Height of p-adic holomorphic functions and applications

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MPI / 91-46

HEIGHT OF p-ADIC HOLOMORPHIC FUNCTIONS AND APPLICATIONS*

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§1. INTRODUCTION

1.1. This paper continues the study of value distribution of p-adic holomorphic functions (see [Ha1]-[Ha5], [H-M]). Our purpose is to construct a p-adic analogue of Nevanlinna theory. As it is mentioned in earlier papers , the study is motivated by the works concerning the relation between number theory and value distribution theory (see [La1], [La2], [La3], [No1], [No2], [Vo]).

1.2. One of most essential differences between complex holomorphic functions and p-adic ones is that the modulus of a p-adic holomorphic function depends only on the modulus of arguments, except on a "critical set". This fact led us to introduce the notion of height of a p-adic holomorphic function. Using the height one can reduce in many cases the study of the zero set of a holomorphic function to the study a real convex parallelopiped. This makes it easier to prove p-adic analogue of statements of Nevanlinna theory.

1.3. It is well-known that the Lelong number plays an important role in the theory of complex entire functions. Here we define the Lelong number of a p-adic entire functions of several variables . In the p-adic case we do not know how to define an analogue of the

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* First version, June 1992.

"volume element", and we use here the notion of local heights. The study of the zero set of an entire function by using the Lelong number will be described in a future paper.

1.4. There are interesting relations between the value distribution theory, Diophantine problems and hyperbolic geometry. Some of them are deep results of Faltings, Vojta, Noguchi and others , while many statements are still conjectural (see [La1], [La2], [No1], [No2], [Vo]). In the p-adic case, because of the total discontinuity it is difficult to define an analogue of the Kobayashi distance. In this paper we propose a definition of p-adic hyperbolicity in the sense of Brody. Namely, a domain X in the projective space $P^n(C_p)$ is called hyperbolic if every holomorphic map from C_p to X is constant. We shall prove some theorems of Borel type on maps with the image lying in the complement of hyperplanes and algebraic hypersurfaces. Our purpose is only to exammine in p-adic case some properties of hyperbolic spaces described in Lang's book [La3].

1.5. The contents of the paper are as follows. The heights of p-adic holomorphic functions are defined in Section 2. We give an analogue of the Poisson-Jensens formula and basic properties of heights. §3 is devoted to p-adic Lelong number. In §4 we are trying to find an analogue of hyperbolicity .

1.6. The author would like to thank the Max-Planck-Institut für Mathematik in Bonn for hospitality and financial supports.

§2. HEIGHTS OF p-ADIC HOLOMORPHIC FUNCTIONS

2.1. Let p be a prime number, Q_p the field of p-adic numbers, and C_p the p-adic completion of the algebraic closure of Q_p. The absolute value in Q_p is normalized so that $|p| = p^{-1}$. We further use the notion $v(z)$ for the additive valuation on C_p which extends ord_p. Let D be the open unit disc in C_p:

$$D_1 = \{z \in C_p; |z| < 1\}$$

and $D = D_1 \times ... \times D_1$ the unit polydisc in C_p^k.

Let $f(z_1, ..., z_k)$ be a holomorphic function in C_p^k represented by the convergent series:

$$f(z_1, ..., z_n) = \sum_{|m|=0}^{\infty} a_{m_1 ... m_k} z_1^{m_1} ... z_k^{m_k}. \tag{1}$$

We set:

$$a_m = a_{m_1 ... m_k},$$

$$z^m = z_1^{m_1} ... z_k^{m_k},$$

$$|m| = m_1 + ... + m_k,$$

$$mt = m_1 t_1 + ... + m_k t_k.$$

Then for every $(t_1, ..., t_k) \in R^k$ we have:

$$\lim_{|m| \to \infty} \{v(a_m) + mt\} = \infty.$$

Hence, there exists an $(m_1, ..., m_k) \in \mathbb{N}^k$ such that $v(a_m) + mt$ is minimal.

2.2. **Definition.** *The height* of the function $f(z_1, ..., z_k)$ is defined by

$$H_f(t_1, ..., t_k) = \min_{o \leq |m| < \infty} \{v(a_m) + mt\}.$$

We use also the notation $H_f(z_1, ..., z_k) = H_f(v(z_1), ..., v(z_k))$.

