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Weakly symmetric Riemannian spaces with reductive isometry group.

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1. Introduction

Let M be a connected Riemannian manifold and $\mathrm{Isom}(M)$ be the full isometry group of M.

Definition 1. The manifold M is said to be *symmetric*, if for every point $x \in M$ there is an isometry $s \in \mathrm{Isom}(M)$ such that $s(x) = x$ and $d_x s = -\mathrm{id}$.

The theory of symmetric manifolds is well developed. Works of Élie Cartan and Helgason, see [12], [13] describe their structure and also deal with harmonic analysis on such manifolds. We will freely use results of these classical works.

The notion of a weakly symmetric manifold is a generalisation of a notion of a symmetric manifold.

Definition 2. The manifold M is said to be *weakly symmetric*, if the following equivalent conditions hold:

i) $\forall x, y \in M \;\; \exists s \in \mathrm{Isom}(M) : s(x) = y, s(y) = x$.

ii) $\forall x \in M \;\; \forall \xi \in T_x(M) \;\; \exists s \in \mathrm{Isom}(M) : s(x) = x, \;\; ds_x(\xi) = -\xi$.

(The equivalence of the given conditions is proved, for example, in [8].)

The geometric meaning of Definition 2 is that for every point $x \in M$ and for every geodesic line containing x there is an isometry stabilising x and reversing the geodesic line.

The notion of a weakly symmetric manifold was introduced by Selberg in [16]. It is well known that there are weakly symmetric Riemannian manifolds which are not symmetric. For example, as was noticed by Selberg, a homogeneous space $(\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SO}_2)/\mathrm{SO}_2$ is weakly symmetric but not symmetric. Recently weakly symmetric Riemannian manifold were intensively studied by several people, see, e.g., [8], [9], [10], [15], [7], [2], [5]. These works show that weakly symmetric spaces possess a fairly interesting geometry. Examples of non-symmetric weakly symmetric manifolds were known before, see [8], [15], but they were constructed as Riemannian manifolds rather then as homogeneous spaces. The proofs of non-symmetry were essentially based on the explicit description of the metric and its curvature tensor.

The main object of the present paper is a consideration of Riemannian manifolds as a homogeneous spaces of Lie groups. This allows us to construct numerous new examples of non-symmetric weakly symmetric Riemannian manifolds. Among weakly symmetric homogeneous spaces of reductive Lie groups, we distinguish non-symmetric Riemannian manifolds. Thereby, we obtain a classification, up to local isomorphism, of all non-symmetric weakly symmetric Riemannian manifolds having reductive isometry group; in particular, the compact ones. Moreover, for each of these manifolds, dimensions of sets of symmetric and non-symmetric Riemannian metrics (see Definition 4) are given. The classification is presented in Table 1. We also prove that an irreducible non-compact non-symmetric homogeneous space of a semisimple Lie group is not a symmetric manifold regardless of the choice of an invariant metric, see Theorem 2.

It is likely that this approach to weakly symmetric Riemannian manifolds can lead to their complete classification.

The main idea of this paper is the following. A weakly symmetric Riemannian manifold having reductive isometry group is a weakly symmetric homogeneous space, and hence, a real form of some spherical homogeneous space, [7]. Spherical homogeneous spaces and their real forms are classified, [11], [3], [6]. If a homogeneous space G/K is a symmetric Riemannian manifold for some choice of a G-invariant Riemannian metric, then there is a decomposition $\mathrm{Isom}(G/K) = GQ$, where Q is a symmetric subgroup of $\mathrm{Isom}(G/K)$. The decompositions of the reductive groups are described in a work by A.L. Onishchick, see [4], that gives us a clue to understanding whether G/K can be a symmetric Riemannian manifold. Via a sequence of reductions, the problem of classifying non-symmetric weakly symmetric Riemannian manifolds is reduced to the consideration of homogeneous spaces from Table 1.

Let us denote Lie groups by capital letters and their Lie algebras by the corresponding lowercase gothic letters. Let G' stand for the commutator group of G and $N_G(K)$ stand for the normaliser of K in G. If it is clear which group G is meant we will write simply $N(K)$.

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2. Premises and results

In this section G is a real Lie group, $K \subset G$ a compact subgroup such that the manifold $M = G/K$ is connected.

Definition 1′. The homogeneous space M is called *symmetric*, if there is an involution σ of G such that $(G^\sigma)^0 \subset K \subset G^\sigma$.

Definition 2′. The homogeneous space M is called *weakly symmetric*, if there is an automorphism $\sigma \in \mathrm{Aut}(G)$, normalising K, such that
iii) $\forall\ x, y \in G\ \exists g \in G:\ \sigma(x)K = gyK,\ \sigma(y)K = gxK.$

If $M = G/K$ is (weakly) symmetric then the pair (G, K) is called (weakly) symmetric.

Let M be a weakly symmetric Riemannian manifold. Then, due to the condition i), $\mathrm{Isom}(M)$ acts on M transitively, in particular, M is a complete Riemannian manifold. Denote by $\mathrm{St}(x) \subset \mathrm{Isom}(M)$ the stabiliser of a point $x \in M$. We have

$$M = \mathrm{Isom}(M)/\mathrm{St}(x) = \mathrm{Isom}(M)^0/(\mathrm{St}(x) \cap \mathrm{Isom}(M)^0),$$

$\mathrm{Isom}(M)$ is a real Lie group, $\mathrm{St}(x)$ is a compact subgroup of $\mathrm{Isom}(M)$. It is easy to verify, that the homogeneous space M of the group $\mathrm{Isom}(M)$ is weakly symmetric with respect to the identity map. The analogues fact concerning symmetric manifolds is well known. A symmetric

Riemannian manifold M is a symmetric homogeneous space of the group $\mathrm{Isom}(M)$ and also of the group $\mathrm{Isom}(M)^0$.

On the other hand, suppose a homogeneous space $M = G/K$ is (weakly) symmetric with respect to some automorphism σ. Let us introduce a G-invariant Riemannian metric on M. (This metric will be also σ-invariant due to condition iii).) Determine an isometry $s \in \mathrm{Isom}(M)$ by the formula $s(gK) = \sigma(g)K$. Then G/K becomes a (weakly) symmetric Riemannian manifold.

Thus, the notions of (weakly) symmetric Riemannian manifolds and (weakly) symmetric homogeneous space are quite close. The difference between them lays in the fact that a weakly symmetric Riemannian manifold M can be a weakly symmetric homogeneous space of several groups.

Let $M = G/K$ be a weakly symmetric homogeneous space. Then M is also a weakly symmetric homogeneous space of G^0, see [2, Lemma 1]. From now on assume that G is connected. This restriction is not important for our goal, i.e., classification of non-symmetric weakly symmetric Riemannian manifolds.

A non-symmetric homogeneous space can be a symmetric Riemannian manifold. For instance, a sphere $\mathrm{S}^{2n-1} = \mathrm{SU}_n/\mathrm{SU}_{n-1} = \mathrm{SO}_{2n}/\mathrm{SO}_{2n-1}$ is a symmetric Riemannian manifold and simultaneously a non-symmetric weakly symmetric homogeneous space of the group SU_n.

Definition 3. Let $M = G/K$ be a homogeneous space, μ be a G-invariant Riemannian metric on M. We say that μ is *(weakly) symmetric*, if the pair (M, μ) is a (weakly) symmetric Riemannian manifold.

To understand whether a given Riemannian metric is symmetric it is sufficient to know the isometry group of the pair (M, μ) or its connected component $\mathrm{Isom}(M)^0$.

Denote by $\mathfrak{m}$ the tangent space $T_{eK}(G/K) \cong \mathfrak{g}/\mathfrak{k}$. The space $\mathfrak{m}$ can be identified with a K-invariant complement of $\mathfrak{k}$ in $\mathfrak{g}$. Let $\mathbb{B}(\mathfrak{m})$ be a set of all positive-definite K-invariant scalar products on the vector space $\mathfrak{m}$. In order to determine a G-invariant Riemannian metric on G/K, it is necessary and sufficient to choose an element of $\mathbb{B}(\mathfrak{m})$.

