TWO CLASSES OF HYPERBOLIC SURFACES IN $\mathbb{P}^3$

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Abstract. We construct two classes of singular Kobayashi hyperbolic surfaces in $\mathbb{P}^3$. The first consists of generic projections of the cartesian square $V = C \times C$ of a generic genus $g \geq 2$ curve C smoothly embedded in $\mathbb{P}^5$. These surfaces have C-hyperbolic normalizations; we give some lower bounds for their degrees and provide an example of degree 32. The second class of examples of hyperbolic surfaces in $\mathbb{P}^3$ is provided by generic projections of the symmetric square $V' = C_2$ of a generic genus $g \geq 3$ curve C. The minimal degree of these surfaces is 16, but this time the normalizations are not C-hyperbolic.

Introduction

In this paper we construct new examples of Kobayashi hyperbolic surfaces in $\mathbb{P}^3$. Concrete examples of Kobayashi hyperbolic smooth surfaces in $\mathbb{P}^3$ of degrees starting with 11 were given in [6, 32, 28, 9, 42]. Moreover, recently Demailly and El Goul [8], and also McQuillan [27], showed that a very generic surface in $\mathbb{P}^3$ of degree $d \geq 42$, resp. $d \geq 36$, is Kobayashi hyperbolic, making the first step towards solving the Kobayashi problem on hyperbolicity of a generic hypersurface in $\mathbb{P}^m$ of high enough degree. Note that examples of smooth hyperbolic hypersurfaces in $\mathbb{P}^m$ (for any $m \geq 3$) were constructed in Masuda-Noguchi [28], and that the set of all hyperbolic hypersurfaces is open in the Hausdorff topology on the space of hypersurfaces [44].

Our first examples of hyperbolic surfaces in $\mathbb{P}^3$ have C-hyperbolic normalizations. A complex space X is called *C-hyperbolic* if there exists a non-ramified holomorphic covering $Y \to X$ with Y being a Carathéodory hyperbolic complex space, i.e. any two points of Y can be separated by a bounded holomorphic function [20]. Since a Carathéodory hyperbolic complex space Y is also Kobayashi hyperbolic, the base X is Kobayashi hyperbolic, too. Thus, C-hyperbolicity implies Kobayashi hyperbolicity.

A degree $d \geq 4$ smooth curve in $\mathbb{P}^2$ is C-hyperbolic. However, for $m \geq 3$, hypersurfaces in $\mathbb{P}^m$ are not C-hyperbolic. Indeed, by the Lefschetz hyperplane section theorem, they are simply connected, and so, do not admit non-trivial coverings. Instead, we construct singular surfaces in $\mathbb{P}^3$ with C-hyperbolic normalization by starting with a projective embedding $X \hookrightarrow \mathbb{P}^N$ of a C-hyperbolic surface X and then considering its general linear projection $\overline{X}$ to $\mathbb{P}^3$. It is classically known (see e.g., [36]) that the normalization of $\overline{X}$ is smooth and so, coincides with X. Thus, the singular surfaces of type $\overline{X}$ provide examples of surfaces in $\mathbb{P}^3$ with C-hyperbolic normalization. In particular, X can be a product $\Gamma_1 \times \Gamma_2$ of smooth projective curves Γ_1, Γ_2 of genera $g_1, g_2 \geq 2$; having the unit bidisc as its universal covering, the variety X is C-hyperbolic. However,

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a singular projective surface $\overline{X}$ with hyperbolic normalization X can be non-hyperbolic even if X has the unit bidisc as universal cover; see [17] for an example.

In §2, we prove that for the cartesian square $V = C \times C$ of a generic genus $g \geq 2$ curve C, the generic projection $\overline{V} \subset \mathbb{P}^3$ is, indeed, Kobayashi hyperbolic (Theorem 2.5). To justify our interest in the singular surface $\overline{V}$, recall that a hyperbolic hypersurface in $\mathbb{P}^m$ possesses a hyperbolic neighborhood, and thus all hypersurfaces sufficiently close to $\overline{V}$ are hyperbolic, too [44]. In §3, we shift our attention to the symmetric square $V' = C_2$ of a curve C of genus ≥ 3, and we show that V' is Kobayashi hyperbolic if C is neither hyperelliptic nor bielliptic (Proposition 3.3). Using the symmetric square V' rather than the cartesian square V as above, we produce an example of a Kobayashi hyperbolic surface $\overline{V}' \subset \mathbb{P}^3$ of degree 16 (Theorem 3.13), which is the smallest possible degree of $\overline{V}'$ when C has general moduli[1] (see below). We observe that this time the normalization of our surface fails to be C-hyperbolic; indeed, the universal cover of C_2 carries no nonconstant bounded holomorphic functions (Corollary 3.9).

We would like to determine the minimal possible degree of the surfaces $\overline{V}$, resp. $\overline{V}'$. With this aim in mind, we let $\delta(Y)$ denote the minimal degree of the projective embeddings of a projective variety Y. In terms of this notation, we want to know the smallest value of $\delta(C \times C)$, resp. $\delta(C_2)$. In §2, resp. §3, we obtain some lower bounds for these minimal degrees in terms of the genus g of C (Proposition 2.2, resp. Proposition 3.15). In particular, we show that (under the assumption of generality of C) $18 \leq \delta(C \times C) \leq 46$ if $g = 2$, $20 \leq \delta(C \times C) \leq 32$ and $\delta(C_2) = 16$ if $g = 3$, and $20 \leq \delta(C_2) \leq 36$ if $g = 4$ (see Proposition 2.2, Example 2.6, Corollary 4.7(a), Proposition 3.15, and Corollary 3.11).

In §4, we investigate the divisors on $C \times C$ and on C_2. In particular, we describe some classes of very ample divisors on symmetric and cartesian squares of curves of genera 2, 3, and 4 (see Theorems 4.4, 4.14 and 4.16). The generic projections of the projective embeddings given by these divisors provide specific examples of the hyperbolic surfaces $\overline{V}$ and $\overline{V}'$ described in §§2–3.

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1. Generic projection of a smooth surface into $\mathbb{P}^3$

Let V be a smooth projective surface in $\mathbb{P}^N$, $N \geq 4$, and let $\overline{V}$ be its image under a generic linear projection to $\mathbb{P}^3$. By the Severi Theorem [40] (see also [30, 36, 13, (4.6)]), the surface $\overline{V}$ has only ordinary singularities. That is, $\overline{S} := \operatorname{sing} \overline{V}$ is a double curve of $\overline{V}$, i.e. a generic point $P \in \overline{S}$ on $\overline{V}$ is a point of transversal intersection of two smooth surface germs. The singularities of the curve $\overline{S}$ itself are ordinary triple points, $P_1, \ldots, P_t$, say, which are also ordinary triple points of the surface $\overline{V}$, i.e. points of transversal intersection of three smooth surface germs. Besides, there is also a certain number $Q_1, \ldots, Q_p \in \overline{S}$ of pinch points of the surface $\overline{V}$. They are smooth points of

[1] As usual, we say that a genus g curve C has *general moduli*, or is *generic*, if in the moduli space $\mathcal{M}_g$ it belongs to the complement of a certain countable union of proper subvarieties.

the double curve $\overline{S}$, and the local equation of $\overline{V}$ around the point Q_i is that of the Whitney umbrella $x^2y - z^2 = 0$.

We will assume that the surface $V \subset \mathbb{P}^N$ is not contained in a hyperplane and does not coincide, up to projective transformations, with the image of $\mathbb{P}^2$ under the second Veronese embedding $\mathbb{P}^2 \hookrightarrow \mathbb{P}^5$. By results of Severi, Moishezon and Mumford (see [40, 30, pp. 60, 72, 114-115]), the double curve $\overline{S}$ is irreducible, as well as its preimage S in the surface V. The curve S has $3t$ ordinary double points over the triple points $P_1, \ldots, P_t$ of $\overline{S}$ as the only singular points. The restriction $\pi \,|\, S \,:\, S \to \overline{S}$ of the projection $\pi \,:\, V \to \overline{V}$ (which is, at the same time, the normalization of $\overline{V}$) is generically $2:1$; it is ramified only over the pinch points $Q_1, \ldots, Q_p \in \overline{S}$, and is an immersion at each of the double points of the curve S (i.e., $\pi \,|\, S$ maps a neighborhood of each double point of S injectively to a neighborhood of a triple point of $\overline{S}$).

The following simple observation reduces the Kobayashi hyperbolicity of the surface $\overline{V}$ to those of the double curve $\overline{S}$.

Proposition 1.1. *Let $\overline{X}$ be a reduced compact complex space, and let $\pi \,:\, X \to \overline{X}$ be the normalization of $\overline{X}$. Assume that the space X is Kobayashi hyperbolic and let $S \subset X$, resp. $\overline{S} := \pi(S) \subset \overline{X}$, be the ramification divisor, resp. the branching divisor, so that the restriction $\pi \,|\, (X \setminus S) \,:\, X \setminus S \to \overline{X} \setminus \overline{S}$ is biholomorphic. Then $\overline{X}$ is Kobayashi hyperbolic if and only if $\overline{S}$ is Kobayashi hyperbolic.*

Proof. Clearly, if the space $\overline{X}$ is hyperbolic, then so is the subspace $\overline{S}$ of $\overline{X}$. By the Brody Theorem [5] (see [18], [20, (3.6.3)] or [45] for the case of complex spaces), the compact complex space $\overline{X}$ is hyperbolic iff any holomorphic mapping $f \,:\, \mathbb{C} \to \overline{X}$ is constant. Assuming that the branching divisor $\overline{S}$ is hyperbolic, we may restrict the consideration to the mappings $f \,:\, \mathbb{C} \to \overline{X}$ with the image not contained in $\overline{S}$. In this case f can be lifted to X (see [34]) and hence, in virtue of the hyperbolicity of the complex space X, it is constant. □

Applying Proposition 1.1 to the case where $V \subset \mathbb{P}^N$, $N \geq 4$, is a smooth Kobayashi hyperbolic surface (e.g., V can be isomorphic to the cartesian product of two smooth projective curves Γ_1 and Γ_2 of genera g_1, $g_2 \geq 2$) and $\overline{V} \subset \mathbb{P}^3$ is a generic projection of V with the irreducible double curve $\overline{S}$, we obtain the following statement:

Corollary 1.2. *Let $\overline{V} \subset \mathbb{P}^3$ be a generic projection of a Kobayashi hyperbolic smooth projective surface. Then $\overline{V}$ is Kobayashi hyperbolic iff the double curve $\overline{S}$ is Kobayashi hyperbolic; i.e., iff the geometric genus of $\overline{S}$ is at least* 2.

We denote by H a generic hyperplane section of V in $\mathbb{P}^N$ regarded as a very ample divisor on V. Let $n = H^2$ be the degree of V, and g_H be the genus of the smooth curve H on V, i.e. the sectional genus of the surface V in $\mathbb{P}^N$. Let c_1, c_2 be the Chern classes of V and $K = K_V$ be the canonical divisor of V. Denote by $d_{\overline{S}}$, resp. $g_{\overline{S}}$, the degree, resp. the geometric genus, of the double curve $\overline{S}$; t, resp. p, denotes, as above, the number of triple points, resp. pinch points, of $\overline{V}$. We also denote by b the number of double points of a generic projection $V \to \mathbb{P}^4$.

We have the following relations [13, (4.6)] [35] [36] [39, IX.5.21] (cf. also [4, (8.2.1)], [30] and [14, A4]):

$$\begin{aligned} c_1^2 &= n(n-4)^2 - (3n-16)d_{\overline{S}} + 3t - b & (1)\\ c_2 &= n(n^2-4n+6) - (3n-8)d_{\overline{S}} + 3t - 2b & (2)\\ \chi(\mathcal{O}_V) = \frac{c_1^2+c_2}{12} &= \binom{n-1}{3} - d_{\overline{S}}(n-4) + g_{\overline{S}} + 2t & (3)\\ p &= 2d_{\overline{S}}(n-4) - 6t - 4(g_{\overline{S}}-1) & (4)\\ b &= \frac{1}{2}[n(n-10) - 5H\cdot K - c_1^2 + c_2] \\ &= \frac{1}{2}[n(n-5) - 10(g_H-1) - c_1^2 + c_2] & (5) \end{aligned}$$

(Note that equations (1)–(4) satisfy a linear relation.)

