = \log|\Delta_E^\Gamma|$$

It is well known that the monodromy along the conjugacy classes determined by a punctures are unipotent. The theorem is proved.

11 Laminations and canonical pairings

In this Section we give a geometric definition of two types of laminations, the $\mathcal{A}$- and $\mathcal{X}$-laminations, with integral, rational and real coefficients, on a punctured surface S. Our rational $\mathcal{A}$-laminations are essentially the same as Thurston's laminations [Th], which are usually considered for a surface S without the boundary. Our definition of $\mathcal{A}$-laminations differs in the treatment of the holes, and more convenient for our purposes.

Our main results about the laminations are the following:

i) The rational and real lamination spaces are canonically isomorphic to the points of the positive moduli spaces $\mathcal{A}_{SL_2,S}$ and $\mathcal{X}_{PSL_2,S}$ with values in the tropical semifields $\mathbb{Q}^t, \mathbb{R}^t$. The same is true for the integral $\mathcal{X}$-laminations, while the set of integral $\mathcal{A}$-laminations contains $\mathcal{A}_{SL_2,S}(\mathbb{Z}^t)$ as an explicitly described subset.

ii) The real $\mathcal{A}$-lamination space is canonically isomorphic to the space of Thurston's transversally measured laminations [Th], [PH].

iii) We define canonical pairing between the real $\mathcal{A}$- (resp. $\mathcal{X}$-) laminations and the $\mathcal{X}$- (resp. $\mathcal{A}$-) Teichmüller spaces.

iv) We define canonical map from the integral $\mathcal{A}$- (resp. $\mathcal{X}$-) laminations to functions on the $\mathcal{X}$- (resp. $\mathcal{A}$-) moduli spaces. Its image consists of functions on the moduli space which are Laurent polynomials with positive integral coefficients in every of our coordinate systems. We conjecture that the image of this map provides a canonical basis in the space of such functions, and that the canonical map admits a natural q-deformation.

It is worth mentioning that the definitions of laminations are similar to the definitions of the singular homology groups, and in a sense is an unoriented version of the latter. There are two different ways to define the notion of laminations for surfaces with boundary, which are analogous to the definition of homology group with compact and closed support respectively.

Throughout this section S is a surface with $n > 0$ holes, Γ is a trivalent graph embedded into S in such a way that S is retractable onto it, and ε_{ij}, where i and j run through the set of edges of Γ, is a skew-symmetric matrix given by

$$\begin{aligned}\varepsilon_{ij} \quad &= \quad \text{(number of vertices where the edge } i \text{ is to the left from the edge } j) \; - \\ &- \quad \text{(number of vertices where the edge } i \text{ is to the right from the edge } j) \; .\end{aligned}$$

The entries of ε_{ij} take five possible values: $\pm 1, \pm 2$ or 0.

1. $\mathcal{A}$- and $\mathcal{X}$-laminations and tropicalisations of the $\mathcal{A}$- and $\mathcal{X}$- moduli spaces.

Definition 11.1 Rational $\mathcal{A}$-lamination *on a surface S is a homotopy class of a collection of finite number of self- and mutually nonintersecting unoriented closed curves with rational weights, subject to the following conditions and equivalence relations.*

1. *Weights of all curves are positive, unless a curve surrounds a hole.*
2. *A lamination containing a curve of zero weight is equivalent to that with this curve removed.*
3. *A lamination containing two homotopy equivalent curves of weights a and b is equivalent to a lamination with one of these curves removed and with the weight $a + b$ on the other.*

The set of all rational $\mathcal{A}$-laminations on a surface S is denoted by $\mathcal{A}_L(S, \mathbb{Q})$. It contains a subset $\mathcal{A}_L(S, \mathbb{Z})$ of *integral $\mathcal{A}$-laminations*, defined as rational $\mathcal{A}$-lamination with positive integral weights.

Any rational $\mathcal{A}$-lamination can be represented by a collection of $3g - 3 + n$ curves. Any integral $\mathcal{A}$-lamination can be represented by a finite collection of curves with unit weight.

Definition 11.2 Rational $\mathcal{X}$-lamination *on a surface S is a pair consisting of*

a) A homotopy class of a collection of finite number of nonselfintersecting and pairwise non-intersecting curves either closed or connecting two boundary components (possibly coinciding) with positive rational weights assigned to each curve and subject to the following equivalence relations:

1. *A lamination containing a curve retractable to a boundary component is equivalent to the one with this curve removed.*
2. *A lamination containing a curve of zero weight is equivalent to the one with this curve removed.*
3. *A lamination containing two homotopy equivalent curves of weights a and b is equivalent to the lamination with one of these curves removed and with the weight $a + b$ on the other.*

b) A choice of orientations of all boundary components but those nonintersecting with curves of the lamination.

Denote the space of rational $\mathcal{X}$-laminations on S by $\mathcal{X}_L(S,\mathbb{Q})$. It contains a subset $\mathcal{X}_L(S,\mathbb{Z})$ of integral $\mathcal{X}$-laminations, defined as rational $\mathcal{X}$-laminations representable by collections of curves with integral weights.

Any rational $\mathcal{X}$-lamination can be represented by a collection of no more than $6g-6+2n$ curves (for the Euler characteristic reasons). Any integral $\mathcal{X}$-lamination can be represented by a finite collection of curves with unit weights.

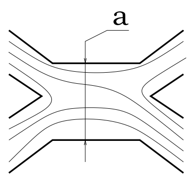
Theorem 11.3 *There are canonical isomorphisms*

$$\mathcal{A}_{SL_2,S}(\mathbb{Q}^t) \stackrel{\sim}{=} \mathcal{A}_L(S,\mathbb{Q}), \qquad \mathcal{X}_{PSL_2,S}(\mathbb{Q}^t) \stackrel{\sim}{=} \mathcal{X}_L(S,\mathbb{Q})$$

The proof of this theorem will occupy the rest of this subsection. The idea is this. We are going to introduce natural coordinates on the space of $\mathcal{A}$- and $\mathcal{X}$-laminations corresponding to a given isotopy class of a trivalent graph Γ on S, which is homotopy equivalent to S. Our construction is a modification of Thurston's "train tracks" ([Th], section 9 and [PH]), and borrows from [F] with some modifications. Then we observe that the transformation rules for this coordinates under the flips are precisely the tropicalisations of the corresponding transformation rules for the canonical coordinates on the $\mathcal{A}$- and $\mathcal{X}$-moduli spaces. This proves the theorem. Let us start implementation of this plan.

Construction of coordinates for $\mathcal{A}$-laminations. We are going to assign to a given rational $\mathcal{A}$-lamination rational numbers on the edges of the graph Γ and show that these numbers are coordinates on the space of laminations.

Retract the lamination to the graph in such a way that each curve retracts to a path without folds on edges of the graph, and no curve goes along an edge and then, without visiting another edge, back. Assign to each edge i the sum of weights of curves, going through it (Picture (139)), divided by two. The collection of these numbers, one for each edge of Γ, is the desired set of coordinates.



(139)

Remark. We divided the natural coordinates of $\mathcal{A}$-laminations by two to simplify some formulas for the multiplicative canonical pairing, see below.

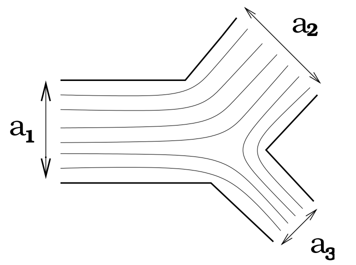
Reconstruction of $\mathcal{A}$-laminations. We need to prove that these numbers are coordinates indeed. For this purpose we describe an inverse construction which gives a lamination starting from the numbers on edges.

First of all note that if we are able to reconstruct a lamination corresponding to a set of numbers $\{a_i\}$, we can do this as well for the set $\{ra_i\}$ and $\{a_i+t\}$ for any rational $r \geq 0$ and t. Indeed, multiplication of all numbers by r can be achieved by multiplication of all weights by r and adding t is obtained by adding loops with weight $t/2$ around each hole. Therefore we can reduce our problem to the case when $\{a_i\}$ are positive integers and any three numbers a_1, a_2, a_3 on the edges incident to each given vertex satisfy the following triangle and parity conditions

$$|a_1 - a_2| \leq a_3 \leq a_1 + a_2 \qquad (140)$$

$$a_1 + a_2 + a_3 \text{ is an integer} \qquad (141)$$

Now the reconstruction of the lamination is almost obvious. Draw a_i lines on the i-th edge and connect these lines at vertices in an nonintersecting way (Picture (142)), which can be done unambiguously.

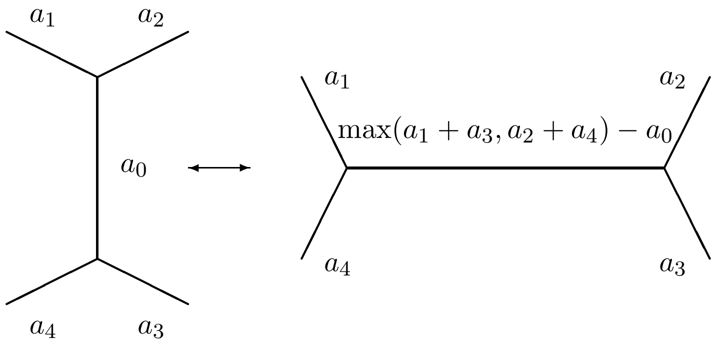


(142)

Graph change for $\mathcal{A}$-laminations. The flip of the edge k changes the coordinates according to the formula:

$$a_i \to \begin{cases} a_i & \text{if} \quad k \neq i, \\ -a_k + \max\left(\sum_{j|\varepsilon_{kj}>0} \varepsilon_{kj} a_j, -\sum_{j|\varepsilon_{kj}<0} \varepsilon_{kj} a_j\right) & \text{if} \quad k = i, \end{cases}$$

Or in a graphical form:

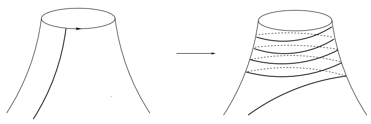


(143)

(Only part of the graph is shown here, the numbers on the other edges remain unchanged.)

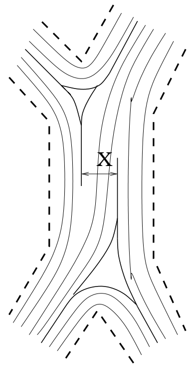
Construction of coordinates for $\mathcal{X}$-laminations. We are going to assign for a given point of the space $\mathcal{X}_L(S, \mathbb{Q})$ a set of rational numbers on edges of the graph Γ, and show that these numbers are global coordinates on this space.

Straightforward retraction of an $\mathcal{X}$-lamination onto Γ is not good because some curves may shrink to points or finite segments. To avoid this problem, let us first rotate each oriented boundary component infinitely many times in the direction prescribed by the orientation as shown on Picture (144).



(144)

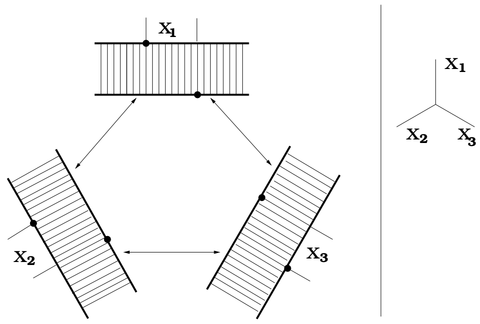
The resulting lamination can be retracted on Γ without folds. However we can get infinitely many curves going through an edge. Now assign to the edge the sum with sign of weights of curves that goes diagonally, i.e., such that being oriented turn to the left at one end of the edge and to the right at another one (in this case we choose the plus sign), or first to the right and then to the left (in this case we choose the minus sign), as shown on Picture (145).



(145)

The collection of such rational numbers, one for each edge, is the desired coordinate system on $\mathcal{X}_L(S, \mathbb{Q})$. Note that the number of curves going diagonally is always finite and therefore the numbers assigned to an edge are well defined. Indeed, consider the curves retracted on the graph. We can mark a finite segment of each non-closed curve in such a way that each of two unmarked semi-infinite rays goes only around a single face and therefore never goes diagonally along the edges. Therefore only the finite marked parts of curves contribute to the numbers on the edges.

Reconstruction of $\mathcal{X}$-laminations. We need to prove that these numbers are coordinates indeed. We will do it by describing an inverse construction. Note that if we are able to construct a lamination corresponding to the set of numbers $\{x_i\}$, we can equally do it for the set $\{rx_i\}$ for any rational $r > 0$. Therefore we can reduce our task to the case when all numbers on edges are integral. Now draw $\mathbb{Z}$-infinitely many lines along each edge. In order to connect these lines at vertices we need to split them at each of the two ends into two $\mathbb{N}$-infinite bunches to connect them with the corresponding bunches of the other edges. Let us make it at the i-th edge, such that $x_i \leq 0$ (resp. $x_i \geq 0$), in such a way that the intersection of the right (resp. left) bunches at the both ends of the edge consist of x_i lines (resp. $-x_i$ lines). Here the left and the right side are considered from the centre of the edge toward the corresponding end. The whole procedure is illustrated on Picture (146). The resulting collection of curves may contain infinite number of curves surrounding holes, which should be removed in accordance with the definition of an unbounded lamination.