As was proved in [2], a connected homogeneous space that is locally isomorphic to a weakly symmetric one is also weakly symmetric. Unfortunately, the analogous statement is not true for symmetric spaces.

Let $\widetilde{M} = \widetilde{G}/\hat{K}$ be a simply connected covering of M. Here $\widetilde{G}$ is a simply connected covering of G and $\hat{K}$ is a connected subgroup, whose Lie algebra is $\mathfrak{k}$. Any G-invariant Riemannian metric μ on M can be lifted to a $\widetilde{G}$-invariant Riemannian metric $\widetilde{\mu}$ on $\widetilde{M}$. Now we will show how to decide whether M is symmetric, assuming that $\mathrm{Isom}(\widetilde{M})$ is known.

Suppose (M, μ) is a symmetric Riemannian manifold. Then $(\widetilde{M}, \widetilde{\mu})$ is also symmetric. Moreover, M is a symmetric homogeneous space of $\mathrm{Isom}(\widetilde{M})^0$.

On the other hand, suppose we know that $(\widetilde{M}, \widetilde{\mu})$ is symmetric. One can easily check if (M, μ) is symmetric or not. Denote by F the group $\mathrm{Isom}(\widetilde{M})$; by σ an involution of F, determining a symmetric space structure on $\widetilde{M}$. Denote by N the ineffective kernel of an action of F on M. We have $\mathrm{Isom}(M)^0 = F/N$. Evidently, M is symmetric if and only if $\sigma(N) = N$.

Thus, the problem of finding all non-symmetric Riemannian manifold among the weakly symmetric homogeneous spaces is reduced to the same problem for simply connected homogeneous spaces.

Definition 4. A linear Lie group $G \subset GL(V)$, having finitely many connected components, is called *reductive*, if it is completely reducible. A Lie algebra of a reductive group is called a reductive algebra.

By the definition, any compact group is reductive.

We will deal with homogeneous spaces of reductive groups. But before we restrict ourself to this case, note that it is quite possible that $\widetilde{M}$ is not a homogeneous space of any reductive group even if M is. For example, there is no reductive group acting transitively on a real line $\mathbb{R} = \widetilde{\mathrm{U}_1}$.

In particular, having restricted the area of our consideration to homogeneous spaces of reductive groups, we might loose some weakly symmetric spaces. To avoid this unhappy event, we prove the following lemma.

Lemma 1. *Let $M = G/K$ be a homogeneous space of a reductive group G and $F := G'K \neq G$. Then $\widetilde{M}$ decomposes, as a Riemannian manifold, into a product $\widetilde{M} = M_s \times M_0$, where M_s is a simply connected covering of the homogeneous space $G'/(G' \cap K)$ and M_0 is a locally euclidian symmetric manifold.*

Proof. Let μ be a G-invariant Riemannian metric on M, determined by an element $b \in \mathbb{B}(\mathfrak{m})$. Denote by $\mathfrak{m}_1$ an orthogonal complement of $\mathfrak{f}/\mathfrak{k}$ in $\mathfrak{m}$. Evidently, $\mathfrak{m}_1 \subset \mathfrak{z}(\mathfrak{g})$. There is an equality of Riemannian manifolds $\widetilde{M} = M_s \times M_0$, where M_0 is a locally euclidian symmetric Riemannian manifold and M_s is a simply connected covering of F/K endowing with a F-invariant Riemannian metric determined by a scalar product $b|_{\mathfrak{f}/\mathfrak{k}}$. To conclude the proof, note that because $F = G'K$, we have $F/K = G'/(G' \cap K)$. □

Corollary. *A Riemannian manifold $\widetilde{M}$ is symmetric if and only if M_s is symmetric; moreover, if M_s is symmetric, then* $\mathrm{Isom}(M_s)$ *is semisimple and there is an equality* $\mathrm{Isom}(\widetilde{M}) = \mathrm{Isom}(M_s) \times \mathrm{Isom}(M_0)$.

Proof. The product of two Riemannian manifolds is symmetric if and only if each of them is symmetric, hence the first statement is true.

Suppose M_s is symmetric. Assume that $\mathrm{Isom}(M_s)$ is not semisimple, i.e., M_s is locally isomorphic to the product $M_1 \times \mathbb{R}^n$, where M_1 is a symmetric homogeneous space of a semisimple group $\mathrm{Isom}(M_1)$ and $n \geq 1$. According to the conditions of the lemma, the group $\widetilde{G'}$ acts on M_s transitively. Hence, $\widetilde{G'}$ has a non-trivial connected centre. This leads to a contradiction, because G' is semisimple. Since M_s is a symmetric homogeneous space os a semisimple group, each isometry of $\tilde{M}$ preserves M_s and M_0, see [13]. □

From now on, we assume that G is connected and reductive, G/K is simply connected, and $G = G'K$.

Let $G = G_1 \times G_2$, $K = K_1 \times K_2$, where K_i is a compact subgroup of G_i. Then one says that the homogeneous space $M = G/K$ is reducible and writes $(G, K) = (G_1, K_1) \times (G_2, K_2)$.

Definition 5. An affine complex F-variety X is called *spherical* if a Borel subgroup $B(F) \subset F$ has an open orbit in X. If F/H is spherical, then the pair (F, H) and the subgroup H are also called spherical.

As was proved in [7], the complexification of a weakly symmetric homogeneous space is a spherical homogeneous space. Furthermore, a real space is irreducible if and only if the complexification is irreducible, see [6]. Affine spherical homogeneous spaces are classified in [11], [3], [6], and their real forms are found in [6].

Theorem 1. *Suppose $(G,K) = (G_1,K_1) \times (G_2,K_2)$ is a weakly symmetric pair, and set $M_i = G_i/K_i$. Then, regardless of the choice of a G-invariant symmetric metric on $M = G/K$, there is a decomposition $M = M_1 \times M_2$ in the sense of Riemannian manifolds.*

If $N_{G_1}(K_1)^0 = K_1$ or $N_{G_2}(K_2)^0 = K_2$, then it is true for any G-invariant metric. In the general case this statement is more complicated and will be proved below. As a corollary of Theorem 1 we have: each symmetric metric on M is a product of symmetric metrics on M_1 and M_2. In particular, a symmetric metric on M exists if and only if such metrics exist on both M_1 and M_2. Thus, in order to classify all weakly symmetric non-symmetric Riemannian manifolds, it suffices to consider only irreducible weakly symmetric homogeneous spaces.

It will be proved in the next section that there is no symmetric metric on a non-compact irreducible non-symmetric homogeneous space.

Denote by $Z(G)$ the centre of G. Set $K_r := K/(K \cap Z(G))$. A homogeneous space G'/K_r is called the *central reduction* of G/K.

Recall the definition of a principal pair given in [3, n. 5].

Definition 6. A pair (G,K) is called *principal*, if $Z(K)^0 = Z = (G^1 \cap Z) \times ... \times (G^m \cap Z)$, where G^i are the simple factors of G and $Z(K)$ is the centre of K.

A complexification of a principal weakly symmetric pair is a principal spherical pair. All principal spherical pairs are classified by Brion, see [11, p. 190].

In Table 1, we give the list of all simply connected compact irreducible weakly symmetric homogeneous spaces of reductive Lie groups whose central reductions are principal irreducible homogeneous spaces.

If a homogeneous space M in Table 1 admits a G-invariant symmetric metric, then we also indicate P, the identity component of the isometry group, and Q, the stabiliser in P of a point in M. The pair (P,Q) is called a *symmetric extension* of (G,K).

The dimensions of the cones of K-invariant and Q-invariant (in brackets) positive-definite scalar products on the space $\mathfrak{m} = \mathfrak{g}/\mathfrak{k}$, i.e., the dimensions of the sets of weakly symmetric and symmetric metrics are given in the fifth column of Table 1.

Note that if a truncation $G'/(G' \cap K)$ (the definition is given in the next section) of a weakly symmetric homogeneous space G/K of a non-semisimple group G is still weakly symmetric, then only one of these two homogeneous spaces is listed in Table 1, usually the former. In rows 5, 7, 17, 23b and 24 the group $\mathrm{Isom}(G/K)^0$ is equal not to P but to its quotient by some central subgroup.

