Kronecker double series and the dilogarithm

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Abstract

In this article we give an explicit expression for the value of a certain Kronecker double series at any point of complex multiplication as a sum of dilogarithms whose arguments are values of some modular unit of higher level at the corresponding points. This result can be interpreted in the spirit of the Zagier conjecture. The special value of the Kronecker double series is equal to the value of the partial ζ-function of an ideal class for an order in an imaginary quadratic field. The values of the above mentioned modular unit belong to the ray class field corresponding to this order. So we get an explicit formula for the value of a partial ζ-function at $s = 2$ as a combination of dilogarithms of algebraic numbers.

1 Introduction.

We start by fixing notations.

1.1. Modular part.

1.1.1. First, we recall some standard facts about modular curves. We set $\mathcal{H} = \{\tau \in \mathbb{C} | \Im(\tau) > 0\}$, $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}_2(\mathbb{Z}) \setminus \{0\}$ acts by $M(\tau) = \frac{a\tau+b}{c\tau+d}$. Denote by $L = L_\tau$ the lattice in $\mathbb{C}$ generated by τ and 1, and denote by $E = E_\tau$ the corresponding elliptic curve $\mathbb{C}/L_\tau$. The matrix M defines a map of lattices: $M^\tau : L_{M(\tau)} \to L_\tau; w \to (c\tau + d)w$ and an isogeny $M_\tau : E_\tau \to E_{M(\tau)};\ \xi \to (ad - bc)(c\tau + d)^{-1}\xi$. If τ is a fixed point of M, then M_τ is a map of the curve E_τ onto itself. In this case we will omit the subscript τ in our notation.

A point ξ of E_τ defines a character χ_ξ of the lattice L_τ: $\chi_\xi(w) = \exp\left(\frac{2\pi i(w\overline{\xi}-\overline{w}\xi)}{\tau-\overline{\tau}}\right)$. This pairing is $\mathrm{Mat}_2(\mathbb{Z})$-invariant: $\chi_{M(\xi)}(w) = \chi_\xi(M(w))$.

1.1.2. Definition. *The second Kronecker double series $\mathcal{K}_2(\xi;\tau)$ is the C^∞-function on $\mathbb{C}\times\mathcal{H}$ defined by the convergent series*

$$\mathcal{K}_2(\xi;\tau) = \left(\frac{\tau-\overline{\tau}}{2\pi i}\right)^2 \sum_{w\in L}{}' \frac{\chi_\xi(w)}{|w|^4}. \qquad (1)$$

It is easy to check that this function is invariant with respect to the above-defined action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{C}\times\mathcal{H}$.

1.1.3. The Weierstrass $\wp$-function is defined by the convergent series

$$\wp(\xi;\tau) = \frac{1}{\xi^2} + \sum_{w\in L_\tau}{}' \left(\frac{1}{(w+\xi)^2} - \frac{1}{w^2}\right).$$

This function is elliptic.

1.2. Dilogarithms.

1.2.1. Definition. *The Euler dilogarithm $Li_2(z)$ is the multi-valued analytic function on $\mathbb{P}^1\setminus\{0,1,\infty\}$ defined as the analytic continuation of the series $\sum_{j\geq 1}\frac{z^j}{j^2}$ (this series converges if $|z|<1$).*

1.2.2. Definition. *The Bloch-Wigner dilogarithm $D_2(z)$ is the (single-valued) continuous function on $\mathbb{P}^1$, smooth on $\mathbb{P}^1\setminus\{0,1,\infty\}$, defined by the formula $D_2(z) = \Im(Li_2(z)) + \arg(1-z)\cdot\log|z|$ for $z\neq 0,1,\infty$, $D_2(0) = D_2(1) = D(\infty) = 0$.*

We can extend the function D_2 linearly to a function on the $\mathbb{Q}$-vector space $\mathbb{Q}[\mathbb{C}]$.

1.2.3. Define a map δ from $\mathbb{Q}[\mathbb{C}\setminus\{0,1\}]$ to $\bigwedge^2\mathbb{C}^*$ by the formula: $[x]\to x\wedge(1-x)$.

1.3. Results.

1.3.1. Main Theorem. *Let τ be a fixed point of $M = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \neq 0,1$. Set $m = \det(M)$, $n = \det(M-1)$. Then*

$$\begin{aligned} &-(m+1)(n+1)\frac{c(\tau-\overline{\tau})}{i}\mathcal{K}_2(0;\tau) \\ &= 4mnD_2\left(\frac{ad-bc}{c\tau+d}\right) + \sum_{\substack{\alpha\in\mathrm{Ker}(M)\setminus 0\\ \beta\in\mathrm{Ker}(M-1)\setminus 0}} \sum_{k=1}^m\sum_{l=1}^n D_2\left(\frac{\wp(k\alpha+l\beta)-\wp(\alpha)}{\wp(\beta)-\wp(\alpha)}\right); \end{aligned}$$

and the argument of the dilogarithm belongs to the kernel of the map δ:

$$\delta\left(4mn\times[c\tau+d]+\sum_{\alpha,\beta,k,l}\left[\frac{\wp(k\alpha+l\beta)-\wp(\alpha)}{\wp(\beta)-\wp(\alpha)}\right]\right)=0\in\wedge^2(\mathbb{C}^*\otimes_{\mathbb{Z}}\mathbb{Q}).$$

1.3.2. Our proof is based on introducing some new function $\mathcal{L}_{1,1}$, which we call the *elliptic* $(1,1)$*-logarithm*, which will be defined in Section 2. The Main Theorem follows from Theorems A and B below.

1.3.3. Theorem A. *Let τ, M, m , n be as in the Theorem* 1.3.1. *Then*

$$\frac{c(\tau-\overline{\tau})}{4i}\frac{(m+1)(n+1)}{mn}\mathcal{K}_2(0;\tau)=-D_2\left(\frac{ad-bc}{c\tau+d}\right)+\sum_{\substack{\alpha\in\mathrm{Ker}M\setminus 0\\ \beta\in\mathrm{Ker}(M-1)\setminus 0}}\mathcal{L}_{1,1}(\alpha,\beta,0). \tag{2}$$

1.3.4. Theorem B. *Let α and β be two nontrivial different torsion points on an elliptic curve E_τ, say $m\alpha=n\beta=0$ for some $m,n\in\mathbb{N}$. Then*

$$\sum_{k=1}^{m}\sum_{l=1}^{n}D_2\left(\frac{\wp(k\alpha+l\beta)-\wp(\alpha)}{\wp(\beta)-\wp(\alpha)}\right)=-2mn(\mathcal{L}_{1,1}(\alpha,\beta,0)+\mathcal{L}_{1,1}(-\alpha,\beta,0)). \tag{3}$$

1.4. General remarks, structure of the article and acknowledgement.

1.4.1. The fact that the value of the Kronecker double series $\mathcal{K}_2$ at a CM-point can be expressed as a combination of dilogarithms is not new. It can be derived from a result of C. Deninger in the following way. Any CM-point τ defines the ring R of endomorphisms of the corresponding lattice L_τ; this ring is an order in some imaginary quadratic extension F $(=R\otimes_{\mathbb{Z}}\mathbb{Q})$ of $\mathbb{Q}$. Extend the field F by the value of the j-invariant at the point τ. Deninger constructed [D1] an element in the third algebraic K-group of the field $F(j(\tau))$ such that the regulator maps this element to the value of the $\mathcal{K}_2$ at this CM-point. Due to Suslin and Bloch, we know that the regulator on K_3 of a number field is given by the Bloch-Wigner dilogarithm. So we can conclude that the value of the $\mathcal{K}_2$ equals a combination of the values of the dilogarithm at numbers in $F(j(\tau))$. The arguments of the dilogarithm in our formula belong to some extension of the field $F(j(\tau))$, but the collection of them is Galois invariant.

Our proof is in some sense parallel to the Deninger construction.

1.4.2. We can also interpret the Main Theorem independently of algebraic K-theory. The value of $\mathcal{K}_2(0;\tau)$ for a CM-point τ is (a rational multiple of) the value at $s=2$ of the partial zeta function $\zeta_{F,\mathcal{A}}(s)=\sum_{\mathfrak{a}\in\mathcal{A}}N(\mathfrak{a})^{-s}$ for some ideal class $\mathcal{A}$. The fact that this value can be written as a combination of dilogarithms with arguments belonging to the associated class field over F is a special case of the refined version of Zagier's conjecture according to which the values of all partial zeta functions at an arbitrary integer argument $s=m$ can be expressed in terms of m-logarithms.

For the case at hand, the same result, of course, also follows from the theorems of Deninger (existence of elements in K_2 with required values of regulator) and Bloch-Suslin (structure of K_2 and description of regulator from K_2). Our proof, as well as giving an explicit formula, also has the advantage of avoiding algebraic K-theory.

It is possible in principle that this method could be applied for higher values of m where it is not known that the regulator may be expressed in terms of polylogarithms.

1.4.3. Theorem A reflects a general phenomenon. Another reflection of this phenomenon is the following fact. For any elliptic modular curve over $\mathbb{Q}$ the value of its L-function at $s=2$ can be expressed as a combination of the values of a special function (the elliptic $(1,2)$-logarithm) [G2], which is a "relative" of our elliptic $(1,1)$-logarithm. On the other hand, for a CM-curve the value of the L-function is equal to a combination of the values of a certain Kronecker double series [D1]. Therefore this Kronecker double series must be equal to a combination of the values of the elliptic $(1,2)$-logarithm.