Proposition 1.3. *In the notation as above, we have*

$$p = c_1^2 - c_2 + 2n(n-5) - 8d_{\overline{S}} = c_1^2 - c_2 + 2n + 8(g_H-1) = c_1^2 - c_2 + 6n + 4H\cdot K \tag{6}$$

where

$$d_{\overline{S}} = \binom{n-1}{2} - g_H\,. \tag{7}$$

Proof. Subtracting (2) from (1) gives the first equality in (6), the second one is provided by plugging in the value of d from (7), and the third one by the Adjunction Formula

$$2g_H - 2 = H\cdot(H+K) = n + H\cdot K\,. \tag{8}$$

To prove (7), consider a generic hyperplane $h \subset \mathbb{P}^N$ which contains the center of the projection $\pi : \mathbb{P}^N \to \mathbb{P}^3$. Then the image $l := \pi(h)$ is a generic plane in $\mathbb{P}^3$. Respectively, the image $L := \pi(H) = l\cdot\overline{V}$ of the hyperplane section $H = h\cdot V$ is a generic plane section of the surface $\overline{V} = \pi(V) \subset \mathbb{P}^3$. Hence, L is a degree n plane curve with $d_{\overline{S}}$ nodes, which are smooth points of the double curve $\overline{S}$, and no other singularities. Now (7) is the genus formula for the plane nodal curve L. □

Corollary 1.4. *We have the following expressions for $g_{\overline{S}}$ and t :*

$$\begin{aligned} g_{\overline{S}} &= \frac{1}{2}(n^2-7n+26) + (n-12)g_H - \frac{5c_1^2-3c_2}{4} & (9)\\ t &= \frac{1}{6}(n^2-3n+8)(n-6) - (n-8)g_H + \frac{2c_1^2-c_2}{3} & (10) \end{aligned}$$

Proof. From (4) and (6), resp. (3), we obtain the equations

$$\begin{aligned} 3t + 2g_{\overline{S}} &= (n-4)d_{\overline{S}} - p/2 + 2 = nd_{\overline{S}} - n(n-5) - \frac{c_1^2-c_2}{2}\,, & (11)\\ 2t + g_{\overline{S}} &= (n-4)d_{\overline{S}} - \binom{n-1}{3} + \frac{c_1^2+c_2}{12}\,, & (12) \end{aligned}$$

which together with (7) give us (9)–(10). □

2. Generic projection of a surface $V = C \times C$ into $\mathbb{P}^3$

From now on we let $V = C \times C$ where C is a smooth projective curve of genus $g \geq 2$. Fix a point $p_0 \in C$ and consider the horizontal, resp. vertical, divisor $E = C \times \{p_0\}$, resp. $F = \{p_0\} \times C$. The canonical divisor $K = K_V$ is numerically equivalent to the divisor $(2g-2)E + (2g-2)F$ on V, and we have

$$c_1^2 = K^2 = 2(2g-2)^2, \quad c_2 = e(V) = e(C)^2 = (2g-2)^2 = c_1^2/2 \tag{13}$$

where e stands for the Euler characteristic. Hence, the signature $\tau = \tau(V)$ vanishes: $\tau = \frac{1}{3}(c_1^2 - 2c_2) = 0$. By the Noether Formula for the holomorphic Euler characteristic, we have

$$\chi(\mathcal{O}_V) = \frac{c_1^2 + c_2}{12} = (g-1)^2\,.$$

Furthermore, for the geometric genus resp. the irregularity of the surface V we have

$$p_g(V) = h^{0,2}(V) = g^2, \qquad q(V) = h^{0,1}(V) = 2g\,.$$

Denote by Δ the diagonal of the cartesian square $V = C \times C$. Let Σ be the subgroup of the Neron-Severi group $\mathrm{NS}(V)$ generated by the classes of E, F, Δ modulo numerical equivalence. By a theorem of Hurwitz [15] (see also [13, (2.5)]), for a genus $g \geq 2$ curve C with generic moduli we have $\Sigma = \mathrm{NS}(V)$. Set $D_{a,a',k} = (a+k)E + (a'+k)F - k\Delta$. Using the fact that $\Delta^2 = 2 - 2g$ we obtain the following standard formulas from the theory of correspondences [11, 13, 39, 43]:

$$D_{a,a',k} \cdot D_{c,c',l} = ac' + a'c - 2gkl, \;\; D^2_{a,a',k} = 2(aa' - gk^2)\,,$$

$$D_{a,a',k} \cdot E = a', \;\; D_{a,a',k} \cdot F = a\,.$$

(The pair $(a,\, a')$ is called the *bidegree* of $D := D_{a,a',k}$, and k is called the *valence* of D.) Furthermore, for $a = a'$ we denote $D_{a,a',k}$ as $D_{a,k}$; thus, $\Delta \equiv D_{1,-1}$ and $K \equiv D_{2g-2,0}$ where $\equiv$ stands for numerical equivalence. We have:

$$\Delta^2 = 2 - 2g, \;\; K^2 = 2(2g-2)^2\,,$$

$$D_{a,k} \cdot D_{a',l} = 2(aa' - gkl); \;\; D^2_{a,k} = 2(a^2 - gk^2)\,,$$

$$D_{a,a',k} \cdot \Delta = a + a' + 2gk, \;\; D_{a,a',k} \cdot K = 2(g-1)(a+a')\,,$$

in particular,

$$D_{a,k} \cdot \Delta = 2(a + gk), \;\; D_{a,k} \cdot K = 4(g-1)a\,.$$

Proposition 2.1. *Let the notation be as in Section 1 above. For a very ample divisor* $H \equiv D_{a,a',k} \in \mathrm{NS}(V)$ *on the surface* $V = C \times C$ *we have:*

$$\begin{aligned}
n &= 2(aa' - gk^2) && (14)\\
g_H &= \frac{n}{2} + 1 + (g-1)(a+a') && (15)\\
d_{\overline{S}} &= \frac{1}{2}n(n-4) - (g-1)(a+a') && (16)\\
g_{\overline{S}} &= \frac{1}{2}(2n^2 - 17n + 2) + (n-12)(g-1)(a+a') - 7(g-1)^2 && (17)\\
p &= 2\left[3n + 4(g-1)(a+a') + 2(g-1)^2\right] && (18)\\
t &= \frac{1}{6}n(n^2 - 12n + 44) - (n-8)(g-1)(a+a') + 4(g-1)^2 && (19)\\
b &= \frac{1}{2}n(n-10) - 5(g-1)(a+a') - 2(g-1)^2\\
&= \frac{1}{2}n(n-5) - 5(g_H - 1) - 2(g-1)^2 && (20)
\end{aligned}$$

Proof. Equation (14) follows from the fact that $n = \deg V = H^2$. By the Adjunction Formula (8),

$$2g_H - 2 = H \cdot K + n = 2(g-1)(a+a') + n\,,$$

and (15) follows. Substituting (13) and (15) into (5)–(7), we obtain (16)–(20). □

We use these formulas to prove the following inequality:

Proposition 2.2. *Let* C *be a genus* $g \geq 2$ *curve with generic moduli. The minimal degree* $n = \delta(C \times C)$ *of a projective embedding* $C \times C \hookrightarrow \mathbb{P}^N$ *satisfies the inequality*

$$n(n-10) \geq 4(g-1)(g-1+5\delta(C)) \tag{21}$$

In particular, $\delta(C \times C) \geq 18$ *for* $g = 2$, $\delta(C \times C) \geq 20$ *for* $g = 3$, $\delta(C \times C) \geq 2g + 16$ *for* $g \geq 4$, *and* $\delta(C \times C) \geq 2(g-1) + 5\sqrt{2g}$ *for* $g \gg 0$.

Proof. Let $H \equiv D_{a,a',k}$ be a hyperplane section of the minimal degree embedding $V = C \times C \hookrightarrow \mathbb{P}^N$. Since $a = H \cdot F = \deg(F \hookrightarrow \mathbb{P}^N) \geq \delta(C)$ and similarly $a' = H \cdot E \geq \delta(C)$, it follows from (15) that

$$g_H \geq \frac{1}{2}(n+2) + 2\delta(C)(g-1)\,. \tag{22}$$

On the other hand, from (20) we have the inequality

$$2b = n(n-5) - 10(g_H - 1) - 4(g-1)^2 \geq 0\,,$$

or equivalently

$$g_H \leq \frac{n^2 - 5n + 10 - 4(g-1)^2}{10}\,. \tag{23}$$

Combining (22) and (23), we obtain (21).

We recall the standard lower bounds for $\delta(C)$. If $g = 2$, then $\delta(C) = 5$ since the genus of a smooth curve in $\mathbb{P}^3$ of degree ≤ 4 is at most 1 unless it is a plane curve of genus 3 (and degree 4). In general, the geometric genus g of a projective curve of degree δ satisfies the inequality $g \le \frac{1}{2}(\delta-1)(\delta-2)$; thus

$$\delta(C) \ge \frac{3}{2} + \sqrt{2g+\tfrac{1}{4}} \ \text{ for } \ g \ge 3\,, \quad \delta(C) = 5 \ \text{ for } \ g = 2\,. \tag{24}$$

The bounds for $g=2$, $g=3$, $g \ge 4$ follow from (21), the fact that n is even, and the estimate $\delta(C) \ge 5$ except for $g=3$ in which case $\delta(C) \ge 4$. To obtain the estimate for large g, we note that by (21) and (24),

$$\begin{aligned} n(n-10) \ &\ge \ 4(g-1)\left(g + 5\sqrt{2g+\tfrac{1}{4}} + \frac{13}{2}\right) \\ &= \ 4g^2\left(1-g^{-1}\right)\left(1 + 5\sqrt{2}\,g^{-\frac{1}{2}} + \frac{13}{2}g^{-1} + O(g^{-\frac{3}{2}})\right)\,. \end{aligned}$$

Therefore

$$\left(\frac{n-5}{2g}\right)^2 \ge 1 + 5\sqrt{2}\,g^{-\frac{1}{2}} + \frac{11}{2}g^{-1} + O(g^{-\frac{3}{2}})\,,$$

which gives us

$$n - 5 \ge 2g + 5\sqrt{2}\,g^{\frac{1}{2}} - 7 + O(g^{-\frac{1}{2}})\,.$$

Since n is an integer, the large g estimate follows. $\square$

Remark 2.3. For generic C of genus $g \ge 4$, we have values of $\delta(C)$ larger than those given by (24) (see [14], [13]). For instance, $\delta(C) = 6$ for a non-hyperelliptic genus 4 curve C [3, p.40, (D-4, D-7)]. These higher bounds in turn give higher lower bounds for n. Anyhow, the bounds for $\delta(C \times C)$ obtained from (21) are probably far from being sharp.

Being a constructive integer-valued function on the moduli space $\mathcal{M}_g$ of genus g curves, in general, $\delta(C)$ is not semi-continuous, i.e. under specialization it can increase as well as decrease. For example, if $g = 3$ then $\delta(C) = 4$ outside of the locus of hyperelliptic curves, whereas by Halphen's Theorem [14, IV.6.1], $\delta(C) = 6$ if C is hyperelliptic. On the other hand, if $g = 6$ then $\delta(C) \ge 8$ at a generic point $C \in \mathcal{M}_6$, $\delta(C) = 7$ precisely at the locus L_1 of smooth curves of type (3, 4) on a smooth quadric $Q \simeq \mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^3$, and $\delta(C) = 5$ precisely at the locus L_2 of smooth plane quintics (see [14, IV.6.4.2]); we have $\dim \mathcal{M}_6 = 15$, $\dim L_1 = 11$ and $\dim L_2 = 13$.

Corollary 2.4. *Let $V = C \times C \hookrightarrow \mathbb{P}^N$ be as in Proposition 2.2. Then $g_{\overline{S}} \ge 225$ for $g = 2$, $g_{\overline{S}} \ge 331$ for $g = 3$, and*

$$g_{\overline{S}} \ge 17g^2 + 81g + 74 \quad \textit{for} \quad g \ge 4\,.$$

In fact,

$$g_{\overline{S}} > 4\sqrt{2}g^{\frac{5}{2}} \quad \textit{for} \quad g \gg 0\,. \tag{25}$$

Proof. It follows from (17) that

$$g_{\overline{S}} \ge \frac{1}{2}(2n^2 - 17n + 2) + 2\delta(C)(n-12)(g-1) - 7(g-1)^2\,. \tag{26}$$

The result for $g=2$, $g=3$, $g\geq 4$, resp., follows by substituting the bounds $n\geq 18,\ 20,\ 2g+16$, resp., and $\delta(C)\geq 5,\ 4,\ 5$, resp., into (26). To verify (25), we substitute $n\geq 2g+5\sqrt{2g}-2$ and (24) into (26) to obtain

$$g_{\overline{S}}\geq 4\sqrt{2}g^{\frac{5}{2}}+17g^2+O(g^{\frac{3}{2}})\,.$$

□

Corollaries 1.2 and 2.4 lead to the following conclusion.

Theorem 2.5. *Let $V=C\times C$ where C is a smooth projective curve of genus $g\geq 2$. Then a generic projection $\overline{V}\subset\mathbb{P}^3$ of the image of any projective embedding $V\hookrightarrow\mathbb{P}^N$ is a Kobayashi hyperbolic surface with C-hyperbolic normalization.*

Example 2.6. Let C be a non-hyperelliptic smooth projective curve of genus $g=3$. It is well known that the canonical divisor K_C of degree $2g-2=4$ is very ample, and it provides the canonical embedding $C\hookrightarrow\mathbb{P}^2$ onto a smooth plane quartic. In turn, the canonical divisor $K=K_V\equiv 4E+4F\equiv D_{4,0}$ of the cartesian square $V=C\times C$ yields the Segre embedding $V\hookrightarrow\mathbb{P}^8$ onto a smooth surface of degree $n=K^2=32$.

In the case where C has genus 2, any effective divisor of degree 5 gives an embedding $C\hookrightarrow\mathbb{P}^3$. Thus, the divisor $D_{5,0}$ on the cartesian square $V=C\times C$ yields the embedding $V\hookrightarrow\mathbb{P}^{15}$ onto a smooth surface of degree $n=50$.

In each of these two cases, formulas (14)–(20) give us the following numerical data for a generic projection of $V=C\times C$ to $\mathbb{P}^3$:

$$g=2, H\equiv D_{5,0}\Rightarrow n=50, g_H=36, d_{\overline{S}}=1140, g_{\overline{S}}=2449, p=384, t=15784, b=948$$
$$g=3, H\equiv D_{4,0}\Rightarrow n=32, g_H=33, d_{\overline{S}}=432,\ \ g_{\overline{S}}=1045, p=336, t=3280,\ \ b=264$$

Remark 2.7. A very ample divisor $H\equiv D_{a,a',k}\in \mathrm{NS}(V)$ defines an embedding $V\hookrightarrow\mathbb{P}^4$ iff $b=0$, i.e. in view of (20), iff

$$n(n-10)=10(g-1)(a+a')+4(g-1)^2\,. \tag{27}$$

But we don't know whether there is a smooth surface in $\mathbb{P}^4$ isomorphic to $V=C\times C$.[2] Indeed Van de Ven conjectured that the irregularity of a smooth surface in $\mathbb{P}^4$ can be at most 2 [33, Problem 8], which would imply that V cannot be embedded in $\mathbb{P}^4$ and thus $b>0$. Any better estimate of b from below would lead to a better lower estimate for $\delta(C\times C)$.

The next proposition shows the nonexistence of such an embedding at least in certain cases (cf. [7, 16]).

Proposition 2.8. *Let C be a generic curve of genus $g\geq 2$, and let $H\equiv D_{a,a',k}$ be a very ample divisor on the cartesian square $V=C\times C$. Then H cannot provide an embedding $V\hookrightarrow\mathbb{P}^4$ in each of the following cases:*

(a) *H is a non-special divisor (i.e., $h^1(\mathcal{O}_V(H))=0$) and $g\leq 13$;*

(b) *H is a non-special divisor and $a+a'\geq g+1$;*

(c) *$a=a'$ and $g=2$.*

[2] For $g=2$, the first integer solution of (27) which also satisfies the necessary conditions for very ampleness given in Remark 4.5 below is $a=511,\ a'=79,\ |k|=142$; but the divisor $D_{511,79,\pm 142}$ is probably not very ample.

