A PARAMETRIZATION OF THE THETA DIVISOR OF THE QUARTIC DOUBLE SOLID

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ABSTRACT. Let $M_X(2;0,3)$ be the moduli space of rank-2 stable vector bundles with Chern classes $c_1=0, c_2=3$ on the Fano threefold X, the double solid $\mathbb{P}^3$ of index two. We prove that the vector bundles obtained by Serre's construction from smooth elliptic quintic curves on X form an open part of an irreducible component M of $M_X(2;0,3)$ and that the Abel-Jacobi map $\phi : M \to J(X)$ into the intermediate Jacobian $J(X)$ of X defined by the second Chern class is generically finite of degree 84 onto a translate $\Theta+\mathrm{const}$ of the theta-divisor. We also prove that the family of elliptic quintics on a general X is irreducible and of dimension 10.

0. INTRODUCTION

After the famous paper of Clemens and Griffiths [CG], in which they determined the intermediate Jacobian and its theta divisor for the cubic threefold, the next Fano threefold to study was naturally the quartic double solid, that is the double cover $\pi : X_2 \xrightarrow{2:1} \mathbb{P}^3$ ramified in a quartic surface $W \subset \mathbb{P}^3$. In 1981, Welters [We] found a family of curves parametrizing the intermediate Jacobian $J(X_2)$, the septics of genus 4, and this is the simplest known family of generically irreducible curves with this property. He developed also techniques that allow one to decide whether the Abel-Jacobi images of certain families of curves lie in a translate $\Theta+\mathrm{const}$ of the theta divisor, but failed to find one parametrizing the whole of $\Theta+\mathrm{const}$. Such a family was found in 1986 (with a use of the techniques of Welters) by Tikhomirov [T-2]: these are the sextics of genus 3, or the so called Reye sextics. In 1991, Clemens [C-2] re-proved Tikhomirov's result in degenerating the double solid into a pair of $\mathbb{P}^3$'s meeting each other transversely along a quadric, the technique he developed earlier in [C-1]. Several other papers have contributed, since 1981, into the study of the quartic double solids: [T-1], [SV], [De], [V].

In 2001, Tikhomirov [T-3] found a new and simpler parametrization of Θ, the one by the family of elliptic quintics $\mathcal{C}_5^1(X)$. Its advantage is in that the elliptic quintics define stable vector bundles on X via Serre's construction, and the Abel-Jacobi map factors through the moduli space of vector bundles.

The study of moduli spaces of stable vector bundles on Fano threefolds of indices 1 and 2 is quite a recent topic. Remind that the *index* of a Fano threefold X is the maximal integer ν dividing K_X in the Picard group of X. The cubic X_3 and the quartic double solid X_2 are Fano of index 2. The nature of the results obtained depends strongly on the index. In the index-2 case, the answers are given in terms of the Abel–Jacobi map defined by the second Chern class c_2 modulo rational equivalence. In the index-1 case, the Abel–Jacobi map does not bring much new information about the moduli spaces.

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The results which are known so far include the desciption of one component of a moduli space $M_X(2; c_1, c_2)$ of rank-2 vector bundles with $c_1 = 0$ or 1 and small c_2 on each one of the following five Fano threefolds: the cubic X_3 [MT-1], [IM-1], [Dr], [B], the quartic [IM-2], the Fano threefolds X_{2g-2} of the main series of genus 7 [IM-3] and 9 [IR], and the double solid X_2, treated in the present paper and in [T-3]. It is noteworthy that in most cases the interesting components of moduli are those parametrizing the vector bundles obtained by Serre's construction from elliptic quintics. For the cubic X_3, the relevant moduli space is $M_{X_3}(2; 0, 2)$, and the Abel-Jacobi map identifies it with an open set in the intermediate Jacobian $J(X_3)$. For the double solid, the corresponding moduli space $M_X(2; 0, 3)$ is reducible: it has at least two components, the one, say, M, associated to elliptic quintics, and the other to disjoint unions of two elliptic curves in X whose images in $\mathbb{P}^3$ are a plane cubic and a line. Leaving apart the question whether there are any other components, we restrict ourselves to the study of M.

The main result of the paper states that the Abel-Jacobi map Φ sends the family of elliptic quintics $\mathcal{C}^1_5(X)$ onto an open subset of $\Theta + \text{const}$, and that the Serre construction defines a factorization $\mathcal{C}^1_5(X) \xrightarrow{\text{Serre}} M \xrightarrow{g} \Theta + \text{const}$ with g generically finite of degree 84 (Theorems 2.7 and 5.1). We also prove the irreducibility of $\mathcal{C}^1_5(X)$ (Theorem 1.4), so that M is really an irreducible component of the moduli space, rather than a union of components. The facts that the image of Φ is dense in $\Theta + \text{const}$ and that g is generically finite were stated in [T-3], so the main goal of our paper is to determine the degree of g. Nonetheless, we provide a complete proof of these facts too, for their proof was only sketched in loc. cit.

We use Welters' criterion (Proposition 2.4) which gives sufficient conditions for a certain subfamily $\{C_t\}_{t\in T_l}$ of a given family $\{C_t\}_{t\in T}$ of curves of degree $k(k+1)-3$ to be mapped by the Abel-Jacobi map into a translate of the theta divisor. Here l is a line in X and

$$T_l = \{t \in T \mid C_t \cap l = \varnothing \ \text{ and } C_t \cup l \ \text{ lies on a surface from } |\mathcal{O}_X(k)|\},$$

where $\mathcal{O}_X(k) := \pi^*\mathcal{O}_{\mathbb{P}^3}(k)$. We construct such a family T with $k = 4$ as a partial compactification of the one of reducible curves $C = C_1 \cup C_2$, where C_1 is a genus-4 septic, C_2 a pullback of a genus-2 quintic $\underline{C}_2 \subset \mathbb{P}^3$, and C_1, C_2 meet each other quasi-transversely in 13 points. We show that T_l contains a subfamily of curves of the form $C' \cup C'' \cup C_2$ such that $C' \in \mathcal{C}^1_5(X)$ and $C'' = \pi^{-1}(m)$, where $m \subset \mathbb{P}^3$ is a trisecant of $\pi(C')$, whose Abel-Jacobi images differ from those of C' by a constant translation. Hence $\Phi_{\mathcal{C}^1_5(X)}(\mathcal{C}^1_5(X)) \subset \Phi_{T_l}(T_l) + \text{const} \subset \Theta + \text{const}$. The opposite inclusion follows from the computation of the differential of the Abel-Jacobi map. The generic fiber of $\Phi_{\mathcal{C}^1_5(X)}$ is a union of ν copies of $\mathbb{P}^1$, which can be interpreted either as pencils of elliptic quintics $\{C_\lambda\}_{\lambda\in\mathbb{P}^1}$ in the K3 surfaces $S(C_\lambda)$ from the linear system $|\mathcal{O}_X(2)|$, or as pencils of the zero loci of sections $s \in H^0(X, \mathcal{E})$, where $\mathcal{E}$ is the vector bundle obtained by Serre's construction from C_λ.

To compute the number ν of copies of $\mathbb{P}^1$ in the generic fiber of $\Phi_{\mathcal{C}^1_5(X)}$, we use the degeneration $\{X_t\}_{t\in\Delta}$ of X into the union $X_0 = \mathbb{P}^{3'} \cup \mathbb{P}^{3''}$ of two copies of $\mathbb{P}^3$ meeting each other in a smooth quadric $Q = \{G = 0\}$, where Δ is a disc in $\mathbb{C}$. The fiber X_t is defined as the double cover $\pi_t : X_t \xrightarrow{2:1} \mathbb{P}^3$ ramified in $W_t = \{tF + G^2 = 0\}$, where $W = \{F = 0\}$ is the branch locus of $X \longrightarrow \mathbb{P}^3$. The Hodge theory and the Neron model of the relative intermediate Jacobian of such a degeneration were studied by Clemens in [C-1]. He made also the following important observation: let $C_0 \in X_0$ be an irreducible curve of degree d which is the limit of a family of curves $C_s \subset X_t$, $t = s^e$. Assume that C_s for $s \neq 0$ is projected birationally onto its image $\underline{C}_s \subset \mathbb{P}^3$; this means that $\underline{C}_s$ is tangent to W_t in $2d$ points. Then C_0 is a $2d$-secant to the octic curve $B = W \cap Q$. Thus the limits of elliptic quintics from $\mathcal{C}^1_5(X_t)$ are elliptic quintics in $\mathbb{P}^{3'}$ or $\mathbb{P}^{3''}$ which are 10-secant to B. The work of Clemens allows us to extend the Abel-Jacobi map to such curves in X_0 and the question on the number of copies of $\mathbb{P}^1$ in the generic fiber of the Abel-Jacobi map Φ_0 at $t = 0$ is reduced to that on the number of elliptic quintics in $\mathbb{P}^3$ which are 10-secant to a given octic B.

By a result of Getzler [Ge], there are 42 elliptic quintics passing through 10 generic points in $\mathbb{P}^3$. We show that the 10-uples of points lying in the smooth complete intersections of a quartic and a quadric in $\mathbb{P}^3$ form a divisor in the main component $\mathcal{H}$ of $\mathrm{Hilb}^{10}(\mathbb{P}^3)$ and that this divisor is not contained in the branch locus of the 42-sheeted covering defined by the elliptic quintics, hence there are $2 \cdot 42 = 84$ copies of $\mathbb{P}^1$ in the generic fiber of Φ_0 (42 copies in each one of the two $\mathbb{P}^3$'s constituting X_0). We show also that the elliptic quintics, 10-secant to B, are acquired with multiplicity 1 in the relative Hilbert scheme of the family $\{X_t\}_{t\in\Delta}$, hence the number of copies of $\mathbb{P}^1$ in the fiber of Φ_t is invariant under the deformation and $\nu = 84$.

To this end, we show that the family of 10-secant elliptic quintics to a given generic B is irreducible and construct a family $\{C_t\}_{t\in\Delta}$ of cycles of 5 lines, or pentagons in X_t, such that C_0 is 10-secant to B and is strongly smoothable into an elliptic quintic in X_0, 10-secant to B. This implies that a generic 10-secant elliptic quintic in X_0 admits a local cross-section passing through it in the relative Hilbert scheme of $\{X_t\}_{t\in\Delta}$, hence the fiber over $t = 0$ is acquired with multiplicity 1.

Describe now the contents of the paper by sections.

The main result of Section 1 is the proof of the irreducibility of the family of elliptic quintics $\mathcal{C}^1_5(X)$. We start by proving the irreducibility of the families $\mathcal{C}^0_2(X)$ of conics and $\mathcal{C}^0_3(X)$ of twisted cubics. The proofs are based on the splitting of the curves under consideration into two components of smaller degrees. As concerns elliptic quintics, we split them into a twisted cubic and an "elliptic conic", that is, $\pi^{-1}(m)$ for a line $m \subset \mathbb{P}^3$. As soon as the irreducibility of the family of split curves is established, we get a distinguished component of $C^1_5(X)$, namely the one containing the smoothings of the split curves. Then we use the irreducibility of the monodromy action on the set of all the components of $C^1_5(X)$ to deduce that the number of components is 1.

In Section 2 we prove the irreducubility of the family $C^4_7(X)$ of genus-4 septics in X, which dominate $J(X)$ according to Welters, and construct the above mentioned family T of curves of degree $k(k+1) - 3 = 17$ for $k = 4$, to which the criterion of Welters is applied. We conclude by the proof of Theorem 2.7, stating that the Abel-Jacobi image of the elliptic quintics is a translate of Θ.

In Section 3 we explain the Clemens' degeneration technique and describe the limiting Abel-Jacobi map Φ_0 on the degenerate double solid, which consists of two copies of $\mathbb{P}^3$. We also show that the central fiber in the domain of the Abel-Jacobi map is reduced by studying the deformations of 10-secant pentagons. This implies that the number of connected components in the generic fiber of Φ_t is the same as for Φ_0.

In Section 4 we prove that the number of elliptic quintics in $\mathbb{P}^3$ passing through 10 generic points remains unchanged if we specialize the 10-uple of points to a one lying on a generic complete intersection of a quadric and a quartic. This implies that the number of connected components in the generic fiber of Φ_0 is 84.

In Section 5 we introduce the Serre construction and state the main theorem in terms of moduli of vector bundles.

1. Conics, twisted cubics and elliptic quintics on X

1.1. Notations.

- X, a general quartic double solid.
- $\pi : X \longrightarrow \mathbb{P}^3$, the natural double covering map.
- $G(1,3)$, the Grassmannian of lines in $\mathbb{P}^3$.
- $\mathcal{C}_2^0 = \mathcal{C}_2^0(X) := \{C \in \mathrm{Hilb}(X) |\ C$ is a smooth irreducible conic in $X\}$.
- $\mathcal{C}_2^1(X) := \{C \in \mathrm{Hilb}(X) \mid C = \pi^{-1}(l),\ l \in G(1,3)\}$; we call the curves from $\mathcal{C}_2^1(X)$ elliptic conics on X.
- $\mathcal{C}_3^0 = \mathcal{C}_3^0(X) := \{C \in \mathrm{Hilb}(X) |\ C$ is a smooth irreducible twisted cubic in $X\}$.
- $\mathcal{C}_5^1 = \mathcal{C}_5^1(X) := \{C \in \mathrm{Hilb}(X) |\ C$ is a smooth irreducible elliptic quintic in $X\}$.
- $\mathcal{F} = \mathcal{F}(X)$, the Fano surface of X, that is the base of the family of lines on X.

The main result of this section is the irreducibility of the family of elliptic quintics on X (Theorem 1.4), but before we need to prove that the families $\mathcal{C}_2^0(X)$ and $\mathcal{C}_3^0(X)$ of conics and, respectively, twisted cubics on X are irreducible.

1.2. Conics. First recall the well known facts about lines on a general X (see, e.g., [T-1], [We]).

Lemma 1.1. *Let X be a general quartic double solid. Then the following assertions hold:*

(i) The Fano surface $\mathcal{F} = \mathcal{F}(X)$ of lines on X is an irreducible surface having at most isolated quadratic points as singularities, and the incidence divisor $D_{\mathcal{F}} = \{(l_1,l_2) \in \mathcal{F} \times \mathcal{F} \mid l_1 \cap l_2 \neq \varnothing\}$ on $\mathcal{F} \times \mathcal{F}$ is irreducible.

(ii) For a general line $l \in \mathcal{F}(X)$, the normal bundle $N_{l/X}$ is isomorphic to $2\mathcal{O}_{\mathbb{P}^1}$.

Next we will prove the following statement:

Lemma 1.2. *Let X be a general quartic double solid. Then we have:*

(i) $\mathcal{C}_2^0(X)$ is irreducible of dimension 4.

(ii) For a general conic $C \in \mathcal{C}_2^0(X)$, the normal bundle $N_{C/X}$ is isomorphic either to $2\mathcal{O}_{\mathbb{P}^1}(1)$ or to $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2)$.

Proof. Let $(l_1,l_2) \in D_{\mathcal{F}}$ be generic. Then l_1 meet each other l_2 quasi-transversely (that is, with different tangents) at some point $x \in X$ and, by Lemma 1.1, $N_{l_i/X} \simeq 2\mathcal{O}_{\mathbb{P}^1}$. Let $C = l_1 \cup l_2$. Then we have the exact sequences

$$0 \to N_{l_i/X} \to N_{C/X}|l_i \xrightarrow{\varepsilon_i} \mathbf{C}_x \to 0, \quad i = 1,2.$$

So

$$N_{C/X}|l_i \simeq \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1), \quad i = 1,2, \tag{1}$$

and the exact triple

$$0 \to N_{C/X} \to \bigoplus_{i=1}^{2} N_{C/X}|l_i \to N_{C/X}|_x \to 0, \qquad N_{C/X}|_x \simeq \mathbf{C}_x^2,$$

implies that

$$h^1(N_{C/X}) = 0. \tag{2}$$

Also, (1) and the exact triple

$$0 \to N_{C/X}|l_2(-x) \to N_{C/X} \xrightarrow{R} N_{C/X}|l_1 \to 0,$$

in which R is the usual restriction map, imply the surjectivity of the map $H^0R : H^0(N_{C/X}) \to H^0(N_{C/X}|l_1)$. Hence the composition of the natural maps

$$\tau_x : H^0(N_{C/X}) \xrightarrow{H^0R} H^0(N_{C/X}|l_1) \xrightarrow{H^0\varepsilon_1} H^0(\mathbf{C}_x)$$

is surjective. Hence (2) and [HH, Prop. 1.1] imply that the conic $C = l_1 \cup l_2$ is strongly smoothable in X, that is there exists an analytic family $\pi : \mathcal{S} \overset{i}{\hookrightarrow} X \times \Delta \xrightarrow{pr_2} \Delta$ over the disk Δ such that $\mathcal{S}$ is a smooth complex surface, π is a flat morphism, $\pi^{-1}(0) = C$ and $C_t = i(\pi^{-1}(t)) \simeq \pi^{-1}(t)$ is a smooth conic in X for any $t \in \Delta \smallsetminus \{0\}$. Hence we can take a section $s : \Delta \hookrightarrow \mathcal{S}, t \mapsto y_t$, of π such that $y_0 \neq x, y_0 \in l_1$. It follows now from (1) and the exact triple

$$0 \to N_{C/X}|l_2(-x) \to N_{C/X}(-y_0) \to N_{C/X}|l_1(-y_0) \to 0$$

that $h^1(N_{C/X})(-y_0) = 0$, so that by semicontinuity (possibly after shrinking Δ) we obtain $h^1(N_{C_t/X})(-y_t) = 0$ for $t \in \Delta \smallsetminus \{0\}$. Combining this with the observation that $\det(N_{C_t/X}) \simeq \mathcal{O}_{\mathbb{P}^1}(2)$ for $t \in \Delta \smallsetminus \{0\}$, we see that the statement (ii) of the lemma holds for a general conic from the irreducible component $\mathcal{C}_2^{0*}(X)$ of $\mathcal{C}_2^0(X)$ containing smoothings of reducible conics. It follows that

$$\dim \mathcal{C}_2^{0*}(X) = 4. \tag{3}$$

To prove (i), recall that there is a natural embedding (see, e.g., [T-1], [We])

$$X \hookrightarrow Y = \mathrm{Proj}(\mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(2)) \tag{4}$$

of X as a divisor of the linear series $|\mathcal{L}^{\otimes 2}|$:

$$X \in |\mathcal{L}^{\otimes 2}| \simeq \mathbb{P}^{45}, \tag{5}$$

where $\mathcal{L} = \mathcal{O}_{Y/\mathbb{P}^3}(1)$ is the Grothendieck sheaf. Recall that

$$p_{Y*}\mathcal{L} = \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(2), \qquad p_{Y*}(\mathcal{L}^{\otimes 2}) = \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(2) \oplus \mathcal{O}_{\mathbb{P}^3}(4), \tag{6}$$

where $p_Y : Y \to \mathbb{P}^3$ is the natural projection. The divisors of the linear series (5) constitute the family $\mathbf{X} \hookrightarrow Y \times \mathbb{P}^{45}$ with natural projections $Y \overset{p}{\leftarrow} \mathbf{X} \overset{q}{\to} \mathbb{P}^{45}$. Consider the following open subset in $\mathbb{P}^{45}$:

$$U = \{t \in \mathbb{P}^{45} \mid X_t = p(q^{-1}(t)) \text{ is smooth}\}. \tag{7}$$

Denote by $\rho : Y \to \mathbb{P}^3$ the natural projection. Then for any $t \in U$, any conic $C \in \mathcal{C}_2^0(X_t)$ is projected by ρ to a conic $C_0 \subset \mathbb{P}^3$, and $\rho|C$ factors into the composition $\rho|C : C \overset{i_C}{\hookrightarrow} F_C \longrightarrow C_0$, where

$$F_C := \rho^{-1}(C_0) = \mathrm{Proj}((\mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(2))|_{C_0}), \qquad i_C(C) \in |\mathcal{L}|F_C|\,. \tag{8}$$

Consider the varieties

$$\mathcal{C}_2^0(\mathbb{P}^3) = \{C_0 \in \mathrm{Hilb}(\mathbb{P}^3) \mid C_0 \text{ is a smooth conic in } \mathbb{P}^3\}$$

and

$$\mathcal{C}_2^0(Y) = \{C \in \mathrm{Hilb}(Y) \mid \rho|_C : C \to \rho(C) \text{ is an isomorphism onto a conic } \rho(C) \in \mathcal{C}_2^0(\mathbb{P}^3)\}.$$

