DOUBLE PANTS DECOMPOSITIONS OF 2-SURFACES

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Abstract. To study geometric structures on surfaces and their moduli spaces, one usually supplies the surface with an additional one-dimensional marking (such as a basis of he fundamental group, triangulation, etc). We introduce a new class of such markings: admissible double pants decompositions, which seems to be very convenient for study of moduli spaces. We define a groupoid generated by simple transformations of double pants decompositions (each generating transformation changes only one curve of a decomposition) and prove that this groupoid acts transitively on the set of all admissible double pants decompositions. We also show that the same groupoid contains a group isomorphic to a mapping class group. Our approach fits for all compact orientable surfaces with finite (possibly, zero) number of marked points except for a sphere with less than 3 marked points and a torus without marked points.

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Introduction

In this paper we suggest a new marking on topological surfaces, which can be useful for study of moduli spaces arising from Riemann surfaces. These moduli spaces (moduli of Riemann surfaces, Hurwitz spaces, moduli spaces of bundles, moduli spaces of connections, etc) are classical objects of study in a meeting point of geometry (algebraic,

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symplectic, and differential), topology and combinatorics. During the last decades, the question has become especially important due to its tight connection to mathematical physics.

Moduli spaces are rather complicated, and it is convenient to study them through covering spaces, where the leafs of the covering are specified by some additional geometric marking on the surface. In case of compact surfaces, for this additional marking one usually chooses a conjugation class of bases in the fundamental group of the surface (with generators satisfying standard algebraic relations). Then one considers a restriction of the structure under consideration to the basis, so that a huge part of problems is reduced to the structure transformations under exchange of basis (or, in other words, to investigation of the action of the mapping class group on the restriction of the structure). For the moduli space of Riemann surfaces this is a classical approach initiated by Fricke and Klein [8] and Teichmüller [20]. One of the difficulties of this approach arises from the fact that a change of one generator hugely affects the other generators. In addition, this way is not convenient for investigation of compactifications of moduli spaces.

A recent cluster approach allows to eliminate these obstacles by consideration of triangulations of surface instead of bases (see works of Fock and Goncharov [4, 5], Gekhtman, Shapiro, Vainshtein [9, 10], Penner [17, 18], Fomin, Shapiro, Thurston [6], Fomin, Thurston [7] and many others). However, this method works for punctured surfaces only. Furthermore, the standard cluster construction based on triangulation has no direct connections to homotopy and homology classes of curves (which are important ingredients of problems concerning moduli spaces).

Another stream in the study of moduli spaces is based on the idea of pants decompositions, i.e. decompositions of the surfaces into several "pairs of pants", where a "pair of pants" means a sphere with 3 holes (see papers of Hatcher, Thurston [13], Hatcher [12], Penner [19], Bakalov, Kirillov [2], and many other works of various authors). In this approach the state of the surface is encoded by the state of some pants decomposition considered as a union of curves on the surface and changed under transformations concerning only one of the curves. Pants decompositions are extremely convenient for discussing questions concerning homology and homotopy classes of curves, but an individual pants decomposition is not sufficient to carry complete information about geometric structure on a surface. Also, an individual pants decomposition is not sufficient for a work with mapping class group: namely, any given pants decomposition is preserved by a large subgroup of the mapping class group. To work with the whole mapping class group either one considers a curve complex including information from the totality of all pants decompositions (as in works of Ivanov [15], Margalitt [16], Irmak, Korkmaz [14], Andersen, Bene, Penner [1] and others) or one introduces some additional markings or "seems", or angular twist parameters (and then, again, a half of the structure does not fit for homology and homotopy considerations).

In this paper, we enrich the structure of pants decomposition in a most symmetric way, namely, by *another* pants decomposition. So, the main object of our study is a *pair* of pants decompositions considered as a union of curves. Their exchanges are generated by very simple elementary transformations called "flips" and "handle twists", each elementary transformation affecting only one curve. A general pair of pants decompositions is convenient to encode complete information on the geometric structures carried by the surface.

We consider a special class of pairs of pants decomposition which we call "admissible double pants decompositions". We say that a pair of pants decompositions DP of a surface S is a *double pants decomposition* if homology classes of the curves contained in DP generate the whole homology lattice $H_1(S,\mathbb{Z})$. *Admissible decompositions* are distinguished by the property that flips and handle twists are sufficient to transform DP into a pair of pants decomposition containing only $4g+n-3$ curves, where g is a genus and n is a number of marked points on S. This is the minimal possible number of different curves in a pair of pants decompositions whose homology classes generate $H_1(S,\mathbb{Z})$ (a general double pants decomposition consists of $6g+2n-6$ curves). It is not always easy to recognize whether a given double pants decomposition is admissible or not. On the other hand, all double pants decomposition we have ever met turn out to be admissible.

An efficiency of our approach is based on the following nontrivial theorem:

Main Theorem. *Let S be a topological surface of genus g with n marked points, where $2g+n>2$. Then the groupoid generated by flips and handle twists acts transitively on the set of admissible double pants decompositions of S.*

It is possible to show that the mapping class group acts effectively on admissible double pants decompositions of some special combinatorial class. Together with the Main Theorem this implies the following result:

Corollary. *The category of double pants decompositions of a topological surface with morphisms generated by flips and handle twists contains a subcategory isomorphic to a category of topological surfaces with mapping class groups as the set of morphisms.*

The paper is organized as follows. Section 1 is mostly devoted to definitions and basic properties of double pants decompositions. We prefer to work with surfaces containing no marked points and postpone all details concerning marked points till Section 4. In Section 2, we prove transitivity theorem for the case of surfaces of genus $g=2$ without marked points (Theorem 2). In Section 3, we prove the Main Theorem for the case of surfaces of any genus (Theorem 3.19). In Section 4, we extend basic definitions to the case of surfaces with marked points and complete the proof of Main Theorem. Finally, in Section 5, we prove the Corollary (Theorem 5.3).

Our approach may be naturally extended to the stable Riemann surfaces and allow to study compactifications of Deligne-Mumford type [3]. We will return to this problem in a sequel to this paper.

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1. Double pants decompositions

1.1. **Zipped pants decompositions.** Let $S = S_{g,n}$ be an oriented surface of genus g with n marked points, where $2g + n > 2$. Throughout Sections 1–3 we assume in addition $n = 0$ (this assumption will be removed in Section 4).

A *curve* c on S is an embedded closed non-contractible curve considered up to a homotopy of S.

Given a set of curves we always assume that there are no "unnecessary intersections", so that if two curves of this set intersect each other in k points then there are no homotopy equivalent pair of curves intersecting in less than k points.

For a pair of curves c_1 and c_2 we denote by $|c_1 \cap c_2|$ the number of (geometric) intersections of c_1 with c_2.

Definition 1.1 (*Pants decomposition*)**.** A *pants decomposition* of S is a system of (non-oriented) mutually disjoined curves $P = P_S = \langle c_1, \dots, c_n \rangle$ decomposing S into pairs of pants (i.e. into spheres with 3 holes).

It is easy to see that any pants decomposition of a surface of genus g consists of $3g-3$ curves. Note, that we do allow self-folded pants, two of whose boundary components are identified in S as shown in Fig. 1.1.

A surface which consists of one self-folded pair of pants will be called *handle*.

We say that a curve $c \in P$ is *non-regular* if c is contained in a handle $\mathfrak{h}$ and $c_1 \in P$ where c_1 is a boundary or $\mathfrak{h}$ (see Fig. 1.1). Otherwise, we say that c is *regular*.

Remark 1.2. A set of curves forming a pants decomposition is maximal in sense that any larger set of mutually disjoined non-oriented curves contains a pair of homotopy equivalent curves.

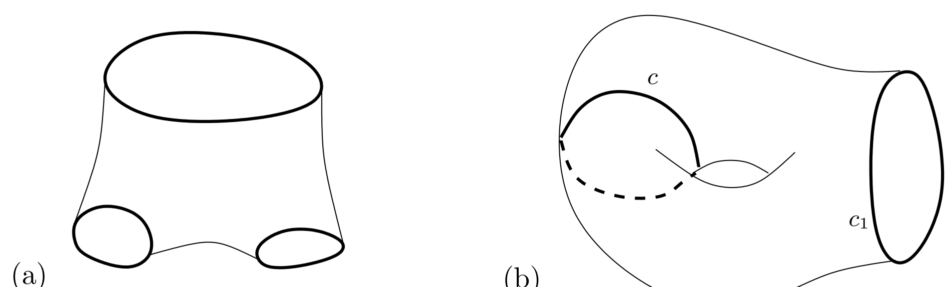

Figure 1.1. (a) A pair of pants; (b) a pair of self-folded pants composing a handle (the handle contains a non-regular curve c).

Definition 1.3 (*Zipper system*)**.** A union $Z = \langle z_1, \dots z_{g+1}\rangle$ of mutually disjoined curves is a *zipper system* if Z decomposes S into two spheres with $g+1$ holes.

Definition 1.4 (Z *compatible with* P)**.** A zipper system Z is *compatible with a pants decomposition* $P = \langle c_1, \dots, c_{3g-3}\rangle$ if $|c_i \bigcap (\cup_{j=1}^{g+1} z_j)| = 2$ for each $i = 1, \dots, 3g-3$.

Fig. 1.2 contains an example of a zipper system Z compatible with a pants decomposition P.

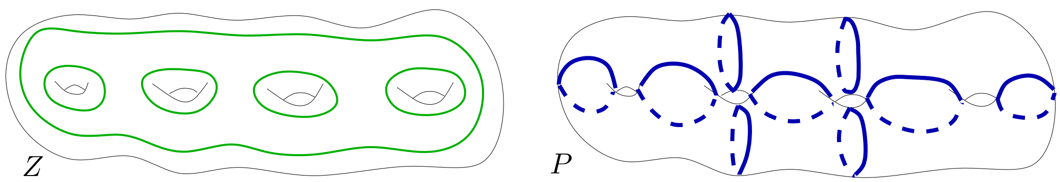

FIGURE 1.2. A zipper system Z compatible with a pants decomposition P.

Lemma 1.5. *If Z is a zipper system compatible with a pants decomposition P then $\cup_{j=1}^{g+1} z_j$ decomposes each pair of pants in P into two hexagons.*

Proof. Suppose that a curve z_j intersects a curve c_i contained in the boundary of a pair of pants p_1. The curves of the pants decomposition cut z_j into segments. Let l be a segment of z_j (or one of such segments) contained in p_1. Since curves do not have unnecessary intersections, l looks as shown in Fig. 1.3(a) or (b). If l looks as in Fig. 1.3(a) then for some of the three boundary curves of p_1 the condition $|c_i \bigcap (\cup_{j=1}^{g+1} z_j)| = 2$ is broken. This implies that it is as one shown in Fig. 1.3(b). Therefore, p_1 looks like in Fig. 1.3(c), i.e. p_1 is decomposed into two hexagons.

□

Remark 1.6. If Z is a zipper system compatible with a pants decomposition P then there exists an involution σ such that σ preserves Z pointwise and such that $\sigma(c_i) = c_i$ for each $c_i \in P$. To build this involution one needs only to switch the pairs of hexagons described in Lemma 1.5.

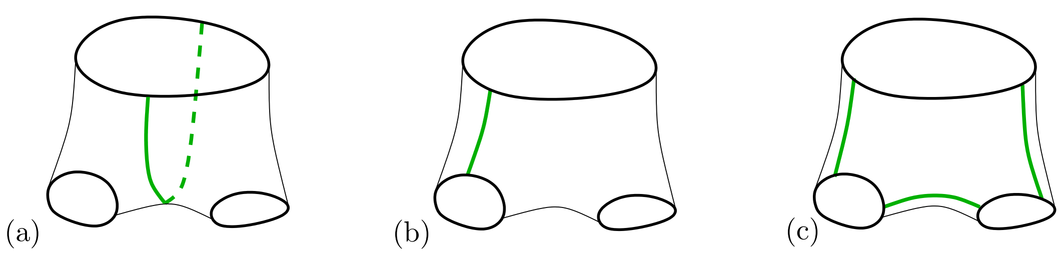

FIGURE 1.3. Zipper system Z decomposes each pair of pants into two hexagons.

