Differential algebras with Banach-algebra coefficients I: From C*-algebras to the K-theory of the spectral curve

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Abstract

We make use of the Burchnall-Chaundy classification of rings of commuting ordinary differential operators (1920's), rediscovered in the 1970's and deformed into the KP hierarchy, and of the Krichever map, to construct a more (general and) analytic version which yields a C*-algebra: our version admits a parametrization over a compact metric space. To this C*-algebra, we associate a KK-class in the K-homology of the spectral curve. Motivated by the fact that all isospectral Burchnall-Chaundy rings make up the Jacobian of the curve, we produce an identification of the K-theory of the curve with that of its Jacobian, implementing the Brown-Douglas-Fillmore theorem on extension groups of operator algebras.

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1 Introduction

The discovery of "integrable equations"[1] in the 1970's prompted a rediscovery of the Burchnall-Chaundy theory (1920's). This result of differential algebra gives a classification of the rank-one[2] commutative rings of Ordinary Differential Operators (ODOs) with fixed spectral curve, as the Jacobi variety of the curve (line bundles of fixed degree, up to isomorphism). We note here that the Burchnall-Chaundy theory allows for singular curves of a certain type, but we always restrict ourselves to the case of Riemann surfaces which is the one of interest in our constructions. The Krichever map sends a quintuple of holomorphic data associated to the ring to Sato's infinite-dimensional Grassmann manifold, where the time flows of the Kadomtsev-Petviashvili (KP) equation acting on isospectral rings become linear.

In [11], we considered an operator-valued function approach to this theory in the setting of a certain type of Banach Grassmannian. In particular, we were able to produce the notion of an operator-valued Baker function which is applicable when the ring of operators is extended to accommodate various algebra coefficients. As part of the Sato correspondence we implemented a conjugating action by an integral operator and showed how the conjugated Burchnall-Chaundy ring $\mathbb{A}$ of pseudodifferential operators can be represented as a commutative subring of a certain Banach *-algebra A. The latter algebra is a subalgebra of the bounded linear operators $\mathcal{L}(H_{\boldsymbol{\mathcal{A}}})$ where $H_{\boldsymbol{\mathcal{A}}}$ is a Hilbert module over a Hilbert *-algebra $\boldsymbol{\mathcal{A}}$. The Banach Grassmannian in question, denoted $\mathrm{Gr}(p, A)$, is a Banach manifold naturally diffeomorphic to the similarity orbit of projections in A modulo an isotropy subgroup. The space $\mathrm{Gr}(p, A)$ resembles in a more general sense the restricted Grassmannians formerly introduced in [19, 22] and these latter spaces can be recovered as subspaces of $\mathrm{Gr}(p, A)$. Some of our results require the algebra $\boldsymbol{\mathcal{A}}$ to be commutative.

The commutative ring $\mathbb{A}$ has as its joint spectrum $X' = \mathrm{Spec}(\mathbb{A})$ an irreducible complex curve whose one-point compactification is a non-singular algebraic curve X of genus $g_X \geq 1$. When $\boldsymbol{\mathcal{A}}$ is commutative, we obtain a parametrized version of the Krichever correspondence [16, 22] (originally for the case $\boldsymbol{\mathcal{A}} = \mathbb{C}$). We show that naturally associated to the ring $\mathbb{A}$ is a C*-algebra $\underline{\mathbb{A}}$ (to which we refer to as the *Burchnall-Chaundy C*-algebra*) which, by the Gelfand-Neumark-Segal theorem, can be realized as a C*-algebra of operators on an associated Hilbert space $H(\boldsymbol{\mathcal{A}})$. For any generating s-tuple of commuting operators in $\underline{\mathbb{A}}$ we consider its joint spectrum and show that this injects into the joint spectrum X'.

The latter part of this paper is devoted to constructing from certain points in $\mathrm{Gr}(p, A)$, namely the image of the Krichever correspondence, classes in the K-homology of $C(X)$ using the identification with Kasparov's bivariant K-theory, specifically the isomorphism $KK(C(X), \mathbb{C}) \cong K^*(C(X))$. In particular, by the Brown-Douglas-Fillmore (BDF) theorem, the group K^1 of a C*-algebra $C(X)$ can be related with the extension group of the compact operators $\mathcal{K}(H)$ by $C(X)$, in the sense that there is a natural transformation of covariant functors

$$\mathrm{Ext}(X) \longrightarrow \mathrm{Hom}(K^1(X), \mathbb{Z}),$$

and this Ext group, in turn, can be identified with the C*-isomorphisms of $C(X)$ into the Calkin algebra $\mathcal{Q}(H)$. Using only the commutative subalgebra we constructed from the Burchnall-Chaundy

[1]The KP hierarchy is the model we use; variations would utilize slightly different objects of abstract algebra, e.g. matrices instead of sclars.

[2]Namely, such that the orders of the operators in the ring are not all divisible by some number $r > 1$; if they are, the ring corresponds to a rank-r vector bundle.

ring, for the case $\boldsymbol{\mathcal{A}} = \mathbb{C}$, we obtain some of these C*-isomorphisms, parametrizing in fact the Jacobian $J(X)$ of X (which is the group $\mathrm{Ext}^1(\mathcal{O}_X, \mathcal{O}_X^*) \cong H^1(X, \mathcal{O}^*)$). Finally, using the Abel map which embeds $X \longrightarrow J(X)$ we fit the three C*-algebras $\underline{\mathbb{A}}, C(X)$ and $C(J(X))$ into the framework of BDF theory [5]. We show, using the basic instrumentation of K-homology, that there exists a natural injection of Ext groups $\mathrm{Ext}(X) \longrightarrow \mathrm{Ext}(J(X))$ (an isomorphism when $g_X = 1$).

This paper is the first of two parts of our work devoted to linking the KP-theory with operator algebras. In Part II we will extend our methods to the τ-function of integrable systems (see e.g. [22]); in particular we pull back the tautological bundle over the Grassmannian, and obtain, much as in the algebraic setting of [1], a Poincaré bundle over a product of homogeneous varieties for operator group flows, whose fibres will be related by the BDF theorem to the Jacobians of the spectral curves. An operator cross-ratio on the fibres will provide the τ-function, whose defining equation is equivalent to the flatness of a universal connection on the bundle. We will investigate the Schwarzian derivative associated to the cross-ratio, as well as an extension of the theory to vector bundles.