In section 4 we prove that (P,Q) is really an extension of (G,K) and that there are no other extensions. It will be shown that having Table 1 one can decide whether a given G-invariant Riemannian metric on a simply connected irreducible weakly symmetric homogeneous space G/K of a reductive group G is symmetric or not.

It can be seen that for almost all homogeneous spaces listed in Table 1 there are G-invariant metrics on M, which are not P-invariant for any P. So these homogeneous spaces are not symmetric manifolds under proper choice of a metric. Exceptions are the cases 14 and 16.

Table 1.

	$M = G/K$	$P = Isom(M)^0$	Q	$\dim \mathbb{B}(\mathfrak{m})$
1a	$\mathrm{SU}_n/\mathrm{SU}_{n-1}$	SO_{2n}	SO_{2n-1}	2(1)
1b	$\mathrm{SU}_n/\mathrm{SU}_{n-k}\times\mathrm{SU}_k\ \ (n\neq n-k)$			
1c	$\mathrm{U}_{2n}/\mathrm{U}_n\times\mathrm{SU}_n$			
2	$\mathrm{SU}_{2n+1}/\mathrm{Sp}_n\cdot\mathrm{U}_1$			
3	$\mathrm{SU}_{2n+1}/\mathrm{Sp}_n$	SU_{2n+2}	Sp_{n+1}	3(1)
4	$\mathrm{U}_1\cdot\mathrm{Sp}_n/\mathrm{U}_n$			
5	$\mathrm{Sp}_n/\mathrm{Sp}_{n-1}\cdot\mathrm{U}_1$	SU_{2n}	$\mathrm{SU}_{2n-1}\cdot\mathrm{U}_1$	2(1)
6	$\mathrm{U}_1\cdot\mathrm{Sp}_n/\mathrm{Sp}_{n-1}\cdot\mathrm{U}_1$			
7	$\mathrm{SO}_{2n+1}/\mathrm{U}_n$	SO_{2n+2}	U_{n+1}	2(1)
8	$\mathrm{U}_1\cdot\mathrm{SO}_{2n+1}/\mathrm{U}_n$			
9	$\mathrm{U}_1\cdot\mathrm{SO}_{4n+2}/\mathrm{U}_{2n+1}$			
10	$\mathrm{SO}_{10}/\mathrm{Spin}_7\times\mathrm{SO}_2$			
11	$\mathrm{U}_1\cdot\mathrm{SO}_{10}/\mathrm{Spin}_7\times\mathrm{SO}_2$			
12	$\mathrm{SO}_9/\mathrm{Spin}_7$	SO_{16}	SO_{15}	2(1)
13	Spin_8/G_2	$\mathrm{SO}_8\times\mathrm{SO}_8$	$\mathrm{SO}_7\times\mathrm{SO}_7$	3(2)
14	Spin_7/G_2	SO_8	SO_7	1(1)
15	E_6/Spin_{10}			
16	G_2/SU_3	SO_7	SO_6	1(1)
17	$(\mathrm{U}_{n+1}\times\mathrm{SU}_n)/\mathrm{U}_n$	$\mathrm{SU}_{n+1}\times\mathrm{SU}_{n+1}$	SU_{n+1}	2(1)
18	$(\mathrm{SU}_n\times\mathrm{Sp}_m)/(\mathrm{U}_{n-2}\times\mathrm{SU}_2\times\mathrm{Sp}_{m-1})$			
19a	$(\mathrm{Sp}_n\times\mathrm{Sp}_l\times\mathrm{Sp}_m)/$ $(\mathrm{Sp}_{n-1}\times\mathrm{Sp}_1\times\mathrm{Sp}_{l-1}\times\mathrm{Sp}_{m-1})$			
19b	$(\mathrm{Sp}_n\times\mathrm{Sp}_1\times\mathrm{Sp}_m)/$ $(\mathrm{Sp}_{n-1}\times\mathrm{Sp}_1\times\mathrm{Sp}_{m-1})$	$\mathrm{SO}_{4n}\times\mathrm{SO}_{4m}$	$\mathrm{SO}_{4n-1}\times\mathrm{SO}_{4m-1}$	6(2)
19b′	$(\mathrm{Sp}_1\times\mathrm{Sp}_n\times\mathrm{Sp}_1)/$ $(\mathrm{Sp}_1\times\mathrm{Sp}_{n-1}\times\mathrm{Sp}_1)$	$\mathrm{SO}_{4n}\times\mathrm{SO}_{4n}$	$\mathrm{SO}_{4n-1}\times\mathrm{SO}_{4n-1}$	5(2)
20	$(\mathrm{Sp}_n\times\mathrm{Sp}_2)/(\mathrm{Sp}_{n-2}\times\mathrm{Sp}_2)$			
21	$(\mathrm{SU}_n\times\mathrm{Sp}_m)/(\mathrm{SU}_{n-2}\times\mathrm{SU}_2\times\mathrm{Sp}_{m-1})$			
22a	$(\mathrm{Sp}_n\times\mathrm{Sp}_2\times\mathrm{Sp}_m)/$ $(\mathrm{Sp}_{n-1}\times\mathrm{Sp}_1\times\mathrm{Sp}_1\times\mathrm{Sp}_{m-1})$			
22b	$(\mathrm{Sp}_1\times\mathrm{Sp}_2\times\mathrm{Sp}_1)/(\mathrm{Sp}_1\times\mathrm{Sp}_1)$	$\mathrm{Sp}_2\times\mathrm{Sp}_2$	Sp_2	3(1)
23	$(\mathrm{SO}_{n+1}\times\mathrm{SO}_n)/\mathrm{SO}_n$	$\mathrm{SO}_{n+1}\times\mathrm{SO}_{n+1}$	SO_{n+1}	2(1)
24a	$(\mathrm{Sp}_n\times\mathrm{Sp}_m)/(\mathrm{Sp}_{n-1}\times\mathrm{Sp}_1\times\mathrm{Sp}_{m-1})$			
24b	$(\mathrm{Sp}_n\times\mathrm{Sp}_1)/(\mathrm{Sp}_{n-1}\times\mathrm{Sp}_1)$	SO_{4n}	SO_{4n-1}	3(1)

3. Extensions of pairs and invariant metrics

Definition 7. Let P, Q, G, K be Lie groups, where $Q \subset P$ and $K \subset G$. Then the pair (P, Q) is called *an extension* of (G, K), if

$$G \subsetneqq P, \;\; P = GQ, \;\; K = Q \cap G.$$

If (P, Q) is a symmetric pair, we call it *a symmetric extension* of (G, K).

Let $M = G/K$ be a homogeneous space. Suppose the pair (G, K) is *effective*, i.e. K contains no non-trivial normal subgroups of G. (Note that a pair $(\mathrm{Isom}(M), \mathrm{St}(x))$ is always effective.) For each G-invariant Riemannian metric μ on M, the pair $(\mathrm{Isom}(M), \mathrm{St}(x))$ is an extension of (G, K), if of course $G \neq \mathrm{Isom}(M)$. In order to find all symmetric G-invariant Riemannian metrics on M it is necessary and sufficient to describe all symmetric extensions of (G, K). Or equivalently, to each symmetric pair find out all weakly symmetric pair which it extends. Recall that as was proved in the previous section, if M is a symmetric manifold, then $\mathrm{Isom}(M)$ is semisimple.

Remark 1. It can be shown, that $\mathrm{Isom}(M)$ is always reductive. Moreover, if M is a symmetric homogeneous space of G, then, due to results of Cartan and Helgason we have $G = \mathrm{Isom}(M)^0$.

If N is a normal subgroup of G and $N \subset K$, then $G/K = (G/N)/(K/N)$ and N acts on M trivially. Therefore, we may consider in the sequel only effective pairs. Recall that a G-invariant Riemannian metric on M is determined by an element of the set $\mathbb{B}(\mathfrak{m})$, i.e., by a K-invariant scalar product on $\mathfrak{m}$. Note that the normaliser $N_G(K)$ naturally acts on $\mathfrak{m} = \mathfrak{g}/\mathfrak{k}$.