Definition 1.7 (*Zipped flip*)**.** Given a pants decomposition $P = \langle c_1, \dots, c_{3g-3} \rangle$ and a zipper system Z compatible with P we define a *zipped flip* of pants decomposition as it is shown in Fig. 1.4. Formally speaking, a *zipped flip* f_i of a pants decomposition $P = \langle c_1, \dots, c_{3g-3} \rangle$ *in the curve* c_i is a replacing of a regular curve $c_i \subset P$ by a unique curve c_i' satisfying the following properties:

- c_i' does not coincide with any of $c_1, \dots, c_{3g-3}$;
- c_i' intersects Z exactly in two points;
- $c_i' \cap c_j = \emptyset$ for all $j \neq i$.

Clearly, $f_i(P)$ is a new pants decomposition of S, and Z is a zipper system compatible with $f_i(P)$. The uniqueness of the curve c_i' satisfying the properties in Definition 1.7 verifies trivially. In particular, it is easy to see that $f_i \circ f_i(c_i) = c_i$.

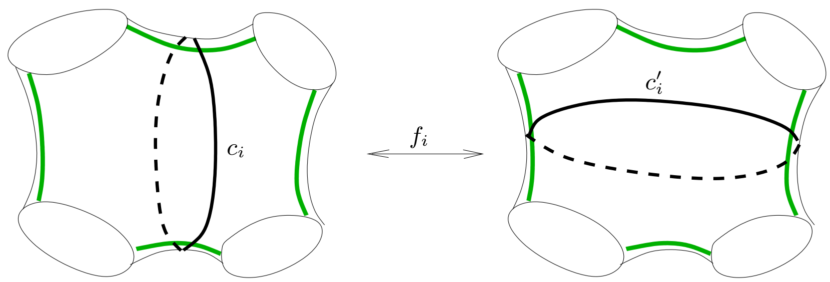

FIGURE 1.4. A zipped flip of a pants decomposition.

Definition 1.8 (*Lagrangian plane of pants decomposition*)**.** Let $P = \langle c_1, \dots, c_{3g-3} \rangle$ be a pants decomposition. A Lagrangian plane $\mathcal{L}(P) \subset H_1(S, \mathbb{Z})$ is a sublattice spanned by the homology classes $h(c_i)$, $i = 1, \dots, 3g-3$ (here c_i is taken with any orientation).

An obvious computation shows that any flip preserves the Lagrangian plane defined by the pants decomposition.

Definition 1.9 (*Category of zipped pants decompositions*)**.** The category $\mathfrak{P}_g(Z, \mathcal{L})$ of *zipped pants decompositions* of a genus g surface S depending on a given zipper system Z and a given Lagrangian plane $\mathcal{L} \subset H_1(S, \mathbb{Z})$ is the following:
Objects: all pants decompositions P of S compatible with Z and such that $\mathcal{L}(P) = \mathcal{L}$;
Elementary morphisms: zipped flips of regular curves (defined with respect to Z).
All other **morphisms** are compositions of elementary ones.

A similar construction gives a *category of ideal triangulations*. Namely, let (S, M) be a closed surface S with a finite set of marked points M. An *ideal triangulation* of (S, M) is a decomposition of S into triangles with vertices in M. We allow triangles whose two or even three vertices coincide; we also allow self-folded triangles (i.e. ones two of whose sides coincide), see Fig. 1.5 for the list of possible triangles and [6] for the detailed exposition.

A *flip of a triangulation* is an exchange of the diagonal in a quadrilateral (see Fig. 1.5). Note that some edges are not flippable: namely, an edge is flippable unless it is an inner edge of a self-folded triangle. An edge e of a triangulation will be called *non-regular* if it is an inner edge of a self-folded triangle, otherwise e will be called *regular*. It is well-known that flips act transitively on the set of all ideal triangulations of a given surface.

Definition 1.10 (*Category of ideal triangulations*)**.** The category $\mathfrak{T}_{g,n}$ of *ideal triangulations* is the following:

Objects: ideal triangulation of a genus g surface S with n marked points;

Elementary morphisms: flips of triangulations.

All other **morphisms** are compositions of elementary ones.

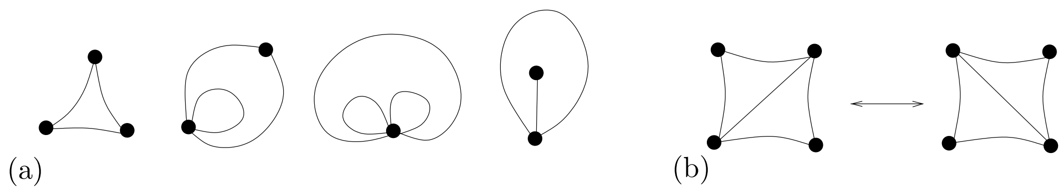

FIGURE 1.5. Ideal triangulations: (a) types of triangles admissible for an ideal triangulation; (b) flip inside a quadrilateral.

Lemma 1.11. *The category $\mathfrak{P}_g(Z, \mathcal{L})$ is isomorphic to the category $\mathfrak{T}_{0,g+1}$.*

Proof. Let P be a pants decomposition of S compatible with a zipper system Z. Consider the zipper system Z. By Definition 1.3 Z decomposes S into two spheres S^+ and S^- with $g+1$ holes. Let

$$S' = S_+/\sim,$$

where $x \sim y$ if both x and y are points of the same boundary component of S_+. Then S' is a sphere with $g+1$ punctures (one puncture for each equivalence class of boundary points).

Now, let P be any pants decomposition compatible with Z. Then the curves of P decompose S_+ into hexagons, one hexagon in S_+ for each pair of pants of P (see Lemma 1.5). The factorization by the equivalence relation takes each hexagon into an ideal triangle at S' (each curve c_i turns into an edge e_i of a triangle), and we obtain an ideal triangulation of S'.

So, for each pants decomposition P compatible with Z we build an ideal triangulation $T = \theta(P)$ of S'. It is easy to see that θ takes regular curves of pants decompositions to regular edges of triangulations (and non-regular ones to non-regular). Moreover, if f_i is a flip of P in the curve c_i (defined with respect to Z), then

$$\theta(f_i(P)) = f_i'(\theta(P)), \tag{1.1}$$

where f_i' is the flip of T in the edge e_i. Restricting θ to pants decompositions P satisfying $\mathcal{L}(P) = \mathcal{L}$ we obtain a functor from $\mathfrak{P}_g(Z, \mathcal{L})$ to $\mathfrak{T}_{0,g+1}$.

Furthermore, if $\theta(P) = T$ then for each elementary morphism f_i' of the triangulation T there exists an elementary morphism f_i of the pants decomposition P satisfying condition 1.1. Since flips act transitively on triangulation of the same surface, θ is surjective. It is clear the θ is also injective. Therefore, θ is an equivalence of the categories.

□

Corollary 1.12. *Morphisms of $\mathfrak{P}_g(Z, \mathcal{L})$ act transitively on the objects.*

Proof. It is well known that flips act transitively on the ideal triangulations of surface. In view of Lemma 1.11 this implies that morphisms of $\mathfrak{P}_g(Z, \mathcal{L})$ act transitively on the objects of the same category.

□

Remark 1.13. In [6], Fomin, Shapiro, Thurston described a wider category of *tagged ideal triangulations* which allows flips in each side of any triangle, not only in regular ones. One can easily reproduce the same construction in the context of pants decompositions, so that the zipped flips of the "tagged" pants decompositions would be in correspondence with mutations of quivers arising from triangulations of a sphere.

1.2. **Unzipped pants decompositions.**

Definition 1.14 (*Unzipped flip*)**.** Let $P = \langle c_1, \dots, c_n \rangle$ be a pants decomposition. Define an *unzipped flip of P in the curve c_i* (or just a *flip*) as a replacing of a regular curve $c_i \subset P$ by any curve c_i' satisfying the following properties:

- c_i' does not coincide with any of $c_1, \dots, c_n$;
- $|c_i' \cap c_i| = 2$;
- $c_i' \cap c_j = \emptyset$ for all $j \neq i$.

Proposition 1.15. *$\mathcal{L}(P) = \mathcal{L}(f(P))$ for any unzipped flip f of P.*

Proof. Let $p_1 \cup p_2$ be two pairs of pants glued along a curve e affected by the flip f. Let a, b, c, d be four curves cutting $p_1 \cup p_2$ out of S. Suppose that $f(e)$ separates a and b from c and d in $p_1 \cup p_2$. Denote by $h(c)$ the homology class of c. Then

$$h(f(e)) = h(a) + h(b) = h(c) + h(d) \in \mathcal{L}(P).$$

The similar relation holds when $f(e)$ separates a and c from b and d or a and d from b and c.

□

Lemma 1.16. *Let P be a pants decomposition and $c \in P$ be a curve. A flip f of P in the curve c is defined uniquely by a homology class of $f(c)$ up to a Dehn twist along c.*

Proof. Consider two pairs of pants p_1 and p_2 adjacent to c, let $M = p_1 \cup p_2$. Since $|(f(c) \cap c)| = 2$ and $f(c) \cap \partial M = \emptyset$, the segment $f(c) \cap p_i$, $i = 1, 2$ looks as shown in Fig. 1.3(a). The homology class of $f(c)$ defines which of the ends of $f(c) \cap p_1$ are glued to which of the ends of $f(c) \cap p_2$. So, the only freedom in gluing of p_1 to p_2 is generated by a Dehn twist along c.

□

The result of Lemma 1.16 may be restated as follows.

Lemma 1.17. *Let P be a pants decomposition compatible with a zipper system Z. Let f_c be a flip of P in a curve $c \in P$. Then f_c is a zipped flip for some $Z' = T_c^m(Z)$, where $m \in \mathbb{Z}$ and T_c is a Dehn twist along c.*

Remark 1.18. In particular, Lemma 1.17 implies that if P_0 is a pants decomposition compatible with a zipper system Z then after any sequence of flips $f_1, \dots, f_k$ we obtain a decomposition P_k compatible with some zipper system Z_k. Moreover, one can choose zipper systems $Z_1, \dots, Z_k$, $Z_0 = Z$ so that f_i is a zipped flip with respect to Z_i and $Z_{i+1} = T_{c_i}^{m_i}(Z_i)$ for the curve $c_i \in P_i$ affected by f_i.

A Dehn twist along a curve $c \in P$ is a composition of two flips (see Fig. 1.6).

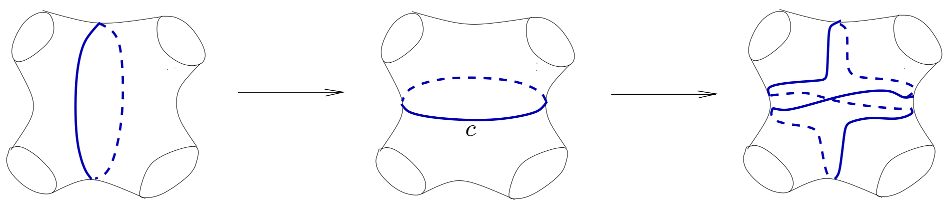

FIGURE 1.6. Dehn twist as a composition of two flips.

Definition 1.19 (*Category of unzipped pants decompositions*)**.** A category $\mathfrak{P}_g(\mathcal{L})$ for the given Lagrangian plane $\mathcal{L}$ is the following:

Objects: pants decompositions P of a genus g surface S satisfying $\mathcal{L}(P) \in \mathcal{L}$;

Elementary morphisms: unzipped flips.

Other **morphisms** are compositions of elementary ones.

Remark 1.20 (A. Hatcher, [11]). *Morphisms of $\mathfrak{P}_g(\mathcal{L})$ do not act transitively an the objects of the same category.* To see this suppose that the surface S is embedded in $\mathbb{R}^3$ and a pants decomposition P is such that each curve of P is contractible inside the inner handlebody defined by $S \subset \mathbb{R}^3$. Then each flip preserves this property of P. On the other hand there exists a pants decomposition $P' \in \mathcal{L}(P)$ with non-contractible (inside the handlebody) curves (see Fig. 1.7 for a non-contractible curve c such that $h(c) \in \mathcal{L}(P)$).