2 Algebraic preliminaries

To work in a fairly general setting (though some conditions could be further relaxed to extend it to more general topological algebras), let A be a unital Banach algebra with group of units $G(A)$ and space of projections $P(A)$; for basic facts about Banach algebras we refer to [8]. Recall that the right Green's relation is $p\mathcal{R}q$ if and only if $pA = qA$ for $p, q \in A$. Then let $\mathrm{Gr}(A) = P(A)/\mathcal{R}$ be the set of equivalence classes in $P(A)$ under $\mathcal{R}$. As the set of such equivalence classes, $\mathrm{Gr}(A)$ may be called *the Grassmannian of* A. Relative to a given topology on A, $\mathrm{Gr}(A)$ is a space with the quotient topology resulting from the natural quotient map

$$\Pi : P(A) \longrightarrow \mathrm{Gr}(A), \tag{2.1}$$

which can be shown to be an open map [10, 18]. Recall that two elements $x, y \in A$ are *similar* if x and y are in the same orbit under the inner automorphic action $*$ of $G(A)$ on A. For $p \in P(A)$, we say that the orbit of p under the inner automorphic action is *the similarity class of* p and denote the latter by $\mathrm{Sim}(p, A)$, whereby it follows that $\mathrm{Sim}(p, A) = G(A) * p$. Following [10] there exists a space $V(p, A)$ modeled on the space of proper partial isomorphisms of A upon which the restriction of (2.1) gives a locally trivial principal $G(pAp)$–bundle

$$G(pAp) \hookrightarrow V(p, A) \longrightarrow \mathrm{Gr}(p, A), \tag{2.2}$$

where the base is the Grassmannian $\mathrm{Gr}(p, A)$ viewed as the image of $\mathrm{Sim}(p, A)$ under this restriction of (2.1). The construction of the bundle in (2.2) generalizes the usual Stiefel bundle construction in finite dimensions. An alternative way of describing the analytic structure of (2.2) was outlined [10, §7]. Let $A[p]$ denote the commutant of $p \in A$ and let $G[p] = G(A[p])$. As shown in [10], the homogeneous space $G(A)/G[p]$ provides the analytic structure for $\mathrm{Gr}(p, A)$:

$$\mathrm{Gr}(p, A) \cong G(A)/G(\Pi(p)) \cong G(A)/G[p]. \tag{2.3}$$

If we further assume that A has the unital associative *-algebra property, there is the corresponding unitary group $U(A) \subset G(A)$ of A with its Lie algebra $\mathfrak{u}(A)$. Thus on setting $U[p] = U(A) \cap G[p]$, we can immediately specialize to representing $\mathrm{Gr}(p,A)$ as the Banach homogeneous space $\mathrm{Gr}(p,A) \cong U(A)/U[p]$. Further, on setting $U_1[p] = G(pAp) \cap U(A)$, we thus obtain a representation of the Stiefel bundle (2.2) as an analytic, principal $U_1[p]$-bundle

$$U_1[p] \hookrightarrow V(p,A) \longrightarrow \mathrm{Gr}(p,A) \cong U(A)/U[p]. \tag{2.4}$$

The following lemma is straightforward and is obtained from [11, §2]:

Lemma 2.1. *Let A and B be unital Banach algebras. Given a unital homomorphism $h : B \longrightarrow A$, the diagram below commutes*

$$\begin{array}{ccc} V(q,B) & \xrightarrow{V(p,h)} & V(p,A) \\ {\scriptstyle \Pi_B}\big\downarrow & & \big\downarrow{\scriptstyle \Pi_A} \\ \mathrm{Gr}(q,B) & \xrightarrow{\mathrm{Gr}(q,h)} & \mathrm{Gr}(p,A) \end{array} \tag{2.5}$$

where for $q \in P(B)$, we have set $p = h(q) \in P(A)$.

In our setting, $\boldsymbol{\mathcal{A}}$ a given topological algebra and E some $\boldsymbol{\mathcal{A}}$-module, then $A = \mathcal{L}_{\boldsymbol{\mathcal{A}}}(E)$ could be taken to be the ring of $\boldsymbol{\mathcal{A}}$ -linear transformations of E. An example is when E is a complex Banach space and $A = \mathcal{L}(E)$ is the Banach algebra of bounded linear operators on E. In order to understand the relationship between spaces such as $\mathrm{Gr}(p,A)$ and the usual Grassmannians of subspaces of E, we will describe a 'spatial correspondence'.

Suppose further that, for a topological algebra $\boldsymbol{\mathcal{A}}$, E is an $\boldsymbol{\mathcal{A}}$-module admitting a decomposition

$$E = F \oplus F^c, \qquad F \cap F^c = \{0\}, \tag{2.6}$$

where F, F^c are fixed closed subspaces of E. Now $p \in P(E) = P(\mathcal{L}(E))$ is chosen such that $F = p(E)$, and consequently $\mathrm{Gr}(A)$ consists of all such closed splitting subspaces. The assignment of pairs $(p, \mathcal{L}(E)) \mapsto (F, E)$, is called a *spatial correspondence*, and so leads to a commutative diagram

$$\begin{array}{ccc} V(p,\mathcal{L}(E)) & \xrightarrow{\ \varphi\ } & V(F,E) \\ {\scriptstyle \Pi}\big\downarrow & & \big\downarrow{\scriptstyle \Pi} \\ \mathrm{Gr}(p,\mathcal{L}(E)) & \xrightarrow{\ =\ } & \mathrm{Gr}(F,E) \end{array} \tag{2.7}$$

where $V(F,E)$ consists of linear homomorphisms of $F = p(E)$ onto a closed splitting subspace of E similar to F. If $u \in V(p, \mathcal{L}(E))$, then $\varphi(u) = u|F$ and if $T : F \longrightarrow E$ is a linear homeomorphism onto a closed complemented subspace of E similar to F, then $\varphi^{-1}(T) = Tp : E \longrightarrow F$. In particular, the points of $\mathrm{Gr}(p, \mathcal{L}(E))$ are in a bijective correspondence with those of $\mathrm{Gr}(F,E)$. The spatial correspondence is thus a convenient way of encoding the geometric/vector space structure of the latter into the former.

Remark 2.1. In the Banach manifold setting similar constructions of spaces such as $\mathrm{Gr}(p,A)$ (or $\mathrm{Gr}(F,E)$) are described in e.g. [3, 7, 14, 25].