2.3. Let us now give a geometric interpretation of heights. For every $(m_1, ..., m_k)$ we construct the graph $\Gamma_{m_1 ... m_k}$ representing $v(a_m z^m)$ as function of $(t_1, , ..., t_k)$. Then we obtain a hyperplane in R^{k+1}:

$$\Gamma_{m_1 ... m_k} : t_{k+1} = v(a_m) + mt.$$

Since $\lim_{|m| \to \infty} \{v(a_m) + mt\} = \infty$ for every $(t_1, ..., t_k) \in R^k$ there exists a hyperplane realizing

$$t_{k+1}(\Gamma_{m_1 ... m_k}) \leq t_{k+1}(\Gamma_{m'_1 ... m'_k})$$

for all $\Gamma_{m'_1...m'_k}$. We denote by H the boundary of the intersection in $R^k \times R$ of half-spaces of R^{k+1} lying under the hyperplanes $\Gamma_{m_1...m_k}$. It is easy to show that if $(t_1, ..., t_k, t_{k+1})$ is a point of H, then $t_{k+1} = H_f(t_1, ..., t_k)$.

2.4. To study of the zero set of a holomorphic function we need the following definition of local heights.

We set:

$$I_f(t_1, ..., t_k) = \{(m_1, ..., m_k) \in N^k, v(a_m) + \sum_{j=1}^{k} m_i t_i = H_f(t_1, ..., t_k)\}$$
$$n_i^+(t_1, ..., t_k) = \min\{m_i | \exists (m_1, ..., m_i, .., m_k) \in I_f(t_1, ..., t_k)\}$$
$$n_i^-(t_1, ..., t_k) = \max\{m_i | \exists (m_1, ..., m_i, .., m_k) \in I_f(t_1, ..., t_k)\}$$

It is easy to see that there exists a number T such that for $(t_1, ..., t_k) \geq (T, ..., T)$ (this means $t_i \geq T$ for all i), the numbers $n_i^+(t_1, ..., t_k)$ and $n_i^-(t_1, ..., t_k)$ are constants. Then we set:

$$h_i^+(t_1, ..., t_k) = n_i^+(t_1, ..., t_k)(T - t_i)$$
$$h_i^-(t_1, ..., t_k) = n_i^-(t_1, ..., t_k)(T - t_i)$$
$$h_i(t_1, ..., t_k) = h_i^-(t_1, ..., t_k) - h_i^+(t_1, ..., t_k)$$
$$h_f(t_1, ..., t_k) = \sum_{i=1}^{k} h_i(t_1, ..., t_k).$$

2.5. **Definition.** $h_f(t_1, ..., t_k)$ is said to be the *local height* of the function $f(z_1, .., z_k)$ at $(t_1, ..., t_k) = (v(z_1), ..., v(z_k))$.

2.6. One can prove basic properties of the height and local height by using the geometric interpretation 2.3. For our purpose we need some of them, namely, the following.

2.7. H is the boundary of a convex polyedron in R^{k+1}.

2.8. If we denote by $\Delta(H)$ the set of the edges of the polyedron H then the set of the critical points is exactly the image of $\Delta(H)$ by the projection:

$$\pi_k : R^k \times R \longrightarrow R^k.$$

2.9. We can show that for every finite parallelopiped in R^{k+1}, $P = \{-\infty < r_i < t_i < +\infty, i = 1, ..., k+1\}$, $H \cap P \times R$ consists of parts of a finite number of hyperplanes $\Gamma_{m_1 ... m_k}$. Indeed, these are the hyperplanes such that at least for an index i we have $m_i = n_i^+(t_1, ..., t_k)$ or $m_i = n_i^-(t_1, ..., t_k)$ for a point $(t_1, ..., t_k) \in P$.

2.10. For every finite parallelopiped and every hyperplane L in general position with respect to H, $L \cap H \cap P$ is a part of a hyperplane of dimension $k-1$.

2.11. If for $i \leq k$ the hyperplane $t_i = s_i$ =const is not in general position, then the hyperplane $t_i = s_i \pm \epsilon$ are in general position for small enough ϵ . Moreover we have:

$$\lim_{\epsilon \to 0} H_f(..., s_i \pm \epsilon, ...) = H_f(..., s_i, ...).$$

2.12.The set of critical points $\pi_k \Delta(H)$ is an union of hyperplanes of dimensions less or equal $k-1$.

2.13. Suppose that $S = S_1 \cap ... \cap S_{k-1}$, where S_i is the hyperplane $t_i = s_i$, $i = 1, ..., k-1$. Replacing S_i by $S_i^{\pm\epsilon}$: $t_i = s_i \pm \epsilon$ if necessary, one can suppose that the hyperplanes S_i are in general position. Then the intersection $S \cap \pi_k \Delta(H) \cap P$ is a finite set of points.

