Present state of the Bloch-Kato conjecture

compiled from notes of the talk delivered by A. Suslin

Ever since Quillen introduced higher K-theory, attempts have been made to understand its relationships with other cohomology theories, such as étale cohomology, chow rings and also Milnor K-theory, which in the case of fields is given by an explicit presentation

$$K_*^M(F) = \coprod_{n \geq 0} (F^*)^{\otimes n} / \langle a \otimes (1-a) \rangle.$$

In the beginning of the eighties, Beilinson produced very explicit conjectures predicting what one should expect in general. For practical purposes, let k be a field of characteristic zero (since resolution of singularities is needed in some proofs).
Let Sm/F be the category of smooth schemes over F.

Beilinson conjecture. *There exist a complexes* $\mathbf{Z}(n), n \geq 0$ *of abelian sheaves on the big Zariski site* $(\mathrm{Sm}/F)_{\mathrm{Zar}}$ *satisfying*
(i) $\mathbf{Z}(0) = \mathbf{Z}, \mathbf{Z}(1) = \mathcal{O}^*[-1]$;
(ii) (The Beilinson-Soulé vanishing conjecture.) *For* $n > 0$, $\mathbf{Z}(n)$ *is acyclic outside* $[1, n]$ *and* $H^n(\mathbf{Z}(n)) = K_n^M$, *the sheaf of Milnor K-groups;*
(iii) There exists a strongly convergent spectral sequence

$$E_2^{p,q} = H^p(X, \mathbf{Z}(q)) \Rightarrow K_{2q-p}(X);$$

(iv) (The Beilinson-Lichtenbaum conjecture.) *For any prime p (different from char k), there is an isomorphism*

$$\mathbf{Z}/p(n) \cong \tau_{\leq n} R\pi_*(\mu_p^{\otimes n}),$$

where $\pi : (Sm/F)_{\text{ét}} \to (Sm/F)_{\mathrm{Zar}}$; *in particular*

$$H^i(F, \mathbf{Z}/p(n)) = \begin{cases} H^i_{\text{ét}}(F, \mu_p^{\otimes n}), & i \leq n, \\ 0, & i > n. \end{cases}$$

One special case of the Beilinson-Lichtenbaum conjectures (for $i = n$) is known as the:

Bloch-Kato conjecture. *For a field F, there is a natural isomorphism*

$$K_n^M(F)/p \cong H^n_{\text{ét}}(F, \mu_p^{\otimes n}).$$

The first successful attempt to define the motivic cohomology theory (though not the complexes $\mathbf{Z}(n)$) was the work of S. Bloch [B]. The motivic complexes $\mathbf{Z}(n)$ were defined by Voevodsky [V], using the theory of sheaves with transfers.

The sheaves $\mathbf{Z}(n)$ of Voevodsky satisfy the Beilinson property (i) and part of property (ii) for trivial reasons. However, the other part – acyclicity in degrees ≤ 0 is still open.
The spectral sequence relating algebraic K-theory to motivic cohomology was constructed in the case of fields by S. Bloch and S. Lichtenbaum [B-L] and generalized to the case of arbitrary smooth schemes by E. Friedlander and A. Suslin [F-S].

A. Suslin and V. Voevodsky [S-V] have shown that the Bloch-Kato conjecture implies the Beilinson-Lichtenbaum conjecture. Finally, V. Voevodsky has proved in his famous paper on the Milnor conjecture [V2] that the Bloch-Kato conjecture holds in the case $p = 2$. More generally, Voevodsky proved in [V2] the following theorem, which may be used also for primes $p \neq 2$.

Theorem (Voevodsky). *Assume the Beilinson-Lichtenbaum conjecture holds in weights $< n$. Assume further that for any field extension F'/F and any n-tuple $(a_1, ..., a_n) \in (F')^*$, there exists a universal splitting variety $X = X_a$ for the symbol $a = \{a_1, ..., a_n\}$ with the following properties:*
(0) X is smooth and projective over F';
(1) X has dimension $d = p^{n-1} - 1$;
(2) $S_d(X)/p$ is prime to p (here, $S_d(X)$ is an appropriate characteristic number of X, which is known to be divisible by p);
(3) The norm map $N : A_0(X_a, K_1) \to (F')^*$ is injective;
then the Bloch-Kato conjecture and hence also the Beilinson-Lichtenbaum conjecture holds also in weight n.

For $p = 2$, one can take $X = X_a$ to be the Pfister (neighbour) quadric, in which case (0),(1) and (2) are trivial and (3) was proved previously by M. Rost.
In the general case, M. Rost [R] has proved that one can always find $X = X_a$ satisfying (0),(1) and (2), and moreover proved half of (3).

References

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