Let E be a complex Banach space admitting a decomposition of the type (2.6). One basic ingredient of our development entails considering a class of Banach Lie groups of the type

$$\widehat{G}(E) \subset \{ \begin{bmatrix} T_1 & S_1 \\ S_2 & T_2 \end{bmatrix} : T_1 \in \mathrm{Fred}(F),\ T_2 \in \mathrm{Fred}(F^c),\ S_1, S_2 \in \mathcal{K}(E)\}, \tag{2.8}$$

that generates a Banach algebra A acting on E, but with possibly a different norm. Here we mention that both compact and Fredholm operators are well-defined in the general category of complex Banach spaces; reference [27] provides the necessary details.

3 Hilbert modules and the Banach *-algebra A

3.1 Hilbert *-modules

Take H to be a separable (infinite dimensional) Hilbert space. Given a unital separable C*-algebra $\boldsymbol{\mathcal{A}}$ one may consider the standard (free countable dimensional) Hilbert module $H_{\boldsymbol{\mathcal{A}}}$ over $\boldsymbol{\mathcal{A}}$ as defined by

$$H_{\boldsymbol{\mathcal{A}}} = \{\{\zeta_i\},\ \zeta_i \in \boldsymbol{\mathcal{A}}\ ,\ i \geq 1 : \sum_{i=1}^{\infty} \zeta_i \zeta_i^* \in \boldsymbol{\mathcal{A}}\ \} \cong \oplus \boldsymbol{\mathcal{A}}_i, \tag{3.1}$$

where each $\boldsymbol{\mathcal{A}}_i$ represents a copy of $\boldsymbol{\mathcal{A}}$. We can form the algebraic tensor product $H \otimes_{\mathrm{alg}} \boldsymbol{\mathcal{A}}$ on which there is an $\boldsymbol{\mathcal{A}}$-valued inner product

$$\langle x \otimes \zeta, y \otimes \eta \rangle = \langle x, y \rangle \zeta^* \eta, \qquad x, y \in H,\ \zeta, \eta \in \boldsymbol{\mathcal{A}}. \tag{3.2}$$

Thus $H \otimes_{\mathrm{alg}} \boldsymbol{\mathcal{A}}$ becomes an inner-product $\boldsymbol{\mathcal{A}}$-module whose completion is denoted by $H \otimes \boldsymbol{\mathcal{A}}$. Given an orthonormal basis for H, we have the following identification (a unitary equivalence) given by $H \otimes \boldsymbol{\mathcal{A}} \approx H_{\boldsymbol{\mathcal{A}}}$ (see e.g. [17, 9]). For properties of compact and Fredholm operators over Hilbert modules, see e.g. [4]. We take $\mathcal{L}(H_{\boldsymbol{\mathcal{A}}})$ to denote the C*-algebra of adjointable linear operators on $H_{\boldsymbol{\mathcal{A}}}$ and $\mathrm{Fred}(H_{\boldsymbol{\mathcal{A}}})$ to denote the space of Fredholm operators (see e.g. [4, 15]).

When $H_{\boldsymbol{\mathcal{A}}}$ is a *Hilbert *-module* there is a generalization of the well-known (nested) sequence of the Schatten ideals of operators on a Hilbert space. The Banach spaces $\mathcal{L}_\ell(H_{\boldsymbol{\mathcal{A}}})$ are defined as the subspaces of $\mathcal{L}(H_{\boldsymbol{\mathcal{A}}})$ consisting of operators S with norm satisfying $\|S\|_\ell^\ell = \mathrm{Tr}(S^*S)^{\ell/2} < \infty$, (for $0 \leq \ell \leq \infty$, where $\mathcal{L}_\infty(H_{\boldsymbol{\mathcal{A}}}) = \mathcal{K}(H_{\boldsymbol{\mathcal{A}}})$ is the compact operators) [24]. We use this definition in §3.2. Since this will turn out to be an essential ingredient (for example, in the Gelfand transform §3.4) we will assume henceforth that $H_{\boldsymbol{\mathcal{A}}}$ is a separable Hilbert *-module.

Remark 3.1. If ϕ denotes a state of $\boldsymbol{\mathcal{A}}$, then we can produce a positive semi-definite pre-Hilbert space structure on $H_{\boldsymbol{\mathcal{A}}}$ via an induced inner product $\langle v|w\rangle_\phi = \phi(\langle v|w\rangle_{\boldsymbol{\mathcal{A}}})$. From the Gelfand-Neumark-Segal (GNS) Theorem there is an associated Hilbert space H_ϕ for which $\ell^2 \otimes H_\phi$ is the completion of $H_{\boldsymbol{\mathcal{A}}}$ under this induced inner product (see e.g. [4, 12]). When ϕ is understood we shall simply denote the latter by $H(\boldsymbol{\mathcal{A}})$. Observe that $H(\boldsymbol{\mathcal{A}})$ contains $H_{\boldsymbol{\mathcal{A}}}$ as a dense vector subspace which is a right $\boldsymbol{\mathcal{A}}$-module.

The assignment of Hilbert spaces $H \mapsto H(\boldsymbol{\mathcal{A}})$ and use of the $\boldsymbol{\mathcal{A}}$-valued inner product in (3.2), thus allows us to modify various results that were originally established for the various objects $(H, \langle\ ,\ \rangle)$ (such as the bounded linear operators, the compact operators, etc. in the case $\boldsymbol{\mathcal{A}} = \mathbb{C}$).

3.2 The Banach *-algebra A and its C*-norm closure

Next we consider decompositions of the type (2.6). We say that a *polarization* of $H_{\mathcal{A}}$ given by a pair of submodules $(\mathsf{H}_+, \mathsf{H}_-)$, such that

$$H_{\mathcal{A}} = \mathsf{H}_+ \oplus \mathsf{H}_- \ , \text{ and } \mathsf{H}_+ \cap \mathsf{H}_- = \{0\}. \tag{3.3}$$

Specific to our situation we take a unitary $\mathcal{A}$–module map J satisfying $J^2 = 1$, thus giving rise to an induced eigenspace decomposition $H_{\mathcal{A}} = \mathsf{H}_+ \oplus \mathsf{H}_-$, for which $\mathsf{H}_- \cong \mathsf{H}_+$.

