RECONSTRUCTION OF HAMILTONIAN T^n-SPACES IN QUATERNIONIC RIEMANNIAN GEOMETRY

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Abstract. Let M be a connected hyperkähler $4n$-manifold with an effective and tri-Hamiltonian T^n-action. We show that, under some additional assumptions, the equivariant homeomorphism type of M is determined by the image $\mu(M)$ of the hyperkähler moment map and by the image $\mu(M_{\rm sing})$ of the union of singular T^n-orbits. It follows that, if μ is proper and the action has a fixed point, then M is T^n-equivariantly homeomorphic to a hyperkähler quotient of a flat quaternionic vector space by a torus. Similarily we show that a compact connected $(4n-1)$-dimensional 3-Sasakian manifold with an effective action of T^n preserving the structure is T^n-equivariantly homeomorphic to a 3-Sasakian quotient of a sphere by an abelian group. Finally, a compact connected $4m$-dimensional quaternion-Kähler manifold of positive scalar curvature admitting an effective and isometric action of T^{m+1} is isometric to $\mathbb{H}P^m$ or $\mathrm{Gr}_2(\mathbb{C}^{m+2})$.

A $4n$-dimensional manifold is hyperkähler if it possesses a Riemannian metric g which is Kähler with respect to three complex structures J_1, J_2, J_3 satisfying the quaternionic relations $J_1J_2 = -J_2J_1 = J_3$ etc. A construction of hyperkähler manifolds admitting an isometric action of the torus T^n, Hamiltonian with respect to all three Kähler forms, was given in [LR] and in [HKLR]. This construction, based on Legendre transform, is purely local and the global properties of resulting manifolds have not been investigated in general. Particular cases of complete hyperkähler $4n$-manifolds with an isometric and tri-Hamiltonian T^n-action include ALE-spaces of type A_n [GH,Hi], toric hyperkähler manifolds [Go1,2,BD], the Taub-NUT metric and its generalizations [GR].

Let M be a $4n$-dimensional hyperkähler manifold with an effective tri-Hamiltonian action of T^n. In this work we shall address the following question:

To what extent does the image of the hyperkähler moment map determine the manifold M?

The corresponding question in symplectic geometry was considered by Delzant [De] who showed that a compact symplectic manifold of dimension $2n$ with an effective Hamiltonian action of T^n is determined, as a Hamiltonian T^n-space, by the image of the moment map (which is a convex polytope). In the hyperkähler case, a tri-Hamiltonian action is automatically isometric and so we cannot expect the tri-symplectic rigidity. In addition, as we show, the hyperkähler moment map $\mu : M \to \mathbb{R}^{3n}$ is open and so its image does not carry much information. In the symplectic case, however, $\mu(M)$ carries additional information - its boundary is the image of the subset:

$$M_{\mathrm{sing}} = \{m \in M;\ \dim \mathrm{Stab}_{T^n}(m) \geq 1\}.$$

In the hyperkähler case $\mu(M_{\mathrm{sing}})$ is union of codimension 3 affine subspaces and we shall show that $(\mu(M), \mu(M_{\mathrm{sing}}))$ determines (M, μ):

Theorem 1. *Let M, M' be two connected $4n$-dimensional hyperkähler manifolds with an effective action of the torus T^n, Hamiltonian with respect to all three Kähler forms, and let μ, μ' be the corresponding hyperkähler moment maps. Suppose that $\mu(M) = \mu(M')$, $\mu(M_{\mathrm{sing}}) = \mu(M'_{\mathrm{sing}})$, $\mu(M)$ is 2-connected and the maps $\mu, \mu' : M \to \mu(M)$ are proper. Then there exists a T^n-equivariant homeomorphism $h : M \to M'$ such that $\mu = \mu' \circ h$.*

Remark. The examples of $\mathbb{H}^*/\mathbb{Z}_k$ shows that $H^2(\mu(M), \mathbb{Z}) = 0$ is necessary. Example 4.2 below shows that both $\pi_1(\mu(M)) = 0$ and the properness of μ are necessary.

The homeomorphism type of an M satisfying assumptions of Theorem 1 is that of an open subset of a (possibly singular) hyperkähler quotient of a (possibly infinite-dimensional) flat quaternionic vector space by a compact abelian group. While we shall not prove this statement in full generality, we shall show:

Theorem 2. *Let M be a connected $4n$-dimensional hyperkähler manifold with an effective tri-Hamiltonian action of T^n such that the moment map $\mu : M \to \mathbb{R}^{3n}$ is proper. Then M is T^n-equivariantly homeomorphic to the product of $T^i \times \mathbb{R}^{3i}$, for some $i \leq n$, and a hyperkähler quotient of a flat quaternionic vector space by a torus.*

Remarks. 1. This vector space will be infinite-dimensional if M has infinite topological type. The torus, by which we quotient, will then be a Hilbert group (see [Go2] and section 4 below).
2. The topology of nonsingular toral hyperkähler quotients of flat quaternionic vector spaces has been investigated in detail in [BD]. Such a quotient is T^n-equivariantly homeomorphic to the product of an $\mathbb{H}^i$ and union of cotangent bundles of projective toric varieties glued together.

Theorems 1 and 2 have a number of consequences. For one they determine the topology of several hyperkähler manifolds constructed in [GR] as hyperkähler quotients of flat quaternionic spaces by (non-compact) abelian groups. These manifolds admit a T^n-action as in Theorem 1 and the moment map is proper to $\mathbb{R}^{3n}$. Theorem 2 implies that these manifolds are homeomorphic to the product of one of the manifolds considered in [BD] and $\left(S^1 \times \mathbb{R}^3\right)^i$ for some i.

Theorem 1 also leads quickly to a topological classification of compact 3-Sasakian $(4n-1)$-dimensional manifolds with an action of T^n preserving the 3-Sasakian structure. We recall that a $(4n-1)$-dimensional manifold S is 3-Sasakian if it possesses a Riemannian metric g with three orthonormal Killing fields defining a local $SU(2)$-action and satisfying a curvature condition. An equivalent condition is that the Riemannian cone $C^*(S) = (\mathbb{R}_{>0} \times S, dr^2 + r^2 g)$ is hyperkähler. Such a manifold is necessarily Einstein with scalar curvature $2(2n-1)(2n+1)$. Moreover the local action extends to a global action of $SO(3)$ or $Sp(1)$ and the quotient of S is a quaternion-Kähler orbifold. Many examples of compact 3-Sasakian manifolds were constructed in [BGM] using the 3-Sasakian quotient construction, analogous to the hyperkähler quotient construction. The toral quotients of spheres were investigated in [BGMR,BD,Bi].