Lemma 2. *Let G/K be a weakly symmetric homogeneous space. Then each K-invariant scalar product on $\mathfrak{m}$ is also $N_G(K)$-invariant.*

Proof. The orbits of K and $N(K)$ in $\mathfrak{m}$ coincide, see [5]. Hence, these groups have the same invariants in $\mathbb{R}[\mathfrak{m}]$. In particular, $S^2(\mathfrak{m})^K = S^2(\mathfrak{m})^{N(K)}$. □

Corollary 1. *The set $\mathbb{B}(\mathfrak{m})$ depends only on the pair $(\mathfrak{g}, \mathfrak{k})$, i.e., the set of weakly symmetric invariant Riemannian metrics on a simply connected homogeneous space G/K coincides with the analogous set on any homogeneous space locally isomorphic to G/K.*

Suppose (G, K) is a weakly symmetric pair such that $N(K)^0 \neq K$. Then $G/K = (N \times G)/N(K)$, where $N = N(K)/K$ acts on G/K by right multiplications.

Corollary 2 (Of Lemma 2). *Each G-invariant Riemannian metric on G/K is also $(N \times G)$-invariant.*

Let $M = G/K$ be a weakly symmetric homogeneous space. Suppose G has a non-trivial connected centre, i.e., $G' \neq G$. A pair $(G', G' \cap K)$ is called a *truncated* weakly symmetric. Denote by K" a group $G' \cap K$. Note that a truncated weakly symmetric pair might be or might be not weakly symmetric. For example, a pair $(\mathrm{SU}_{n+1}, \mathrm{SU}_n)$ is a truncation of $(\mathrm{U}_{n+1}, \mathrm{U}_n)$ and a non-weakly symmetric pair $(\mathrm{SU}_{2n}, \mathrm{SU}_n \times \mathrm{SU}_n)$ is a truncation of a weakly symmetric pair $(\mathrm{U}_{2n}, \mathrm{U}_n \times \mathrm{SU}_n)$.

Corollary 3 (Of Lemma 2). *If $(G', K")$ is a weakly symmetric pair, then the sets of G- and G'-invariant Riemannian metrics on M coincide.*

The last statement means that the homogeneous spaces G/K and G'/K" correspond exactly to the one and the same weakly symmetric Riemannian manifold.

As proved in [7], N is an Abelian group. In particular, replacing G by $G \times N$ is, in a sense, a process inverse to a truncation.

Lemma 3. *Let (G, K) be an irreducible weakly symmetric pair, where $\mathfrak{z}(\mathfrak{g}) \neq 0$. Suppose the corresponding truncated pair is reducible, i.e., $(G', G' \cap K) = (G_1, K_1) \times (G_2, K_2)$. Then there are weakly symmetric pairs $(\check{G}_1, \check{K}_1)$ and $(\check{G}_2, \check{K}_2)$ such that their product is an extension of (G, K) and the sets of G- and $(\check{G}_1 \times \check{G}_2)$-invariant Riemannian metrics on G/K coincide.*

Proof. Set $(\check{G}_1, \check{K}_2) := (G/G_2, K/(K \cap G_2))$ and $(\check{G}_2, \check{K}_2) := (G/G_1, K/(K \cap G_1))$. Note that there is a decomposition $\check{G}_1 \times \check{G}_2 = G'(\check{K}_1 \times \check{K}_2)$. The group G is embedded in a natural way into $\check{G}_1 \times \check{G}_2$, here the centre of G is embedded diagonally into the product of centres of G/G_2 and G/G_1. Evidently, $(G/G_2) \times (G/G_1) = G(\check{K}_1 \times \check{K}_2)$, hence, the product $(\check{G}_1, \check{K}_2) \times (\check{G}_2, \check{K}_2)$ is really an extension of (G, K).

The pair $(G/G_2 \times G/G_1, K)$ is weakly symmetric, because its central reduction coincides with the central reduction of (G, K). Evidently, $K/(K \cap G_2) \times K/(K \cap G_1) \subset N(K)$ (here we consider the normaliser in $\check{G}_1 \times \check{G}_2$). In particular, the orbits of K and $\check{K}_1 \times \check{K}_2$ in $\mathfrak{m} = \mathfrak{g}_1/\mathfrak{k}_1 \oplus \mathfrak{g}_1/\mathfrak{k}_2$ are the same. To conclude the proof, note that $T_{eK}(G/K)$ is isomorphic to $\mathfrak{m}$ as a K-, $N(K)$- and, hence, $\check{K}_1 \times \check{K}_2$-module. $\square$

Corollary. *Every symmetric metric on G/K is a product of symmetric metrics on G_1/K_1 and G_2/K_2.*

In what follows we will consider only those weakly symmetric spaces G/K, whose truncated pairs $(G', K")$ are irreducible. As was already shown, $(G', N_{G'}(K"))$ is a weakly symmetric pair, moreover, it is irreducible if $(G', K")$ is irreducible. In particular, we can assume that the list of the irreducible truncated pairs is known.

Recall that a weakly symmetric space G/K under consideration is also a homogeneous space of semisimple group G'.

Theorem 2. *An irreducible (as a homogeneous space) non-symmetric non-compact homogeneous space of a semisimple group G is not a symmetric Riemannian manifold regardless of the choice of a G-invariant metric.*

Proof. Assume that $M = G/K$ is symmetric. Decompose it into a product of irreducible Riemannian manifolds. Let M_n be a product of all non-compact factors, i.e., a symmetric space of negative curvature. Suppose a semisimple group $H \subset \mathrm{Isom}(M_n)$ acts transitively on M_n. By the Karpelevich theorem, see [1, Theorem 1], there is a Cartan involution σ of $\mathrm{Isom}(M_n)$ such that $\sigma(H) = H$ and $\sigma|_{\mathfrak{h}}$ is a Cartan involution of $\mathfrak{h}$. In particular, $\mathfrak{h} = \mathfrak{k}_1 \oplus \mathfrak{m}_1$, $\mathfrak{m}_1 \subset \mathfrak{m}$. But then $\mathfrak{m}_1 = \mathfrak{m}$. Because $[\mathfrak{m}, \mathfrak{m}] \oplus \mathfrak{m} = \mathfrak{isom}(M_n)$, we have $\mathfrak{h} = \mathfrak{isom}(M_n)$. Hence, any connected semisimple subgroup of $\mathrm{Isom}(M_n)$ acting transitively on M_n coincides with $\mathrm{Isom}(M_n)^0$. The group $\mathrm{Isom}(M_n)$ can not act non-trivially on a compact or locally euclidian symmetric Riemannian manifold. This means that G contains $\mathrm{Isom}(M_n)^0$ as a factor. Thus, if $M = M_n$, then the homogeneous space G/K is symmetric and if $M \neq M_n$, then G/K is reducible (as a homogeneous space). $\square$

From now on, assume that G/K is compact or, equivalently, that G is compact. It is well known, that the isometry group of a compact Riemannian manifold is compact.

Definition 7. Let $\mathfrak{p}$, $\mathfrak{q}$, $\mathfrak{g}$, $\mathfrak{k}$ be Lie algebras, where $\mathfrak{q} \subset \mathfrak{p}$ and $\mathfrak{k} \subset \mathfrak{g}$. A pair $(\mathfrak{p}, \mathfrak{q})$ is called *an extension* of $(\mathfrak{g}, \mathfrak{k})$, if

$$\mathfrak{g} \subsetneqq \mathfrak{p}, \quad \mathfrak{p} = \mathfrak{g} + \mathfrak{q}, \quad \mathfrak{k} = \mathfrak{q} \cap \mathfrak{g}.$$

If $\mathfrak{q} = \mathfrak{p}^\sigma$, where σ is an involution of $\mathfrak{p}$, then we call $(\mathfrak{p}, \mathfrak{q})$ *a symmetric extension* of $(\mathfrak{g}, \mathfrak{k})$.

Let us call a pair of algebras $(\mathfrak{g}, \mathfrak{k})$, corresponding to a weakly symmetric pair (G, K), weakly symmetric.