Remark 1.21. It is not clear if flips act transitively on the pants decompositions whose curves are contractible in a given handlebody of a $S \subset \mathbb{R}^3$.

Example 1.22. (An orbit of a pants decomposition of S_2.) In this example we describe an orbit of an arbitrary pants decomposition of a surface of genus 2. The description is in terms of a graph Γ where a vertex $v_P \in \Gamma$ correspond to a pants decompositions P and an edge $e \in \Gamma$ connecting v_P to $v_{P'}$ correspond to a flip f such that $P' = f(P)$.

A pants decomposition P of S_2 may be of one of two types:

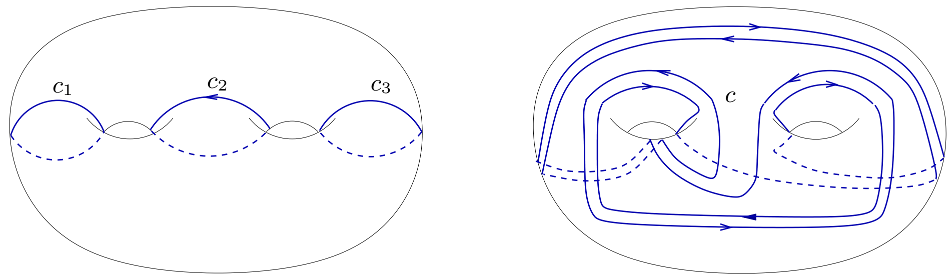

FIGURE 1.7. Pants decomposition $P = \langle c_1, c_2, c_3 \rangle$ and a curve c such that $h(c) = h(c_2)$ and c is not contractible inside the inner handlebody (the curve c is linked non-trivially with c_3).

- (a) "non-self-folded": P consists of two non-self-folded pairs of pants (the pants decompositions of this type are denoted by squares in Fig. 1.8);
- (b) "self-folded": P consists of two handles glued along the holes (the pants decompositions of this type are denoted by circles in Fig. 1.8).

For each of the two types of vertices of Γ we need to understand how many edges are incident to the vertex. In fact, the number of such edges is always infinite: if $c \in P$ is a regular curve, T_c is a Dehn twist along c and $f(c)$ is a flip of the curve c then $T_c(f(c))$ is also a flip of c (see Fig. 1.6). On the other hand, Lemma 1.16 states that modulo Dehn twist T_c there are exactly two possibilities for the flip $f(c)$. Therefore, instead of Γ we will draw the simplified graph $\overline{\Gamma}$ obtained from Γ after factorizing by Dehn twists T_c for each flipped curve c. Then for each regular curve $c \in P$ there are exactly 2 edges emanating from vertex $v_P \in \overline{\Gamma}$. If P is of non-self-folded type, there are 3 regular curves in P, so there are 6 edges incident to $v_P \in \overline{\Gamma}$. If P is of self-folded type, there is a unique regular curve in P, so there are only two edges incident to $v_P \in \overline{\Gamma}$. The graph $\overline{\Gamma}$ is shown in Fig. 1.8.

We will say that a path $\gamma \in \overline{\Gamma}$ is *alternating* if any edge of γ connects two vertices of different types. It is easy to see that for each path $\gamma \in \overline{\Gamma}$ there exists an alternating path $\gamma' \in \overline{\Gamma}$ with the same endpoints. Indeed, each edge of $\overline{\Gamma}$ connecting two vertices of the same type (those are always "square" vertices) may be substituted by an alternating path of two edges. Since a Dehn twist along a curve of pants decompositions is a composition of two flips (each changing the type of a pants decomposition in case of S_2), we obtain the same property for an arbitrary path in Γ:

For each path $\gamma \in \Gamma$ there exists an alternating path $\gamma' \in \Gamma$ with the same endpoints.

Example 1.22 shows that the pants decompositions containing the curves separating handles play special role among other pants decompositions.

Definition 1.23 (Standard pants decomposition)**.** A pants decomposition P is *standard* is P contains g curves $c_1, \dots, c_g$ such that c_i cuts out of S a handle $\mathfrak{h}_i$.

1.3. **Admissible double pants decompositions.**

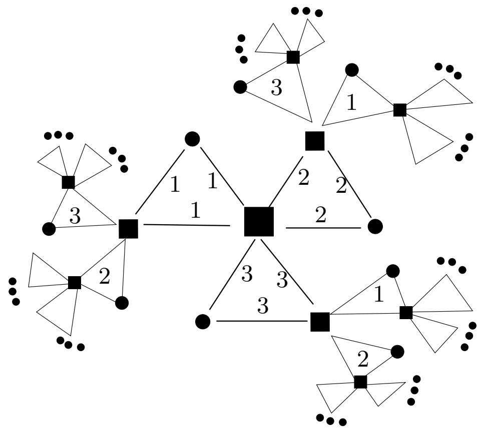

FIGURE 1.8. An orbit of a pants decomposition of S_2 (modulo Dehn twists). Vertices marked by squares and circle correspond to non-self-folded and self-folded pants decompositions respectively. Edges correspond to flips. The labels $1, 2, 3$ on the edges show which of the three curves of the decomposition is flipped along this edge.

Definition 1.24 (*Lagrangian planes in general position*)**.** Two *Lagrangian planes* $\mathcal{L}_1$ *and* $\mathcal{L}_2$ *are in general position* if $H_1(S, \mathbb{Z}) = \langle \mathcal{L}_1, \mathcal{L}_2 \rangle$.

See Fig. 1.9 for an example of two pants decompositions spanning a pair of Lagrangian planes in general position.

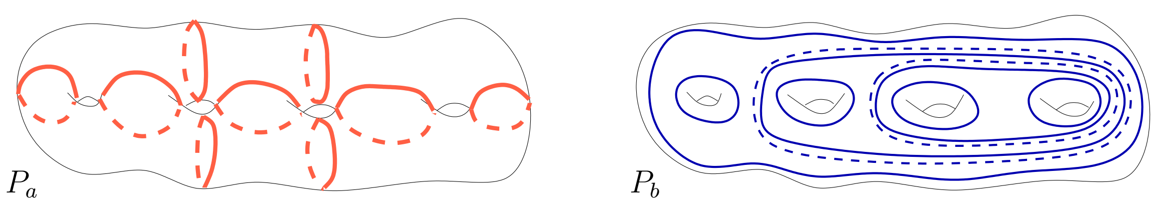

FIGURE 1.9. Pair of pants decompositions (P_a, P_b).

Definition 1.25 (*Double pants decomposition*)**.** A *double pants decomposition* $DP = (P_a, P_b)$ is a pair of pants decompositions P_a and P_b of the same surface such that the Lagrangian planes $\mathcal{L}_a = \mathcal{L}(P_a)$ and $\mathcal{L}_b = \mathcal{L}(P_b)$ spanned by these pants decompositions are in general position.

Definition 1.26 (*Handle twists*)**.** Given a double pants decomposition $DP = (P_a, P_b)$ we define an additional transformation which may be performed if P_a and P_b contain the same curve $a_i = b_i$ separating the same handle $\mathfrak{h}$, see Fig. 1.10. Let $a \in \mathfrak{h}$ and $b \in \mathfrak{h}$ be the only curves of P_a and P_b contained in $\mathfrak{h}$. Then a *handle twist in* $\mathfrak{h}$ is a Dehn twist along a or along b in any of two directions (see Fig. 1.10(b)).

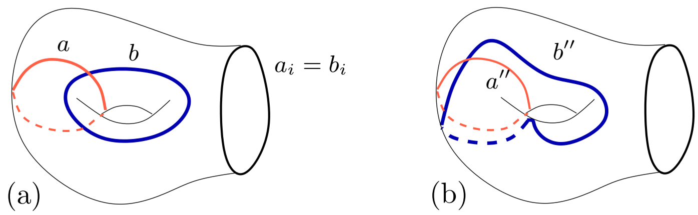

FIGURE 1.10. Handle twists: (a) handle (double self-folded pair of pants); (b) the same handle after one of the four possible handle twists.

Definition 1.27 (*Category of double pants decompositions*)**.** A category $\mathfrak{DP}_{g,0}$ of *double pants decompositions* of a genus g surface $S = S_{g,0}$ is the following:

Objects: double pants decompositions $DP = (P_a, P_b)$ of S.

Elementary morphisms:

- unzipped flips of P_i ($i \in \{a, b\}$);
- handle twists.

Other **morphisms** are compositions of elementary ones.

Remark 1.28. The index "$g, 0$" in the notation $\mathfrak{DP}_{g,0}$ is to underline that this category concerns surfaces of genus g *without* marked points.

Definition 1.29 ($\mathfrak{DP}$-*equivalence*)**.** Two double pants decompositions are $\mathfrak{DP}$-equivalent if there exists a morphism of $\mathfrak{DP}_{g,0}$ taking one of them to another.

Definition 1.30 (*Standard double pants decomposition, principle curves*)**.** A double pants decomposition (P_a, P_b) is *standard* if there exist g curves $c_1, \dots, c_g$ such that the following two conditions hold:

- $c_i \in P_a \cap P_b$;
- c_i cut out of S a handle $\mathfrak{h}_i$.

The set of curves $\{c_1, \dots, c_g\}$ in this case is a *set of principle curves* of (P_a, P_b).

See Fig. 1.11 for an example of a standard double pants decomposition.

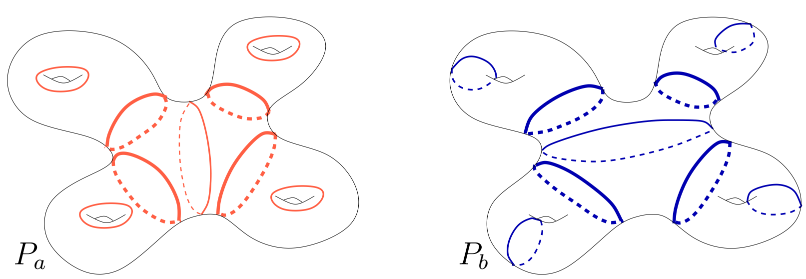

FIGURE 1.11. Standard double pants decomposition (P_a, P_b). The principle curves are shown by bold lines.

Definition 1.31 (*Admissible decomposition*)**.** A double pants decomposition (P_a, P_b) is *admissible* if it is $\mathfrak{DP}$-equivalent to some standard double pant decomposition.

Example 1.32. It is easy to check that a double pants decomposition (P_a, P_b) shown in Fig. 1.9 is admissible.

Remark 1.33. It is not clear if the set of all admissible double pants decompositions is smaller than the set of all double pants decompositions.

The set of admissible double pants decompositions is closed under the action of flips and handle twists, so we may define a subcategory of $\mathfrak{DP}_{g,0}$:

Definition 1.34 (*Category of admissible double pants decompositions*)**.** A category $\mathfrak{ADP}_{g,0}$ of *admissible double pants decompositions* of a genus g surface is the following:

Objects: admissible double pants decompositions $DP = (P_a, P_b)$ of a genus g surface.

Elementary morphisms:

- unzipped flips of P_i ($i \in \{a, b\}$);
- handle twists.

Other **morphisms** are compositions of elementary ones.

Our next aim is to prove that morphisms of $\mathfrak{ADP}_{g,0}$ act transitively on the objects of $\mathfrak{ADP}_{g,0}$. This is done in Section 2 for the case of $g = 2$ and in Section 3 for a general case.

2. Transitivity of morphisms in case $g = 2$

In this section we prove the Main Theorem for the case of surface of genus $g = 2$ containing no marked points. The proof is based on the following result 2.2 of Hatcher and Thurston [13].

Definition 2.1 (*S-moves*)**.** Let P be a pants decomposition of S and $a, c \in P$ be two curves such that c cuts out of S a handle $\mathfrak{h}$ and $a \in \mathfrak{h}$. An *S-move* of a pants decomposition P in a curve a is a substitution of a by a curve a', where $a' \in \mathfrak{h}$ is an arbitrary curve such that $|a \cap a'| = 1$.