We now have:

Theorem 3. *Let S be a compact connected 3-Sasakian manifold of dimension $4n-1$ with an effective action of T^n preserving the structure. Then there is an integer d such that S is T^n-equivariantly homeomorphic to a 3-Sasakian quotient of the $(4d-1)$-dimensional sphere by a subgroup of T^d.* $\square$

As we explain in section 5, the homeomorphism type of S is determined by the image of singular orbits under the 3-Sasakian moment map.

Finally we have a much stronger statement for quaternion-Kähler manifolds:

Theorem 4. *Let M be a compact connected $4m$-dimensional quaternion-Kähler manifold of positive scalar curvature. If M admits an effective and isometric action of T^{m+1}, then M is isometric to $\mathbb{H}P^m$ or $Gr_2(\mathbb{C}^{m+2})$ with their symmetric quaternion-Kähler metrics.*

1. A local-global principle for classifying T^n-spaces

In this section we shall extend the usual classification of locally trivial T^n-bundles to a class of T^n-spaces.

Let M be a T^n-space and let $\pi : M \to B$ be a continuous map which factors through M/T^n so that the induced map $\bar{\pi} : M/T^n \to B$ is a homeomorphism. We shall refer to $\pi : M \to B$ as a *piecified T^n-bundle* (cf. [GM]). Two piecified T^n-bundles $\pi : M \to B$ and $\pi' : M' \to B$ over the same B are equivalent (resp. locally equivalent) if there exists an equivariant homeomorphism $\phi : M \to M'$ such that

$\pi' \circ \phi = \pi$ (resp. every point in B has a neighbourhood U such that $\pi^{-1}(U)$ and $(\pi')^{-1}(U)$ are equivalent). Recall that an action is locally smooth if it has linear slices (i.e. it is locally smoothable). We have:

Theorem 1.1. *Let M be a topological manifold with an effective locally smooth T^n-action which has connected stabilizers and suppose that $M_{\text{sing}})/T^n$ does not disconnect M/T^n locally. Then the equivalence classes of piecified T^n-bundles which are locally equivalent to a given $\pi : M \to B$ are parameterized by $H^2(B, \mathbb{Z}^n)$ (Čech cohomology).*

Example 1.2. If M is a $2n$-dimensional compact Hamiltonian T^n-space, then the moment map $\mu : M \to \mu(M)$ is a piecified T^n-bundle. Here $\mu(M)$ is a convex polytope $\Delta \subset \mathbb{R}^n$. Furthermore, the isotropy groups and isotropy representations are determined by Δ, and so, by the existence of normal form for the moment map [GSt, Ch.41], any two such Hamiltonian T^n-spaces are locally equivalent. Therefore, by Theorem 1.1, the T^n-equivariant homeomorphism type of M (and the moment map μ) is determined by Δ. This is the topological content of Delzant's theorem [De].

The remainder of the section is devoted to a proof of Theorem 1.1. First we establish some terminology.

Let $\pi : X \to B$ be a piecified torus bundle. Then B together with the canonical map $\tau(b) = Stab_{T^n}(x)$, where x is any element of X with $\pi(x) = b$, will be called the T^n-structure of the bundle X. More abstractly, let $\mathcal{T}(n)$ denote the set of all closed subgroups of T^n. We topologize $\mathcal{T}(n)$ by taking the smallest topology for which the sets $\{N; H \leq N\}$, $H \leq T^n$, are closed. Then (cf. [Br]) a *T^n-structure* (B, τ) is a Hausdorff space B together with a continuous map $\tau : B \to \mathcal{T}(n)$.
If $\pi : X \to B$ is a piecified T^n-bundle and (B, τ) is its T^n-structure, we shall refer to X as a T^n-bundle over (B, τ). We shall denote the group of auto-equivalences of $\pi : X \to B$ by $\mathrm{Aut}(X)$.

Definition 1.3. *Let (B, τ) be a T^n-structure. The trivial T^n-bundle over (B, τ) is the T^n-space $X = B \times T^n / \sim$, where $\sim$ is an equivalence relation defined by $(b, t) \sim (b', t')$ if and only if $b = b'$ and $t't^{-1} \in \tau(b)$. We shall write $\theta(B, \tau)$ for this X.*

We observe that X is a trivial T^n-space if and only if there exists a global section $\sigma : B \to X$ to the T^n-action. We have the simple

Lemma 1.4. *The group of automorphisms of a trivial T^n-bundle is given by: $Aut(\theta(B, \tau)) = \big\{\{f_H : \tau^{-1}(H) \to T^n/H;\ H \in \mathcal{T}(n)\}\big\}$ where each f_H is continuous* *and if $b_k \to b$, with $b_k \in \tau^{-1}(H), b \in \tau^{-1}(K)$, then $\pi(f_H(b_k)) \to f_K(b)$. Here $\pi : T^n/H \to T^n/K$ denotes the natural projection.* $\square$

The space of all $f = \{f_H\}$ defined above will be denoted by $\mathrm{Map}((B,\tau);T^n)$. Observe that if $f = \{f_H\} \in \mathrm{Map}((B,\tau);T^n)$, then f defines an automorphism of any T^n-bundle X over (B,τ) by

$$x \mapsto f(\pi(x)) \cdot x \tag{1.1}$$

where $f(\pi(x))$ is any element of T^n projecting to $f_H(\pi(x))$, $H = \tau(\pi(x))$.
The natural notion of locally trivial T^n-bundles is too strong - for example the diagonal S^1-action on $\mathbb{C}^2$ does not admit local sections at the origin. Instead, we adopt the following definitions:

Definition 1.5. *A T^n-bundle X over (B,τ) is* weakly trivial *if the map F : $Map((B,\tau);T^n) \to Aut(X)$, given by* (1.1)*, is a bijection;*
X is weakly locally trivial *if the family of locally finite coverings $\mathfrak{U} = \{U_i\}_{i\in I}$ of B, such that $\pi^{-1}(U)$ is weakly trivial for each $U \in N(\mathfrak{U})$ (the nerve of $\mathfrak{U}$), is cofinal in the family of all coverings.*

Implicit in the definition of a weakly locally trivial T^n-bundle is paracompactness of B. The reason for the above definition is the following