1.4.4. The paper is organized as follows. In the second section we define the elliptic $(1,1)$-logarithm for an arbitrary elliptic curve and study its elementary properties. In the third section we realize the Kronecker series as an integral over the square of the elliptic curve, and reduce this integral to an integral over the elliptic curve itself if this curve has complex multiplication. In the fourth section we compare the elliptic $(1,1)$-logarithm and the dilogarithm. In the fifth section we check the vanishing of the map δ for the arguments of dilogarithms on the right hand side of the Main Theorem, thus completing its proof. In the last section we prove a more general formula, relating values of $\mathcal{K}_2$ at torsion points to the dilogarithm.

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2 Elliptic $(1,1)$-logarithm.

In this section we define and study properties of the elliptic $(1,1)$-logarithm. Some motivation for considering this function is Goncharov's integral representation of the Bloch-Wigner dilogarithm. This representation uses the special case of a very general differential operation which we call A_*.

2.1. The operations A_n.

Let $\phi_1,\cdots,\phi_n$ be smooth functions on a complex variety X.

2.1.1. Definition.

$$A_n(\phi_1,\cdots,\phi_n) = \mathrm{Alt}_n\left(\sum_{j=0}^{n-1}(-1)^j\phi_1\partial\phi_2\wedge\cdots\wedge\partial\phi_{n-j}\wedge\overline{\partial}\phi_{n-j+1}\wedge\cdots\wedge\overline{\partial}\phi_n\right), \tag{4}$$

where Alt_n *denotes the alternation with respect to the action of the symmetric group* S_n *with a factor* $(n!)^{-1}$ *:*

$$\mathrm{Alt}_n(F(x_1,\cdots,x_n) = \frac{1}{n!}\sum_{\sigma\in S_n}\mathrm{sign}(\sigma)F(x_{\sigma(1)},\cdots,x_{\sigma(n)}).$$

2.1.2. Remark. If $\phi_j = \log|f_j|^2$ for *analytic* functions f_j, then A_n is (up to a factor 2 for odd n and $2i$ for even n) the so-called Beilinson-Deligne product of the f_j's.

2.1.3. Remark. The (p,q)-component of A_n equals $(-1)^q\frac{p!q!}{(p+q)!}$ times the (p,q)-component of $\mathrm{Alt}_n(\phi_1 d\phi_2\wedge\cdots\wedge d\phi_n)$.

An important property of the operations A_n is the following:

2.1.4. Proposition. *For $n>1$*

$$\begin{aligned} dA_n(\phi_1,\cdots,\phi_n) &= \partial\phi_1\wedge\partial\phi_2\wedge\cdots\wedge\partial\phi_n + (-1)^{n-1}\overline{\partial}\phi_1\wedge\overline{\partial}\phi_2\wedge\cdots\wedge\overline{\partial}\phi_n \\ &\quad + \sum_{j=1}^{n}(-1)^j\overline{\partial}\partial\phi_j\wedge A_{n-1}(\phi_1,\cdots,\hat{\phi}_j,\cdots,\phi_n). \end{aligned} \tag{5}$$

The proof is a straightforward computation.

2.2. Goncharov's integral representation of the Bloch-Wigner dilogarithm.

2.2.1. Lemma. *[G1] The value of the Bloch-Wigner dilogarithm at a point $a \neq 0, 1, \infty$ is equal to the following convergent integral:*

$$\frac{1}{4\pi}\int_{\mathbb{P}^1} A_3(\log|z|^2, \log|1-z|^2, \log|a-z|^2).$$

Proof. As we integrate over a curve, only the $(1,1)$-component gives a contribution. So we can replace A_3 by

$$-\frac{1}{6}\left(\log|z|^2 d\,\log|1-z|^2 \wedge d\,\log|a-z|^2 - \log|1-z|^2 d\,\log|z|^2 \wedge d\,\log|a-z|^2 \right.$$
$$\left. + \log|a-z|^2 d\,\log|z|^2 \wedge d\,\log|1-z|^2\right).$$

First, we will prove that the integral converges. The integrand is smooth outside the points 0, 1, a and ∞. Let (r,ϕ) be a polar coordinate system near one of the first three points. Then any term of the integrand has an asymptotic either $\log|r| r\,dr \wedge d\,\phi$, or $r^{-1} r\,dr \wedge d\,\phi$ and is integrable. As A_3 is trilinear and totally antisymmetric, we can replace the collection of arguments $(\log|z|^2, \log|1-z|^2, \log|a-z|^2)$ by the collection $(\log|z|^2, \log|1-z|^2 - \log|z|^2, \log|a-z|^2 - \log|z|^2) = (\log|z|^2, \log|1-z^{-1}|^2, \log|az^{-1}-1|^2)$; for this collection the convergence at ∞ can be checked by the same consideration as above.

By Stokes's formula we reduce the integral to

$$-\frac{1}{16\pi}\int \left(\log|z|^2 d\,\log|1-z|^2 \wedge d\,\log|a-z|^2 - \log|1-z|^2 d\,\log|z|^2 \wedge d\,\log|a-z|^2\right).$$

Evidently, the $(1,1)$-component of $d\,\phi_1 \wedge d\,\phi_2$ is the negative of the $(1,1)$-component of $(\partial - \overline{\partial})\phi_1 \wedge (\partial - \overline{\partial})\phi_2$. So the integral is equal to

$$\frac{1}{16\pi}\int \left(\log|z|^2(\partial-\overline{\partial})\log|1-z|^2 - \log|1-z|^2(\partial-\overline{\partial})\log|z|^2\right)\wedge(\partial-\overline{\partial})\log|a-z|^2.$$

An easy calculation shows that

$$d\,D_2(z) = -i\left(\log|z|(\partial-\overline{\partial})\log|1-z| - \log|1-z|(\partial-\overline{\partial})\log|z|\right).$$

So, the integral is equal to

$$\frac{i}{4\pi}\int d\,D_2(z)\wedge(\partial-\overline{\partial})\log|a-z|^2=\frac{1}{4\pi i}\int D_2(z)d\,(\partial-\overline{\partial})\log|a-z|^2.$$

As $d\,(\partial-\overline{\partial})\log|a-z|^2=2\overline{\partial}\partial\log|a-z|^2=4\pi i(\delta_a(z)-\delta_\infty(z))$ and $D_2(\infty)=0$, the proof is finished.

2.2.2. Remark. As follows from the proof, for any compact curve X and any two meromorphic functions f and g on it we have the integral representation:

$$\sum_{p\in X}\mathrm{ord}_p(g)D_2(f(p))=\frac{1}{4\pi}\int_X A_3(\log|f|^2,\log|1-f|^2,\log|g|^2).\qquad(6)$$

2.2.3. Remark. For any curve C

$$\int_C A_3(\phi_1,\phi_2,\phi_3)=\int_C\frac{1}{2}\phi_1 d\,\phi_2\wedge d\,\phi_3,$$

so this integral is equal to zero if $\phi_j=\mathrm{const}$ for some j.

2.2.4. Lemma. *For any three different points $a,b,c\in\mathbb{C}$, we have*

$$\int_{\mathbb{P}^1}A_3(\log|z-a|^2,\log|z-b|^2,\log|z-c|^2)=4\pi D_2\left(\frac{c-a}{b-a}\right).\qquad(7)$$

Proof. Denote the ratio $\frac{c-a}{b-a}$ by r. Make the change of variable $x=\frac{z-a}{b-a}$. Then $\log|z-a|^2=\log|x|^2+\log|b-a|^2$, $\log|z-b|^2=\log|x-1|^2+\log|b-a|^2$ and $\log|z-c|^2=\log|x-r|^2+\log|b-a|^2$. According to the previous Remark, terms including the constant $\log|b-a|^2$ vanish. So, we reduce the integral to the integral from the previous Lemma.

2.2.5. Remark. The integrand in (7) formally vanishes if any of a, b and c are the same by antisymmetry (we say formally because we are not really allowed to multiply the 1-forms if their singularities coincide). If only two of a, b, c are equal, then the r.h.s. of (7) is also zero, since $D_2(0)=D_2(1)=D_2(\infty)=0$. Along the triple diagonal the expression in (7) has a discontinuity. However, it is continuous on the blow-up of $\mathbb{C}^3$ along the triple diagonal.

2.3. The elliptic $(1,1)$-logarithm.

We will define a real-valued function $\mathcal{L}_{1,1}(\alpha,\beta,\gamma;\tau)$ on the third power of an elliptic curve (more precisely, on the fibered third power of the universal elliptic curve over the modular curve), invariant with respect to the diagonal action of the elliptic curve by shifts and antisymmetric with respect to permutations of the variables. This function is called the elliptic $(1,1)$-logarithm.

2.3.1. A natural generalization of the function $\log|x-a|^2$ for an elliptic curve E_τ is the *Kronecker double series*

$$\mathcal{L}_1(\xi;\tau) := \log|\tilde{\theta}(\xi,\tau)|^2 + \frac{1}{2}\frac{2\pi i}{\tau-\overline{\tau}}(\xi-\overline{\xi})^2 = -\frac{\tau-\overline{\tau}}{2\pi i}\sum_{w\in L}{}'_e \frac{\chi_\xi(w)}{|w|^2}$$

where $\sum_e$ denotes Eisenstein summation; see [W],

$$\tilde{\theta}(\xi,\tau) = \frac{\theta(\xi,\tau)}{\eta(\tau)} = q^{1/12}(z^{\frac{1}{2}} - z^{-\frac{1}{2}})\prod_{j=1}^{\infty}(1-q^j z)(1-q^j z^{-1}).$$

$q = \exp(2\pi i\tau)$ (more precisely, $q^{\frac{1}{12}} = \exp(\frac{1}{6}\pi i\tau)$), and $z = \exp(2\pi i\xi)$ (more precisely, $z^{\pm\frac{1}{2}} = \exp(\pm\pi i\xi)$). The notation $\mathcal{L}_1$ for this function is not standard. It is meant to emphasize that this function is the elliptic 1-logarithm.