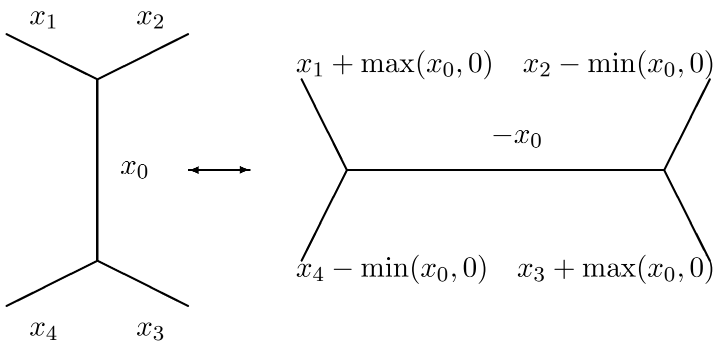
(146)

Note that though we have started with infinite bunches of curves the resulting lamination is finite. All these curves glue together into a finite number of connected components and possibly infinite number of closed curves surrounding punctures. Indeed any curve of the lamination is either closed or goes diagonally along at least one edge. Since the total number of pieces of curves going diagonally $I = |\sum_i x_i|$ is finite the resulting lamination contain no more than this number of connected components. (In fact the number of connected components equals I provided all numbers x_i are all non-positive or all nonnegative.)

Graph change for $\mathcal{X}$-laminations. The flip of the edge k changes the coordinates according to the formula

$$x(i) \to \begin{cases} -x_i & \text{if} \quad i = k, \\ x_i & \text{if} \quad \varepsilon_{ki} = 0 \text{ and } k \neq i, \\ x_i + \max(0, x_k) & \text{if} \quad \varepsilon_{ki} > 0, \\ x_i - \max(0, -x_k) & \text{if} \quad \varepsilon_{ki} < 0, \end{cases}$$

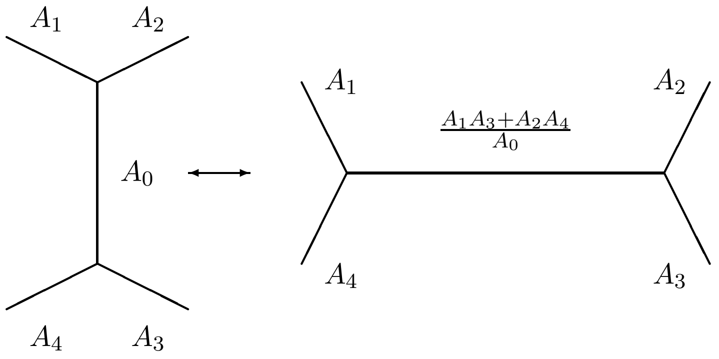
or, in graphical form:



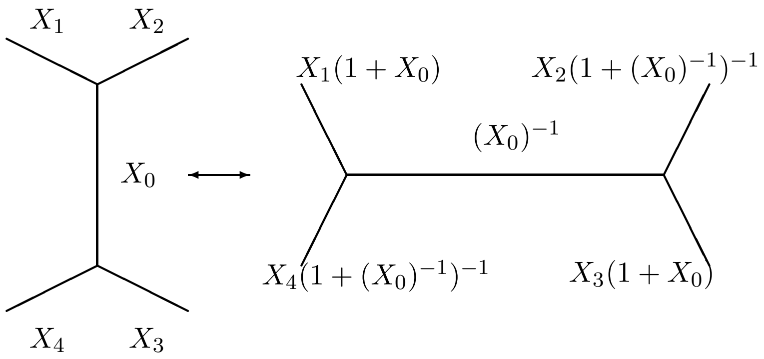
(147)

(Only part of the graph is shown here, the numbers on the other edges remain unchanged.)

End of the proof of the theorem 11.3. Recall that the positive structure on the moduli spaces $\mathcal{A}_{SL_2,S}$ and $\mathcal{X}_{PSL_2,S}$ has been defined using the coordinate systems corresponding to trivalent ribbon graphs Γ embedded to S. The coordinate systems on the lamination spaces are defined using the same graphs. The transformation rules (143) and (147) are tropical limits of the ones for the respective moduli spaces $\mathcal{A}_{SL_2,S}$ and $\mathcal{X}_{PSL_2,S}$, as one sees comparing the pictures (143) and (147) with the pictures (148) and (149) where these rules are exhibited. The theorem is proved.



(148)



(149)

Relations and common properties of $\mathcal{X}_L(S, \mathbb{Q})$ and $\mathcal{A}_L(S, \mathbb{Q})$. **1.** Since the transformation rules for coordinates (143) and (147) are continuous w.r.t. the standard topology of $\mathbb{Q}^n$ the coordinates define a natural topology on the lamination spaces. One can now define the spaces of *real laminations* as a completion of the corresponding spaces of rational laminations. These spaces are denoted as $\mathcal{A}_L(S,\mathbb{R})$ and $\mathcal{X}_L(S,\mathbb{R})$, respectively. Of course we have the coordinate systems on these spaces automatically. Evidently there are canonical isomorphisms

$$\mathcal{A}_{SL_2,S}(\mathbb{R}^t) \stackrel{\sim}{=} \mathcal{A}_L(S,\mathbb{R}), \qquad \mathcal{X}_{PSL_2,S}(\mathbb{R}^t) \stackrel{\sim}{=} \mathcal{X}_L(S,\mathbb{R})$$

The constructed spaces of real laminations coincide with the spaces of transversally measured laminations introduced by W.Thurston [Th]. We'll prove this statement below.

2. The space of $\mathcal{A}$-laminations (resp. $\mathcal{X}$) is covariant (resp. contravariant) functor on the category of embeddings of surfaces.

3. An $\mathcal{X}$-lamination is integral if and only if it has integral coordinates. However an $\mathcal{A}$-lamination is integral if and only if it has half-integral coordinates and their sum is an integer. So we have

$$\mathcal{X}_{PSL_2,S}(\mathbb{Z}^t) \stackrel{\sim}{=} \mathcal{X}_L(S,\mathbb{Z}), \qquad \mathcal{A}_{SL_2,S}(\mathbb{Z}^t) \subset \mathcal{A}_L(S,\mathbb{Z}),$$

4. There is a canonical map

$$p : \mathcal{A}_L(S,\mathbb{Q}) \to \mathcal{X}_L(S,\mathbb{Q})$$

which removes all the curves retractable to boundary components and provide each of the boundary components with canonical orientation induced by the orientation of the surface. In coordinates this map is given by

$$x_i = \sum_j \varepsilon_{ij} a_j, \tag{150}$$

5. For any given $\mathcal{X}$-lamination and any given puncture one can calculate the sum

$$\sum_{i\in\gamma} x_i, \tag{151}$$

where γ is the set of edges surrounding the puncture. The absolute value of this sum gives the total weight of the curves entering the puncture, and its sign gives the orientation of the puncture.

6. For any given $\mathcal{A}$-lamination and any given puncture consider the expression

$$\max_{i\in\gamma}(a_i + a_{i'} - a_{i''}), \tag{152}$$

where i' is the edge next to i in counterclockwise direction and i'' is the edge between i and i'. This expression gives the total weight of the horocycles surrounding the puncture.

2. Additive canonical pairing and intersection pairing. Additive canonical pairing is a function of two arguments. One argument is a rational (or, later on, real) lamination, which can be either $\mathcal{A}$- or $\mathcal{X}$-lamination, and the other is a point of the opposite type Teichmüller space, $\mathcal{X}^+ := \mathcal{X}^+_{PSL_2,S}$ or $\mathcal{A}^+ := \mathcal{A}^+_{SL_2,S}$:

$$I : \mathcal{A}_L(\mathbb{Q}) \times \mathcal{X}^+ \to \mathbb{R}, \quad I : \mathcal{X}_L(\mathbb{Q}) \times \mathcal{A}^+ \to \mathbb{R},$$

We will also define the *intersection pairing*

$$\mathcal{I} : \mathcal{A}_L(\mathbb{Q}) \times \mathcal{X}_L(\mathbb{Q}) \to \mathbb{R}$$

which should be thought of as a degeneration of additive pairing. Abusing notation, we shall denote the first two of them by a single letter I. We shall also denote the canonical maps $p : \mathcal{A}^+ \to \mathcal{X}^+$ and $p : \mathcal{A}_L \to \mathcal{X}_L$ by one letter p.

Let us define these pairings. Below by the length of a curve on a surface with metric of curvature -1 we always mean the length of the unique geodesic isotopic to this curve.

Definition 11.4 *1. Let $l \in \mathcal{X}_L$ be a single closed curve. Then $I(l,m)$ is its length w.r.t the hyperbolic metric on S defined by m.*

2. *Let $l \in \mathcal{A}_L$ be a single closed curve. Then $I(l,m)$ is equal to $\pm$ its length w.r.t the hyperbolic metric on S defined by m. The sign is $+$ unless l surrounds a negatively oriented hole.*

3. *Let l be a curve connecting two punctures and $m \in \mathcal{A}^+$. Then $I(l,m)$ is the signed length of the curve l between the horocycles.*

4. *Let l_1 and l_2 be nonintersecting collections of curves. Then $I(\alpha l_1 + \beta l_2, m) = \alpha I(l_1, m) + \beta I(l_2, m)$, and similarly for $\mathcal{I}$.*

5. *Let l_1 and l_2 be two curves. Then $\mathcal{I}(l_1, l_2)$ is the minimal number of intersection points between l_1 and l_2.*

In the proposition below by coordinates we always mean the coordinates corresponding to a certain given trivalent graph Γ such that S is retractable on Γ. All the statements are assumed to be valid for every such graph Γ. Recall the logarithms $x_i := \log X_i$ and $a_i := \log A_i$ of the canonical positive coordinates on the $\mathcal{X}$- and $\mathcal{A}$- Teichmüller spaces. The coordinates on the lamination spaces are also denoted by x_i and a_i.

Proposition 11.5

1. *The pairings I and $\mathcal{I}$ are continuous.*

2. *$I(l, p(m)) = I(p(l), m)$, where $l \in \mathcal{A}_L$ and $m \in \mathcal{A}^+$.*

3. *If a point l of the space $\mathcal{X}_L$ has positive coordinates $x_1, \ldots, x_n$ and a point m of the Teichmüller $\mathcal{A}$-space has coordinates $a_1, \ldots, a_n$, then $I(l,m) = \sum_\alpha a_\alpha x_\alpha$.*

4. *Let m be a point of a Teichmüller space with coordinates $u_1, \ldots, u_n$ with respect to a graph Γ. Let C be a positive real number. Denote by $C \cdot m$ a point with coordinates $Cu_1, \ldots, Cu_n$. Let m_L be a lamination with the coordinates $u_1, \ldots, u_n$. Then $\lim\limits_{C \to \infty} I(l, C\cdot m)/C = \mathcal{I}(l, m_L)$.*

Observe that the last property implies the properties 2 and 3 for $\mathcal{I}$.

Proof. The property 2 follows from the definition and the property 1. The property 3 follows from the observation that if all coordinates of an $\mathcal{X}$-laminations are positive w.r.t. a given graph Γ, all its leaves can be deformed to paths across the edges of Γ. The proof of the property 4 will be postponed to our consideration of the multiplicative pairing. Since the definition describes explicitly the value of the pairing if one argument is a rational lamination, to prove existence one needs to only to check that it is continuous.

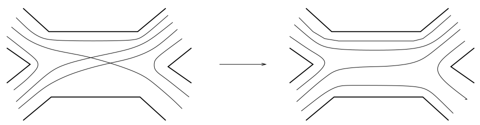
Let us first prove the continuity of the pairing between $\mathcal{A}$-laminations and the $\mathcal{X}$-Teichmüller space. To do this it suffices to prove it for laminations without curves with negative weights. Indeed, if we add such curve to a lamination the length obviously changes continuously. We are going to show that the length of integral lamination is a convex function of its coordinates, i.e. that

$$I(l_1, m) + I(l_2, m) \geq I(l_1 +_\Gamma l_2, m) \tag{153}$$

were by $l_1 +_\Gamma l_2$ we mean a lamination with coordinates being sums of the respective coordinates of l_1 and l_2. Taking into account the homogeneity of I, one sees that the inequality (153) holds for all rational laminations and therefore can be extended by continuity to all real laminations.

Let us prove the inequality (153). Draw both laminations l_1 and l_2 on the surface and deform them to be geodesic. These laminations in general intersect each other in finite number of points. Then retract the whole picture to the ribbon graph in such a way that no more intersection points appear and the existing ones are moved to the edges. Now it becomes obvious that at each intersection point we can rearrange our lamination cutting both intersecting curves at the intersection point and gluing them back in another order in a way to make the resulting set of curves homotopically equivalent to a nonintersecting collection. We can do it at each intersection point in two different ways, and we use the retraction to the ribbon graph to choose one of them.