Proposition 1.6. *Let X be a weakly locally trivial T^n-bundle over (B,τ) such that $\tau^{-1}(1)$ is dense and $B - \tau^{-1}(1)$ does not disconnect B locally. Then the isomorphism classes of T^n-spaces over (B,τ) which are locally equivalent to X are parameterized by $H^2(B,\mathbb{Z}^n)$.*

Proof. For any locally finite covering $\mathfrak{U} = \{U_i\}_{i\in I}$ of B, the equivalence classes of T^n-bundles over (B,τ) which are equivalent over each $U_i \in \mathfrak{U}$ to $\pi^{-1}(U_i)$ are parameterized by $H^1(B,\mathcal{A}(\mathfrak{U}))$, where $\mathcal{A}(\mathfrak{U})$ is the presheaf on $\mathfrak{U}$ which attaches to each face $U_{i_1} \cap \ldots \cap U_{i_s}$ of $N(\mathfrak{U})$ the (abelian) group $A(U_{i_1} \cap \ldots \cap U_{i_s})$ of automorphisms of $\pi^{-1}(U_{i_1} \cap \ldots \cap U_{i_s})$. We shall construct a fine resolution $\mathcal{B}(\mathfrak{U}) \to \mathcal{A}(\mathfrak{U}) \to 0$. Choose $\mathfrak{U}$ to satisfy the condition of weak local triviality. Then, for any $U \in N(\mathfrak{U})$, $A(U) = \mathrm{Map}((U,\tau);T^n)$ and is described by Lemma 1.4. We define the presheaf $\mathcal{B}(\mathfrak{U})$ on $\mathfrak{U}$ by replacing tori in this description with their Lie algebras, viewed as groups under addition. Choosing a partition of unity subordinate to $\mathfrak{U}$ shows that $H^m(B,\mathcal{B}(\mathfrak{U}))$ vanishes for $m \geq 1$. The kernel of the resolution is the sheaf $\mathcal{Z}$ defined as $\mathcal{A}$ or $\mathcal{B}$ but with corresponding sublattices of $\mathbb{Z}^n$ replacing the subgroups of T^n or $\mathbb{R}^n$. The functions are now locally constant and as $U \cap \tau^{-1}(1)$ is open dense and connected, the sheaf $\mathcal{Z}$ is just the constant sheaf $\mathbb{Z}^n$. Taking a direct limit over coverings of B proves the proposition. $\square$

To prove Theorem 1.1 it is now enough to show that any T^n-manifold with a locally smooth action, satisfying the assumptions, is weakly locally trivial. First of all, by the definition of local smoothness, the T^n-action can be linearized in a neighbourhood of any point $p \in M$. More precisely, since the isotropy group H_p of p is a connected subtorus of T^n, there exists another subtorus $\hat{H}_p$ such that $T^n = H_p \times \hat{H}_p$. The orbit $T^n p$ is isomorphic to $\hat{H}_p$ and an invariant neighbourhood S of p in M is equivariantly homeomorphic to a T^n-vector bundle ζ over $\hat{H}_p$. Since the group is abelian, the action of $\hat{H}_p$ provides a trivialization and so $\zeta \simeq \hat{H}_p \times V$, where V is a vector space with a linear action of H_p on it. Theorem 1.1 follows now from

Proposition 1.7. *Let V be a vector space equipped with a linear T^n-action which has connected stabilizers. Then every invariant open subset of V is a weakly trivial T^n-bundle.*

Let us show first

Lemma 1.8. *Let X be a T^n-space over (B,τ) and suppose that there is a finite closed covering $\{C_i\}$ of B such that $\pi^{-1}(C_i)$ is a trivial T^n-bundle for each i. Then every open or closed subset of X is weakly trivial.*

Proof. The assumption is hereditary and so it is enough to show that X itself is a weakly trivial T^n-bundle. Let $g_{ij} \in \operatorname{Aut}(\theta(C_i \cap C_j, \tau))$ be the transition functions. Let h be an automorphism of X. Then, like for ordinary locally trivial principal bundles, h is uniquely defined by a collection of $u_{i,a} \in \operatorname{Map}((C_a \cap C_i, \tau); T^n)$ which satisfy

$$u_{j,b}(z) = g_{ji}(z) u_{i,a}(z) g_{ab}(z) \tag{1.2}$$

for any $z \in C_a \cap C_b \cap C_i \cap C_j$ and any i, j, a, b. Observe that all objects in (1.2) are elements of $T^n/\tau(z)$. Now (1.2) implies that $u_{j,j}(z) = u_{i,i}(z)$ for any $z \in C_j \cap C_i$, and so h is given by a globally defined $u \in \operatorname{Map}((B,\tau); T^n)$. $\square$

To finish the proof of Theorem 1.1 we have to show that, if T^n acts linearly and with connected stabilizers on a vector space V, then the assumption of Lemma 1.8 is satisfied. We can assume that V is complex, the action is unitary and the only fixed point is 0. We proceed by induction on n. The statement is trivial for $n = 0$. Choose a point $p \neq 0$ in V and let H_0 be the isotropy group of p. Let V_0 be the subspace of V fixed by H_0 and let V_1 be the orthogonal complement to V_0. By the inductive assumption we have a finite closed H_0-invariant covering P_i of V_1 which trivializes the H_0-action on V_1. Let H_1 be such that $T^n = H_0 \times H_1$. We can find an (open or closed) neighborhood P of p in the sphere in V which is T^n-equivariantly homeomorphic to $H_1 \times U \times V_1$, where U is an (open or closed) ball in some Euclidean space and H_0 acts linearly on V_1. The T^n-action is trivial on each $H_1 \times U \times P_i$. Let ϕ be the homeomorphism between P and $H_1 \times U \times V_1$ and let $Q_i = \phi^{-1}(H_1 \times U \times P_i)$. The Q_i form a closed invariant covering of P trivializing the T^n-action. The action is then also trivial on each of the sets $\bigcup_{r\geq 0} rQ_i$. Thus we can find a finite closed invariant covering of V trivializing the T^n-action. This proves Theorem 1.1.

2. Local structure of moment map

We recall that a complex-symplectic Kähler manifold is a Kähler manifold (M, J, ω_1) together with a non-degenerate closed complex-valued 2-form ω which is of type $(2,0)$ for the complex structure J. Such a form ω can be written as $\omega_2 + \sqrt{-1}\omega_3$ for some real symplectic forms ω_2, ω_3 and we say that a group action is Hamiltonian for ω if it is Hamiltonian for ω_2 and ω_3. Such an action must be holomorphic and the moment map $\mu_\omega = \mu_2 + \sqrt{-1}\mu_3$ for ω is then also holomorphic.