Clearly

$$\dim \mathcal{C}_2^0(\mathbb{P}^3) = 8, \tag{9}$$

and for any $C_0 \in \mathcal{C}_2^0(\mathbb{P}^3)$ we have $(\mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(2))|_{C_0} \simeq \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(4)$. This together with (8) implies that there exists a rank-6 vector bundle E over $\mathcal{C}_2^0(\mathbb{P}^3)$ with fiber $E_{\{C_0\}} = H^0(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(4))$ over an arbitrary point $C_0 \in \mathcal{C}_2^0(\mathbb{P}^3)$ (we identify C_0 with $\mathbb{P}^1$), and $\mathcal{C}_2^0(Y)$ is a dense open subset of $\mathrm{Proj}(E)$ such that the natural projection $\mathcal{C}_2^0(Y) \to \mathcal{C}_2^0(\mathbb{P}^3), C \mapsto \rho(C)$, coincides with the composition $\mathcal{C}_2^0(Y) \hookrightarrow \mathrm{Proj}(E) \xrightarrow{\mathrm{pr}} \mathcal{C}_2^0(\mathbb{P}^3)$, where pr is the structure morphism. It is obvious that $\mathcal{C}_2^0(\mathbb{P}^3)$ is irreducible, hence so is $\mathcal{C}_2^0(Y)$, and from (9) we obtain

$$\dim \mathcal{C}_2^0(Y) = 13. \tag{10}$$

Consider the incidence variety $\Gamma \hookrightarrow U \times \mathcal{C}_2^0(Y)$:

$$\Gamma = \{(t, C) \in U \times \mathcal{C}_2^0(Y) \mid C \subset X_t\}, \qquad U \xleftarrow{p_1} \Gamma \xrightarrow{p_2} \mathcal{C}_2^0(Y).$$

It follows from (5) that

$$p_2^{-1}(C) = U \cap |\mathcal{I}_{C/Y} \otimes \mathcal{L}^{\otimes 2}| \overset{open}{\hookrightarrow} |\mathcal{I}_{C/Y} \otimes \mathcal{L}^{\otimes 2}|. \tag{11}$$

Any conic $C \in \mathcal{C}_2^0(Y)$ is a zero-set of a regular section of the sheaf $\mathcal{G} = p_Y^*(\mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}(2)) \oplus \mathcal{L}$. Hence the ideal sheaf $\mathcal{I}_{C/Y}$ has the standard Koszul resolution

$$0 \to \wedge^3 \mathcal{G}^\vee \to \wedge^2 \mathcal{G}^\vee \to \mathcal{G}^\vee \to \mathcal{I}_{C/Y} \to 0. \tag{12}$$

Now a simple computation using (6) and the projection formula for p_Y proves that the only nonzero numbers $h^i(\wedge^j \mathcal{G}^\vee \otimes \mathcal{L}^{\otimes 2})$ are $h^0(\mathcal{G}^\vee \otimes \mathcal{L}^{\otimes 2}) = 46$ and $h^0(\wedge^2 \mathcal{G}^\vee \otimes \mathcal{L}^{\otimes 2}) = 9$. Twisting (12) by $\mathcal{L}^{\otimes 2}$, we deduce that $h^0(\mathcal{I}_{C/Y} \otimes \mathcal{L}^{\otimes 2}) = 37$, that is $|\mathcal{I}_{C/Y} \otimes \mathcal{L}^{\otimes 2}| = \mathbb{P}^{36}$. Now (10), (11) imply that Γ is an irreducible variety of dimension 49. Next, the fibre $p_1^{-1}(t) = \mathcal{C}_2^0(X_t),\ t \in U$, contains a distinguished irreducible component $\mathcal{C}_2^{0*}(X_t)$ of dimension 4 (see (3)), in accordance with the surjectivity of the morphism $p_1 : \Gamma \to U$ and the equalites $\dim U = 45$, $\dim \Gamma = 49$. Since both Γ and U are irreducible, it follows that $\mathcal{C}_2^0(X_t) = \mathcal{C}_2^{0*}(X_t)$. □

1.3. Twisted cubics. Now we will prove the irreducibility of the family of rational cubics on X.

Lemma 1.3. *Let X be a general quartic double solid. Then the following assertions hold:*

(i) $\mathcal{C}_3^0(X)$ is irreducible of dimension 6.

(ii) For a general smooth cubic $C \in \mathcal{C}_3^0(X)$, the normal bundle $N_{C/X}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(k) \oplus \mathcal{O}_{\mathbb{P}^1}(4-k),\ 0 \le k \le 4$.

Proof. The proof is similar to that of Lemma 1.2. Let us start by proving (ii). Using Lemmas 1.1 and 1.2 pick a general line $l \in \mathcal{F}$ and a conic $C' \in \mathcal{C}_2^0(X)$ meeting each other quasi-transversely in a unique point x and such that

$$N_{l/X} \simeq 2\mathcal{O}_{\mathbb{P}^1}, \quad N_{C'/X} \simeq \mathcal{O}_{\mathbb{P}^1}(k) \oplus \mathcal{O}_{\mathbb{P}^1}(2-k), \quad 0 \leq k \leq 1.$$

Let $C = l \cup C'$. Then we have the exact sequences

$$0 \to N_{l/X} \to N_{C/X}|l \xrightarrow{\varepsilon} \mathbf{C}_x \to 0, \quad 0 \to N_{C'/X} \to N_{C/X}|C' \to \mathbf{C}_x \to 0.$$

Hence

$$N_{C/X}|C' \simeq \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b), \quad 0 \leq a \leq b, \quad a+b=3, \tag{13}$$

and the exact triple

$$0 \to N_{C/X} \to N_{C/X}|l \oplus N_{C/X}|C' \to N_{C/X}|_x \to 0, \qquad N_{C/X}|_x \simeq \mathbf{C}_x^2,$$

implies that

$$h^1(N_{C/X}) = 0. \tag{14}$$

Further, (13) and the restriction triple

$$0 \to N_{C/X}|C'(-x) \to N_{C/X} \xrightarrow{R_l} N_{C/X}|l \to 0$$

imply the surjectivity of the map $H^0R_l : H^0(N_{C/X}) \to H^0(N_{C/X}|l)$. Hence the composition $\tau_x : H^0(N_{C/X}) \xrightarrow{H^0R_l} H^0(N_{C/X}|l) \longrightarrow H^0(\mathbf{C}_x)$ is surjective. Similarly to the case of conics, we deduce from the surjectivity of τ_x and (14) the existence of a flat family of cubic curves $C_t \subset X$ over a disc Δ together with a section $t \in \Delta \mapsto y_t \in C_t$ such that $C_0 = C$, C_t is smooth for $t \neq 0$ and $h^1(N_{C_t/X}(-y_t)) = 0$ for all t. The fact that $\det(N_{C_t/X}) \simeq \mathcal{O}_{\mathbb{P}^1}(4)$ for $t \in \Delta \smallsetminus \{0\}$ together with (i) of Lemma 1.2 imply the wanted assertion for a generic cubic from any irreducible component of $\mathcal{C}_3^0(X)$ which contains smoothings of a generic decomposable curve. Let us show that there is only one such irreducible component.

By [We], Corollary (5.4), the curve $D_l \subset \mathcal{F}$ of lines in X meeting a given generic line l is irreducible and is a 6-sheeted covering of l, because there are 6 lines passing through the generic point of X. So, if we take a generic reducible conic $C = l_1 + l_2 \subset X$, then the curve D_C of lines meeting C is a 6-sheeted covering of C, which is irreducble as a covering, that is there is no subcurve $D' \subset D_C$ which is a d-sheeted covering of C for some $d < 6$. The irreducibility of a covering is stable under small deformations, so we can conclude that D_C stays irreducible when we deform C into a smooth conic. The family of conics in X being irreducible, the incidence variety $\mathcal{D}$ of pairs $(l, C') \in \mathcal{F} \times \mathcal{C}_2^0$ such that $l \cap C' \neq \varnothing$ is irreducible. By the above, $l + C'$ is a smooth point of $\mathrm{Hilb}(X)$ for generic $(l, C') \in \mathcal{D}$, so there is only one component $\mathcal{C}_3^{0*}(X)$ of $\mathcal{C}_3^0(X)$ containing the smoothings of the curves $l + C'$ for generic $(l, C') \in \mathcal{D}$.

Now we can prove the assertion (i); it implies, in particular, that $\mathcal{C}_3^{0*}(X)$ is the whole of $\mathcal{C}_3^0(X)$ and hence (ii) is verified for a generic cubic C from $\mathcal{C}_3^0(X)$.

We have $\dim \mathcal{C}_3^{0*}(X) = 6$, and any irreducible component $\mathcal{C}_3^{0'}(X)$ of $\mathcal{C}_3^0(X)$ has dimension at least $\chi(N_{C/X}) = 6$. This together with an easy dimension count implies that

$$\check{\mathcal{C}}_3^0(X) = \{C \in \mathcal{C}_3^0(X) \mid \rho|C : C \to \rho(C) \text{ is an isomorphism}\}$$

is a dense open subset of $\mathcal{C}_3^0(X)$.

In the notations of (5) – (7), any cubic $C \in \check{\mathcal{C}}_3^0(X_t)$ for $t \in U$ is projected by ρ to a twisted cubic $C_0 \subset \mathbb{P}^3$. Consider the varieties

$$\mathcal{C}_3^0(\mathbb{P}^3) = \{C_0 \in \mathrm{Hilb}(\mathbb{P}^3) \mid C_0 \text{ is a smooth cubic in } \mathbb{P}^3\}$$

and

$$\check{\mathcal{C}}_3^0(Y) = \{C \subset Y \mid \rho|_C : C \to \rho(C) \text{ is an isomorphism onto a cubic } \rho(C) \in \mathcal{C}_3^0(\mathbb{P}^3)\}.$$

We have

$$\dim \mathcal{C}_3^0(\mathbb{P}^3) = 12. \tag{15}$$

As in the proof of Lemma 1.2, we obtain $(\mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(2))|_{C_0} \simeq \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(6)$ for any $C_0 \in \mathcal{C}_3^0(\mathbb{P}^3)$. Hence there exists a rank-8 vector bundle E over $\mathcal{C}_3^0(\mathbb{P}^3)$ with fiber $E_{\{C_0\}} = H^0(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(6))$ over $C_0 \in \mathcal{C}_3^0(\mathbb{P}^3)$, and $\check{\mathcal{C}}_3^0(Y)$ is a dense open subset of $\mathrm{Proj}(E)$ such that the natural projection $\mathcal{C}_3^0(Y) \to \mathcal{C}_3^0(\mathbb{P}^3)$, $C \mapsto \rho(C)$, coincides with the composition $\check{\mathcal{C}}_3^0(Y) \hookrightarrow \mathrm{Proj}(E) \xrightarrow{\mathrm{pr}} \mathcal{C}_3^0(\mathbb{P}^3)$, where pr is the structure morphism. $\mathcal{C}_3^0(\mathbb{P}^3)$ being irreducible, $\mathcal{C}_3^0(Y)$ is irreducible too and by (15)

$$\dim \check{\mathcal{C}}_3^0(Y) = 19. \tag{16}$$

Consider the incidence variety $\Gamma \hookrightarrow U \times \check{\mathcal{C}}_3^0(Y)$:

$$\Gamma = \{(t, C) \in U \times \check{\mathcal{C}}_3^0(Y) \mid C \subset X_t\}, \qquad U \xleftarrow{p_1} \Gamma \xrightarrow{p_2} \check{\mathcal{C}}_3^0(Y). \tag{17}$$

From (5) it follows that

$$p_2^{-1}(C) = U \cap |\mathcal{I}_{C/Y} \otimes \mathcal{L}^{\otimes 2}| \overset{open}{\hookrightarrow} |\mathcal{I}_{C/Y} \otimes \mathcal{L}^{\otimes 2}|. \tag{18}$$

Now any twisted cubic $C_0 \in \mathcal{C}_3^0(\mathbb{P}^3)$ has the following well known resolution of its structure sheaf: $0 \to 2\mathcal{O}_{\mathbb{P}^3}(-3) \to 3\mathcal{O}_{\mathbb{P}^3}(-2) \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_{C_0} \to 0$. Hence the twisted ideal sheaf $\mathcal{I}_{C/Y} \times \mathcal{L}^{\otimes 2}$ of any $C \in \check{\mathcal{C}}_3^0(Y)$ has a resolution of the form

$$0 \to 2\mathcal{L}(-3h) \to 3\mathcal{L}(-2h) \oplus 2\mathcal{L}^{\otimes 2}(-3h) \to \mathcal{L} \oplus 3\mathcal{L}^{\otimes 2}(-2h) \to \mathcal{I}_{C/Y} \times \mathcal{L}^{\otimes 2} \to 0.$$

An easy computation with this resolution gives $|\mathcal{I}_{C/Y} \otimes \mathcal{L}^{\otimes 2}| = \mathbb{P}^{32}$. An argument similar to that used in the end of the proof of Lemma 1.2 shows then the irreducibility of $\mathcal{C}_3^0(X)$. □

1.4. Elliptic quintics. Now we proceed to the study of the family $\mathcal{C}_5^1(X)$ of elliptic quintics on X. The following theorem is the main result of Section 1.

Theorem 1.4. *Let X be a general quartic double solid. Then $\mathcal{C}_5^1(X)$ is irreducible and is of dimension* 10.

Proof. The proof goes along the lines of those of Lemmas 1.2 and 1.3.

Using Lemma 1.3, pick a general cubic $C_1 \in \mathcal{C}_3^0$ and a smooth elliptic conic $C_2 \in \mathcal{C}_2^1(X)$ meeting each other quasi-transversely at a unique point x and such that

$$N_{C_1/X} \simeq \mathcal{O}_{\mathbb{P}^1}(k) \oplus \mathcal{O}_{\mathbb{P}^1}(4-k), \quad 0 \le k \le 2, \qquad N_{C_2/X} \simeq 2\mathcal{O}_{C_2}(x+y),$$
$$\{x, y\} = \pi^{-1}(\pi(x)).$$

Let $C = C_1 \cup C_2$. Then we have the exact sequences

$$0 \to N_{C_i/X} \to N_{C/X}|C_i \xrightarrow{\varepsilon_i} \mathbf{C}_x \to 0, \quad i = 1, 2.$$

Exactly as in the proofs of Lemmas 1.2 and 1.3, one shows that there exists an analytic family $\pi : \mathcal{S} \overset{i}{\hookrightarrow} X \times \Delta \xrightarrow{pr_2} \Delta$ over a disk Δ such that $\mathcal{S}$ is a smooth complex surface, π is a flat morphism, $\pi^{-1}(0) = C$ and $C_t = i(\pi^{-1}(t)) \simeq \pi^{-1}(t)$ is a smooth elliptic quintic in X for any $t \in \Delta \smallsetminus \{0\}$. Let $\mathcal{C}_5^{1*}(X)$ denote the unique irreducible component of $\mathrm{Hilb}(X)$ which contains the of generic reducible curve $C = C_1 \cup C_2$ as above. The uniqueness is proved exactly as in the proof of Lemma 1.3. From the values of $h^i(\mathcal{N}_{C/X})$, we see that $\dim \mathcal{C}_5^{1*}(X) = 10$. For any other irreducible component $\mathcal{C}_5^{1*}(X)$ of $\mathcal{C}_5^1(X)$ we have $\dim \mathcal{C}_5^{1*}(X) \geq 10$. An easy dimension count shows that

$$\check{\mathcal{C}}_5^1(X) = \{C \in \mathcal{C}_5^1(X) \mid \rho|C : C \to \rho(C) \text{ is an isomorphism}\} \tag{19}$$

is a dense open subset of $\mathcal{C}_5^1(X)$.

Now for $t \in U$ any quintic $C \in \check{\mathcal{C}}_5^1(X_t)$ is projected by ρ to an elliptic quintic $C_0 \subset \mathbb{P}^3$. Consider the varieties

$$\mathcal{C}_5^1(\mathbb{P}^3) = \{C_0 \in \mathrm{Hilb}(\mathbb{P}^3) \mid C_0 \text{ is a smooth elliptic quintic in } \mathbb{P}^3\}$$

and

$$\check{\mathcal{C}}_5^1(Y) = \{C \in \mathrm{Hilb}(Y) \mid \rho|_C : C \to \rho(C) \text{ is an isomorphism onto its image } \rho(C) \in \mathcal{C}_5^1(\mathbb{P}^3)\}.$$

Here $\mathcal{C}_5^1(\mathbb{P}^3)$ is an irreducible variety of dimension 20 and, arguing as in the proof of Lemma 1.2, for any $C_0 \in \mathcal{C}_5^1(\mathbb{P}^3)$ we have $(\mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(2))|_{C_0} \simeq \mathcal{O}_{C_0} \oplus \mathcal{O}_{C_0}(D)$, $\deg D = 10$, so that $h^0((\mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(2))|_{C_0}) = 11$. This implies that there exists a rank-11 vector bundle E over $\mathcal{C}_5^1(\mathbb{P}^3)$ with fiber $E_{\{C_0\}} = H^0((\mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(2))|_{C_0})$ over $C_0 \in \mathcal{C}_5^1(\mathbb{P}^3)$, and $\check{\mathcal{C}}_5^1(Y)$ is a dense open subset of $\mathrm{Proj}(E)$ such that the natural projection $\check{\mathcal{C}}_5^1(Y) \to \mathcal{C}_5^1(\mathbb{P}^3)$, $C \mapsto \rho(C)$, coincides with the composition $\check{\mathcal{C}}_5^1(Y) \hookrightarrow \mathrm{Proj}(E) \xrightarrow{\mathrm{pr}} \mathcal{C}_5^1(\mathbb{P}^3)$, where pr is the structure morphism. Hence $\check{\mathcal{C}}_5^1(Y)$ is irreducible and $\dim \check{\mathcal{C}}_5^1(Y) = 30$.

Consider the incidence variety $\Gamma \hookrightarrow U \times \check{\mathcal{C}}_5^1(Y)$:

$$\Gamma = \{(t, C) \in U \times \check{\mathcal{C}}_5^1(Y) \mid C \subset X_t\}, \qquad U \xleftarrow{p_1} \Gamma \xrightarrow{p_2} \check{\mathcal{C}}_5^1(Y).$$

From (5) it follows that

$$p_2^{-1}(C) = U \cap |\mathcal{I}_{C/Y} \otimes \mathcal{L}^{\otimes 2}| \overset{open}{\hookrightarrow} |\mathcal{I}_{C/Y} \otimes \mathcal{L}^{\otimes 2}|.$$

Now take any quintic $C \in \check{\mathcal{C}}_5^1(Y)$. Let $\rho_C = \rho|_C : C \to C_0$ be the natural isomorphic projection of C onto a quintic $C_0 \in \mathcal{C}_5^1(\mathbb{P}^3)$. We have $p_{Y*}(\mathcal{L}^{\otimes 2} \otimes \mathcal{O}_C) \simeq \mathcal{O}_{\mathbb{P}^3}(4)|C_0$. Consider the standard exact triple

$$0 \to \mathcal{I}_{C/Y} \otimes \mathcal{L}^{\otimes 2} \to \mathcal{L}^{\otimes 2} \xrightarrow{\varepsilon} \mathcal{L}^{\otimes 2} \otimes \mathcal{O}_C \to 0. \tag{20}$$

Applying ρ_{C*}, we obtain the following one:

$$0 \to p_{Y*}\mathcal{I}_{C/Y} \otimes \mathcal{L}^{\otimes 2} \to \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(2) \oplus \mathcal{O}_{\mathbb{P}^3}(4) \xrightarrow{\varepsilon_0} \mathcal{O}_{\mathbb{P}^3}(4)|C_0 \to 0. \tag{21}$$

The map $\varepsilon_0|\mathcal{O}_{\mathbb{P}^3}(4) : \mathcal{O}_{\mathbb{P}^3}(4) \to \mathcal{O}_{\mathbb{P}^3}(4)|C_0$ coincides with the restriction one in the triple

$$0 \to \mathcal{I}_{C_0,\mathbb{P}^3}(4) \to \mathcal{O}_{\mathbb{P}^3}(4) \xrightarrow{\mathrm{res}} \mathcal{O}_{\mathbb{P}^3}(4)|C_0 \to 0.$$

Hence, by (20) and (21), the equality $h^1(\mathcal{I}_{C/Y} \otimes \mathcal{L}^{\otimes 2}) = 0$ is a consequence of $h^1(\mathcal{I}_{C_0,\mathbb{P}^3}(4)) = 0$. To prove the last one, take two general cubic surfaces S_1, S_2 through C_0 such that the residual curve of their intersection is a smooth rational quartic C_1 meeting C_0 in a scheme Z of length 10. Then we have two exact triples

$$0 \to \mathcal{I}_{C_0 \cup C_1,\mathbb{P}^3}(4) \to \mathcal{I}_{C_0,\mathbb{P}^3}(4) \to \mathcal{O}_{C_1}(4)(-Z) \to 0,$$

where $\mathcal{O}_{C_1}(4)(-Z) = \mathcal{O}_{\mathbb{P}^1}(6)$, and

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-2) \to 2\mathcal{O}_{\mathbb{P}^3}(1) \to \mathcal{I}_{C_0 \cup C_1,\mathbb{P}^3}(4) \to 0.$$

Their long cohomology exact sequences imply the vanishing of $h^1(\mathcal{I}_{C_0,\mathbb{P}^3}(4)) = 0$. Hence $h^0(\mathcal{I}_{C/Y} \otimes \mathcal{L}^{\otimes 2}) = 26$ and $|\mathcal{I}_{C/Y} \otimes \mathcal{L}^{\otimes 2}| = \mathbb{P}^{25}$. The remaining part of the proof is the same as in Lemma 1.2. □

In Section 2 we will use the following auxiliary result about quintics on X.