2.3.2. The function $\mathcal{L}_1$ is the Green function for the operator $\overline{\partial}\partial$ on an elliptic curve

$$\overline{\partial}\partial(\mathcal{L}_1(\xi;\tau)) = 2\pi i\delta_0 + \frac{2\pi i}{\tau-\overline{\tau}}d\,\xi\wedge d\,\overline{\xi}.$$

Here δ_0 denotes the delta function at zero.

2.3.3. The function $\mathcal{L}_1$ satisfies the distribution relation

$$\sum_{\alpha:M_\tau(\alpha)=\beta}\mathcal{L}_1(\alpha;\tau) = \mathcal{L}_1(\beta;M(\tau)).$$

2.3.4. We have the "theta decomposition" of any elliptic function f ($\mathrm{ord}_\beta(f)=0$)

$$\log|f(\xi)|^2 - \log|f(\beta)|^2 = \sum \mathrm{ord}_\alpha(f)\left(\mathcal{L}_1(\xi-\alpha) - \mathcal{L}_1(\beta-\alpha)\right). \quad (8)$$

So the natural elliptic generalization of the function $D_2\left(\frac{c-a}{b-a}\right)$ is the following function

2.3.5. Definition. *The elliptic* $(1,1)$*-logarithm* $\mathcal{L}_{1,1}(\alpha,\beta,\gamma;\tau)$ *is the following* convergent *integral:*

$$\frac{1}{4\pi}\int_{E_\tau} A_3\left(\mathcal{L}_1(\xi-\alpha),\mathcal{L}_1(\xi-\beta),\mathcal{L}_1(\xi-\gamma)\right).$$

The convergence of the integral can be checked by the same consideration as for $D_2(a)$.

2.3.6. Remark. As above, only the $(1,1)$-component of the integrand gives a nontrivial contribution, so

$$\mathcal{L}_{1,1}(\alpha,\beta,\gamma;\tau) = -\frac{1}{8\pi}\int_{E_\tau}\mathcal{L}_1(\xi-\alpha)d\,\mathcal{L}_1(\xi-\beta)\wedge d\,\mathcal{L}_1(\xi-\gamma).$$

We can give another definition of this function based on the Fourier expansion.

2.3.7. Lemma. *The function $\mathcal{L}_{1,1}(\alpha,\beta,\gamma;\tau)$ considered as a distribution is equal to the following series*

$$\frac{(\tau-\overline{\tau})^2}{16\pi^2 i}\times\sum_{\substack{w_1+w_2+w_3=0\\ w_i\neq 0}}\frac{\chi_\alpha(w_1)\chi_\beta(w_2)\chi_\gamma(w_3)(w_2\overline{w}_3-\overline{w}_2w_3)}{|w_1|^2|w_2|^2|w_3|^2}.$$

A series of this kind was introduced by Deninger [D2]. In contrast to his case, however, our series is not absolutely convergent.

Proof. According to the previous Remark we can compute the following integral

$$-\frac{1}{8\pi}\int\left(-\frac{\tau-\overline{\tau}}{2\pi i}\sum_{w_1\neq 0}\frac{\chi_{\xi-\alpha}(w_1)}{|w_1|^2}\right)\left(\sum_{w_2\neq 0}\frac{\chi_{\xi-\beta}(w_2)(\overline{w}_2d\,\xi-w_2d\,\overline{\xi})}{|w_2|^2}\right)$$
$$\wedge\left(\sum_{w_3\neq 0}\frac{\chi_{\xi-\gamma}(w_3)(\overline{w}_3d\,\xi-w_3d\,\overline{\xi})}{|w_3|^2}\right)$$

$$=-\frac{1}{8\pi}\frac{\tau-\overline{\tau}}{2\pi i}\int\sum_{w_i\neq 0}\frac{\chi_{\xi-\alpha}(w_1)\chi_{\xi-\beta}(w_2)\chi_{\xi-\gamma}(w_3)(w_2\overline{w}_3-\overline{w}_2w_3)}{|w_1|^2|w_2|^2|w_3|^2}d\,\xi\wedge d\,\overline{\xi}.$$

The integrals of the terms with $w_1+w_2+w_3\neq 0$ vanish, as integrals of nontrivial harmonics over a torus, so we get

$$-\frac{1}{8\pi}\frac{\tau-\overline{\tau}}{2\pi i}\sum_{\substack{w_1+w_2+w_3=0\\ w_i\neq 0}}\frac{\chi_\alpha(w_1)\chi_\beta(w_2)\chi_\gamma(w_3)(w_2\overline{w}_3-\overline{w}_2w_3)}{|w_1|^2|w_2|^2|w_3|^2}\int d\,\xi\wedge d\,\overline{\xi}$$

$$= -\frac{1}{8\pi}\frac{\tau-\overline{\tau}}{2\pi i} \sum_{\substack{w_1+w_2+w_3=0 \\ w_i \neq 0}} \frac{\chi_\alpha(w_1)\chi_\beta(w_2)\chi_\gamma(w_3)(w_2\overline{w}_3 - \overline{w}_2 w_3)}{|w_1|^2|w_2|^2|w_3|^2} \times (-(\tau-\overline{\tau})).$$

2.3.8. Remark. This Fourier series is antisymmetric with respect to the permutations of α, β and γ, since for $w_1+w_2+w_3=0$

$$w_2\overline{w}_3 - \overline{w}_2 w_3 = \frac{1}{3}\begin{vmatrix} 1 & 1 & 1 \\ w_1 & w_2 & w_3 \\ \overline{w}_1 & \overline{w}_2 & \overline{w}_3 \end{vmatrix}.$$

2.3.9. The function $\mathcal{L}_{1,1}$ is smooth outside the diagonals; this follows from the general formula for differentiation of an integral with respect to a parameter. On the diagonals the function $\mathcal{L}_{1,1}$ is equal to zero, as A_3 is antisymmetric.

2.3.10. Lemma. *The function $\mathcal{L}_{1,1}$ is continuous on the complement of the triple diagonal $\alpha=\beta=\gamma$.*

Proof. We will prove that for $\gamma \neq 0$

$$\lim_{t\to 0} \mathcal{L}_{1,1}(t\alpha, t\beta, \gamma; \tau) = 0.$$

Let us choose some rather small ϵ and represent the integral as the sum of the integral over the disk of radius ϵ around 0 and the integral over the complement of this disk inside the elliptic curve. The second integral tends to zero, as the integral of any term of A_3 converges and A_3 is antisymmetric. Inside the disk, $\mathcal{L}_1(\xi - t\alpha)$ equals $\log|\xi - t\alpha|^2 + \phi(\xi, t\alpha)$, where $\phi(\xi, t\alpha)$, is a smooth function, the same is true for $\mathcal{L}_1(\xi - t\beta)$. Substitute this decomposition into A_3; we get several types of summands: 1) all arguments of A_3 are smooth, 2) one argument is singular and 3) two arguments are singular. In the first two cases the integral tends to zero by the same reason as above. In order to estimate the last summand, perform the change of variable $z = t^{-1}\xi$

$$\int_{|\xi|<\epsilon} A_3(\log|\xi - t\alpha|^2, \log|\xi - t\beta|^2, \mathcal{L}_1(\xi-\gamma))$$

$$= \int_{|\xi|<t^{-1}\epsilon} A_3(\log|z-\alpha|^2, \log|z - t\beta|^2, \mathcal{L}_1(tz-\gamma) - \mathcal{L}_1(-\gamma))$$

$$+ \int_{|\xi|<t^{-1}\epsilon} A_3(\log|z-\alpha|^2, \log|z-t\beta|^2, \mathcal{L}_1(-\gamma)),$$

the second integral tends to zero as $\mathcal{L}_1(-\gamma)$ is a constant.

For rather small t the first integral can be represented as a sum of the integral over the disk of radius $\sqrt{t^{-1}\epsilon}$ and the integral over the annulus $\sqrt{t^{-1}\epsilon} < |z| < t^{-1}\epsilon$. In the second integral replace $\log|z-\beta|^2$ by $\log|z-\beta|^2 - \log|z-\alpha|^2$. The integral over the disk is small since $\mathcal{L}_1(tz-\gamma)$ and also its derivatives are small; the integral over the annulus is small as $|\log|z-\beta|^2 - \log|z-\alpha|^2|$ and also its derivatives are small.

2.3.11. Lemma. *Let α, β and γ be points on an elliptic curve such that not all three of them coincide. Then the limit of $\mathcal{L}_{1,1}(t\alpha, t\beta, t\gamma; \tau)$ at $t \to 0$ is equal to $D_2\left(\frac{\gamma-\alpha}{\beta-\alpha}\right)$.*

Proof. Let us fix some rather small ϵ and represent the integral as the sum of the integral over the disk of radius ϵ around 0 and the integral over the complement of this disk inside the elliptic curve. The second integral tends to zero, as A_3 is antisymmetric. Inside the disk, $\mathcal{L}_1(\xi - t\alpha)$ equals $\log|\xi - t\alpha|^2 + \phi(\xi, t\alpha)$, where $\phi(\xi, t\alpha)$ is a smooth function, and the same is true for $\mathcal{L}_1(\xi - t\beta)$ and for $\mathcal{L}_1(\xi - t\gamma)$. As in the proof of the previous Lemma, only the summand $A_3(\log|\xi - t\alpha|^2, \log|\xi - t\beta|^2, \log|\xi - t\gamma|^2)$ gives a nontrivial contribution in the limit. Hence

$$\lim_{t\to 0} \mathcal{L}_{1,1}(t\alpha, t\beta, t\gamma; \tau) = \frac{1}{4\pi} \lim_{t\to 0} \int_{|\xi|<\epsilon} A_3(\log|\xi - t\alpha|^2, \log|\xi - t\beta|^2, \log|\xi - t\gamma|^2)$$

$$= \frac{1}{4\pi} \lim_{t\to 0} \int_{|\xi|<t^{-1}\epsilon} A_3(\log|t^{-1}\xi - \alpha|^2, \log|t^{-1}\xi - \beta|^2, \log|t^{-1}\xi - \gamma|^2)$$

$$= \frac{1}{4\pi} \int_{\mathbb{C}} A_3(\log|z-\alpha|^2, \log|z-\beta|^2, \log|z-\gamma|^2) = D_2\left(\frac{\gamma-\alpha}{\beta-\alpha}\right).$$

2.3.12. Remark. Evidently, $\mathcal{L}_{1,1}(\alpha, \beta, \gamma; \tau)$ is invariant with respect to "diagonal" shift of the arguments and to changing the sign of the arguments, so we get that the elliptic $(1,1)$-logarithm is a function on the moduli space $\overline{\mathcal{M}}_{1,3}$ of curves of genus 1 with three marked points.