Indeed, connect them as shown on Picture (154).



(154)

One can easily see that the numbers on edges, corresponding to the new lamination l are exactly the sums of the numbers, corresponding to l_1 and l_2. On the other hand, the lamination on the original surface is no longer geodesic, because the curves of it have breaks. But its length is exactly $I(l_1, m) + I(l_2, m)$. When we deform l to a geodesic lamination, its length can only decrease, what proves the inequality (153).

The proof of the continuity when the first argument is an $\mathcal{X}$-lamination and the second is a point of the $\mathcal{A}$-Teichmüller space is is similar. Call a pair of laminations l_1 and l_2 *similar* if at each hole both laminations l_1 and l_2 have the same orientation or at least one of them does not touch the hole. Once the inequality (153) is proved for similar laminations l_1 and l_2, the continuity follows since the set of laminations similar to a given one is a closed subset in the space $\mathcal{X}_L$. Conjecturally the inequality (153) holds for any pair l_1 and l_2. But for a pair of similar laminations the arguments used above works: one can first forget about orientations and move the leaves in order to minimize the number of intersection points. Then one can first spiralize the leaves according to the orientations, retract the resulting collection of curves to the graph and then resolve the intersection points just as above (Picture (154)). The inequality (153) follows.

Using 3 we deduce for this the convexity of the pairing $\mathcal{I}$.

The continuity implies the following

Corollary 11.6 *Our real $\mathcal{A}$-laminations are canonically identified with Thurston's transversally measured laminations.*

Proof. Thurston proved that rational laminations are dense in the space of all transversally measured laminations w.r.t. the topology given by the length function. Our rational $\mathcal{A}$-laminations coincide with Thurston's. Our canonical pairing of $\mathcal{A}$-laminations with the Teichmüller space is nothing but Thurston's length function. Therefore the continuity shows that our topology coincides with Thurston's. The corollary is proved.

Corollary 11.7 *The additive canonical pairings and the intersection pairing provide continuous pairings between the real lamination and Teichmuller/lamination spaces:*

$$I : \mathcal{A}_L(\mathbb{R}) \times \mathcal{X} \to \mathbb{R}, \quad I : \mathcal{X}_L(\mathbb{R}) \times \mathcal{A} \to \mathbb{R}, \quad \mathcal{I} : \mathcal{A}_L(\mathbb{R}) \times \mathcal{X}_L(\mathbb{R}) \to \mathbb{R}$$

We say that an element of $\mathcal{X}_{PSL_2,S}(\mathbb{C})$ is quasi-Fuchsian if the corresponding representation of $\pi_1(S)$ is quasi-Fuchsian. The set of all quasi-Fuchsian elements form an open domain of $\mathcal{X}_{PSL_2,S}(\mathbb{C})$ which we call the quasi-Fuchsian domain.

Conjecture 11.8 *Let l be a real $\mathcal{A}$-lamination. Then the function $I(l, *)$ is analytically continuable exactly to the quasi-Fuchsian domain of $\mathcal{X}_{PSL_2,S}(\mathbb{C})$.*

3. Multiplicative canonical pairing. The multiplicative canonical pairing $\mathbb{I}$ is a pairing between integral $\mathcal{A}_{SL_2,S}$-laminations and $\mathcal{X}_{PSL_2,S}$-moduli space or between integral $\mathcal{X}_{PSL_2,S}$-laminations and $\mathcal{A}_{SL_2,S}$-moduli space. These pairings are better understood as the maps

$$\mathbb{I}_A : \mathcal{A}_{SL_2,S}(\mathbb{Z}^t) \longrightarrow \mathbb{Q}(\mathcal{X}_{PSL_2,S}), \qquad \mathbb{I}_X : \mathcal{X}_{PSL_2,S}(\mathbb{Z}^t) \longrightarrow \mathbb{Q}(\mathcal{A}_{SL_2,S}),$$

Before defining the map $\mathbb{I}_A$ we will define an auxiliary map $\widetilde{\mathbb{I}}_A : \mathcal{A}_{SL_2,S}((\mathbb{Z}^t)/2) \longrightarrow \mathbb{Q}(\widehat{\mathcal{X}_{PSL_2,S}})$, where the hat denotes certain covering of the space $\mathcal{X}_{PSL_2,S}$. Afterwards we shall check that the restriction of this pairing to $\mathcal{A}_{SL_2,S}(\mathbb{Z}^t)$ comes from $\mathcal{X}_{PSL_2,S}$, thus defining the desired pairing.

Here is our strategy. Let us say that a Laurent polynomial is sign definite if all its non zero coefficients are of the same sign. Given a sign definite Laurent polynomial we can multiply it by ± 1 to make a positive Laurent polynomial out of it. We are going to define first the related maps $\mathbb{I}'_{\mathcal{A}}$ and $\mathbb{I}'_{\mathcal{X}}$. Then we will show that $\mathbb{I}'_{\mathcal{A}}(l)$ and $\mathbb{I}'_{\mathcal{X}}(l)$ are sign definite Laurent polynomials. By definition the elements $\mathbb{I}_{\mathcal{A}}$ and $\mathbb{I}_{\mathcal{X}}$ are the corresponding positive Laurent polynomials. We will produce our sign definite Laurent polynomials as products of sign definite Laurent polynomials, so we allow them to be well defined only up to a sign. For example a trace of a matrix from $PSL_2(F)$ is a defined up to a sign function.

Recall that a curve l connecting two punctures on S provides a well defined regular function $\Delta(l)$ on the space $\mathcal{A}_{SL_2,S}$. Further, let l be a closed curve surrounding a puncture on S. Then the monodromy operator for an framed local system $\mathcal{L}$ on S around l has a distinguished eigenvalue λ_l determined by the framed structure: the eigenvector with this eigenvalue λ_l generates the invariant flag assigned to this puncture.

Definition 11.9 *1. Let l be a single closed curve with the weight k which does not surround a puncture. Then*

i) the value of the function $\pm\widetilde{\mathbb{I}}'_{\mathcal{A}}(l)$ on the decorated PSL_2-local system $\mathcal{L}$ is the trace of the k-th power of the monodromy of $\mathcal{L}$ along l.

ii) the value of the function $\mathbb{I}'_{\mathcal{X}}(l)$ on the framed SL_2-local system $\mathcal{L}$ is the trace of the k-th power of the monodromy of $\mathcal{L}$ along l.

2. Let l be a closed curve surrounding a puncture on S, and $k \cdot l$ the $\mathcal{A}$-lamination determined by this curve and a weight k. Then $\widetilde{\mathbb{I}}'_{\mathcal{A}}(k \cdot l) = \lambda_l^k$.

3. Let l be a curve connecting two punctures with weight k. Let us assume that orientations of these punctures are induced by the orientation of the surface. Then $\mathbb{I}'_{\mathcal{X}}(l) = \Delta(l)^k$.

4. Let l_1 and l_2 be nonintersecting collections of curves, such that no isotopy class of a curve enters both l_1 and l_2. Then $\mathbb{I}'_*(l_1 + l_2) = \mathbb{I}'_*(l_1)\mathbb{I}'_*(l_2)$, where $*$ stays for $\mathcal{A}$ or $\mathcal{X}$, and $\widetilde{\mathbb{I}}'$ is needed in the $\mathcal{A}$-case.

Here we assume that we have reduced the lamination to the form without equivalent curves.

Observe that in this definition the map $\mathbb{I}'_{\mathcal{A}}(l)$ is defined for all integral laminations. In the definition of the map $\mathbb{I}'_{\mathcal{X}}(l)$ we are missing the laminations corresponding to curves connecting two punctures in the case when at least one of the punctures is oriented differently then the surface S. We do not know how to define the map $\mathbb{I}'_{\mathcal{X}}(l)$ for those laminations.

In the theorem below by coordinates we always mean the coordinates corresponding to a certain fixed trivalent graph Γ such that S is retractable on Γ. And all the statements are assumed to be valid for every such graph Γ. The function $\mathbb{I}'_{\mathcal{X}}(l)$ is restricted to those laminations l where it has been defined.

Theorem 11.10

1. *Let $a_1, \ldots, a_n \in \mathbb{Z}$ be coordinates of an integral $\mathcal{A}$-lamination l. Then the function $\widetilde{\mathbb{I}}'_{\mathcal{A}}(l)$ is a sign definite Laurent polynomial with integral coefficients in the coordinates $X_1, \ldots, X_n$ on $\mathcal{X}$. Its highest term is equal to $\prod_i X_i^{a_i}$ and the lowest one is $\prod_i X_i^{-a_i}$.*

2. *The function $\mathbb{I}'_{\mathcal{X}}(l)$ is a sign definite Laurent polynomial with integral coefficients in the coordinates $A_1, \ldots, A_n$ on $\mathcal{A}$.*

3. *Let $C \in \mathbb{Z}$. Then $\lim_{C\to\infty} \log \mathbb{I}(l, C \cdot m)/C = \mathcal{I}(l, m_L)$, where the definitions of $C \cdot m$ and m_L are the same as for the property 4 in the additive case.*

4. *$\mathbb{I}_{\mathcal{A}}(l, p(m)) = \mathbb{I}_{\mathcal{X}}(p(l), m)$ where $l \in \mathcal{A}_{SL_2,S}(\mathbb{Z}^t)$ and $m \in \mathcal{A}_{SL_2,S}$.*

5. *$\mathbb{I}_*(l_1)\mathbb{I}_*(l_2) = \sum_l c_*(l_1, l_2; l)\mathbb{I}_*(l)$, where $*$ stays for either $\mathcal{X}$ or $\mathcal{A}$, and $c_*(l_1, l_2; l)$ are integral (and conjecturally positive) constants, such that the sum is always finite.*

Observe that the property 1 implies that if an $\mathcal{A}$-lamination is integral, the corresponding function is lifted from $\mathcal{X}_{PSL_2,S}$, the map $\mathbb{I}_{\mathcal{A}}(l)$ is defined.

Proof. The statement 3 immediately follows from the analogous additive statement and the fact that for a curve l connecting punctures $\exp(I(l, m)) = \mathbb{I}(l, m)$ and for a closed curve l we have $\cosh(I(l, m)) = \mathbb{I}(l, m)$.

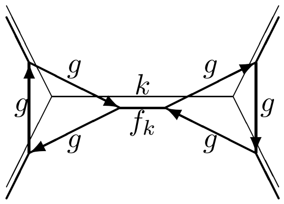
As a corollary one deduces the property 4 of the proposition 11.5. Indeed, the properties 1 and 3 shows that the identity $\lim\limits_{C\to\infty} I(l, C \cdot m)/C = \mathcal{I}(l, m)$ holds for an integral lamination l. By continuity it can be extended to any laminations.

The statement 4 is obvious if l is a loop: if this loop does not surround a puncture, both sides of the equality are equal to the trace of the monodromy of the corresponding local system; if l surrounds a puncture, it is 1 for both sides. The general case follows by the multiplicativity.

The properties 1 and 2 for the $\mathcal{X}$-moduli space follow essentially from Theorem 8.13 and Corollary 8.14, valid for PSL_m. However for the sake of completeness of this Section, we prove them below.

Before presenting the proof of the properties 1 and 2 recall the construction of monodromy matrices from coordinates of our moduli spaces given in Sections 7 and 8, specified to the case of $G = PGL_2$ and $G = SL_2$.

Let us do it first for the $\mathcal{X}$-moduli space. Replace the graph Γ by another one Γ' by gluing small triangles into each vertex and define a graph connection on it as shown on Picture (155) (The graph Γ is shown by the thin lines).



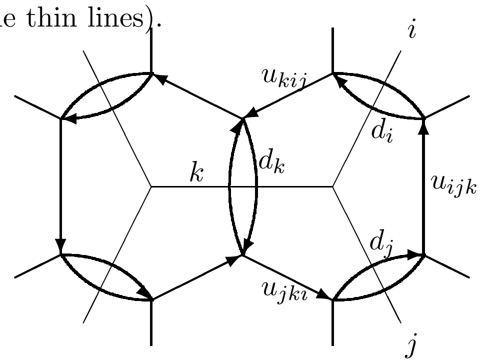
(155)

Here f_k and g are given by

$$f_k = \begin{pmatrix} 0 & -X_k^{1/2} \\ X_k^{-1/2} & 0 \end{pmatrix}, \qquad g = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}.$$

Here $X_k^{\pm 1/2}$ are natural coordinates on a covering. The monodromy around small triangles is the unity since $g^3 = -1$ in SL_2, i.e. $g^3 = 1$ in PSL_2. The orientations of the edges with the matrices f_i do not play any role since $f_i = f_i^{-1}$ in PSL_2.

Here is a similar construction for the $\mathcal{A}$-moduli space. Replace the graph Γ by another one by replacing each vertex by a hexagon and define a graph connection on it as shown on Picture (156). (The graph Γ is shown by the thin lines).