Lemma 1.5. *Let X be general so that $\mathcal{C}_5^1(X)$ is irreducible and let $\overline{\mathcal{C}_5^1(X)}$ be the closure of $\mathcal{C}_5^1(X)$ in* Hilb(X). *Then there exists a strongly smoothable reduced reducible quintic $C_0 \in \overline{\mathcal{C}_5^1(X)}$ of the form*

$$C_0 = l_1 \cup ... \cup l_5, \quad l_i \in \mathcal{F}, \quad i = 1, ..., 5, \tag{22}$$

such that

$$a_1 = l_1 \cap l_2, \quad a_2 = l_2 \cap l_3, \quad a_3 = l_3 \cap l_4, \quad a_4 = l_4 \cap l_1, \quad a_5 = l_1 \cap l_5, \tag{23}$$

are distinct points satisfying the following conditions:

$$N_{l_i/X} = 2\mathcal{O}_{l_i}, \; i = 1, ..., 5, \tag{24}$$

$$N_{C_0/X}|l_1 \simeq \mathcal{O}_{l_1}(1) \oplus \mathcal{O}_{l_1}(2), \quad N_{C_0/X}|l_i \simeq 2\mathcal{O}_{l_i}(1), \; i = 2, 3, 4, \\ N_{C_0/X}|l_5 \simeq \mathcal{O}_{l_1} \oplus \mathcal{O}_{l_1}(1). \tag{25}$$

Proof. For a general line $l \in \mathcal{F}$ on X, we have $N_{l/X} \simeq 2\mathcal{O}_l$. Hence, for a general plane $\mathbb{P}^2$ in $\mathbb{P}^3$ the del Pezzo surface $S = \pi^{-1}(\mathbb{P}^2)$ is smooth and all the lines in it have trivial normal bundle in X. One easily finds a configuration C_0 of five lines in S satisfying (22) and (23). Let $C_1 = l_1 \cup ... \cup l_4$. Since each line $l_i \in C_0$ is a (-1)-curve on S, it follows that $N_{C_1/S}|l_i \simeq \mathcal{O}_{l_i}(1), \; i = 1, ..., 4$. Hence the exact triple

$$0 \to N_{C_1/S}|l_i \to N_{C_1/X}|l_i \to N_{S/X}|l_i \to 0$$

and the obvious equality $\det(N_{C_1/X}|l_i) \simeq \mathcal{O}_{l_i}(2)$ imply $N_{C_1/X}|l_i \simeq 2\mathcal{O}_{l_i}(1), \; i = 1, ..., 4$. Adding l_5 to C_1 yields C_0, and one immediately obtains (25).

To show that C_0 is strongly smoothable, denote

$$Z_a = \{a_1, ..., a_5\} \tag{26}$$

and consider the natural exact triples

$$0 \to N_a \to N_{C_0/X} \to \mathcal{O}_{Z_a} \to 0, \qquad 0 \to N'_a \to N_a \to N_{l_1/X} \oplus N_{l_3/X} \to 0, \tag{27}$$

where $N'_a := (N_{C_0/X}|l_2 \cup l_4 \cup l_5)(-Z_a)$; then the strong smoothability of C_0 follows from the equality $h^1(N_a) = 0$ (see [HH]). To prove the last equality, remark that, by (24), one has $N'_a \simeq 2\mathcal{O}_{l_2}(-1) \oplus 2\mathcal{O}_{l_4}(-1) \oplus \mathcal{O}_{l_5} \oplus \mathcal{O}_{l_5}(-1)$, hence $h^1(N'_a) = 0$. Also, (25) implies $h^1(N_{l_1/X} \oplus N_{l_3/X}) = 0$, so the second triple (27) implies $h^1(N_a) = 0$. □

2. The Abel-Jacobi map of the family of elliptic quintics

In this section we will study the Abel-Jacobi maps of three families of curves on a general double solid X: genus-4 septics $\mathcal{C}_7^4(X)$, elliptic quintics $\mathcal{C}_5^1(X)$ and a certain family T of reducible curves of degree 17 and arithmetic genus 29 which will set a connection between the two other families.

2.1. Notations. We will use the following symbols, in addition to those introduced in 1.1:

- $\mathcal{C}_7^4(X) := \{C \in \mathrm{Hilb}(X) | \ C$ is a smooth irreducible septic of genus 4 in $X\}$.
- $\overline{\mathcal{C}_7^4(X)}$, the closure of $\mathcal{C}_7^4(X)$ in $\mathrm{Hilb}(X)$.
- $\overline{\mathcal{C}_7^4(X)}^* := \{C \in \overline{\mathcal{C}_7^4(X)} \mid C$ is a reduced curve with at worst ordinary double points$\}$, a dense open subset of $\overline{\mathcal{C}_7^4(X)}$ containing $\mathcal{C}_7^4(X)$.
- $\mathcal{C}_7^4(\mathbb{P}^3) := \{C \in \mathrm{Hilb}(\mathbb{P}^3) \mid C$ is a smooth irreducible septic of genus 4 in $\mathbb{P}^3\}$.
- $\mathcal{C}_5^2(\mathbb{P}^3) := \{C \in \mathrm{Hilb}(\mathbb{P}^3) | \ C$ is a reduced quintic curve of arithmetic genus 2 on a smooth quadric surface $Q \subset \mathbb{P}^3$ of bidegree (3,2) $\}$.
- $\pi^*\mathcal{C}_5^2(\mathbb{P}^3) := \{C \in \mathrm{Hilb}(X) | \ C = \pi^{-1}(\underline{C}), \ \underline{C} \in \mathcal{C}_5^2(\mathbb{P}^3)\}$.
- $\mathcal{C}_{(3,3)}(\mathbb{P}^3) := \{C \in \mathrm{Hilb}(\mathbb{P}^3) | \ C$ is a curve of arithmetic genus 4 on a smooth quadric surface $Q \subset \mathbb{P}^3$ of bidegree (3,3) (that is C is a complete intersection of Q and a cubic surface)$\}$.
- $\pi^*\mathcal{C}_{(3,3)}(\mathbb{P}^3) := \{C \in \mathrm{Hilb}(X) | \ C = \pi^{-1}(\underline{C}), \ \underline{C} \in \mathcal{C}_{(3,3)}(\mathbb{P}^3)\}$.
- $\overline{\mathcal{C}_5^1(X)}$, the closure of $\mathcal{C}_5^1(X)$ in $\mathrm{Hilb}(X)$.
- $\overline{\mathcal{C}_5^1(X)}^* = \{C \in \overline{\mathcal{C}_5^1(X)} \mid C$ is a reduced curve with at most ordinary double points$\}$, an open subset of $\overline{\mathcal{C}_5^1(X)}$.
- $\mathcal{C}_{5+2} := \{C \in \mathrm{Hilb}(X) \mid C$ is a reduced curve of the form $C = C' \cup C''$, where $C' \in \overline{\mathcal{C}_5^1(X)}^*, C'' \in \mathcal{C}_2^1(X)$ and C' meets C'' quasi-transversely in 3 points$\}$.
- $\check{\mathcal{C}}_{5+2} := \{C' \cup C'' \in \mathcal{C}_{5+2} \mid C' \in \check{\mathcal{C}}_5^1(X)\}$, where $\check{\mathcal{C}}_5^1(X)$ is defined by (19).
- $\Phi_{\mathcal{B}} : \mathcal{B} \to J(X)$, the Abel-Jacobi map of a given subvariety $\mathcal{B}$ of the Hilbert scheme of curves on X (defined uniquely up to a choice of a reference point in $\mathcal{B}$).

2.2. Septics of genus 4. We are interested in the family of septics $\mathcal{C}_7^4(X)$ because of the following result of Welters [We, 6.18]:

Proposition 2.1. *For a general quartic double solid X, the Abel-Jacobi map $\Phi_{\mathcal{C}_7^4(X)}$: $\mathcal{C}_7^4(X) \to J(X)$ is dominant.*

We begin with the following lemma.

Lemma 2.2. *Let X be a general quartic double solid. Then:*

(i) $\check{\mathcal{C}}_{5+2}$ is irreducible of dimension 21 and the natural projection p'' : $\check{\mathcal{C}}_{5+2} \to \check{\mathcal{C}}_5^1(X)$, $C' \cup C'' \mapsto C'$, is surjective.

(ii) Let $\overline{\check{\mathcal{C}}_{5+2}}$ be the closure of $\check{\mathcal{C}}_{5+2}$ in $\mathcal{C}_{5+2}$. There exists a strongly smoothable curve $C_0' \cup C_0'' \in \overline{\check{\mathcal{C}}_{5+2}} \smallsetminus \check{\mathcal{C}}_{5+2}$ such that

$$C_0' = l_1 \cup ... \cup l_5 \in \overline{\mathcal{C}_5^1(X)} \smallsetminus \mathcal{C}_5^1(X) \tag{28}$$

is the curve (22) from Lemma 1.5 and $C_0'' \in \mathcal{C}_2^1(X)$ is a smooth elliptic conic such that

$$Z_b := C_0' \cap C_0'' = \{b_1, b_2, b_3\}, \quad b_1 \in l_2, \quad b_2 \in l_4, \quad b_3 \in l_5, \quad Z_b \cap Z_a = \varnothing, \tag{29}$$

where $Z_a = \mathrm{Sing} C_0'$ was introduced in (26).

(iii) The set

$$\mathcal{C}^*_{5+2} := \{C \in \check{\mathcal{C}}_{5+2} \mid C \text{ is strongly smoothable}\} \subset \mathcal{C}_{5+2} \cap \overline{\mathcal{C}^4_7(X)}^* \tag{30}$$

is an irreducible dense open subset of $\mathcal{C}_{5+2}$ *and the projection* $p'' : \mathcal{C}^*_{5+2} \to \check{\mathcal{C}}^1_5(X)$, $C' \cup C'' \mapsto C'$, *is dominant.*

(iv) Let $\mathcal{C}^{4*}_7(X)$ *be an irreducible component of* $\overline{\mathcal{C}^4_7(X)}$ *containing* $\mathcal{C}^*_{5+2}$. *Then* $\dim \mathcal{C}^{4*}_7(X) = 14$ *and* $\mathcal{C}^{4*}_7(X)^0 = \{C \in \mathcal{C}^{4*}_7(X) \mid C$ is smooth and $\pi|C : C \to \pi(C)$ is an isomorphism$\}$ *is a nonempty open subset of* $\mathcal{C}^{4*}_7(X)$.

Proof. (i) The curve $\underline{C'} = \pi(C')$ is an elliptic quintic in $\mathbb{P}^3$. For any elliptic quintic $B \subset \mathbb{P}^3$ and for any $x \in B$, there are at most two trisecant lines of B passing through x, and the family Z of trisecant lines of B is an irreducible curve. To prove this, consider the linear projection π_x of B with center x to a plane $\mathbb{P}^2$. The image of B is a plane quartic of geometric genus 1, so we have the following two possibilities for its singular locus: 1) two singular points, each one of which is either a node, or a cusp, 2) one tacnode. The inverse image in B of a singular point of $\pi_x(B)$ is a pair of points, say, z_1, z_2, and the line $\overline{z_1 z_2}$ is a trisecant of B, which meets B in three distinct points x, z_1, z_2. Hence the family Z of trisecant lines of B is a curve. To prove its irreducibility, consider the incidence curve $D = \{(z,l) \in B \times Z \mid z \in l\}$ in $B \times Z$ with its natural projections $B \xleftarrow{p} D \xrightarrow{q} Z$. The above argument shows that p is a degree-two map. It suffices to verify that it has at least one simple ramification point to see that both D and Z are irreducible.

The points x with only one inverse image l in D are those for which $\pi_x(B)$ has a tacnode. Equivalently, the tangent lines $T_{z_1}B$ and $T_{z_2}B$ are coplanar. In this case l is either a point of intersection of two local branches of D, or a cusp, or a simple ramification point of p. To eliminate the first two possibilities, it suffices to see that l is a nonsingular point of D. Assume that it is singular, then all the three curves $\pi_{z_1}(B), \pi_{z_2}(B), \pi_x(B)$ are also tacnodal. This implies that the tangent lines T_xB, $T_{z_1}B$ and $T_{z_2}B$ of B are coplanar. This is absurd, because the intersection index of B with a plane in $\mathbb{P}^3$ is equal to 5.

So, p has a simple ramification point if and only if it possesses a trisecant l such that the tangent lines $T_{z_1}B$ and $T_{z_2}B$ are coplanar for a pair z_1, z_2 of points of $l \cap B$; let us call such a trisecant special. One can deduce the existence of special trisecants from the Chasles-Cayley-Brill formula (see [SR], Chapter XII, Corollary 3.52) which gives the number N of fixed points of a correspondence $G \subset C \times C$ on a smooth curve C of genus $g(C) > 0$:

$$N = n_1 + n_2 + 2g(C)\gamma(G),$$

where n_i is the degree of the projection $p_i : G \longrightarrow C$ to the i-th factor, and $\gamma(G)$ is the valence of the correpondence, defined as the integer m such that the linear equivalence class of $mu + p_1^{-1}(u)$ does not depend on $u \in C$.

We will now describe a correspondence G in $B \times B$ for our elliptic curve B, whose fixed points correspond to the special trisecants of B. We will specify a divisor $G_u = p_1^{-1}(u)$ on B for every $u \in B$. So, let $u \in B$ be any point, and m_1, m_2 the two trisecant lines of B passing through u. Let $m_i \cap B = \{u, a_i, b_i\}$, $i = 1, 2$. Let h_1 be the section of B by the plane containing m_1 and the tangent line $T_{a_1}B$. We have

$$h_1 = u + 2a_1 + b_1 + c_1.$$

Similarly, define the hyperplane sections h_2, h_3, h_4 of B such that

$$h_2 = u + a_1 + 2b_1 + d_1,\ h_3 = u + 2a_2 + b_2 + c_2,\ h_4 = u + a_2 + 2b_2 + d_2.$$

Define $G_u = c_1 + c_2 + d_1 + d_2$. For such G, we have $n_1 = 4$, $n_2 \geq 1$. Moreover, since $h_0 = x + a_1 + a_2 + b_1 + b_2$ is also a hyperplane section of B, we have

$$h_1 + h_2 + h_3 + h_4 - 3h_0 = u + G_u \sim h_0,$$

and hence G is of valence $\gamma(G) = 1$. Then the Chasles-Cayley-Brill formula gives the estimate $N \geq 7$. Now, it is easy to see that all the fixed points of G provide special trisecants of B. Indeed, by definition, u is a fixed point of G if and only if u coincides with one of the points of G_u. Let, for example, $u = c_1$. Then $h_1 = 2u + 2a_1 + b_1$, and m_1 is a special trisecant of B.

Thus, we have proved that B has at least one special trisecant, hence D and Z are irreducible. Apply this to $B = \underline{C'}$. The fiber ${p''}^{-1}(C')$ is identified with a nonempty open subset of the family of trisecant lines of $\underline{C'}$, so it is 1-dimensional and irreducible, which implies the assertion (i).

(ii) Remark that, in Lemma 1.5, one can choose the curve $C'_0 = l_1 \cup ... \cup l_5$ in such a way that the lines $m_i = \pi(l_i)$, $i = 2, 4, 5$ in $\mathbb{P}^3$ are disjoint. Then it is obvious that a small (in the classical topology over $\mathbb{C}$) smooth deformation C'_t of C'_0 can be lifted to a small deformation $C'_t \cup C''_t \in \check{\mathcal{C}}_{5+2}$ of $C'_0 \cup C''_0$. By the strong smoothability of C'_0 proved in Lemma 1.5, $C := C'_0 \cup C''_0 \in \overline{\check{\mathcal{C}}_{5+2}}$. To see that C is strongly smoothable, we consider the natural exact triples similar to (27):

$$0 \to N_a \to N_{C/X} \to \mathcal{O}_{Z_a} \to 0, \qquad 0 \to N'_a \to N_a \to N_{C_1/X} \to 0, \tag{31}$$

$$0 \to N_b \to N_{C_0/X} \to \mathcal{O}_{Z_b} \to 0, \qquad 0 \to N'_b \to N_b \to N_{C''_0/X} \to 0, \tag{32}$$

where $C_1 := l_1 \cup l_2 \cup l_4 \cup C''$, $N'_a := (N_{C/X}|l_3 \cup l_5)(-Z_a)$, $N'_b := (N_{C/X}|C'_0)(-Z_b)$. Then the strong smoothability of C will follow from the equalities

$$h^1(N_a) = 0, \tag{33}$$

$$h^1(N_b) = 0. \tag{34}$$

To prove the first of them, one deduces from Lemma 1.5 that $N'_a = 2\mathcal{O}_{l_3}(-1) \oplus 2\mathcal{O}_{l_5}(-1)$. Then the exact triples

$$0 \to 2\mathcal{O}_{l_2}(-1) \oplus 2\mathcal{O}_{l_4}(-1) \to N_{C_1/X} \to N_{C_1/X}|C''_0 \oplus 2\mathcal{O}_{l_1}(1) \to 0,$$

$$0 \to N_{C''_0/X} \to N_{C_1/X}|C''_0 \to \mathbb{C}_{b_1} \oplus \mathbb{C}_{b_2} \to 0$$

together with the second triple (31) give (33).

To prove (34), we include N'_b into the series of exact triples following from the construction of C:

$$0 \to N_1 \to N'_b \to \mathcal{O}_{l_5} \oplus \mathcal{O}_{l_5}(-1) \to 0, \quad 0 \to 2\mathcal{O}_{l_3}(-1) \to N_1 \to N_2 \to 0,$$

$$0 \to N_3 \to N_2 \to \mathcal{O}_{l_2} \oplus \mathcal{O}_{l_2}(1) \to 0, \quad 0 \to \mathcal{O}_{l_4} \oplus \mathcal{O}_{l_4}(-1) \to N_3 \to \mathcal{O}_{l_4} \oplus \mathcal{O}_{l_4}(-1) \to 0.$$

As $C''_0 = \pi^{-1}(m)$, where m denotes a line in $\mathbb{P}^3$, one immediately verifies that $h^1(N_{C''_0/X}) = 0$. This together with the second triple (32) gives (34).

(iii) follows from (i) and (ii) since the strong smoothability is an open condition in $\overline{\check{\mathcal{C}}_{5+2}}$.