If G is embedded into P, then (P, Q) is a symmetric extension of (G, K) if and only if $(\mathfrak{p}, \mathfrak{q})$ is a symmetric extension of $(\mathfrak{g}, \mathfrak{k})$. Therefore, to solve our problem we have to describe only the symmetric extensions of weakly symmetric pairs of algebras.

We shall say that $(\mathfrak{g}, \mathfrak{h})$ is effective if $\mathfrak{h}$ does not contains any ideals of $\mathfrak{g}$. If (G, H) is locally effective then $(\mathfrak{g}, \mathfrak{h})$ is effective.

We will deal with complex Lie algebras for the convenience of the reader. Let $\mathfrak{g}_\mathbb{C}$, $\mathfrak{k}_\mathbb{C}$, $\mathfrak{p}_\mathbb{C}$, $\mathfrak{q}_\mathbb{C}$ be the complexifications of $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{p}$, $\mathfrak{q}$, respectively. A pair $(\mathfrak{p}, \mathfrak{q})$ is a symmetric extension of $(\mathfrak{g}, \mathfrak{k})$ if and only if $(\mathfrak{p}_\mathbb{C}, \mathfrak{q}_\mathbb{C})$ is a symmetric extension of $(\mathfrak{g}_\mathbb{C}, \mathfrak{k}_\mathbb{C})$.

4. Symmetric extensions of spherical pairs

4.0. In this section all algebras are complex and reductive and all pairs of Lie algebras are effective.

We introduce an addition on the set of pairs of algebras:

$$(\mathfrak{g}^1, \mathfrak{h}^1) \oplus (\mathfrak{g}^2, \mathfrak{h}^2) = (\mathfrak{g}^1 \oplus \mathfrak{g}^2, \mathfrak{h}^1 \oplus \mathfrak{h}^2).$$

A pair $(\mathfrak{g}, \mathfrak{h})$ is said to be irreducible if it can not be realised as a sum of two pairs.

Let $(\mathfrak{g}, \mathfrak{h})$ be a spherical pair. Then we call $(\mathfrak{g}', \mathfrak{g}' \cap \mathfrak{h})$ a *truncated spherical* pair. Denote by $\mathfrak{h}$" an algebra $\mathfrak{g}' \cap \mathfrak{h}$.

In what follows we assume that $(\mathfrak{g} = \bigoplus \mathfrak{g}_j, \mathfrak{h})$ is an irreducible spherical or truncated spherical pair, $\mathfrak{g}$ is a reductive algebra, $\mathfrak{g}_j$ is a simple ideal of $\mathfrak{g}$, $\mathfrak{h}_j$ is a projection of $\mathfrak{h}$ on $\mathfrak{g}_j$, $\mathfrak{h}_{i,j}$ is a projection of $\mathfrak{h}$ on an ideal $\mathfrak{g}_i \oplus \mathfrak{g}_i$.

A pair $(\mathfrak{p}, \mathfrak{q}) = \bigoplus(\mathfrak{p}_i, \mathfrak{q}_i)$ is symmetric, $(\mathfrak{p}_i, \mathfrak{q}_i)$ is an irreducible symmetric pair, $\mathfrak{p}$ is a semisimple Lie algebra; π_i is a projection operator on $\mathfrak{p}_i$.

We call a pair of algebras $(\mathfrak{g}, \mathfrak{h})$ simple (semisimple), if $\mathfrak{g}$ is simple (semisimple).

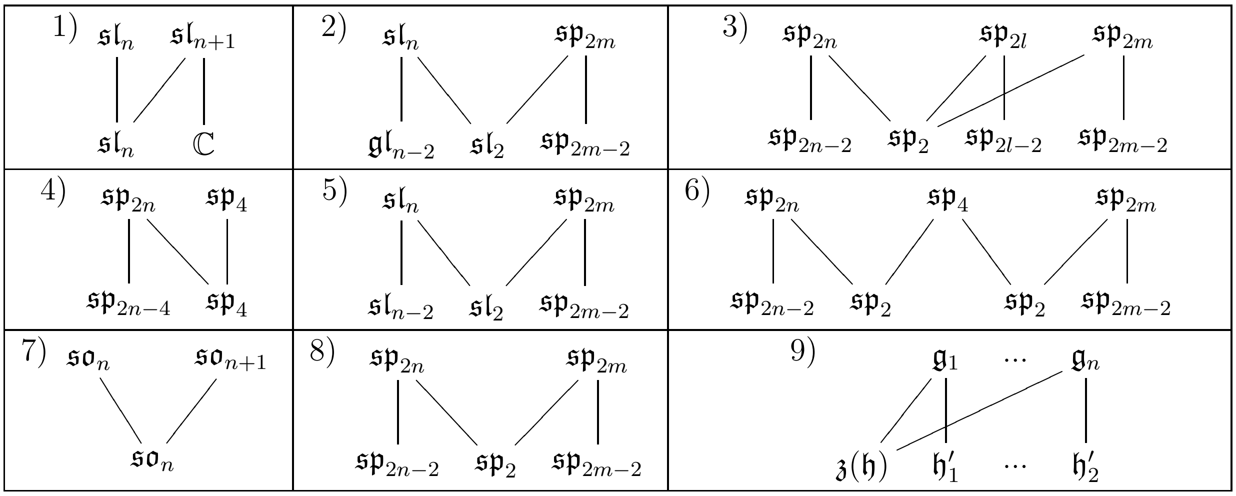
The purpose of this section is to describe all symmetric extensions of reductive spherical pairs. Along with other things, we use the fact that any extension of a reductive spherical pair is also an extension of the corresponding semisimple truncated spherical pair.

Note that $(\mathfrak{g}, 0)$ is an extension of truncated spherical pair only if $\mathfrak{g} = \mathfrak{sl}_2$.

For the clarity of our presentation all irreducible non-symmetric spherical pairs in which $\mathfrak{g}$ is not simple are listed in Table 2.

In each case $\mathfrak{g}$ equals the direct sum of algebras in the upper row, $\mathfrak{h}$ equals the direct sum of algebras in the lower row; the embeddings are denoted by line segments. Case 9 is a scheme of a non-principal spherical pair (see [3]). As proved in Theorem 3, non-principal pairs have no symmetric extensions, therefore, no details concerning their structure will be given. Notice only that $\mathfrak{z}(\mathfrak{h}) \neq 0$ for a non-principal pair $(\mathfrak{g}, \mathfrak{h})$.

Table 2.



4.1. The decompositions of reductive Lie algebras into sums of two reductive subalgebras are described in [4]. The decompositions of simple algebras are listed in Table 3. Analysing it, we will describe all symmetric extensions of spherical pairs. Decompositions of non-simple algebras are more intricate. The dealing with all decomposition would only made our paper more complicated.

Table 3.

$\mathfrak{g}$	$\mathfrak{g}^1$	φ^1	$\mathfrak{g}^2$	φ^2	$\mathfrak{u}$
$\mathfrak{sl}_{2n}$	$\mathfrak{sp}_{2n}$	ϖ_1	$\mathfrak{sl}_{2n-1}$	$\varpi_1 + \mathbf{1}$	$\mathfrak{sp}_{2n-2}$
			$\mathfrak{sl}_{2n-1} \oplus \mathbb{C}$		$\mathfrak{sp}_{2n-2} \oplus \mathbb{C}$
$\mathfrak{so}_7$	G_2	ϖ_1	$\mathfrak{so}_5$	$\varpi_1 + \mathbf{1} + \mathbf{1}$	$\mathfrak{sl}_2$
			$\mathfrak{so}_5 \oplus \mathbb{C}$		$\mathfrak{sl}_2 \oplus \mathbb{C}$
			$\mathfrak{so}_6$	$\varpi_1 + \mathbf{1}$	$\mathfrak{sl}_3$
$\mathfrak{so}_{2n+4}$	$\mathfrak{so}_{2n+3}$	$\varpi_1 + \mathbf{1}$	$\mathfrak{sl}_{n+2}$	$\varpi_1 + \varpi_{n+1}$	$\mathfrak{sl}_{n+1}$
			$\mathfrak{sl}_{n+2} \oplus \mathbb{C}$		$\mathfrak{sl}_{n+1} \oplus \mathbb{C}$
$\mathfrak{so}_{4n}$	$\mathfrak{so}_{4n-1}$	$\varpi_1 + \mathbf{1}$	$\mathfrak{sp}_{2n}$	$\varpi_1 + \varpi_1$	$\mathfrak{sp}_{2n-2}$
			$\mathfrak{sp}_{2n} \oplus \mathbb{C}$		$\mathfrak{sp}_{2n-2} \oplus \mathbb{C}$
			$\mathfrak{sp}_{2n} \oplus \mathfrak{sl}_2$		$\mathfrak{sp}_{2n-2} \oplus \mathfrak{sl}_2$
$\mathfrak{so}_{16}$	$\mathfrak{so}_{15}$	$\varpi_1 + \mathbf{1}$	$\mathfrak{so}_9$	ϖ_4	$\mathfrak{so}_7$
$\mathfrak{so}_8$	$\mathfrak{so}_7$	ϖ_3	$\mathfrak{so}_7$	$\varpi_1 + \mathbf{1}$	G_2