Throughout, the main algebra we consider is the Banach *-algebra

$$A = \mathcal{L}_J(H_{\mathcal{A}}) := \{\mathsf{T} \in \mathcal{L}_{\mathcal{A}}(H_{\mathcal{A}}) : \mathsf{T} \text{ is adjointable and } [J, \mathsf{T}] \in \mathcal{L}_2(H_{\mathcal{A}})\}, \tag{3.4}$$

(by definition $\mathcal{L}_2(H_{\mathcal{A}})$ denotes the Hilbert-Schmidt operators) where for $\mathsf{T} \in A$, we assign the norm (cf. [19, §6.2]):

$$\|\mathsf{T}\|_J = \|\mathsf{T}\| \ + \ \|[J, \mathsf{T}]\|_2. \tag{3.5}$$

The algebra A can thus be seen as a Banach *-algebra with isometric involution (when $\mathcal{A} \cong \mathbb{C}$ we simply write $\mathcal{L}_J(H)$). This is our restricted algebra which we will use henceforth. Together with the topology induced by $\| \ \|_J$, the group of units $G(A)$ is a complex Banach Lie group for which we have the unitary Lie subgroup $U(A) \subset G(A)$.

It is convenient to denote by $\bar{A}$ the C*-algebra norm closure of A in the C*-algebra $\mathcal{L}(H_{\mathcal{A}})$. Thus A is a Banach *-algebra with isometric involution dense in the C*-algebra $\bar{A}$. In particular, as A is a Banach algebra, it is closed under the holomorphic functional calculus, and as it is a *-subalgebra of the C*-algebra $\bar{A}$, it is therefore a (unital) *pre-C*-algebra* in the sense of [4, 6].

Let us recall the definition of a *Fredholm module* as generalized in [4, §17.5]. For any C*-algebra B, and any (right) Hilbert $\mathcal{A}$–module E, we call E a (left) Fredholm B-module over $\mathcal{A}$ provided there is given an involutive representation $\Pi : B \longrightarrow \mathcal{L}_{\mathcal{A}}(E)$ together with an essentially (that is, modulo compact operators) unitary operator S which essentially commutes with $\Pi(B)$. Such a module then determines a member of the Kasparov group $KK(B, \mathcal{A})$ (this latter concept will be used in §7). In particular, we are interested in the case where B is simply a C*-subalgebra of $\mathcal{L}_{\mathcal{A}}(E)$ which is generated by an ordinary algebra homomorphism. In fact we will be usually taking $E = H_{\mathcal{A}}$. In this situation, given a completely positive state (in the commutative case, a measure giving positive mass to each open subset of the spectrum), we can then pass (as in Remark 3.1) to the Hilbert space $H(\mathcal{A})$ to obtain a Fredholm module in the sense of [6, IV].

3.3 The Grassmannians $\mathrm{Gr}(p, A)$

We proceed with A as defined in (3.4). In (2.8) we take E to be the complex Banach space underlying A where we consider the former as now admitting a decomposition of the type (2.6) compatible with the polarization (3.3). That is, we identify $F \cong \mathsf{H}_+$ and $F^c \cong \mathsf{H}_-$ (this implicitly makes use of the spatial correspondence). In the following we restrict to the $\ell = 2$ (Hilbert-Schmidt) case and thus (2.8) specializes to

$$\widehat{G}(A) = \{ \begin{bmatrix} T_1 & S_1 \\ S_2 & T_2 \end{bmatrix} : T_1 \in \mathrm{Fred}(\mathsf{H}_+), \ T_2 \in \mathrm{Fred}(\mathsf{H}_-), \ S_1 \in \mathcal{L}_2(\mathsf{H}_-, \mathsf{H}_+), \ S_2 \in \mathcal{L}_2(\mathsf{H}_+, \mathsf{H}_-)\}. \tag{3.6}$$

Granted that $\widehat{G}(A)$ acts analytically on $\mathrm{Gr}(A)$, a typical orbit is the restricted Grassmannian $\mathrm{Gr}(p, A) = \widehat{G}(A)/G[p]$ (see [10, 11]). More specifically, from §2, we have $\mathrm{Gr}(p, A) \cong U(A)/U[p]$, where

$$U[p] \cong U(A) \cap G[p] \cong U(A) \cap (U(\mathsf{H}_+) \times U(\mathsf{H}_-)). \tag{3.7}$$

Directly from Lemma 2.1 we deduce:

Lemma 3.1. *Let $B \subset A$ be a Banach *-subalgebra of A, with inclusion $h : B \longrightarrow A$. Then there is an induced inclusion of Grassmannians $\mathrm{Gr}(q, B) \subset \mathrm{Gr}(p, A)$ where for $q \in P(B)$ we have set $p = h(q) \in P(A)$.*

Example 3.1. Let $\boldsymbol{\mathcal{B}} \subset \boldsymbol{\mathcal{A}}$ be a C*-subalgebra. Then we have an inclusion

$$\mathcal{L}_J(H) \otimes \boldsymbol{\mathcal{B}} \longrightarrow \mathcal{L}_J(H) \otimes \boldsymbol{\mathcal{A}} \cong A = \mathcal{L}_J(H_{\boldsymbol{\mathcal{A}}}). \tag{3.8}$$

In particular, when $\boldsymbol{\mathcal{B}} \cong \mathbb{C}$ and $H = L^2(S^1, \mathbb{C}^r)$ for which there is a polarization $H = H_+ \oplus H_-$ ($H_+ \cap H_- = \{0\}$), the inclusion $\mathcal{L}_J(H) \longrightarrow \mathcal{L}_J(H) \otimes \boldsymbol{\mathcal{A}}$ along with the spatial correspondence induces an inclusion $\mathrm{Gr}(H_+, H) \subset \mathrm{Gr}(p, A)$ where $\mathrm{Gr}(H_+, H)$ is the 'restricted' Grassmannian as used in [19, 22] (see also §4.2 below).

The space $\mathrm{Gr}(p, A)$ may be realized more specifically in the following way. Suppose that a fixed $p \in P(A)$ acts as the projection of $H_{\boldsymbol{\mathcal{A}}}$ on H_+ along H_-. Therefore $\mathrm{Gr}(p, A)$ is the Grassmannian consisting of subspaces $W = r(H_{\boldsymbol{\mathcal{A}}})$ for $r \in P(A)$ such that:

(1) the projection $p_+ = pr \; : \; W \longrightarrow \mathsf{H}_+$ is in $\mathrm{Fred}(H_{\boldsymbol{\mathcal{A}}})$, and

(2) the projection $p_- = (1-p)r \; : \; W \longrightarrow \mathsf{H}_-$ is in $\mathcal{L}_2(\mathsf{H}_+, \mathsf{H}_-)$.

Alternatively, for (2) we may take projections $q \in P(A)$ such that for the fixed $p \in P(A)$, the difference $q - p \in \mathcal{L}_2(\mathsf{H}_+, \mathsf{H}_-)$. Further, we define the *big cell* $C_{\mathrm{b}} = C_{\mathrm{b}}(p_1, A) \subset \mathrm{Gr}(p, A)$ as the collection of all subspaces $W \in \mathrm{Gr}(p, A)$ such that the projection $p_+ \in \mathrm{Fred}(H_{\boldsymbol{\mathcal{A}}})$ is an isomorphism.