Proof. (a) Suppose H were to give an embedding $V \hookrightarrow \mathbb{P}^4$. By Severi's Theorem [40] (see also [14, ex. IV.3.11.b]), a smooth surface X in $\mathbb{P}^4$ is linearly normal, i.e. $h^0(\mathcal{O}_X(1)) = 5$, unless it is a projection of the second Veronese embedding $\mathbb{P}^2 \hookrightarrow \mathbb{P}^5$. Hence $h^0(\mathcal{O}_V(H)) = 5$, and thus by the Riemann-Roch and Adjunction Formulas together with (15) and (27), we obtain:

$$\begin{aligned}\chi(\mathcal{O}(H)) &= 5 - h^1(\mathcal{O}(H)) + h^2(\mathcal{O}(H)) = \frac{1}{2}H\cdot(H-K) + \frac{c_1^2+c_2}{12}\\ &= n - g_H + 1 + (g-1)^2 = \frac{1}{2}n - (g-1)(a+a') + (g-1)^2\\ &= \frac{15n - n^2 + 14(g-1)^2}{10}\,. \qquad (28)\end{aligned}$$

Assuming that the embedding $V \hookrightarrow \mathbb{P}^4$ is non-special, i.e. $h^1(\mathcal{O}(H)) = 0$ (cf. [16]), from (28) we get the inequality

$$(n-5)(n-10) \le 14(g-1)^2\,. \qquad (29)$$

On the other hand, from (21) and (24), we obtain:

g	$\delta(C) \ge$	$n \ge$
2	5	18
3	4	20
4	5	24
5	5	28
6	5	30
7	6	36

g	$\delta(C) \ge$	$n \ge$
8	6	38
9	6	40
10	6	44
11	7	48
12	7	50
13	7	54

The lower bounds for n from this table contradict (29). This proves (a).

(b) If H is an embedding and (b) were to hold, then by (27) and (29), we would have

$$5(n-10) = n(n-10) - (n-5)(n-10) \ge 10(g-1)(a+a'-g+1) \ge 20(g-1) \qquad (30)$$

and thus by (29) and (30),

$$14(g-1)^2 \ge (n-10)^2 \ge 16(g-1)^2\,,$$

a contradiction.

For the proof of (c) see Corollary 4.7(d) below. □

3. The symmetric square of a curve of genus ≥ 3

As in §2, we let C be a curve of genus g, except we now assume that $g \ge 3$. We still let $V = C \times C$ denote the cartesian square, and we let $V' := C_2 = C\times C/\{1,\sigma\}$ denote the symmetric square, where $\sigma : V \to V$ is the involution $\sigma(z,w) = (w,z)$. More generally, consider the $d-$th symmetric power C_d, $d > 0$, of C. Recall that C_d can be identified with the space of effective divisors of degree d on C, and the fibres of the Abel-Jacobi morphism $u_d : C_d \to J_C$ into the Jacobian variety J_C can be identified with the complete degree d linear systems on C.

By a *Brody curve* in a compact hermitian complex manifold M we mean a non-constant holomorphic map $f : \mathbb{C} \to M$ with bounded derivative $f' : \mathbb{C} \to TM$ (see e.g. [45]). Recall that by the Brody Theorem [5], M is hyperbolic iff it does not admit any Brody curve. In the next proposition, we use the same approach as in M. Green [12] to describe Brody curves in the symmetric powers C_d of a smooth projective curve C.

Proposition 3.1. *Let $f : \mathbb{C} \to C_d$ be a Brody curve. Then either*

(a) *$u_d \circ f(\mathbb{C})$ =const, and then there exists a linear pencil g^1_d on C, or*

(b) *the image $u_d \circ f(\mathbb{C})$ lies on a smooth abelian subvariety of $W_d := u_d(C_d) \subset J_C$.*

Proof. (a) If $u_d \circ f(\mathbb{C})$ =const, i.e. $f(\mathbb{C})$ is contained in a fiber of u_d, then this fiber represents a complete linear system g^r_d on C of positive dimension r; in particular, it contains a linear pencil g^1_d.

(b) Otherwise, $u_d \circ f : \mathbb{C} \to J_C$ is a Brody curve, and so is the lift $\widetilde{f} : \mathbb{C} \to \mathbb{C}^g$. As noted in [12], the derivative $\widetilde{f}'$, being bounded, must be constant; that is, $\widetilde{f}$ is an embedding onto an affine line in $\mathbb{C}^g$, and the closure of the image $u_d \circ f(\mathbb{C})$ in J_C is a shifted subtorus contained in the subvariety $W_d = u_d(C_d)$. □

Corollary 3.2. *For a generic curve C of genus $g \geq 3$ and for any $d < g/2 + 1$ the symmetric power C_d is Kobayashi hyperbolic, while for any genus g curve C and any $d \geq g/2 + 1$ it is not hyperbolic.*

Proof. By Theorems 1.1 and 1.5 in [3, Ch. V], for $d \geq \frac{g}{2} + 1$ any genus g curve C possesses a linear pencil g^1_d (indeed, for the Brill-Noether number $\rho = \rho(g, d, 1) = 2d - g - 2$, we have $\rho \geq 0$). Hence, the symmetric power C_d contains a smooth rational curve, and so, it fails being Kobayashi hyperbolic. While for $d < \frac{g}{2} + 1$ (i.e. for $\rho < 0$) a genus g curve C with generic moduli has no g^1_d. Furthermore, its Jacobian variety is simple, i.e. it does not contain proper abelian subvarieties. Indeed, it is known that in the moduli space $\mathcal{M}_g$ of genus g curves, the locus of curves with non-simple Jacobian varieties is a countable union of subvarieties of codimension at least $g-1$; see [7, (3.4)-(3.7)]. Hence by Proposition 3.1 and Brody's Theorem, the variety C_d, having no Brody curve, must be Kobayashi hyperbolic. □

In the case of the symmetric square $V = C_2$ of a genus g curve C, the next proposition provides more precise information. Recall that $V' \simeq \mathbb{P}^2$ if $g = 0$, V' is a ruled surface over C if $g = 1$, and V' is the Jacobian J_C blown up at a point if $g = 2$, so that the symmetric square of a genus $g \leq 2$ curve cannot be hyperbolic.

Proposition 3.3. *Let C be a smooth projective curve. Then the following are equivalent:*

(i) *the surface C_2 is Kobayashi hyperbolic;*

(ii) *C_2 does not contain any rational or elliptic curves;*

(iii) *the curve C is neither hyperelliptic nor bielliptic.*

Proposition 3.3 is an immediate consequence of Brody's Theorem, Proposition 3.1 and the following two lemmas, the first of which is well known and the second due to Abramovich and Harris [1, Th. 3].

Lemma 3.4. *The symmetric square $V' = C_2$ contains a rational curve Γ iff the curve C is hyperelliptic.*

Proof. Since the Jacobian variety J_C has no rational curve, Γ should coincide with a fibre of the Abel-Jacobi map $u_2 : C_2 \to J_C$, thus representing a pencil g^1_2 on C. Therefore, C is a hyperelliptic curve. The converse implication is easy. □

Lemma 3.5. (Abramovich-Harris) *The symmetric square $V' = C_2$ of a genus $g \geq 3$ curve C contains an elliptic curve Γ iff the curve C is bielliptic, i.e. there exists a $2:1$ morphism $f : C \to T$ of C onto a smooth elliptic curve T.*

Proof. [1, Th. 3]. □

Remark 3.6. (a) We easily see that the curve Γ in Lemma 3.4 is given as $\Gamma = f^\vee(\mathbb{P}^1)$, where $f : C \to \mathbb{P}^1$ is a $2:1$ morphism and $f^\vee : \mathbb{P}^1 \ni z \longmapsto f^*(z) \in C_2$. It can also be shown using [2, (3.2)] that for $g \geq 4$, the curve Γ in Lemma 3.5 is similarly given as $\Gamma = f^\vee(T)$. In Proposition 3.18 below, we show that for a curve Γ as above, $\Gamma^2 = 1 - g < 0$.

Observe that for a genus 2 curve C, the symmetric square $V' = C_2$ contains an elliptic curve iff V' is an elliptic surface (and hence, iff the Jacobian J_C is isogenous to a product of two elliptic curves). Thus, if $g = 2$ there may exist smooth elliptic curves Γ on C_2 with $\Gamma^2 = 0$ and so these curves are not of the form $f^\vee(T)$.

(b) Since the hyperelliptic involution is unique, there can be only one rational curve on V', for $g \geq 2$. In contrast, if C is bielliptic and $2 \leq g \leq 5$, there can be several elliptic curves on V'. For instance, the Fermat quartic $C = \{x^4 + y^4 + z^4 = 0\}$ in $\mathbb{P}^2$ admits 15 different involutions ($(x : y : z) \longmapsto (-x : y : z)$, $(x : y : z) \longmapsto (\zeta y : \zeta^{-1}x : z)$, $\zeta^4 = 1$, etc.), which are elements of order 2 in the automorphism group $\operatorname{Aut} C \simeq (\mathbb{Z}/4\mathbb{Z})^2 \rtimes S_3$ (see [22, pp. 274–275]) and which provide 15 different elliptic curves on $V' = C_2$.

Another example is the Klein quartic $C = \{xy^3 + yz^3 + x^3z = 0\}$ in $\mathbb{P}^2$ [19, Ch. 8]. The automorphism group $\operatorname{Aut} C \simeq \mathrm{PSL}_3(\mathbb{Z}/2\mathbb{Z}) \simeq \mathrm{PSL}_2(\mathbb{Z}/7\mathbb{Z})$ being a simple group of order 168 (which is the maximal possible one for a genus 3 curve) [23], it is generated by 21 reflections, i.e. elements of order 2, which form a conjugacy class. Each of them defines a bielliptic involution on C and whence, an elliptic curve of the above type on the surface $V' = C_2$.

See also [3, VI.F12-F14] for an example of a genus 5 curve bielliptic in ten different ways, so that $V' = C_2$ possesses 10 different elliptic curves. However [3, VIII.C-2], a genus $g \geq 6$ curve C can have only one bielliptic structure, and hence $V' = C_2$ may possess only one elliptic curve Γ.

(c) As was noted in [1, Th. 3] (see also [3, VIII.C-1]), a bielliptic hyperelliptic curve C is a type $(2, 4)$ curve on a smooth quadric, and therefore of genus at most 3. (Indeed, a birational embedding $C \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ is given by the pencils g^1_2 and g^1_4 on C where the first one is the hyperelliptic one, and the second one is provided by a hyperelliptic pencil g^1_2 on an elliptic curve T under a $2:1$ morphism $C \to T$.) For instance [3, I.H-6], $C = \{y^2z^4 - x^6 + z^6 = 0\} \subset \mathbb{P}^2$ is such a curve of genus 2.

If R is a genus 2 curve, then to any index 2 subgroup of the fundamental group $\pi_1(R)$ there corresponds a nonramified Galois $2:1$ covering $C \to R$ where C is a genus 3 curve. Let $\sigma \in \operatorname{Aut} C$ be the generator of the Galois group $\mathbb{Z}/2\mathbb{Z}$. Since σ acts freely on C, and any involution of a smooth plane quartic has fixed points, C is hyperelliptic; denote by i the hyperelliptic involution on C. Then the involution $\sigma' := \sigma \circ i = i \circ \sigma \in \mathrm{Aut}\, C$ has 4 fixed points (that is, the union of two common orbits of i and σ; they come from the two fixed points of the induced action of σ on the canonical model of C). Thus σ' is a bielliptic involution. Therefore (Farkas-Accola; see [22, Lemma 8]), any genus 3 curve which dominates a genus 2 curve is hyperelliptic and bielliptic. For example [22], $C = \{y^2 = (x^2-1)(x^2-\lambda^2)(x^2-\mu^2)(x^2-\lambda^2/\mu^2)\},\ \ \lambda^2 \neq \mu^2,\ \lambda^2,\, \mu^2 \in \mathbb{C} \setminus \{0,\, 1\}$, is such a curve.

Example 3.7. A genus 6 nodal plane sextic C is neither hyperelliptic, nor bielliptic [3, V.A-12]. Hence by Proposition 3.3, $V' = C_2$ is a hyperbolic surface.

As for examples of plane quartics with the same property, a genus 3 curve C is hyperelliptic or bielliptic iff it possesses a holomorphic involution, as noted in Remark 3.6(c) above. Or equivalently, iff the automorphism group $\mathrm{Aut}\, C$ contains an element of order 2, that is, iff $\mathrm{Aut}\, C$ is of even order. In fact, in [22] all the genus 3 curves were classified according to the order of the group $\mathrm{Aut}\, C$. E.g., this order equals 9 for the plane quartic $C = \{x^4 - xz^3 - y^3 z = 0\}$ and whence, C is neither hyperelliptic nor bielliptic.

Next we show that the symmetric power $C_d,\ d \geq 2$, of a curve C is never C-hyperbolic and indeed is just the opposite in the sense of the following definition.

Definition: A complex space X is called *Liouville* if it carries no nonconstant bounded holomorphic functions [25]. We call it *super-Liouville* if the universal cover over X is Liouville.

Evidently, any compact complex space is Liouville; in fact, this property is opposite to being Carathéodory hyperbolic. In turn, being super-Liouville is opposite of being C-hyperbolic. By Lin's Theorem [25], any compact complex space with nilpotent (or nilpotent-by-finite) fundamental group is super-Liouville.

In particular, this is so for complex tori, as well as for manifolds birational to complex tori. Thus, any cover over the symmetric power C_g of a genus g curve C is Liouville (indeed, the Abel-Jacobi map $u_g : C_g \to J_C$ is birational).

Another example of a super-Liouville variety is provided by the theta-divisor $\Theta \subset J_C$ of the Jacobian of a genus $g \geq 3$ curve C with general moduli. Indeed, the divisor Θ being ample (see [31]), by the Lefschetz hyperplane section theorem (see for instance, [29]), the embedding $\Theta \hookrightarrow J_C$ induces an isomorphism of the fundamental groups. As for the symmetric powers, we have the following statement:

Lemma 3.8. *For any curve C of genus g and for any $d \geq 2$ we have $\pi_1(C_d) \simeq \mathbb{Z}^{2g}$.*

Proof. It is known that $H_1(C_d; \mathbb{Z}) \simeq \mathbb{Z}^{2g}$ [26]. Therefore, it suffices to show that the fundamental group $\pi_1(C_d)$ is abelian. Let $\pi : C \times \cdots \times C \to C_d$ be the natural projection. We first show:

Claim: $\pi_* : \pi_1(C \times \cdots \times C) \to \pi_1(C_d)$ is a surjection.