The purpose of this section is the following result:

Proposition 2.1. *Let (M, J, ω_1, ω) be a complex-symplectic Kähler manifold of dimension $4n$ equipped with an effective action of T^n, Hamiltonian for ω_1 and ω, with $\mu_1, \mu_\omega = \mu_2 + \sqrt{-1}\mu_3$ the corresponding moment maps. Then the moment map $\mu = (\mu_1, \mu_2, \mu_3) : M \to \mathbb{R}^{3n}$ is open. Furthermore, every point $p \in M$ has a T^n-invariant neighbourhood U such that:*

(i) *$\mu_{|_U}$ has connected fibers and induces a homeomorphism between U/T^n and $\mu(U)$.*
(ii) *$\mu(U_{\text{sing}})$ is the intersection of $\mu(U)$ with the union of codimension 3 affine subspaces H_k of the form $H_k = \{x = (x^1, x^2, x^3) \in \mathbb{R}^n \otimes \mathbb{R}^3; \langle x^i, u_k\rangle = \lambda_k^i,\ i = 1, 2, 3\}$, passing through $\mu(m)$, for some finite collection of vectors $\{u_k\} \subset \mathbb{R}^n$ and some scalars λ_k^i, $i = 1, 2, 3$;*
(iii) *the isotropy group T_p of p is the connected subtorus of T^n whose Lie algebra is spanned by $\{u_k\}$. Furthermore, the set $\{u_k\}$ (after rescaling) is part of a $\mathbb{Z}$-basis of $\mathbb{Z}^n$ and the isotropy representation of T_p is determined by this set.*

We shall need the following isotropic embedding theorem for complex-symplectic forms:

Proposition 2.2. *Let (M, J, ω_1, ω) be a complex-symplectic Kähler manifold equipped with an action of T^n, Hamiltonian for ω_1. Suppose further that the action extends to a holomorphic action of $(\mathbb{C}^*)^n$, Hamiltonian for ω. Let H be the isotropy group of a point $p \in M$ and let X be the $(\mathbb{C}^*)^n$-orbit through p, which we identify with $(\mathbb{C}^*)^n/H^{\mathbb{C}}$. Then there exists a $(\mathbb{C}^*)^n$-invariant neighbourhood of p which is isomorphic, as a complex-symplectic $(\mathbb{C}^*)^n$-manifold, to an invariant neighbourhood of X in $T^*X \times W$ where $W \simeq (T_pX)^{\perp}/T_pX$.*

Here the orthogonal is taken with respect to ω. The above result has an appropriate extension for reductive group actions, providing that the G-orbit through p is isotropic with respect to ω_1 and that the $G^{\mathbb{C}}$-orbit is isotropic with respect to ω.

Proof. Since X is an ω-isotropic (as the action is Hamiltonian and the group is abelian) submanifold of M, we have that, as a complex-symplectic vector space, $T_pM = \left((T_pX)^{\perp}/T_pX\right) \oplus (T_pX \oplus T_pX^*)$. In addition, since ω is $H^{\mathbb{C}}$-invariant and $H^{\mathbb{C}}$ acts trivially on T_pX, it acts trivially on T_pX^*. Therefore we can equivariantly identify T_XM, as a complex-symplectic bundle with $T_X(T^*X \times W) = (TX \oplus T^*X) \times W$. Hence, over X, ω agrees with the canonical complex symplectic form of $T^*X \times W$. Since the T^n-orbit through p is isotropic for ω_1, a result of Sjamaar [Sj] implies that an open $(\mathbb{C}^*)^n$-invariant neighbourhood of p is equivariantly biholomorphic to $X \times S$, where $S = H^{\mathbb{C}}B$ with B an open ball in T_pM/T_pX. Therefore an open $(\mathbb{C}^*)^n$-invariant neighbourhood of X is equivariantly biholomorphic to an invariant neighbourhood of X in $T^*X \times W$. To finish the proof we have to show that the complex-symplectic form pulled back from M agrees with the standard form in a neighbourhood of X in $T^*X \times W$, i.e. we need the equivariant Darboux theorem for complex-symplectic forms. For this we observe that, since this neighbourhood U can be taken to be a convex open neighbourhood of the zero section in a vector bundle, there is a holomorphic retraction of U onto X which commutes with the action of $(\mathbb{C}^*)^n$. Now the proof of the ordinary Darboux theorem, as given in [GSt], goes through for complex symplectic forms. □

As we do not assume that the T^n-action extends to a global $(\mathbb{C}^*)^n$-action we need a slight modification of this result.

Proposition 2.3. *Let (M, J, ω_1, ω) be as in the previous proposition with the exception that we do not assume that the action of T^n extends to a global holomorphic action of $(\mathbb{C}^*)^n$. Let Y be the T^n-orbit through p, $Y \simeq T^n/H$ and let $X = (\mathbb{C}^*)^n/H^{\mathbb{C}} \simeq T^*Y$. Then a T^n-invariant neighbourhood of p is isomorphic, as a complex-symplectic T^n-manifold, to a neighbourhood of Y in $T^*X \times W$ where*

$W \simeq (T_pX)^{\perp}/T_pX$.

Proof. Examination of Sjamaar's proof shows that the construction of a slice can be done for a local $(\mathbb{C}^*)^n$-action in a neighborhood of a T^n-orbit isotropic for ω_1. The remainder of the proof of 2.2 also goes through. $\square$

We shall also need the following two lemmata.