3.4 The case where $\boldsymbol{\mathcal{A}}$ is commutative

Henceforth we will take $\boldsymbol{\mathcal{A}}$ to be a commutative separable C*-algebra. The Gelfand transform implies there exists a compact metric space Y such that $Y = \mathrm{Spec}(\boldsymbol{\mathcal{A}})$ and $\boldsymbol{\mathcal{A}} \cong C(Y)$. Setting $B = \mathcal{L}_J(H)$, we can now express the Banach *-algebra A in the form

$$A \cong B \otimes \boldsymbol{\mathcal{A}} \cong \{\text{continuous functions } Y \longrightarrow B\}, \tag{3.9}$$

for which the $\| \; \|_2$-trace in (3.5) is regarded as continuous as a function of Y. In view of Example 3.1 several results obtainable in the case of $\mathrm{Gr}(H_+, H)$ can be extended by Y-parametrization in a straightforward way. In view of Example 3.1 it will be convenient to set the usual restricted Grassmannian $\mathrm{Gr}(H_+, H) = \mathrm{Gr}(q, B)$.

4 The Burchnall-Chaundy ring

We briefly recall how in the case $\boldsymbol{\mathcal{A}} = \mathbb{C}$ the KP flows act on $\mathrm{Gr}(p, A)$.

4.1 The Burchnall-Chaundy ring and the formal Baker function

Let $\mathbb{B}$ denote the algebra of analytic functions $U \longrightarrow \mathbb{C}$ where U is a connected open neighborhood of the origin in $\mathbb{C}$. The (noncommutative) algebra $\mathbb{B}[\partial]$ of linear differential operators with coefficients in $\mathbb{B}$, consists of expressions

$$\sum_{i=0}^{N} a_i\ \partial^i,\ \ (a_i \in \mathbb{B},\quad \text{for some } N \in \mathbb{Z}). \tag{4.1}$$

Here $\partial := \partial/\partial x$ and the a_i can be regarded as operators on functions, with multiplication

$$[\partial, a] = \partial a - a\partial = \partial a/\partial x\ . \tag{4.2}$$

More generally, we pass to the algebra $\mathbb{B}[[\partial^{-1}]]$ of formal pseudodifferential operators with coefficients in $\mathbb{B}$. This algebra is obtained by formally inverting the operator ∂ (see e.g. [21]) and taking Laurent series as in (4.1), with $-\infty < i \leq N$.

Recall that the n-th generalized KdV-hierarchy, for each n a reduction of the KP hierarchy, consists of all evolution operators for $n-1$ unknown functions $u_0(x,t), \ldots, u_{n-2}(x,t)$ that can be expressed as

$$\frac{\partial L}{\partial t} = [P, L], \tag{4.3}$$

where $L \in \mathbb{B}[\partial^{-1}]$ is an n-th order differential operator

$$L = \partial^n + u_{n-2}\ \partial^{n-2} + \ldots + u_1\ \partial + u_0, \tag{4.4}$$

and P is a differential operator for which the order $\mathrm{ord}[P, L] \leq (n-2)$. Later we will see that the evolution of eigenfunctions of L via comparison with the constant-coefficients operator ∂^n is related to an integral operator K conjugating L such that $K(L)K^{-1} = \partial^n$. The construction of [21] enables the correspondence '$(\frac{\partial}{\partial x})^{-1} \leftrightarrow$ multiplication by z' to realize elements of a commutative $\mathbb{A} \subset \mathbb{B}[[\partial^{-1}]]$ as elements of $\mathbb{B}[[z]][z^{-1}]$. Note, although this is not an issue in the present paper, that for formal objects in $\mathbb{B}[[\partial^{-1}]]$ to make sense in the analytic (in z^{-1}) setting, convergence properties will have to be imposed.

4.2 Holomorphic data over the spectral curve

The ring $\mathbb{A}$ is assumed to be a commutative subring of $\mathbb{B}[\partial]$, whose joint spectrum is a complex irreducible curve $X' = \mathrm{Spec}(\mathbb{A})$ with completion a non-singular algebraic curve $X = X' \cup \{x_\infty\}$ of genus $g_X \geq 1$. We recall from [22] the following associated quintuple of data $(X, x_\infty, \mathcal{L}, z, \varphi)$: $\mathcal{L} \longrightarrow X$ is a holomorphic line bundle, x_∞ is a smooth point of X, z the inverse of a local parameter on X at x_∞, where z is used to identify a closed neighborhood X_∞ of x_∞ in X with a neighborhood of the disk $D_\infty = \{z : |z| \geq 1\}$ in the Riemann sphere, and φ is a holomorphic trivialization of $\mathcal{L}$ over $X_\infty \subset X$. Subsequently, we consider L^2–boundary values of $\mathcal{L}$ over X/D_∞ and φ identifies sections of $\mathcal{L}$ over S^1 with $\mathbb{C}$-valued functions. Thus we arrive at the (separable) Hilbert space $H = L^2(S^1, \mathbb{C})$ together with a special case of (2.6), that is, on setting $F = H_+$ and $F^c = H_-$, we have as before a polarization

$$H = H_+ \oplus H_-,\ \text{ and } H_+ \cap H_- = \{0\}, \tag{4.5}$$

in the case $B = \mathcal{L}_J(H)$. Further the quintuple $(X, x_\infty, \mathcal{L}, z, \varphi)$ can be mapped, by the *Krichever correspondence*, to a point $W \in \mathrm{Gr}(q, B)$ (cf [16, 22]). Following [22], the properties of the projections $p_\pm$ in §3.3 apply and the kernel (respectively, cokernel) of the projection $p_+ : W \longrightarrow H_+$ is isomorphic to the sheaf-cohomology group $H^0(X, \mathcal{L}_\infty)$ (respectively, $H^1(X, \mathcal{L}_\infty)$) where $\mathcal{L}_\infty = \mathcal{L} \otimes \mathcal{O}(-x_\infty)$. Henceforth, we shall work with the Hilbert space $H = L^2(S^1, \mathbb{C})$.

This remarkable correspondence [16] (see also [22] Proposition 6.2) links the data of the quintuple to a flow of multiplication operators on H inducing a linear flow on the Jacobian torus $J(X)$ of X (recall that $J(X)$ is the commutative group of holomorphic line bundles of zero degree over X). Furthermore, this flow corresponds to the evolution in (4.3) of solutions of the generalized KdV flows. In a later section we will view this correspondence in a 'parametrized' setting.