Note that we are using "general position" in an evident sense.

2.14. Now we are able to formulate and prove an analogue of the Poisson-Jensen formula. For any $(t_1, ..., t_k) \in R^k$ we set:

$$h_f(t_1, ..., t_i^{\pm}, ..., t_k) = \lim_{\epsilon \to 0} h_f(t_1, ..., t_i \pm \epsilon, ..., t_k)$$

and for two points $(t_1, ..., t_k)$ and $(T_1, ..., T_k)$:

$$\begin{aligned}\delta_i = & h_i^{-\epsilon_i}(t_1^{\epsilon_1}, ..., t_{i-1}^{\epsilon_{i-1}}, T_i^{\epsilon_i}, ..., T_k^{\epsilon_k}) \\ & - h_i^{\epsilon_i}(t_1^{\epsilon_1}, ..., t_{i-1}^{\epsilon_{i-1}}, t_i, T_{i+1}^{-\epsilon_{i+1}}, ..., T_k^{-\epsilon_k}) \\ & + \sum_{s_i} h_i^{\epsilon_i}(t_1^{\epsilon_1}, ..., t_{i-1}^{\epsilon_{i-1}}, s_i, T_{i+1}^{-\epsilon_{i+1}}, ..., T_k^{-\epsilon_k})\end{aligned}$$

where $\epsilon_i = \text{sign}(T_i - t_i)$ and the sum takes all $s_i \in (T_i, t_i)$. Note that by 2.4 the h_i are vanishing , except possibly on a finite set of values s_i, and δ_i does not depends on the choice of T.

2.15. THEOREM. *(The Poisson-Jensen formula).*

$$H_f(T_1, ..., T_k) - H_f(t_1, ..., t_k) = \sum_{i=1}^{k} \epsilon_i \delta_i.$$

PROOF: By using 2.3 - 2.14, it suffices to prove Theorem 2.15 for holomorphic functions of one variable.

Let $f(z)$ be an entire function on C_p and let $t_o > t > 0$. Then the formula in Theorem 2.15 takes the following form:

$$H_f(t_o) - H_f(t) = h_f^-(t_o) - h_f^+(t) + \sum_{t_o > s > t} h_f(s) \tag{2}$$

Suppose that $t_o > t_1 > t_2 > ... > t_n > t$ are all the critical points of the function $f(z)$. Note that the height $H_f(s)$ is a linear function of s in every segment $[t_{k+1}, t_k]$ and we have:

$$\begin{aligned}& n_f^-(t_k) = n_f^+(t_{k+1}) \\ & H_f(s) = v(a_{n_f^+(t_{k+1})}) + n_f^+(t_{k+1})s = v(a_{n_f^-(t_k)}) + n_f^-(t_k)s.\end{aligned}$$

From this it follows that :

$$\begin{aligned}H_f(t_k) - H_f(t_{k+1}) & = [v(a_{n_f^-(t_k)}) + n_f^-(t_k)t_k] - [v(a_{n_f^+(t_{k+1})}) + n_f^+ t_(k+1)t_k] \\ & = n_f^-(t_k)(t_k - t_{k+1})\end{aligned}$$

$$\begin{aligned} H_f(t_o) - H_f(t) &= H_f(t_o) - H_f(t_1) + H_f(t_1) - H_f(t_2) + \ldots \\ &+ H_f(t_n) - H_f(t) \\ &= (n_f^-(t_o)t_o - n_f^-(t_o)t_1) + (n_f^-(t_1)t_1 - n_f^-(t_1)t_2) + \ldots \\ &+ (n_f^-(t_n)t_n - n_f^-(t_n)t) \\ &= h_f^-(t_o) + t_1(n_f^-(t_1) - n_f^-(t_o)) + t_2(n_f^-(t_2) - n_f^-(t_1)) + \ldots \\ &+ t_n(n_f^-(t_n) - n_f^-(t_{n-1})) - h_f^+(t) \\ &= h_f^-(t_o) - h_f^+(t) + \sum_{t_o > s > t} h_f(s). \end{aligned}$$

Theorem 2.15 is proved.