We denote by $\Delta = \bigcup_{i \neq j} \Delta_{ij}$ the union of the diagonal hypersurfaces in the cartesian power $C^d = C \times \cdots \times C$, and we let $\Delta' = \pi(\Delta),\ \ \pi^0 = \pi \,|\, (C^d \setminus \Delta) : C^d \setminus \Delta \to C_d \setminus \Delta'$.

Since π^0 is a non-ramified covering, we have the exact sequence

$$\mathbf{1} \longrightarrow \pi_1(C^d \setminus \Delta) \xrightarrow{\pi^0_*} \pi_1(C_d \setminus \Delta') \xrightarrow{\rho} S_d \longrightarrow \mathbf{1}$$

where S_d stands for the symmetric group.[3] This sequence can be extended to the commutative diagram

$$\begin{tikzcd}
 & & \pi_1(C^d \setminus \Delta) \arrow[r, "i_*"] \arrow[d, "\pi^0_*"] & \pi_1(C^d) \arrow[r] \arrow[d, "\pi_*"] & \mathbf{1} \\
\mathbf{1} \arrow[r] & N' \arrow[r] \arrow[d, "\rho|N'"] & \pi_1(C_d \setminus \Delta') \arrow[r, "i'_*"] \arrow[d, "\rho"] & \pi_1(C_d) \arrow[r] \arrow[d] & \mathbf{1} \\
\mathbf{1} \arrow[r] & S_d \arrow[r, "\mathrm{id}"] \arrow[d] & S_d \arrow[r] \arrow[d] & \mathbf{1} & \\
 & \mathbf{1} & \mathbf{1} & &
\end{tikzcd}$$

where $i : C^d \setminus \Delta \hookrightarrow C^d$, $i' : C_d \setminus \Delta' \hookrightarrow C_d$ are the natural embeddings and $N' = \ker i'_*$. This kernel N' is generated, as a normal subgroup, by a vanishing loop α' of the diagonal Δ' in the manifold C_d. We note that $\rho(\alpha')$ is a transposition in S_d. Indeed, the kernel of i_* is generated, as a normal subgroup, by vanishing loops α_{ij} of the diagonals Δ_{ij} in C^d; if we choose α' so that, for instance, $\pi^0_*(\alpha_{12}) = (\alpha')^2$, then $\rho(\alpha') = (1\,2)$. Since $\rho(N')$ is a normal subgroup of S_d containing a transposition, we conclude that $\rho(N') = S_d$; i.e., the first two columns, as well as the rows, of the diagram are exact. By the usual diagram chasing (as in the "nine lemma"), we conclude that the last column is exact, too; i.e., π_* is surjective. This proves the claim.

Now we can prove that the group $\pi_1(C_d)$ is abelian. Indeed, let $\{a_i^{(k)}\}_{1\le i\le 2g}$ denote the set of standard generators of the k-th factor of the fundamental group $\pi_1(C^d) \simeq \pi_1(C) \times \cdots \times \pi_1(C)$; we have $[a_i^{(k)}, a_j^{(l)}] = 1$ for all $i, j = 1, \ldots, 2g$, if $k \ne l$. Hence, since $d \ge 2$ and $\pi_*(a_i^{(k)}) = \pi_*(a_i^{(l)}) =: a'_i \in \pi_1(C_d)$, $i = 1, \ldots, 2g$, we have $1 = [\pi_*(a_i^{(k)}), \pi_*(a_j^{(l)})] = [a'_i, a'_j]$ for all $i, j = 1, \ldots, 2g$. But by the above claim,

$$\{a'_i = \pi_*(a_i^{(k)})\}_{1\le i\le 2g}$$

is a set of generators of the group $\pi_1(C_d)$. Thus, this group is, indeed, abelian. □

Corollary 3.9. *For any curve C and any $d \ge 2$ the symmetric power C_d of C is super-Liouville, that is, any non-ramified covering over C_d is Liouville.*

[3]The group $\pi_1(C_d \setminus \Delta') = B_{g,d}$ is called the d-th braid group of a genus g compact Riemann surface.

Next we show how to produce examples of hyperbolic surfaces in $\mathbb{P}^3$ starting with the symmetric square of a generic genus $g \geq 3$ curve C. To obtain these examples, we need to know the relations between the divisor theories on the cartesian square $V = C \times C$ and on the symmetric square $V' = C_2$. We let $\pi : V \to V'$ denote the projection. If D' is a divisor on V', it follows from the push-pull formula that

$$\pi_* \pi^* D' = 2D' \qquad \text{and} \qquad \pi^* D' \cdot A = D' \cdot \pi_* A\,, \tag{31}$$

for any divisor A on V. Thus, if D_1, D_2 are divisors on V', we have

$$\pi^* D_1 \cdot \pi^* D_2 = 2 D_1 \cdot D_2\,. \tag{32}$$

We let E', Δ' be the divisors on V' given by $\pi^* E' = E + F$, $\pi^* \Delta' = 2\Delta$. It is well known that (when C has general moduli) the Neron-Severi group $\mathrm{NS}(V')$ is generated by the classes of E' and Θ' where Θ' is the pull-back of the theta divisor Θ on J_C by the Abel-Jacobi map u_2 (see [3, 13]).[4] Furthermore [26] (see also [3, VIII.(5.4)], [21]),

$$\Delta' \equiv (2g+2)E' - 2\Theta'\,, \tag{33}$$

and hence E', $\frac{1}{2}\Delta'$ generate $\mathrm{NS}(V')$, where $\frac{1}{2}\Delta' \equiv (g+1)E' - \Theta'$. We write

$$D'_{a,k} = (a+k)E' - \frac{k}{2}\Delta'\,, \tag{34}$$

so that $E' \equiv D'_{1,0}$, $\Delta' \equiv 2D'_{1,-1}$, $\Theta' \equiv D'_{g,1}$, and also

$$\pi^* D'_{a,k} = D_{a,k}\,. \tag{35}$$

By (32), we have

$$D'_{a,k} \cdot D'_{c,l} = ac - gkl\,. \tag{36}$$

We now compute $K := K_{V'}$. Since Δ is the ramification divisor of the branched cover $\pi : C \times C \to C_2$ we have

$$\pi^* K = K_V - \Delta \equiv D_{2g-3,1}\,,$$

and so by (34) (cf. [3, VIII.(5.4)]),

$$K \equiv D'_{2g-3,1} \equiv (2g-2)E' - \frac{1}{2}\Delta'\,. \tag{37}$$

As for the topological invariants of the surface V', we have:

$$\begin{aligned} c_1^2 = K^2 &= (2g-3)^2 - g = 4g^2 - 13g + 9 && (38)\\ c_2 = e(V') &= \frac{1}{2}[e(V) + e(\Delta)] \\ &= \frac{1}{2}[e(C)^2 + e(C)] \\ &= (2g-3)(g-1) = 2g^2 - 5g + 3\,. && (39) \end{aligned}$$

Hence,

$$\chi(\mathcal{O}_{V'}) = \frac{c_1^2 + c_2}{12} = \frac{(g-1)(g-2)}{2}\,. \tag{40}$$

[4] Recall that by Lefschetz' Theorem (see [37] for a modern proof), for a curve C with general moduli, the Neron-Severi group $\mathrm{NS}(J_C)$ is a free abelian group generated by the class of the theta divisor Θ.

Next we describe some of the very ample divisors on the surface V'. For the projection $\pi : V \to V'$ we write $\pi(z,w) = \{z,w\}$. If A is a divisor on C, we let $A^{(2)}$ be the (unique) divisor on V' given by

$$\pi^* A^{(2)} = A \times C + C \times A\,.$$

If $\widetilde{A}$ is linearly equivalent to A, then $\widetilde{A}^{(2)}$ is linearly equivalent to $A^{(2)}$; thus for a holomorphic line bundle $L \to C$, we have a unique line bundle $L^{(2)} \to V'$, and $\pi^* L^{(2)} = L \widehat{\otimes} L := \mathrm{pr}_1^* L + \mathrm{pr}_2^* L$.

It follows from the Nakai-Moishezon criterion and (32), (31) that D is ample iff $\pi^* D$ is. Thus if L is a line bundle on C, then L is ample iff $L \widehat{\otimes} L$ is ample iff $L^{(2)}$ is ample. We further note that the latter holds also for "very ample":

Lemma 3.10. *$L \to C$ is very ample iff $L^{(2)} \to V'$ is very ample.*

Proof. Let $\iota : C \to V'$ be given by $\iota(z) = \{z, x_0\}$. If $L^{(2)}$ is very ample, then so is $L = \iota^* L^{(2)}$. Conversely, suppose that L is very ample. We first show that $L^{(2)}$ separates points. For $s,t \in H^0(C,L)$, we let $s \odot t$ denote the section of $L^{(2)}$ given by

$$s \odot t(\{z,w\}) = s(z)t(w) + s(w)t(z)\,.$$

Let $\zeta \neq \eta \in V'$ be two arbitrary points. We must find a section $\lambda \in H^0(V', L^{(2)})$ such that $\lambda(\zeta) = 0,\ \lambda(\eta) \neq 0$. First suppose that $\zeta = \{z_1, w_1\},\ z_1 \notin \{z_2, w_2\} = \eta$. Since L is very ample, we can find a section $s \in H^0(C,L)$ such that $s(z_1) = 0,\ s(z_2) \neq 0,\ s(w_2) \neq 0$. Then $\lambda = s \odot s$ is our desired section. The other possibility is that $\zeta = \{z_1, z_1\},\ \eta = \{z_1, z_2\}$. In this case, we let $\lambda = s_1 \odot s_2$ where $s_i(z_j)$ vanishes if $i = j$ and is nonzero if $i \neq j$. Thus, $L^{(2)}$ separates points.

Next consider an arbitrary nonzero tangent vector $X \in T_\zeta(V'),\ \zeta = \{z_0, w_0\} \in V'$. To complete the proof, we must find a section $\lambda \in H^0(V', L^{(2)})$ such that $\lambda(\zeta) = 0$ but the 1-jet $\mathcal{J}_1\lambda$ of λ does not vanish in the X direction. First assume that $z_0 \neq w_0$ and write $X = c_1 \frac{\partial}{\partial z}|_{z_0} + c_2 \frac{\partial}{\partial w}|_{w_0}$; we may assume that $c_1 \neq 0$. We can then let $\lambda = s \odot s$, where s is a section in $H^0(C,L)$ such that $s(z_0) = 0,\ s(w_0) \neq 0$, and $\mathcal{J}_1 s|_{z_0} \neq 0$. Now consider the case $\zeta = \{z_0, z_0\}$. Choose a local frame e for L at z_0, and use a local coordinate centered at z_0 so that we may write $z_0 = 0$. Since $(z+w, zw)$ are local coordinates at $\zeta \in V'$, it suffices to find $\lambda \in H^0(V', L^{(2)})$ such that

$$\lambda = (0 + c_1(z+w) + c_2 zw + \cdots)e \odot e \tag{41}$$

for arbitrary $c_1, c_2 \in \mathbb{C}$. Choose $s,t \in H^0(C,L)$ with $\mathcal{J}_1 s = ze$, $\mathcal{J}_1 t = (c_1 + \frac{c_2}{2}z)e$. Then the section $s \odot t \in H^0(V', L^{(2)})$ satisfies (41). Thus $L^{(2)}$ is very ample. □

Corollary 3.11. *Let C be a non-hyperelliptic curve of genus $g \geq 3$. Then the divisor $K_C^{(2)} \equiv D'_{2g-2,0}$ on the symmetric square $V' = C_2$ of C is very ample.*

Proof. Indeed, since the canonical divisor K_C is very ample, by 3.10, $K_C^{(2)}$ is very ample, too. □

Proposition 3.12. *For a very ample divisor $H \equiv D'_{a,k}$ on the surface V', in the notation from Section 1, we have:*

$$n = a^2 - gk^2 \tag{42}$$

$$g_H = \frac{1}{2}[n + 2 + (2g-3)a - gk] \tag{43}$$

$$g_{\overline{S}} = \frac{1}{2}(n^2 - 7n - 7g^2 + 25g + 8) + (n-12)g_H \tag{44}$$

$$b = \frac{1}{2}n(n-5) - 5(g_H - 1) - (g-1)(g-3) \tag{45}$$

Proof. These follow as before from (5), (8), (9) and the above formulas (34)–(40). For example, (43) follows from the adjunction formula $2g_H - 2 = H \cdot K + n$, where

$$H \cdot K = D'_{a,k} \cdot D'_{2g-3,1} = (2g-3)a - gk\,.$$

□

Theorem 3.13. *Let C be a genus 3 curve that is neither hyperelliptic nor bielliptic, and consider the divisor $H = K_C^{(2)}$ on $V' = C_2$. Then H is very ample and a generic projection $\overline{V}' \subset \mathbb{P}^3$ of the image of the projective embedding $V' \hookrightarrow \mathbb{P}^N$ given by H is a Kobayashi hyperbolic surface of degree 16 in $\mathbb{P}^3$.*

Proof. By Corollary 3.11, the divisor $K_C^{(2)} \equiv D'_{4,0}$ is very ample. Furthermore by (42)–(44), we have $n = 16$, $g_H = 15$, $g_{\overline{S}} = 142$. Now the conclusion follows from Corollary 1.2 and Proposition 3.3. □

Remark 3.14. By [44], to obtain a *smooth* hyperbolic surface in $\mathbb{P}^3$ of degree 16, it is enough to perturb a little the coefficients of the equation which defines the singular hyperbolic surface provided by Theorem 3.13.

The following proposition tells us that actually, 16 is the lowest possible degree of a projective embedding of the symmetric square $V' = C_2$ of a generic genus $g \geq 3$ curve C.