Lemma 2.4. *Let U be a neighbourhood of T^n in $(\mathbb{C}^*)^n$ which is of the form $T^n \times \exp(B(0,\epsilon))$, where $B(0,\epsilon)$ is an open ball in $\mathbb{R}^n$ of radius ϵ, $0 < \epsilon \leq +\infty$. Let ω be a T^n-invariant Kähler form on U such that the T^n-action is Hamiltonian. Then any moment map for ω has connected fibers and induces a homeomorphism between $B(0,\epsilon)$ and $\mu(U)$.*

Proof. Cf. [Gu, section A2.3]. Since the action is Hamiltonian for ω, there exists a globally defined invariant Kähler potential $K : B(0,\epsilon) \to \mathbb{R}$ for ω which is a strictly convex function. The moment map μ is then (up to addition of a constant) the Legendre transform of K, i.e. $\mu(x) = \frac{\partial K}{\partial x}\big|_x$. Since K is strictly convex, the claim follows. $\square$

Lemma 2.5. *Let T^n act on $\mathbb{H}^n$ via an isomorphism with the maximal torus of $Sp(n)$. Then the orbit space $\mathbb{H}^n/T^n$ is homeomorphic to $\mathbb{R}^{3n}$.*

Proof. It is enough to consider the standard action of the maximal torus of $Sp(n)$. In coordinates $z_1, \dots, z_n, w_1, \dots, w_n$ on $\mathbb{C}^n \times \mathbb{C}^n \simeq \mathbb{H}^n$, this action of $T^n = \{t = (t_1, \dots, t_n) \in \mathbb{C}^n \,;\, |t_k| = 1\}$ is given by:

$$t \cdot (z, w) = (t_1 z_1, \dots, t_n z_n, t_1^{-1} w_1, \dots, t_n^{-1} w_n). \tag{2.1}$$

The ring of invariant polynomials is generated by $p_k = |z_k|^2$, $q_k = |w_k|^2$, $r_k = z_k w_k$ with relations $p_k q_k = |r^k|^2$. Hence $\mathbb{H}^n/T^n$ is diffeomorphic (by a theorem of Schwarz [Sch] which is the C^∞ version of a classical theorem of Hilbert on invariant theory) to the subspace of $(\mathbb{R}^4)^n$ of the form X^n, where $X = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4; x_1 \geq 0, x_2 \geq 0, x_1 x_2 = x_3^2 + x_4^2\}$. Now the map $X \to \mathbb{R}^3$ given by $(x_1, x_2, x_3, x_4) \mapsto (x_1 - x_2, x_3, x_4)$ defines a homeomorphism. $\square$

We now can prove Proposition 2.1. First, we shall show that T^n acts with connected stabilizers. Let p be a point in M and H_p its stabilizer, $\dim H_p = i$. Let H_0 be the identity component of H_p. There is an $(n-i)$-dimensional subtorus H_1 of T^n such that $T^n = H_0 \times H_1$. Let $\Gamma = H_p/H_0 = H_1 \cap H_p$, so that the T^n-orbit of p is isomorphic to H_1/Γ. By Proposition 2.3, an invariant neighbourhood of p can be identified with a neighbourhood of H_1/Γ in $T^*(H_1/\Gamma)^{\mathbb{C}} \times W$. It follows that $\dim_{\mathbb{C}} W \leq 2i$. Moreover $(H_p)^{\mathbb{C}}$ acts on W via a homomorphism into $Sp(i,\mathbb{C})$, and, as the action is effective, the representation is faithful, i.e. $(H_0)^{\mathbb{C}}$ acts via an isomorphism with the maximal complex torus of $Sp(i,\mathbb{C})$. Thus $\Gamma = 1$.

We observe that this argument shows that the isotropy representation of H_p is given by an isomorphism with the standard maximal torus of $Sp(i)$ acting on $\mathbb{H}^i$. For the purpose of showing part (i) we can assume, without loss of generality, that this isomorphism is the identity.

From Proposition 2.3 we can identify (U,J,ω) with a neighbourhood of H_1 in $T^*(H_1^{\mathbb{C}}) \times \mathbb{C}^{2i}$. The complex moment map is of the form $\mu_\omega(u,v) = (\phi_1(u), \phi_2(v))$, where $u \in T^*(H_1^{\mathbb{C}})$ and $v \in \mathbb{C}^{2i}$, ϕ_1 is the moment map for $H_1^{\mathbb{C}}$ and ϕ_2 for the complexification $(\mathbb{C}^*)^i$ of T^i. If we trivialize $T^*(H_1^{\mathbb{C}})$, using the $H_1^{\mathbb{C}}$-action, as $H_1^{\mathbb{C}} \times \mathbb{C}^{n-i}$, then ϕ_1 is just the projection on the second factor. On the other hand $\phi_2(z,w) = \sum_{k=1}^{i}(z_k w_k)e_k$, where $z = \sum z_k e_k$ and $w = \sum w_k e_k$ (the formulas for ϕ_1 and ϕ_2 are valid up to addition of constants).

Consider a point $(u,v) \in U_1 \times U_2$, where $v \in \mathbb{C}^{2i}$ is of the form $v = (z,w)$. We observe that as long as $\phi_2(z,w)$ has all coordinates nonzero, the set of points mapping to $\mu_\omega(u,v)$ is the $(\mathbb{C}^*)^n$-orbit through (u,v). This orbit X (or at least a neighbourhood of T^n in it) is a Kähler submanifold of U and, by Lemma 3.4, μ_1 separates T^n-orbits on X. Hence μ induces a $1-1$ mapping between V/T^n and $\mu(V)$, where $V = \{(u,(z,w)) \in U; z_k w_k \neq 0,\ \ k = 1,\dots,i\}$. Let $\bar{\mu}$ denote the induced mapping on U/T^n. We shall show that it is a homeomorphism.

By changing U if necessary we can identify U/T^n with an open ball B in $\mathbb{R}^{3n}$ (Lemma 2.5) and write $\bar{\mu}$ as $\bar{\mu}(x^1,x^2,x^3) = (\bar{\mu}_1(x^1,x^2,x^3), x^2, x^3)$. Then, since $\bar{\mu}$ restricted to V/T^n is a continuous $1-1$ map from an open subset of $\mathbb{R}^{3n}$ into $\mathbb{R}^{3n}$, it must be a homeomorphism (on V/T^n). The complement S of V/T^n are the hyperplanes $x_k^2 = x_k^3 = 0$, $k > n-i$. The map $\bar{\mu}$ sends each of them into itself. Moreover $\mu_\omega^{-1}(x)$, $x \in \mu_\omega(U) - \mu_\omega(V)$, is a finite union of $(\mathbb{C}^*)^n$-orbits and Lemma 2.4 implies that $\bar{\mu}$ is finite-to-one on B. Suppose, in order to get a contradiction, that $\bar{\mu}$ is not $1-1$. Let x be a point in B such that $\{x\} \neq \bar{\mu}^{-1}(\bar{\mu}(x))$ and such that the dimension of the T^n-orbit projecting to x is maximal among all such x. We consider intersections of B with $(\mathbb{R}_{>0})^n$-orbits (i.e. projections of $(\mathbb{C}^*)^n$-orbits). The $(\mathbb{R}_{>0})^n$-orbits intersecting some neighbourhood W of x are of one of three types: 1) the orbit of x, 2) those contained in V, 3) those that have dimension higher than the orbit through x. It follows, because of our choice of x and Lemma 2.4, that $\bar{\mu}$ is $1-1$ on W and so a homeomorphism into. Thus $\bar{\mu}(B)$ contains arbitrarily small balls around $\bar{\mu}(x)$. Since S is union of codimension 2 hyperplanes, there is an arbitrarily small neighbourhood Y of $\bar{\mu}(x)$ (in $\bar{\mu}(B)$) such that $Y - S$ is connected. Since $\bar{\mu}$ maps S to itself and is a homeomorphism on $V = B - S$, $\bar{\mu}^{-1}(Y - S)$ must be connected. This is possible only if x is the only point mapping to $\bar{\mu}(x)$ (as $\bar{\mu}$ is finite-to-one) and so we obtain a contradiction. Therefore $\bar{\mu}$ is a continuous $1-1$ map from an open subset of $\mathbb{R}^{3n}$ into $\mathbb{R}^{3n}$, and so a homeomorphism.