2.16.**Remark.** Note that the formula 2.15 is ananlogous to the classical Poisson-Jensen formula. In fact, suppose that $t_o = \infty$, $f(0) \neq 0$ and t is not a critical point of the function $f(z)$. Then we have $H_f(t_o) = -\log_p |f(0)|$, $H_f(t) = \log_p |f(z)|$ on the circle $|z| = p^{-t}$, $h_f(t_o) = 0$, $\sum_{t_o > s > t} h_f(s) - h_f^+(t) = \sum -\log_p |z_i|$, where the sum extends over all the zeros z_i of the function $f(z)$ in the disc $|z| \leq p^{-t}$. Then the formula 2.15 takes the following form:

$$\log_{v(z)=t} |f(z)| - \log_p |f(0)| = \sum -\log_p |z_i|.$$

Recall that the classical Poisson-Jensen formula is the following:

$$\frac{1}{2\pi} \int_0^{2\pi} \log |f(e^{i\theta})| d\theta - \log |f(0)| = \sum_{a \in D, a \neq 0} -(\mathrm{ord}_a f) \log |a|,$$

where D is the unit disc in C and $\mathrm{ord}_a f$ is the order of $f(z)$ at a.

2.17.**Remark.** The formula 2.15 is not symmetry with respect to variables $t_1, \ldots, t_k$, and then one obtain a number of formulas of the height via local heights. Then it follows many equalities relating local heights. This fact has an analogue in the case of holomorphic functions of two complex variables (see [Ca]).

2.18.**Remark.** In [Ro] Robba gave an "approximation formula" , from which follows the Schwarz lemma for p-adic holomorphic functions of several variables. One can also obtain the Schwarz lemma by using the formula 2.15.

Let us finish this section with the following important theorem, the proof of which is easy by using the geometric interpretation of height.

2.19. THEOREM. *Every non-constant holomorphic function on* C_p^k *is a surjective map onto* C_p.

§3 . LELONG NUMBER

3.1. **Definition.** The *Lelong number* of a holomorphic function $f(z_1,...,z_k)$ at the point $(z_1,...,z_k)$ is defined by:

$$\nu_f(z_1,...,z_k)=\sum_{i=1}^{k}\{n_i^-(t_1,...,t_k)-n_i^+(t_1,...,t_k)\},$$

where $t_i=v(z_i)$.

3.2. **Example** . In the case of $n=1$, $\nu_f(z)$ is the number of zeros of f at $v(z)=t$ with counting multiplicity (see [Ma]).

3.3 **Remark.** The Lelong number of a holomorphic function $f(z)$ depends only on the modulus of the arguments.

3.4. LEMMA. $\nu_f(z_1,...,z_k)\neq 0$ *if and only if* $v(z_1,...,z_k)\in\pi_k\Delta_{H(f)}$, *where* $\pi_k\Delta_{H(f)}$ *is the projection of* $\Delta_H(f)\subset R^k\times R$ *on* R^k.

PROOF: In fact, suppose $\nu_f(z_1,..,z_k)\neq 0$ and denote $t_i=v(z_i)$. Then for every i, $n_i^+(t_1,...,t_k)=n_i^-(t_1,...,t_k)$ and there exists an unique n_i such that the set $\{(m_1,...,m_k)\in I_f, m_i=n_i\}$ is not empty. From this it follows that I_f contains an unique element

$(n_1, ..., n_k)$, and we have $H_f(t_1, ..., t_k) = v(a_n) + nt$, $|f(z_1, ..., z_k)| = p^{-H_f(t_1,...,t_k)}$. Hence, $(t_1, ..., t_k) \notin \pi_k \Delta_{H(f)}$.

Conversely, suppose $\nu_f(z_1, ..., z_k) \neq 0$. Then there exist at least one indexe i such that $n_i^-(t_1, ..., t_k) \neq n_i^+(t_1, ..., t_k)$. Therefore by using Remark 2.9 one can see that there exist at least two faces of $H(f)$ containing the point $(t_1, ..., t_k)$. This means that $v(z_1, ..., v_k) \in \pi_k \Delta_{H(f)}$. Lemma 3.4 is proved.

3.5. THEOREM. *A holomorphic function $f(z_1, ..., z_k)$ is a polynomial if and only if the Lelong number $\nu_f(z_1, ..., z_k)$ is constant for large enough $\|z\|$.*

PROOF: From the properties 2.7-2.13 of height one can show that $\nu_f(z_1, ..., z_k) =$ const for large enough $\|z\|$ if and only if there exist finitely many hyperplanes $\Gamma_{m_1 ... m_k}$ appear in the construction of H_f. This is equivalent to that f is a polynomial.