3 From the Kronecker series to the elliptic $(1,1)$-logarithm.

The reduction of the Kronecker series $\mathcal{K}_2$ to the elliptic $(1,1)$-logarithm for an elliptic curve with complex multiplication splits up into two steps.

First, we represent $\mathcal{K}_2(\xi;\tau)$ as an integral over the square of the elliptic curve. This representation is valid for any elliptic curve and for any point ξ on it.

Second, we reduce the integral over the square of the elliptic curve to an integral over the elliptic curve itself. This is possible for a curve with complex multiplication and uses the existence of an extra projection of the square of the curve onto itself .

3.1. An integral representation of the Kronecker series.

We start from the simplest example which illustrates the main idea of this representation.

3.1.1. Proposition.

$$\mathcal{K}_2(\alpha;\tau) = \frac{\tau-\overline{\tau}}{2\pi i}\int\limits_{E_\tau\times E_\tau}\mathcal{L}_1(\alpha-\eta_1-\eta_2)\frac{\partial\,\mathcal{L}_1(\eta_1)}{\partial\,\eta_1}\frac{\partial\mathcal{L}_1(\eta_2)}{\partial\,\overline{\eta}_2}\frac{d\,\eta_1\wedge d\,\overline{\eta}_1}{\tau-\overline{\tau}}\wedge\frac{d\,\eta_2\wedge d\,\overline{\eta}_2}{\tau-\overline{\tau}}. \tag{9}$$

Proof. Let us use the Fourier expansions of $\mathcal{L}_1$ and its derivatives

$$\frac{\tau-\overline{\tau}}{2\pi i}\int\left(-\frac{\tau-\overline{\tau}}{2\pi i}\sum_{w_1\in L}{}'_e\frac{\chi_{\alpha-\eta_1-\eta_2}(w_1)}{|w_1|^2}\right)\left(\sum_{w_2\in L}{}'_e\frac{\chi_{\eta_1}(w_2)}{w_2}\right)\left(-\sum_{w_3\in L}{}'_e\frac{\chi_{\eta_2}(w_3)}{\overline{w}_3}\right)$$

$$\times\frac{d\,\eta_1\wedge d\,\overline{\eta_1}}{\tau-\overline{\tau}}\wedge\frac{d\,\eta_2\wedge d\,\overline{\eta_2}}{\tau-\overline{\tau}}.$$

The integral of the term with $w_1\neq w_2, w_1\neq w_3$ vanishes, as an integral of a nontrivial harmonic over a torus, so we get the integral

$$\frac{\tau-\overline{\tau}}{2\pi i}\int\left(-\frac{\tau-\overline{\tau}}{2\pi i}\sum_{w_1\in L}{}'_e\frac{\chi_{\alpha}(w_1)}{|w_1|^2}\right)\left(\frac{1}{-w_1}\right)\left(-\frac{1}{-\overline{w}_1}\right)\frac{d\,\eta_1\wedge d\,\overline{\eta_1}}{\tau-\overline{\tau}}\wedge\frac{d\,\eta_2\wedge d\,\overline{\eta_2}}{\tau-\overline{\tau}}$$

$$=\left(\frac{\tau-\overline{\tau}}{2\pi i}\right)^2\sum_{w_1\in L}\frac{\chi_\alpha(w_1)}{|w_1|^4}(-1)^2=\mathcal{K}_2(\xi;\tau).$$

At the end of this subsection we will discuss a more symmetric representation of $\mathcal{K}_2$ for any τ and ξ; this result will not be used further.

Now we formulate the main result of this subsection.

3.1.2. Lemma. *Let τ be a fixed point of $M = \begin{pmatrix} a & b \\ c & d\end{pmatrix}\neq 0,1$; $m=\det M, n=\det(M-1)$. Denote by η_3 the expression $\eta_1+M_\tau(\eta_2)-\alpha$*

and by η_4 *the expression* $\eta_1+\eta_2-\beta$. *Then*

$$\begin{aligned}
\int\limits_{E_\tau^2} A_3\left(\mathcal{L}_1(\eta_1),\mathcal{L}_1(\eta_2),\mathcal{L}_1(\eta_3)\right)\times\left(\frac{d\eta_4\wedge d\overline{\eta}_4}{\tau-\overline{\tau}}\right) &= \frac{\pi i c(\tau-\overline{\tau})}{m}\mathcal{K}_2(-\alpha;\tau);\\
\int\limits_{E_\tau^2} A_3\left(\mathcal{L}_1(\eta_1),\mathcal{L}_1(\eta_2),\mathcal{L}_1(\eta_4)\right)\times\left(\frac{d\eta_3\wedge d\overline{\eta}_3}{\tau-\overline{\tau}}\right) &= -\pi i c(\tau-\overline{\tau})\mathcal{K}_2(-\beta;\tau);\\
\int\limits_{E_\tau^2} A_3\left(\mathcal{L}_1(\eta_1),\mathcal{L}_1(\eta_3),\mathcal{L}_1(\eta_4)\right)\times\left(\frac{d\eta_2\wedge d\overline{\eta}_2}{\tau-\overline{\tau}}\right) &= \frac{\pi i c(\tau-\overline{\tau})}{mn}\mathcal{K}_2(M_\tau(\beta)-\alpha;\tau);\\
\int\limits_{E_\tau^2} A_3\left(\mathcal{L}_1(\eta_2),\mathcal{L}_1(\eta_3),\mathcal{L}_1(\eta_4)\right)\times\left(\frac{d\eta_1\wedge d\overline{\eta}_1}{\tau-\overline{\tau}}\right) &= -\frac{\pi i c(\tau-\overline{\tau})}{n}\mathcal{K}_2(\beta-\alpha;\tau).
\end{aligned} \tag{10}$$

Proof. We will prove the third equation; the other equations can be proved by the same consideration.

The last factor of the integrand has type $(1,1)$ and is closed, so we can replace the expression $A_3\left(\mathcal{L}_1(\eta_1),\mathcal{L}_1(\eta_3),\mathcal{L}_1(\eta_4)\right)$ by $\frac{1}{2}\mathcal{L}_1(\eta_3)d\mathcal{L}_1(\eta_1)\wedge d\mathcal{L}_1(\eta_4)$.

So we can calculate the integral

$$\frac{1}{2}\int\left(-\frac{\tau-\overline{\tau}}{2\pi i}\sum_{w_3\in L}{}'_e\frac{\chi_{\eta_1+M_\tau(\eta_2)-\alpha}(w_3)}{|w_3|^2}\right)$$

$$\times\left(\left(\sum_{w_1\in L}{}'_e\frac{\chi_{\eta_1}(w_1)d\eta_1}{w_1}\right)\wedge\left(-\sum_{w_4\in L}{}'_e\frac{\chi_{\eta_1+\eta_2-\beta}(w_4)(d\overline{\eta}_1+d\overline{\eta}_2)}{\overline{w}_4}\right)\right.$$

$$\left.+\left(\sum_{w_1\in L}{}'_e\frac{\chi_{\eta_1}(w_1)d\overline{\eta}_1}{\overline{w}_1}\right)\wedge\left(-\sum_{w_4\in L}{}'_e\frac{\chi_{\eta_1+\eta_2-\beta}(w_4)(d\eta_1+d\eta_2)}{w_4}\right)\right)\wedge\frac{d\eta_2\wedge d\overline{\eta_2}}{\tau-\overline{\tau}}.$$

As $\chi_{M_\tau}(\eta_2)(w)=\chi_{\eta_2}(M(w))$, only the terms with $w_3+w_1+w_4=0$ and $M(w_3)+w_4=0$ give nontrivial contributions in the integral. So we get

$$\frac{1}{2}\frac{(\tau-\overline{\tau})^2}{2\pi i}\sum_{w_3\in L}{}'\frac{\chi_{-\alpha}(w_3)}{|w_3|^2}\times\left(\frac{1}{(M-1)(w_3)}\times\frac{\chi_{-\beta}(-M(w_3))}{-\overline{M(w_3)}}\right.$$

$$- \frac{1}{\overline{\overline{(M-1)(w_3)}}} \times \frac{\chi_{-\beta}(-M(w_3))}{-M(w_3)} \Bigg) \times \int \frac{d\eta_1 \wedge d\overline{\eta_1}}{\tau - \overline{\tau}} \wedge \frac{d\eta_2 \wedge d\overline{\eta_2}}{\tau - \overline{\tau}}$$

$$= -\frac{1}{2} \frac{(\tau-\overline{\tau})^2}{2\pi i} \sum_{w_3 \in L}{}' \frac{\chi_{M(\beta)-\alpha}(w_3)}{|w_3|^4} \times \left(\frac{1}{(c\tau + d - 1)(c\overline{\tau} + d)} - \frac{1}{(c\overline{\tau} + d - 1)(c\tau + d)} \right)$$

$$= \frac{1}{2} \frac{(\tau-\overline{\tau})^2}{2\pi i} \frac{c(\tau - \overline{\tau})}{nm} \sum_{w_3 \in L}{}' \frac{\chi_{M(\beta)-\alpha}(w_3)}{|w_3|^4} = \frac{\pi i c(\tau - \overline{\tau})}{mn} \mathcal{K}_2(M_\tau(\beta) - \alpha; \tau).$$

We used above the following simple formula: for any auto-isogeny $|c\tau + d|^2 = ad - bc = \det M$.