(156)

Here the matrices d_k and u_{ijk} are given by:

$$d_i = \begin{pmatrix} 0 & -A_i \\ \frac{1}{A_i} & 0 \end{pmatrix}, \qquad u_{ijk} = \begin{pmatrix} 1 & 0 \\ \frac{A_k}{A_iA_j} & 1 \end{pmatrix}.$$

The monodromy around the hexagons is the unity since $d_j u_{jki} d_k u_{kij} d_i u_{ijk} = 1$ in SL_2. The monodromy around the 2-gons is -1. The monodromy along a face path fixes vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Now proceed with the proof. Due to the multiplicativity it suffices to prove statements 1 and 2 for a connected curve l. Consider first the statement 1. In this case l must be a loop. Let α be a loop on the graph Γ' homotopic, on S, to the loop l. We may assume that the loop α contains no consecutive edges of little triangles on Γ'. Observe that

$$f_k g = \begin{pmatrix} X_k^{1/2} & 0 \\ X_k^{-1/2} & X_k^{-1/2} \end{pmatrix}, \qquad f_k g^{-1} = \begin{pmatrix} X_k^{1/2} & X_k^{1/2} \\ 0 & X_k^{-1/2} \end{pmatrix} \tag{157}$$

So calculation of the monodromy around such a loop reduces to taking products of matrices of type (157), just one of them per each edge of the original graph Γ which appears in the loop α. These matrices are positive and Laurent. Evidently the trace of such monodromy operator is a positive Laurent polynomial, and the coefficients at the highest and the lowest monomial equal to 1.

Remark. In general the coordinates of an $\mathcal{A}$-lamination l are half-integers $a_1, ..., a_n \in \mathbb{Z}/2$, and the function $\widetilde{\mathbb{I}}'_{\mathcal{A}}(l)$ is defined only on a covering of the space $\mathcal{X}$. However if $a_i \in \mathbb{Z}$, the proof shows that it is defined on the space $\mathcal{X}$ itself. So we get the function $\mathbb{I}'_{\mathcal{A}}(l)$.

Consider now the property 2. Call a matrix with Laurent-polynomial coefficients *sign definite* if each its entry is a Laurent polynomials with all positive or all negative coefficients. Let the type of a *sign definite* matrix M be a matrix $|M|$ of the same size as M with entries equal to $+, -$ or 0 indicating the sign of the monomials of the corresponding entries of M. Define a partial product on the set of types by the requirement, that $ab = c$ if for any two matrices M_1 and M_2, such that $|M_1| = a$ and $|M_2| = b$, we have $|M_1M_2|$.

Consider a path l connecting two holes, retracted to the graph. The desired intersection index is given by the bottom left element of the monodromy matrix. The sequence of matrices along the path is composed out of the matrices u, u^{-1} and d. Since it connects two holes, we may assume that it starts and ends with a d. The path can be deformed to make the sequence to avoid the subsequences $uu^{-1}, u^{-1}u, udu, d^2$ and $u^{-1}du^{-1}$. To investigate the sign properties of the product, let us replace every matrix in the sequence by its type and investigate the type of the product. Let

$$D := |d_i| = \begin{pmatrix} 0 & - \\ + & 0 \end{pmatrix}, \quad U^+ = |u_{ijk}| = \begin{pmatrix} + & 0 \\ + & + \end{pmatrix}, \quad U^- = |(u_{ijk})^{-1}| = \begin{pmatrix} + & 0 \\ - & + \end{pmatrix}$$

Now observe that $(U^+)^n = U^+$ and $(U^-)^n = U^-$ for any $n \in \mathbb{N}$, and replace the sequences of U^+ and U^- by single letters. the resulting sequence has one of the following four forms:

$$DU^-D(U^+DU^-D)^kU^+D, \quad D(U^+DU^-D)^kU^+D, \quad DU^-D(U^+DU^-D)^k, \quad D(U^+DU^-D)^k$$

The crucial observation is this:

$$\text{the products } U^+DU^-D \text{ and } DU^-DU^+ \text{ are of the type } -P = \begin{pmatrix} - & - \\ - & - \end{pmatrix} \tag{158}$$

In particular the type of $(U^+DU^-D)^k$ is either $(-1)^kP$. Our problem now is reduced to the verification that in each of the four cases the lower left entry is sign definite. Since $DP = PD = D$, this reduces to the cases D, DU^+D, DU^-D, D, which is an easy exercise.

Let us prove the statement for a closed loop l. In this case $\mathbb{I}'_{\mathcal{X}}(l)$ is the trace of a cyclic word of type $...DU^{\pm}DU^{\mp}DU^{\pm}DU^{\mp}DU^{\pm}$. We will assume that its length is minimal possible for a given loop l. So the following cases may occur:

i) The word is U^+, U^- or D. These are trivial cases.

ii) The cyclic word is DU^+ or DU^-. One easily checks that in this case the trace is sign-definite.

iii) The cyclic word has the length greater then two. Then the length is even: otherwise the length of the cyclic word can be reduced. Thus the word contains U^+. Since the word is cyclic, we may assume that it starts from U^+. Now the claim follows from (158) and ii).

Let us prove the property 5. We start with the the second part of the property. For this we introduce the notion of *quasi-lamination*. A quasi-lamination is a collection of curves, connecting punctures or closed. Unlike the laminations, they can intersect or have self-intersections. One can define the canonical multiplicative pairings for quasi-laminations by the same rules as for laminations for single curves and extend it for the collections of curves by multiplicativity, i.e., by definition $\mathbb{I}_*(l_1)\mathbb{I}_*(l_2) = \mathbb{I}_*(l)$, where l is a quasi-lamination given by the union of l_1 and l_2. The functions on the moduli spaces corresponding to quasi-laminations satisfy the so-called linear *skin relations*. Using these relations one can represent any function corresponding to a quasi-lamination as an integral linear combination of similar functions corresponding to laminations, thus proving the property. Let us describe the relations. Let x be an intersection or self-intersection point of a quasi-lamination l. The lamination in the vicinity of x looks like a cross $+$. Let l_1 and l_2 be the quasi-laminations obtained by replacing this cross by and respectively. The relation is

$$\mathcal{I}(m, l) \pm \mathcal{I}(m, l_1) \pm \mathcal{I}(m, l_1) = 0,$$

for some choice of signs depending on l, l_1, l_2. The verification of these relations is a straightforward corollary of the identities for 2×2 matrices:

$$\begin{aligned} \mathrm{tr}A\mathrm{tr}B &= \mathrm{tr}AB + \mathrm{tr}AB^{-1} \\ \mathrm{tr}AC\ B_{21} &= (ABC)_{21} + (AB^{-1}C)_{21} \\ (AC)_{21}(BD)_{21} &= (AD)_{21}(BC)_{21} - (AB^{-1})_{21}(D^{-1}C)_{21} \end{aligned}$$

Here A_{21} is the left bottom matrix element of a 2×2 matrix A.

Since l_1 as well as l_2 have one intersection point less than l has, repeating this procedure one can express the pairing with any quasi-lamination by a linear combinations of pairings with quasi-laminations without intersection points. Now to reduce the pairing to the pairing with a lamination we need to remove contractible curves and replace equivalent curves by one curve with multiplicity. The first procedure can be done by the obvious statement that the pairing with a closed curve is equal to the constant 2. The second procedure is a corollary of the fact that $\mathrm{tr}A^k = P_k(\mathrm{tr}A)$, where P_k is the k-th Tchebychev polynomial.

The proof of the second part of the property is absolutely similar and even simpler, since we don't need to consider curves connecting punctures. The theorem is proved.

5. Conjectures. Let L be a set. Denote by $\mathbb{Z}_+\{L\}$ the abelian semigroup generated by L. Its elements are expressions $\sum_i n_i\{l_i\}$ where $l_i \geq 0$, the sum is finite, and $\{l_i\}$ is the generator corresponding to an element $l_i \in L$. Similarly denote by $\mathbb{Z}\{L\}$ the abelian group generated by L.

Let $\mathcal{X}$ be a positive space. Recall from Section 4.3 the ring $\mathbb{L}[\mathcal{X}]$ of all good Laurent polynomials, the semiring $\mathbb{L}_+[\mathcal{X}]$ of positive good Laurent polynomials and the set of extremal elements $\mathbf{E}(\mathcal{X})$.

It follows from Definition 11.9 and Proposition 11.10 that the canonical multiplicative pairings provide canonical maps

$$\mathbb{I}^+_{\mathcal{A}} : \mathbb{Z}_+\{\mathcal{A}_{SL_2,S}(\mathbb{Z}^t)\} \longrightarrow \mathbb{L}_+[\mathcal{X}_{PSL_2,S}], \qquad \mathbb{I}^+_{\mathcal{X}} : \mathbb{Z}_+\{\mathcal{X}_{PSL_2,S}(\mathbb{Z}^t)\} \longrightarrow \mathbb{L}_+[\mathcal{A}_{SL_2,S}], \tag{159}$$

and hence

$$\mathbb{I}_{\mathcal{A}} : \mathbb{Z}\{\mathcal{A}_{SL_2,S}(\mathbb{Z}^t)\} \longrightarrow \mathbb{L}[\mathcal{X}_{PSL_2,S}], \qquad \mathbb{I}_{\mathcal{X}} : \mathbb{Z}\{\mathcal{X}_{PSL_2,S}(\mathbb{Z}^t)\} \longrightarrow \mathbb{L}[\mathcal{A}_{SL_2,S}], \tag{160}$$

which are multiplicative in the sense that property 4. of Definition 11.9 holds. Conjecturally the polynomials provided by laminations in both cases are *minimal*, i.e., any difference of such polynomials have at least one negative coefficient. Moreover, we conjecture the following.

Conjecture 11.11 *The maps (159) and (160) are isomorphisms. They provide the isomorphisms*

$$\mathcal{A}_{SL_2,S}(\mathbb{Z}^t) = \mathbf{E}(\mathcal{X}_{PSL_2,S}), \qquad \mathcal{X}_{PSL_2,S}(\mathbb{Z}^t) = \mathbf{E}(\mathcal{A}_{SL_2,S})$$

In particular this means that, both in $\mathcal{A}$ and $\mathcal{X}$ cases, any good Laurent polynomial is a linear combination with integer coefficients of the images of the corresponding type laminations under the canonical map. For positive good Laurent polynomials all the coefficients are positive.

Observe that if G is simply-connected then the Langlands dual ${}^LG$ always has trivial center, so the corresponding moduli space $\mathcal{X}_{{}^LG,S}$ has the positive structure defined in Section 6.

Conjecture 11.12 *Let G be a connected, simply-connected, split semisimple algebraic group. Then*

a) There exist canonical additive pairings between the real lamination and Teichmüller spaces

$$I : \mathcal{A}_{G,S}(\mathbb{R}^t) \times \mathcal{X}^+_{{}^LG,S} \to \mathbb{R}, \quad I : \mathcal{X}_{{}^LG,S}(\mathbb{R}^t) \times \mathcal{A}^+_{G,S} \to \mathbb{R},$$

as well as the intersection pairing between the real lamination spaces

$$\mathcal{I} : \mathcal{A}_{G,S}(\mathbb{R}^t) \times \mathcal{X}_{{}^LG,S}(\mathbb{R}^t) \to \mathbb{R},$$

All of them are continuous.

b) There exist canonical isomorphisms

$$\mathbb{I}^+_{\mathcal{A}} : \mathbb{Z}_+\{\mathcal{A}_{G,S}(\mathbb{Z}^t)\} \longrightarrow \mathbb{L}_+[\mathcal{X}_{{}^LG,S}], \qquad \mathbb{I}^+_{\mathcal{X}} : \mathbb{Z}_+\{\mathcal{X}_{{}^LG,S}(\mathbb{Z}^t)\} \longrightarrow \mathbb{L}_+[\mathcal{A}_{G,S}], \tag{161}$$

$$\mathbb{I}_{\mathcal{A}} : \mathbb{Z}\{\mathcal{A}_{G,S}(\mathbb{Z}^t)\} \longrightarrow \mathbb{L}[\mathcal{X}_{{}^LG,S}], \qquad \mathbb{I}_{\mathcal{X}} : \mathbb{Z}\{\mathcal{X}_{{}^LG,S}(\mathbb{Z}^t)\} \longrightarrow \mathbb{L}[\mathcal{A}_{SL_2,S}], \tag{162}$$

They provide the isomorphisms

$$\mathcal{A}_{G,S}(\mathbb{Z}^t) = \mathbf{E}(\mathcal{X}_{{}^LG,S}), \qquad \mathcal{X}_{{}^LG,S}(\mathbb{Z}^t) = \mathbf{E}(\mathcal{A}_{G,S})$$

When $G = SL_2$ we have ${}^LG = PSL_2$, so Conjecture 11.12 reduces to Conjecture 11.11.