Proposition 3.15. *Let C be a genus $g \geq 3$ curve with general moduli. Then $\delta(C_2) = 16$ for $g = 3$, $\delta(C_2) \geq 20$ for $g = 4$, and*

$$\delta(C_2) > \sqrt{2g(g+11)} + 5 \quad \textit{for} \quad g \geq 5\,. \tag{46}$$

Proof. Let $H \equiv D'_{a,k}$ be a hyperplane section of $V' = C_2 \hookrightarrow \mathbb{P}^N$. By (42), $|k| < a/\sqrt{g}$. Since $a = \deg \iota^* H \geq \delta(C)$, where $\delta(C)$ is as in Proposition 2.2, it then follows from (43) and (24) that

$$g_H > \frac{1}{2}\left[n + 2 + (2g - \sqrt{g} - 3)\,\delta(C)\right]\,. \tag{47}$$

On the other hand, from (45) and the fact that b is nonnegative, we have the inequality

$$g_H \leq \frac{n^2 - 5n + 10 - 2(g-1)(g-3)}{10}\,. \tag{48}$$

Combining (47) and (48), we obtain

$$(n-5)^2 > 2g^2 - 8g + 31 + 5\delta(C)(2g - \sqrt{g} - 3)\,. \tag{49}$$

We begin with the case $g = 3$. Then $\delta(C) = 4$, and (49) yields $(n-5)^2 > 50$ or $n \geq 13$. But the only solutions of the diophantine equation $3k^2 = a^2 - n$ (provided by (42)) with $n = 13, 14, 15$ and $0 < a \leq 11$ are $n = 13$, $(a,k) = (4,1), (5,2)$ and $(11,6)$. In the first two cases $g_H = 12$ and in the third case $g_H = 15$. On the other hand, (48) with $n = 13$, $g = 3$ yields $g_H \leq 11$. Therefore, if $n \leq 15$, we can replace $\delta(C)$ with 12 in (49), which yields

$$100 \geq (n-5)^2 > 25 + 60(3 - \sqrt{3}) > 100\,,$$

a contradiction. Hence $\delta(C \times C) \geq 16$. However, Theorem 3.13 implies that $\delta(C \times C) \leq 16$.

If $g \geq 4$, then $\delta(C) \geq 5$ and (49) yields

$$(n-5)^2 > 2g^2 + 42g - 25\sqrt{g} - 44\,. \tag{50}$$

For $g = 4$, (50) gives $(n-5)^2 > 106$ or $n \geq 16$. But for $g = 4$, equation (42) becomes

$$(a - 2k)(a + 2k) = n\,,$$

which has no integer solutions for $16 \leq n \leq 19$. Thus $n \geq 20$.

For $g \geq 5$, (50) and the inequality $25\sqrt{g} + 44 < 20g$ yields (46). □

Corollary 3.16. *Let C be a genus $g \geq 3$ curve with general moduli. Then for any projective embedding $C_2 \hookrightarrow \mathbb{P}^N$, we have $g_{\overline{S}} \geq 130$.*

Proof. For the case $g = 3$, using (47) with $n \geq 16$ and $\delta(C) = 4$, we get $g_H > 15 - 2\sqrt{3} > 11.5$ (i.e., $g_H \geq 12$) and thus by (44), $g_{\overline{S}} \geq 130$. Similarly for $g = 4$, using $n \geq 20$ and $\delta(C) \geq 5$, we get $g_H \geq 19$ and $g_{\overline{S}} \geq 280$.

We now assume that $g \geq 5$. From (46), we obtain

$$n > \sqrt{2}\, g + 10\,. \tag{51}$$

The inequalities (47), (24) and (51) yield

$$g_H > \sqrt{2}\, g^{3/2} + \frac{3}{2}g - \frac{3}{4}(1 + 2\sqrt{2})g^{1/2} + \frac{15}{4} > \sqrt{2}\, g^{3/2} + 4\,. \tag{52}$$

It then follows from (44), (51), and (52) that

$$g_{\overline{S}} > 2g^{5/2} - \frac{5}{2}g^2 - 2\sqrt{2}\, g^{3/2} + \frac{21\sqrt{2} + 25}{2}g + 11 \geq 165\,.$$

□

Finally, from Corollaries 1.2, 3.16 and Propositions 3.3, 3.15, we obtain the following theorem.

Theorem 3.17. *Let $V' = C_2$ where C is a smooth projective curve of genus $g \geq 3$ with general moduli. Then a generic projection $\overline{V}' \subset \mathbb{P}^3$ of the image of any projective embedding $V' \hookrightarrow \mathbb{P}^N$ is a Kobayashi hyperbolic surface of degree $n \geq 16$.*

As an aside, we have the following observation related to Lemma 3.4 and Remark 3.6 above:

Proposition 3.18. *Let $f : C \to R$ be a $2:1$ morphism from a curve C of genus $g \geq 2$ to another curve R, and let $\Gamma = f^\vee(R) \subset C_2$, where $f^\vee$ is as in Remark 3.6 above. Then $\Gamma^2 = 1 - g$, and if R is nonrational, then the Picard number of the surface C_2 is at least 3. This is the case, in particular, if C is bielliptic.*

Proof. It is clear that Γ is a smooth curve isomorphic to R. Thus by the adjunction formula and (37),

$$\Gamma^2 = 2g_R - 2 - K \cdot \Gamma = 2g_R - 2 - (2g-2)\Gamma \cdot E' - \frac{1}{2}\Gamma \cdot \Delta' \,, \tag{53}$$

where g_R denotes the genus of R. Here $\Gamma \cdot E' = 1$ and by the Riemann-Hurwitz formula,

$$\Gamma \cdot \Delta' = 2 + 2g - 4g_R \,. \tag{54}$$

Hence by (53) and (54), $\Gamma^2 = 1 - g$.

To obtain the second statement, suppose on the contrary that E', $\frac{1}{2}\Delta'$ generate $\mathrm{NS}(V')$, and write $\Gamma = D'_{a,k}$. Then $a = \Gamma \cdot E' = 1$ and

$$2 + 2g - 4g_R = \Gamma \cdot \Delta' = D'_{1,k} \cdot D'_{2,-2} = 2 + 2gk \,,$$

so that $k = (g - 2g_R)/g$. Then $1 - g = \Gamma^2 = 1 - (g - 2g_R)^2/g$, which gives $g = g_R$ (recall that $g_R > 0$). But then by (54), $g \geq 2g_R - 1 = 2g - 1$ or $g \leq 1$, contradicting our assumption. $\square$

Remark 3.19. Let C be a genus 3 curve. It is well known (see [3, VI.§4]) that the theta-divisor $\Theta \subset J_C$ is singular iff C is hyperelliptic. According to Lemma 3.4 and Proposition 3.18, if C is hyperelliptic then there is a unique (smooth) rational curve Γ on the surface $V' = C_2$, and $\Gamma^2 = -2$, so that the Abel-Jacobi morphism

$$u_2 \,:\, C_2 \to W_2 \simeq \Theta \subset J_C$$

contracts Γ to a (unique) singular point of type A_1 of the surface $W_2 = \Theta + p,\ p \in J_C$.

4. Divisors on cartesian and symmetric squares of curves of genus ≥ 2

To provide more examples of the hyperbolic surfaces in $\mathbb{P}^3$ given by Theorems 2.5 and 3.17, we must find projective embeddings of V, resp. V', i.e., we must find sufficient conditions for a divisor $H \equiv D_{a,a',k}$, resp. $H \equiv D'_{a,k}$, to be very ample. In this section we use a description of the nef cones on symmetric squares given in Kouvidakis [21] together with Reider's characterization of very ampleness [38] to show that H is very ample in the following cases:

i) genus $C = 2$, $H \equiv D_{a,k} \in \mathrm{NS}(C \times C)$, $a \geq 5$ and $2|k| \leq a - 3$ (Theorem 4.4(e)).
ii) C is a generic curve of genus 3, $H \equiv D_{a,k} \in \mathrm{NS}(C \times C)$, $a \geq 7$, $(a,k) \neq (8,2)$ and $-\frac{1}{3}(a-4) \leq k \leq \frac{5}{9}(a-4)$ (Theorem 4.14).
iii) C is a generic curve of genus 3, $H \equiv D'_{a,k} \in \mathrm{NS}(C_2)$, $a \geq 7$, $(a,k) \neq (7,3)$ or $(9,4)$, and $4 - a \leq 3(k-1) \leq \frac{1}{3}(5a - 16)$ (Theorem 4.16(b$'$)).
iv) C is a generic curve of genus 4, $a \geq 9$ and $9 - a \leq 4k \leq 2a - 10$ (Theorem 4.16(b$''$)).

We let Σ' denote the subgroup of the Neron-Severi group $\mathrm{NS}(V')$ generated by the classes of E', $\frac{1}{2}\Delta'$; for a curve with general moduli, $\Sigma' = \mathrm{NS}(V')$. We begin by restating (using the basis $\{D'_{1,0} = E',\ D'_{0,1} \equiv \Theta' - gE'\}$ of $\Sigma' \otimes \mathbb{Q}$) Kouvidakis's description [21, Thm. 2] of the effective and nef cones on the symmetric square of a generic curve:

Theorem 4.1. (Kouvidakis) *Let $V' = C_2$ be the symmetric square of a genus $g \geq 2$ curve with general moduli. Denote by $\mathrm{EFF}(V')$, resp. $\mathrm{NEF}(V')$, the cone of (the classes of) quasi-effective, resp. nef, $\mathbb{Q}$-divisors in $\Sigma' \otimes \mathbb{Q}$. We also set*

$$\mathcal{E} = \left\{ D'_{a,k} \in \Sigma' \otimes \mathbb{Q} \;\middle|\; \begin{array}{lllll} -a & \leq & k \leq a & \text{if} & g = 2 \\ -a & \leq & k \leq 3a/5 & \text{if} & g = 3 \\ -a & \leq & k \leq a/\sqrt{g} & \text{if} & g \geq 4 \end{array} \right\}$$

and

$$\mathcal{E}' = \left\{ D'_{a,k} \in \Sigma' \otimes \mathbb{Q} \;\middle|\; -a \leq k \leq \frac{a}{\sqrt{g} - 1} \right\}$$

for $g \geq 5$; further,

$$\mathcal{N} = \left\{ D'_{a,k} \in \Sigma' \otimes \mathbb{Q} \;\middle|\; \begin{array}{lllll} -a/2 & \leq & k \leq a/2 & \text{if} & g = 2 \\ -a/3 & \leq & k \leq 5a/9 & \text{if} & g = 3 \\ -a/g & \leq & k \leq a/\sqrt{g} & \text{if} & g \geq 4 \end{array} \right\}$$

and

$$\mathcal{N}' = \left\{ D'_{a,k} \in \Sigma' \otimes \mathbb{Q} \;\middle|\; -\frac{a}{g} \leq k \leq a\frac{\sqrt{g} - 1}{g} \right\}$$

for $g \geq 5$. Then we have:

$$\mathcal{E} \subseteq \mathrm{EFF}(V') \subseteq \mathcal{E}', \qquad \text{resp.} \qquad \mathcal{N}' \subseteq \mathrm{NEF}(V') \subseteq \mathcal{N},$$

and

$$\mathrm{EFF}(V') = \mathcal{E}, \qquad \text{resp.} \qquad \mathrm{NEF}(V') = \mathcal{N},$$

if $g = 2,\ 3$ or if $\sqrt{g} \in \mathbb{Z}$.

Here we call a divisor D *quasi-effective* iff $mD \equiv G$ for some $m > 0$ and for some effective divisor G. Recall that the ample cone (sometimes called the *Kähler cone*) is the interior of the nef cone, and hence it can be described by making the inequalities in the definition of $\mathcal{N}$, $\mathcal{N}'$ strict.

We let $\Sigma^{\mathrm{sym}} = \pi^*\Sigma'$ denote the subgroup of $\Sigma \subset \mathrm{NS}(V)$ generated by the "symmetric" divisors $\{D_{a,k}\}$. The following elementary observation allows us to transfer Kouvidakis's description of the effective and nef (as well as ample) cones to the case of symmetric divisors $D_{a,k} \in \Sigma^{\mathrm{sym}} \subset \mathrm{NS}(V)$.

Proposition 4.2. *Let V, resp. V' be the cartesian, resp. symmetric, square of a smooth projective curve C. Then the class $D_{a,k} \in \Sigma^{\mathrm{sym}} \subset NS(V)$ is quasi-effective, resp. nef, resp. ample, iff $D'_{a,k} \in \Sigma' \subset NS(V')$ is likewise.*

Proof. The proposition is an immediate consequence of the push-pull formulas (31), (35) and the Nakai-Moishezon criterion. □

Let C be a smooth projective curve of genus $g = 2$. Then C is a hyperelliptic curve, and there is a unique hyperelliptic involution $i : C \to C$. The orbits $(p,\ i(p))$ of the involution i are divisors of the canonical class, and the six fixed points of i are the Weierstrass points of the curve C. Let $\varphi = \varphi_{|K_C|} : C \to \mathbb{P}^1$ be the quotient by the involution i. The involution i or, equally, the morphism φ defines the symmetric correspondence (the graph of i)

$$D(i) \subset V = C \times C,\ D(i) := \{(p,\, i(p)) \,|\, p \in C\}$$

of valence 1 and of bidegree (1, 1). Hence,

$$D(i) \equiv D_{1,1} = 2E + 2F - \Delta \in \Sigma\,,$$

whereas $K = K_V \equiv D_{2,0} = 2E + 2F$. We have $\Delta^2 = D(i)^2 = -2$.

To obtain a description of the very ample (as well as some of the globally generated) divisors of type $D_{a,k}$ on the surface V, resp. V', we shall use Reider's characterization of global generation and of very ampleness [38, Theorem 1(ii)] (see also [24]):

Theorem 4.3. (Reider) *Let L be a nef line bundle on a smooth projective surface V such that $L^2 \geq 5$, resp. $L^2 \geq 10$. Then the adjoint line bundle $K + L$ is globally generated, resp. very ample, unless there exists an effective divisor Γ on V which verifies one of the following conditions (i)–(ii), resp. (i′)–(iii′):*

(i) $L \cdot \Gamma = 0$ and $\Gamma^2 = -1$;
(ii) $L \cdot \Gamma = 1$ and $\Gamma^2 = 0$.

(i′) $L \cdot \Gamma = 0$ and $\Gamma^2 = -1$ or -2;
(ii′) $L \cdot \Gamma = 1$ and $\Gamma^2 = -1$ or 0;
(iii′) $L \cdot \Gamma = 2$ and $\Gamma^2 = 0$.