(iv) Take the curve $C = C_0' \cup C_0'' \in \overline{\check{\mathcal{C}}_{5+2}} \setminus \check{\mathcal{C}}_{5+2}$ from (ii) above. From (31)-(34) one immediately obtains $h^1(N_{C/X}) = 0$, $h^0(N_{C/X}) = 14$, hence $\dim \mathcal{C}_7^{4*}(X) = 14$. Next, consider C as the union $C = C_3 \cup C_4$, where $C_3 := l_5 \cup C_0''$ and $C_4 := l_1 \cup l_2 \cup l_3 \cup l_4$. One easily verifies that the reducible elliptic cubic C_3 is strongly smoothable and

$$\dim_{\{C_3\}} \mathrm{Hilb}(X) = 6 \tag{35}$$

and thus there exists a unique 6-dimensional irreducible component $\mathcal{C}_3^{1*}(X)$ of $\mathrm{Hilb}(X)$ containing C_3. Denote $R = \{\tilde{C} \in \mathcal{C}_3^{1*}(X) \mid \tilde{C}$ is reducible of the form $\tilde{C} = l \cup \pi^{-1}(m)$, where l is a line on X and m is a line on $\mathbb{P}^3\}$. By construction, $C_3 \in R$. It follows easily that: a) for any $\tilde{C} \in \mathcal{C}_3^{1*}(X) \setminus R$ the map $\pi|\tilde{C} : \tilde{C} \to \pi(\tilde{C})$ is an isomorphism of $\tilde{C}$ onto a plane cubic $\pi(\tilde{C})$; b) $\dim R = 5$, and $\dim_{\{C_3\}} R(l_1, l_2, l_4) = 2$, where $R(l_1, l_2, l_4) = \{\tilde{C} \in R \mid \tilde{C}$ intersects each one of the lines l_1, l_2, l_4 in one point$\}$. Thus, by (35), there exists $\tilde{C} \in \mathcal{C}_3^{1*}(X) \setminus R$ intersecting each of the lines l_1, l_2, l_4 in one point, so that π maps $\tilde{C} \cup C_4$ isomorphically onto its image. By construction, $\tilde{C} \cup C_4 \in \mathcal{C}_7^{4*}(X)$. The property of a curve in X to be mapped isomorphically onto its image in $\mathbb{P}^3$ is open and a general curve in $\mathcal{C}_7^{4*}(X)$ is smooth as a deformation of the strongly smoothable curve $C_3 \cup C_4$, so $\mathcal{C}_7^{4*}(X)^0$ is open in $\mathcal{C}_7^{4*}(X)$, and this ends the proof. □

Lemma 2.3. *Let X be a general quartic double solid. Then $\mathcal{C}_7^4(X)$ is irreducible of dimension* 14.

Proof. This is proved in the same way as Lemmas 1.2, 1.3 and 1.4. First, remark that $\mathcal{C}_7^4(\mathbb{P}^3)$ is an irreducible variety of dimension 28 (see, e.g., [H, Ch.4]), and for any $C_0 \in \mathcal{C}_7^4(\mathbb{P}^3)$ we have $(\mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(2))|_{C_0} \simeq \mathcal{O}_{C_0} \oplus \mathcal{O}_{C_0}(D)$, $\deg D = 14$, so that $h^0((\mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(2))|_{C_0}) = 9$. This implies that there exists a rank-9 vector bundle E over $\mathcal{C}_7^4(\mathbb{P}^3)$ with fiber $E_{\{C_0\}} = H^0((\mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(2))|_{C_0})$ over $C_0 \in \mathcal{C}_7^4(\mathbb{P}^3)$. Thus, denoting $\check{\mathcal{C}}_7^4(Y) = \{C \subset Y \mid \rho|_C : C \to \rho(C)$ is an isomorphism onto its image $\rho(C) \in \mathcal{C}_7^4(\mathbb{P}^3)\}$, we see that $\check{\mathcal{C}}_7^4(Y)$ is a dense open subset of $\mathrm{Proj}(E)$ such that the natural projection $\check{\mathcal{C}}_7^4(Y) \to \mathcal{C}_7^4(\mathbb{P}^3)$, $C \mapsto \rho(C)$, coincides with the composition $\check{\mathcal{C}}_7^4(Y) \hookrightarrow \mathrm{Proj}(E) \xrightarrow{\mathrm{pr}} \mathcal{C}_7^4(\mathbb{P}^3)$, where pr is the structure morphism. Hence $\mathcal{C}_7^4(Y)$ is irreducible and

$$\dim \check{\mathcal{C}}_7^4(Y) = 36. \tag{36}$$

Consider the incidence variety $\Gamma \hookrightarrow U \times \check{\mathcal{C}}_7^4(Y)$:

$$\Gamma = \{(t, C) \in U \times \check{\mathcal{C}}_7^4(Y) \mid C \subset X_t\}, \qquad U \xleftarrow{p_1} \Gamma \xrightarrow{p_2} \check{\mathcal{C}}_7^4(Y).$$

From (5) it follows that

$$p_2^{-1}(C) = U \cap |\mathcal{I}_{C/Y} \otimes \mathcal{L}^{\otimes 2}| \overset{open}{\hookrightarrow} |\mathcal{I}_{C/Y} \otimes \mathcal{L}^{\otimes 2}|. \tag{37}$$

Now take any curve $C \in \check{\mathcal{C}}_7^4(Y)$. Let $\rho_C = \rho|_C : C \to C_0$ be the natural isomorphic projection of C onto a quintic $C_0 \in \mathcal{C}_7^4(\mathbb{P}^3)$. We have $p_{Y*}(\mathcal{L}^{\otimes 2} \otimes \mathcal{O}_C) \simeq \mathcal{O}_{\mathbb{P}^3}(4)|C_0$. Consider the standard exact triple

$$0 \to \mathcal{I}_{C/Y} \otimes \mathcal{L}^{\otimes 2} \to \mathcal{L}^{\otimes 2} \xrightarrow{\varepsilon} \mathcal{L}^{\otimes 2} \otimes \mathcal{O}_C \to 0. \tag{38}$$

Apply ρ_{C*}:

$$0 \to p_{Y*}\mathcal{I}_{C/Y} \otimes \mathcal{L}^{\otimes 2} \to \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(2) \oplus \mathcal{O}_{\mathbb{P}^3}(4) \xrightarrow{\varepsilon_0} \mathcal{O}_{\mathbb{P}^3}(4)|C_0 \to 0. \tag{39}$$

The map $\varepsilon_0|\mathcal{O}_{\mathbb{P}^3}(4) : \mathcal{O}_{\mathbb{P}^3}(4) \to \mathcal{O}_{\mathbb{P}^3}(4)|C_0$ is iedntified with the restriction one in the triple

$$0 \to \mathcal{I}_{C_0,\mathbb{P}^3}(4) \to \mathcal{O}_{\mathbb{P}^3}(4) \xrightarrow{\mathrm{res}} \mathcal{O}_{\mathbb{P}^3}(4)|C_0 \to 0.$$

Hence, by (38), (39), the vanishing of $h^1(\mathcal{I}_{C/Y}\otimes\mathcal{L}^{\otimes 2})$ follows from that of $h^1(\mathcal{I}_{C_0/\mathbb{P}^3}(4))$. To prove the last one, take two general cubic surfaces S_1, S_2 through C_0 (such cubics exist since, by Riemann-Roch, $h^0(\mathcal{O}_{C_0}(3)) = 18$.) There are two possibilities for S_1 and S_2: 1) both S_1 and S_2 are irreducible, and 2) at least one of S_1, S_2 is reducible. Consider them separately.

1) In this case $S_1\cap S_2 = C_0\cup C'$, where C' is a curve of degree 2. We have the following two subcases: 1.a) C' a plane conic, or 1.b) C' is a pair of two skew lines, possibly coincident, that is a double line on a smooth quadric. Consider the scheme $Z = C_0\cap C'$. One easily verifies that $\xi(\mathcal{O}_Z) = 7$ in case 1.a) and $\xi(\mathcal{O}_Z) = 8$ in case 1.b), and that $\xi(\mathcal{O}_{Z\cap l}) = 4$ for any line $l\subset C'$. Thus, in both cases one obtains $h^1(\mathcal{I}_{Z,C'}(4)) = 0$. Then we have two exact triples $0\to\mathcal{I}_{C_0\cup C_1/\mathbb{P}^3}(4)\to\mathcal{I}_{C_0/\mathbb{P}^3}(4)\to\mathcal{I}_{Z/C'}(4)\to 0$, and $0\to\mathcal{O}_{\mathbb{P}^3}(-2)\to 2\mathcal{O}_{\mathbb{P}^3}(1)\to\mathcal{I}_{C_0\cup C'/\mathbb{P}^3}(4)\to 0$. Their long cohomology exact sequences imply $h^1(\mathcal{I}_{C_0/\mathbb{P}^3}(4)) = 0$.

2) In this case a smooth curve C_0 lies on an irreducible quadric, say, Q, since C_0 is not a plane curve. Then Q is either a smooth quadric, or a cone. Consider the case of a smooth quadric Q, the case of a cone being treated similarly. Then C_0 is a curve of bedegree $(5,2)$ on Q, so that the sheaf $\mathcal{I}_{C_0/\mathbb{P}^3}(4)$ is included into the exact triple $0\to\mathcal{I}_{Q/\mathbb{P}^3}(4)\longrightarrow\mathcal{I}_{C_0/\mathbb{P}^3}(4)\longrightarrow\mathcal{I}_{C_0/Q}(4)\to 0$, where clearly $\mathcal{I}_{Q/\mathbb{P}^3}(4) = \mathcal{O}_{\mathbb{P}^3(2)}$ and $\mathcal{I}_{C_0/Q}(4) = \mathcal{O}_Q(-1,3)$, so that $h^1(\mathcal{I}_{Q/\mathbb{P}^3}(4)) = h^1(\mathcal{I}_{C_0/Q}(4)) = 0$ and, as a corollary, $h^1(\mathcal{I}_{C_0/\mathbb{P}^3}(4)) = 0$.

Hence $h^0(\mathcal{I}_{C/Y}\otimes\mathcal{L}^{\otimes 2}) = 24$ and $|\mathcal{I}_{C/Y}\otimes\mathcal{L}^{\otimes 2}| = \mathbb{P}^{23}$, so that by (36)-(37) we deduce that Γ is irreducible of dimension 59. The remaining part of the proof goes along the same lines as the end of the proofs of Lemmas 1.2, 1.3 and 1.4. Namely, by Lemma 2.2 (iv), for a general double solid $X\in U$, there is a distinguished component $\mathcal{C}_7^{4*}(X)$ of the fibre $p_1^{-1}(X) = \mathcal{C}_7^4(X)$ which plays the role of $\mathcal{C}_2^{0*}(X)$, $\mathcal{C}_3^{0*}(X)$, $\mathcal{C}_5^{1*}(X)$ in the proofs of the above mentioned lemmas. (Remark, that, since $\dim\Gamma = 59 = 14+45 = \dim\mathcal{C}_7^{4*}(X)+\dim U$, it follows that $p_1:\Gamma\to U$ is dominant.) $\square$

2.3. Curves of degree 17 and arithmetic genus 29.

Let

$$M := \left\{ (Q,l,Z,Z_l) \;\middle|\; \begin{array}{l} Q \text{ is a smooth quadric in } \mathbb{P}^3,\ l \text{ a line on Q, } Z = \\ \{y_4,\dots,y_{10}\} \text{ a 7-uple of distinct points on } Q\setminus l, \\ Z_l = \{y_1,y_2,y_3\} \text{ a triple of distinct points on } l \end{array} \right\}$$

and

$$M^* = \{(Q,l,Z,Z_l)\in M \mid \dim|\mathcal{J}_{Z\cup Z_l/Q}(3,2)| = 1\},$$

where we assume that l is of bidegree (0,1) on Q. Clearly, M is irreducible and M^* is nonempty, hence dense in M. Next, let

$$\Pi = \{(C,(Q,l,Z,Z_l))\in\check{\mathcal{C}}_5^1(X)\times M \mid \underline{C}\cap Q = Z\cup Z_l, \text{ where } \underline{C} = \pi(C)\}.$$

It has two natural projections: $\check{\mathcal{C}}_5^1(X)\xleftarrow{p_\Pi}\Pi\xrightarrow{q_\Pi} M$. It is easy to see that Π is nonempty. Indeed, for any $C\in\check{\mathcal{C}}_5^1(X)$, consider the smooth quintic $\underline{C} = \pi(C)$, take any trisecant line l of $\underline{C}$ and any smooth quadric Q through l and let $Z = C\cap(Q\setminus l)$ and $Z_l = \underline{C}\cap l$; then, by construction, $(C,(Q,l,Z,Z_l))\in\Pi$. We can factor p_Π as

$$p_\Pi:\Pi\xrightarrow{p'}\mathcal{C}_{5+2}\xrightarrow{p''}\check{\mathcal{C}}_5^1(X), \tag{40}$$
$$p':(C',(Q,l,Z,Z_l))\mapsto C'\cup\pi^{-1}(l),\quad p'':C'\cup C''\mapsto C',$$

and our construction shows that p' is surjective with an irreducible and rational fiber of dimension 7. Besides, as it was already mentioned in the proof of Lemma 2.2, for $C' \in \check{\mathcal{C}}_5^1(X)$, the fiber ${p'}^{-1}(C')$ is isomorphic to a dense open subset of the 1-dimensional family of trisecant lines of the smooth quintic $\pi(C')$ in $\mathbb{P}^3$. This description of the fibres of p' and p'' together with (40) implies, in particular, that Π is irreducible for general X.

Let us show now that

$$\Pi^* := {p'}^{-1}(\mathcal{C}_{5+2}^*) \cap q_\Pi^{-1}(M^*) \neq \varnothing. \tag{41}$$

Choose any $C \in \check{\mathcal{C}}_5^1(X)$ and a trisecant line l of $\underline{C}$ such that $C \cup \pi^{-1}(l) \in \mathcal{C}_{5+2}^*$, where $\underline{C} = \pi(C)$. Then $Z_l = \underline{C} \cap l = \{x_1^0, x_2^0, x_3^0\}$ is a triple of distinct points. Through each x_i^0 there is, in addition to l, one more trisecant l_0^i of $\underline{C}$, and we denote $\{x_{2i+2}^0, x_{2i+3}^0\} = l_i^0 \cap \underline{C} \setminus \{x_i^0\}, \quad i = 1, 2, 3$. Since $\underline{C}$ has degree 5, it follows immediately that the lines l_1^0, l_2^0, l_3^0 are disjoint, hence there exists a unique smooth quadric Q_0 containing l, l_1^0, l_2^0 and l_3^0. Let x_{10}^0 be the residual point of intersection of $\underline{C}$ and Q^0: $\underline{C} \cap Q_0 \smallsetminus \{x_1^0, ..., x_{10}^0\}$. One can easily verify that for sufficiently general l, the 10 points $x_1^0, ..., x_{10}^0$ are distinct. Denote $Z^0 = \{x_4^0, ..., x_{10}^0\}$ and let l_4^0 be a line on Q^0 through x_{10}^0 in the same ruling as l. Let it be the ruling $|\mathcal{O}_{Q_0}(0,1)|$. Then

$$|\mathcal{J}_{Z^0 \cup Z_l/Q_0}(3,2)| = \bigcup_{i=1}^4 l_i^0 + |\mathcal{O}_{Q_0}(0,1)| \simeq \mathbb{P}^1.$$

Now consider the triple $x_0 = (x_1^0, x_2^0, x_3^0)$ as a point of $\mathrm{Sym}^3(l) \simeq \mathbb{P}^3$ and take a sufficiently small neighborhood U of x_0 in the classical topology of $\mathrm{Sym}^3(l)$ so that for any $x = (x_1, x_2, x_3) \in U$ the following properties are satisfied: (i) for each $i = 1, 2, 3$, there is a unique trisecant line l_i of $\underline{C}$ passing through x_i and close to l_i^0 in the sense that $\{x_{2i+2}, x_{2i+3}\} = l_i \cap \underline{C} \setminus \{x_i^0\}$ is close to the pair $\{x_{2i+2}^0, x_{2i+3}^0\}$ in $\mathrm{Sym}^2(C)$; (ii) there exists a unique quadric Q containing l, l_1, l_2, l_3 and close to Q_0 in the space of quadrics; for this quadric, the residual point $x_{10} = \underline{C} \cap Q \smallsetminus \{x_1, ..., x_9\}$ is close to x_{10}^0.

Denote $Z_l = \{x_1, x_2, x_3\}, \quad Z = \{x_4, ..., x_{10}\}$ and let l_4 be the line on Q through x_{10} belonging to the same ruling as l. These data define a point $m = (Q, l, Z, Z_l) \in M$. Now we will determine the exact sense of the condition that U is sufficiently small:

$$|\mathcal{J}_{Z \cup Z_l/Q}(3,2)| \simeq \mathbb{P}^1. \tag{42}$$

Remark that, by construction, $\bigcup_{i=1}^4 l_i + |\mathcal{O}_Q(0,1)| \subset |\mathcal{J}_{Z \cup Z_l/Q}(3,2)|$, so that

$$|\mathcal{J}_{Z \cup Z_l/Q}(3,2)| = \bigcup_{i=1}^4 l_i + |\mathcal{O}_Q(0,1)|. \tag{43}$$

The condition (42) is open and clearly nonempty. So, for any $x_0 \in \mathrm{Sym}^3(l)$, we obtain a point $m = m(x_0) = (Q, l, Z, Z_l) \in M^*$. Then $(C, m) \in \Pi^*$, and (41) follows. As Π is irreducible and M^* is open in M, this implies that Π^* is dense and open in Π and, consequently,

$$p' : \Pi^* \to \mathcal{C}_{5+2}^* \text{ is dominant.} \tag{44}$$

Hence (40) and Lemma 2.2(i) show that

$$p_\Pi : \Pi^* \to \check{\mathcal{C}}_5^1(X), \ (C', (Q, l, Z, Z_l)) \mapsto C', \text{ is dominant.} \tag{45}$$

Remark that for any $C_2 \in \pi^*|\mathcal{J}_{Z\cup Z_l/Q}(3,2)|$, we have :

$$C'' \cup C_2 \in \pi^*\mathcal{C}_{(3,3)}(\mathbb{P}^3). \tag{46}$$

Define, for any given $z = (C', (Q, l, Z, Z_l)) \in \Pi^*$,

$$\begin{gathered} C_1 = p'(z) = C' \cup C'', \quad C'' = \pi^{-1}(l), \\ f_z = \{C_2 \in \pi^*\mathcal{C}_5^2(\mathbb{P}^3) \mid C_2 = \pi^{-1}(\underline{C}_2),\ \underline{C}_2 \in |\mathcal{J}_{Z\cup Z_l/Q}(3,2)|\},\ f_z \simeq \mathbb{P}^1, \\ f_z^* = \{C_2 \in f_z \mid C_2 \text{ is a reduced curve}\}, \end{gathered} \tag{47}$$

Clearly, f_z^* is a nonempty open subset of f_z. By construction, for any $C_2 \in f_z^*$, the curves C_1 and C_2 are reduced and meet each other quasi-transversely in 13 points:

$$C_1 \cap C_2 = \{y_1, y_2, y_3, y_1', y_2', y_3', y_4, ..., y_{10}\}, \tag{48}$$

$$\begin{aligned} \tilde{Z}_l = \pi^{-1}(Z_l) = \{y_1, y_2, y_3, y_1', y_2', y_3'\} = C'' \cap C_2,\ \tilde{Z} = (\pi|C)^{-1}(Z) \\ = \{y_4, ..., y_{10}\} = C' \cap C_2. \end{aligned}$$

Hence

$$C_0 = C' \cup C'' \cup C_2 \tag{49}$$

is a curve on X of degree 17 and arithmetic genus 29.

Remark also that $\pi^*\mathcal{C}_5^2(\mathbb{P}^3) \simeq \mathcal{C}_5^2(\mathbb{P}^3)$ is clearly a rational irreducible variety of dimension 20:

$$\dim \pi^*\mathcal{C}_5^2(\mathbb{P}^3) = 20. \tag{50}$$

Now consider the incidence variety

$$\Gamma = \left\{ (C_1, C_2) \in \overline{\mathcal{C}_7^4(X)}^* \times \pi^*\mathcal{C}_5^2(\mathbb{P}^3) \;\middle|\; \begin{array}{l} C_1 \text{ and } C_2 \text{ are reduced and meet each} \\ \text{other quasi-transversely in 13 points} \end{array} \right\}$$

with projections $\overline{\mathcal{C}_7^4(X)}^* \xleftarrow{p_1} \Gamma \xrightarrow{p_2} \pi^*\mathcal{C}_5^2(\mathbb{P}^3)$ and define its locally closed subset $\Sigma^* = p_1^{-1}(p'(\Pi^*))$. From (40), (44) and (45) it follows immediately that the natural projection

$$p_{\Sigma^*} := p'' \circ p_1 : \Sigma^* \to \check{\mathcal{C}}_5^1(X),\ (C' \cup C'', C_2) \mapsto C',\ \text{is dominant.} \tag{51}$$

Moreover, for any $z \in \Pi^*$, by (47) and by the description of p' in (40), we obtain for $C_1 = p'(z)$:

$$p_1^{-1}(C_1) \text{ is a rational 7-dimensional variety.} \tag{52}$$

This implies that, for general X, Σ^* is an irreducible variety of dimension 17, and there is an isomorphism

$$\psi : \Sigma^* \xrightarrow{\sim} \Sigma', (C' \cup C'', C_2) \mapsto C' \cup C'' \cup C_2, \tag{53}$$

where

$$\Sigma' = \{C_0 \in \mathrm{Hilb}(X) \mid C_0 \text{ is defined by (48)-(49) for some } z \in \Pi^*\}.$$

In view of (53), we rewrite (51) as follows:

$$p_{\Sigma'} : \Sigma' \to \mathcal{C}_5^1(X),\ C_0 = (C' \cup C'' \cup C_2) \mapsto C',\ \text{is dominant.}$$

Also, by (46), $\mathrm{Im}(q_{\Sigma'}) \subset \pi^*\mathcal{C}_{(3,3)}(\mathbb{P}^3)$, where $q_{\Sigma'}$ is the projection $C_0 = (C' \cup C'' \cup C_2) \mapsto C'' \cup C_2$.