Quantum canonical map. We use notations from [FG2]. So $\mathcal{X}^q_{PSL_2,S}$ denotes the q-deformation of the moduli space $\mathcal{X}_{PSL_2,S}$, defined by gluing the quantum tori given by the generators X_i and relations

$$q^{-\varepsilon_{ij}} X_i X_j = q^{-\varepsilon_{ji}} X_j X_i$$

Conjecture 11.13 *There exists quantum canonical map*

$$\widehat{\mathbb{I}}^q : \mathcal{A}_{SL_2,S}(\mathbb{Z}^t) \longrightarrow \mathcal{X}^q_{PSL_2,S}$$

In other words for any positive integral A-lamination l one can associate a Laurent polynomial $\widehat{\mathbb{I}}^q(l)$ of the variables $X_1, \ldots, X_n$, satisfying the following properties:

1. $\widehat{\mathbb{I}}^1(l) = \mathbb{I}_{\mathcal{A}}(l)$
2. *The highest term of $\widehat{\mathbb{I}}^q(l)$ is $q^{-\sum_{i<j} \varepsilon_{ij} a_i a_j} \prod_i X_i^{a_i}$.*
3. *All coefficients of $\widehat{\mathbb{I}}^q(l)$ are positive Laurent polynomials of q.*
4. *Let $*$ be the canonical involutive antiautomorphism on the fraction field $\mathbb{Q}(\mathcal{X}^q_{PSL_2,S})$. Then $*\widehat{\mathbb{I}}^q(l) = \widehat{\mathbb{I}}^q(l)$.*

5. $\widehat{\mathbb{I}}^q(l_1)\widehat{\mathbb{I}}^q(l_2) = \sum_l c^q(l_1, l_2; l)\widehat{\mathbb{I}}^q(l)$, *where the* $c^q(l_1, l_2; l)$ *are Laurent polynomials of* q *with positive integral coefficients and such that the sum is finite.*

Of course, we expect a similar conjecture for an arbitrary G. The most general and precise conjectures are formulated in the set up of cluster ensembles, see Section 4 of [FG2]. However if S is any hyperbolic surface but the punctured torus, the positive atlases defined in this paper for the pair of moduli spaces are different then the ones for the cluster ensemble, and therefore the corresponding conjectures describe different objects. All conjectures formulated in Section 4 of [FG2] in the framework of cluster ensembles are easily rephrased in the framework of the pair of positive spaces $(\mathcal{X}_{G,S}, \mathcal{A}_{G,S})$, and expected to be valid in this framework.

The exists a generalization of these conjectures where S is replaced by $\widehat{S}$.

12 Positive coordinate systems for $\mathcal{X}$-laminations

0. Extended positive atlas on $\mathcal{X}_{G,S}$.

1. Positive coordinate systems: definitions and the main result. Let $\mathcal{C}_S$ be the set of pairs (an ideal triangulation of S considered modulo isotopy, a choice of orientations of boundaries of all holes of S). The direct product of the mapping class group of S and the group $(\mathbb{Z}/2\mathbb{Z})^n$ acts naturally on the set $\mathcal{C}_S$: the factor $(\mathbb{Z}/2\mathbb{Z})^n$ acts by changing orientations of holes. The coordinate systems on the space of $\mathcal{X}$-laminations on S are parametrized by the set $\mathcal{C}_S$. So there are 2^n coordinate systems corresponding to an ideal triangulation of S.

Definition 12.1 *Let l be an $\mathcal{X}$-lamination on S. A coordinate system corresponding to an ideal triangulation of S is* positive (non-negative) for the lamination l, *if all the coordinates of l in this coordinate system are* > 0 *(respectively* ≥ 0*).*

Theorem 12.2 *a) There exists a dense (resp. dense open) subset in the set of all real $\mathcal{X}$-laminations on S such that every lamination from it admits a non-negative (resp. positive) coordinate system.*

b) Let T be an ideal triangulation of S corresponding to a non-negative coordinate system for a finite real $\mathcal{X}$-lamination l. Then the isotopy class of the collection of edges of T with > 0 *coordinates is isotopic to l, and hence uniquely determined by l.*

c) A positive coordinate system for an $\mathcal{X}$-lamination is unique if exists.

To get the part a) of Theorem 12.2 we need the following result: for generic real $\mathcal{X}$-lamination all curves of the lamination go between the punctures. For this we use some results from the theory of Morse-Smale foliations.

2. Morse-Smale foliations and real laminations. Let us recall the notion of a Morse-Smale foliation on S and some results about them.

Given a foliation on a punctured disc, the *winding number* of the foliation at the puncture is an integer given by the the rotation number of the fibers of the foliation restricted to a small circle near the puncture. For example the radial foliation has the winding number $+1$, and the foliation tangent to the family of concentric circles also has the winding number $+1$.

Let S be a surface with holes. A *Morse-Smale foliation* on S is a foliation $\mathcal{F}$ on S minus a finite number of points, called singular points of the foliation, with the following structures/properties:

i) $\mathcal{F}$ is equipped with a real transversal measure.

ii) The fibers of $\mathcal{F}$ are transversal to the boundary of S.

iii) The winding number of the foliation at each singular point is < 0.

A fiber of a Morse-Smale foliation on S is *singular* if it ends at a singular point of the foliation. The condition iii) implies that the number of singular fibers of a Morse-Smale foliation ending in a singular point is finite. The condition ii) easily implies that a Morse-Smale foliation does not have fibers contractible onto the boundary.

Lemma 12.3 *The number of homotopy types of the leaves of a Morse-Smale foliation going between the boundary components is finite.*

Proof. The above remarks show that we may assume that the fibers can not be contracted to the boundary. Cutting the surface along a non-singular leaf going between the boundary components we decrease either the number of holes, or the genus. The lemma follows.

We need the following result.

Theorem 12.4 *There exists an open dense domain $U_{MS}(S)$ in the space of all Morse-Smale foliations on S such that every non-singular curve of a foliation from this domain joins two boundary components.*

Now let us replace the homotopic non-singular fibers of a Morse-Smale foliation $\mathcal{F}$ by a single curve with the weight provided by the transversal measure. According to a well known theorem of Thurston, this way we get a bijection between the Morse-Smale foliation on S and Thurston's $\mathbb{R}$-laminations: every $\mathbb{R}$-lamination is representable as a Morse-Smale foliation on S, and this representation is unique: if two laminations correspond to two Morse-Smale foliations, these foliations are isotopic. It follows from Lemma 12.3 that if $\mathcal{F}$ was from the set $U_{MS}(S)$, we will get a Thurston $\mathbb{R}$-lamination $l_{\mathcal{F}}$ with a finite number of curves.

3. Proof of Theorem 12.2. Obviously c) is a particular case of b). Moreover clearly that in b) not only the set of the edges of T carrying positive coordinates, but also the set of the orientations of the holes serving as the endpoints of these edges is determined by l.

a) We will assume without the loss of generality that an $\mathcal{X}$-lamination l is defined using the orientations of the holes induced by the orientation of the surface S. The general case is reduced to this using the action of the group $(\mathbb{Z}/2\mathbb{Z})^n$ changing the orientations of the holes.

A real $\mathcal{X}$-lamination is called *finite* if it can be presented by a finite collection l of mutually and self non-intersecting curves with real weights on them. Recall that to get an $\mathcal{X}$-lamination out of such a collection of curves with real weights l we have to choose in addition orientations of all holes. As we already said, we do it by taking the orientations induced by the orientation of the surface S, getting an $\mathcal{X}$-lamination l^+. To define the $\mathcal{X}$-coordinates of the finite $\mathcal{X}$-lamination l^+ we have to perform the following procedure at every hole of S: we replace a segment of the lamination going to the hole by a ray winding infinitely many times around the puncture in the direction prescribed by the orientation of S, and then retract the winded lamination to a trivalent graph on S.

Let us call finite real $\mathcal{X}$-laminations such that every leaf of the lamination joins two boundary component *special real $\mathcal{X}$-laminations.* Theorem 12.4 and Lemma 12.3 imply that the space of special finite real $\mathcal{X}$-laminations is an open dense subset in the space of all real $\mathcal{X}$-laminations on S.

Now, given a special $\mathcal{X}$-lamination l^+, let us exhibit a non-negative triangulation of S for it. The curves of the finite lamination l provide a decomposition D_l of S. Adding a finite number of curves to this decomposition we obtain a triangulation T of S containing the decomposition D_l. Let Γ be the dual graph for this triangulation. We claim that it provides a non-negative coordinate system for the $\mathcal{X}$-lamination l^+: all coordinates assigned to the edges of D_l are > 0, while the rest of the coordinates equal to zero. Indeed, a curve of the lamination, being retracted onto the graph Γ, will cross the only one edge of Γ, the one dual to the edge of the triangulation T given by this curve. Figure 48 illustrates this claim, by showing the winding procedure for a curve of the lamination on S traveling between the holes. The part a) of the theorem is proved.

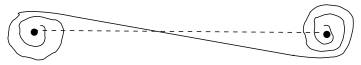


Figure 48: Calculating a coordinate of an $\mathcal{X}$-lamination assigned to an edge of the dual graph

b) Given a special real $\mathcal{X}$-lamination l, there is the following correspondence

$$\mathcal{I}_l \subset \{\text{edges of } \Gamma\} \times \{\text{connected components of the lamination } l\} \tag{163}$$

Namely, a pair (an edge e of Γ, a connected component α of l) belongs to $\mathcal{I}_l$ if and only if there exists a part of α which goes diagonally along the edge e. Let us denote by π_2 the natural projection of the correspondence $\mathcal{I}_l$ to the second factor in (163):

$$\pi_2 : \mathcal{I}_l \longrightarrow \{\text{connected components of the lamination } l\}$$

Lemma 12.5 *Let l be a special real $\mathcal{X}$-lamination on S. Then:*
a) The map π_2 is always surjective.
b) If $\pi_2^{-1}(\alpha)$ is a single edge e, then this edge is isotopic to an edge of the dual triangulation T.
c) If all coordinates of l are ≥ 0, then the map π_2 is injective.

Proof. a) Take a connected component β of l which is not in the image of π_2. This means that there is no edge of Γ such that β goes diagonally along this edge. Thus β can be shrunken into the boundary of a hole. Indeed, β either turn right at every edge of Γ, or it turns left at every edge of Γ, and thus it is isotopic to a curve which does not intersect Γ.

b) If $\pi_2^{-1}(\alpha) = e$, then α is homotopic to a curve which intersects Γ just once. Indeed, it either goes left before e and right after e, or right before and left after e. The claim follows from this.

c) Assume the opposite. Then there is a connected component α of l such that $|\pi_2^{-1}(\alpha)| > 1$. But this is impossible: after the curve α crossed diagonally the first edge on its way it can not cross the other edge since all its coordinates are ≥ 0.

The lemma is proved.

Corollary 12.6 *Let l be a special real $\mathcal{X}$-lamination on S. Suppose that all coordinates of l in the coordinate system corresponding to an ideal triangulation T are ≥ 0. Then the set of the edges of T which carry > 0 coordinates of l is isotopic to the lamination l.*

Proof. Follows immediately from the parts b) and c) of Lemma 12.5.

The part b) of the theorem is an immediate consequence of Corollary 12.6. The theorem is proved.

Corollary 12.7 *a) Any rational $\mathcal{X}$-lamination without loops admits a non-negative coordinate system.*

b) The set of rational $\mathcal{X}$-laminations without loops is dense in the space $\mathcal{X}_L(S; \mathbb{Q})$ of all rational $\mathcal{X}$-laminations.

Proof. a) It has been proved during the proof of Theorem 12.2.

b) The intersection of the domain described in Theorem 12.2 with the space of all rational $\mathcal{X}$-laminations is described as the set of the ones without closed loops. The corollary is proved.

So a little perturbation destroys loops in a lamination.

It is tempting to avoid the use of the Morse-Smale theory in the proof, making the proof self-contained. Here is perhaps a first step in this direction.

Lemma 12.8 *Let l be a real $\mathcal{X}$-lamination and α be a closed leaf of it. Take any 3-valent graph and let now x_i be the coordinates of l and a_i be the coordinates of α considered as an $\mathcal{A}$-lamination. Then $\sum_i x_i a_i = 0$.*

Proof. The intersection pairing of α with l equals zero. On the other hand it is computed by the formula $\sum_i x_i a_i$. The lemma is proved.

This lemma implies that since all a_i's are integers if all x_i's are linearly independent over $\mathbb{Q}$, there are no closed curves. However this does not prove yet that for generic real $\mathcal{X}$-lamination all curves of the lamination go between the punctures: we have to take care of the non-closed curves winding inside of S.

4. An application to the canonical map $\mathbb{I}_{\mathcal{X}}$. It follows from the part a) of Corollary 12.7 that restriction of the canonical map $\mathbb{I}_{\mathcal{X}}$ to the integral laminations without loops is especially simple. Namely, given such a lamination l, in every non-negative coordinate system for l the element $\mathbb{I}_{\mathcal{X}}(l)$ is given by a product of the coordinate functions in this coordinate system. So $\mathbb{I}_{\mathcal{X}}(l)$ is in the (orbi)-cluster algebra corresponding to the moduli space $\mathcal{A}_{\mathcal{SL}_{\in},\mathcal{S}}$. On the other hand, the image of laminations containing loops does not belong to it.