5 The Baker function in the C*-algebra setting

5.1 The Burchnall-Chaundy ring with coefficients in $\boldsymbol{\mathcal{A}}$

Continuing with $\boldsymbol{\mathcal{A}} \cong C(Y)$ a commutative (separable) C*-algebra, we now consider the algebra $\mathbb{B} \otimes \boldsymbol{\mathcal{A}}$ of analytic functions $U \longrightarrow \boldsymbol{\mathcal{A}}$ where U is a connected open neighborhood of the origin in $\mathbb{C}$. As before we have the algebra $\mathbb{B}[\partial]$ of linear differential operators with coefficients in $\mathbb{B}$, consisting of expressions (4.1) where essentially the same discussion in §4.1 applies verbatim. In particular, the coefficients a_i are now thought of as $\boldsymbol{\mathcal{A}}$-valued functions.

5.2 A flow of multiplication operators

In the following we take D to denote the closed unit disk centered at the origin in $\mathbb{C}$.

Definition 5.1. Let $\Gamma_+(Y) \subset \widehat{G}(A)$ be the group of multiplication operators on $H_{\boldsymbol{\mathcal{A}}}$ given by the group of maps $g : S^1 \times Y \longrightarrow \mathbb{C}^*$ such that g is continuous in y for each $y \in Y$, $g(x, y)$ is real analytic in $x \in S^1$ extending to $g(z, y)$ holomorphic in $z \in D$, and $g(0, y) = 1$, for each $y \in Y$.

An action of $\Gamma_+(Y)$ on $\mathrm{Gr}(p, A)$ is induced via its restriction to the subspace H_+. In the special case $Y = \{pt\}$, we set $\Gamma_+(Y) = \Gamma_+$. In view of (3.9) and $\mathrm{Gr}(q, B) = \mathrm{Gr}(H_+, H)$, we consider the restriction $\Gamma_+(Y)|\mathrm{Gr}(q, B)$ acting as a Y-parametrized family $\{(\Gamma_+)_y\}_{y \in Y}$. Effectively, we then have a flow of multiplication operators as given by

$$\Gamma_+(Y) = \{(\exp(\sum_a t_a z^a))_y\}_{y \in Y}. \tag{5.1}$$

Definition 5.2. Let $\Gamma_-(Y) \subset \widehat{G}(A)$ be the group of multiplication operators on $H_{\boldsymbol{\mathcal{A}}}$ given by the group of maps $g : \mathbb{C} \backslash \mathrm{Int}(D) \times Y \longrightarrow \mathbb{C}^*$ such that g is continuous in y for each $y \in Y$, $g(x, y)$ is real analytic in $x \in \mathbb{C} \backslash \mathrm{Int}(D)$ extending to $g(z, y)$ holomorphic in $z \in \mathbb{C} \backslash \mathrm{Int}(D)$, and $g(\infty, y) = 1$, for each $y \in Y$.

5.3 The abstract operator-valued Baker function

Recalling Definitions 5.1 and 5.2 above, let us consider subspaces $W \in \mathrm{Gr}(p, A)$ of the form

$$W = g h_g \mathsf{H}_+, \tag{5.2}$$

with $g \in \Gamma_+(Y)$ and $h_g \in \Gamma_-(Y)$. Also for $g \in \Gamma_+(Y)$, we consider projections

$$p_1^g : \ g^{-1}(W) \longrightarrow \mathsf{H}_+, \ p_1^g \in \mathrm{Fred}(H_{\boldsymbol{\mathcal{A}}}), \tag{5.3}$$

and define in relationship to the big cell, the subgroup of $\Gamma_+(Y)$ as given by

$$\Gamma_+^W(Y) = \{g \in \Gamma_+(Y) : p_1^g \text{ is an isomorphism}\}. \tag{5.4}$$

Definition 5.3. *The operator-valued Baker function ψ_W* associated to the subspace $W \in C_{\mathrm{b}} \subset \mathrm{Gr}(p, A)$ in (5.2), is defined formally as:

$$\psi_W = (p_1^g)^{-1}(\mathbf{1}) = (\sum_{s=0}^{\infty} a_s(g) \ \zeta^{-s}) \ g(\zeta) \in W g(\zeta), \tag{5.5}$$

where $g \in \Gamma_+^W(Y)$ and the a_s are analytic $\boldsymbol{\mathcal{A}}$-valued (operator-matrix) functions on $\Gamma_+^W(Y)$ extending to all $\boldsymbol{\mathcal{A}}$-valued functions $g(z, y)$ in $\Gamma_+(Y)$ meromorphic in z (cf. [11]).

5.4 A formal integral operator

Relative to $W \in \mathrm{Gr}(p, A)$, the set

$$B_W = \{f(z) = \sum_{s=-\infty}^{N} c_s z^{-s} : s \in \mathbb{N}, \ c_s \in \boldsymbol{\mathcal{A}}, \ f(z)W \subset W\}, \tag{5.6}$$

contains the coordinate ring of the curve $X \setminus \{x_\infty\}$. As in [11, §6], following [22], there exists a formal integral operator $K \in \mathbb{B}[[\partial^{-1}]]$ given by

$$K = 1 + \sum_{s=1}^{\infty} a_s(x) \ \partial^{-s}, \tag{5.7}$$

(where the a_s are $\boldsymbol{\mathcal{A}}$-valued functions) unique up to a constant coefficient operator such that $L = K(\partial^n)K^{-1}$ belongs to $\mathbb{A}$. Under the above correspondence the (formal) Baker function ψ_W is defined as $\psi_W = Ke^{xz}$, the main point being that the function ψ_W will be an eigenfunction for the operator $L^{1/n} = \partial +$ [lower-order terms], that is,

$$L^{1/n}\psi_W = z \ \psi_W, \tag{5.8}$$

and accordingly

$$\psi_W(x, z) = (1 + \sum_{s=1}^{\infty} a_s(x) \ z^{-s}) \ e^{xz}. \tag{5.9}$$

Using a form of the Sato correspondence [21], we established in [11] (for $\boldsymbol{\mathcal{A}}$ not necessarily commutative):

Theorem 5.1. [11, Theorem 6.1] *Given the Baker function ψ_W associated to a subspace $W \in \mathrm{Gr}(p, A)$, the Burchnall-Chaundy ring $\mathbb{A}$ is conjugated into $A \subset \mathcal{L}_J(H_{\boldsymbol{\mathcal{A}}})$ as a commutative subring, the conjugating integral operator K being unique up to constant coefficient operators.*