Next, looking at the local equations of Γ in $W = \overline{\mathcal{C}_7^4(X)}^* \times \pi^*\mathcal{C}_5^2(\mathbb{P}^3)$ at a point $w = (C_1, C_2) \in \Sigma^*$, we see that the transversal intersection of the curves C_1 and C_2 in each one of the 13 points (48) gives one local equation for Γ in Y; hence by (50), the dimension of the fiber of p_1 at w is at least 20-13=7. Since by (52) this dimension is precisely 7, it follows that there exists an irreducible component, say, Γ_0 of Γ, which contains w and dominates $\overline{\mathcal{C}_7^4(X)}^*$ via the projecion p_1:

$$\overline{p_1(\Gamma_0)} = \overline{\mathcal{C}_7^4(X)}^*, \tag{54}$$

where the closure is taken in $\overline{\mathcal{C}_7^4(X)}^*$ and we use the irreducibility of $\overline{\mathcal{C}_7^4(X)}^*$ by Lemma 2.3. Hence, taking into account the irreducibility of Σ^*, (52) and (53), it follows that $\Sigma_0 = \psi(\Gamma_0 \cap \Sigma^*)$ is a dense open subset of Σ; also, by (51), the natural projection

$$p_{\Sigma_0} : \Sigma_0 \to \mathcal{C}_5^1(X),\ \ C_0 = (C' \cup C'' \cup C_2) \mapsto C',\ \text{ is dominant.} \tag{55}$$

Now consider in $\mathrm{Hilb}(X)$ the following subset:

$$\tilde{\Gamma}_0 = \{C \in \mathrm{Hilb}(X) \mid C \text{ is a reducible curve of the form } C = C_1 \cup C_2, \\ \text{where } (C_1, C_2) \in \Gamma_0\}.$$

There is an isomorphism $\psi : \Gamma_0 \xrightarrow{\sim} \tilde{\Gamma}_0,\ (C_1, C_2) \mapsto C_1 \cup C_2$, similar to (53).

2.4. Criterion of Welters.

Now we invoke the following important technical result of G.Welters [We, Prop. 6.17], giving a sufficient condition for a family of curves to parametrize, via the Abel-Jacobi map, a translate of the theta divisor of $J(X)$.

Proposition 2.4. *Let T be a smooth irreducible (not necessarily complete) variety parametrizing a family $\{C_t | t \in T\}$ of reduced curves of degree $d = k(k+1) - 3$ on X with the following properties:*

(i) The Abel-Jacobi map $\Phi_T : T \to J(X)$ is dominant;

(ii) for all $t \in T$, the linear system of surfaces $|\mathcal{J}_{C_t/X}(k)|$ has dimension k and the surfaces of this linear system do not contain all the lines of X;

(iii) the incidence divisor in $T \times \mathcal{F}$ is reduced.

Then for any line $l \in \mathcal{F}$, the subvariety of T defined by

$$T_l = \{t \in T \mid C_t \cap l = \varnothing \text{ and } C_t \cup l \text{ lies on a surface from } |\mathcal{O}_X(k)|\}$$

is sent by the Abel-Jacobi map into a translate of the theta divisor $\Theta \subset J(X)$.

Let us construct such a T from the curves introduced in the previous subsection. Consider the Abel-Jacobi map $\Phi_{\tilde{\Gamma}_0} : \tilde{\Gamma}_0 \to J(X)$ and the induced map

$$\Phi_0 : \Gamma_0 \underset{\sim}{\xrightarrow{\psi}} \tilde{\Gamma}_0 \xrightarrow{\Phi_{\tilde{\Gamma}_0}} J(X),\ (C_1, C_2) \mapsto \Phi_{\tilde{\Gamma}_0}(C_1 \cup C_2), \tag{56}$$

where

$$\Phi_{\tilde{\Gamma}_0}(C_1 \cup C_2) = \Phi_{\overline{\mathcal{C}_7^4(X)}^*}(C_1) + \Phi_{\pi^*\mathcal{C}_5^2(\mathbb{P}^3)}(C_2), \quad C_1 \in \overline{\mathcal{C}_7^4(X)}^*,\ \ C_2 \in \pi^*\mathcal{C}_5^2(\mathbb{P}^3). \tag{57}$$

As $\pi^*\mathcal{C}_5^2(\mathbb{P}^3) \simeq \mathcal{C}_5^2(\mathbb{P}^3)$ is rational,

$$\Phi_{\pi^*\mathcal{C}_5^2(\mathbb{P}^3)} = \text{const},\ \ \mathrm{Im}(\Phi_{\pi^*\mathcal{C}_5^2(\mathbb{P}^3)}) = \{c\}. \tag{58}$$

Hence, taking the closure of the image of the Abel-Jacobi map in $J(X)$ and using (54), (56)-(58) and Proposition (2.1), we obtain:

$$\begin{aligned}(59)\quad \overline{\mathrm{Im}(\Phi_{\tilde{\Gamma}_0})} = \overline{\mathrm{Im}(\Phi_0)} &= \overline{c + \mathrm{Im}(\Phi_{\overline{\mathcal{C}^4_7(X)}^*}(\overline{p_1(\Gamma_0)}\,))} = \overline{c + \mathrm{Im}(\Phi_{\overline{\mathcal{C}^4_7(X)}^*})} \\ &= c + \overline{\mathrm{Im}(\Phi_{\mathcal{C}^4_7(X)})} = c + J(X) = J(X),\end{aligned}$$

that is the Abel-Jacobi map $\Phi_{\tilde{\Gamma}_0} : \tilde{\Gamma}_0 \to J(X)$ is dominant. Now, consider the Abel-Jacobi map $\Phi_{\Sigma_0} : \Sigma_0 \to J(X)$. As $\pi^*\mathcal{C}_{(3,3)}(\mathbb{P}^3)$ is rational, $\mathrm{Im}(\Phi_{\pi^*\mathcal{C}_{(3,3)}(\mathbb{P}^3)}) = \{c'\}$ is a point, and we obtain:

$$(60)\qquad \overline{\mathrm{Im}(\Phi_{\Sigma_0})} = \overline{\mathrm{Im}(\Phi_{\mathcal{C}^1_5(X)})} + c'.$$

Now let $\sigma : T \overset{bir}{\longrightarrow} \tilde{\Gamma}_0$ be any desingularization of $\tilde{\Gamma}_0$. Define

$$(61)\qquad \Sigma \subset \sigma^{-1}(\Sigma_0),\ \text{an irreducible component of } \sigma^{-1}(\Sigma_0) \text{ such that } \sigma : \Sigma \twoheadrightarrow \Sigma_0.$$

Consider the Abel-Jacobi maps $\Phi_T = \Phi_{\tilde{\Gamma}_0} \circ \sigma : T \to J(X)$ and $\Phi_\Sigma = \Phi_{\Sigma_0} \circ \sigma : \Sigma \to J(X)$. Then (60) gives

$$(62)\qquad \overline{\mathrm{Im}(\Phi_{\Sigma})} = \overline{\mathrm{Im}(\Phi_{\mathcal{C}^1_5(X)})} + c'.$$

We obtain the following proposition:

Proposition 2.5. *Let X be a general quartic double solid. Then there exists a smooth irreducible variety T parametrizing some family of curves of degree 17 and arithmetic genus 29 such that:*

(i) the Abel-Jacobi map $\Phi_T : T \to J(X)$ is dominant;

(ii) T contains the subset Σ, defined by (61) and satisfying (62), such that the curves C_t for $t \in \Sigma$ split into three components as in (48)-(49);

(iii) the incidence divisor $\mathcal{D} = \{(C, l) \in T \times \mathcal{F} \mid C \cap l \neq \varnothing\}$, endowed with its natural scheme-theoretic structure, is reduced.

Proof. (i) and (ii) folow from (59), (55) and (62). Let us prove (iii). There are 6 distinct lines, counted with multiplicity 1 in the Fano scheme $\mathcal{F}$, which pass through a general point of X. So, it is sufficient to show that there exists a curve C from T which pass through a generam point of X: in this case, the curve $\mathcal{D}_C$ is a reduced 6-sheeted covering of C, and hence $\mathcal{D}$ is reduced. But this is clear, since by the construction of Σ, the components of the reduced curves $C_0 \in \Sigma$ sweep all of X as C_0 runs through Σ. □

We proceed now to the proof of the remaining hypotheses of the Welters criterion for our family T.

Lemma 2.6. *Let X be a general quartic double solid. Then the following assertions are verified:*

(i) For any curve $C \in \Sigma$, $h^0(\mathcal{J}_{C/X}(4)) = 5$;

(ii) for a general (hence any) line $l \in \mathcal{F}$ such that $C \cap l = \varnothing$, there exists a surface $S \in |\mathcal{O}_X(4)|$ containing $C \cup l$;

(iii) for general $C \in \Sigma$, the surfaces of the linear system $|\mathcal{J}_{C/X}(4)|$ do not contain all the lines of X.

Proof. (i) First, take a curve $C \in \Sigma$. By the definition of Σ we have $C = C_1 \cup C_2$, where $C_1 \in \check{\mathcal{C}}_5^1(X)$, $\pi : C_1 \to \underline{C}_1 := \pi(C_1)$ is an isomorphism, $C_2 = \pi^{-1}(\underline{C}_2)$, where $\underline{C}_2 \in \mathcal{C}_{(3,3)}(\mathbb{P}^3)$, and $Z := C_1 \cap C_2 = \{y_1, ..., y_{10}\}$ is a subset of the 13-uple of points (48). This means, in particular, that $\underline{C}_2$ lies in a uniquely defined quadric, say, Q. Let $Z_0 := \pi(Z)$, $\deg(Z_0) = \deg(Z) = 10$. Since by construction $\underline{C}_1 \cap Q \supset Z_0$ (as a scheme-theoretic inclusion) and $\deg(\underline{C}_1 \cap Q) = 10 = \deg(Z_0)$, it follows that, scheme-theoretically, $Z_0 = \underline{C}_1 \cap Q$. Using the fact that $\mathcal{O}_Q(\underline{C}_2) = \mathcal{O}_Q(3)$, we get the following exact triple:

$$0 \to \mathcal{J}_{Q\cup\underline{C}_1/\mathbb{P}^3}(4) \to \mathcal{J}_{\underline{C}_1\cup\underline{C}_2/\mathbb{P}^3}(4) \to \mathcal{O}_Q(1) \to 0. \tag{63}$$

Let $S \subset \mathbb{P}^3$ be any smooth cubic surface passing through the smooth quintic curve $\underline{C}_1$. We have the exact triple

$$0 \to \mathcal{J}_{Q\cup S/\mathbb{P}^3} \to \mathcal{J}_{Q\cup\underline{C}_2/\mathbb{P}^3} \to \mathcal{J}_{(S\cap Q)\cup\underline{C}_1/S} \to 0.$$

Here $\mathcal{J}_{(S\cap Q)\cup\underline{C}_1/S} \simeq \mathcal{O}_S(-2)(-\underline{C}_1)$, because S is smooth, hence twisting by $\mathcal{O}_{\mathbb{P}^3}(4)$, we obtain:

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-1) \to \mathcal{J}_{Q\cup\underline{C}_2/\mathbb{P}^3}(4) \to \mathcal{O}_S(2)(-\underline{C}_1) \to 0. \tag{64}$$

One easily verifies that the natural map $H^0(\mathcal{O}_S(2)) \to H^0(\mathcal{O}_{\underline{C}_1}(2))$ is surjective, so that $h^1(\mathcal{O}_S(2)(-\underline{C}_1)) = 0$. Besides, $h^1(\mathcal{O}_{\mathbb{P}^3}(-1) = 0$. Hence (64) implies $h^1(\mathcal{J}_{Q\cup\underline{C}_1/\mathbb{P}^3}(4)) = 0$. This together with (63) and the equality $h^1(\mathcal{O}_Q(1)) = 0$ gives

$$h^1(\mathcal{J}_{\underline{C}_1\cup\underline{C}_2/\mathbb{P}^3}(4)) = 0. \tag{65}$$

Hence the exact triple $0 \to \mathcal{J}_{\underline{C}_1\cup\underline{C}_2/\mathbb{P}^3}(4) \to \mathcal{J}_{\underline{C}_2/\mathbb{P}^3}(4) \to \mathcal{J}_{Z_0/\underline{C}_1}(4) \to 0$ yields the surjectivity $H^0(\mathcal{J}_{\underline{C}_2/\mathbb{P}^3}(4)) \twoheadrightarrow H^0(\mathcal{J}_{Z_0/\underline{C}_1}(4))$.

Now, since $\pi : C_1 \to \underline{C}_1$ is an isomorphism, we have $\pi_*\mathcal{J}_{Z/C_1}(4) \simeq \mathcal{J}_{Z_0/\underline{C}_1}(4)$, which provides the exact triple

$$0 \to \mathcal{J}_{Z_0/\underline{C}_1}(4) \to \pi_*\mathcal{O}_C(4) \to \pi_*\mathcal{O}_{C_2}(4) \to 0.$$

From here, one immediately obtains:

$$h^0(\pi_*\mathcal{O}_C(4)) = \chi(\mathcal{O}_C(4)) = 40. \tag{66}$$

Next, consider the natural restriction map $g : \pi_*\mathcal{O}_X(4) \twoheadrightarrow \pi_*\mathcal{O}_C(4) \twoheadrightarrow \pi_*\mathcal{O}_{C_2}(4)$. Here $\ker(g) = \pi_*\mathcal{J}_{C_2/X}(4)$, so passing to the map on global sections $f : H^0(\pi_*\mathcal{O}_X(4)) \to H^0(\pi_*\mathcal{O}_C(4))$, we obtain:

$$\operatorname{coker}(f) = \operatorname{coker}(H^0(\pi_*\mathcal{J}_{C_2/X}(4)) \xrightarrow{\text{res}} H^0(\mathcal{J}_{Z_0/\underline{C}_1}(4))). \tag{67}$$

By the projection formula for π, we have $\pi_*\mathcal{J}_{C_2/X}(4) = \pi_*\pi^*\mathcal{J}_{\underline{C}_2/\mathbb{P}^3}(4) = \mathcal{J}_{\underline{C}_2/\mathbb{P}^3}(4) \oplus \mathcal{J}_{\underline{C}_2/\mathbb{P}^3}(2)$, and the composition $H^0(\mathcal{J}_{\underline{C}_2/\mathbb{P}^3}(4)) \hookrightarrow H^0(\pi_*\mathcal{J}_{C_2/X}(4)) \xrightarrow{\text{res}} H^0(\mathcal{J}_{Z_0/\underline{C}_1}(4))$ coincides with the surjective map (65), so (67) implies $\operatorname{coker}(f) = 0$. Hence (66) and the equality $h^0(\mathcal{O}_X(4)) = h^0(\mathcal{O}_{\mathbb{P}^3}(4)) \oplus h^0(\mathcal{O}_{\mathbb{P}^3}(2)) = 45$ imply that

$$H^0(\mathcal{J}_{C/X}(4)) = H^0(\pi_*\mathcal{J}_{C/X}(4)) = \ker(f) = \mathbb{C}^5.$$

(ii) This is a standard dimension count. Take a cubic surface S containing the above smooth quintic $\underline{C}_1$ and take for the above curve $\underline{C}_2$ the complete intersection $\underline{C}_2 = F \cap Q$, where Q is a general quadric. Then $C_1 \cup C_2 \subset \pi^{-1}(S)$. Now take a general plane $\mathbb{P}^2$ in $\mathbb{P}^3$ and take a line l in the del Pezzo surface $\pi^{-1}(\mathbb{P}^2)$ which does not intersect $C_1 \cup C_2$. Clearly such a line exists for a general choice of $\mathbb{P}^2$. Then $S = \pi^{-1}(F \cup \mathbb{P}^2) \in |\mathcal{O}_X(4)|$ is the desired surface.

(iii) Again, this is a standard dimension count. One should embed the curve C into $Y = \mathrm{Proj}(\mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(2))$ (see (4)) and consider the linear system $L = |\mathcal{J}_{C/Y} \otimes \mathcal{O}_{Y/\mathbb{P}^3}(2)|$ on Y. From (i), we deduce that $\dim L = 5$. Take now a line $l \subset Y$ mapped isomorphically onto a line in $\mathbb{P}^3$ by the natural projection $p_Y : Y \to \mathbb{P}^3$. If it is sufficiently general, then there exists a unique smooth $X \in L$ containing l; this X is the wanted double solid, and the lemma is proved. $\square$

We are ready now to prove the main result of this section:

Theorem 2.7. $\overline{\mathrm{Im}(\Phi_{\mathcal{C}_5^1(X)})} = \Theta + \mathrm{const}.$

Proof. The family of curves T constructed in Proposition 2.5 satisfies the hypotheses of Proposition 2.4 with $k = 4$. According to Lemma 2.6, for general $l \in \mathcal{F}$, we have $T_l \supset \Sigma$, where we replace Σ by a nonempty open subset, if necessary. Hence Proposition 2.4 and (62) imply:

$$\overline{\mathrm{Im}(\Phi_{\mathcal{C}_5^1(X)})} \subset \Theta + \mathrm{const}. \tag{68}$$

To prove the opposite inclusion, let us remind a well known description of the image of the differential of the Abel-Jacobi map (see, e.g., [We, (2.11.0) and (6.19)]), in applying it specifically to the map $\Phi := \Phi_{\mathcal{C}_5^1(X)} : \mathcal{C}_5^1(X) \to J(X)$. Take any unobstructed quintic $C \in \check{\mathcal{C}}_5^1(X)$, so that $T_C\mathcal{C}_5^1(X) = H^0(N_{C/X}) = \mathbb{C}^{10}$. Assume, moreover, that C does not lie on the ramification quartic W of X and denote $C_0 = \pi(C)$, $\;Z_0 = \pi(C \cap W)$. Consider the composition

$$\phi :\ \mathbb{C}^{10} = T_C\check{\mathcal{C}}_5^1(X) \xrightarrow{d\Phi} T_{\Phi(C)}J(X) \xrightarrow{\sim} T_0J(X) \simeq (H^0(\mathcal{O}_{\mathbb{P}^3}(2)))^\vee.$$

Then the kernel of the dual map $\phi^\vee$ is described as follows:

$$\mathrm{Ker}(\phi^\vee) = \mathrm{Ker}\{\mathbb{C}^{10} = H^0(\mathcal{O}_{\mathbb{P}^3}(2)) \xrightarrow{H^0(r_{C_0})} H^0(\mathcal{O}_{C_0}(2)) \xrightarrow{H^0(r_{Z_0})} H^0(\mathcal{O}_{Z_0}(2))\}, \tag{69}$$

where $r_{C_0} : \mathcal{O}_{\mathbb{P}^3}(2) \to \mathcal{O}_{C_0}(2)$, $r_{Z_0} : \mathcal{O}_{C_0}(2) \to \mathcal{O}_{Z_0}(2)$ are the restriction maps, and $\mathrm{Im}\,\phi$ is the orthogonal complement of $\mathrm{Ker}(\phi^\vee)$. By definition, C_0 is a smooth elliptic quintic in $\mathbb{P}^3$, hence it does not lie on a quadric, so $H^0(r_{Z_0})$ is an isomorphism, and $\mathrm{Ker}(H^0(r_{Z_0})) \simeq H^0(\mathcal{O}_{C_0}) = \mathbb{C}$, as follows from the exact triple $0 \to \mathcal{O}_{C_0} \xrightarrow{\cdot Z_0} \mathcal{O}_{C_0}(2) \xrightarrow{r_{Z_0}} \mathcal{O}_{Z_0}(2) \to 0$. Hence $\dim \mathrm{Ker}(\phi^\vee) = 1$ and $\dim \mathrm{Im}\,\Phi = 9$. This together with (68) and the irreducibility of Θ (see [We]) implies the wanted assertion. $\square$

3. Pentagons on a degenerate double solid

3.1. Degeneration into a pair of $\mathbf{P}^3$'s.

In this section, we will use the technique of degeneration of the double solid into a reducible variety, developed by Clemens in [C-1], [C-2]. Let $\boldsymbol{\pi} : \mathbf{X} \longrightarrow \Delta$ be a family of double covers $\pi_t : X_t \longrightarrow \mathbb{P}^3$ branched in the quartics $W_t = \{tF + G^2 = 0\} \subset \mathbb{P}^3$ $(t \in \Delta)$, where Δ is an open disc in $\mathbb{C}$, the equation $F = 0$ defines a smooth quartic W and $G = 0$ a smooth quadric Q such that $W \cap Q = B$ is a smooth octic curve. For $t \in \Delta^* = \Delta \setminus \{0\}$, the fiber X_t is a smooth double solid, and for $t = 0$ a union of two copies of $\mathbb{P}^3$, say, $\mathbb{P}', \mathbb{P}''$, meeting each other transversely along Q.