In Theorem 4.4 below, we describe all the quasi-effective, nef, and very ample, as well as some globally generated, divisors of type $D_{a,k}$ on the surface $V = C \times C$, for an *arbitrary* genus 2 curve C. For a *generic* curve C, statements (a)–(c) of Theorem 4.4 and Corollary 4.6 can be obtained (in view of Proposition 4.2) from the genus 2 case of Kouvidakis's Theorem 4.1, but we provide a direct proof for the reader's convenience.

Theorem 4.4. *Let $V = C \times C$ be the cartesian square of a genus 2 curve C. Then a divisor $H \equiv D_{a,k} \in \Sigma^{\mathrm{sym}} \subset \mathrm{NS}(V)$ is*

(a) *quasi-effective iff*

$$|k| \leq a\,; \tag{55}$$

(b) *nef iff*

$$2|k| \leq a\,; \tag{56}$$

under this condition it is also big unless $H \equiv 0$;

(c) *ample iff*

$$2|k| \leq a - 1\,; \tag{57}$$

(d) *ample, globally generated, and non-special if $(a, k) = (4, 0)$ or $a \geq 5$ and*

$$2|k| \leq a - 2\,; \tag{58}$$

(e) *very ample iff $a \geq 5$ and*

$$2|k| \leq a - 3\,. \tag{59}$$

Proof. For a divisor class $\Gamma \in \mathrm{NS}(V)$, we write $\Gamma = \Gamma^\Sigma + \Gamma^\perp$, where $\Gamma^\Sigma \in \Sigma \otimes \mathbb{Q}$ and $\Gamma^\perp \in \mathrm{NS}(V) \otimes \mathbb{Q}$ is such that $\Gamma^\perp \cdot D = 0\ \forall D \in \Sigma$. We first establish the following

Claim: Suppose $\Gamma \in \mathrm{NS}(V)$ is effective and non-zero, and let $\Gamma^\Sigma \equiv D_{c,c',l}$. Then

(i) $c, c', 4l \in \mathbb{Z}$, $c, c' \geq 0$ and $c + c' \geq 1$;

(ii) if $\Gamma \neq \Delta$ or $D(i)$ then

$$4|l| \leq c + c' \, ; \tag{60}$$

(iii) furthermore, $c + c' \geq 2$ if $\Gamma \not\equiv E$ or F.

To verify the claim, we can assume without loss of generality that Γ is an irreducible curve different from $\Delta \equiv D_{1,-1}$, $D(i) \equiv D_{1,1}$, $\mathbb{C} \times \{p\} \equiv E$ and $\{p\} \times C \equiv F$. First, we note that $\Gamma \cdot E = \Gamma^\Sigma \cdot E = c' \in \mathbb{Z}$. Since $\Gamma \not\equiv E$, $c' \geq 1$; indeed, if $c' = 0$, then $\Gamma \cap E = \emptyset$, which can only happen if $\Gamma = C \times \{p\}$, contrary to our assumption. Likewise $\Gamma \cdot F = c \in \mathbb{Z}^+$, and furthermore, $\Gamma \cdot D_{0,-1} = 4l \in \mathbb{Z}$. To show (60), we note that

$$0 \leq \Gamma \cdot D_{1,\pm 1} = D_{c,c',l} \cdot D_{1,\pm 1} = c + c' \mp 4l \, ,$$

which gives (60). Thus the claim is established.

(a): If (55) holds, then we easily see that $D_{a,k}$ is an effective linear combination of $E + F$, Δ and $D(i)$. The converse is an immediate consequence of the above claim.

(b): If H is nef, then

$$0 \leq H \cdot D_{1,\pm 1} = 2(a \mp 2k)$$

and hence (56) holds. Conversely, assume (56), and let $\Gamma \in \mathrm{NS}(V)$ be an effective divisor. Write $\Gamma^\Sigma = D_{c,c',l}$; by the claim,

$$H \cdot \Gamma = H \cdot \Gamma^\Sigma = H \cdot D_{c,c',l} = (c + c')a - 4kl \geq (c + c')(a - 2|k|) \geq 0 \, , \tag{61}$$

and thus H is nef. It is well known that a nef divisor A on a projective manifold X is big if and only if $A^{\dim X} > 0$ (see for example, [41, pp. 146–147]). Under the condition (56) we have $H^2 = 2a^2 - 4k^2 \geq a^2 > 0$ unless $a = k = 0$, and thus H is big unless $H \equiv 0$.

(c): If H is ample, then (57) follows from (b) and the fact that the ample cone is the interior of the nef cone (or simply by noting that $H \cdot D_{1,\pm 1} > 0$). Conversely, assume (57). Then as we observed in (b), $H^2 > 0$. Now let $\Gamma \neq 0$ be effective, with $\Gamma^\Sigma \equiv D_{c,c',l}$. Then by (57) and (61), $H \cdot \Gamma \geq c + c' > 0$. The ampleness of H now follows by the Nakai-Moishezon criterion.

(d): Assume that the conditions in (d) hold. By (c), H is ample. To show that H is globally generated, represent $H = K + L$ where $K \equiv D_{2,0}$ and $L \equiv D_{a-2,k}$. Then, in virtue of (58) and (b), the divisor L is nef. Furthermore, $L^2 = 2(a-2)^2 - 4k^2 \geq (a-2)^2 \geq 9$ if $a \geq 5$ and $L^2 = 8$ if $(a, k) = (4, 0)$. So, by Reider's Theorem, H is globally generated unless there is an effective divisor $\Gamma = D_{c,c',l} + \Gamma^\perp$ such that (i) or (ii) holds. First suppose we have (ii); by the Hodge Index Theorem, $(\Gamma^\perp)^2 \leq 0$ and so,

$$\Gamma^2 = D^2_{c,c',l} + (\Gamma^\perp)^2 \leq D^2_{c,c',l} = 2(cc' - 2l^2) \, . \tag{62}$$

Since $\Gamma^2 = 0$ by (ii), it follows from (62) that

$$cc' \geq 2l^2 \, . \tag{63}$$

Furthermore by (ii),

$$1 = L \cdot \Gamma = L \cdot \Gamma^\Sigma = D_{a-2,k} \cdot D_{c,c',l} = (a-2)(c + c') - 4kl \, . \tag{64}$$

Then by (64), $kl > 0$. Hence, by (58),

$$4kl = 4|k||l| \leq 2|l|(a-2) \, , \tag{65}$$

and therefore by (64)–(65),

$$1 \geq (a-2)(c+c'-2|l|) \geq 2(c+c'-2|l|)\,. \tag{66}$$

Thus

$$2|l| \geq c+c'-\frac{1}{2}\,, \tag{67}$$

and hence by (63) and (67),

$$\left(2|l|+\frac{1}{2}\right)^2 \geq (c+c')^2 \geq 4cc' \geq 8l^2\,,$$

or $2l^2-|l| \leq \frac{1}{8}$. Since $4l \in \mathbb{Z}$, it follows that $|l| \leq \frac{1}{2}$, and then by (67), $c+c' \leq 1$. According to the claim, the only such possibility is $\Gamma \equiv E$ or F, but in that case, $L\cdot\Gamma = a-2 \geq 2$, contradicting (ii).

Next suppose that (i) holds. This time, since $\Gamma^2=-1$, (62) yields

$$cc' \geq 2l^2-\frac{1}{2}\,. \tag{68}$$

Also by (i), we have

$$0 = L\cdot\Gamma = L\cdot\Gamma^{\Sigma} = D_{a-2,k}\cdot D_{c,c',l} = (a-2)(c+c')-4kl\,. \tag{69}$$

Now, as in case (ii), using (69) in place of (64), we obtain

$$0 \geq (a-2)(c+c'-2|l|) \geq 2(c+c'-2|l|)\,. \tag{70}$$

and hence

$$2|l| \geq c+c'\,. \tag{71}$$

This time by (68) and (71),

$$4l^2 \geq c^2+2cc'+c'^2 \geq c^2+c'^2+4l^2-1$$

and thus $c^2+c'^2 \leq 1$. Again the only possibility is $\Gamma \equiv E$ or F, but that contradicts (i). Therefore, by Reider's Theorem, H is globally generated. The non-speciality of H follows from the Ramanujam (or Kawamata-Viehweg) Vanishing Theorem (see for example, [41, Ch. VII]) applied to L.

(e): Suppose H is a very ample divisor. Then the restrictions $H\,|\,E$ and $H\,|\,F$ are very ample, too, and hence, $a = \deg H\,|\,E = \deg H\,|\,F \geq 5$. (Indeed, there is no very ample divisor on C of degree ≤ 4, because the genus of a smooth non-plane curve in $\mathbb{P}^3$ of degree ≤ 4 is at most 1.) Furthermore, $2a-4k = \deg H\,|\,D(i) \geq 5$ and $2a+4k = \deg H\,|\,\Delta \geq 5$, and so, $4|k| \leq 2a-6$, which gives (59).

Conversely, suppose $a \geq 5$ and (59) is fulfilled. As above, represent $H=K+L$. In virtue of (59), we have $2|k| \leq (a-2)-1$ and so, by (c), L is an ample divisor. We also have:

$$L^2 = D^2_{a-2,\,k} = 2(a-2)^2-4k^2 \geq 2(a-2)^2-(a-3)^2 = a^2-2a-1 \geq 14\,,$$

since $a \geq 5$.

Hence by Reider's Theorem, H is very ample unless there is an effective divisor $\Gamma = D_{c,c',l}+\Gamma^{\perp}$ such that (i'), (ii') or (iii') holds. However, (i') cannot hold since L is ample. Now suppose that (ii') holds. Since in this case, $\Gamma^2 \geq -1$, it follows from

(62) as in case (i) of (d) that (68) again holds. Also by (ii'), we again have (64), and repeating the argument in case (ii) of (d), we obtain

$$1 \geq (a-2)(c+c'-2|l|) \geq 3(c+c'-2|l|)\,.$$

Since $4l \in \mathbb{Z}$, we again obtain (71). However, we showed in (d) that (68) and (71) imply that $\Gamma \equiv E$ or F, and this contradicts (ii').

Finally, suppose that (iii') holds. Since in this case, $\Gamma^2 = 0$, the Hodge Index Theorem again yields (63) as in (d). This time

$$2 = L\cdot\Gamma = L\cdot\Gamma^{\Sigma} = (a-2)(c+c') - 4kl\,. \tag{72}$$

Using (72) as in the proof of (d), we obtain

$$2 \geq (a-2)(c+c'-2|l|) \geq 3(c+c'-2|l|)\,. \tag{73}$$

Since $4l \in \mathbb{Z}$, (73) yields (67). In (d), we showed that (63) and (67) imply that $\Gamma \equiv E$ or F, and hence $L\cdot\Gamma = a-2 \geq 3$, contradicting ($iii'$). Therefore by Reider's Theorem, H is very ample. □

Remark 4.5. It follows from the proof of parts (c) and (e) of Theorem 4.4 that if a divisor $H \equiv D_{a,a',k} \in \Sigma$ is ample, then

$$2k^2 < aa' \quad \text{and} \quad 4|k| \leq a+a'-1\,;$$

if H is very ample, then $a, a' \geq 5$ and

$$4|k| \leq a+a'-5\,.$$

Corollary 4.6. *Let $V' = C_2$ be the symmetric square of a genus 2 curve C. Then a divisor $H \equiv D'_{a,k} \in \Sigma' \subset \mathrm{NS}(V')$ is quasi-effective, resp. nef, resp. ample, iff the inequality (55), resp. (56), resp. (57), holds.*

Proof. This follows by Kouvidakis [21] in the case where C is a generic curve, and by Theorem 4.4 and Proposition 4.2 in the general case. □

Corresponding (less precise) descriptions of the globally generated and very ample divisors on V' are given in Theorem 4.16 below. The next corollary provides some further consequences of Theorem 4.4.

Corollary 4.7. *As above, let $V = C \times C$ where C is a genus 2 curve. Then the following statements hold.*

(a) *If $H \equiv D_{a,k} \in \Sigma^{\mathrm{sym}} \subset \mathrm{NS}(V)$ is a very ample divisor, then the degree $n = H^2$ of the projective embedding $V \hookrightarrow \mathbb{P}^N$ defined by H is at least $a^2+6a-9 \geq 46$. This lower bound is achieved by the very ample divisor $H = 6(E+F) - \Delta \equiv D_{5,1} \in \Sigma^{\mathrm{sym}}$.*

(b) *Let $H \equiv D_{a,k} \in \Sigma^{\mathrm{sym}} \subset \mathrm{NS}(V)$, $k \neq 0$, be an ample divisor. Then the divisor $2H$ is globally generated, and $3H$ is very ample[5]. More precisely, let $m_0 > 0$ be such that the divisor m_0H is very ample but $(m_0-1)H$ is not. If $2|k| = a-1$ then $m_0 = 3$, and if $2|k| = a-2$, then $m_0 = 2$.*

[5]Recall [31, II.6.1, III.7] that if A is a simple abelian variety, then any effective divisor H on A is ample, $2H$ is globally generated and $3H$ is very ample.

(c) *Very ampleness of a divisor $H \equiv D_{a,k}$ on the surface V is a numerical condition. Moreover, if $H = K + L \equiv D_{a,k}$, $a \geq 5$, then H is ample and globally generated if L is nef, and H is very ample if and only if L is ample.*

(d) *There is no embedding $V \hookrightarrow \mathbb{P}^4$ defined by a symmetric divisor $H \equiv D_{a,k} \in \Sigma^{\mathrm{sym}} \subset \mathrm{NS}(V)$; in particular, if C has general moduli so that $\Sigma = \mathrm{NS}(V)$, then there is no embedding $V \hookrightarrow \mathbb{P}^4$ in which the images of the generators E and F are of the same degree.*

Proof. (a) By (59) we have:

$$n = H^2 = D_{a,k}^2 = 2a^2 - 4k^2 \geq 2a^2 - (a-3)^2 = a^2 + 6a - 9 \geq 46\,,$$

since $a \geq 5$. By Theorem 4.4, the divisor $H = 6E + 6F - \Delta \equiv D_{5,1}$ is very ample. It defines a projective embedding of V of degree $n = 46$.

(b) The statement follows from Theorem 4.4.

(c) Theorem 4.4(e) tells us that very ampleness of a divisor in Σ^{sym} is numerical. Inequalities (56)–(59) immediately imply the next statement.