5.5 The Y-parametrized holomorphic data

Since we have effectively tensored the coefficients of $\mathbb{A}$ by $\boldsymbol{\mathcal{A}}$, we can modify the discussion in §4.1 and §4.2 using the expression for A in (3.9). Specifically, the same construction involving the data in §4.2 yields a Y*-parametrized map* to $\mathrm{Gr}(q, B) = \mathrm{Gr}(H_+, H)$ where we recall $B = \mathcal{L}_J(H)$ from §3.4. Consequently, for a Y-parametrized quintuple in §4.2, $y \in Y$, we obtain the assignment

$$\{(X, x_\infty, \mathcal{L}_y, z_y, \varphi_y)\}_{y \in Y} \longrightarrow \text{ certain points } W_y \in Y \times \mathrm{Gr}(q, B). \tag{5.10}$$

As we have noted in §5.2, the restriction $\Gamma_+(Y)|\mathrm{Gr}(q, B)$ acts as a family of multiplication operators $\{(\Gamma_+)_y\}_{y \in Y}$ on subspaces $W \in \mathrm{Gr}(q, B)$. Following [22, §6], an element $g \in \Gamma_+$ serves as a transition function for a line bundle over X (that is, $g \in \Gamma_+$ determines a point in $\mathrm{Pic}^{g_X}(X)$ which is then twisted into a point of $J(X)$). The restricted action $\Gamma_+(Y)|\mathrm{Gr}(q, B)$ gives a parametrized version of this result:

Proposition 5.1. *Let $\mathcal{J}_0(X \times Y)$ denote the space of topologically trivial line bundles on $X \times Y$. Then there exists a well-defined map*

$$\Gamma_+(Y)|\mathrm{Gr}(q, B) \longrightarrow \mathcal{J}_0(X \times Y), \tag{5.11}$$

given by $g_y \longrightarrow \mathcal{L}_{g_y}$, and an induced action of $\Gamma_+(Y)|\mathrm{Gr}(q, B)$ *on* $\mathcal{J}_0(X \times Y)$.

Proof. We follow [22, Proposition 6.9] closely. Let $X_0 = X \backslash X_\infty$, and let $\mathcal{L}_{g_y} \longrightarrow X \times Y$ be the line bundle obtained by taking topologically trivial line bundles over $X_0 \times Y$ and $X_\infty \times Y$ and glueing them by $g_y = (g, y)$ over an open neighborhood of $S^1 \times Y$, where $g \in \Gamma_+$. This line bundle has degree g_X, so it is not topologically trivial, but by changing g_y by an element of Γ_- we achieve degree zero. Thus we obtain a map

$$\begin{gathered} \Gamma_+(Y)|\mathrm{Gr}(q, B) \longrightarrow \mathcal{J}_0(X \times Y), \\ g_y \mapsto \mathcal{L}_{g_y} \end{gathered} \tag{5.12}$$

where $\mathcal{L}_{g_y}$ has a φ_g-induced trivialization

$$\varphi_{g_y} = (\varphi_g, y) : \mathcal{L}_{g_y}|X_\infty \times Y \longrightarrow \mathbb{C} \times X_\infty \times Y \ . \tag{5.13}$$

Consequently, $(\Gamma_+)_y$ acts on (X, x_∞, z) and on $(\mathcal{L}, \varphi)$ via the tensor product with $(\mathcal{L}_{g_y}, \varphi_{g_y})$. □

In this way the action of $\Gamma_+(Y)|\mathrm{Gr}(q, B)$ on Y-parametrized solutions of the generalized n-th KdV equations corresponds to $\mathcal{L} \longrightarrow \mathcal{L} \otimes \mathcal{L}_{g_y}$. For a fixed $y \in Y$, the assignment $g \mapsto \mathcal{L}_g$ defines a surjective group homomorphism $\Gamma_+ \longrightarrow J(X)$, as in the case $\boldsymbol{\mathcal{A}} \cong \mathbb{C}$ [22, Remark 6.8].

6 The Burchnall-Chaundy C* algebra

Motivated by Theorem 5.1 our next aim is to obtain a natural C*-algebra associated to $\mathbb{A}$. Firstly, in view of Theorem 5.1, let

$$i_K : \mathbb{A} \longrightarrow A, \tag{6.1}$$

be the inclusion map induced via conjugation by the integral operator K in (5.7). We also recall the C*-algebra $\bar{A}$ of §3.2.

Definition 6.1. The *Burchnall-Chaundy C* algebra* denoted $\underline{\mathbb{A}}$ is the C*-subalgebra of $\bar{A}$ generated by $i_K(\mathbb{A})$.

Thus $\underline{\mathbb{A}}$ is a separable C*-subalgebra of $\bar{A}$ (this uses the fact that $\bar{A}$ is separable). We also endow $\underline{\mathbb{A}}$ with an identity as induced from that of $\bar{A}$. Further, let

$$\underline{i}_K : \mathbb{A} \longrightarrow \underline{\mathbb{A}}, \tag{6.2}$$

be the induced map of (6.1) into the C*-algebra $\underline{\mathbb{A}}$. Note that both maps in (6.1), (6.2) factor through the tensor product $\mathbb{A} \otimes \boldsymbol{\mathcal{A}}$.

Remark 6.1. Note that the relevant algebras considered in §3.2 and above, are already inside the C*-algebra $\mathcal{L}_{\boldsymbol{\mathcal{A}}}(H_{\boldsymbol{\mathcal{A}}})$. In view of Remark 3.1, applying a completely positive state to pass to $H(\boldsymbol{\mathcal{A}})$ serves to make $\mathcal{L}_{\boldsymbol{\mathcal{A}}}(H_{\boldsymbol{\mathcal{A}}})$ a C*-subalgebra of $\mathcal{L}(H(\boldsymbol{\mathcal{A}}))$. Thus we may, if we wish, identify $\underline{\mathbb{A}}$ as a particular C*-subalgebra of $\mathcal{L}(H(\boldsymbol{\mathcal{A}}))$.