One can associate to $\boldsymbol{\pi}$ the compactified family $\mathbf{F} \longrightarrow \Delta$ of Fano surfaces, whose fiber F_t over $t \neq 0$ parametrizes lines in X_t, and the Neron model $\mathbf{J} \longrightarrow \Delta$ of the family of the intermediate Jacobians J_t of the threefolds X_t. The compactifications of both families are described by Clemens. Namely, F_0 is the union of two components $F' \cup F''$, where F' (resp. F'') parametrizes the bisecant lines to C in $\mathbb{P}'$ (resp. $\mathbb{P}''$). As follows from a remark of Clemens on p. 211 in [C-1], the fiber F_0 is acquired in the family $\mathbf{F}/\Delta$ with multiplicity 1. The lines parametrized by F_0, that is, the bisecants to C in either one of the two copies of $\mathbb{P}^3$, will be called *lines in* X_0. The Neron model $\mathbf{J}$ is a family of abelian Lie groups such that the fiber J_0 over $t = 0$ is the union of two components J^+, J^-, both isomorphic to a $\mathbb{C}^*$-extension of the Jacobian $J(B)$ of B. The Neron model has a natural compactification $\overline{\mathbf{J}}$, obtained by pasting in two copies of $J(B)$, so that the central fiber $\overline{J}_0$ is the union of two components $\overline{J}^{\pm}$ which are both $\mathbb{P}^1$-bundles over $J(B)$ and which meet each other transversely along two disjoint sections of the $\mathbb{P}^1$-bundles.

In [C-2], Clemens studies the family of sextics of genus 3 in the varieties X_t. We will treat in the same way the one of elliptic quintics.

Let $\mathbf{H} \longrightarrow \Delta$ be the subscheme of the relative Hilbert scheme $\mathrm{Hilb}(\mathbf{X}/\Delta)$, such that the fiber H_t over $t \neq 0$ is the component of $\mathrm{Hilb}(X_t)$ whose generic point represents an elliptic quintic in X_t, and the points of H_0, the fiber of $\mathbf{H}$ over $t = 0$, represent the curves which are limits of elliptic quintics in X_t as $t \to 0$, that is, $\mathbf{H}$ is the closure of $\bigcup_{t\in\Delta^*} H_t$ in $\mathrm{Hilb}(\mathbf{X}/\Delta)$. Let $\tilde{\Delta} \longrightarrow \Delta$, $s \mapsto t = s^e$, be some base change and $C_s \in H_{s^e}$ a family of curves in X_{s^e} which are smooth elliptic quintics for $s \neq 0$. Their images $\underline{C}_s = \pi_t(C_s) \subset \mathbb{P}^3$ are elliptic quintics, possibly with self-intersections, which are totally tangent to W_t. Each C_s for $s \neq 0$ defines a unique K3 surface $S(C_s) \in |\mathcal{O}_{X_t}(2)|$ such that $C_s \subset S(C_s)$, where we denote by $\mathcal{O}_{X_t}(k)$ the sheaf $\pi_t^*(\mathcal{O}_{\mathbb{P}^3}(k))$. The image $\pi_t(S(C_s))$ is a quartic $\underline{S(C_s)}$ in $\mathbb{P}^3$, containing $\underline{C}_s$ and tangent to W_t along the curve $\underline{S(C_s)} \cap W_t$, so that $\pi_t^{-1}(\underline{S(C_s)})$ is the union of two components: $S(C_s)$ and $\iota S(C_s)$, where ι denotes the Galois involution of π_t. For $s = 0$, there is a unique limit quartic in $\mathbb{P}^3$, possibly singular, which we denote by $\underline{S(C_0)}$. So the family $S(C_s)$ extends over $s = 0$ with fiber $S(C_0) \simeq \underline{S(C_0)}$ lying in one of the two components $\mathbb{P}', \mathbb{P}''$ of X_0, and the limiting curve C_0 lies entirely in $\mathbb{P}'$ or in $\mathbb{P}''$. As remarks Clemens, a component of C_0 of degree d which does not lie in $Q = \mathbb{P}' \cap \mathbb{P}''$ has to meet $B = W \cap Q$ in $2d$ points counting multiplicities, because otherwise the deformed curve $\underline{C}_s$ for small s will not be totally tangent to W_t, which is absurd. We will call such components 2d-secants to B.

Let H', resp. H'' be the subset of H_0 consisting of the curves $C \subset X_0$ such that: (1) C is smoothable in X_0, (2) C_{red} has no components contained in Q, and (3) all the components of C lie in $\mathbb{P}'$, resp. in $\mathbb{P}''$. Denote $H^0 = H' \cup H''$.

We want now to define a relative Abel-Jacobi map $\boldsymbol{\alpha} = \boldsymbol{\alpha}_{\mathbf{H}} : \mathbf{H} \longrightarrow \overline{\mathbf{J}}$. To do this, we need a local analytic family of algebraic 1-cycles $\{Z_t\}_{t\in\Delta}$ in X_t (we can shrink Δ if necessary) such that Z_t is homologous in X_t to the curves $C_t \in H_t$ for any $t \in \Delta^*$. As $H^2(X_t, \mathbb{Z}) \simeq \mathbb{Z}$, we can choose for Z_t a multiple of a line in X_t. We have mentioned the fact that F_0 is a reduced fiber of $\mathbf{F}$, so there exists a local analytic family of lines $l_t \in F_t$ near $t = 0$, and we can use $Z_t = 5l_t$ as the section of reference points for the definition of $\boldsymbol{\alpha}$. According to [C-1], Sect. 8, $\boldsymbol{\alpha}$ extends as a regular map to all of $H^0 \subset H_0$, and the image of H' (resp. H'') lies in one component J' (resp. J'') of the smooth locus $J_0 = J^+ \cup J^-$ of $\overline{J}_0$ (thus, $J' = J^+$ or J^-, and J'' may a priori coincide either with J', or with the other component of J_0). If we consider the composition ϕ of $\boldsymbol{\alpha}|_{H^0}$ with the natural projection $\eta : J_0 \longrightarrow J(B)$, we obtain the following description modulo a constant translation by a fixed divisor on B depending on the choice of the reference point: $\phi(C)$ is the class of $B \cdot C$ considered as a degree-10 divisor on B. In particular, H' dominates $J(B)$ via ϕ if and anly if H'' does.

We will use these considerations to prove the following proposition.

Proposition 3.1. *Let $\boldsymbol{\pi} : \mathbf{X} \longrightarrow \Delta$ be a family of degenerating quartic double solids as above which is generic, that is, F and G are generic. Then there exists a morphism*

$$\boldsymbol{\alpha} = \boldsymbol{\alpha}_{\mathbf{H}} : \mathbf{H} \setminus A \longrightarrow \overline{\mathbf{J}}$$

over a possibly shrinked disc Δ, where $A = H_0 \setminus H^0$ is a proper closed subset of H_0, such that the following properties are verified:

(i) The restriction $\alpha_t : H_t \longrightarrow J_t$ of $\boldsymbol{\alpha}$ to any fiber H_t with $t \in \Delta^*$ is, modulo a constant translation in J_t, the Abel–Jacobi map of H_t.

(ii) The generic fiber of α_t for any $t \in \Delta$ is the union of finitely many copies of $\mathbb{P}^1$ corresponding to elliptic pencils in the K3 surfaces $S(C_t)$.

(iii) The closure $\mathbf{D}$ of $\boldsymbol{\alpha}(\mathbf{H} \setminus A)$ is a relative divisor in $\overline{\mathbf{J}}$ over Δ, that is $D_t = \mathbf{D} \cap \overline{J}_t$ is a divisor in $\overline{J}_t$ for every $t \in \Delta$, and D_t is a translate of the theta divisor of J_t if $t \in \Delta^*$.

(iv) The smooth locus J_0 of the central fiber $\overline{J}_0$ of $\overline{\mathbf{J}}$ consists of two components $J^{\pm}$, and only one of them, say, J' contains $\alpha_0(H^0)$. Moreover, $\alpha_0(H^0) = \alpha_0(H') = \alpha_0(H'') \subset J'$.

(v) $\alpha_0(H^0)$ is a section of the natural projection $J' \longrightarrow J(C)$ over an open subset of $J(C)$.

Proof. We have already proved (i) and (ii); (iii) is a consequence of Theorem 2.7. To prove (iv) and (v), we will use the characterization of the divisors in $\overline{J}_0$ which can be limits of a family of principal polarizations in the neighboring fibers J_t, given by Clemens in [C-2], (4.2.1)-(4.2.2). It states that: 1) only one of the irreducible components of D_0, say, D', dominates $J(B)$, and 2) if $\overline{J}^{\epsilon}$ ($\epsilon = +$ or $-$) is the component of $\overline{J}_0$ containing D', then D' is a section of the natural projection $\overline{J}^{\epsilon} \longrightarrow J(B)$ over an open subset of $J(B)$.

As we have remarked, H' dominates $J(B)$ via ϕ if and only if H'' does. Hence, if we assume that H' dominates $J(B)$, then $\alpha_0(H')$ and $\alpha_0(H'')$ both dominate $J(B)$, hence $J' = J''$ and $\alpha_0(H') = \alpha_0(H'')$ is one and the same rational section of $J' \longrightarrow J(B)$.

It remains to see that H' dominates $J(B)$. This follows from the fact that for generic W, Q and for a generic 10-uple of points Z in $B = W \cap Q$, there is at least one elliptic quintic in $\mathbb{P}^3$ passing through Z (see Corollary 4.6). Alternatively, one can prove, by an argument similar to that of Clemens on p. 97 of [C-2], that the adherence values of $\boldsymbol{\alpha}$ along A project to a proper closed subset of $J(B)$, and thus at least one of the components of $H_0 \setminus A = H' \cup H''$ has to project to a rational section of $\overline{J}^{\epsilon} \longrightarrow J(B)$. □

3.2. Deforming pentagons from X_0 to X_t.

We want now to prove that the fiber H_0 is reduced at a generic point of H'. To this end, we will construct a local cross-section of the projection $\mathbf{H} \longrightarrow \Delta$ in the neighborhood of 0 taking its values in the subschemes of H_t parametrizing cycles of five lines, or pentagons in X_t.

Definition 3.2. Let X be a quartic double solid. A pentagon in X is a reducible curve $\Gamma = \bigcup_{i=1}^{5} \ell_i$, where ℓ_i are lines in X such that $\ell_i \cap \ell_{i+1} \neq \varnothing$ (addition of subscripts modulo

5). If ℓ_i meets ℓ_{i+1} quasi-transversely at one point and these are the only intersection points of ℓ_i, ℓ_j $(i \neq j)$, we will call Γ a good pentagon.

Consider the 5-uple fibered product of $\mathbf{F}$ with itself over Δ:

$$\mathbf{Y} = \mathbf{F}^5_\Delta = \mathbf{F} \times_\Delta \mathbf{F} \times_\Delta \mathbf{F} \times_\Delta \mathbf{F} \times_\Delta \mathbf{F}$$

Let Y_t denote the fiber of $\mathbf{Y}$ over $t \in \Delta$ and $\mathbf{D}_i$ the subvariety of Y parametrizing the 5-uples $(\ell_1, \dots, \ell_5)$ of lines in the X_t defined by the incidence relation $\ell_i \cap \ell_{i+1} \neq \varnothing$ $(i = 1, \dots, 5,\ 5+1 = 1)$. Then the variety parametrizing all the pentagons in X_t, $t \in \Delta$ can be represented as the intersection $\mathbf{\Sigma} := \bigcap_{i=1}^5 \mathbf{D}_i$.

Proposition 3.3. *Let the quartic W and the quadric Q defining the family $\mathbf{X} \longrightarrow \Delta$ be generic. Let Σ_0 be the fiber of $\mathbf{\Sigma}$ over $t = 0$. Then there exists a component of Σ_0, such that its generic point y represents a good pentagon in X_0 whose five vertices are not in Q, the 6 subvarieties Y_0, $\mathbf{D}_1, \dots, \mathbf{D}_5$ are nonsingular at y and their intersection is transversal at y.*

Proof. It suffices to prove the assertions of the proposition for special W, Q with a degenerate octic $C = W \cap Q$. Choose them in such a way that C decomposes into four smooth conics C_k, $k = 1, \dots, 4$, so that the planes $\Pi_k = \langle C_k \rangle$ of these conics are in general position in $\mathbb{P}^3$. Consider the component Σ' of Σ_0 parametrizing the pentagons $\bigcup_{i=1}^5 \ell_i$ in $\mathbb{P}'$ such that ℓ_k is a bisecant of C_k for $k = 1, 2, 3, 4$ and ℓ_5 meets C_2 and C_3. Choose affine coordinates (u_i, v_i), $i = 1, \dots, 5$, on an affine open set of Y_0 as follows. For $i = 1, \dots, 4$, (u_i, v_i) are affine coordinates in the plane $\mathbb{P}^2_i$ parametrizing lines in Π_i, and u_5, v_5 are the parameters of some birational parametrizations of the conics C_2, C_3 respectively. As F_0 is acquired in $\mathbf{F}$ with multiplicity 1, these ten rational functions extend to local parameters on an affine open set of the neighboring fibers F_t.

We will show that the equations defining $\mathbf{D}_i$ with respect to (u_i, v_i), $i = 1, \dots, 5$ form a system of maximal rank at some point $y_0 \in \Sigma'$. The transversality of the $\mathbf{D}_i$ with Y_0 follows automatically from the fact that the 11 functions $t, u_1, v_1, \dots, u_5, v_5$ form a local (analytic) coordinate system in $\mathbf{Y}$ near y_0, and from the observation that the equations of $\mathbf{D}_i$ do not depend on t. In fact, for our purposes it suffices to see that the linearizations of the equations at y_0 are linearly independent.

In order to assure that Σ' is nonempty and to choose a point y_0 in such a way that the lines of the corresponding pentagon have particularly simple coordinates, let us fix a pentagon in $\mathbb{P}^3$ first: $\ell_i^0 = \overline{P_i P_{i+1}}$ (addition of subscripts modulo 5), where $P_1 = (1:0:0:0)$, $P_2 = (0:1:0:0)$, $P_3 = (0:0:1:0)$, $P_4 = (0:0:0:1)$, $P_5 = (1:1:1:1)$. Then choose W as the union of 4 generic planes Π_k subject to the conditions $\ell_k^0 \subset \Pi_k$, $k = 1, \dots, 4$, and afterwards choose a generic quadric Q passing through the two points $A_1^0 = \Pi_2 \cap \ell_5^0$ and $A_2^0 = \Pi_3 \cap \ell_5^0$. Choose for y_0 the point of Y_0 representing this pentagon and assign to it the values of parameters $u_i = v_i = 0$ for all i.

Now, let the plane of the first conic Π_1 be $\{x_3 - \lambda x_4 = 0\}$. Then we can define the line ℓ_1 corresponding to parameters u_1, v_1 in the neighborhood of ℓ_1^0 as the one passing by the two points $(1:0:\lambda u_1 : u_1), (0:1:\lambda v_1 : v_1) \in \Pi_1$. Its Plücker coordinates are

$$p_{12} = 1, p_{13} = \lambda v_1, p_{14} = v_1, p_{23} = -\lambda u_1, p_{24} = -u_1, p_{34} = O(2),$$

where $O(2)$ stands for the terms of second order in $(u_1, v_1, \dots, u_5, v_5)$ or smaller. Similarly, Π_2 can be given by $\{x_1 - \mu x_4 = 0\}$ and $\ell_2 = \overline{(\mu u_2 : 1 : 0 : u_2)(\mu v_2 : 0 : 1 : v_2)}$ has Plücker coordinates

$$q_{12} = -\mu v_2, q_{13} = \mu u_2, q_{14} = O(2), q_{23} = 1, q_{24} = v_2, q_{34} = -u_2.$$

The incidence relation between two lines in $\mathbb{P}^3$ represented by their Plücker coordinates $(p_{ij}), (q_{ij})$ is given by a single equation:

$$p_{12}q_{34} + p_{34}q_{12} - p_{13}q_{24} - p_{24}q_{13} + p_{14}q_{23} + p_{23}q_{14} = 0.$$

Thus the equation of $\mathbf{D}_1$, linearized at y_0, has the form

$$\mathbf{D}_1 \ : \qquad -u_2 + v_1 = 0.$$

Continuing the calculation in the same manner, we put:

$$\Pi_3 = \{x_1 - \nu x_2 = 0\} \ , \quad \ell_3 = \overline{(\nu u_3 : u_3 : 1 : 0)(\nu v_3 : v_3 : 0 : 1)} \ ,$$

$$\Pi_4 = \{x_1 - x_3 - \eta(x_2 - x_3) = 0\} \ , \quad \ell_4 = \overline{(\eta u_4 : u_4 : 0 : 1)(1 + \eta v_4 : 1 + v_4 : 1 : 1)} \ .$$

To determine the dependence of ℓ_5 on u_5, v_5, remark first that $\ell_5^0 = \overline{A_1^0 A_2^0}$, where $A_1^0 = (\mu : 1 : 1 : 1)$, $A_2^0 = (\nu : 1 : 1 : 1)$. So, to deform ℓ_5^0, we will deform A_i^0 to A_i along the conic C_{1+i} and represent the deformed line ℓ_5 as the one joining A_1 and A_2. We have:

$$\ell_5 = \overline{A_1 A_2} \ , \quad A_1 = (\mu : 1 + (1+\alpha)u_5 + O(2) : 1 + u_5 + O(2) : 1) \ ,$$
$$A_2 = (\nu : 1 : 1 + (1-\beta)v_5 + O(2) : 1 + v_5 + O(2)) \ ,$$

where α, β are the parameters corresponding to tangent directions of C_1, resp. C_2 at A_1, resp. A_2, and we have replaced the conics by their tangent lines, which explains the appearance of the terms $O(2)$ in the formulas for A_i.

Finally, the linearized system of equations of $\mathbf{D}_1, \dots, \mathbf{D}_5$ with respect to variables $u_1, v_1, \dots, u_5, v_5$ has the following matrix:

$$\begin{bmatrix} 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mu & \nu & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1-\nu & \eta - 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \nu - \mu & \alpha(1-\nu) & (1-\mu)(1-\beta) \\ (1-\nu)(\nu-\mu) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -\beta \end{bmatrix}$$

One immediately sees that its rank is 5 as soon as $\mu \neq 0, \nu - \mu \neq 0, \nu - 1 \neq 0$. Hence the wanted transversality is proved. $\square$

3.3. Smoothability of good pentagons.

Now we will check that the good pentagons can be deformed to elliptic quintics.

Proposition 3.4. *Under the hypotheses and in the notation of Proposition 3.3, there exists a local cross-section $\Delta \longrightarrow \mathbf{H}$, $t \mapsto y(t)$, over a possibly shrinked disc Δ, such that $y_0 \in H'$ and the corresponding curves $\Gamma_{y(t)} \subset X(t)$ are good pentagons, strongly smoothable into elliptic quintics in X_t for all $t \in \Delta$.*

Remind that $\Gamma_0 = \Gamma$ is called *smoothable* if there exists a flat family of curves Γ_s such that its generic member is smooth. If there exists such a deformation whose base and the total space are both smooth, then Γ_0 is *strongly smoothable.*

Proof of Proposition 3.4. As above, we specialize our family $\mathbf{X}\longrightarrow\Delta$ to the case when C is the union of four conics C_i. Let $\Gamma = \bigcup_{i=1}^5 \ell_i^0$ be a good pentagon constructed in the proof of Proposition 3.3.