Theorem 2.2 (A. Hatcher, W. Thurston [13], [12])**.** *Let $S_{g,n}$ be a surface of genus g with n holes. Any pants decomposition of $S_{g,n}$ can be transformed to any other pants decomposition of $S_{g,n}$ via flips and S-moves.*

Remark 2.3. In the initial paper of Hatcher and Thurston [13] the surface $S_{g,n}$ is supposed to be closed surface containing no marked points. This assumption is removed in [12].

Remark 2.4. Theorem 2.2 does not imply immediately transitivity of morphisms in $\mathfrak{ADP}_{g,0}$ (Theorem 3.19) since the set of handle twists in $\mathfrak{ADP}_{g,0}$ is much smaller than the set of S-moves in Theorem 2.2 (the former depends on the relative position of two decompositions).

Now we will prove several lemmas: Lemmas 2.6 and 2.7 will be used for the proof of transitivity both in case of $g = 2$ and in general case. Lemma 2.8 is specific for genus 2, its generalization requires more work for general genus.

Definition 2.5 (*Double S-move*)**.** Under the conditions of definition of a handle twist (Definition 1.26), a *double S-move* in $\mathfrak{h}$ is the move switching the curves a and b.

Lemma 2.6. *Double S-move is a morphism of $\mathfrak{D}\mathfrak{P}_{g,0}$.*

Proof. Any double S-move is a composition of 3 handle twists, see Fig. 2.1.

□

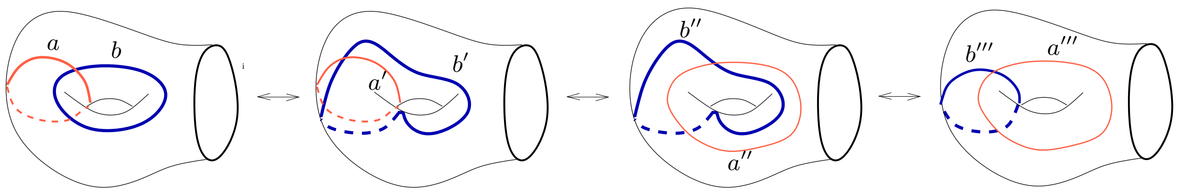

FIGURE 2.1. Double S-move as a composition of three handle twists.

Lemma 2.7. *Let (P_a, P_b) and (P'_a, P'_b) be two standard double pants decomposition containing the same handle $\mathfrak{h}$. Then $(P_a, P_b)|_{\mathfrak{h}}$ may be transformed to $(P'_a, P'_b)|_{\mathfrak{h}}$ by a sequence of handle twists in $\mathfrak{h}$ (where $(P_1, P_2)|_{\mathfrak{h}}$ is a restriction of the double pants decomposition to the handle $\mathfrak{h}$).*

Proof. Let a, b, a', b' be the curves of P_a, P_b, P'_a, P'_b contained in $\mathfrak{h}$. We need to find a composition $\overline{\psi}$ of handle twists in $\mathfrak{h}$ such that $\overline{\psi}(a) = a'$, $\overline{\psi}(b) = b'$.

First, suppose that $a = a'$. Then $\overline{\psi}$ is a composition of Dehn twists along a (this follows from the fact that $\langle h(a), h(b)\rangle = \langle h(a), h(b)\rangle$, where $\langle x, y\rangle \subset H_1(S, \mathbb{Z})$ is a sublattice spanned by x and y).

Next, suppose that $a' = b$. Then a double S-move interchanging a with b reduces the question to the previous case.

Now, suppose that $a' \neq a, b$. Then we have

$$h(a') = l_a h(a) + l_b h(b),$$

where $l_a, l_b \in \mathbb{Z}$ are coprime. Clearly, a non-zero homology class of a curve in $\mathfrak{h}$ defines a homotopy class. So, we only need to find a sequence $\overline{\psi}$ of morphisms of $\mathfrak{D}\mathfrak{P}_{g,0}$ taking a to any curve $x \in S'$ such that $h(x) = l_a h(a) + l_b h(b)$. Since l_a and l_b are coprime and a handle twist transforms $(h(a), h(b))$ into either $(h(a) \pm h(b), h(b))$ or $(h(a), h(b) \pm h(a))$, this sequence of handle twists do exists.

□

Lemma 2.8. *Let (P_a, P_b) be a standard double pants decomposition of $S_{2,0}$. Let $\varphi = \varphi_k \circ \cdots \circ \varphi_1$ be a sequence of flips of P_a such that $\varphi(P_a)$ is a standard decomposition. Then there exists a morphism η of $\mathfrak{A}\mathfrak{D}\mathfrak{P}_{2,0}$ such that $\eta((P_a, P_b))$ is a standard double pants decomposition and $\eta((P_a, P_b)) = (\varphi(P_a), P'_b)$ (where P'_b is arbitrary).*

Proof. Recall from Example 1.22 that for each two pants decompositions P_1 and P_2 connected by a sequence of flips there exists a sequence of flips connecting these pants decompositions and such that each flip in this sequence changes the type of pants decomposition from "self-folded" into "non-self-folded" or back (in the other word takes a standard pants decomposition to a non-standard one and back). This implies that it is sufficient to show the lemma for compositions $\varphi = \varphi_2 \circ \varphi_1$ of two flips.

If φ_2 changes the same curve as φ_1 does, then φ is a twist and the required composition η is shown in Fig. 2.2.

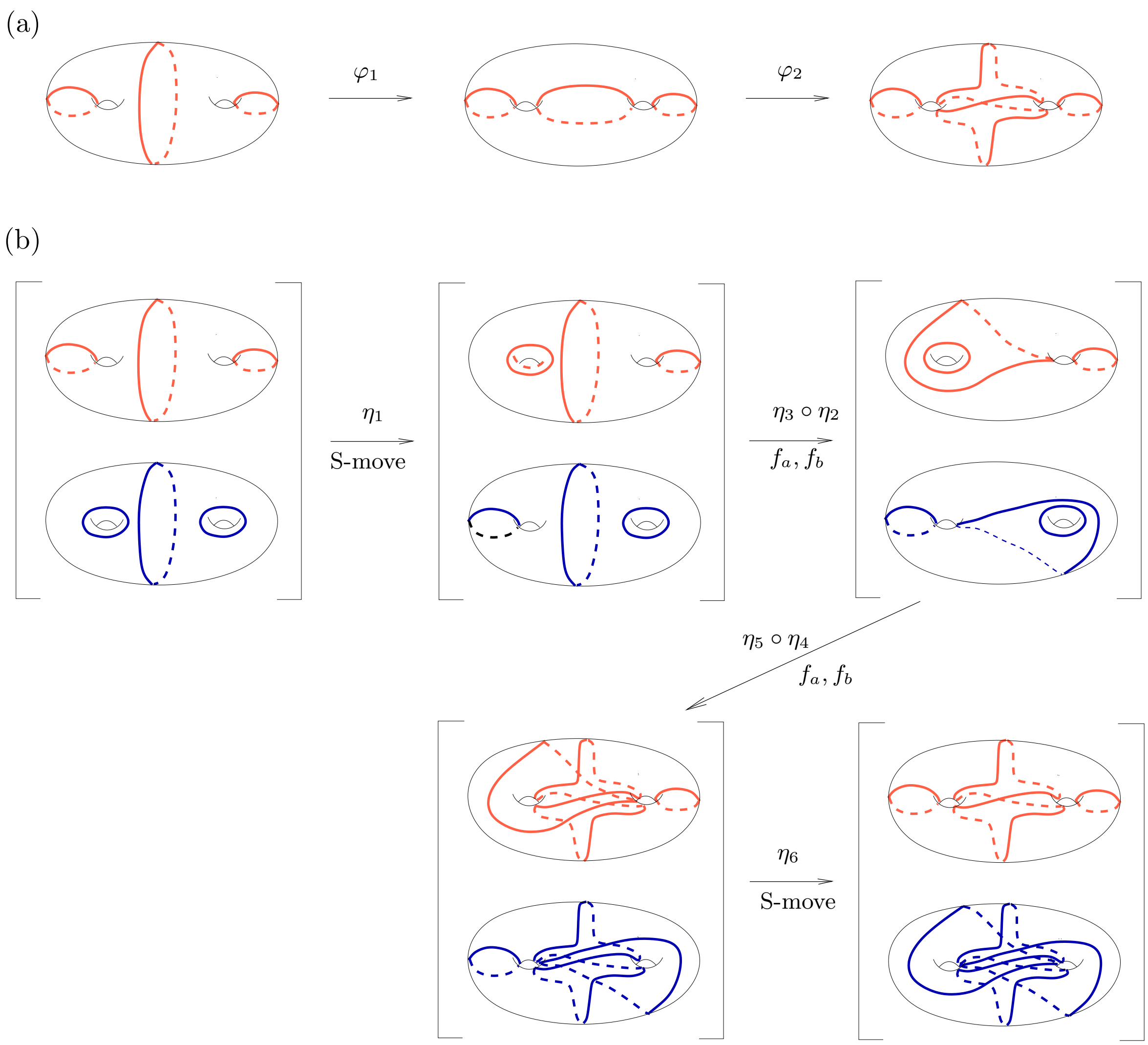

FIGURE 2.2. (a) Twist $\varphi(P_a)$: $\varphi = \varphi_2 \circ \varphi_1$; (b) Composition $\eta(P_a, P_b)$ for the twist φ.

If φ_1 and φ_2 change different curves then (modulo twists) φ looks like in Fig. 2.3(a) and the required composition η is shown in Fig. 2.3(b).

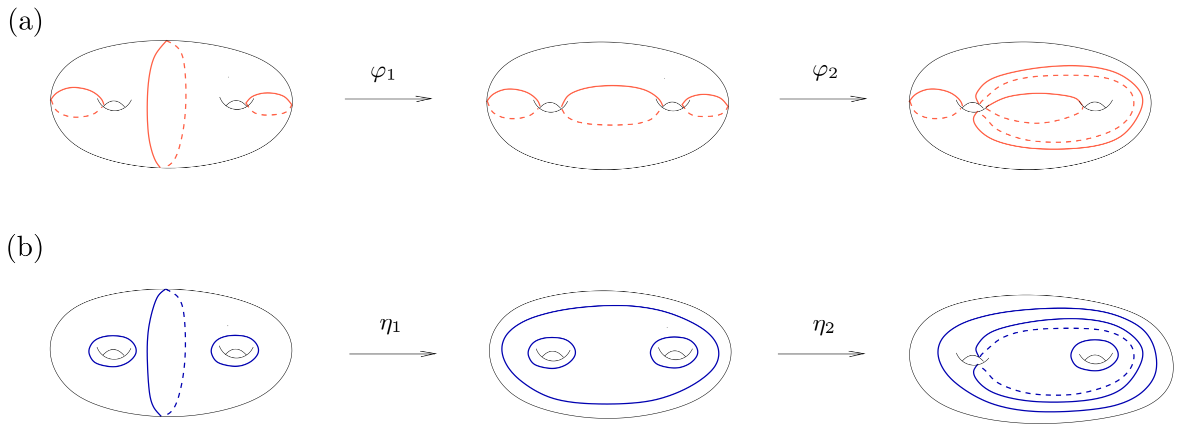

FIGURE 2.3. (a) Composition $\varphi(P_a)$ of two flips $\varphi = \varphi_2 \circ \varphi_1$; (b) Composition $\eta = \eta((P_a, P_b))$ for φ: $\eta(P_a) = \varphi(P_a)$, $\eta(P_b)$ is shown.

This completes the proof of Lemma 2.8. □

Theorem 2.9. *Morphisms of* $\mathfrak{ADP}_{2,0}$ *act transitively on the objects of* $\mathfrak{ADP}_{2,0}$.

Proof. By Definition 1.27, the objects of $\mathfrak{ADP}_{2,0}$ are admissible double pants decompositions, so, it is sufficient to prove the transitivity on the set of standard pants decompositions.

Let (P_a, P_b) and (P'_a, P'_b) be two standard double pants decompositions. If the principle curve of (P_a, P_b) coincides with one of (P'_a, P'_b) then Lemma 2.7 implies that (P_a, P_b) is $\mathfrak{DP}$-equivalent to (P'_a, P'_b).