(d) Suppose that $H \equiv D_{a,k} \in \mathrm{NS}(V)$ is a very ample divisor which defines an embedding $V \hookrightarrow \mathbb{P}^4$. Then $a \geq 5$, and by (27) we get

$$n^2 - 10n = 20a + 4\,. \tag{74}$$

From (74) we obtain

$$n = 2(a^2 - 2k^2) \leq \sqrt{20a + 29} + 5 < 4a < a^2\,,$$

where the last two inequalities are consequences of the bound $a \geq 5$. Hence $a^2 < 4k^2$, which contradicts (59). □

Example 4.8. A generic linear system g_3^1 on a genus 2 curve C gives rise to a symmetric correspondence, say, T on C, which represents an effective divisor $T \equiv D_{2,1}$ on $V = C \times C$. By Theorem 4.4(b), this divisor is nef and big, but not ample; indeed, $T \cdot D(i) = 0$.

Remark 4.9. Suppose $H \equiv D_{a,k} \in \mathrm{NS}(V)$, where V is as in Theorem 4.4, $a \geq 5$ and (58) holds, so that by part (d) of the theorem, H is ample and globally generated. Then we easily see that H fails to be very ample iff $2|k| = a - 2$ (and $a \geq 6$) iff equation (i') of Reider's Theorem holds with $\Gamma = D(i)$ or Δ (the former when $k > 0$, the latter when $k < 0$). By the proof of Theorem 4.4(e), neither (ii') nor (iii') of Reider's Theorem can hold for such H. Furthermore, (i') can hold only for the above choices of Γ. (Indeed, (i') implies that, in the notation from the proof of Theorem 4.4, $c + c' \leq 2|l|$, $cc' \geq 2l^2 - 1$, and hence $(c, c', l) = (1, 0, \pm\frac{1}{2})$, $(0, 1, \pm\frac{1}{2})$ or $(1, 1, \pm 1)$. By (60), the first two cases cannot occur and the last happens only when $\Gamma = D(i)$ or Δ.)

Remark 4.10. It is well known that the correspondences on a curve C form a ring with the unit $\Delta \equiv D_{1,-1}$. For $g(C) = 2$, multiplication by $D(i) \equiv D_{1,1}$ is an involution in this ring. It follows that the effective cone in the Neron-Severi group $\mathrm{NS}(C \times C)$ is invariant under this transform. Therefore, the same holds for the ample and the nef cones, as well. Since multiplication by $D(i)$ transforms the divisor class $D_{a,a',k}$ into $D_{a,a',-k}$, we see why the inequalities in Theorem 4.4 are symmetric with respect to the sign change of k. Clearly, the same is true for any hyperelliptic curve.

We now consider the case where C is a non-hyperelliptic smooth curve of genus $g = 3$ (as in Example 2.6) and we let $V = C \times C$. Again we assume that C has general periods; in particular, we assume that the Neron-Severi group $\mathrm{NS}(V) = \Sigma$ is generated by the classes of E, F, Δ.

Lemma 4.11. *The class $D_{10,6} \in \mathrm{NS}(V)$ contains a unique irreducible effective curve B.*

Proof. Consider the canonical embedding $C \hookrightarrow \mathbb{P}^2$ of degree 4. Denote by B the symmetric correspondence on C given as the closure in $V = C \times C$ of the set of pairs (p, q) where $\{p, q\} = L_x \cap C \setminus \{x\}$ for the line $L_x \in C^*$ tangent to C at some non-flex point $x \in C$ different from p. Write $B \equiv D_{a,k}$. To compute a, k, we first note that by the Class Formula, the dual curve C^* has degree $2(g + d - 1) = 12$. The pencil of lines through a generic point $p \in C$ represents the line $H = p^*$ in the dual projective plane $\mathbb{P}^{2*}$ that is tangent to C^* at the point $L_p^* \in \mathbb{P}^{2*}$. By Bezout's theorem, $H \cap C^*$ consists of 12 points, including the point L_p^* of multiplicity 2. Thus, there are 10 lines through p tangent to C, excluding the tangent L_p, and hence there are 10 choices of points q with $(p, q) \in B$; i.e., $a = B \cdot F = 10$. To compute k, we recall that the smooth quartic C has 28 bitangent lines, and thus $D \cdot \Delta = 56$, since each bitangent gives two points of $D \cap \Delta$. From the equality $56 = D \cdot \Delta = 2a + 2gk = 20 + 6k$ we obtain $k = 6$. Therefore,

$$B \equiv D_{10,6} \,.$$

In fact, B is an irreducible curve. To see this, consider the $2:1$ map $\varphi : B \to C$ given by $\varphi(p, q) = x$. If B were reducible, then $B = B_1 \cup B_2$ and φ^{-1} would have global branches $\psi_j : C \to B_j$, $j = 1, 2$. The projection to the first factor $\pi_1 : B \to C$ of degree 10 has simple critical points over the 24 flexes of C. On the other hand, by the Riemann-Hurwitz Formula, the composition $\pi_1 \circ \psi_j : C \to C$, $j = 1, 2$, must be an isomorphism, a contradiction.

Since $B^2 < 0$, B is the only effective divisor in the numerical class $D_{10,6}$. □

Remark 4.12. Some other natural correspondences that we do not use here are the tangent correspondence $T \equiv D_{2,10,2}$ given by the set of pairs (x, q) in the above construction, its inverse $T^{-1} \equiv D_{10,2,2}$, and the correspondence $G \equiv D_{3,1} \equiv \frac{1}{4}(T + T^{-1})$ given by the closure of the set of pairs of distinct points in the same fiber of a linear projection $C \to \mathbb{P}^1$ (see e.g. [13]).

The above correspondence B can be generalized to higher genera in two different ways. First of all, we may define it in the same way as above for a generic plane nodal curve of degree d and of genus g; then we get $B \equiv 2D_{a,k}$ where $a = (d - 3)(d + g - 2)$ and $k = d + g - 4$. It is easily seen that $B^2 < 0$ only for $g = 3,\ d = 4$.

On the other hand, following a suggestion by C. Ciliberto, for a genus g curve with general moduli we can consider the correspondence

$$B = \{(p,\, q) \in C \times C \mid p + q + (g-1)r + D \sim K_C \quad \text{for some} \quad r \in C,\ D \in \mathrm{Div}\,(C),\ D \geq 0\}$$

(geometrically, that means that p and q lie on a cut of the canonical model $\varphi_K(C) \subset \mathbb{P}^{g-1}$ of C by the highest osculating hyperplane say, H_r at some other point $r \in \varphi_K(C)$). Using de Jonquères' formula [3, VIII.5] one can verify that $B \equiv (g-1)(g-2)D_{a,k}$ with $a = g^2 - g - 1$ and $k = g$, and once again, $B^2 > 0$ for any $g \geq 4$.

The next proposition (in the case of a generic curve) also follows from Kouvidakis' Theorem 4.1 and Proposition 4.2. For the convenience of the reader, we provide a direct proof below.

Proposition 4.13. *Let C be a non-hyperelliptic genus 3 curve. Then a class $D_{a,k} \in \Sigma^{\rm sym} \subset \mathrm{NS}(C\times C)$ is*
(a) *quasi-effective if and only if*

$$-a \le k \le \frac{3}{5}a \qquad \Longleftrightarrow \qquad |5k+a| \le 4a\,; \tag{75}$$

(b) *nef if and only if*

$$-\frac{1}{3}a \le k \le \frac{5}{9}a \qquad \Longleftrightarrow \qquad |9k-a| \le 4a\,; \tag{76}$$

(c) *ample if and only if*

$$-\frac{1}{3}a < k < \frac{5}{9}a \qquad \Longleftrightarrow \qquad |9k-a| < 4a\,. \tag{77}$$

Proof. We first show (c): Suppose that $D\equiv D_{a,k}$ is ample. Then

$$D\cdot\Delta = D_{a,k}\cdot D_{1,-1} = 2(a+3k) > 0\,, \tag{78}$$

$$D\cdot B = D_{a,k}\cdot D_{10,6} = 4(5a-9k) > 0\,, \tag{79}$$

which yields (77). Conversely, suppose that (77) holds, and let $D\equiv D_{a,k}$. By (77), $D^2 = 2(a^2-3k^2)>0$. Thus by the Nakai-Moishezon criterion, it suffices to show that $D\cdot\Gamma>0$ for all effective curves Γ on V. As above, in virtue of (77), we have $D\cdot\Delta>0$ and $D\cdot B>0$. Let Γ be an irreducible effective divisor different from B and Δ, and let as above, $\Gamma=\Gamma^\Sigma+\Gamma^\perp$ where $\Gamma^\Sigma\equiv D_{c,c',l}\in\Sigma\otimes\mathbb{Q}$ and $\Gamma^\perp=\Gamma-\Gamma^\Sigma\perp\Sigma\otimes\mathbb{Q}$. Then $\Gamma\cdot B=\Gamma^\Sigma\cdot B = 10s-36l\ge 0$ and $\Gamma\cdot\Delta=\Gamma^\Sigma\cdot\Delta = s+6l\ge 0$ where $s:=c+c'$, i.e.

$$-\frac{1}{6}s \le l \le \frac{5}{18}s. \tag{80}$$

From (77) and (80), we obtain $kl\le |k||l| < \frac{25}{162}as$. Furthermore, since by (77), $a>0$, and clearly $s=c+c'>0$, we have

$$D\cdot\Gamma = D\cdot\Gamma^\Sigma = as-6kl > \frac{2}{27}as > 0\,.$$

Thus, D is ample.

To show (b), we proceed exactly as above, first noting that if D is nef, then the inequalities $D\cdot\Delta\ge 0,\ D\cdot B\ge 0$ yield (76). Conversely, (76) implies that $D\cdot\Delta\ge 0,\ D\cdot B\ge 0$, and $D\cdot\Gamma>0$ for irreducible curves Γ different from B and Δ.

We now show (a): First suppose that $k\ge 0$. If $D_{a,k}$ is quasi-effective, then by (76), we have $D_{a,k}\cdot D_{c,l} = 2(ac-3kl)\ge 0$ whenever $l=\frac{5}{9}c>0$, and thus $k\le\frac{3}{5}a$. Conversely, if $0\le k\le\frac{3}{5}a$, then $D_{a,k}\equiv\frac{1}{3}D_{3a-5k,0}+\frac{k}{6}B$ is quasi-effective. Now suppose that $k\le 0$. If $D_{a,k}$ is quasi-effective then $ac-3kl\ge 0$ for $l=-\frac{1}{3}c<0$, which gives $k\ge -a$. Conversely, if $-a\le k\le 0$, then $D_{a,k}\equiv D_{a+k,0}+|k|\Delta$ is quasi-effective. $\square$

To simplify the discussion, we assume in the sequel that the curve C has general moduli so that $\mathrm{NS}(V)=\Sigma$, resp. $\mathrm{NS}(V')=\Sigma'$.

Theorem 4.14. *Let C be a non-hyperelliptic genus 3 curve with $\mathrm{NS}(V) = \Sigma$. Then a divisor $H \equiv D_{a,k} \in \Sigma^{\mathrm{sym}} \subset \mathrm{NS}(V)$ is very ample if $a \geq 7$, $(a,k) \neq (8,2)$ and*

$$-\frac{1}{3}(a-4) \leq k \leq \frac{5}{9}(a-4)\,; \tag{81}$$

H is ample and globally generated for the additional values $(6,0)$ and $(8,2)$ of (a,k). In all these cases H is non-special.

Proof. Suppose $a \geq 8$ and (81) holds; represent $H = K + L$, where $K = K_V \equiv D_{4,0}$ and $L = H - K \equiv D_{a-4,k}$. By Proposition 4.13, the divisor L is nef. If $a \geq 13$, then by the inequalities (81), we have

$$L^2 = 2\left((a-4)^2 - 3k^2\right) \geq \frac{4}{27}(a-4)^2 \geq 12\,.$$

One can easily check that also $L^2 \geq 12$ for all the integer solutions of (81) with $7 \leq a \leq 12$, except for $(a,k) = (8,2)$. Thus, Reider's criterion can be applied; that is, the divisor $H = K + L$ is very ample unless there exists an effective divisor $\Gamma \equiv D_{c,c',l}$ such that

$$\Gamma^2 = 2(cc' - 3l^2) = 0 \quad \text{and} \quad 1 \leq L \cdot \Gamma = (a-4)s - 6kl \leq 2 \tag{82}$$

or

$$\Gamma^2 = 2(cc' - 3l^2) = -2 \quad \text{and} \quad L \cdot \Gamma = (a-4)s - 6kl = 0\,, \tag{83}$$

where $c, c' \geq 0$, $s = c + c' > 0$. (The fact that Γ^2 is even eliminates the other possibilities in Reider's Theorem.)

We first consider the case $a = 7$. By (81), $|k| \leq 1$ when $a = 7$. If $k = 0$, then $L \cdot \Gamma = 3s$ and thus neither (82) nor (83) can hold. Now suppose that $|k| = 1$, so that $L \cdot \Gamma = 3s \pm 6l$. Hence (82) cannot hold; suppose further that (83) is satisfied, i.e.,

$$cc' = 3l^2 - 1 \qquad \text{and} \qquad s = 2|l|\,.$$

But then,

$$4l^2 = s^2 = (c + c')^2 \geq 4cc' = 12l^2 - 4$$

or $2l^2 \leq 1$. This implies that $l = s = 0$, a contradiction, so (83) also cannot hold. Thus in the sequel we assume that $a \geq 8$.

First suppose there is an effective divisor Γ satisfying (82). The inequality $(a-4)s - 6kl \leq 2$ implies that $kl > 0$, and hence from (81) and (82) we get

$$\frac{10}{3}(a-4)|l| \geq 6|k||l| = 6kl \geq (a-4)s - 2$$

or

$$3s - 10|l| \leq \frac{6}{a-4}\,.$$

Since $a \geq 8$, this yields

$$3s - 10|l| \leq 1\,. \tag{84}$$

By (82),

$$s^2 = (c + c')^2 \geq 4cc' = 12l^2\,. \tag{85}$$

Combining (84) and (85), we get

$$\frac{25}{3}s^2 \geq 100l^2 \geq (3s-1)^2\,,$$

or

$$2s^2 - 18s + 3 \leq 0\,.$$

Hence $s \leq 8$ and thus by (85) and the fact that $kl > 0$, we have $1 \leq |l| \leq 2$. If $l = \pm 1$, then $cc' = 3$ and thus $c = 1$, $c' = 3$ (or vice versa) so that $s = 4$. But then $3s - 10|l| = 2$, contradicting (84).