3.1.3. Remark. For general τ any auto-isogeny of E_τ is multiplication by some integer; and the r.h.s. vanishes. Hence the result is interesting only for elliptic curves with complex multiplication.

3.1.4. The function $\frac{2\pi i}{\tau - \overline{\tau}} \mathcal{K}_2$ can be treated as a component of the vector valued function (the elliptic trilogarithm):

$$\mathcal{L}_3(\xi; \tau) = \frac{\tau - \overline{\tau}}{2\pi i} \begin{pmatrix} \displaystyle\sum_{w \in L}{}' \frac{\chi_\xi(w)}{w^3 \overline{w}} \\ \\ \displaystyle\sum_{w \in L}{}' \frac{\chi_\xi(w)}{w^2 \overline{w}^2} \\ \\ \displaystyle\sum_{w \in L}{}' \frac{\chi_\xi(w)}{w \overline{w}^3} \end{pmatrix},$$

which takes values in the symmetric square $S^2(\mathcal{H})$ of the homology group $\mathcal{H}$ of the elliptic curve E_τ with complex coefficients. This space is isomorphic to a direct summand of the second cohomology group of the square of the elliptic curve. A "natural" basis of this space is

$$f_1 = \frac{d\overline{\eta}_1 \wedge d\overline{\eta}_2}{(\tau - \overline{\tau})^2}; \quad f_2 = \frac{d\eta_1 \wedge d\overline{\eta}_2 - d\eta_2 \wedge d\overline{\eta}_1}{(\tau - \overline{\tau})^2}; \quad f_3 = \frac{d\eta_1 \wedge d\eta_2}{(\tau - \overline{\tau})^2}.$$

3.1.5. Proposition.

$$\mathcal{L}_3(\xi; \tau) = \int_{E_\tau^2} A_3\left(\mathcal{L}_1(\eta_1), \mathcal{L}_1(\eta_2), \mathcal{L}_1(\eta_1 + \eta_2 - \xi)\right) \wedge \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}. \qquad (11)$$

The proof is a straightforward calculation.

3.2. Reduction to the elliptic $(1,1)$-logarithm.

3.2.1. Let us consider the current $\Phi = \frac{1}{2\pi^2} A_4 \left(\mathcal{L}_1(\eta_1), \mathcal{L}_1(\eta_2), \mathcal{L}_1(\eta_3), \mathcal{L}_1(\eta_4)\right)$ on E_τ^2, where, as above, $\eta_3 = \eta_1 + M_\tau(\eta_2) - \alpha$ and $\eta_4 = \eta_1 + \eta_2 - \beta$ for some auto-isogeny $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0, 1$. The wedge product of currents is not defined, but for general α and β the divisors of the singularities of $\mathcal{L}_1$'s are in general position and the wedge product is well defined; the formula for the differential of the wedge product is also true.

3.2.2. The differential $d\,\Phi$ of Φ is equal to

$$\frac{1}{2\pi^2}\times\big(-\overline{\partial}\partial\mathcal{L}_1(\eta_1)\wedge A_3\left(\mathcal{L}_1(\eta_2), \mathcal{L}_1(\eta_3), \mathcal{L}_1(\eta_4)\right) + \overline{\partial}\partial\mathcal{L}_1(\eta_2)\wedge A_3\left(\mathcal{L}_1(\eta_1), \mathcal{L}_1(\eta_3), \mathcal{L}_1(\eta_4)\right)$$

$$-\overline{\partial}\partial\mathcal{L}_1(\eta_3)\wedge A_3\left(\mathcal{L}_1(\eta_1), \mathcal{L}_1(\eta_2), \mathcal{L}_1(\eta_4)\right) + \overline{\partial}\partial\mathcal{L}_1(\eta_4)\wedge A_3\left(\mathcal{L}_1(\eta_1), \mathcal{L}_1(\eta_2), \mathcal{L}_1(\eta_3)\right)\big),$$

as the $(4,0)$-part and the $(0,4)$-part vanish on a surface. We split the $\overline{\partial}\partial\mathcal{L}_1$'s into a δ-part and a smooth part. Integrals with δ-functions are integrals over the elliptic curve and are equal to sums of values of the elliptic $(1,1)$-logarithm. Integrals of smooth parts of $\partial\overline{\partial}\mathcal{L}_1$'s are calculated in the previous Lemma.

3.2.3. Let us call by $B_1, \cdots, B_4$ the δ-parts of the four components of $d\,\Phi$ and by $B_1', \cdots, B_4'$ the smooth ones. Let us first calculate the integrals with δ-functions.

$$\int B_1 = -\frac{2\pi i}{2\pi^2}\int\limits_{E_\tau} A_3\left(\mathcal{L}_1(\eta_2), \mathcal{L}_1(\eta_3), \mathcal{L}_1(\eta_4)\right)\big|_{\eta_1=0}$$

$$= \frac{1}{\pi i}\int\limits_{E_\tau} A_3\left(\mathcal{L}_1(\eta_2), \mathcal{L}_1(M_\tau\eta_2 - \alpha), \mathcal{L}_1(\eta_2 - \beta)\right)$$

$$= \frac{1}{\pi i}\int\limits_{E_\tau}\sum_{\alpha':M_\tau(\alpha')=\alpha} A_3\big(\mathcal{L}_1(\eta_2), \mathcal{L}_1(\eta_2-\alpha'), \mathcal{L}_1(\eta_2-\beta)\big) = -4i\sum_{\alpha':M_\tau(\alpha')=\alpha}\mathcal{L}_{1,1}(0,\alpha',\beta).$$

The same consideration shows that

$$B_2 = 4i\mathcal{L}_{1,1}(0,\alpha,\beta), \quad B_3 = -4i\sum_{\substack{\alpha':M_\tau(\alpha')=\alpha \\ \gamma':(M_\tau-1)(\gamma')=\alpha-\beta}}\mathcal{L}_{1,1}(\alpha',0,\gamma'),$$

$$B_4 = 4i \sum_{\gamma':(M_\tau - 1)(\gamma') = \alpha - \beta} \mathcal{L}_{1,1}(0, \beta, \gamma').$$

3.2.4. As the smooth part of $\overline{\partial}\partial\mathcal{L}_1(\xi;\tau)$ equals $\dfrac{2\pi i}{\tau - \overline{\tau}} d\,\xi \wedge d\,\overline{\xi}$, the integrals of B'_j were already calculated in (10).

3.2.5. As the integral of the differential of a current over a compact variety equals zero, we get $\sum B_i = -\sum B'_i$, so we have proved the following result:

3.2.6. Proposition. *Let τ be a fixed point of $M \neq 0, 1$;$m = \det M, n = \det(M - 1)$. Let $\alpha \neq 0$ and let $\beta \neq 0, \alpha, M_\tau(\alpha)$. Then*

$$\frac{c(\tau - \overline{\tau})}{4i} \left(\frac{1}{m}\mathcal{K}_2(-\alpha;\tau) + \mathcal{K}_2(-\beta;\tau) + \frac{1}{mn}\mathcal{K}_2(M_\tau(\beta) - \alpha;\tau) + \frac{1}{n}\mathcal{K}_2(\beta - \alpha;\tau) \right)$$

is equal to

$$- \sum_{\alpha' \in M^{-1}(\alpha)} \mathcal{L}_{1,1}(0, \alpha', \beta) + \mathcal{L}_{1,1}(0, \alpha, \beta) - \sum_{\substack{\alpha' \in M^{-1}(\alpha) \\ \gamma' \in (M-1)^{-1}(\alpha - \beta)}} \mathcal{L}_{1,1}(\alpha', 0, \gamma')$$

$$+ \sum_{\gamma' \in (M-1)^{-1}(\alpha - \beta)} \mathcal{L}_{1,1}(0, \beta, \gamma').$$

As the function $\mathcal{K}_2$ is continuous, we can "degenerate" this formula:

3.2.7. Theorem. *Let τ be a fixed point of $M \neq 0, 1$; m and n as above. Then*

$$\frac{c(\tau - \overline{\tau})}{4i} \left(\frac{m + n + 1}{mn}\mathcal{K}_2(-\alpha;\tau) + \mathcal{K}_2(0;\tau) \right) = - \sum_{\substack{\alpha' \in M^{-1}(\alpha) \\ \gamma' \in (M-1)^{-1}(\alpha)}} \mathcal{L}_{1,1}(\alpha', 0, \gamma'),\ \alpha \neq 0; \tag{12}$$

$$\frac{c(\tau - \overline{\tau})}{4i} \frac{(m + 1)(n + 1)}{mn}\mathcal{K}_2(0;\tau) = -D_2\left(\frac{ad - bc}{c\tau + d}\right) - \sum_{\substack{\alpha' \in \mathrm{Ker}(M) \backslash 0 \\ \gamma' \in \mathrm{Ker}(M-1) \backslash 0}} \mathcal{L}_{1,1}(\alpha', 0, \gamma'). \tag{13}$$

Proof. The first formula is the result of substituting $\beta = 0$; and the second one is the limit of the first as $\alpha \to 0$. This completes the proof of Theorem A.