Suppose that the principle curves of (P_a, P_b) and (P'_a, P'_b) are different. It is left to show that there exists a sequence $\eta = \eta_n \circ \cdots \circ \eta_1$ of morphisms of $\mathfrak{ADP}_{2,0}$ such that $\eta((P_a, P_b) = (P'_a, P''_b)$ and P''_b is arbitrary pants decomposition turning the pair (P'_a, P''_b) into a standard pants decomposition.

By Theorem 2.2, there exists a sequence $\psi = \psi_k \circ \cdots \circ \psi_1$ of flips and S-moves taking P_a to P'_a. By definition, in case of $g = 2$ an S-move is applicable only to standard double pants decompositions. This implies that the sequence ψ is a composition of several subsequences of two types:

- subsequences of flips, each subsequence takes a standard pants decomposition to another standard one;
- S-moves.

By Lemma 2.8, a subsequence of the first type may be extended to a morphism of $\mathfrak{ADP}_{2,0}$ taking a standard pant decomposition to another a standard one. By Lemma 2.7 any S-move of the component P_a may be realized as a sequence of morphisms of $\mathfrak{ADP}_{2,0}$ taking a standard double pants decomposition to a standard one. This implies that ψ may be extended to a morphism of $\mathfrak{ADP}_{2,0}$ and the theorem is proved.

□

3. Transitivity of morphisms in case of surfaces without marked points

In this section we adjust the proof of Theorem 2.9 to the case of higher genus.

3.1. **Preparatory lemmas.**

Lemma 3.1. *Let P be a pants decomposition and $\mathfrak{h}$ be a handle cut out by some $c \in P$. Let φ be an S-move in $\mathfrak{h}$. Then $\varphi = (\overline{\varphi}_1)^{-1} \circ \varphi_2 \circ \overline{\varphi}_1$ where $\overline{\varphi}_1$ is a sequence of flips preserving $\mathfrak{h}$ and φ_2 is an S-move of a standard pants decomposition.*

Proof. Consider the surface $S \setminus \mathfrak{h}$. By Theorem 2.2 it is possible to transform any pants decomposition of $S \setminus \mathfrak{h}$ to a standard one (clearly it may be done using flips only: one may use zipped flips with respect to any zipper system compatible with P). This defines the sequence $\overline{\varphi}_1$. Then we apply S-move in the handle $\mathfrak{h}$ and apply φ_1^{-1} to bring the pants decomposition of $S \setminus \mathfrak{h}$ into initial position.

□

In the proof of Theorem 2.9 we used the fact that in case of $g = 2$ S-moves are defined for standard decompositions only. This does not hold for $g > 2$. However, in view of Lemma 3.1 the following holds.

Lemma 3.2. *Let P_a and P'_a be standard pants decompositions and there exists a sequence $\varphi = \varphi_k \circ \cdots \circ \varphi_1$ of flips and S-moves such that $\varphi(P_a) = P'_a$. Then it is possible to choose φ in such a way that all S-moves in φ are applied to standard decompositions.*

Lemma 3.3. *Let $S_{0,g}$ be a sphere with g holes. Then any pants decomposition of $S_{0,g}$ may be transformed to any other pants decomposition of $S_{0,g}$ by a sequence of flips.*

Proof. The statement follows immediately from Theorem 2.2 since S-moves could not be applied to $S_{0,g}$.

□

3.2. **Transitivity in case of the same zipper system.**

Definition 3.4 (*(P_a, P_b) compatible with Z*)**.** A double pants decomposition (P_a, P_b) is *compatible* with a zipper system Z if P_a is compatible with Z and P_b is compatible with Z.

In this section we will prove that if (P_a, P_b) an (P'_a, P'_b) are two admissible double pants decompositions compatible with the same zipper system then (P_a, P_b) is $\mathfrak{DP}$-equivalent to (P'_a, P'_b).

Definition 3.5 (*Strictly standard decomposition*)**.** A double pants decomposition (P_a, P_b) is *strictly standard* if it is standard and $c \in \{P_a \cup P_b\} \setminus \{P_a \cap P_b\}$ if and only if c is contained inside some handle.

Example 3.6. The double pants decomposition in Fig. 1.11 is standard but not strictly standard.

Remark 3.7 (Strictly standard decompositions are minimal). Strictly standard decompositions could be also characterized by any of the following equivalent minimal properties:

- double pants decompositions with minimal possible number of distinct curves (i.e. with $4g-3$ curves);
- double pants decompositions with minimal possible number of intersections of curves (i.e. with g intersections).

We will not use these characterizations below.

To prove transitivity of morphisms of $\mathfrak{ADP}_{g,0}$ on the objects of $\mathfrak{ADP}_{g,0}$ it is sufficient to prove $\mathfrak{DP}$-equivalence of all standard double pants decompositions (since the objects of $\mathfrak{ADP}_{g,0}$ are admissible ones). Furthermore, any standard double pants decomposition is $\mathfrak{DP}$-equivalent to some strictly standard one in view of Lemma 3.3. So, it is sufficient to prove the transitivity of morphisms of $\mathfrak{ADP}_{g,0}$ on strictly standard double pants decompositions.

To prove the transitivity of morphisms of $\mathfrak{ADP}_{g,0}$ on strictly standard double pants decompositions we do the following:

- we show that each strictly standard double pants decomposition is compatible with some zipper system (see Proposition 3.8);
- we prove that morphisms of $\mathfrak{ADP}_{g,0}$ act transitively on strictly standard double pants decompositions compatible with a given zipper system (see Lemma 3.16);
- Finally, we show that for two different zipper systems Z and Z' we may find a sequence of zipper systems $Z = Z_1, Z_2 \dots, Z_k = Z'$, in which Z_i differs from Z_{i+1} by a twist along some curve c, $|c \cap Z_i| = 2$. We show (Lemma 3.17) that in this case morphisms of $\mathfrak{ADP}_{g,0}$ are sufficient to change Z_i to Z_{i+1}.

Proposition 3.8. *For any strictly standard double pants decomposition (P_a, P_b) there exists a zipper system Z compatible with (P_a, P_b).*

Proof. An intersection of the required zipper system $Z = \langle z_0, z_1, \dots, z_g \rangle$ with a handle looks as shown in Fig. 3.1.(a): if a_i and b_i are curves of P_a and P_b contained in a given handle $\mathfrak{h}_i$, $i = 1, \dots, g$, than $\mathfrak{h}_i$ contains a zipper z_i such that $|z_i \cap a_1| = 1$ and $|z_i \cap b_1| = 1$. The curve z_0 visits each of the handles and goes in $\mathfrak{h}_i$ along z_i.

The condition that (P_a, P_b) is strictly standard leads to the existence of appropriate z_0 outside of handles: to show this we build a *dual graph* of (P_a, P_b) substituting each pair of pants by an Y-shaped figure and each handle by a point (see Fig. 3.1.(b)). Since (P_a, P_b) is strictly standard we obtain a tree. Then z_0 is built as any curve on S which projects to a curve $\bar{z}_0$ going around the tree (more precisely, $\bar{z}_0$ determines the order in which z_0 visits the handles of (P_a, P_b)).

□

Proposition 3.9. *Let (P_a, P_b) be a standard double pants decomposition and Z be a zipper system compatible with (P_a, P_b). Then*

- *each of the handles of (P_a, P_b) contains exactly one curve of Z,*

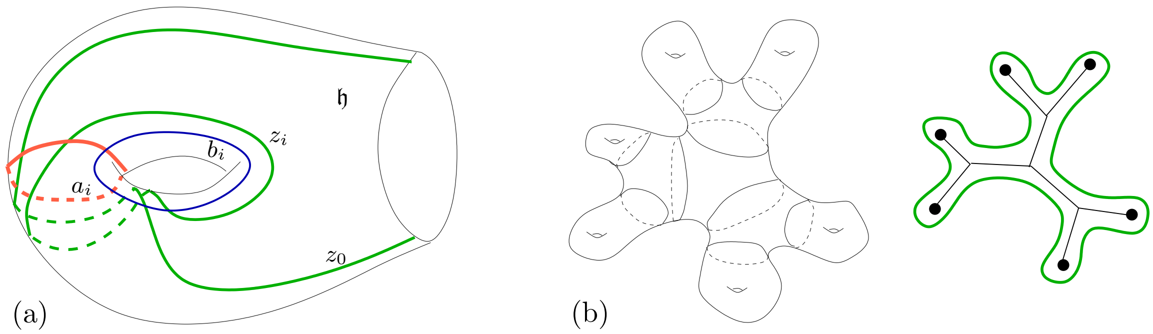

FIGURE 3.1. A zipper system for a given standard double pants decomposition: (a) behavior in a handle; (b) outside of the handles: a dual graph of a strictly standard pants decomposition.

- *if $z_0 \in Z$ does not belong entirely to any handle then z_0 visits each of g handles exactly once.*

Proof. Let c be a curve separating a handle $\mathfrak{h}$ in (P_a, P_b). Then each curve on S intersects c even number of times. By definition of a zipper system compatible with a pants decomposition, c is intersected by exactly one of the curves z_i. Notice that a pairs of pants dissected along a connected curve does not turn into a union of simply-connected components, which implies that there is a curve $z_j \in Z$, $z_j \neq z_i$ which intersects the handle $\mathfrak{h}$. Since $z_j \cap c = \emptyset$, z_j is contained in $\mathfrak{h}$. This proves the first statement of the proposition. The second statement follows from the fact that each curve c_i separating a handle in (P_a, P_b) should be intersected by some of z_i.

□

Definition 3.10 (*Principle zipper, cyclic order*)**.** Let (P_a, P_b) be a standard double pants decomposition and $Z = \langle z_0, z_1, \dots, z_g \rangle$ be a zipper system compatible with (P_a, P_b). Suppose that z_0 is the curve visiting all handles of (P_a, P_b). Then z_0 is a *principle zipper* of Z.

A *cyclic order* of Z is $(z_1, z_2, \dots, z_g)$ if an orientation of z_0 goes from $\mathfrak{h}_i$ to $\mathfrak{h}_{i+1}$, where $\mathfrak{h}_i$ is the handle containing z_i and i is considered modulo $g+1$ (more precisely, Z decomposes S into two $(g+1)$-holed spheres S_+ and S_-, so that we may choose a positive orientation of z_0 as one which goes in positive direction around S_+; for a definition of a cyclic order we choose the positive orientation of z_0).

Remark 3.11. The definition of cyclic order depends on the choice of S_+ among two subsurfaces. However, the definition of *the same cyclic order* in two standard double pants decompositions is independent of this choice (provided that the choice of S_+ is the same for both decompositions).

Proposition 3.12. *Let (P_a, P_b) and (P'_a, P'_b) be two standard pants decompositions compatible with the same zipper system $Z = \langle z_0, z_1, \dots, z_g \rangle$. Suppose that z_0 is the principle zipper of Z both for (P_a, P_b) and (P'_a, P'_b) and the cyclic order of Z is $(z_1, z_2, \dots, z_n)$ both for (P_a, P_b) and (P'_a, P'_b).*

Then the set of principle curves of (P_a, P_b) coincides with the set of principle curves of (P'_a, P'_b).

Proof. Let $c_1, \dots, c_g$ be the principle curves of (P_a, P_b) and $c'_1, \dots, c'_g$ be the principle curves of (P'_a, P'_b). Since the cyclic order of Z is the same for both double pants decompositions, we may assume that $z_0 \cap c_i = z_0 \cap c'_i$. Let S^+ and S^- be the connected components of $S \setminus Z$. Then $c_i \cap S^+$ separates from S^+ an annulus containing z_i as a boundary component. Clearly, the same holds for $c_i \cap S^-$ as well as for $c'_i \cap S^+$ and $c'_i \cap S^-$. This implies that c_i is homotopy equivalent to c'_i (more precisely, there exists an isotopy of c_i to c'_i with the fixed points $c_i \cap z_0 = c'_i \cap z_0$).