3.6. Remark. In the case of functions of one variable $\nu_f(z) =$ const is equivalent to that $\nu_f(z) = 0$ for large enough $|z|$.

§4. HYPERBOLICITY

4.1. **Definition.** A subset X of the projective space $P^n(C_p)$ is called *hyperbolic* if every holomorphic map from C_p into $P^n(C_p)$ with the image in X is constant.

Note that by a holomorphic map from C_p into $P^n(C_p)$ we mean a collection $f = (f_o, f_1, ..., f_n)$ where $f_i(z)$ are holomorphic functions having no zeros in common.

4.2. **Examples.** 4.2.1. The unit disc $D \in C_p$ is hyperbolic. Indeed, every holomorphic function on C_p with values in D is a bounded entire function , and therefore, is constant (Theorem 2.19).

4.2.2 . If X, Y are hyperbolic, the $X \times Y$ is hyperbolic. Hence, a polydisc $D \times ... \times D$ in $P^n(C_p)$ is hyperbolic.

4.2.3. From Theorem 2.19 it follows that the sets $C_p\backslash\{$ one points $\}$ and $P^1\backslash\{$ two points $\}$ are hyperbolic.

4.3. **Remark**. For any hyperbolic set $X \in C_p^n$, $C_p^n\backslash X$ is not bounded. Indeed, if $C_p^n\backslash X$ is bounded, then $C_p^n\backslash X \subset B_r$ for a ball of radius r. For a constant a with $|a| > r$ the following map

$$f : C_p \longrightarrow C_p^n, \quad z \mapsto (z, z+a, ..., z+a)$$

has the image lying in $C_p^n\backslash B_r$, and hence X is not hyperbolic.

4.4. Let H_k $,(k = 0, 1, ..., m)$ be hyperplanes of $P^n(C_p)$, then they said to be *in general position* if any l $(l \leq n+1)$ these hyperplanes are linearly independent.

4.5. THEOREM. *The complement in* $P^n(C_p)$ *of* $n+1$ *hyperplanes in general position is a hyperbolic space.*

Indeed, let $f : C_p \longrightarrow P^n$ be a holomorphic map with image lies in the complement of $n+1$ hyperplanes in genenral position.. Let $(x_o, ..., x_n)$ be the coordinates of $P^n(C_p)$. Then there is a projective change of coordinates such that these hyperplanes are defined by the equations $x_o = 0, ..., x_n = 0$. Now we can write f in homogeneous coordinates

$$f = (f_o, ..., f_n).$$

By the hypothesis the functions $f_o, ..., f_n$ are non-zero entire functions in C_p, and then they are constant.

4.6. THEOREM. *Let* $X_1, ..., X_{n+1}$ *be* $n+1$ *hyperplanes in* $P^n(C_p)$ *in general position. Let*

$$X = X_1 \cup X_2 \cup ... \cup X_{n+1}$$

be their union. Then

1) $P^n(C_p)\backslash X$ *is hyperbolic.*

2)for every $\{i_1, ..., i_k, j_1, ..., j_r\} = \{1, ..., n+1\}$ *the space*

$$(X_{i_1} \cap ... \cap X_{i_k}) \backslash (X_{j_1} \cup ... \cup X_{j_r})$$

is hyperbolic.

PROOF: 1) Theorem 4.5.

2) Let

$$f : C_p \longrightarrow X_{i_1} \cap ... \cap X_{i_k} \backslash X_{j_1} \cup ... \cup X_{j_r}$$

be a holomorphic map. Since the hyperplanes are in general position , $X = X_{i_1} \cap ... \cap X_{i_k}$ can be identified with P^{n-k}. Then $\{X_{j_m} \cap X\}$ are in general position in X. We have $r = (n-k)+1$, and 2) is a corollary of Theorem 4.5

4.7. THEOREM. *Let* $X \longrightarrow Y$ *be a holomorphic map of p-adic analytic spaces. Suppose that* Y *is hyperbolic, and for every* $y \in Y$ *there exists a neighborhood* U *of* y *such that* $\pi^{-1}(U)$ *is hyperbolic. Then* X *is a hyperbolic space.*

PROOF: Let $f : C_p \longrightarrow X$ be a holomorphic map. then $\pi.f$ is holomorphic, and is constant, since Y is hyperbolic. We set $y_o = \pi.f(C_p)$. Let U_o is a neighborhood of y_o such that $\pi^{-1}(U_o)$ is hyperbolic. Since the image of f lies in $\pi^{-1}(U_o)$, f is constant.