13 A Weil–Petersson form on $\mathcal{A}_{G,\widehat{S}}$ and its motivic avatar

1. The K_2-invariant of a G–local system on a compact surface. Let S be a compact smooth oriented surface of genus $g > 1$. A representation $\rho : \pi_1(S, x) \to G(F)$ of $\pi_1(S, x)$ to the group of F–points of a reductive group G induces a map of the classifying spaces $\rho_B : B\pi_1(S, x) \to BG(F)$. Since S is a $K(\pi_1(S, x)$-space, applying the functor H_2 we get a map

$$\rho_* : H_2(S) = H_2(B\pi_1(S, x)) \longrightarrow H_2BG(F) = H_2G(F)$$

According to Suslin's stabilization theorem [Sus] the natural map $H_2GL_2(F) \to H_2GL_n(F)$ is an isomorphism for any infinite field F provided $n \geq 2$. There is a natural map $H_2GL_2(F) \to K_2(F)$ (see e.g. loc. cit.). Thus if $n \geq 2$ there is a map $\pi_n : H_2GL_n(F) \to K_2(F)$. A finite dimensional representation V of G provides a map $BG \to BGL(V)$. Composing it with $\pi_{\rm dimV}$ we get a homomorphism $\pi_{G,V} : H_2G(F) \to K_2(F)$. Let

$$\pi_G : H_2G(F) \to K_2(F)$$

be the homomorphism corresponding to the adjoint representation of G. Composing it with ρ_* we get a map

$$\pi_G \circ \rho_* : H_2(S) \longrightarrow K_2(F)$$

Let $[S] \in H_2(S)$ be the fundamental class of S.

Definition 13.1 *Let G be a split reductive group and F is a field. The K_2–invariant*

$$w_\rho \in K_2(F)$$

of a representation of $\rho : \pi_1(S) \to G(F)$ is the element element $\pi_G \circ \rho_*([S]) \in K_2(F)$.

Since the mapping class group Γ_S preserves the fundamental class of S, w_ρ is invariant under the action of Γ_S, i.e. $w_g\rho = w_\rho$ for $g \in \Gamma_S$.

Applying this to the field $\mathbb{F}$ of rational functions on the moduli space $\mathcal{L}_{G,S}$ and the corresponding tautological (or universal) representation $\pi_1(S, x) \to G(\mathbb{F})$ we get a Γ_S-invariant class $W_{G,S} \in K_2(\mathbb{F})$. The restriction of this class to any point $\rho \in \mathcal{L}_{G,S}$ is the class w_ρ. Therefore the class $W_{G,S}$ is regular on $\mathcal{L}_{G,S}$, i.e. provides an element

$$W_{G,S} \in H^2(\mathcal{L}_{G,S}, \mathbb{Q}_{\mathcal{M}}(2))^{\Gamma_S} \subset K_2(\mathbb{F})^{\Gamma_S}$$

Applying to it the map $d\log$ we a regular 2-form $\Omega_{G,S}$ on $\mathcal{L}_{G,S}$.

2. The weight two motivic complex. Let F be a field. Recall the Bloch complex of F

$$\delta : B_2(F) \longrightarrow \Lambda^2F^*; \qquad B_2(F) := \frac{\mathbb{Z}[F^*]}{R_2(F)}$$

The subgroup $R_2(F)$ is generated by the elements

$$\sum_{i=1}^{5}(-1)^i\{r(x_1,...,\widehat{x}_i,...,x_5)\}, \qquad x_i \in P^1(F), \quad x_i \neq x_j$$

where $\{x\}$ is the generator of $\mathbb{Z}[F^*]$ corresponding to $x \in F^*$. Let $\{x\}_2$ be the projection of $\{x\}$ to $B_2(F)$. Then $\delta\{x\}_2 := (1-x)\wedge x$. One has $\delta(R_2(F) = 0$, (see [G1]).

Let X_k be the set of all codimension k irreducible subvarieties of X. The weight two motivic complex $\Gamma(X;2)$ of a regular irreducible variety X with the field of functions $\mathbb{Q}(X)$ is the following cohomological complex

$$\Gamma(X;2) := \qquad B_2(\mathbb{Q}(X)) \xrightarrow{\delta} \Lambda^2\mathbb{Q}(X)^* \xrightarrow{\text{Res}} \prod_{Y\in X_1}\mathbb{Q}(Y)^* \xrightarrow{\text{div}} \prod_{Y\in X_2}\mathbb{Z} \tag{164}$$

where the first group is in the degree 1 and Res is the tame symbol map (10). If $Y \in X_1$ is normal, the last map is given by the divisor $\text{div}(f)$ of f. In Y is not normal, we take its normalization $\widetilde{Y}$, compute $\text{div}(f)$ on $\widetilde{Y}$, and then push it down to Y.

3. The second motivic Chern class. According to Milnor the universal G–bundle over the classifying space for an algebraic group G can be thought of as the following (semi)simplicial variety $EG_\bullet$:

$$\overrightarrow{\longrightarrow} G^4 \overrightarrow{\longrightarrow} G^3 \overrightarrow{\longrightarrow} G^2 \overrightarrow{\longrightarrow} G$$

The group G acts diagonally from the left on $EG_\bullet$, and the quotient $BG_\bullet$ is the simplicial model for the classifying space of G. The simplicial structure includes the maps

$$s_i^n : G^n \longrightarrow G^{n-1}; \quad (g_1,...,g_n) \longmapsto (g_1,...,\widehat{g}_i,...,g_n)$$

Let $\mathcal{F}^\bullet(X)$ be a complex of sheaves on X which depends functorially on X for the smooth projections. So a smooth projection $f : X \to Y$ provides a morphism of complexes $f^* : \mathcal{F}^\bullet(Y) \longrightarrow \mathcal{F}^\bullet(X)$. Then the hypercohomology $H^*(BG_\bullet, \mathcal{F}^\bullet)$ are the cohomology of the total complex associated with the following bicomplex:

$$\begin{array}{ccccccccc}
 & ... & & ... & & ... & & ... & \\
 & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
\xleftarrow{d} & \mathcal{F}^3(G^3) & \xleftarrow{d} & \mathcal{F}^3(G^2) & \xleftarrow{d} & \mathcal{F}^3(G) & \xleftarrow{d} & \mathcal{F}^3(*) & \\
 & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
\xleftarrow{d} & \mathcal{F}^2(G^3) & \xleftarrow{d} & \mathcal{F}^2(G^2) & \xleftarrow{d} & \mathcal{F}^2(G) & \xleftarrow{d} & \mathcal{F}^2(*) & \\
 & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
\xleftarrow{d} & \mathcal{F}^1(G^3) & \xleftarrow{d} & \mathcal{F}^1(G^2) & \xleftarrow{d} & \mathcal{F}^1(G) & \xleftarrow{d} & \mathcal{F}^1(*) &
\end{array}$$

The differential $d : \mathcal{F}^*(G\backslash G^{n-1}) \longrightarrow \mathcal{F}^*(G\backslash G^n)$ is defined by $d := \sum_{i=1}^n(-1)^{i-1}(s_i^n)^*$. So $H^n(BG_\bullet, \mathcal{F}^\bullet)$ is represented by cocycles in $\oplus_{p+q=n}\mathcal{F}^p(G^q)$.

It is well known that

$$H^*(BG_\bullet, \mathbb{Q}_\mathcal{M}(2)) = H^*(BG_\bullet, \Gamma(X;2)\otimes\mathbb{Q})$$

where the right hand side is computed via the bicomplex above. Moreover it is known that if $* = 4$, this $\mathbb{Q}$–vector space can be identified with the one of G–invariant quadratic forms on the Lie algebra of G. In particular the Killing form provides a distinguished cohomology class

$$c_{2,G}^\mathcal{M} = c_2^\mathcal{M} \in H^4(BG_\bullet, \mathbb{Q}_\mathcal{M}(2)) \tag{165}$$

A cocycle representing the class $c_2^\mathcal{M}$ is given by the following data:

$$C_4 \in B_2(\mathbb{Q}(G^4)^*)^G, \quad C_3 \in \Big(\Lambda^2\mathbb{Q}(G^3)^*\Big)^G, \quad C_2 \in \Big(\prod_{Y\in(G^2)_1}\mathbb{Q}(Y)^*\Big)^G \tag{166}$$

Lemma 13.2 *There exists a cocycle $C^U_\bullet = (C^U_4, C^U_3, C^U_2)$ with $C^U_2 = 0$ representing the class $c_2^{\mathcal{M}}$ such that its component on G^k is invariant under the right action of U^k, and skew symmetric under the permutations of the factors of G^k modulo 2-torsion.*

Proof. Let $Q(X_*(H))^W$ be the group of W-invariant quadratic forms on the group of cocharacters of the Cartan group H. Recall ([BrD], Section 4; [EKLV], 3.2, 4.7, 4.8) that for a split semi-simple group G there exists canonical isomorphism, functorial in G,

$$H^2(BG, \mathbb{Z}_{\mathcal{M}}(2)) \stackrel{\sim}{\longrightarrow} H^2(BH, \mathbb{Z}_{\mathcal{M}}(2))^W = Q(X_*(H))^W \qquad (167)$$

Remark. The work [BrD] employs the group $H^0(X, \mathbf{K}_2)$, where $\mathbf{K}_2$ is the sheaf of K_2 groups in the Zariski topology. The Gersten resolution shows that it is canonically isomorphic to our group $H^2(X, \mathbb{Z}_{\mathcal{M}}(2))$. In ([BrD], 4.1) the isomorphism (167) is formulated for simply-connected G, but it is valid for all G.

If $G = GL_n$ a cocycle satisfying all the conditions of the lemma was constructed in [G3]. We will recall its construction below. We may assume that G is simple. In particular it provides a class in $H^2(BGL_n, \mathbb{Z}_{\mathcal{M}}(2))$ denoted by w_{GL_n}.

Let $\rho : G \longrightarrow GL_n$ be a nontrivial homomorphism. The Killing form on GL_n restricts to λ_ρ times the Killing form on G. Since $\lambda_\rho \neq 0$, the class $\rho^* w_{GL_n}$ is not zero. The class $\rho^* w_{GL_n}$ is represented by an element in $C'_3 \in K_2(\mathbb{Q}(G^3))$. Since $\rho(U)$ lies in a maximal unipotent subgroup U' of GL_n, it is invariant under the right action of U^3. Therefore there exists an U^3-invariant lift $C^U_3 \in \Lambda^2(\mathbb{Q}(G^3))$ of the class $\rho^* w_{GL_n}$. Consider the element

$$\sum_{i=1}^{4} (-1)^{i-1} (s_i^4)^* C^U_3 \qquad (168)$$

It lies in $\Lambda^2\mathbb{Q}((G/U)^4)^*$. We claim that its projection to K_2 is zero. This means that there exists an U^4-invariant element $C^U_4 \in B_2(\mathbb{Q}(G^4))$ such that $\delta(C^U_4) = \sum_{i=1}^4 (-1)^{i-1}(s_i^4)^* C^U_3$. It remains to skewsymmetrize it. To check the claim observe that the projection of the element (168) to K_2 lies in the subspace $H^2((G/U)^4, \mathbb{Z}_{\mathcal{M}}(2))$ of the elements of K_2 with zero tame symbols. On the other hand we know that its pull back to G^4 is zero. Since $H^2(X, \mathbb{Z}_{\mathcal{M}}(2))$ is homotopy invariant, and the variety U is isomorphic to an affine space, we are done. The lemma is proved.

Observe that $G\backslash G^3/U^3$ is birationally isomorphic to $\mathrm{Conf}_3(\mathcal{A})$. So it comes from a class in K_2 of the field of rational functions on $\mathrm{Conf}_3(\mathcal{A})$. Set $\mathbb{F}_{G,k} := \mathbb{Q}(\mathrm{Conf}_k(\mathcal{A}))$. Lemma 13.2 can be reformulated as follows: there exist elements

$$C^U_4 \in B_2(\mathbb{F}_{G,4}), \quad C^U_3 \in \Lambda^2 \mathbb{F}^*_{G,3}$$

which satisfies the following three conditions:

$$\text{i)} \sum_{i=1}^{5} (-1)^{i-1} (s_i^5)^* C^U_4 = 0; \quad \text{ii)} \sum_{i=1}^{4} (-1)^{i-1} (s_i^4)^* C^U_3 = \delta C^U_4; \quad \text{iii)} \mathrm{Res}(C^U_3) = 0$$

4. The motivic avatar of the Weil–Petersson form. The mapping class group Γ_S acts on the space $\mathcal{A}_{G,\widehat{S}}$. Thanks to Proposition 3.2 we can define the second rational Γ_S*–equivariant weight two motivic cohomology* of $\mathcal{A}_{G,\widehat{S}}$ explicitly by

$$H^2_{\Gamma_S}(\mathcal{A}_{G,\widehat{S}}, \mathbb{Q}_{\mathcal{M}}(2)) := H^2(\mathbb{G}_\bullet(S) \times_{\Gamma_S} \mathcal{A}_{G,\widehat{S}}, \mathbb{Q}_{\mathcal{M}}(2)) := H^2(\mathbb{G}_\bullet(S) \times_{\Gamma_S} \mathcal{A}_{G,\widehat{S}}, \Gamma_{\mathcal{M}}(2))$$

Remarks. 1. $\mathcal{A}_{G,\widehat{S}}$ and $\mathbb{G}_\bullet(S)$ are objects of different nature: the first is a stack whose generic point is an algebraic variety, while the second is a polyhedral complex. Their product is mixed object. To define its hypercohomology with coefficients in a complex of sheaves $\mathcal{F}^\bullet$ one proceeds similarly to the definition of hypercohomology of a simplicial algebraic variety.