□

Corollary 3.13. *In assumptions of Proposition 3.12, (P_a, P_b) may be transformed to (P'_a, P'_b) by morphisms of $\mathfrak{ADP}_{g,0}$ preserving the principle curves of the standard pants decomposition.*

Proof. This follows from Proposition 3.12, Lemma 2.7 and Lemma 3.3. □

Corollary 3.13 implies that a standard pants decomposition is determined (modulo action of morphisms of $\mathfrak{ADP}_{g,0}$) by the set of principle curves. Thus, it makes sense to consider a set of principle curves itself, independently of the complete pants decomposition.

Definition 3.14. A zipper system Z is *compatible with a set of principle curves* if $Z = \langle z_0, z_1, \dots, z_g \rangle$ where z_0 visits each handle exactly once and each of the handles contain exactly one of z_i, $i = 1, \dots, g$.

In other words, Z is compatible with a set of principle curves if and only if it is compatible with some standard pants decomposition containing this set of principle curves.

Proposition 3.15. *Let (P_a, P_b) and (P'_a, P'_b) be two standard pants decompositions compatible with the same zipper system $Z = \langle z_0, z_1, \dots, z_g \rangle$. Suppose that z_0 is the principle zipper of Z both for (P_a, P_b) and (P'_a, P'_b). Then (P_a, P_b) is $\mathfrak{DP}$-equivalent to (P'_a, P'_b).*

Proof. By Proposition 3.12 together with Corollary 3.13 the proposition is trivial unless the cyclic order of Z is different for the cases of (P_a, P_b) and (P'_a, P'_b). It is shown in Fig. 3.2 that the transposition of two neighboring zippers z_i and z_{i+1} in the cyclic order of Z may be realized by morphisms of $\mathfrak{ADP}_{g,0}$.

In more detail, in Fig. 3.2, left we show (a part of) a zipper system Z compatible with a set of principle curves. In view of Lemma 2.7, all double pants decompositions of a handle are $\mathfrak{DP}$-equivalent, so we may choose any of them, say one denoted by $(P_a^{(1)}, P_b^{(1)})$ in Fig. 3.2 (we draw only the front, " visible" part of the decomposition, the non-visible part completely repeats it). Applying two flips (one for P_a and one for P_b) we obtain a double pants decomposition $(P_a^{(2)}, P_b^{(2)})$, and then after two more flips we obtain a standard double pants decomposition $(P_a^{(3)}, P_b^{(3)})$. Choosing appropriate curves in the handles of $(P_a^{(3)}, P_b^{(3)})$ (we use handle twists for that) we turn it into a standard double pants decomposition compatible with Z. Notice that the cyclic order in Z is changed: in the initial decomposition (an orientation of) the principle zipper z_0 visits first the handle containing z_1 and then the handle containing z_2, while in the final decomposition the same orientation of z_0 visits the handles in reverse order. So, the transposition of two adjacent (in the cyclic order) handles is realizable by the morphisms of $\mathfrak{ADP}_{g,0}$.

Since the permutation group is generated by transpositions of adjacent elements, the proposition is proved.

□

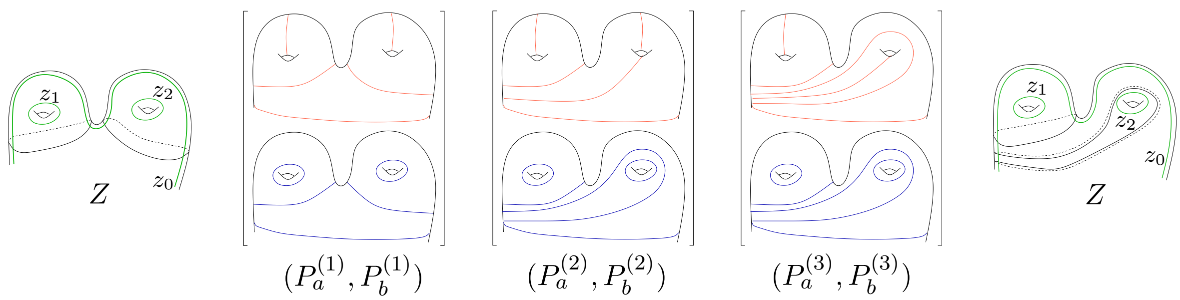

FIGURE 3.2. Transposition in a cyclic order is realizable by morphisms of $\mathfrak{ADP}_{g,0}$.

Lemma 3.16. *Let (P_a, P_b) and (P'_a, P'_b) be two standard pants decompositions compatible with the same zipper system Z. Then (P_a, P_b) is $\mathfrak{DP}$-equivalent to (P'_a, P'_b).*

Proof. By Proposition 3.15, the lemma is trivial if the principle zipper of Z is the same for (P_a, P_b) and (P'_a, P'_b). So, we only need to show that the morphisms of $\mathfrak{DP}_{g,0}$ allow to change the principle zipper in Z. We will show that it may be done by flips only.

Let Z be a zipper system (see Fig. 3.3) compatible with a set $\bar{c}$ of principle curves. Let (P_a, P_b) be a standard double pants decomposition containing this set of principle curves. Let $\bar{c}'$ be another set of principle curves compatible with Z and such that z_0 is not a principle zipper (see Fig. 3.3, down). We choose a standard double pants decomposition (P'_a, P'_b) with a set of principle curves $\bar{c}'$, see Fig. 3.3 (to keep the figure readable we do not draw the curves of (P'_a, P'_b) decomposing $S \setminus \cap_{i=1}^{g} \mathfrak{h}_g$). To prove the lemma it is sufficient to show that P_a is flip-equivalent to P'_a and P_b is flip-equivalent to P'_b.

The fact that P_a is flip-equivalent to P'_a follows from Corollary 1.12. Indeed, both P_a and P'_a are compatible with Z and clearly belong to the same Lagrangian plane, so P_a is flip-equivalent to P'_a as objects of $\mathfrak{P}_g(Z, \mathcal{L})$.

It is left to show that P_b is flip-equivalent to P'_b. We will do that by induction based on the following statement:

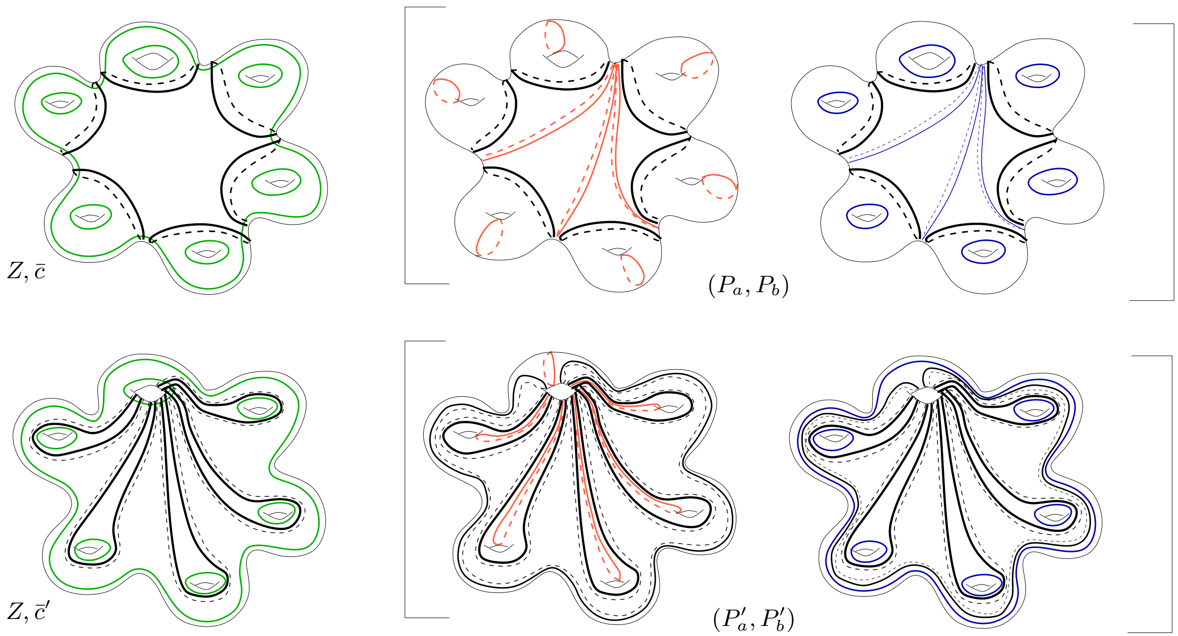

FIGURE 3.3. Principle zipper may be changed by morphisms of $\mathfrak{ADP}_{g,0}$.

Claim. *Let P and P' be two pants decompositions in the same Lagrangian plane, let $\{c_1, \dots, c_g\} \subset P'$ be homologically non-trivial curves. Let P'' be a pants decompositions flip-equivalent to P and containing $c_1, \dots, c_g$. Then P' is flip-equivalent to P.*

To prove the claim consider $S' = S \setminus \{\cup_{i=1}^{g} c_i\}$. Since $h(c_i) \neq 0$ the surface S' is a sphere with $2g$ holes. Thus, by Lemma 3.3 flips act transitively on pants decomposition of S'. This implies that P' is flip equivalent to P'' which is by assumption flip equivalent to P, and the claim is proved.

Denote by d_i a curve shown in Fig. 3.4(a), so that $d_0 = c_g$. To show that P_b is flip-equivalent to P'_b we demonstrate that some pants decomposition P_i containing the curves $c_1, c_2, \dots, c_{g-1}, d_i$ is flip-equivalent to some pants decomposition P_{i+1} containing $c_1, c_2, \dots, c_{g-1}, d_{i+1}$, where $i = 0, 1, \dots, g-1$. Then applying the Claim several times we will see that P_b is flip-equivalent to P'_b. A pants decomposition shown in Fig. 3.4(b) contains $c_1, c_2, \dots, c_{g-1}$ and both d_i and d_{i+1}, so any pants decomposition containing $c_1, c_2, \dots, c_{g-1}, d_i$ is flip-equivalent to any pants decomposition containing $c_1, c_2, \dots, c_{g-1}, d_{i+1}$, and the lemma is proved.

□

3.3. **Proof of transitivity in general case.**

Lemma 3.17. *Let Z be a zipper system, let σ be an involution preserving Z pointwise and let c be a curve satisfying $|Z \cap c| = 2$, $\sigma(c) = c$. Denote by T_c a Dehn twist along*

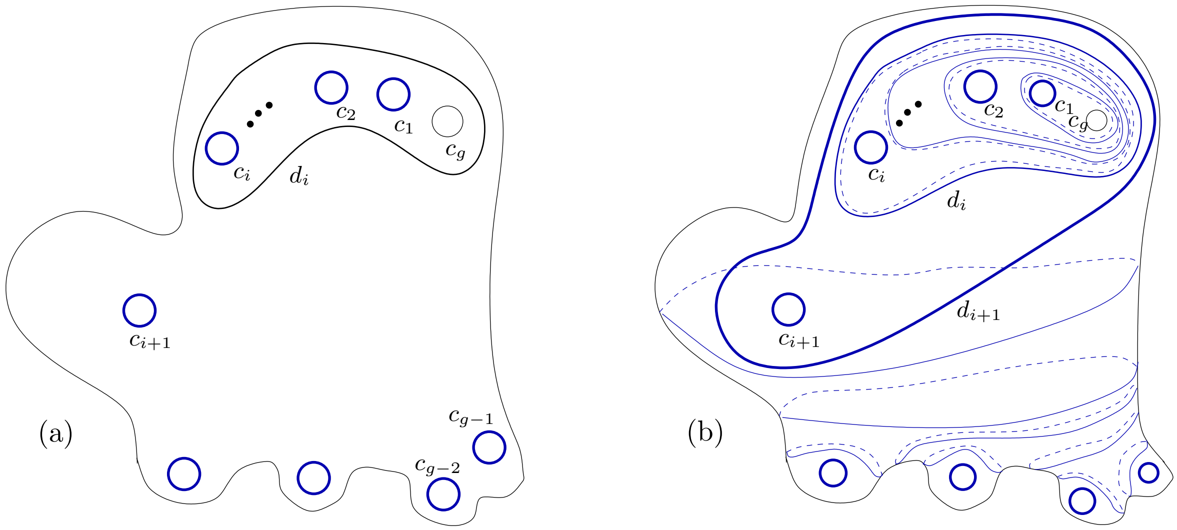

FIGURE 3.4. Proving that P_b is flip-equivalent to P'_b.

c. Then there exist standard pants decompositions (P_a, P_b) *and* (P'_a, P'_b) *compatible with* Z *and* $Z' = T_c(Z)$ *respectively and* $\mathfrak{DP}$*-equivalent to each other.*

The same statement holds for Z *and* $Z'' = T_c^m(Z)$ *for any positive integer degree* m.