4 From the elliptic $(1,1)$-logarithm to the dilogarithm.

In this section we relate the elliptic $(1,1)$-logarithm to the dilogarithm. Specifically, we express the combination $\mathcal{L}_{1,1}(0,\alpha,\beta;\tau)+\mathcal{L}_{1,1}(0,\alpha,-\beta;\tau)$ as a sum of dilogarithms for any torsion points α and β on *any* elliptic curve E_τ. The proof uses the representation of an elliptic curve as a covering of degree 2 of the projective line.

4.1. The dilogarithm as a combination of elliptic $(1,1)$-logarithms.

We start by expressing the dilogarithm as a combination of elliptic $(1,1)$-logarithms.

4.1.1. The Weierstrass $\wp$-function maps the elliptic curve onto $\mathbb{P}^1$ and this covering has two branches. Let the points $\pm\alpha$ on the elliptic curve be the preimages of a point a on $\mathbb{P}^1$: $\wp(\pm\alpha)=a$; similarly, $\wp(\pm\beta)=b$, $\wp(\pm\gamma)=c$. Then by 2.2.4.,

$$D_2\left(\frac{c-a}{b-a}\right)=\frac{1}{4\pi}\int_{\mathbb{P}^1}A_3(\log|z-a|^2,\log|z-b|^2,\log|z-c|^2)$$

$$=\frac{1}{2}\frac{1}{4\pi}\int_{E_\tau}A_3(\log|\wp(\xi)-\wp(\alpha)|^2,\log|\wp(\xi)-\wp(\beta)|^2,\log|\wp(\xi)-\wp(\gamma)|^2).$$

The extra factor of $1/2$ reflects the number of branches.

4.1.2. The "theta decomposition" implies

$$\log|\wp(\xi)-\wp(\alpha)|^2=\mathcal{L}_1(\xi+\alpha)+\mathcal{L}_1(\xi-\alpha)-2\mathcal{L}_1(\xi)-2\mathcal{L}_1(\alpha).$$

Let us substitute this expression into A_3 and integrate. After a straightforward computation we get

4.1.3. Lemma.

$$D_2\left(\frac{\wp(\gamma)-\wp(\alpha)}{\wp(\beta)-\wp(\alpha)}\right)=\mathcal{L}_{1,1}(\alpha,\beta,\gamma)+\mathcal{L}_{1,1}(-\alpha,\beta,\gamma)+\mathcal{L}_{1,1}(\alpha,-\beta,\gamma)$$
$$+\mathcal{L}_{1,1}(\alpha,\beta,-\gamma)-2\left(\mathcal{L}_{1,1}(0,\beta,\gamma)+\mathcal{L}_{1,1}(0,-\beta,\gamma)+\mathcal{L}_{1,1}(\alpha,0,\gamma)\right.$$
$$\left.+\mathcal{L}_{1,1}(-\alpha,0,\gamma)+\mathcal{L}_{1,1}(\alpha,\beta,0)+\mathcal{L}_{1,1}(-\alpha,\beta,0)\right). \tag{14}$$

4.2. The elliptic $(1,1)$-logarithms as a combination of dilogarithms. Now we combine the expressions from the previous Lemma in order to cancel almost all terms on the r.h.s.

4.2.1. Theorem. *For any two nontrivial torsion points $\alpha \neq \pm\beta$ on an elliptic curve E_τ, say $m\alpha = n\beta = 0$ for some $m, n \in \mathbb{N}$, we have*

$$\sum_{k=1}^{m}\sum_{l=1}^{n} D_2\left(\frac{\wp(k\alpha+l\beta)-\wp(\alpha)}{\wp(\beta)-\wp(\alpha)}\right) = -2mn(\mathcal{L}_{1,1}(\alpha,\beta,0)+\mathcal{L}_{1,1}(-\alpha,\beta,0)). \tag{15}$$

Sketch of proof. We must check that all except the two last terms on the r.h.s. of (14) cancel after summation. We show that the first one cancels:

$$\mathcal{L}_{1,1}(\alpha,\beta,k\alpha+l\beta) = \mathcal{L}_{1,1}(\alpha-(\alpha+\beta),\beta-(\alpha+\beta),k\alpha+l\beta-(\alpha+\beta))$$
$$= \mathcal{L}_{1,1}(-\beta,-\alpha,(k-1)\alpha+(l-1)\beta) = \mathcal{L}_{1,1}(\beta,\alpha,(1-k)\alpha+(1-l)\beta)$$
$$= -\mathcal{L}_{1,1}(\alpha,\beta,(1-k)\alpha+(1-l)\beta);$$

hence the first summands with arguments (k,l) and $(1-k,1-l)$ differ only by the sign and after averaging the first summands cancel. The arguments for the other terms are similar.

This completes the proof of Theorem B and hence of the formula for $\mathcal{K}_2(0,\tau)$ in the Main Theorem.

4.2.2. Remark. We have proved that the *combination* $\mathcal{L}_{1,1}(\alpha,\beta,0)+\mathcal{L}_{1,1}(-\alpha,\beta,0)$ for torsion points α and β is equal to a sum of dilogarithms. It is even true that the single term $\mathcal{L}_{1,1}(\alpha,\beta,0)$ is equal to a combination of dilogarithms; but we do not need this expression for the Main Theorem and it is rather complicated. We will derive it in Section 6.

5 Vanishing of the map δ.

In this section we prove that the argument of the dilogarithm in the Main Theorem belongs to the kernel of the map δ.

5.1. Values of the θ-function at torsion points.

5.1.1. The theta function $\tilde{\theta}(\xi,\tau) = \frac{\theta(\xi,\tau)}{\eta(\tau)} = q^{1/12}(z^{\frac{1}{2}}-z^{-\frac{1}{2}})\prod_{j=1}^{\infty}(1-q^jz)(1-q^jz^{-1})$ is not elliptic, it is only *quasiperiodic*

$$\tilde{\theta}(\xi+1,\tau) = -\tilde{\theta}(\xi,\tau); \quad \tilde{\theta}(\xi+\tau,\tau) = -z^{-1}q^{-1/2}\times\tilde{\theta}(\xi,\tau);$$

but for any torsion point $\xi = r\tau + s; r, s \in \frac{1}{N}\mathbb{Z}$ of order N, we can redefine the *value* of the theta function at this point by the formula:

$$\tilde{\theta}[\xi](\tau) = z^{1/2+r} q^{-r^2/2-r} \tilde{\theta}(\xi, \tau).$$

The shift of the argument by a point of the lattice multiplies this value by some root of unity. Let us denote by $\equiv$ the equality modulo multiplication by a root of unity.

5.1.2. Remark. If τ is imaginary quadratic over $\mathbb{Q}$, then the numbers $\tilde{\theta}[\xi](\tau)$ are *algebraic.*

5.1.3. For any two torsion points α and β the theta decomposition holds

$$\wp(\alpha;\tau) - \wp(\beta;\tau) \equiv \frac{\tilde{\theta}'(0,\tau)^2 \times \tilde{\theta}[\alpha-\beta](\tau) \times \tilde{\theta}[\alpha+\beta](\tau)}{\tilde{\theta}[\alpha](\tau)^2 \times \tilde{\theta}[\beta](\tau)^2}.$$

5.1.4. Let α be a torsion point on the elliptic curve $E_{M(\tau)}$. Then

$$\prod_{\beta \in M^{-1}\alpha} \tilde{\theta}[\beta](\tau) \equiv \tilde{\theta}[\alpha](M(\tau)), \text{ if } \alpha \neq 0;$$

$$\tilde{\theta}'(0,\tau) \times \prod_{\beta \in \mathrm{KerM}\setminus 0} \tilde{\theta}[\beta](\tau) \equiv \frac{ad-bc}{c\tau+d} \times \tilde{\theta}'(0, M(\tau)).$$

5.2. Computations.

5.2.1. First, we calculate the value of the map δ on $\frac{\wp(\gamma)-\wp(\alpha)}{\wp(\beta)-\wp(\alpha)}$

$$\delta\left(\frac{\wp(\gamma)-\wp(\alpha)}{\wp(\beta)-\wp(\alpha)}\right) = \left(\frac{\wp(\gamma)-\wp(\alpha)}{\wp(\beta)-\wp(\alpha)}\right) \wedge \left(\frac{\wp(\gamma)-\wp(\beta)}{\wp(\alpha)-\wp(\beta)}\right)$$

$$\equiv \left(\frac{\tilde{\theta}[\alpha-\gamma] \times \tilde{\theta}[\alpha+\gamma] \times \tilde{\theta}[\beta]^2}{\tilde{\theta}[\alpha-\beta] \times \tilde{\theta}[\alpha+\beta] \times \tilde{\theta}[\gamma]^2}\right) \wedge \left(\frac{\tilde{\theta}[\beta-\gamma] \times \tilde{\theta}[\beta+\gamma] \times \tilde{\theta}[\alpha]^2}{\tilde{\theta}[\alpha-\beta] \times \tilde{\theta}[\alpha+\beta] \times \tilde{\theta}[\gamma]^2}\right).$$

So the answer is the same as the result of the following procedure:

1. Define the map ν by the formula

$$\{\alpha,\beta,\gamma\} \to -\left(\{\alpha-\beta\} \wedge \{\beta-\gamma\} + \{\beta-\gamma\} \wedge \{\gamma-\alpha\} + \{\gamma-\alpha\} \wedge \{\alpha-\beta\}\right).$$

2. Apply the map ν to the arguments of the function $\mathcal{L}_{1,1}$ on the r.h.s. of (14).

3. Apply the map $\wedge^2\tilde{\theta} : \xi_1 \wedge \xi_2 \to \tilde{\theta}[\xi_1] \wedge \tilde{\theta}[\xi_2] \in \bigwedge^2 \mathbb{C}^*$ to the result of the previous step.