4.8. THEOREM. *Let f be a holomorphic map from* C_p *into* $P^n(C_p)$ *with image lies in the complement of* $k \geq 2$ *different hypersurfaces. Then there exist proper algebraic subspaces* $X_1, ..., X_m$, $m = \dfrac{k(k-1)}{2}$, *such that the image of f lies in the intersection of* $X_1, .., X_m$.

PROOF: Let $P_1, ..., P_k$ be the homogeneous polynomials defining the hypersurfaces $Y_1, ..., Y_k$.∎ For every $i, 1 \leq i \leq k$, $P_i.f$ is constant. We can find numbers α_i such that $\alpha_i(P_i.f) - \alpha_j(P_j.f) \equiv 0$ on C_p. We set

$$Q_{ij} = \alpha_i P_i - \alpha_j P_j.$$

Then Q_{ij} are homogeneous polynomials , which define the algebraic subspaces $X_1, ..., X_m$, $m = \frac{k(k-1)}{2}$. Note that X_i's are proper algebraic subspaces, and the image of f lies in their intersection.

3.8.. **Remark.** The theorem can be regarded as an analogue of the Green theorem in the complex case (see [La3])

REFERENCES

[Ca] H. Cartan. *Sur la notion de croissance attachée à une fonction méromorphe de deux variables , et ses applications aux fonctions méromorphes d'une variable.* C. R. A. Sc. Paris, 189, 1929, 521-523.

[C-S] G. Cornell, J. H. Silverman (Ed.). Arithmetic Geometry. Springer-Verlag: New York-Berlin-Heidelberg-London-Paris-Tokyo, 1986.

[Ha1] Ha Huy Khoai. *On p-adic meromorphic functions.* Duke Math. J., Vol 50, 1983, 695-711.

[Ha2] Ha Huy Khoai. *Sur la théorie de Nevanlinna p-adique.* Groupe d'Etude d'Analyse ultramétrique, 15-ème année, Paris, 1987-1988, 35-39.

[Ha3] Ha Huy Khoai. *Heights for p-adic meromorphic functions and value distribution theory.* Max-Planck-Institut für Mathematik Bonn, MPI/89-76.

[Ha4] Ha Huy Khoai. *La hauteur des fonctions holomorphes p-adiques de plusieurs variables.* C. R. A. Sc. Paris, 312, 1991, 751-754.

[Ha5] Ha Huy Khoai. *La hauteur d'une suite de points dans* C_p^k *et l'interpolation des fonctions holomorphes de plusieurs variables.* C. R. A. Sc. Paris, 312, 1991, 903-905.

[H-M] Ha Huy Khoai and My Vinh Quang. *p-adic Nevanlinna theory.* Lecture Notes in Math. 1351, 1988, 138-152.

[Ko] S. Kobayashi. Hyperbolic Manifolds and Holomorphic mappings. Marcel Deckker: New York, 1970.

[La1] S. Lang. *Higher dimensional Diophantine problems.* Bull. Amer. Math. Soc. 80, 1974, 779-787.

[La2] S. Lang. *Hyperbolic and Diophantine analysis.* Bull. Amer. Math. Soc. 14, 1986, 159-205.

[La3] S. Lang. Introduction to Complex Hyperbolic Spaces. Springer-Verlag: New York-Berlin-Heidelberg, 1987.

[Ma] Y. Manin. *p-adic Automorphic functions.* In: Current Problems of Mathematics. Mir: Moscow, 1974.

[No] J. Noguchi. *Hyperbolic manifolds and Diophantine Geometry.* Sugaku Expositions, Vol 4, 63-81. Amer. math. Soc., Providence, Rhode Island 1991.

[No2] J. Noguchi. *Meromorphic mappings into compact hyperbolic complex spaces and geometrc Diophantine problems.* International J. of math, vol 3, n.2, 1992, 277-289.

[Ro] Ph. Robba. *Prolongement analytique pour les fonctions de plusieurs variables sur un corps valué complet.* Bull. Soc math. France 101, 1973, 193-217.

[Ro] Ph. Robba. *Lemmes de Schwarz et lemmes d'approximations p-adiques en plusieurs variables.* Inv. Math. 48, 1978, 245-277.

[Vo] P. Vojta. Diophantine Approximation and Value Distribution Theory. Lecture Notes in Math 1239. Springer-Verlag: Berlin-Heidelberg-New York-London, 1987.