Lemma 3.5. *The Hilbert scheme of $X_0 = \mathbb{P}' \cup \mathbb{P}''$ is smooth of local dimension* 20 *at* Γ.

Proof. We will denote by $\mathcal{N}_Z$ the normal sheaf of a curve Z in $\mathbb{P}' \simeq \mathbb{P}^3$. The normal sheaf of a line in $\mathbb{P}^3$ is $2\mathcal{O}(1)$. Using the relation between the restriction of the normal sheaf of a reducible curve to its component with the normal sheaf of this component (see [HH]), we see that $\mathcal{N}_\Gamma|_{\ell_i}$ is obtained by two positive elementary tranformations from $\mathcal{N}_{\ell_i} \simeq 2\mathcal{O}(1)$. Hence it is either $2\mathcal{O}(2)$, or $\mathcal{O}(1) \oplus \mathcal{O}(3)$. The second case occurs only if the proper transforms of ℓ_{i-1}, ℓ_{i+1} in the blowup of $\mathbb{P}'$ with center ℓ_i meet the exceptional divisor $E = \mathbb{P}(\mathcal{N}_{\ell_i}) \simeq \mathbb{P}^1 \times \mathbb{P}^1$ in two points lying on one cross-section of the natural projection $E\longrightarrow\ell_i$. This may happen only if $\ell_{i-1}, \ell_i, \ell_{i+1}$ lie in one plane, which is impossible in a *good* pentagon. So, $\mathcal{N}_\Gamma|_{\ell_i} \simeq 2\mathcal{O}(2)$ for any i.

Now, denote $\mathcal{N}_\Gamma$ by $\mathcal{N}_1$ and consider the following chain of restriction exact sequences:

$$0\longrightarrow\mathcal{N}_{i+1}\longrightarrow\mathcal{N}_i\longrightarrow\mathcal{N}_i|_{\ell_i}\longrightarrow 0\ ,\ \ i = 1, \dots, 4,$$

in which $\mathrm{Supp}\mathcal{N}_i = \bigcup_{j=i}^5 \ell_j$. We have already determined $\mathcal{N}_1|_{\ell_1} \simeq 2\mathcal{O}_{\ell_1}(2)$. For $i = 2, \dots, 5$ one finds successively:

$$\mathcal{N}_i|_{\ell_i} \simeq 2\mathcal{O}_{\ell_i}(1) \ \ (i = 2, 3, 4),$$

whilst

$$\mathcal{N}_5 = \mathcal{N}_5|_{\ell_5} \simeq 2\mathcal{O}\ .$$

Now, reading the same chain of exact sequences backwards, one easily calculates $h^k(\mathcal{N}_i)$ for all i, k and concludes that $h^0(\mathcal{N}_\Gamma) = 20$, $h^1(\mathcal{N}_\Gamma) = 0$. This ends the proof of the lemma. □

Now we will impose on the curves parametrized by $\mathrm{Hilb}(X_0)$ the incidence conditions with the four conics C_i.

Lemma 3.6. *In the situation of the preceding lemma, let U_C denote the locally closed subset of* $\mathrm{Hilb}(X_0)$ *representing the curves meeting transversely C at 10 points. Then the point y_0 representing Γ is a smooth point of U_C, $\dim_{y_0} U_C = 10$, and the generic point of the component of U_C contaning y_0 represents a nonsingular curve in $\mathbb{P}'$.*

Proof. By Lemma 3.5, y_0 is a smooth point of $\mathrm{Hilb}(X_0)$. It has a neighborhood U consisting only of smooth points and contained entirely in $\mathrm{Hilb}(\mathbb{P}')$. Let $F \subset \mathbb{P}'$ be any reduced curve meeting Γ transversely at r points $x_1, \dots, x_r$ which are nonsingular both on F and on Γ, and let ξ_i for $i = 1, \dots, r$ denote the point of $\mathbb{P}(\mathcal{N}_\Gamma)$ corresponding to the direction of the branch of F passing through x_i. Denote by $\mathrm{elm}^-_{\xi_1,\dots,\xi_r}\mathcal{N}_\Gamma$ the negative elementary transformation of the normal bundle $\mathcal{N}_\Gamma$ w.r.t. the ξ_i (see [HH]). Let $U_F \subset U$ be the locus of curves meeting F transversely at r distinct points and represented by elements of U. Then U_F is smooth at y_0 if $h^1(\Gamma, \mathrm{elm}^-_{\xi_1,\dots,\xi_r}\mathcal{N}_\Gamma) = 0$ and in this case we have for the tangent space $T_{y_0}U_F = H^0(\Gamma, \mathrm{elm}^-_{\xi_1,\dots,\xi_r}\mathcal{N}_\Gamma)$ (see [Ran], Remark 5.1).

Let us apply this observation to $F = C = \bigcup_{i=1}^4 C_i$, $r = 10$, $C_i \cap \ell_i = \{x_{2i-1}, x_{2i}\}$, $i = 1, \ldots, 5$. Let $\mathcal{N} = \mathrm{elm}^-_{\xi_1,\ldots,\xi_{10}} \mathcal{N}_\Gamma$. As we saw in the proof of Lemma 3.5, $\mathcal{N}_\Gamma|_{\ell_i} \simeq 2\mathcal{O}(2)$ for any i. As the branches of C passing through x_{2i-1}, x_{2i} lie in one plane for each $i = 1, \ldots, 4$ and they do not for $i = 5$, we have $\mathcal{N}|_{\ell_i} \simeq \mathcal{O} \oplus \mathcal{O}(2)$ for $i = 1, \ldots, 4$ and $\mathcal{N}|_{\ell_5} \simeq 2\mathcal{O}(1)$. Similarly to the proof of Lemma 3.5, one can easily show that $h^0(\mathcal{N}) = 10$, $h^1(\mathcal{N}) = 0$. It remains to prove that Γ is smoothed out by a generic deformation inside U_C. It suffices to show that the natural map $T_{y_0} U_C \longrightarrow T^1_{p_i}$ to the Schlesinger's space T^1 is surjective for any $i = 1, \ldots, 5$, where $p_i = \ell_i \cap \ell_{i+1}$ (the addition of subscripts modulo 5). We do it in the same way as the authors of [HH] in the proof of Theorem 4.1.

Let $F = \ell_i$, $G = \bigcup_{j \neq i} \ell_j$. We have the following natural exact sequences:

$$0 \longrightarrow \mathcal{N} \longrightarrow \mathcal{N}|_F \oplus \mathcal{N}|_G \longrightarrow \mathcal{N}|_{\{p_{i-1}, p_i\}} \longrightarrow 0 \ ,$$

$$0 \longrightarrow \mathrm{elm}^+_{\{p'_{i-1}\}} \mathrm{elm}^-_{\{\xi_{2i-1}, \xi_{2i}\}} \mathcal{N}_F \longrightarrow \mathcal{N}|_F \longrightarrow T^1_{p_i} \longrightarrow 0 \ ,$$

where p'_i is the point of $\mathbb{P}(\mathrm{elm}^-_{\{\xi_{2i-1},\xi_{2i}\}} \mathcal{N}_F)$ corresponding to the tangent direction of l_{i+1} at p_i. It follows that if the natural maps $H^0(\mathcal{N}|_G) \longrightarrow \mathcal{N}|_{\{p_i\}}$ and $H^0(\mathcal{N}|_F) \longrightarrow T^1_{p_i}$ are both surjective, then $H^0(\mathcal{N}) \longrightarrow T^1_{p_i}$ is surjective as well. We have $\mathcal{N}|_F \simeq \mathcal{O} \oplus \mathcal{O}(2)$ or $2\mathcal{O}(1)$, so the surjectivity of the second map is obvious. Next, G is a chain of four lines $\ell_{i+1}, \ldots, \ell_{i+4}$ and

$$H^0(\mathcal{N}|_G) = \{(s_{i+1}, s_{i+2}, s_{i+2}, s_{i+4}) \in \sum_{k=i+1}^{i+4} H^0(\mathcal{N}|_{\ell_k}) \mid s_k(p_k) = s_{k+1}(p_k) \quad \text{for } k = i+1, i+2, i+3\}$$

Each restriction $\mathcal{N}|_{\ell_k}$ is generated by global sections, so for arbitrary s_{i+1}, we can choose a s_{i+2} such that $s_{i+2}(p_{i+1}) = s_{i+1}(p_{i+1})$, then choose a s_{i+3} such that $s_{i+3}(p_{i+2}) = s_{i+2}(p_{i+2})$, and finally, choose a s_{i+4} such that $s_{i+4}(p_{i+3}) = s_{i+3}(p_{i+3})$. Thus any $s_{i+1} \in H^0(\mathcal{N}|_{\ell_{i+1}})$ extends to a section of $H^0(\mathcal{N}|_G)$, hence the map $H^0(\mathcal{N}|_G) \longrightarrow \mathcal{N}|_{\{p_i\}}$ is surjective and we are done. □

Now we are coming back to the proof of Proposition 3.4. By Proposition 3.3, we can find a local cross-section of Σ representing a flat family of good pentagons in the varieties X_t, whose fiber over 0 is a good pentagon in $\mathbb{P}'$. By the universal property of the Hilbert scheme, this family is induced via a cross-section $\Delta \longrightarrow \mathrm{Hilb}(\mathbf{X}/\Delta)$ over Δ, which we will denote by $t \mapsto y(t)$. It suffices to show that $y_0 \in H'$. We have verified in Lemma 3.6 that a good pentagon Γ in $\mathbb{P}'$ becomes strongly smoothable when C is specialized into a union of four conics. By the semi-continuity of $h^i(\mathcal{N})$, and because the surjectivity of the maps $H^0(\mathcal{N}) \longrightarrow T^1_{p_i}$ is an open condition, a generic good pentagon in some component of good pentagons in $\mathbb{P}'$ is strongly smoothable for generic $\mathbf{X} \longrightarrow \Delta$. In fact, we do not check that the family of good pentagons in $\mathbb{P}'$ is irreducible, so we are speaking here about one of its components.

As Γ is a connected curve of degree 5 and arithmetic genus 0, so is its smoothing, hence it is an elliptic quintic in $\mathbb{P}'$. By construction, it meets transversely C at 10 points and is an element of H'. This ends the proof of Proposition 3.4. □

This immediately implies

Corollary 3.7. *With the same assumptions as above, there is a component of H' which is acquired with multiplicity 1 in the fiber H_0 of $\mathbf{H} \longrightarrow \Delta$.*

We will prove in Section 4 that H' is irreducible, hence H' and H'' are irreducible components of H_0 acquired with multiplicity 1.

4. Elliptic quintics through ten points

4.1. Elliptic quintics, 10-secant to $B = W \cap Q$.

In this section we will prove the irreducibility of the subscheme H' of $\mathrm{Hilb}(X_0)$ introduced in Sect. 3.1.

Lemma 4.1. *Let Q (resp. S) be a generic quadric (resp. quartic) in $\mathbb{P}^3$, and $B = Q \cap S$. Let $\mathcal{C}^1_5[B]_{10} \subset \mathcal{C}^1_5(\mathbb{P}^3)$ be the base of the family of elliptic quintics meeting B in a subscheme of length 10. Then $\mathcal{C}^1_5[B]_{10}$ is irreducible.*

Proof. Let $\mathbb{P}^{34}$ be the space of quartic surfaces in $\mathbb{P}^3$ and consider the incidence variety $\Gamma = \{(S,C) \in \mathcal{C}^1_5(\mathbb{P}^3) \times \mathbb{P}^{34} \mid C \subset S\}$ with projections $\mathbb{P}^{34} \xleftarrow{p_1} \Gamma \xrightarrow{p_2} \mathcal{C}^1_5(\mathbb{P}^3)$. First remark that for any quintic $C \in \mathcal{C}^1_5(\mathbb{P}^3)$ one has $h^0(\mathcal{I}_{C/\mathbb{P}^3}(4)) = 15$. In fact, since C does not lie on a quadric, it follows that the restriction map $H^0(\mathcal{O}_{\mathbb{P}^3}(2)) \to H^0(\mathcal{O}_C(2))$ is surjective, hence the exact triple $0 \to \mathcal{I}_{C/\mathbb{P}^3}(2) \to \mathcal{O}_{\mathbb{P}^3}(2) \to \mathcal{O}_C(2) \to 0$ gives $h^1(\mathcal{I}_{C/\mathbb{P}^3}(2)) = 0$ and a fortiori $h^1(\mathcal{I}_{C/\mathbb{P}^3}(4)) = 0$, so that we obtain the desired equality from the exact triple $0 \to H^0(\mathcal{I}_{C/\mathbb{P}^3}(4)) \to H^0(\mathcal{O}_{\mathbb{P}^3}(4)) \to H^0(\mathcal{O}_C(4)) \to 0$. Hence $p_2^{-1}(C) \simeq \mathbb{P}^{14}$, and as $\mathcal{C}^1_5(\mathbb{P}^3)$ is irreducible of dimension 20, Γ is irreducible as well and $\dim \Gamma = 34$. Moreover, denoting $\mathcal{C}^1_5(\mathbb{P}^3)^* := \mathcal{C}^1_5(\mathbb{P}^3) \smallsetminus \mathrm{Sing}(\mathcal{C}^1_5(\mathbb{P}^3))$, $\Gamma^* = p_2^{-1}(\mathcal{C}^1_5(\mathbb{P}^3)^*)$, we see that Γ^* is smooth and is a projective bundle over $\mathcal{C}^1_5(\mathbb{P}^3)^*$.

Next, clearly, a general quartic S in the fibre $p_2^{-1}(C)$ is smooth for general $C \in \mathcal{C}^1_5(\mathbb{P}^3)$. Hence, denoting $D = p_1(\Gamma)$, we obtain that D is irreducible. Besides, for any smooth quartic $S = p_1(S,C) \in D$ the fibre $p_1^{-1}(S)$ is a dense open subset of the linear series $|\mathcal{O}_S(C)| \simeq \mathbb{P}^1$; hence $\dim D = 33$, i.e. D is an irreducible divisor on $\mathbb{P}^{34}$. Moreover, $p_1 : \Gamma \to D$ is a smooth morphism of relative dimension 1.

Next, consider the closure $\overline{D}$ of D in $\mathbb{P}^{34}$ and prove that

$$\operatorname{codim}_{\overline{D}} \mathrm{Sing}\, \overline{D} \geq 2. \tag{70}$$

In fact, if $\operatorname{codim}_{\overline{D}} \mathrm{Sing}\, \overline{D} = 1$, i.e. $\dim \mathrm{Sing}\, \overline{D} = 32$, then, taking any $C \in \mathcal{C}^1_5(\mathbb{P}^3)^*$, we see that $p_1(p_2^{-1}(C)) \cap \mathrm{Sing}\, \overline{D} = \mathbb{P}^{14} \cap \mathrm{Sing}\, \overline{D} \neq \varnothing$; hence, since $p_1|\Gamma^*$ is a smooth morphism, $p_2^{-1}(C) \cap \mathrm{Sing}\, \Gamma^* \subset p_2^{-1}(C) \cap p_1^{-1}(\mathrm{Sing}\, \overline{D}) \neq \varnothing$, which contradicts the smoothness of Γ^*.

Next let $\mathcal{C}_{2\cdot 4}(\mathbb{P}^3) := \{C \in \mathrm{Hilb}(X) \mid C$ is a smooth complete intersection curve of a smooth quartic surface and a smooth quadric surface in $\mathbb{P}^3\}$. Clearly, $\mathcal{C}_{2\cdot 4}(\mathbb{P}^3)$ is a smooth irreducible variety of dimension 33. Let U be the dense open subset of smooth quartics in $\mathbb{P}^{34}$ and consider the incidence variety $\Pi = \{(C,F) \in \mathcal{C}_{2\cdot 4}(\mathbb{P}^3) \times \mathbb{P}^{34} \mid C \subset F\}$ with projections $\mathcal{C}_{2\cdot 4}(\mathbb{P}^3) \xleftarrow{q_1} \Pi \xrightarrow{q_2} \mathbb{P}^{34}$. One immediately verifies, using the resolution $0 \to \mathcal{O}_{\mathbb{P}^3}(-2) \to \mathcal{O}_{\mathbb{P}^3}(2) \oplus \mathcal{O}_{\mathbb{P}^3} \to \mathcal{I}_{C/\mathbb{P}^3}(4) \to 0$ for $C \in \mathcal{C}_{2\cdot 4}(\mathbb{P}^3)$, that $h^0(\mathcal{I}_{C/\mathbb{P}^3}(4)) = 11$. Besides, by the definition of $\mathcal{C}_{2\cdot 4}(\mathbb{P}^3)$, a general quartic in $q_1^{-1}(C)$ is smooth, hence $q_1^{-1}(C)$ is a dense open part of the projective space $\mathbb{P}^{10} = P(H^0(\mathcal{I}_{C/\mathbb{P}^3}(4)))$. Thus, Π is irreducible of dimension 43. Moreover, for any point $(C,S) \in \Pi$, we have $(C,S) \in q_2^{-1}(S')$ with $S' \in U$, and $q_2^{-1}(S')$ is an open subset of the linear series $|\mathcal{O}_{S'}(C)| = |\mathcal{O}_{S'}(2)| \simeq \mathbb{P}^9$ consisting of smooth curves. Clearly, $D' := U \cap D \neq \varnothing$, hence D' is an open subset of D, so that $D_\Pi := q_2^{-1}(D')$ is an irreducible divisor in Π.

Let us look now at the general fiber of the projection $q_D := q_1|D_\Pi : D_\Pi \to \mathcal{C}_{2\cdot 4}(\mathbb{P}^3)$. Let $\overline{\Pi}$ be the closure of Π in $\overline{\mathcal{C}} \times \mathbb{P}^{34}$ with natural projections $\mathcal{C} \xleftarrow{\overline{q}_1} \overline{\Pi} \xrightarrow{\overline{q}_2} \mathbb{P}^{34}$, where $\overline{\mathcal{C}}$ is the closure of $\mathcal{C}_{2\cdot 4}(\mathbb{P}^3)$ in $\mathrm{Hilb}(\mathbb{P}^3)$, and $\overline{D}_\Pi = \overline{q}_2^{-1}(\overline{D})$. Let $Z \subset \overline{D}$ be the subset of $\overline{D}$ which consists of reducible quartics, $\mathrm{codim}_{\overline{D}} Z \geq 11$, $D^* = \overline{D} \smallsetminus Z$, $D^*_\Pi := \overline{q}_2^{-1}(D^*)$. Clearly, for any $S \in D^*$, the linear series $|\mathcal{O}_S(2)| \simeq \mathbb{P}^9$ lies in $\overline{\mathcal{C}}$. Hence $\overline{q}_2|_{D^*_\Pi} : D^*_\Pi \to D^*$ is a projective bundle with fiber $\mathbb{P}^9$, in particular, it is a smooth morphism of relative dimension 9. Now by construction the fiber of the projection $\overline{q}_D = \overline{q}_1|\overline{D} : \overline{D}_\Pi \to \overline{\mathcal{C}}$ over any point $C \in \mathcal{C}_{2\cdot 4}(\mathbb{P}^3)$ is a divisor in $\overline{q}_1^{-1}(C) = \mathbb{P}^{10}$ which contains $q_D^{-1}(C)$ as an open subset. Assume that $q_D^{-1}(C)$ is reducible. Then its closure $\overline{q_D^{-1}(C)}$ is also a reducible divisor in $\mathbb{P}^{10}$, hence it has singularities in codimension 1. Since $C \in \mathcal{C}_{2\cdot 4}(\mathbb{P}^3)$ is smooth, it follows that $\mathrm{codim}_{\overline{D}_\Pi} \mathrm{Sing}\, \overline{D}_\Pi = 1$. As $\mathrm{codim}_{\overline{D}} Z \geq 11$, this implies that $\mathrm{codim}_{D^*_\Pi} \mathrm{Sing}\, D^*_\Pi = 1$. Hence, since $\overline{q}_2|_{D^*_\Pi} : D^*_\Pi \to D^*$ is a smooth morphism, it follows that $\mathrm{codim}_{D^*} \mathrm{Sing}\, D^* = 1$ and a fortiori $\mathrm{codim}_{\overline{D}} \mathrm{Sing}\, \overline{D} \geq 2$, which contradicts (70).