Here $\mathfrak{g}^1$, $\mathfrak{g}^2$ are subalgebras of $\mathfrak{g}$, $\mathfrak{g} = \mathfrak{g}^1 + \mathfrak{g}^2$, $\mathfrak{u} = \mathfrak{g}^1 \cap \mathfrak{g}^2$. In all cases $n > 1$, φ^1 and φ^2 are the restrictions of the defining representation of $\mathfrak{g}$ on $\mathfrak{g}^1$ and $\mathfrak{g}^2$ (highest weights are indicated), ϖ_m are the fundamental weights, $\mathbf{1}$ is a trivial representation.

Notice that in Table 3 all algebras $\mathfrak{g}^1$ are simple; if $\mathfrak{g}^2$ is not simple, then $\mathfrak{g}^2 = \mathfrak{r} \oplus \mathfrak{a}$, where $\mathfrak{r}$ is simple, $\mathfrak{a} = \mathbb{C}$ or $\mathfrak{a} = \mathfrak{sl}_2$.

Note also that a pair $(\mathfrak{g}, \mathfrak{u})$ is spherical or truncated spherical only in one case, namely then $\mathfrak{g} = \mathfrak{so}_8,\ \ \mathfrak{u} = G_2$.

Table 3 gives us all simple symmetric extensions. One has to choose those rows in which $(\mathfrak{g}, \mathfrak{g}_1)$ (or $(\mathfrak{g}, \mathfrak{g}_2)$) is symmetric and another pair $(\mathfrak{g}_1, \mathfrak{u})$ (or, respectively $(\mathfrak{g}_1, \mathfrak{u})$) is spherical. These extensions are listed in rows 1a, 3, 5, 7, 12, 14, 16 and 24b of Table 1.

4.2. Here we describe non-simple symmetric extensions.

Lemma 4. *Let a symmetric pair* $(\mathfrak{q} \oplus \mathfrak{q}, \mathfrak{q})$ *be an extension of a spherical pair* $(\mathfrak{g}, \mathfrak{h})$, *where* $\mathfrak{q}$ *is a simple Lie algebra. Then* $(\mathfrak{g}, \mathfrak{h}) = (\mathfrak{q} \oplus \mathfrak{h}, \mathfrak{h})$ *and the schemes of the embeddings* $\mathfrak{g} \subset (\mathfrak{q} \oplus \mathfrak{q})$, $\mathfrak{h} \subset \mathfrak{g}$ *have the following form.*

$$\begin{tikzcd} \mathfrak{q} \arrow[d, dash] & \mathfrak{q} \arrow[d, dash] & \mathfrak{q} \arrow[dr, dash] & & \mathfrak{h} \arrow[dl, dash] \\ \mathfrak{q} & \mathfrak{h} & & \mathfrak{h} & \end{tikzcd}$$

Moreover, for any proper subalgebra $\mathfrak{f} \subset \mathfrak{h}$ *that does not contain* $\mathfrak{h}'$, *a pair* $(\mathfrak{g}, \mathfrak{f})$ *is not spherical or either truncated spherical.*

Proof. Denote by $\mathfrak{q}_1$ and $\mathfrak{q}_2$ the projections of $\mathfrak{g}$ on the first and second summand of $\mathfrak{p} = \mathfrak{q} \oplus \mathfrak{q}$, respectively. Evidently, $\mathfrak{q}_1 + \mathfrak{q}_2 = \mathfrak{q}$. The pair $(\mathfrak{q}_1 \oplus \mathfrak{q}_2, \mathfrak{q}_1 \cap \mathfrak{q}_2)$ is an extension of $(\mathfrak{g}, \mathfrak{h})$, hence it is spherical.

Assume that neither $\mathfrak{q}_1$ nor $\mathfrak{q}_2$ is equal to $\mathfrak{q}$, i.e., there is a non-trivial decomposition of $\mathfrak{q}$. It can be seen form Table 3 that $\mathfrak{q}_1$ is a simple algebra and $\mathfrak{q}_2$ is a sum of at most two ideals (or vise versa). We also can say that $\mathfrak{q} = \mathfrak{q}_1 + \mathfrak{q}_2'$. As a corollary of the classification of the spherical pairs, see Table 2, we have $\mathfrak{q}_2' \subset \mathfrak{q}_1$. But in that case $\mathfrak{q} = \mathfrak{q}_1 + \mathfrak{q}_2' = \mathfrak{q}_1$. If $\mathfrak{g} = \mathfrak{q}$, then $\mathfrak{h} = 0$, but it is impossible because $(\mathfrak{g}, \mathfrak{h})$ is spherical.

Thus, we have proved that $\mathfrak{g}$ contains one of the summands $\mathfrak{q}$. Hence, the structure of $(\mathfrak{g}, \mathfrak{h})$ and the embedding schemes described in the lemma settings are entirely determined. The second statement of the lemma can be deduced from the classification of the spherical pair. One can easily check it using Table 2. $\square$

The rows 17, 22b and 23 of Table 1 correspond to a symmetric extensions of a type described in Lemma 4.

Lemma 5. *If a simple symmetric pair* $(\mathfrak{p}, \mathfrak{q})$ *is an extension of a semisimple truncated spherical (or spherical) pair* $(\mathfrak{g}, \mathfrak{h})$, *where* $\mathfrak{g}$ *is not simple, then*

$$\begin{cases} (\mathfrak{p}, \mathfrak{q}) = (\mathfrak{so}_{4n}, \mathfrak{so}_{4n-1}), \\ (\mathfrak{g}, \mathfrak{h}) = (\mathfrak{sp}_{2n} \oplus \mathfrak{sl}_2, \mathfrak{sp}_{2n-2} \oplus \mathfrak{sl}_2). \end{cases}$$

Moreover, a pair $(\mathfrak{g}, \mathfrak{f})$ *is truncated spherical (or spherical) for neither proper subalgebra* $\mathfrak{f} \subset \mathfrak{h}$.

Proof. The first statement can be deduced from Table 3 and the second one from Table 2. $\square$