Proof. Since $|Z \cap c| = 2$, the curve c either intersects twice the same curve $z_0 \in Z$ or have single intersections with two distinct curves $z_1, z_2 \in Z$. Consider this two cases.

Suppose that $|c \cap z_0| = 2$ and $z_0 \in Z$. Since $\sigma(c) = c$, the curve c decomposes S into two parts, as in Fig. 3.5(a). Choose a strictly standard double pants decomposition (P_a, P_b) containing c, compatible with Z and such that z_0 is a principle zipper (see Fig. 3.5(b)). Then (P_a, P_b) is also compatible with $Z' = T_c(Z)$.

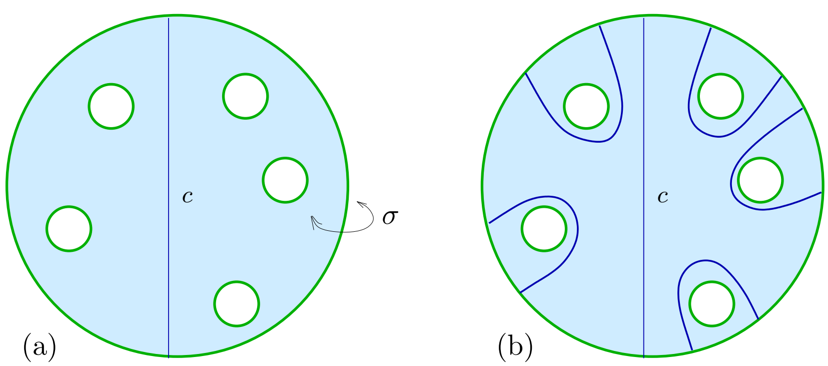

FIGURE 3.5. Case $|c \cap z_0| = 2$: (a) c decomposes S; (b) strictly standard double pants decomposition (only principle curves and c are drown).

Suppose that $|c \cap z_1| = |c \cap z_2| = 1$, $z_1, z_2 \in Z$. Choose (P_a, P_b) so that z_1 and z_2 are not principle zippers and z_1 and z_2 are two neighboring curves in the cyclic order.

Fig. 3.6 contains a sequence of morphisms of $\mathfrak{DP}_{g,0}$ taking (P_a, P_b) to a standard double pants decomposition compatible with $Z' = T_c(Z)$.

The statement concerning $Z'' = T_c^m(Z)$ follows immediately after multiple application of the initial statement.

□

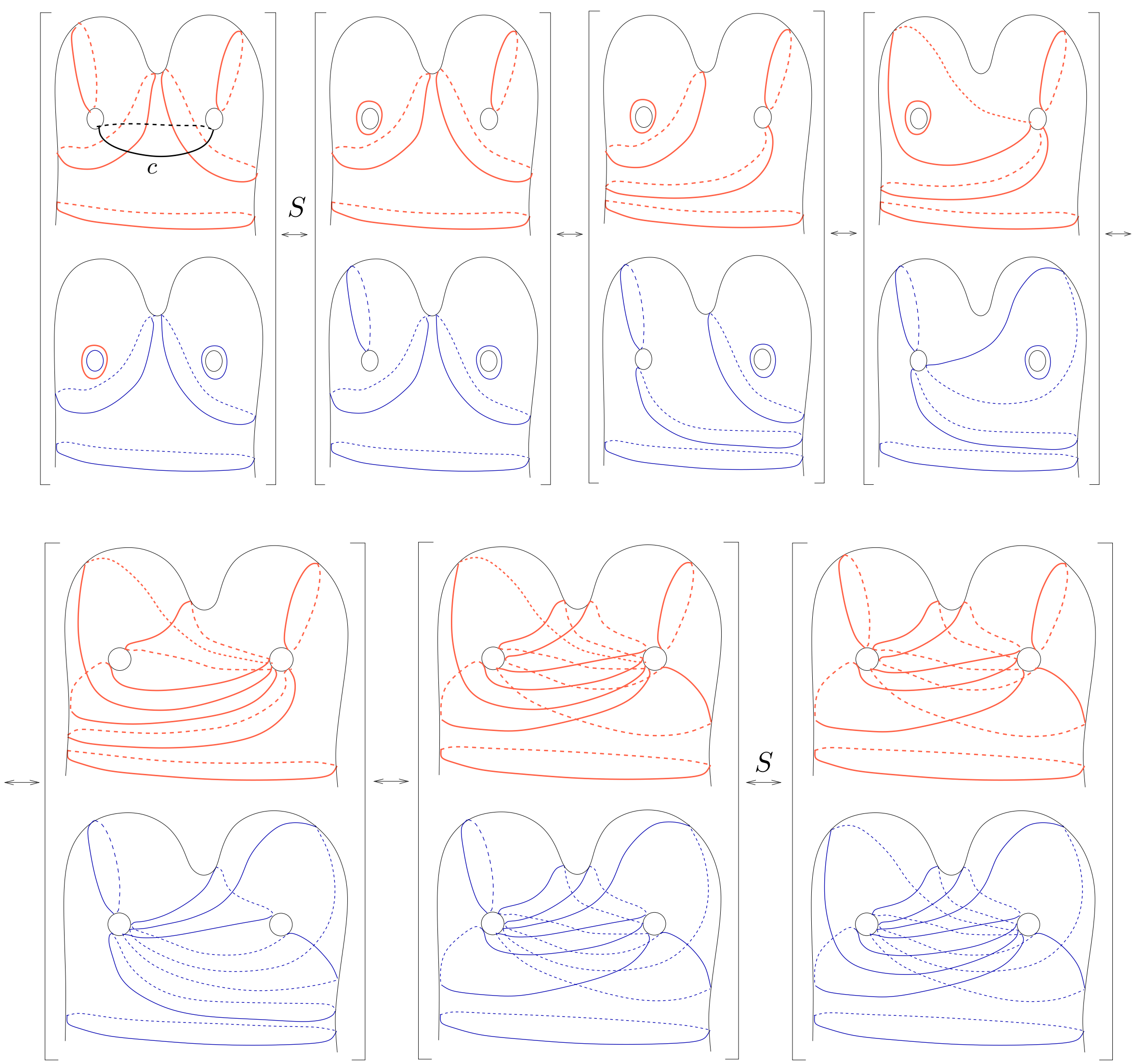

FIGURE 3.6. Twist T_c along c as a composition of elementary morphisms of $\mathfrak{DP}_{g,0}$: the steps labeled by "S" are double S-moves, other steps are compositions of two flips, one in P_a another in P_b. The final figure coincides with the initial one twisted around c.

Definition 3.18 ($\mathfrak{DP}$-*equivalent standard pants decompositions*)**.** A standard pants decomposition P_a is $\mathfrak{DP}$-*equivalent* to a standard pants decomposition P_a' if there exist standard pants decompositions P_b and P'_b such that (P_a, P_b) and (P'_a, P'_b) are standard and $\mathfrak{DP}$-equivalent to each other.

Theorem 3.19. *Let $S = S_{g,0}$ a surface without marked points. Then morphisms of $\mathfrak{ADP}_{g,0}$ act transitively on the objects of $\mathfrak{ADP}_{g,0}$.*

Proof. It is clear from the definition of admissible pants decomposition that it is sufficient to prove transitivity for standard pants decompositions only. By Lemmas 2.7 and 3.3 morphisms of $\mathfrak{DP}_{g,0}$ act transitively on standard pants decompositions including the same principle curves. This implies that it is sufficient to show that any two standard pants decompositions P_a and P'_a are $\mathfrak{DP}$-equivalent.

By Theorem 2.2 there exists a sequence $\{\varphi_i\}$ of flips and S-moves taking P_a to P'_a . In view of Lemma 3.1 we may assume that in this sequence S-moves are applied only to the standard double pants decompositions. Lemma 2.7 treats the S-moves in standard pants decompositions, thus, we may assume that $\{\varphi_i\}$ consists entirely of flips of P_a.

If Z is a zipper system compatible with (P_a, P_b) and all flips in $\{\varphi_i\}$ are zipped flips (with respect to Z) then there is nothing to prove. Our idea is to decompose the sequence $\{\varphi_i\}$ into several subsequences

$$\{\varphi_i\} = \{\{\varphi_i\}_1, \dots, \{\varphi_i\}_k\}$$

such that in j-th subsequence all flips are zipped flips with respect to the same zipper system Z_j. Denote by P_a^j the pants decomposition obtained from P_a after application of the first j subsequences of flips, $P_a^0 = P_a$, $P_a^k = P'_a$. Clearly, P_a^j is compatible both with Z_j and Z_{j+1}. So, we may use zipped flips (with respect to Z_j) to transform P_a^j to some standard pants decomposition $P_a^j(Z_j)$ compatible with Z_j. Similarly, we may transform P_a^j to some standard pants decomposition $P_a^j(Z_{j+1})$ compatible with Z_{j+1} (see Fig. 3.7).

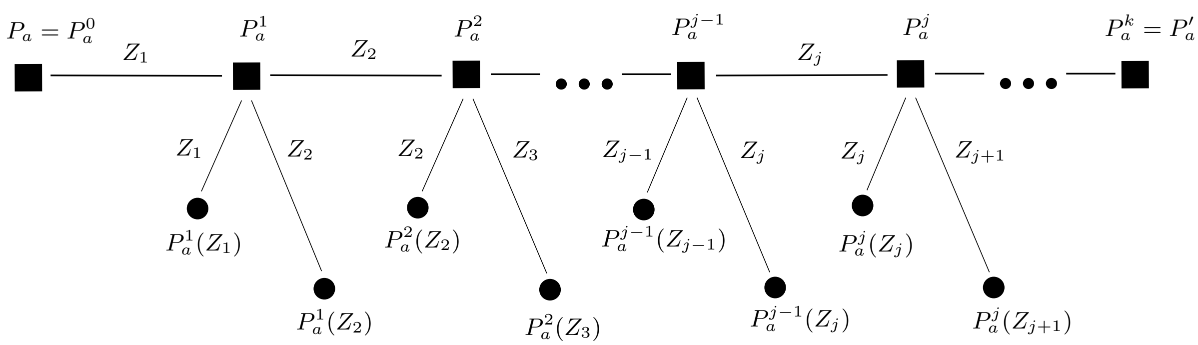

FIGURE 3.7. Proof of transitivity for $g > 2$.

Now, for proving $\mathfrak{DP}$-equivalence of P_a and P'_a we only need to show two facts:

(1) $P_a^{j-1}(Z_j)$ is $\mathfrak{DP}$-equivalent to $P_a^j(Z_j)$ for $0 < j \le k$;
(2) $P_a^j(Z)$ is $\mathfrak{DP}$-equivalent to $P_a^j(Z_{j+1})$ for $0 < j < k$.

The first of this facts follows immediately from Lemma 3.16 since both $P_a^{j-1}(Z_j)$ and $P_a^j(Z_j)$ are compatible with the same zipper system Z_j.

So, we are left to prove the second fact. By Lemma 1.18, we may assume $Z_{j+1} = T_c^m(Z_j)$, where T_c is a Dehn twist along a curve $c \in P_a^j$. By Remark 1.6, there exists an involution σ preserving Z pointwise and such that $\sigma(c_i) = c_i$ for each curve $c_i \in P_a$, so we may apply Lemma 3.17. By Lemma 3.17 there exist a pair of $\mathfrak{DP}$-equivalent double pants decompositions, one compatible with Z_j and another compatible with Z_{j+1}. In view of Lemma 3.16, this implies that $P_a^j(Z)$ is $\mathfrak{DP}$-equivalent to $P_a^j(Z_{j+1})$.

□

4. Surfaces with marked points

In this section we generalize Theorem 3.19 and Theorem 5.3 to the case of surfaces with boundary or for surfaces with marked points.