2. Our cocycles for the classes in $H^2_{\Gamma_S}$ will sit at the generic point of $\mathcal{A}_{G,\widehat{S}}$. Therefore we can forget about the complications related to the stack nature of $\mathcal{A}_{G,\widehat{S}}$.

Theorem 13.3 *A choice of W-invariant quadratic form Q on $X_*(H)$ provides a class*

$$\mathbb{W}_G(S) \in H^2_{\Gamma_S}(\mathcal{A}_{G,\widehat{S}}, \mathbb{Q}_{\mathcal{M}}(2))$$

It depends linearly on the form Q.

Here is a refined version of Theorem 13.3. Let $\mathbb{F}$ be the field of rational functions on $\mathcal{A}_{G,\widehat{S}}$. So the generic $\mathbb{F}$–point of the variety $\mathcal{A}_{G,\widehat{S}}$ describes generic decorated unipotent G–local system on Γ.

Theorem 13.4 *A cocycle $C^U_\bullet$ as in Lemma 13.2 representing the class $c_2^{\mathcal{M}}$ provides a homomorphism of complexes*

$$\begin{array}{ccccccc}
G_5(S) & \longrightarrow & G_4(S) & \longrightarrow & G_3(S) & \longrightarrow & 0 \\
\downarrow & & \downarrow w_4 & & \downarrow w_3 & & \downarrow \\
0 & \longrightarrow & B_2(\mathbb{F}) & \stackrel{\delta}{\longrightarrow} & \Lambda^2\mathbb{F}^* & \stackrel{\partial}{\longrightarrow} & \prod_{Y \in \mathcal{A}_m(S)_1} \mathbb{Q}(Y)^*
\end{array}$$

Different cocycles lead to homotopic homomorphisms.

Proof of theorem 13.4. A local system on a graph Γ is the following data:

i) a collection $\{L_v\}$ of vector spaces attached to the vertices v of Γ.

ii) For any edge e an orientation of e gives rise to an operator $M_{\vec{e}} : L_{v_1} \longrightarrow L_{v_2}$. Here v_1 and v_2 are the vertices of e, and the chosen orientation is from v_1 to v_2. Reversing the orientation of e we get the inverse operator.

Let Γ be a graph embedded to S, so that S retracts to Γ. Then there is a natural bijection between the local systems on S and Γ.

We picture a valence k vertex v of a ribbon graph by a little oriented circle S^1_v with k marked points on it corresponding to the edges shared by v. The orientation of the circle is compatible with the cyclic order at v. The marked points cut the circle into k arcs $\alpha_1, ..., \alpha_k$. A decorated unipotent G–local system on a ribbon graph Γ is given by a collection of affine flags X_{v,α_i} over v. The affine flag X_{v,α_i} is invariant under the monodromy around the face path provided by α_i.

Take a generator $(\Gamma, \varepsilon_\Gamma)$ of $G_3(S)$. We assign to it an element of $\Lambda^2\mathbb{F}^*$ as follows. For each vertex v of Γ there are three cyclically ordered affine flags X^1_v, X^2_v, X^3_v over v. Evaluating the component C^U_3 on them we get an element $C^U_3(X^1_v, X^2_v, X^3_v) \in \Lambda^2\mathbb{F}^*$. Observe that a trivalent ribbon graph has canonical orientation provided by the cyclic structure at the edges. We define the sign $\mathrm{sgn}(\varepsilon_\Gamma)$ as the ratio of this orientation to the one ε_Γ. We define the map w_3 by setting

$$w_3(\Gamma, \varepsilon_\Gamma) := \sum_{v \in V(\Gamma)} \mathrm{sgn}(\varepsilon_\Gamma) C^U_3(X^1_v, X^2_v, X^3_v) \in \Lambda^2\mathbb{F}^*$$

Let $(\Gamma, \varepsilon_\Gamma) \in G_4(S)$. Then Γ is represented by a ribbon graph with exactly one 4–valent vertex v_0. Let us enumerate the edges at this vertex in a way compatible with the cyclic order. Then the graph inherits canonical orientation, and we will assume that ε_Γ is this orientation. One has

$$\partial : (\Gamma, \varepsilon_\Gamma) \longmapsto (\Gamma_1, \varepsilon_{\Gamma_1}) - (\Gamma_2, \varepsilon_{\Gamma_2})$$

where Γ_1 and Γ_2 are the two trivalent ribbon graphs shown on Figure 49, and ε_{Γ_i} is the canonical orientation of the trivalent ribbon graph Γ_i.

A decorated unipotent G–local system on Γ provides us four affine flags $X^1, ..., X^4$ in the domains near the vertex v_0. We enumerate them using the chosen order of the edges at v_0 and set

$$w_4(\Gamma, \varepsilon_\Gamma) := C^U_4(X^1, X^2, X^3, X^4) \in B_2(\mathbb{F})$$

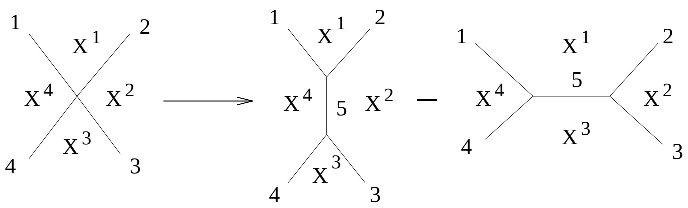


Figure 49:

The skew symmetry property of the component C_4^U guarantees that this definition does not depend on the ordering of the edges at v_0. The condition $\delta \circ w_4(\Gamma, \varepsilon_\Gamma) = w_3 \circ \delta(\Gamma, \varepsilon_\Gamma)$ is just equivalent to the cocycle condition ii) for the cycle $C_\bullet^U$.

Let $(\Gamma, \varepsilon_\Gamma) \in G_5(S)$. We assign to $(\Gamma, \varepsilon_\Gamma)$ zero. One needs to check that

$$w_4\Big(\partial(\Gamma, \varepsilon_\Gamma)\Big) = 0 \quad \text{in } B_2(\mathbb{F})$$

Observe that Γ has either one 5–valent vertex v_0, or two 4–valent vertices, and all the other vertices are of valence three. Suppose that Γ has a 5–valent vertex v_0. Let us choose an order of the edges at v_0 compatible with the cyclic structure at the vertex. It provides an orientation of Γ, and we can assume that ε_Γ is this orientation. Then $\partial(\Gamma, \varepsilon_\Gamma)$ is given by the five generators shown on the picture. Let us denote by $X^1, ..., X^5$ the five affine flags over the vertex v_0, enumerated by using the numeration of the edges, as shown on Figure 50.

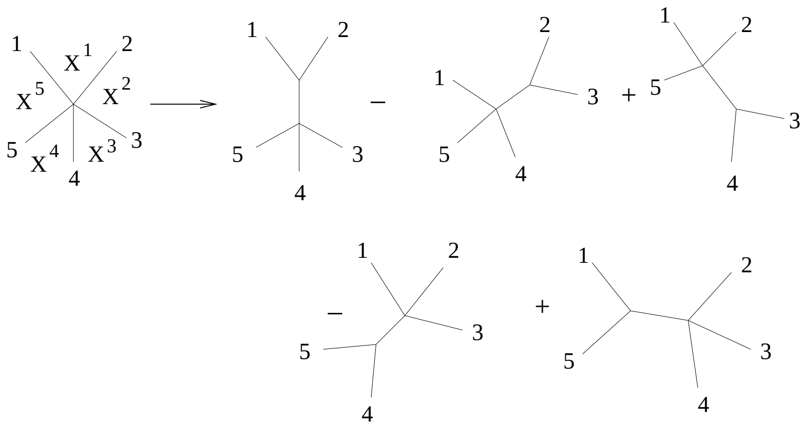


Figure 50: The pentagon relation

Then

$$w_4\Big(\partial(\Gamma, \varepsilon_\Gamma)\Big) = \sum_{i=1}^{5} (-1)^{i-1} C_4^U(X^1, ..., \widehat{X}^i, ..., X^5) = 0$$

The right hand side is zero thanks to the condition i) for the cocycle $C_\bullet^U$.

If Γ has two 4–valent vertices then it is easy to check, using only the skew symmetry of C_4^U, that $w_4\Big(\partial(\Gamma, \varepsilon_\Gamma)\Big) = 0$. We conclude that the properties i)–iii) for the cocycle $C_\bullet^U$ imply that $w_\bullet$ is morphism of complexes. The theorem is proved.

Proof of theorem 13.3. It follows the same lines as the proof of Theorem 13.4.

5. The K_2–avatar of the degenerate symplectic form on $\mathcal{A}_{G,\widehat{S}}$.

Corollary 13.5 *Let G be a split semi-simple algebraic group. Then a choice of a W-invariant quadratic form Q on $X_*(H)$ provides a class*

$$W_{G,\widehat{S}} \in H^2(\mathcal{A}_{G,\widehat{S}}, \mathbb{Q}_{\mathcal{M}}(2))^{\Gamma_S}$$

The form $\Omega_{G,\widehat{S}} := d\log(W_{G,\widehat{S}})$ is a regular Γ_S-invariant 2-form on the nonsingular part of $\mathcal{A}_{G,\widehat{S}}$.

Proof. For a field F one has $K_2(F) = \mathrm{Coker}(\delta)$ by Matsumoto's theorem. Thus for a regular variety X we have $H^2(X, \mathbb{Q}_{\mathcal{M}}(2)) = H^2\Gamma(X,2) \otimes \mathbb{Q}$ where the left hand side was defined in (11).

We define the class $W_{G,\widehat{S}}$ as the image of the class $\mathbb{W}_{G,\widehat{S}}$ under the canonical projection

$$H^2(\mathbb{G}_\bullet(S) \times_{\Gamma_S} \mathcal{A}_{G,\widehat{S}}, \mathbb{Q}_{\mathcal{M}}(2)) \longrightarrow H^2(\mathcal{A}_{G,\widehat{S}}, \mathbb{Q}_{\mathcal{M}}(2))^{\Gamma_S}$$

The properties of the map $d\log$ provide the second statement. The corollary is proved.

A direct construction of the class $W_{G,\widehat{S}}$. Recall the canonical orientation of a trivalent ribbon graph Γ. Let $\Gamma \to \Gamma'$ be a flip defined by an edge e, and $\overline{\Gamma}$ the ribbon graph obtained by shrinking of the edge e. Then $\overline{\Gamma}$ has just one 4–valent vertex, the shrinked edge e. According to the proof of Theorem 13.4 we have

$$w_3(\Gamma) - w_3(\Gamma') = \delta w_4(\overline{\Gamma}) \tag{169}$$

So the projection of $w_3(\Gamma) \in \Lambda^2\mathbb{F}^*$ to $K_2(\mathbb{F})$ does not depend on the choice of Γ. It is the class $W_{G,\widehat{S}}$.

In the next subsection we look at the proof of theorem 13.4 from a bit different perspective, getting a more general result.

5. The graph and the affine flags complexes. Let G be a group and X a G–set. Let $C_n(X)$ be the coinvariants of the diagonal action of G on $\mathbb{Z}[X^n]$. So the generators of $C_n(X)$ are elements $(x_1, ..., x_n)$ such that $(x_1, ..., x_n) = (gx_1, ..., gx_n)$ for any $g \in G$. The groups $C_n(X)$ form a complex $C_\bullet(X)$, called the *configurations complex*, with the differential ∂ given by

$$\partial : (x_1, ..., x_n) \longmapsto \sum_{i=1}^{n} (-1)^{i-1}(x_1, ..., \widehat{x}_i, ...x_n) \tag{170}$$

Applying this construction to the set of $\mathbb{F}$-points of the affine flag variety $\mathcal{A}_G$ for G we get a configuration complex $C_\bullet(\mathcal{A}_G(\mathbb{F}))$. We define the related to $\widehat{S}$ affine flags complex

$$A_\bullet^{\widehat{S}}(G) := \quad ... \xrightarrow{\partial} A_{n+2}^{\widehat{S}}(G) \xrightarrow{\partial} A_{n+1}^{\widehat{S}}(G) \xrightarrow{\partial} A_n^{\widehat{S}}(G) \xrightarrow{\partial} ...$$

is its subcomplex generated by configurations of affine flags over $\mathbb{F}$ in generic position.