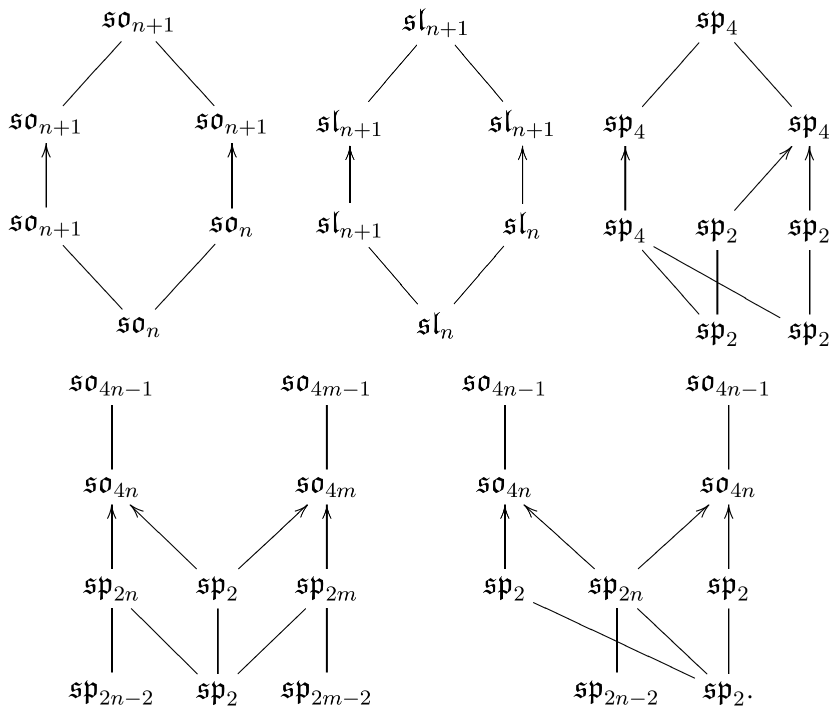
Lemma 6. *If a non-simple symmetric pair* $(\mathfrak{p}, \mathfrak{q})$ *is an extension of a simple spherical pair* $(\mathfrak{g}, \mathfrak{h})$*, then*

$$\begin{cases} (\mathfrak{p}, \mathfrak{q}) = (\mathfrak{so}_8 \oplus \mathfrak{so}_8, \mathfrak{so}_7 \oplus \mathfrak{so}_7), \\ (\mathfrak{g}, \mathfrak{h}) = (\mathfrak{so}_8, G_2). \end{cases}$$

Proof. Suppose $(\mathfrak{p}, \mathfrak{q})$ is a symmetric extension of $(\mathfrak{g}, \mathfrak{h})$. Due to Lemma 4, $(\mathfrak{p}, \mathfrak{q})$ does not contain summands of the type $(\mathfrak{q} \oplus \mathfrak{q}, \mathfrak{q})$. Let $\mathfrak{p}_1$ and $\mathfrak{p}_2$ be a simple ideals of $\mathfrak{p}$. Then there is a decomposition $\mathfrak{g} = \mathfrak{l}_1 + \mathfrak{l}_2$, where $\mathfrak{l}_1 = \pi_1(\mathfrak{g}) \cap \mathfrak{q}_1$, $\mathfrak{l}_2 = \pi_2(\mathfrak{g}) \cap \mathfrak{q}_2$. Because $\mathfrak{h} \subset (\mathfrak{l}_1 \cap \mathfrak{l}_2)$, t pair $(\mathfrak{g}, \mathfrak{u} = \mathfrak{l}_1 \cap \mathfrak{l}_2)$ is spherical. But it can happen only if $\mathfrak{g} = \mathfrak{so}_8$, $\mathfrak{h} = G_2$.

The pair $(\mathfrak{so}_8, G_2)$ has a symmetric extension, namely it is the pair $(\mathfrak{so}_8, \mathfrak{so}_8^{\sigma_1}) \oplus (\mathfrak{so}_8, \mathfrak{so}_8^{\sigma_2})$, where σ_1 and σ_2 are different non-conjugated outer involutions of $\mathfrak{so}_8$ such that $\mathfrak{so}_8^{\sigma_1} = \mathfrak{so}_8^{\sigma_2} = \mathfrak{so}_7$. □

Theorem 3. *There are only five symmetric extensions of irreducible spherical pair that are not described above, namely:*



(In each case $\mathfrak{q}$ *equals the direct sum of the upper algebras,* $\mathfrak{p}$ *equals the direct sum of the algebras in the second row,* $\mathfrak{g}$ *– in the third row,* $\mathfrak{h}$ *– in the lower row.)*

Proof. Suppose $(\mathfrak{p}, \mathfrak{q})$ is a symmetric extension of $(\mathfrak{g}, \mathfrak{h})$. Recall that it is also an extension of a corresponding irreducible truncated spherical pair $(\mathfrak{g}', \mathfrak{h}")$, where $\mathfrak{h}" = \mathfrak{g}' \cap \mathfrak{q}$. The irreducibility of $(\mathfrak{g}', \mathfrak{h}")$ means that for any simple ideal $\mathfrak{g}_i$ there is a simple ideal $\mathfrak{g}_j$ $(i \neq j)$ and an algebra $\mathfrak{p}_l$ such that $\mathfrak{g}_i \oplus \mathfrak{g}_j$ is embedded into $\mathfrak{p}_l$ and $(\mathfrak{g}_i \oplus \mathfrak{g}_j, \mathfrak{q}_l \cap (\mathfrak{g}_i \oplus \mathfrak{g}_j))$ is irreducible.

Due to Lemmas 4 and 5, we know that $(\mathfrak{h}")_i$ and $(\mathfrak{h}")_j$ are semisimple, i.e., $\mathfrak{h}"$ is semisimple. In particular, non-principal symmetric pairs have no symmetric extensions. Since algebras $\mathfrak{h}"$ corresponding to them have a non-trivial centres.

If $\mathfrak{p}_l$ is simple, then by Lemma 5, $(\mathfrak{p}_l, \mathfrak{q}_l)$ is an extension of $(\mathfrak{g}_i \oplus \mathfrak{g}_j, (\mathfrak{h}")_{i,j})$. Suppose $\mathfrak{p}_l = \mathfrak{q}_l \oplus \mathfrak{q}_l$. Then if $(\mathfrak{g}_i \oplus \mathfrak{g}_j, \mathfrak{q}_l \cap (\mathfrak{g}_i \oplus \mathfrak{g}_j))$ is irreducible, then, by Lemma 4, $\mathfrak{g}_i = \mathfrak{q}_l$ or $\mathfrak{g}_j = \mathfrak{q}_l$. In any case $(\mathfrak{p}_l, \mathfrak{q}_l)$ is an extension of $(\mathfrak{g}_i \oplus \mathfrak{g}_j, (\mathfrak{h}")_{i,j})$.

Note that if
$$(\mathfrak{p}, \mathfrak{q}) = (\mathfrak{p}^1, \mathfrak{q}^1) \oplus (\mathfrak{p}^2, \mathfrak{q}^2),$$
and $(\mathfrak{p}^1, \mathfrak{q}^1)$ is an extension of $(\mathfrak{g}, \mathfrak{h})$ (or of $(\mathfrak{g}', \mathfrak{h}")$), then $(\mathfrak{p}^2, \mathfrak{q}^2) = 0$. (This is a corollary of the fact that a symmetric pair has no symmetric extension.) Hence, if $\mathfrak{g}$ has only two simple ideals, then its symmetric extension is an irreducible pair. All such extensions are described above.

If $(\mathfrak{g}', \mathfrak{h}")$ is irreducible, then, as can be seen in Table 2, $\mathfrak{g}'$ is a sum of at most three simple ideals. Moreover, if it is a sum of exactly three ideals, then $(\mathfrak{g}', \mathfrak{h}")$ is spherical and the centre of a normaliser of $\mathfrak{h}"$ in $\mathfrak{g}'$ is trivial. Hence, in the given case $\mathfrak{g} = \mathfrak{g}'$ and $(\mathfrak{g}, \mathfrak{h})$ ia a pair from the third or sixth row of Table 2. So, to conclude we have to describe symmetric extensions of these two spherical pairs.

Suppose that the embedding scheme of $\mathfrak{h} \subset \mathfrak{g}$ has a suspended vertex, corresponding to $\mathfrak{h}$, i.e., there is a simple ideal $\mathfrak{g}$ of $\mathfrak{h}$, which is a subalgebra of some simple ideal of $\mathfrak{g}$ (without loss of generality we can assume that $\mathfrak{f} \subset \mathfrak{g}_1$). Then, by Lemma 4, an ideal $\mathfrak{g}_1$ is embedded into a simple symmetric pair. According to Lemma 5, we have $\mathfrak{g}_1 = \mathfrak{sp}_{2n}$ and there is a simple ideal $(\mathfrak{g}_2)$ such that $(\mathfrak{g}_1 \oplus \mathfrak{g}_2, \mathfrak{h}_{1,2})$ is a pair from the eighth row of Table 2 with $m = 1$. Moreover, there is a pair
$$(\mathfrak{p}_j, \mathfrak{q}_j) = (\mathfrak{so}_{4n}, \mathfrak{so}_{4n-1}),$$
such that $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ is embedded into $\mathfrak{p}_j$. It means that if a pair from the third row of Table 2 has a symmetric extension, then one of the numbers n, m, l is equal to 1. Moreover, the symmetric extension has to be a sum of two pais, where one of the simple ideals of $\mathfrak{g}$ is embedded into both summands. If this ideal is not equal to $\mathfrak{sp}_2$, then the others are.