To finish the proof of 2.1 we have to show (ii) and (iii). Let $\rho \in \mathbb{R}^n$ and let M_ρ be any component of the fixed point set of the 1-PS of T^n generated by ρ, i.e. $X_\rho = 0$ on M_ρ. Then M_ρ is a complex-symplectic Kähler submanifold of M. It follows that, for $m \in M_\rho$ and $v \in T_m M$, we have $\langle d\mu_i(v), \rho\rangle = 0$, $i = 1, 2, 3$. Therefore $\langle \mu_i, \rho\rangle$ is constant on M_ρ, $i = 1, 2, 3$, and this proves (ii) and the first part of (iii). Let us return to an identification of a T^n-invariant neighbourhood of m with a neighbourhood of T^{n-i} in $\left((\mathbb{C}^*)^{n-i} \times \mathbb{C}^{n-i}\right) \times \mathbb{C}^{2i}$. As before, we can demand that the equivariant biholomorphism preserves the complex symplectic form and so the moment map μ_ω. If the weights of the representation of T^i are $\alpha_1, \dots, \alpha_i$ (these must form a $\mathbb{Z}$-basis of $\mathbb{Z}^i$, as T^i acts via an isomorphism with the maximal torus of $Sp(i)$), then the canonical form of μ_ω is (π, μ_ω^0) where π is the projection onto $\mathbb{C}^{n-i}$ and $\mu_\omega^0 : \mathbb{C}^i \times \mathbb{C}^i \to \mathbb{C}^i$ is given by:

$$\mu_\omega^0\left(\sum z_k e_k, \sum w_k e_k\right) = \sum (z_k w_k)\alpha_k.$$

Therefore the vector u_k is orthogonal to $\alpha_1, \dots, \widehat{\alpha_k}, \dots, \alpha_i$ and the second claim of 2.1(iii) follows.

3. Local equivalence of Hamiltonian hyperkähler T^n-manifolds

We are now ready to show that the image of M_{sing} determines M locally.

Proposition 3.1. *Let M, M' be $4n$-dimensional hyperkähler manifolds with effective tri-Hamiltonian actions of T^n, μ, μ' denoting the respective hyperkähler moment maps. Suppose that there are neighbourhoods U, U' of points $m \in M$ and $m' \in M$ such that $\mu(m) = \mu'(m')$ and $\mu(U_{\text{sing}}) = \mu'(U'_{\text{sing}})$. Then there exist smaller invariant neighbourhoods V and V' of m and m' and an equivariant homeomorphism $h : V \to V'$ such that $\mu' \circ h = \mu$.*

Proof. We can consider both U and U' to be neighbourhoods of $(T^{n-i}, 0, 0)$ in $T^{n-i} \times \mathbb{R}^{3n-3i} \times \mathbb{H}^i$ with the same T^n-action (thanks to Proposition 2.1). The point m is identified with $(1, 0, 0)$.

We have the standard flat hyperkähler structure on $T^{n-i} \times \mathbb{R}^{3n-3i} \times \mathbb{H}^i$ which is $T^{n-i} \times Sp(i)$-invariant. Let $\omega_1^0, \omega_2^0, \omega_3^0$ be the three Kähler forms for this structure. It is enough to show that μ is equivalent to the moment map μ^0 for the flat hyperkähler structure. By Proposition 2.3 we can assume that $\omega_2+\sqrt{-1}\omega_3=\omega_2^0+\sqrt{-1}\omega_3^0$ and $\mu_2=\mu_2^0$, $\mu_3=\mu_3^0$. We can even assume that ω_1 coincides with ω_1^0 at the point m (and so at all points of $(T^{n-i},0,0)$). Indeed, we can assume that the hyperkähler structures coincide at the point m to start with. The local diffeomorphism f which gives us the standard complex-symplectic form in a neighbourhood of m does not have to have, a priori, $df_{|m}=1$; it does, however, satisfy $df_{|m}(v)=v$ for any $v\in T_XU$, where X is a neighbourhoood of the T^{n-i}-orbit of m in the $(\mathbb{C}^*)^{n-i}$-orbit. In other words $df_{|m}=1\oplus L$, where L is an element of $Sp(i,\mathbb{C})$ commuting with the maximal torus. If we put $\hat{L}=1\oplus L$, then $\hat{L}^{-1}$ is a global automorphism of the standard complex-symplectic structure on $T^*(\mathbb{C}^*)^{n-i}\times\mathbb{C}^{2i}$ and $\hat{L}^{-1}\circ f$ has all the properties.

Hence our hyperkähler structure is of the form $(\omega_1,\omega_2^0,\omega_3^0)$ on U with $\omega_1=\omega_1^0$ at m. There is a smaller neighbourhood of m, such that the form $\omega_1^t=t\omega_1+(1-t)\omega_1^0$ is Kähler (for J_1) for all $t\in[0,1]$. We use the hyperkähler moment map μ^0 to identify U/T^n with an open ball around 0 in $\mathbb{R}^{3n}$ (Proposition 2.1). In other words the induced map $\bar{\mu}^0$ is the identity on $\mathbb{R}^{3n}$ and we consider the induced map $\bar{\mu}^t=t\bar{\mu}+(1-t)\mathrm{Id}$ for $(\omega_1^t,\omega_2^0,\omega_3^0)$. Proposition 2.1 implies that each $\bar{\mu}^t$ is a homeomorphism (perhaps after further shrinking of U). In addition, since $\{\bar{\mu}^t(x);t\in[0,1]\}$ is a segment, $x\in H_k$ implies that $\bar{\mu}^t(x)\in H_k$ for all t. Therefore, as $\bar{\mu}^t$ is a homeomorphism and $\bar{\mu}^t(0)=0$, $\bar{\mu}^t(x)\in H_k$ implies that $x\in H_k$ (again in some neighbourhood of 0). In other words, $\bar{\mu}^t$ provides a homotopy which preserves the orbit structure (i.e. the T^n-structure of section 1). By a theorem of Palais [Pa] (see also [Br, Thm.II.7.3]), $\bar{\mu}^t$ can be lifted to an equivariant homotopy P_t of $T^{n-i}\times\mathbb{R}^{3n-3i}\times\mathbb{H}^i$ with P_0 being the identity map. Now taking $h=P_1$ proves Proposition 3.1. □