5.2.2. The map $\wedge^2\tilde{\theta}\circ\nu$ satisfies the same properties as the map $\mathcal{L}_{1,1}$ with respect to shifts or permutations of arguments. So after summation over $\gamma = j\alpha + k\beta$ we get

$$\delta\left(\sum_{k=1}^{m}\sum_{l=1}^{n}\left\{\frac{\wp(k\alpha+l\beta)-\wp(\alpha)}{\wp(\beta)-\wp(\alpha)}\right\}\right) \equiv -2mn\wedge^2\tilde{\theta}\circ\nu(\{\alpha,\beta,0\}+\{-\alpha,\beta,0\})$$

$$\equiv 2mn(\beta)\Big(\tilde{\theta}[\alpha-\beta]\wedge\tilde{\theta}[\beta]+\tilde{\theta}[\beta]\wedge\tilde{\theta}[\alpha]+\tilde{\theta}[\alpha]\wedge\tilde{\theta}[\alpha-\beta]$$

$$+\tilde{\theta}[\alpha+\beta]\wedge\tilde{\theta}[\beta]+\tilde{\theta}[\beta]\wedge\tilde{\theta}[\alpha]+\tilde{\theta}[\alpha]\wedge\tilde{\theta}[\alpha+\beta]\Big).$$

5.2.3. Finally, we perform the last summation:

$$\delta\left(\sum_{\substack{\alpha\in\mathrm{Ker}M\backslash 0\\ \beta\in\mathrm{Ker}(M-1)\backslash 0}}\sum_{k=1}^{m}\sum_{l=1}^{n}\left\{\frac{\wp(k\alpha+l\beta)-\wp(\alpha)}{\wp(\beta)-\wp(\alpha)}\right\}\right)$$

$$\equiv 4mn\sum_{\substack{\alpha\in\mathrm{Ker}M\backslash 0\\ \beta\in\mathrm{Ker}(M-1)\backslash 0}}\Big(\tilde{\theta}[\alpha-\beta]\wedge\tilde{\theta}[\beta]+\tilde{\theta}[\beta]\wedge\tilde{\theta}[\alpha]+\tilde{\theta}[\alpha]\wedge\tilde{\theta}[\alpha-\beta]\Big).$$

now sum the first terms over α's, the second terms over α's and β's and the third terms over β's. We get

$$4mn\left(\sum_{\beta\in\mathrm{Ker}(M-1)\backslash 0}\Big(\tilde{\theta}[M\beta]\wedge\tilde{\theta}[\beta]-\tilde{\theta}[\beta]\wedge\tilde{\theta}[\beta]\Big)\right.$$

$$+\left(\frac{(a-1)(d-1)-bc}{c\tau+d-1}\right)\wedge\left(\frac{ad-bc}{c\tau+d}\right)$$

$$\left.+\sum_{\alpha\in\mathrm{Ker}M\backslash 0}\Big(\tilde{\theta}[\alpha]\wedge\tilde{\theta}[(M-1)\alpha]-\tilde{\theta}[\alpha]\wedge\tilde{\theta}[\alpha]\Big)\right)$$

$$=4mn\times\left(\frac{(a-1)(d-1)-bc}{c\tau+d-1}\right)\wedge\left(\frac{ad-bc}{c\tau+d}\right).$$

This expression is the negative of δ of the first term $4mn\left\{\frac{ad-bc}{c\tau+d}\right\}$ in the formula (14).

6 Values of the Kronecker double series at torsion points.

In this section we express the value of the Kronecker double series $\mathcal{K}_2(\alpha;\tau)$ at a CM-point τ and a torsion point α on the elliptic curve E_τ as a sum of dilogarithms. We have proved (12) that this value of the Kronecker double series can be reduced to a combination of $\mathcal{L}_{1,1}(\alpha',0,\gamma')$, ($\alpha' \in M^{-1}(\alpha), \gamma' \in (M-1)^{-1}(\alpha)$) and $\mathcal{K}_2(0,\tau)$. So we will prove that $\mathcal{L}_{1,1}(\alpha',\gamma',0)$ equals to a sum of dilogarithms.

6.1. Reduction to "standard" functions.

6.1.1. Let α be a torsion point of exact order $N(\alpha) = 2^{\rho(\alpha)} \times N_0(\alpha)$, $N_0(\alpha)$ is odd. Denote by f_α a function with divisor $N(\alpha)\times(\alpha - 0)$. Evidently, $\log|f_\alpha(\eta)|^2 = \text{const} + N\mathcal{L}_1(\eta-\alpha) - N\mathcal{L}_1(\eta)$. This yields the formula

6.1.2. Lemma. *For any three different torsion points α, β and γ*

$$\frac{1}{4\pi}\int_{E_\tau} A_3(f_\alpha, f_\beta, f_\gamma) = N(\alpha)N(\beta)N(\gamma)$$
$$\times\left(\mathcal{L}_{1,1}(\alpha,\beta,\gamma) - \mathcal{L}_{1,1}(0,\beta,\gamma) - \mathcal{L}_{1,1}(\alpha,0,\gamma) - \mathcal{L}_{1,1}(\alpha,\beta,0)\right). \tag{16}$$

6.1.3. Lemma. *For any nontrivial non-equal torsion points α and β on the elliptic curve E_τ*

$$\sum_{k=1}^{N(\alpha)}\sum_{l=1}^{N(\beta)} \frac{1}{N(\alpha)^2 N(\beta)^2 N(k\alpha+l\beta)} \int_{E_\tau} A_3(f_\alpha, f_\beta, f_{k\alpha+l\beta}) = -\mathcal{L}_{1,1}(\alpha,\beta,0). \tag{17}$$

The proof is parallel to the proof of formula (15).

We will reduce the integrals in the l.h.s. of the previous equation to "dilogarithmic" integrals.

First, we have a decomposition of $f_\alpha(\eta)$ into a product of "standard" functions $g_\xi = \wp(\eta-\xi) - \wp(\xi)$. Let $\lambda = \lambda(\alpha)$ be a natural number such that $2^\lambda \equiv 1 \bmod (N_0)$ and denote by α_0 the point $2^\rho\alpha$.

6.1.4. Lemma.

$$f_\alpha(\eta)^{-\frac{2^\lambda-1}{N_0}}$$

$$= \text{const} \prod_{i=0}^{\rho-1} \left(\wp(\eta - 2^i\alpha) - \wp(2^i\alpha)\right)^{2^{\rho-i-1}(2^\lambda-1)} \prod_{j=0}^{\lambda-1} \left(\wp(\eta - 2^j\alpha_0) - \wp(2^j\alpha_0)\right)^{2^{\lambda-j-1}} \tag{18}$$

Proof. Compare divisors.

6.1.5. Remark. If F is a function on an elliptic curve and ξ is a torsion point we can define the operation of *averaging with a factor* 2 by the formula

$$\mathrm{Av}_2^\xi(F)(\xi) = \sum_{j=0}^{\infty} 2^{-j}F(2^j\xi) = \sum_{j=0}^{\rho(\xi)-1} 2^{-j}F(2^j\xi) + \sum_{j=\rho(\xi)}^{\rho(\xi)+\lambda(\xi)-1} \frac{2^{-j}}{1-2^{-\lambda(\xi)}}F(2^j\xi). \tag{19}$$

The last equality is nothing more than the formula for the sum of a geometric progression. So we can formally rewrite the statement of the previous lemma as

$$\log(f_\alpha(\eta)) = -\frac{N(\alpha)}{2}\mathrm{Av}_2^\alpha(\log(\wp(\eta-\alpha) - \wp(\alpha)))\,,$$

the superscript α denotes the variable with respect to which we average.

6.2. From standard functions to dilogarithms.

6.2.1. Lemma. *Let $k \subset K$ be an extension of fields of degree 2. Denote by σ the involution of K over k. Then for any elements f,g in $K \setminus k$*

$$f \times \frac{g^\sigma - g}{fg^\sigma - gf^\sigma} + g \times \frac{f^\sigma - f}{gf^\sigma - fg^\sigma} = 1;$$

and $\dfrac{g^\sigma - g}{fg^\sigma - gf^\sigma} \in k$.

The proof is evident.

6.2.2. We apply this lemma to the extension $\mathbb{C}(\mathbb{P}^1) \subset \mathbb{C}(E)$ and "standard" functions. The involution σ is given by changing sign of the argument of a function.

6.2.3. Lemma. *For the "standard" functions $g_\alpha(\eta) = \wp(\eta-\alpha) - \wp(\alpha)$ and $g_\beta(\eta) = \wp(\eta-\beta) - \wp(\beta)$,*

$$\frac{g_\beta^\sigma(\eta) - g_\beta(\eta)}{g_\alpha(\eta)g_\beta^\sigma(\eta) - g_\beta(\eta)g_\alpha^\sigma(\eta)} = K(\alpha,\beta) \times \frac{(\wp(\eta) - \wp(\alpha))^2}{\wp(\eta) - C(\alpha,\beta)},$$

where

$$C(\alpha,\beta)=\frac{\wp(\alpha)\wp'(\beta)\left(\wp(\alpha+\beta)-\wp(\alpha)\right)+\wp(\beta)\wp'(\alpha)\left(\wp(\alpha+\beta)-\wp(\beta)\right)}{\wp'(\beta)\left(\wp(\alpha+\beta)-\wp(\alpha)\right)+\wp'(\alpha)\left(\wp(\alpha+\beta)-\wp(\beta)\right)},$$

$$K(\alpha,\beta)=\frac{-\wp'(\beta)}{\left(\wp(\alpha)-\wp(\beta)\right)\times\left(\wp'(\beta)\left(\wp(\alpha+\beta)-\wp(\alpha)\right)+\wp'(\alpha)\left(\wp(\alpha+\beta)-\wp(\beta)\right)\right)}$$

Proof. Let us consider the numerator and the denominator on the l.h.s. separately.