In the sixth case both n and m have to be equal to 1. Theoretically, besides the given in the theorem extension one more symmetric extension could be possible:
$$\begin{tikzcd}
 & \mathfrak{so}_7 & & \mathfrak{so}_7 & \\
 & \mathfrak{so}_8 \arrow[u, dash] & & \mathfrak{so}_8 \arrow[u, dash] & \\
\mathfrak{sp}_2 \arrow[ur] \arrow[dr, dash] & & \mathfrak{sp}_4 \arrow[ul] \arrow[ur] \arrow[dl, dash] \arrow[dr, dash] & & \mathfrak{sp}_2 \arrow[ul] \arrow[dl, dash] \\
 & \mathfrak{sp}_2 & & \mathfrak{sp}_2 &
\end{tikzcd},$$
but in this case we have not the decomposition of $\mathfrak{so}_8 \oplus \mathfrak{so}_8$, because $\mathfrak{sp}_4$ does not have non-trivial decompositions. $\square$

5. Extensions of reducible pairs

In this section we consider only real Lie algebras. Still we keep the notation of the previous section. Thus, $(\mathfrak{p}, \mathfrak{q}) = \bigoplus (\mathfrak{p}_i, \mathfrak{q}_i)$ is a symmetric pair, $(\mathfrak{p}_i, \mathfrak{q}_i)$ is an irreducible symmetric pair, $\mathfrak{p}$ is a semisimple Lie algebra; π_i is a projection onto $\mathfrak{p}_i$.

Lemma 7. *Suppose a real semisimple symmetric pair* $(\mathfrak{p}, \mathfrak{q})$ *is an extension of a reducible spherical pair* $(\mathfrak{g}, \mathfrak{h}) = (\mathfrak{g}_1, \mathfrak{h}_1) \oplus (\mathfrak{g}_2, \mathfrak{h}_2)$. *Then* $\pi_i(\mathfrak{z}(\mathfrak{g}_1)) = 0$ *or* $\pi_i(\mathfrak{z}(\mathfrak{g}_2)) = 0$ *for any* i.

Proof. Assume that it is not true for some i, i.e., there are two vectors $\xi_1 \in \mathfrak{z}(\mathfrak{g}_1)$, $\xi_2 \in \mathfrak{z}(\mathfrak{g}_2)$ such that $\pi_i(\xi_1) \neq 0$ and $\pi_i(\xi_2) \neq 0$. Note that $\dim \mathfrak{z}(\pi_i(\mathfrak{g})) = \dim \mathfrak{z}((\pi_i(\mathfrak{g}) \cap \mathfrak{q}_i)) = 1$. For a simple pair $(\mathfrak{p}_i, \mathfrak{q}_i)$ it can be seen from Table 3, for a pair $(\mathfrak{q} \oplus \mathfrak{q}, \mathfrak{q})$ it is due to Lemma 4, since a centre of a spherical subalgebra in an irreducible pair is at most one-dimensional.

If $(\mathfrak{p}_i, \mathfrak{q}_i)$ is simple then denote by $\mathfrak{f}$ a simple ideal of $\mathfrak{g}$ isomorphically embedded, as can be seen from Table 3, into $\pi_i(\mathfrak{g})'$; and if $(\mathfrak{p}_i, \mathfrak{q}_i) = (\mathfrak{q} \oplus \mathfrak{q}, \mathfrak{q})$ then denote by $\mathfrak{f}$ a simple ideal isomorphically embedded into $\mathfrak{q} \oplus 0$.

Denote by $\hat{\mathfrak{h}}$ and $\tilde{\mathfrak{h}}$ projections of $\mathfrak{h}$ and $\mathfrak{h} \cap \pi_i^{-1}(\mathfrak{q})$ onto $\mathfrak{f} \oplus \mathbb{R}\xi_1 \oplus \mathbb{R}\xi_2$, respectively. Evidently, $\hat{\mathfrak{h}} \subset \tilde{\mathfrak{h}}$. We can assume without loss of generality that $\mathfrak{f} \subset \mathfrak{g}_1$. Then $\hat{\mathfrak{h}}$ contains the vector ξ_2. But, on the other hand, $\dim \mathfrak{z}(\tilde{\mathfrak{h}}) = 2$ and the embedding scheme of $\tilde{\mathfrak{h}} \subset (\mathfrak{f} \oplus \mathbb{R}\xi_1 \oplus \mathbb{R}\xi_2)$ has the following form.

$$\begin{tikzcd}
\mathfrak{f} \arrow[d, dash] \arrow[dr, dash] & \mathbb{R}\xi_1 \arrow[d, dash] \arrow[dr, dash] & \mathbb{R}\xi_2 \arrow[d, dash] \arrow[dl, dash] \\
\tilde{\mathfrak{h}}' & \mathbb{R}\eta_1 & \mathbb{R}\eta_2
\end{tikzcd}$$

Hence, ξ_2 can not be contained in $\tilde{\mathfrak{h}}$. ◻

The proof of Theorem 1. Suppose an element $b \in \mathbb{B}(\mathfrak{m})$ determined a symmetric Riemannian metric on G/K. It is sufficient to show that $b(\mathfrak{m}_1, \mathfrak{m}_2) = 0$. If $N_{G_1}(K_1) = K_1$ or $N_{G_2}(K_2) = K_2$, then $\mathfrak{g}_1/\mathfrak{k}_1$ and $\mathfrak{g}_2/\mathfrak{k}_2$ have no isomorphic non-trivial $K_1 \times K_2$-submodules, hence, $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are orthogonal with respect to any element of $b \in \mathbb{B}$.

Consider the other case. Let $(G_1', \hat{K}_1)$ be a central reduction of (G_1, K_1), i.e., $\hat{K}_1 = K_1/(K_1 \cap Z(G_1))$. Denote by N the group $N_{G_1'}(\hat{K}_1)/\hat{K}_1$. Instead of (G_1, K_1) we can deal with its extension $(N \times G_1, N_{G_1'}(\hat{K}_1))$. As was already said, the scalar product b is also $N_{G'}(\hat{K}_1)$-invariant and N is Abelian.

Thus, we can assume that the condition $N_{G_i'}(\hat{K}_i) = \hat{K}_i$ for $i = 1, 2$ was satisfied initially. In particular, there is no non-zero K-invariant vector in $\mathfrak{g}_i'/\hat{\mathfrak{k}}_i$. Hence, $b(\mathfrak{g}_1'/\hat{\mathfrak{k}}_1, \mathfrak{g}_2'/\hat{\mathfrak{k}}_2) = 0$. To conclude we have to verify that $(\mathfrak{z}(\mathfrak{g}_1) + \mathfrak{k})/\mathfrak{k}$ and $(\mathfrak{z}(\mathfrak{g}_2) + \mathfrak{k})/\mathfrak{k}$ are orthogonal with respect to b.

Denote by Q the stabiliser of eK in $P = \mathrm{Isom}(M)^0$. Let $\mathfrak{m} = \bigoplus \mathfrak{m}_i$ be a decomposition of $\mathfrak{m}$ into the sum of irreducible representations of Q. It is well known, that $b(\mathfrak{m}_i, \mathfrak{m}_j) = 0$ for $i \neq j$. Due to Lemma 7, we know that the spaces $(\mathfrak{z}(\mathfrak{g}_1) + \mathfrak{q})/\mathfrak{q}$ and $(\mathfrak{z}(\mathfrak{g}_2) + \mathfrak{q})/\mathfrak{q}$ are embedded into the sums $\sum\limits_{i \in I_1} \mathfrak{m}_i$ and $\sum\limits_{i \in I_2} \mathfrak{m}_i$, respectively, where $I_1 \cap I_2 = \varnothing$. Hence, they are orthogonal with respect to b. ◻

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