4. Proof of Theorems 1 and 2

In view of Theorem 1.1 and Proposition 3.1, Theorem 1 will be proved if we can show that under its assumptions μ has connected fibers, i.e. it induces a homeomorphism between M/T^n and $\mu(M)$. This follows from the following

Lemma 4.1. *Let $f:X\to Y$ be a proper local homeomorphism from a connected manifold X to a smooth manifold Y. Then f is a covering map.*

Proof. Using f we can give X a smooth structure in which f is smooth. Let us fix a complete Riemannian metric on Y (this can be done because of a theorem of Whitney which says that any differentiable manifold can be embedded as a closed submanifold into some $\mathbb{R}^N$). Since f is proper, the pulled back metric on X is complete. Now f is a local isometry from a complete Riemannian manifold and so a covering. $\square$

Applying this lemma with $X = M/T^n$, $Y = \mu(M)$ and $f = \bar{\mu}$ proves Theorem 1. We turn to Theorem 2. If μ is proper as a map to $\mathbb{R}^{3n}$, then $\mu(M)$ is all of $\mathbb{R}^{3n}$ (since μ is open). It follows that the image of the union of singular orbits is the union of codimension 3 affine subspaces H_k, $k \in I$, (the fixed point set of any subgroup is a hyperkähler submanifold again satisfying the assumptions). The subspaces H_k must necessarily be discrete, i.e. every point has a neighbourhood intersecting only finitely many of them. Furthermore, it follows from Proposition 2.1 that H_k is of the form

$$H_k = \{x = (x^1, x^2, x^3) \in \mathbb{R}^n \otimes \mathbb{R}^3; \langle x^i, u_k \rangle = \lambda_k^i, \;\; i = 1, 2, 3\} \tag{4.1}$$

for some vectors $\{u_k\} \subset \mathbb{R}^n$ and some scalars λ_k^i, $i = 1, 2, 3$, $k \in I$. Moreover, the following condition is satisfied (Proposition 2.1(iii)):

(4.2) for each $x \in \mathbb{R}^{3n}$ the set $\{u_k; x \in H_k\}$ is part of a $\mathbb{Z}$-basis of $\mathbb{Z}^n$.

The results of [BD] show that as long as I is finite and the vectors $\{u_k\}$ generate $\mathbb{R}^n$, there exists a $4n$-dimensional hyperkähler manifold M' equipped with an effective tri-Hamiltonian action of T^n and such that $\mu(M'_{\text{sing}})$ is the union of the H_k. This hyperkähler manifold can be realized as a hyperkähler quotient of a flat quaternionic space by a torus. It follows from Theorem 1 that M is equivariantly homeomorphic to M'.

In fact, the assumption that I is finite is not necessary. It is not difficult to see that the constructions of [BD] and of [Go2] can be put together to give hyperkähler quotients of certain flat quaternionic Hilbert spaces by infinite dimensional tori (Hilbert groups). These quotients will produce $4n$-dimensional hyperkähler manifolds M with arbitrary $\{H_k\}_{k \in I}$ as long as the H_k are discrete, the vectors u_k generate $\mathbb{R}^n$ and (4.2) is satisfied.

Finally we have to consider the case when the vectors u_k do not span $\mathbb{R}^n$. Then $\mathbb{R}^{3n} = \mathbb{R}^{3i} \times \mathbb{R}^{3n-3i}$ where the H_k are of the form $\mathbb{R}^{3i} \times H'_k$ and the vectors u_k span $\mathbb{R}^{3n-3i}$. Let M' be a $(4n-4i)$-dimensional hyperkähler manifold with homeomorphism type determined by the H'_k. Then the product of M' and $T^i \times \mathbb{R}^{3i}$ with its flat hyperkähler metric has the H_k as the image of singular orbits. Therefore, by Theorem 1, M and $M' \times T^i \times \mathbb{R}^{3i}$ are equivariantly homeomorphic. This finishes the proof of Theorem 2.

We shall now give examples showing the necessity of assumptions in Theorem 1.

Example 4.2. We consider $\mathbb{R}^3 \times S^1$ with its flat S^1-invariant hyperkähler structure for which the hyperkähler moment map μ is simply the projection onto $\mathbb{R}^3$. Let U_1, U_2 be open subsets of $\mathbb{R}^3$ such that U_1 is contractible and $\pi_1(U_2) \neq 0$, $\pi_n(U_2) = 0$ for $n > 1$. Let $f_1 : \tilde{U}_1 \to U_1$ be a local diffeomorphism but not a diffeomorphism (such a $\tilde{U}_1$ can be found in $\mathbb{R}^4$). Let $f_2 : \tilde{U}_2 \to U_2$ be a double covering. Consider $M_i = \tilde{U}_i \times S^1$, $i = 1, 2$, and extend naturally the maps f_i. Now pull back the hyperkähler structure from $U_i \times S^1$ to M_i using f_i. Then the hyperkähler moment map on M_i is $\mu_i = \mu \circ f_i$. We observe that (M_1, μ_1) satisfies all assumptions of Theorem 1 except the properness, while (M_2, μ_2) satisfies all of them except the simple-connectedness of the image of the moment map. The conclusion of Theorem 1 (with $M_i' = U_i \times S^1$) fails in both cases.