Remark 4.1. We prefer to work with surfaces with holes instead of surfaces with marked points (the latter may be obtained from the former by contracting the boundary components). Since we never consider the boundary and the neighbourhood or the boundary, this makes no difference for our reasoning. In case of marked surfaces one need to extend the definition of a "pair of pants": for marked surfaces a pair of pants is a sphere with 3 "features", each of the features may be either a hole or a marked point.

Let $S_{g,n}$ be a surface of genus g with n holes. Definitions of pants decomposition and double pants decomposition remain the same as in case of $n = 0$.

Definition 4.2 (*Standard double pants decomposition of an open surface*)**.** A double pants decomposition (P_a, P_b) of $S_{g,n}$ is *standard* if P_a and P_b contain the same set of g handles (where a *handle* is a surface of genus 1 with 1 hole).

A standard double pants decomposition is *strictly standard* if any curve of (P_a, P_b) either belongs to both of P_a and P_b or is contained in some of g handles.

In the same way as in case of $n = 0$ we define: flips and handle twists, admissible double pants decompositions and the category $\mathfrak{ADP}_{g,n}$ of admissible pants decompositions. We will consider objects of $\mathfrak{ADP}_{g,n}$ as surfaces with holes, but contracting the boundaries of the surface we obtain the equivalent category whose objects are surfaces with marked points.

Before proving the transitivity of morphisms on the objects of $\mathfrak{ADP}_{g,n}$, we reprove Theorem 2.2 for the case of open surfaces (although the result is contained in [12], it is convenient to have also the prove here, since the same idea will work for the case of double pants decompositions). More precisely, we derive the result for open surfaces from the result for closed ones.

Given a pants decomposition of an open surface, S-moves are defined in the same way as for closed surfaces.

Lemma 4.3 (A. Hatcher, [12])**.** *Flips and S-moves act transitively on pants decompositions of $S_{g,n}$.*

Proof. The proof is by induction on the number of holes n. The base ($n = 0$) is Theorem 2.2 for the case of closed surfaces. Suppose that the lemma holds for $n = k$ and consider a surface $S_{g,k+1}$.

We consider simultaneously two surfaces, $S_{g,k+1}$ and $S_{g,k}$, where the latter is thought as a copy of $S_{g,k+1}$ with a disk attached to the boundary of $(k+1)$-th hole. Each curve on $S_{g,k+1}$ turns into a curve on $S_{g,k}$ (but two distinct curves may became the same). Any pants decomposition of $S_{g,k+1}$ turns into a pants decomposition of $S_{g,k}$ containing one pair of pants less than the initial ones. More precisely, each pants decomposition $S_{g,k+1}$ contains a unique pair of pants one of whose boundary components is $(k+1)$-th hole. This pair of pants disappears in $S_{g,k}$ (when the hole is removed, two other boundary components turn in the same curve). To go back from a pants decomposition of $P_{g,k}$ of $S_{g,k}$ to a pants decomposition of $S_{g,k+1}$ we need only to choose one of the curves $c \in P_{g,k}$ and attach in the place of c a thin strip containing a hole.

A flip as in Fig. 4.1 allow to change the curve $c \in P(S_{g,k})$ where the holed strip as attached (this flip in the decomposition of $S_{g,k+1}$ does not change the decomposition of $S_{g,k}$). Applying a sequence of flips we may move the strip to any given curve of the pants decomposition of $S_{g,k}$. Furthermore, for any flip or S-move in the decomposition of $S_{g,k}$ we may apply similar transformation in $S_{g,k+1}$ (we only need to check in advance that the holed strip is not attached to the curve affected by the transformation, in the latter case, first we need to change the "stripped" curve). So the transitivity of flips and S-moves on pants decompositions of $S_{g,k+1}$ follows now from transitivity for $S_{g,k}$ and a fact that flips allow as to choose the stripped curve arbitrary.

□

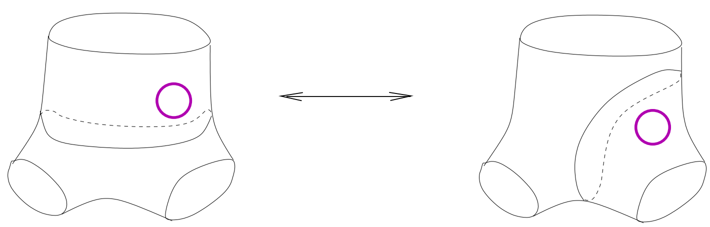

FIGURE 4.1. Flip changing the curve where the holed strip is attached.

Definition 4.4 (Simple double pants decomposition). A double pants decomposition (P_a, P_b) is *simple* if $|a_i \cap b_j| \le 1$ for all curves $a_i \in P_a$, $b_j \in P_b$.

Theorem 4.5. *Morphisms of* $\mathfrak{ADP}_{g,n}$ *act transitively on the elements of* $\mathfrak{ADP}_{g,n}$.

Proof. The proof is by induction on the number of holes n. The base ($n = 0$) is proved in Theorem 3.19. Suppose that the theorem holds for $n = k$ and consider a surface $S_{g,k+1}$.

Following the proof of Lemma 4.3, we consider simultaneously double pants decompositions (P_a, P_b) of $S_{g,k+1}$ and $(\widetilde{P}_a, \widetilde{P}_b)$ of $S_{g,k}$. Each of two pants decompositions of

$S_{g,k+1}$ differs from corresponding pants decomposition of $S_{g,k}$ by a holed strip attached in some of curves (so that $(k+1)$-th hole in $S_{g,k+1}$ is lying in the intersection of two strips). The transitivity for the case of $S_{g,k}$ shows that flips and handle twists are sufficient to transform the double pants decomposition of $S_{g,k+1}$ to one which projects to any given double pants decomposition of $S_{g,k}$. As it is shown in the proof of Lemma 4.3, flips also allow to choose the curves of $(\widetilde{P}_a, \widetilde{P}_b)$ where the holed strips are attached. This implies transitivity of morphisms of $\mathfrak{ADP}_{g,n}$ on all admissible double pants decomposition of $S_{g,k+1}$ which project to simple double pants decomposition of $S_{g,k}$.

The reasoning above does not work for double pants decomposition of $S_{g,k+1}$ which do not project to simple double pants decompositions of $S_{g,k}$: indeed, in this case we may choose the double pants decomposition $(\widetilde{P}_a, \widetilde{P}_b)$ of $S_{g,k}$ and the curves a_i and b_j where the strips are attached, but in the case $|a_i \cap b_j| > 1$ we are not able to choose which of the intersections of the strips contains the hole.

To adjust the proof to this case, notice that a strictly standard double pants decomposition of $S_{g,k+1}$ projects to a strictly standard double pants decomposition of $S_{g,k+1}$, which is simple. This implies transitivity on strictly standard pants decompositions, and hence, on standard ones. In view of definition of admissible double pants decomposition (as one which may be obtained from a standard one), we have transitivity for all admissible double pants decompositions of $S_{g,k+1}$.

Thus, given the statement for $n = k$ we have proved it for $n = k + 1$, hence, the theorem holds for any integer $n \geq 0$.

□

Theorem 4.5 completes the proof of the Main Theorem.

5. Flip-twist groupoid and mapping class group

All morphisms of $\mathfrak{DP}_{g,0}$ are reversible, so the morphisms form a groupoid acting on the objects of $\mathfrak{DP}_{g,0}$. We will call it a *flip-twist groupoid* and denote FT.

In general, elements of FT change the topology of the double pants decomposition, so FT is not a group. However, there are some elements which preserve the topology. Clearly, these elements belong to mapping class group $MCG(S)$ of the surface (recall that a *mapping class group* $MCG(S)$ of a surface S is a group of homotopy classes of self-homeomorphisms of S with a composition as a group operation). In fact, all elements of mapping class group occur to belong to FT.

We consider the curves of a double pants decomposition as *labeled curves*, so that a symmetry of S interchanging the curves would not be trivial.

Lemma 5.1. *Let $S = S_{g,n}$ be a genus g surface with $n \geq 0$ marked points, where $2g + n > 2$. Then there exists an admissible double pants decomposition (P_a, P_b) of P such that if $g \in MCG(S)$ fixes (P_a, P_b) then $g = id$.*

Proof. Let (P_a, P_b) be any admissible double pants decomposition such that no curve belongs to both P_a and P_b. For example, such a decomposition may be constructed as in Fig.1.9 for a closed surface without marked points and as in Fig. 5.1 in general case. We will show that if $g \in MCG(S)$ fixes (P_a, P_b) then $g = id$.

Suppose there exists an element $g \in MCG(S)$ such that $g((P_a, P_b)) = (P_a, P_b)$, $g \neq id$. Since $g(P_a) = P_a$, g is a composition of Dehn twists along the curves contained in P_a. Since the curves do not intersect each other, the Dehn twists do commute. Let $a_1 \in P_a$ be any curve whose twist contributes to g. By assumption, $a_1 \notin P_b$. Hence, there exists a curve $b_1 \in P_b$ such that $b_1 \cap a_1 \neq \emptyset$ (otherwise P_b is not a maximal set of non-intersecting curves in S). Then $g(b_1) \neq b_1$, so $g(P_b) \neq P_b$ which contradicts to the assumption. The contradiction implies the lemma.

□

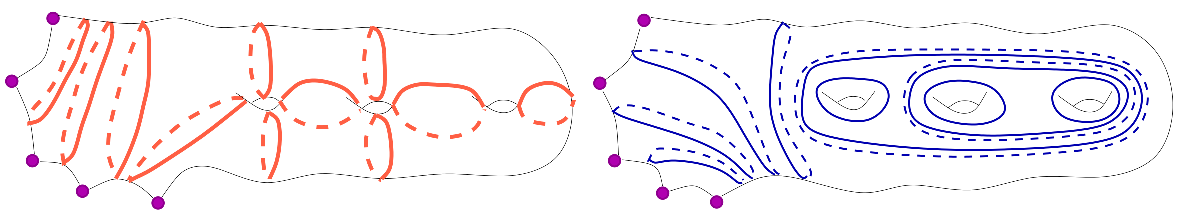

FIGURE 5.1. Double pants decomposition (P_a, P_b) of surface with marked points such that no curve belongs both to P_a and P_b.

Definition 5.2 (*Category of topological surfaces*)**.** A category $\mathfrak{Top}_{g,n}$ of *topological surfaces* is one whose **objects** are topological surfaces of genus g with n marked points and whose **morphisms** are elements of mapping class group $MCG(S)$.

Theorem 5.3. *For any pair (g, n) such that $2g+n > 2$ the category $\mathfrak{ADP}_{g,n}$ contains a subcategory $\mathfrak{TopDP}_{g,n}$ which is isomorphic to $\mathfrak{Top}_{g,n}$.*

Proof. Consider an admissible double pants decomposition DP described in Lemma 5.1. Consider the orbit $MCG(DP)$ of DP under the action of the mapping class group. It follows from Lemma 5.1 that for $g_1, g_2 \in MCG$, $g_1 \neq g_2$ one has $g_1(DP) \neq g_2(DP)$. Let $\mathfrak{TopDP}_{g,n}$ be a subcategory of $\mathfrak{ADP}_{g,n}$ such that the objects of $\mathfrak{TopDP}_{g,n}$ are elements of the orbit $MCG(DP)$. The assumptions of Lemma 5.1 imply that the objects of $\mathfrak{TopDP}_{g,n}$ are in one-to-one correspondence with the objects of $\mathfrak{Top}_{g,n}$. Furthermore, Theorem 3.19 implies that for each two objects $x, y \in \mathfrak{TopDP}_{g,n}$ there exists a morphism $x \to y$. So, the morphisms of $\mathfrak{Top}_{g,n}$ and $\mathfrak{TopDP}_{g,n}$ are in one-to-one correspondence and we have an equivalence of two categories.

□

Remark 5.4. The special choice of the admissible pants decomposition in the proof of Theorem 5.3 is indispensable: for example, if we took a standard double pants decomposition DP then all Dehn twists along principle curves of DP act on DP trivially and the orbit $MCG(DP)$ gives a subcategory of $\mathfrak{ADP}_{g,n}$ isomorphic to some subcategory of $\mathfrak{Top}_{g,n}$.

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