Thus, for general $C \in \mathcal{C}_{2\cdot 4}(\mathbb{P}^3)$, both $q_D^{-1}(C)$ and $\overline{q_D^{-1}(C)}$ are irreducible. The generic quartic S from $\overline{q_D^{-1}(C)}$ is smooth and contains a unique pencil $|E|$ of elliptic curves, and these curves belong to $\mathcal{C}_5^1[B]_{10}$, where $B = Q \cap S$, Q being the unique quadric containing C. Conversely, taking B as in the hypothesis of the lemma, any smooth elliptic quintic E meeting B in 10 points lies in a generically smooth quartic surface S through B, i.e. such that $(B, S) \in \overline{q}_D^{-1}(B)$, and varies on S in a pencil. Hence we have a fibration $H' \to \overline{q}_D^{-1}(C)$ with an open subset of $\mathbb{P}^1$ as a fiber. This implies the irreducibility of $\mathcal{C}_5^1[B]_{10}$. □

Corollary 4.2. *Let $Q \subset \mathbb{P}^3$ be a smooth quadric surface, $W \subset \mathbb{P}^3$ a generic quartic one, $B = Q \cap W$. Then the subscheme H' (resp. H'') of $\mathrm{Hilb}(X_0)$ introduced in Sect. 3.1 and $\mathcal{C}_5^1[B]_{10}$ have a common dense open subset, hence H' (resp. H'') is irreducible. This implies, by Corollary 3.7, that the fiber H_0 of $\mathbf{H}$ over $t = 0$ is reduced at the generic points of H' and H''.*

4.2. 42 elliptic quintics through 10 points

Consider the following open subset of $\mathrm{Hilb}\,\mathbb{P}^3$:

$$U = \{C \mid C \text{ is a smooth irreducible elliptic quintic such that } \{C\} \text{ is a smooth point of } \mathrm{Hilb}\,\mathbb{P}^3\}.$$

There is a universal family of quintics $\mathcal{C} \subset U \times \mathbb{P}^3$ over U with projection $p_{\mathcal{C}} : \mathcal{C} \to U$. Consider the 10-th relative symmetric power $\mathcal{S} := \mathrm{Sym}^{10}(\mathcal{C}/U)$ of $\mathcal{C}$ over U with projection $p : \mathcal{S} \to U$ and fiber

$$p^{-1}(\{C\}) = \mathrm{Sym}^{10} C, \quad \{C\} \in U. \tag{71}$$

Thus $\mathcal{S}$ is a smooth irreducible variety of dimension 30. We can identify it with a subvariety of $U \times \mathrm{Hilb}^{10}\mathbb{P}^3$:

$$\mathcal{S} = \{(\{C\}, \{Z\}) \mid \{C\} \in U, \{Z\} \in \mathrm{Hilb}^{10}\mathbb{P}^3 \text{ and } Z \subset C \text{ as a scheme}\},$$

and we have a natural projection

$$\pi : \mathcal{S} \to \mathrm{Hilb}^{10}\mathbb{P}^3, \ (\{C\}, \{Z\}) \mapsto \{Z\}.$$

There exists at least one quintic from the closure of $\mathcal{C}_7^4(\mathbb{P}^3)$ passing through any 10 points in $\mathbb{P}^3$, and for a general 10-uple of points in $\mathbb{P}^3$, the number of such quintics is finite and they are smooth. In fact, we have:

Lemma 4.3 (E. Getzler, [Ge], Table 2). *For a general 10-uple Z of points in $\mathbb{P}^3$, there are 42 elliptic quintics passing through Z.*

Denote by H^{10} the component of $\mathrm{Hilb}^{10}\,\mathbb{P}^3$ contating a 10-uple of distinct points. It follows that $H := \pi(\mathcal{S})$ is an open subset of H^{10} and that $\pi : \mathcal{S} \to H$ is generically finite of degree 42. We need to replace in the above lemma "a general 10-uple Z of points in $\mathbb{P}^3$" by "a general 10-uple Z of points in a smooth quadric $Q \subset \mathbb{P}^3$", that is, we should prove that the 10-uples of points in a given smooth quadric form a divisor in H which is not contained in the branch locus of π.

Let

$$D = \{(\{C\}, \{Z\}) \in \mathcal{S} \mid Z = C \cap Q \text{ for some quadric } Q \text{ in } \mathbb{P}^3\}.$$

Clearly, D is a divisor in $\mathcal{S}$, and by (71), for any $\{C\} \in U$,

$$D \cap p^{-1}(\{C\}) = \{\{Z\} \in \mathrm{Sym}^{10}\, C \mid \mathcal{O}_C(Z) = \mathcal{O}_{\mathbb{P}^3}(2)|C\} = \big|\mathcal{O}_{\mathbb{P}^3}(2)|C\big|\} \simeq \mathbb{P}^9. \tag{72}$$

As U is irreducible, it follows from (72) that D is irreducible too, hence so is $B := \pi(D)$. It is easy to see that B is a divisor. Indeed, for any smooth elliptic quintic C in $\mathbb{P}^3$ and for any $Z \in\ |\mathcal{O}_C(2)|$, there is a unique quadric Q passing through Z, so the dimension of B is at least that of the family of quadrics Q in $\mathbb{P}^3$ plus that of the the family of the 10-uples of points in Q, which gives 29.

Further, let $R \subset \mathcal{S}$ be the ramification divisor of π, so that

$$K_{\mathcal{S}} = p^* K_U \otimes K_{\mathcal{S}/U} = \pi^* K_H \otimes \mathcal{O}_{\mathcal{S}}(R). \tag{73}$$

Our aim is to prove the following

Proposition 4.4. *(i) D is not contained in R, hence $\pi : \mathcal{S} \to H$ is étale in a general point of D.*

(ii) π is an étale covering over an open subset V of H such that $B \cap V \neq \varnothing$.

Proof. We start by (ii), in assuming (i) already proved.

For a general 10-uple of points Z on a general smooth quadric Q, there exists a quartic surface passing through Z and intersecting Q in a smooth curve, say, C_0. Further, by Corollary 4.2, the set of all the smooth elliptic quintics meeting C_0 in 10-uples of points is irrreducible. Hence $(\pi^{-1}(\mathrm{Sym}^{10}(C_0)))_{\mathrm{red}}$ is irreducible. Hence π has the same ramification indices in all the points of $\pi^{-1}(Z)$, where $\{Z\} \in B$ is the above general point of B. But according to (i), π is unramified in at least one point of $\pi^{-1}(Z)$. Hence π is unramified, that is étale, in all the points of $\pi^{-1}(Z)$.

Prove now (i). To see that D is not contained in R, it suffices to show, by virtue of (72), that

$$R = p^{-1}(R_0) \tag{74}$$

for some divisor R_0 on U. Take any point $\{C\} \in U$. Then π embeds $S_C := \mathrm{Sym}^{10}\, C$ into H. Moreover, H is the union of the subvarieties $\pi(S_C) \simeq S_C$ when $\{C\}$ runs through U. Since S_C is a fiber of p, (74) follows from the relation

$$R \cap S_C \overset{alg}{\sim} 0 \ \text{ on } \ S_C, \tag{75}$$

which we are going to prove now.

First, clearly

$$K_{\mathcal{S}/U}|S_C = K_{S_C}. \tag{76}$$

Next, consider S_C as a subvariety of H, embedded by $\pi|_{S_C}$, and compute the normal sheaf $N_{S_C/H}$. Let $\Gamma \subset \mathcal{S} \times S_C$ be the incidence divisor and $C \xleftarrow{p_1} \mathcal{S} \times S_C \xrightarrow{p_2}$ the natural projections.

Lemma 4.5.

$$N_{S_C/H} = p_{2*}p_1^*(N_{C/\mathbb{P}^3}|\Gamma), \ R^1 p_{2*}p_1^*(N_{C/\mathbb{P}^3}|\Gamma) = 0. \tag{77}$$

Proof. Take any $\{Z\} \in S_C$. Since Z is a curvilinear 0-dimensional subscheme in $\mathbb{P}^3$, hence a locally complete intersection, we have:

$$\begin{aligned} &\mathrm{Ext}^1(\mathcal{I}_{Z/C}, \mathcal{O}_Z) = 0, \quad \mathrm{Hom}(\mathcal{I}_{Z/C}, \mathcal{O}_Z) = T_{\{Z\}}S_C \simeq \mathbb{C}^{10}, \\ &\mathrm{Hom}(\mathcal{I}_{Z/\mathbb{P}^3}, \mathcal{O}_Z) = T_{\{Z\}}H \simeq \mathbb{C}^{30}, \quad \mathrm{Hom}(\mathcal{I}_{C/\mathbb{P}^3}, \mathcal{O}_Z) = H^0(N_{C/\mathbb{P}^3} \otimes \mathcal{O}_Z). \end{aligned} \tag{78}$$

Thus, applying the functor $\mathrm{Hom}(\cdot, \mathcal{O}_Z)$ to the exact triple $0 \to \mathcal{I}_{C/\mathbb{P}^3} \to \mathcal{I}_{Z/\mathbb{P}^3} \to \mathcal{I}_{Z/C} \to 0$, we obtain the exact triple:

$$0 \to T_{\{Z\}}S_C \to T_{\{Z\}}H \to H^0(N_{C/\mathbb{P}^3} \otimes \mathcal{O}_Z) \to 0. \tag{79}$$

Hence

$$N_{S_C/H}|_Z = H^0(N_{C/\mathbb{P}^3} \otimes \mathcal{O}_Z). \tag{80}$$

From here, taking into account (78), we deduce:

$$H^0(N_{C/\mathbb{P}^3} \otimes \mathcal{O}_Z) = p_{2*}p_1^*(N_{C/\mathbb{P}^3}|\Gamma)|_Z, \quad H^1(N_{C/\mathbb{P}^3} \otimes \mathcal{O}_Z) = 0. \tag{81}$$

so that, globalizing (80) over S_C, we get (77) by base change. □

Coming back to the proof of Proposition 4.4, take any cubic surface F in $\mathbb{P}^3$ through C, smooth along C; we can shrink U, if necessary, so that such a cubic exists for all $C \in U$. Then the exact sequence of normal bundles of the embeddings $C \hookrightarrow F \hookrightarrow \mathbb{P}^3$, in view of the equality $\det(N_{C/\mathbb{P}^3}) = \mathcal{O}_C(4)$, can be written in the form $0 \to \mathcal{O}_C(1) \to N_{C/\mathbb{P}^3} \to \mathcal{O}_C(3) \to 0$. Denote $E_{\mathcal{O}_C(1)} = p_{2*}(\mathcal{O}_C(1) \boxtimes \mathcal{O}_{S_C}|\Gamma$, $E_{\mathcal{O}_C(3)} = p_{2*}(\mathcal{O}_C(3) \boxtimes \mathcal{O}_{S_C}|\Gamma$ and apply $R^i p_{2*}(N_{C/\mathbb{P}^3} \boxtimes \mathcal{O}_{S_C} \otimes \cdot)$ to the last exact triple. Using (77), we get

$$c_1(N_{S_C/H}) = c_1(E_{\mathcal{O}_C(1)}) + c_1(E_{\mathcal{O}_C(3)}). \tag{82}$$

By [F, Ex. 14.4.17(b)] we obtain: $c_1(E_{\mathcal{O}_C(1)}) \overset{alg}{\sim} -5x$, $c_1(E_{\mathcal{O}_C(3)}) \overset{alg}{\sim} 5x$, where x is the algebraic equivalence class of the image of Sym^9C under the embedding $Sym^9C \hookrightarrow Sym^{10}C : D \mapsto D + P_0$, and $P_0 \in C$ is a fixed point. Hence

$$c_1(N_{S_C,H}) \overset{alg}{\sim} 0. \tag{83}$$

so that, identifying S_C with $\pi(S_C)$, we get $K_H|\pi(S_C) \overset{alg}{\sim} K_{\pi(S_C)}$, hence also

$$\pi^* K_H^{-1}|S_C \overset{alg}{\sim} (\pi|S_C)^*(K_{\pi(S_C)}^{-1}) = K_{S_C}^{-1}. \tag{84}$$

Substituting (76) and (84) into (73)$|S_C$, we obtain (75), and this ends the proof. □

The following statements are obvious corollaries of what we have proved:

Corollary 4.6. *Let $Q \subset \mathbb{P}^3$ be a smooth quadric surface. Then there are* 42 *elliptic quintics in $\mathbb{P}^3$ passing through* 10 *generic points of Q. There are also* 42 *elliptic quintics in $\mathbb{P}^3$ passing through* 10 *generic points on the curve $B = Q \cap W$ for a generic quartic surface $W \subset \mathbb{P}^3$.*

In fact, the number 42 given by the result of Getzler is that of stable pointed genus-1 curves in $\mathbb{P}^3$ with incidence relation in 10 prescribed generic points. The moduli space of these curves is projective, so the result implies that there are no singular curves in $\overline{\mathcal{C}_5^1(\mathbb{P}^3)}$ which pass through the 10 generic points, because all the 42 solutions to the problem are in the interior part of the moduli space, but not on its boundary, corresponding to singular curves. We thus obtain:

Corollary 4.7. *Let $\alpha_0 : H' \cup H'' \longrightarrow J^+$ be the map introdiced in Proposition 3.1, (ii). Then the number of copies of $\mathbb{P}^1$ in the generic fiber of α_0 is equal to* 84.

As $H' \cup H''$ contains all the elliptic quintics which are the limits of elliptic quintics from the neighboring fibers, we deduce from Corollaries 4.7 and 4.2 that the number of $\mathbb{P}^1$'s in the generic fiber of α_t is 84 for all sufficiently small $t \in \Delta$. This implies the following result.

Theorem 4.8. *Let X be a general quartic double solid. Then the generic fiber of the Abel Jacobi map $\Phi_{\mathcal{C}_5^1(X)} : \mathcal{C}_5^1(X) \longrightarrow \Theta + \mathrm{const}$ is a dense open subset in the union of* 84 *copies of $\mathbb{P}^1$.*

5. Vector bundles

Let X be general, so that $\mathcal{C}_5^1(X)$ is irreducible, and consider the dense open subset $\check{\mathcal{C}}_5^1(X)$ of $\mathcal{C}_5^1(X)$ defined in (19). Take any curve $C \in \check{\mathcal{C}}_5^1(X)$ and use the Serre construction to define the $\mathcal{O}_X$-sheaf $\mathcal{E}_C$ as an extension of $\mathcal{O}_X$-sheaves:

$$0 \to \mathcal{O}_X(-1) \to \mathcal{E}_C \to \mathcal{J}_{C/X}(1) \to 0. \tag{85}$$

Any such extension is determined uniquely up to an isomorphism by a 1-dimensional subspace of the group $\mathrm{Ext}^1(\mathcal{J}_{C/X}(1), \mathcal{O}_X(-1)) \simeq H^0(\mathcal{E}xt^1(\mathcal{J}_{C/X}(1), \mathcal{O}_X(-1))) \simeq H^0(\mathcal{E}xt^2(\mathcal{O}_C(1), \mathcal{O}_X(-1))) \simeq H^0(\mathcal{E}xt^2(\mathcal{O}_C, \omega_X)) \simeq H^0(\omega_C) \simeq \mathbb{C}$ (we are using $h^i\mathcal{O}_X(-2) = 0, \quad i < 3$, and $\omega_C \simeq \mathcal{O}_C$ C). Thus $\mathcal{E}_C$ is determined uniquely up to an isomorphism by C. Besides, by a well known property of the Serre construction, the local triviality of $\mathcal{E}xt^1(\mathcal{J}_{C/X}(1), \mathcal{O}_X(-1)) \simeq \omega_C$ implies that $\mathcal{E}_C$ is locally free.

From (85), one easily computes the Chern classes:

$$c_1(\mathcal{E}_C) = 0, \qquad c_2(\mathcal{E}_C) = 3$$

(here we understand c_2 as an element of the group B_1 of classes of algebraic equivalence of 1-dimensional cycles on X, generated by the class of a line on X). As $\pi : C \to \pi(C)$ is an isomorphism onto a quintic $\pi(C)$ that does not lie in a plane and $H^0(\mathcal{O}_X(1)) = \pi^*H^0(\mathcal{O}_{\mathbb{P}^3}(1))$, it follows that $H^0(\mathcal{J}_{C/X}(1)) \simeq H^0(\mathcal{J}_{\pi(C)/\mathbb{P}^3}(1)) = 0$. Hence the triple (85) gives $h^0(\mathcal{E}_C) = 0$. This means that $\mathcal{E}_C$ is Gieseker stable (see [MT-1, Prop. 2.6]). Thus, denoting by $M_X(2; 0, 3)$ the Gieseker-Maruyama moduli scheme of stable rank-2 vector bundles on X with Chern classes $c_1 = 0$, $c_2 = 3$, we have $[\mathcal{E}_C] \in M(2; 0, 3)$, where

$[\mathcal{E}]$ stands for the isomorphism class of $\mathcal{E}$. Remark that the equality $h^0(\mathcal{J}_{C/X}(2)) = 1$ immediately implies $h^i(\mathcal{J}_{C/X}(2)) = 0, \quad i > 0$; hence by (85) also

$$h^i(\mathcal{E}_C) = 0, \quad i > 0.$$

Let $\tilde{M} = \{[\mathcal{E}] \in M_X(2;0,3) |\, h^i(\mathcal{E}) = 0, \quad i > 0\}$. By semicontinuity, this is an open subset of $M_X(2;0,3)$ containing $[\mathcal{E}_C]$. Let M be the irreducible component of $\tilde{M}$ containing $[\mathcal{E}_C]$. We have a natural map

$$f : \check{\mathcal{C}}^1_5(X) \to M,\ C' \mapsto [\mathcal{E}_{C'}].$$

It is a standard matter to show that f is a dominant morphism (see [MT-1, Lemmas 5.2 and 5.3]), whose fibre $f^{-1}(f(C'))$ is open in $P(H^0(\mathcal{E}_{C'}(1)) \simeq \mathbb{P}^1$:

$$\overline{f^{-1}(f(C'))} = \mathbb{P}^1, \quad C' \in \check{\mathcal{C}}^1_5(X).$$

The closure is taken in $\overline{\mathcal{C}^1_5(X)}$, and we keep in mind that, by Riemann-Roch, $h^0(\mathcal{E}_{C'}(1)) = 2$, since by the definition of $\tilde{M}$, $h^i(\mathcal{E}_{C'}(1)) = 0, \quad i > 0$. The equality $h^0(\mathcal{J}_{C'/X}(2)) = 1$ means that the linear series $|\mathcal{J}_{C'/X}(2))|$ consists of a unique K3 surface $S(C')$, and by Serre construction, the fibre $\overline{f^{-1}(f(C'))} = \mathbb{P}^1$ is just the pencil of elliptic curves on $S(C')$, which are nothing else than the zero loci of sections oh $\mathcal{E}_{C'}(1)$:

$$\overline{f^{-1}(f(C'))} = |\mathcal{O}_{S(C')}(C')| = \mathbb{P}^1. \tag{86}$$

Now consider the Stein factorization of the Abel-Jacobi map $\Phi = \Phi_{\mathcal{C}^1_5(X)}|\check{\mathcal{C}}^1_5(X) : \check{\mathcal{C}}^1_5(X) \to \Theta + \text{const}$. Using the above description (86) of fibres of f and the well-known interpretation of the differential of the Abel-Jacobi map in terms of the branch quartic W of $\pi : X \to \mathbb{P}^3$ (see the end of Section 2, or [T-1, Corollary 1], or [We, Prop.2.13]), we obtain the following

Theorem 5.1. *Let X be a general quartic double solid. Then there exists a quasi-finite dominant morphism $g : M_0 \to \Theta + \text{const}$ of degree 84 from an open part M_0 of the above component M of the moduli space $M_X(2;0,3)$ to a translate of the theta divisor of $J(X)$ which gives the Stein factorization of the Abel-Jacobi map Φ of $\check{\mathcal{C}}^1_5(X)$:*

$$\begin{tikzcd} \check{\mathcal{C}}^1_5(X) \arrow[rr, "\Phi"] \arrow[dr, "f"'] & & \Theta + \text{const} \\ & M_0 \arrow[ur, "g"'] & \end{tikzcd} \tag{87}$$

The fibers of $f : \check{\mathcal{C}}^1_5(X) \to M_0$ are open subsets of $\mathbb{P}^1$.

The proof is similar to that of Theorem 5.6 in [MT-1].

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