5. Compact 3-Sasakian and quaternion-Kähler manifolds

Theorem 4.1 quickly leads to the topological classification of $(4n-1)$-dimensional 3-Sasakian manifolds with an action of T^n preserving the 3-Sasakian structure. Such an action is automatically Hamiltonian [BGM], i.e. there is a 3-Sasakian moment map $\mu : S \to \mathbb{R}^{3n}$. This moment map extends to a hyperkähler moment map on the cone $C^*(S)$: $\mu(r, s) = r^2\mu(s)$ for $s \in S$, $r \in \mathbb{R}_{>0}$. For $n = 1$ the 3-Sasakian manifolds are spherical space forms and so we can assume that $n > 1$. Let $Z = \mu(C^*(S))$. We observe that, by Proposition 2.1, $0 \notin Z$ and $\mu(Z)$ is open. Furthermore, since S is compact, Z is closed (in $\mathbb{R}^{3n} - \{0\}$). Hence Z is both open and closed in $\mathbb{R}^{3n} - \{0\}$ and so $Z = \mathbb{R}^{3n} - \{0\}$. In addition, the equivariance of μ under the $\mathbb{R}_{>0}$-action implies that μ is proper (as a map to $\mathbb{R}^{3n} - \{0\}$). Therefore (as $n > 1$) the assumptions of Theorem 1 are satisfied for $C^*(S)$ and the homeomorphism type of $C^*(S)$ is determined by the image of singular orbits. However, the fixed point set of a 1-PS of T^n is of the form $C^*(S')$ for a 3-Sasakian submanifold S' and so its image H_k is a codimension 3 subspace of $\mathbb{R}^{3n}$. By Proposition 2.1, the subspaces H_k satisfy the condition (4.2) and they determine the homeomorphism type of S. It was shown in [BD] that, as long as $\#\{H_k\} = d \geq n$, then for every choice of subspaces H_k satisfying (4.2) (with $\lambda_k^i = 0$) there exists a compact 3-Sasakian manifold S with a T^n-action such that $\mu(C(S)_{\text{sing}})$ is the union of the H_k. This manifold S was given as 3-Sasakian quotient of the standard sphere S^{4d-1} by a $(d-n)$-dimensional subgroup of T^d. Theorem 3 of the introduction will be therefore proved if we can show that $\#\{H_k\} \geq n$. Suppose that $\#\{H_k\} = m < n$. Then the H_k satisfy (4.2) also at the origin of $\mathbb{R}^{3n}$. By the argument at the end of the previous section there is a hyperkähler manifold of the form $M = M' \times T^{n-m} \times \mathbb{R}^{3n-3m}$ such that $\mu(M_{\text{sing}})$ is union of the H_k. Since all H_k pass through the origin, M' is homeomorphic to

$\mathbb{H}^m$. It follows from Theorem 1 that $C^*(S)$ is T^n-equivariantly homeomorphic to $M - T^{n-m}$. However, by the results of [GS], the odd Betti numbers of $C^*(S)$ vanish which is not the case for $M - T^{n-m}$ if $m < n$. We have proved Theorem 3.

Remark 5.1. The above argument shows that $\mu(S)$ is the boundary of a star-shaped region in $\mathbb{R}^{3n}$. Furthermore the homeomorphism type of S is determined by $\mu(S_{\text{sing}})$ or, equivalently, by the linear subspaces H_k.

We now turn to quaternion-Kähler manifolds of dimension $4n-4$ with an effective isometric action of T^n. Such a manifold M is an $SO(3)$ or $SU(2)$ quotient of a $(4n-1)$-dimensional 3-Sasakian manifold S with an effective action of T^n preserving the 3-Sasakian structure. By the above argument, S is determined by a collection of linear subspaces H_k of $\mathbb{R}^{3n}$ with $\#\{H_k\} \geq n$. It follows from the results of [Bi], that there can be only n or $n+1$ subspaces H_k (the Betti numbers of S are determined by the number d of the H_k; the relation of [GS], which is necessary for Q to be smooth, is satisfied only for $d = n$ or $d = n+1$). Furthermore, if $d = n+1$, then $b_2(M) = b_2(S) = 1$ and it follows from [LS] that M is isometric to $\mathrm{Gr}_2(\mathbb{C}^{n+1})$ with its symmetric quaternion-Kähler metric. In the $d = n$ case, we want to show that M is isometric to $\mathbb{H}P^{n-1}$ with its symmetric metric. By results of [Sa] it is enough to show that M is homeomorphic to $\mathbb{H}P^{n-1}$. We know that S is homeomorphic to S^{4n-1}/Γ, for some finite group Γ, and so the result will follow if we can show that the $Sp(1)$ action on S is the standard one. This, however, is the case for any S. Indeed, the $Sp(1)$ or $SO(3)$ action rotates the complex structures of $C^*(S)$ and so induces the standard $SO(3)$-action on $\mathbb{R}^n \otimes \mathbb{R}^3$. Hence we know the induced action on S/T^n. If there were two actions of $Sp(1)$ (we can always lift an $SO(3)$ action to one by $Sp(1)$), $\pi, \pi' : Sp(1) \times S \to S$, inducing the same $SO(3)$ action on S/T^n, then the composite action $\rho(g,s) = \pi'\left(g^{-1}, \pi(g,s)\right)$ would map fibers to fibers. On each fiber, which is a torus, we obtain an action of $Sp(1)$ which commutes with the action of the torus on itself. Therefore ρ is the trivial action and the claim follows. We have proved Theorem 4.

6. Possible generalizations and refinements

There are several refinements of Theorems 1-3 that one would expect to hold true.

1. The most obvious is to replace "homeomorphism" with "diffeomorphism" in all the statements. In our approach the difficulty with this stems from the fact that the hyperkähler moment map μ on M does not induce a diffeomorphism between M/T^n and $\mu(M)$, but only a homeomorphism (Lemma 2.5).
2. In Theorem 3 we even expect "isometric" in place of "homeomorphic".

3. Delzant's theorem [De] gives classification up to a symplectomorphism. Perhaps it is possible that, in the situation of Theorem 2, the manifolds are actually isomorphic as complex-symplectic manifolds (with respect to one complex structure), as it is the case for the flat metric and the Taub-NUT metric on $\mathbb{H}$. If one could show this, one could probably show 2., since the natural $SO(3)$-action rotates the complex structures on $C^*(S)$ and so the isomorphism would preserve all three symplectic forms.
4. By analogy with four dimensions (ALE-spaces), one should ask under what conditions "isometric" can be put in Theorem 2. The metric on manifolds in [BD] is asymptotic (in some sense) to the cone metric on $C^*(S)$, for a singular 3-Sasakian space S. Perhaps one could show that a hyperkähler manifold satisfying assumptions of Theorem 2 and a suitable curvature decay (in L^2-sense?) is isometric to one of the manifolds in [BD].
5. Finally, it should be possible to extend the results to hyperkähler and 3-Sasakian orbifolds.

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