Let us define a homomorphism $f_k : G_k(S) \longrightarrow A_k^{\widehat{S}}(G)$. If Γ has at least two vertices of valence ≥ 4 we set $f_k(\Gamma, \varepsilon_\Gamma) = 0$. Suppose that Γ has just one vertex v_0 of valence ≥ 4. Then $\mathrm{val}(v_0) = k$. Choose an order of the edges sharing v_0 compatible with the cyclic order at v_0. This order provides an orientation $\widetilde{\varepsilon}_\Gamma$ of Γ. The order of the edges provides an order of the domains near v_0, and hence the affine flags $X^1, ..., X^k$ attached to the domains. We set

$$f_k(\Gamma, \varepsilon_\Gamma) := \widetilde{\varepsilon}_\Gamma / \varepsilon_\Gamma \cdot (X^1, ..., X^k)$$

It clearly does not depend on the chosen order of the edges.

Proposition 13.6 *The map $f_\bullet$ provides a homomorphism of complexes*

$$\begin{array}{ccccccccc}
\longrightarrow & G_5(S) & \longrightarrow & G_4(S) & \longrightarrow & G_3(S) & \longrightarrow & 0 \\
 & \downarrow f_5 & & \downarrow f_4 & & \downarrow f_3 & & \downarrow \\
\longrightarrow & A_5^{\widehat{S}}(G) & \longrightarrow & A_4^{\widehat{S}}(G) & \longrightarrow & A_3^{\widehat{S}}(G) & \longrightarrow & A_2^{\widehat{S}}(G)
\end{array}$$

Proof. Let us check that for a generator $(\Gamma, \varepsilon_\Gamma) \in G_k(S)$ one has

$$\partial \circ f_{k-1}(\Gamma, \varepsilon_\Gamma) = f_k \circ \partial(\Gamma, \varepsilon_\Gamma) \tag{171}$$

Suppose first that $k > 3$. Then obviously both terms in (171) are zero if either there are three vertices in Γ of valence ≥ 4, or there are two vertices of valence ≥ 5. Suppose there are just two vertices, v_1 and v_2, of valence ≥ 4. We may assume that one of them, say for v_1, is of valence 4. Then v_1 provides two terms to $\partial(\Gamma)$. They enter with the opposite signs, and hence f_{k-1} kills their sum.

It remains to consider the case when there is just one vertex v of valence $k > 3$. Then the only component of $\partial(\Gamma)$ which is not immediately killed by f_{k-1} is the one shown on Figure 51. It matches the i-th term in the formula for $\partial \circ f_k(\Gamma)$. So (171) holds. If $k = 3$ then each edge of Γ contributes two

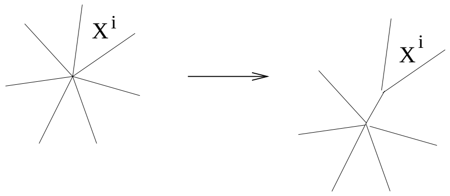


Figure 51:

terms to $f_k \circ \partial(\Gamma, \varepsilon_\Gamma)$, and they appear with the opposite signs, canceling each other. The proposition is proved.

7. Calculation of the $\mathbb{W}$-class for $G = SL_m$. Recall that Γ is a ribbon graph embedded to $\widehat{S}$, and I_m^Γ is the set parametrising the corresponding Δ- and X-coordinates, see Section 6. Recall the canonical map $p : \mathcal{A}_{SL_m, \widehat{S}} \to \mathcal{X}_{PSL_m, \widehat{S}}$.

Theorem 13.7 *The W–class on $\mathcal{A}_{PSL_m, \widehat{S}}$ is given by*

$$W_{SL_m, \widehat{S}} = \sum_{i,j \in \mathrm{I}_m^\Gamma} \varepsilon_{ij} \{\Delta_i^\Gamma, d \log \Delta_j^\Gamma\} = \sum_{i \in \mathrm{I}_m^\Gamma} \{\Delta_i^\Gamma, p^*(X_\alpha^\Gamma)\} \in K_2(\mathbb{F}), \quad \mathbb{F} = \mathbb{Q}(\mathcal{A}_{SL_m, \widehat{S}}) \tag{172}$$

It does not depend on the choice of the graph Γ.

Corollary 13.8 *The Weil-Petersson form on $\mathcal{A}_{PSL_m, \widehat{S}}$ is given by*

$$\Omega_{SL_m, \widehat{S}} = \sum_{i,j \in \mathrm{I}_m^\Gamma} \varepsilon_{ij} d \log \Delta_i^\Gamma \wedge d \log \Delta_j^\Gamma$$

It does not depend on the choice of the graph Γ.

Proof. We are going to apply our construction to the cocycle $C_\bullet^U$ for the second motivic Chern class $c_2^{\mathcal{M}}$ defined in [G3]. This cocycle satisfies the conditions of Lemma 13.2. We will get a cocycle representing the $\mathbb{W}$-class whose $\Lambda^2\mathbb{F}^*$-component is given by the following formula:

$$\sum_{i,j \in \mathrm{I}_m^\Gamma} \varepsilon_{i,j} \Delta_i^\Gamma \wedge \Delta_j^\Gamma = \sum_{i \in \mathrm{I}_m^\Gamma} \{\Delta_i^\Gamma, p^*(X_i^\Gamma)\} \in \Lambda^2\mathbb{F}^* \tag{173}$$

It of course depends on the choice of the graph Γ. Theorem 13.7 follows immediately from this formula. Observe that the equality of the two terms in the formula is a straitforward consequence of the formula ?? for $p^*(X_i^\Gamma)$. To prove formula (173) we will give below an explicit construction of the $w_\bullet$-map.

Denote by $A_\bullet(m)$ the affine flag complex for PSL_m. We will construct the following diagram of morphisms of complexes, where the $A_\bullet^{\widehat S}(m)$ and $BC_\bullet^{\widehat S}(2)$ are complexes defined using $\mathbb{F}$-vector spaces, as explained below:

$$
\begin{array}{ccccccccc}
\longrightarrow & G_5(S) & \longrightarrow & G_4(S) & \longrightarrow & G_3(S) & & \\
& \downarrow f_5 & & \downarrow f_4 & & \downarrow f_3 & & \\
\longrightarrow & A_5^{\widehat S}(m) & \longrightarrow & A_4^{\widehat S}(m) & \longrightarrow & A_3^{\widehat S}(m) & & \\
& \downarrow g_5 & & \downarrow g_4 & & \downarrow g_3 & & \\
\longrightarrow & BC_5^{\widehat S}(2) & \longrightarrow & BC_4^{\widehat S}(2) & \longrightarrow & BC_3^{\widehat S}(2) & & \\
& \downarrow & & \downarrow h_4 & & \downarrow h_3 & & \\
& 0 & \longrightarrow & B_2(\mathbb{F}) & \stackrel{\delta}{\longrightarrow} & \Lambda^2\mathbb{F}^* & \stackrel{\text{Res}}{\longrightarrow} & \prod_{Y\in(\mathcal{A}_{SL_m,\widehat S})_1}\mathbb{Q}(Y)^*
\end{array}
$$

Then $w_\bullet$ is the composition $w_\bullet := h_\bullet \circ g_\bullet \circ f_\bullet$.

The complexes in the diagram are given by horizontal lines, while morphisms are given by vertical arrows. The top complex is the graph complex on S. The next one is the complex of affine flags in the generic position in an m-dimensional vector space. Then follow the rank 2 bigrassmannian complex ([G1]) and the complex $\Gamma(\mathcal{A}_{SL_m,\widehat S};2)$. The homomorphisms $g_\bullet$ and $h_\bullet$ were defined in [G3] and [G1] - [G2]. We recall their definition below.

The grassmannian complex. Take $X = V_m$, where V_m is an F–vector space. The grassmannian complex

$$
C_\bullet(m)_{/F} := \quad ... \stackrel{\partial}{\longrightarrow} C_{m+2}(m) \stackrel{\partial}{\longrightarrow} C_{m+1}(m) \stackrel{\partial}{\longrightarrow} C_m(m)
$$

is the subcomplex of the m–truncated complex $\sigma_{\geq m}C_\bullet(V_m)$ given by the configurations of vectors in the generic position in V_m.

The rank two bigrassmannian complex $BC_\bullet(2)$ ([G1]). Consider a bicomplex:

$$
\begin{array}{ccccccc}
... & & ... & & ... & & ... \\
... & & \downarrow \partial' & & \downarrow \partial' & & \downarrow \partial' \\
... & \stackrel{\partial}{\longrightarrow} & C_6(3) & \stackrel{\partial}{\longrightarrow} & C_5(3) & \stackrel{\partial}{\longrightarrow} & C_4(3) \\
... & & \downarrow \partial' & & \downarrow \partial' & & \downarrow \partial' \\
... & \stackrel{\partial}{\longrightarrow} & C_5(2) & \stackrel{\partial}{\longrightarrow} & C_4(2) & \stackrel{\partial}{\longrightarrow} & C_3(2)
\end{array}
$$

The horizontal differential are given by (170). The vertical ones are

$$
\partial' : C_n(k) \longrightarrow C_{n-1}(k-1); \quad (x_1, ..., x_n) \longmapsto \sum_{i=1}^n (-1)^{i-1}(x_i|x_1, ..., \widehat{x}_i, ...x_n)
$$

Here $(x_i|x_1, ..., \widehat{x}_i, ...x_n)$ denotes the configuration of vectors obtained by projecting x_j, $j \neq i$, to the quotient $V_k/ < x_i >$. The complex $BC_\bullet(2)$ is the total complex associated with this bicomplex.

The homomorphism $g_\bullet$ ([G3]). Let us define its component

$$
g_n^k : A_n(m) \longrightarrow C_n(k); \quad k \leq m
$$

Choose non negative integers $p_1, ..., p_n$ such that $p_1 + ... + p_n = m - k$. Let $X_\bullet^1, ..., X_\bullet^n$ be the flags underlying an n–tuple of affine flags $\widetilde{X}^1, ..., \widetilde{X}^n$. Consider the quotient

$$
\frac{V_m}{X_{p_1} \oplus ... \oplus X_{p_n}} \tag{174}
$$

It is a vector space of dimension k. Our affine flags induce an n–tuple of the affine flags in this quotient. Taking the first vector of each of these affine flags, we get a configuration of n vectors in (174). The corresponding element of the group $C_n(k)$ is, by definition, $g_n^k(\widetilde{X}^1, ..., \widetilde{X}^n)$. According to the Key Lemma from [G3] this way we get a morphism of complexes.

The homomorphism $h_\bullet$. Consider the following diagram:

$$\begin{array}{ccccc} C_5(2) & \stackrel{\partial}{\longrightarrow} & C_4(2) & \stackrel{\partial}{\longrightarrow} & C_3(2) \\ \downarrow l_5 & & \downarrow l_4 & & \downarrow l_3 \\ 0 & \longrightarrow & B_2(\mathbb{F}) & \stackrel{\delta}{\longrightarrow} & \Lambda^2\mathbb{F}^* \end{array}$$

$$l_3 : (l_1, l_2, l_3) \longmapsto \Delta(l_1, l_2) \wedge \Delta(l_2, l_3) + \Delta(l_2, l_3) \wedge \Delta(l_3, l_1) + \Delta(l_3, l_1) \wedge \Delta(l_1, l_2)$$
$$l_4 : (l_1, l_2, l_3, l_4) \longmapsto -\{r(l_1, l_2, l_3, l_4)\}_2$$

Using the Plücker relation we see that, modulo 2–torsion, it provides a morphism of complexes, (see s. 2.3 in [G1]). The rank two grassmannian complex is a subcomplex of the rank two grassmannian bicomplex. Extending it by zero to the rest of the rank two grassmannian bicomplex we get the map $h_\bullet$. By s. 2.4 of [G1] it is a morphism of complexes.

The formula (173) follows immediately from the construction of the map w_3 given above. The Theorem 13.7 and hence the Corollary 13.8 are proved.

The class $\mathbb{W}_{PSL_m,\widehat{S}}$ is lifted from a similar class on the space $\mathcal{U}_{PSL_m,\widehat{S}}$:

Theorem 13.9 *a) There exists unique class*

$$\mathbb{W}^{\mathcal{U}}_{PSL_m,\widehat{S}} \in H^2_{\Gamma_S}(\mathcal{U}_{PSL_m,\widehat{S}}, \mathbb{Q}_{\mathcal{M}}(2)) \quad \textit{such that } \mathbb{W}_{SL_m,\widehat{S}} = p^*\mathbb{W}^U_{PSL_m,\widehat{S}}$$

b) The 2-form $\Omega^U_{SL_m,\widehat{S}} := d\log(W^U_{PSL_m,\widehat{S}})$ is symplectic form on the moduli space $\mathcal{U}_{PSL_m,\widehat{S}}$.

Proof. a) The class $\mathbb{W}_{SL_m,\widehat{S}}$ has two nontrivial components, with values in B_2 and Λ^2. The B_2 component comes from the $\mathcal{U}_{PSL_m,\widehat{S}}$-space by the very definition. For the Λ^2-component this is deduced from the formula (173).

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A.G.: Department of Math, Brown University, Providence RI 02906, USA; sasha@math.brown.edu
V.F.: ITEP, B. Cheremushkinskaya 25, 117259 Moscow, Russia. fock@math.brown.edu