The numerator $g_\beta^\sigma - g_\beta$ is odd, so it vanishes at semi-periods. On the other hand it has double poles at $\eta = \pm\beta$. Hence,

$$g_\beta^\sigma - g_\beta = c\times\frac{\wp'(\eta)}{\left(\wp(\eta)-\wp(\beta)\right)^2}.$$

The constant c can be calculated by considering the leading term at $\eta=\beta$ and it is equal to $-\wp'(\beta)$.

The denominator $g_\alpha g_\beta^\sigma - g_\beta g_\alpha^\sigma$ is also odd. It has a zero of order ≥ 2 at the point 0, and hence (by oddness) ≥ 3. On the other hand it has double poles at $\eta=\pm\alpha,\pm\beta$. Hence,

$$g_\alpha g_\beta^\sigma - g_\beta g_\alpha^\sigma = c'\times\frac{\wp'(\eta)\times\left(\wp(\eta)-C(\alpha,\beta)\right)}{\left(\wp(\eta)-\wp(\alpha)\right)^2\times\left(\wp(\eta)-\wp(\beta)\right)^2}$$

for some constants c' and $C(\alpha,\beta)$. These constants can be calculated by computation of the leading terms at the points α and β;

$$c'=-\left(\wp(\alpha)-\wp(\beta)\right)\times\left(\wp'(\beta)\left(\wp(\alpha+\beta)-\wp(\alpha)\right)-\wp'(\alpha)\left(\wp(\alpha+\beta)-\wp(\beta)\right)\right).$$

6.2.4. Let us denote the function $\frac{g_\beta^\sigma-g_\beta}{g_\alpha g_\beta^\sigma-g_\beta g_\alpha^\sigma}$ by $F_{\alpha\beta}$. So the "non-evident" zeroes $\pm\xi_{\alpha\beta}$ of the function $F_{\alpha\beta}$ are defined by the condition $\wp(\pm\xi_{\alpha\beta})=C(\alpha,\beta)$ (and $\wp'(\pm\xi_{\alpha\beta})=\pm\sqrt{4C(\alpha,\beta)^3-c_4(\tau)C(\alpha,\beta)-c_6(\tau)}$, c_4 and c_6 where are the coefficients of the Weierstrass equation).

6.2.5. Lemma. *For any three points α, β and γ on an elliptic curve*

$$\begin{aligned}g_\alpha\wedge g_\beta\wedge g_\gamma+g_\alpha^\sigma\wedge g_\beta^\sigma\wedge g_\gamma^\sigma &= (g_\alpha F_{\alpha\beta})\wedge(g_\beta F_{\beta\alpha})\wedge g_\gamma+(g_\alpha^\sigma F_{\alpha\beta})\wedge(g_\beta^\sigma F_{\beta\alpha})\wedge g_\gamma^\sigma\\ &-\big((g_\alpha F_{\alpha\gamma})\wedge F_{\beta\alpha}\wedge(g_\gamma F_{\gamma\alpha})+(g_\alpha^\sigma F_{\alpha\gamma})\wedge F_{\beta\alpha}\wedge(g_\gamma^\sigma F_{\gamma\alpha})\\ &-F_{\alpha\beta}\wedge(g_\beta F_{\beta\gamma})\wedge(g_\gamma F_{\gamma\beta})+F_{\alpha\beta}\wedge(g_\beta^\sigma F_{\beta\gamma})\wedge(g_\gamma^\sigma F_{\gamma\beta})\big)\end{aligned}$$

$$-F_{\alpha\beta}\wedge F_{\beta\alpha}\wedge(g_\gamma g_\gamma^\sigma)+\big((g_\alpha g_\alpha^\sigma)\wedge F_{\beta\alpha}\wedge F_{\gamma\alpha}+F_{\alpha\gamma}\wedge F_{\beta\alpha}\wedge(g_\gamma g_\gamma^\sigma)+2\times(F_{\alpha\gamma}\wedge F_{\beta\alpha}\wedge F_{\gamma\alpha})$$
$$+F_{\alpha\beta}\wedge(g_\beta g_\beta^\sigma)\wedge F_{\gamma\beta}+F_{\alpha\beta}\wedge F_{\beta\gamma}\wedge(g_\gamma g_\gamma^\sigma)+2\times(F_{\alpha\beta}\wedge(F_{\beta\gamma})\wedge F_{\gamma\beta})\big)\ .$$

The proof is a straightforward computation.

6.2.6. The first six terms on the r.h.s. of the previous equation have type $\phi\wedge(1-\phi)\wedge\psi$, so the integral of A_3 of any such term equals the sum of dilogarithms by (6). The last seven terms contain only σ-invariant functions, so the corresponding integrals can be reduced to integrals over $\mathbb{CP}^1$ and are equal to combinations of dilogarithms also. On the other hand, the integral of A_3 of the second term on the l.h.s is equal to the corresponding integral of the first one for evident geometrical reasons.

6.3. Results.

If we combine all previous results, we get the following.

6.3.1. Theorem. *For any two different nontrivial torsion points α and β on an elliptic curve*

$$\mathcal{L}_{1,1}(\alpha,\beta,0)=D_2(\Phi(\alpha,\beta)); \tag{20}$$

where

$$\Phi(\alpha,\beta)=\frac{1}{\mathrm{ord}(\alpha)\mathrm{ord}(\beta)}\sum_{\gamma\in<\alpha,\beta>\backslash 0}\mathrm{Av}_2^\alpha\mathrm{Av}_2^\beta\mathrm{Av}_2^\gamma\Phi_1(\alpha,\beta,\gamma);$$

where $<\alpha,\beta>$ denotes the subgroup generated by α and β, $\mathrm{Av}_2^\xi(F)(\xi)$ for a torsion point ξ is defined by (19), and

$$\Phi_1(\alpha,\beta,\gamma)=\{G_{\alpha\beta}(\gamma)\}_2$$
$$+\mathrm{Alt}_{\alpha,\beta}(2\{G_{\alpha\gamma}(\beta)\}_2+2\{G_{\alpha\gamma}(-\beta)\}_2-\{G_{\alpha\gamma}(\xi_{\beta\alpha})\}_2-\{G_{\alpha\gamma}(-\xi_{\beta\alpha})\}_2)$$
$$+8\left\{\frac{\wp(\gamma)-\wp(\alpha)}{\wp(\beta)-\wp(\alpha)}\right\}_2+4\mathrm{Cyc}_{\alpha,\beta,\gamma}\left(\left\{\frac{\wp(2\gamma)-\wp(\alpha)}{\wp(\beta)-\wp(\alpha)}\right\}_2\right)-4\mathrm{Cyc}_{\alpha,\beta,\gamma}\left(\left\{\frac{C(\alpha,\gamma)-\wp(\alpha)}{\wp(\beta)-\wp(\alpha)}\right\}_2\right)$$
$$-2\mathrm{Alt}_{\alpha,\beta,\gamma}\left(\left\{\frac{\wp(2\gamma)-\wp(\alpha)}{C(\beta,\gamma)-\wp(\alpha)}\right\}_2\right)+\mathrm{Alt}_{\alpha,\beta}\left(2\left\{\frac{C(\alpha,\gamma)-\wp(\alpha)}{C(\alpha,\beta)-\wp(\alpha)}\right\}_2\right.$$
$$\left.+2\left\{\frac{\wp(\gamma)-C(\alpha,\gamma)}{C(\alpha,\beta)-C(\alpha,\gamma)}\right\}_2+\left\{\frac{\wp(2\gamma)-C(\alpha,\gamma)}{C(\alpha,\beta)-C(\alpha,\gamma)}\right\}_2+\left\{\frac{C(\alpha,\gamma)-\wp(2\alpha)}{C(\alpha,\beta)-\wp(2\alpha)}\right\}_2\right);$$

where

$$G_{\alpha,\beta}(\eta)=(\wp(\eta-\alpha)-\wp(\alpha)\times K(\alpha,\beta)\frac{(\wp(\eta)-\wp(\alpha))^2}{\wp(\eta)-C(\alpha,\beta)}.$$

Here $C(\alpha,\beta)$ and $K(\alpha,\beta)$ are as in Lemma 6.2.3., the points $\pm\xi_{\alpha\beta}$ are solutions of the equation $\wp(\pm\xi_{\alpha\beta}) = C(\alpha,\beta)$; $\mathrm{Alt}_{\alpha,\beta}$, $\mathrm{Alt}_{\alpha,\beta,\gamma}$, $\mathrm{Cyc}_{\alpha,\beta,\gamma}$ denote the (anti)symmetrization with respect to S_2, S_3, A_3 (with a factor 1).

6.3.2. Theorem. *Let τ be a fixed point of $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0,1$; $m = \det M$, $n = \det(M-1)$. Let α be a torsion point on the curve E_τ. Then*

$$\frac{c(\tau-\overline{\tau})}{i}\left(\frac{m+n+1}{mn}\mathcal{K}_2(\alpha;\tau)+\mathcal{K}_2(0;\tau)\right) = D_2\left(\sum_{\substack{\alpha'\in M^{-1}(\alpha)\\ \gamma'\in(M-1)^{-1}(\alpha)}}\Phi(\alpha',\gamma')\right), \tag{21}$$

where Φ is defined in the previous Theorem. The argument of the dilogarithm belongs to the kernel of the map $\delta : \{x\}_2 \to x\wedge(1-x)$.

We have proved all statements of this theorem except the last one. It can be proved by the same consideration as in Section 5.

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