Now suppose that $l = -2$. Then $k < 0$ and hence by (81) we have $3|k| \leq a - 4$. Thus by (82),

$$3|k|(s-4) = 3|k|s - 12|k| \leq (a-4)s - 12|k| \leq 2\,.$$

Therefore $s \leq 4$, which contradicts the fact that by (85), $s^2 \geq 48$.

It remains to consider $l = 2$. In this case, (84) yields $s \leq 7$. Let

$$\Gamma = \sum_{j=1}^{m} \Gamma_j \equiv \sum_{j=1}^{m} D_{c_j, c'_j, l_j}$$

be the decomposition of Γ into irreducible effective divisors. Note that $c_j, c'_j \geq 0$, $\sum(c_j + c'_j) = s \leq 7$, $\sum l_j = l = 2$, and thus the Γ_j are all distinct from $B \equiv D_{10,10,6}$. Therefore, as in the proof of Proposition 4.13 (see (80)), $B \cdot \Gamma_j \geq 0$ gives

$$l_j \leq \frac{5}{18}(c_j + c'_j)\,. \tag{86}$$

Summing (86) over j yields $2 \leq \frac{5}{18} \cdot 7$, a contradiction.

Next, suppose there is an effective divisor Γ satisfying (83). As before, we conclude that $kl > 0$. Hence from (81) and (83) we get

$$\frac{10}{3}(a-4)|l| \geq 6|k||l| = 6kl = (a-4)s$$

or

$$s \leq \frac{10}{3}|l|\,. \tag{87}$$

By (83),

$$3l^2 = cc' + 1 \tag{88}$$

and hence by (87),

$$\frac{100}{9}l^2 \geq s^2 = (c + c')^2 \geq 4cc' = 12l^2 - 4\,. \tag{89}$$

By (89) and the fact that $kl > 0$, we have again $1 \leq |l| \leq 2$. If $|l| = 2$, then (88) yields $cc' = 11$, which contradicts the fact that by (87), $c + c' = s \leq 6$.

If $|l| = 1$, we have $cc' = 2$ and $s = 3$; therefore $\Gamma \equiv D_{2,1,\pm 1}$ or $D_{1,2,\pm 1}$. If $\Gamma \equiv D_{2,1,-1}$ or $D_{1,2,-1}$, then (83) yields $k = -\frac{1}{2}(a-4)$, which contradicts (81). On the other hand, if $\Gamma \equiv D_{2,1,1}$ or $D_{1,2,1}$, then we again decompose Γ into irreducible effective divisors:

$$\Gamma = \sum_{j=1}^{m} \Gamma_j \equiv \sum_{j=1}^{m} D_{c_j, c'_j, l_j}$$

Since $\sum(c_j + c'_j) = 3$, $\sum l_j = 1$, the Γ_j must all be distinct from $B \equiv D_{10,10,6}$ and thus (86) holds. Summing (86) over j then yields $1 \leq \frac{5}{18} \cdot 3$, a contradiction. To summarize, Γ cannot satisfy (82) or (83); therefore H is very ample by Reider's Theorem 4.3.

To obtain the second statement, suppose $H \equiv D_{6,0}$, resp. $D_{8,2}$, and write $H = K + L$ as before. Then in both cases, $L^2 = 8$, and (81) holds, so by Proposition 4.13(b), L is nef. For any divisor $\Gamma \equiv D_{c,c',l} \in \mathrm{NS}(V)$, we have $\Gamma^2 \neq -1$ and $L \cdot \Gamma = 2s$, resp. $4s - 12l$, and hence $L \cdot \Gamma \neq 1$. Thus, H is globally generated by Reider's Theorem; furthermore, H is ample by Proposition 4.13(c). As above, the non-speciality of H follows from the Ramanujam Vanishing Theorem. □

Remark 4.15. The smallest possible degree of a projective embedding $V \hookrightarrow \mathbb{P}^N$ provided by the above theorem is 92, given by the very ample divisor $D_{7,1}$. Recall (Example 2.6) that the canonical divisor $K_C \equiv D_{4,0}$ is also very ample and gives an embedding of degree 32.

Applying the same methods as in Theorems 4.4(d)–(e) and 4.14 above, we describe in the next theorem some of the non-special globally generated, resp. very ample, divisors on the symmetric square $V' = C_2$ of generic curves C of genera 2, 3 and 4. (Recall that if C has genus 2, then C_2 is an abelian surface with a point blown up and is consequently non-hyperbolic.)

Theorem 4.16. *Let $V' = C_2$ where C is a genus $g \geq 2$ curve with general moduli, so that $\mathrm{NS}(V') = \Sigma'$. Let $H \equiv D'_{a,k} \in \mathrm{NS}(V')$ be a divisor.*

For $g = 2$ the divisor H is

(a) *non-special, nef and globally generated if $a \geq 4$ and*

$$2|k-1| \leq a-2\,; \tag{90}$$

(b) *very ample if $a \geq 5$ and*

$$2|k-1| \leq a-3\,. \tag{91}$$

On the other hand, if H is very ample, then $a \geq 5$ and

$$1-a \leq 2(k-1) \leq a-3\,. \tag{92}$$

For $g = 3$ the divisor H is

(a′) *non-special, ample and globally generated if $a \geq 6$, $(a,k) \neq (7,3)$ and*

$$3-a \leq 3(k-1) \leq \frac{5}{3}a - 5\,; \tag{93}$$

(b′) *very ample if $a \geq 7$, $(a,k) \neq (7,3)$ or $(9,4)$, and*

$$4-a \leq 3(k-1) \leq \frac{5a-16}{3}\,. \tag{94}$$

On the other hand, if H is very ample, then $a \geq 4$ and

$$-1-a \leq 3(k-1) \leq \frac{5a-11}{3}\,. \tag{95}$$

For $g = 4$ the divisor H is

(a$''$) *non-special, ample and globally generated if* $a \ge 8$ *and*

$$9 - a \le 4k \le 2a - 8\,; \tag{96}$$

(b$''$) *very ample if* $a \ge 9$ *and*

$$9 - a \le 4k \le 2a - 10\,. \tag{97}$$

On the other hand, if H *is very ample, then* $a \ge 6$ *and*

$$3 - a \le 4k \le 2a - 6\,. \tag{98}$$

Proof. (a): Let $g = g(C) = 2$. Set $L = H - K \equiv D'_{a-1,k-1}$. By Theorem 4.1, in view of (90) the divisor L is ample. Inequality (90) implies that $2|k| \le a$, and hence by Theorem 4.1, H is nef. By the Kodaira Vanishing Theorem, H is also non-special.

Since $a \ge 4$, we have

$$L^2 = (a-1)^2 - 2(k-1)^2 \ge \frac{a^2}{2} - 1 \ge 7\,.$$

Thus by Reider's Theorem, H is globally generated unless one of the cases (i) or (ii) of this theorem happens. But (i) is impossible since L is ample, and (ii) is impossible since there is no non-zero divisor $\Gamma \equiv D'_{c,l}$ on V' with $\Gamma^2 = c^2 - 2l^2 = 0$. □

(b): Assume first that $a \ge 5$ and (91) holds. Then as above, $L = H - K$ is an ample divisor, and $L^2 \ge a^2/2 - 1 \ge 11$. Thus by Reider's Theorem, H is very ample unless one of the cases (i')-(iii') of this theorem happens. The cases (ii') with $\Gamma^2 = 0$, (i'), and (iii') are excluded by the same reasons as above. Then we are left with the possibility that

$$\Gamma^2 = c^2 - 2l^2 = -1 \quad \text{and} \quad L \cdot \Gamma = (a-1)c - 2(k-1)l = 1 \tag{99}$$

for an effective divisor $\Gamma \equiv D'_{c,l} \in \mathrm{NS}(V')$. By (99), $(k-1)l > 0$, and furthermore by (91) and (99),

$$1 = L \cdot \Gamma \ge (a-1)(c - |l|) + 4|l|\,;$$

hence, $|l| > c$. (If $|l| = c$, the above inequality would yield $1 \ge 4|l| = 4c \ge 4$, a contradiction.) But then $c^2 - 2l^2 < -c^2 \le -1$ contradicting (99). Therefore by Reider's Theorem, H is very ample.

Suppose now that H is very ample. Since E', resp. Δ', is a smooth genus 2 curve in V', and the restriction $H\,|\,E'$, resp. $H\,|\,\Delta'$, is very ample, we have that

$$\deg(H\,|\,E') = H \cdot E' = a \ge 5\,,$$

resp.

$$\deg(H\,|\,\Delta') = H \cdot \Delta' = 2D'_{a,k} \cdot D'_{1,-1} = 2(a + 2k) \ge 5\,.$$

Therefore, $2(k-1) \ge 1 - a$. On the other hand, since H is ample, by Theorem 4.1, we have $2k < a$, or $2(k-1) \le a - 3$. Finally, from these we get the inequality (92). □

(a$'$), (b$'$): For $g = 3$ we have $K = K_{V'} \equiv D_{3,1}$ and $L = H - K \equiv D'_{a-3,k-1}$. In virtue of Theorem 4.1, L is nef, resp. ample, iff (93), resp. (94), holds. The inequality (93) implies that

$$-a < 3k < 5a/3 \tag{100}$$

and hence (again by Theorem 4.1), H is ample. By Ramanujam's Vanishing Theorem, the divisor $H = K + L$ is non-special as soon as L is nef.

Furthermore, assuming (93), resp. (94), we get the inequality

$$L^2 = (a-3)^2 - 3(k-1)^2 \geq \frac{2}{27}(a-3)^2\,, \qquad \text{resp.} \qquad L^2 > \frac{2}{27}(a-3)^2\,, \tag{101}$$

and hence $L^2 \geq 10$ if $a \geq 15$. By checking all possible integer values of a, k with $a \leq 14$ satisfying the conditions of (a$'$), resp. (b$'$), one easily verifies that $L^2 \geq 5$, resp. ≥ 10, for these values. Therefore by Reider's Theorem, under the assumptions on (a, k) of (a$'$), resp. (b$'$), H is globally generated, resp. very ample, unless there is an effective divisor $\Gamma \equiv D_{c,l} \in \mathrm{NS}(V')$ for which one of the conditions (i)–(ii), resp. (i')–(iii'), of this theorem holds. Note that the diophantine equations $\Gamma^2 = c^2 - 3l^2 = 0$ and $\Gamma^2 = c^2 - 3l^2 = -1$ have no solutions (since the latter cannot hold modulo 3). Finally, $L \cdot \Gamma = 0$ is impossible assuming (94), because in that case the divisor L is ample. Thus, (i)–(ii), resp. (i')–(iii'), cannot hold.

Next we assume that the divisor H is very ample. Then so are also the restrictions $H\,|\,\Delta'$ and $H\,|\,B'$ where $B' = \pi(B) \equiv D'_{10,6} \in \mathrm{NS}(V')$. Here Δ' is a smooth reduced curve on V' isomorphic to C, and hence $\deg(H\,|\,\Delta') \geq \delta(\Delta') = \delta(C) = 4$, i.e.

$$H \cdot \Delta' = 2D'_{a,k} \cdot D'_{1,-1} = 2(a+3k) \geq 4\,.$$

By the construction, the curve B' is birationally equivalent to the curve C; in particular, the geometric genus of B' equals 3. Hence, $\deg(H\,|\,B') \geq \delta(B') \geq 4$, or

$$H \cdot B' = D'_{a,k} \cdot D'_{10,6} = 10a - 18k \geq 4.$$

These inequalities provide (95). $\square$

(a$''$), (b$''$): For $g = 4$ we have $K = K_{V'} \equiv D_{5,1}$ and so, $L = H - K \equiv D_{a-5,k-1}$. Thus by the Kouvidakis Theorem 4.1, the divisor L is nef iff (96) holds. Under these inequalities we have $-a < 4k < 2a$, which implies (again due to Theorem 4.1) that the divisor H is ample. By the Ramanujam Vanishing Theorem, it is non-special.

As in the genus 3 case above, we easily verify that $L^2 \geq 5$, resp. $L^2 \geq 10$, under the conditions of (a$''$), resp. (b$''$). Hence, by Reider's Theorem, H is globally generated resp. very ample, unless (for an effective divisor $\Gamma \equiv D_{c,l} \in \mathrm{NS}(V')$) one of the conditions (i)–(ii), resp. (i')–(iii'), holds. We note that the diophantine equations $\Gamma^2 = c^2 - 4l^2 = -1$ and $\Gamma^2 = c^2 - 4l^2 = -2$ have no solutions (since neither can hold modulo 4). Hence, $\Gamma^2 = 0$, that is, $\Gamma \equiv \alpha D_{2,\pm 1}$ where $\alpha > 0$. But then we would have

$$L \cdot \Gamma = \alpha(2(a-5) \mp 4(k-1)) = 1 \quad \text{or} \quad 2\,.$$

Therefore, $\alpha = 1$, that is, $\Gamma \equiv D_{2,\pm 1}$, and $L \cdot \Gamma = 2$, i.e. (iii') holds. Since the cases (i) and (ii) have been eliminated, H is globally generated provided that the assumptions in (a$''$) are fulfilled.

Assume that (iii') holds. Then we have $a + 2k = 8$ if $\Gamma \equiv D_{2,-1}$, and $a = 2k + 4$ if $\Gamma \equiv D_{2,1}$. In the first case (96) yields $6 \leq a \leq 7$ which is excluded by the assumptions of (b$''$); the second case contradicts (97). Thus H is very ample.

To show the last statement of (b$''$), we now suppose that H is very ample. Then so are the restrictions $H\,|\,E'$, $H\,|\,\Delta'$ and $H\,|\,B''$ where $B'' \equiv D_{2,1} \in \mathrm{NS}(V')$ is a smooth curve on V' isomorphic to C, provided by any of the two linear pencils g^1_3 on the curve C. (Recall [14, IV.5.5.2] that a generic genus 4 curve C possesses exactly two such pencils.) Since $E' \simeq \Delta' \simeq B'' \simeq C$ and $\delta(C) = 6$ (see Remark 2.3 above), we have

$$a = H \cdot E' \geq \delta(C) = 6\,,\ 2(a+4k) = H \cdot \Delta' \geq \delta(C) = 6\,,\ 2a - 4k = H \cdot B'' \geq \delta(C) = 6\,.$$

This proves the inequalities $a \geq 6$ and (98). □

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