In the following we will also consider a commutative Banach subalgebra $\underline{\mathbb{B}}$ of $\underline{\mathbb{A}}$ as generated by elements of $i_K(\mathbb{A})$. Suppose we have some finite number s of commuting operators $L_i \in \mathbb{A}$, for $1 \leq i \leq s$. Recalling (6.2) and setting $T_i = \underline{i}_K(L_i) \in \underline{\mathbb{B}}$, leads to

$$0 = \underline{i}_K([L_i, L_j]_{\mathbb{A}}) = [\underline{i}_K(L_i), \underline{i}_K(L_j)]_{\underline{\mathbb{A}}} = [T_i, T_j]_{\underline{\mathbb{A}}}, \tag{6.3}$$

and hence we obtain a commuting s-tuple $\mathcal{T}(s) \equiv (T_1, \ldots, T_s) \in \underline{\mathbb{B}}^s$.

We establish a connection between the joint spectrum of the Burchnall-Chaundy ring $\mathbb{A}$ and the joint spectrum $\sigma(\mathcal{T}(s), \underline{\mathbb{B}})$ of a generating commuting s-tuple $\mathcal{T}(s)$ of operators in the Banach subalgebra $\underline{\mathbb{B}}$: while this is all we wish to say, standard facts of Banach algebra make it possible to see that the spectra are in fact homeomorphic.

Theorem 6.1. *For any s-tuple $\mathcal{T}(s) \in \underline{\mathbb{B}}^s$ of commuting operators that generate $\underline{\mathbb{B}}$, there exists an injective map of spectra*

$$\sigma(\mathcal{T}(s), \underline{\mathbb{B}}) \longrightarrow (X' = \mathrm{Spec}(\mathbb{A})) \times Y. \tag{6.4}$$

Proof. Recall that each $T_i = \underline{i}_K(L_i) \in \underline{\mathbb{B}}$, for $1 \leq i \leq s$. By definition of the spectrum, any point of the latter can be regarded as a non–trivial algebra homomorphism $\underline{\mathbb{B}} \xrightarrow{f} \mathbb{C}$, which by definition restricts to the ring of generators $\mathbb{A}$, that is, we have a restricted homomorphism $f|\mathbb{A} : \mathbb{A} \xrightarrow{f} \mathbb{C}$. Note that if f_1, f_2 are two such homomorphisms, then if $f_1 = f_2$ on $\mathbb{A}$, then likewise, $f_1 = f_2$ on $\underline{\mathbb{B}}$. Finally, by using the contravariance of the Spec–functor applied to the subring inclusion $\underline{i}_K$, an injective map $\sigma(\mathcal{T}(s), \underline{\mathbb{B}}) \longrightarrow X' = \mathrm{Spec}(\mathbb{A})$ thus follows. □

Remark 6.2. The s-tuple $\mathcal{T}(s) = (T_1, \ldots, T_s)$ of commuting operators in $\underline{\mathbb{B}}$, being images $T_i = \underline{i}_K(L_i)$ of operators that commute with L of order n, provide a solution to the n-th generalized KdV-hierarchy in (4.3), simply reproducing the construction in ([22, Proposition 4.12, Corollary 5.18, and §6].

7 Extensions of compact operators and K-homology

7.1 Extensions of the ideal of compact operators

In relationship to $H_{\boldsymbol{\mathcal{A}}}$ we have the Calkin C*-algebra $\mathcal{Q}(H_{\boldsymbol{\mathcal{A}}}) = \mathcal{L}(H_{\boldsymbol{\mathcal{A}}})/\mathcal{K}(H_{\boldsymbol{\mathcal{A}}})$. Note that we have in our case:

$$\begin{aligned}
\mathcal{Q}(H_{\boldsymbol{\mathcal{A}}}) &= \mathcal{L}(H_{\boldsymbol{\mathcal{A}}})/\mathcal{K}(H_{\boldsymbol{\mathcal{A}}}) \\
&\cong \mathcal{L}(H \otimes \boldsymbol{\mathcal{A}})/\mathcal{K}(H \otimes \boldsymbol{\mathcal{A}}) \\
&\cong (\mathcal{L}(H) \otimes \boldsymbol{\mathcal{A}})/(\mathcal{K}(H) \otimes \boldsymbol{\mathcal{A}}) \\
&\cong (\mathcal{L}(H)/\mathcal{K}(H)) \otimes \boldsymbol{\mathcal{A}} \\
&= \mathcal{Q}(H) \otimes \boldsymbol{\mathcal{A}}.
\end{aligned} \tag{7.1}$$

Application of the Brown-Douglas-Fillmore extension theory [5] shows that an extension of the compact operators $\mathcal{K}(H_{\boldsymbol{\mathcal{A}}})$ by $\underline{\mathbb{A}}$ yields a unital *-monomorphism

$$\varrho : \underline{\mathbb{A}} \longrightarrow \mathcal{Q}(H_{\boldsymbol{\mathcal{A}}}). \tag{7.2}$$

Let us also note that by the discussion of §3.2, a Fredholm module $(H_{\boldsymbol{\mathcal{A}}}, S)$ over $\underline{\mathbb{A}}$ determines an element in K–homology $K^*(\underline{\mathbb{A}})$. Using the morphism ϱ in (7.2), we produce a well defined map from the ring $\mathbb{A}$ to $\mathcal{Q}(H) \otimes \boldsymbol{\mathcal{A}}$ as the composition

$$\varrho_K : \mathbb{A} \longrightarrow \mathbb{A} \otimes \boldsymbol{\mathcal{A}} \xrightarrow{i_K} \underline{\mathbb{A}} \xrightarrow{\varrho} \mathcal{Q}(H_{\boldsymbol{\mathcal{A}}}), \tag{7.3}$$

which shows that each element $L \in \mathbb{A}$ determines an extension of $\mathcal{K}(H_{\boldsymbol{\mathcal{A}}})$ by $\underline{\mathbb{A}}$. Furthermore, the contravariance of the "Spec" functor induces a well-defined map

$$\varrho_K^* : \mathrm{Spec}(\mathcal{Q}(H_{\boldsymbol{\mathcal{A}}})) \longrightarrow X' = \mathrm{Spec}(\mathbb{A}) \times Y. \tag{7.4}$$

7.2 Construction of a Kasparov KK-class

Let us now return to the (non-singular) algebraic curve $X = X' \cup \{x_\infty\}$ associated to Spec($\mathbb{A}$). Here we shall consider a fixed $y \in Y$ as in the case $\boldsymbol{\mathcal{A}} \cong \mathbb{C}$, and momentarily replace $H_{\boldsymbol{\mathcal{A}}}$ by $H = L^2(S^1, \mathbb{C})$. We will consider extensions such as (7.2) by the (commutative) C*-algebra $C(X)$. The group $\mathrm{Ext}(X)$ of these extensions is the same as the degree-1 K-homology group $K^1(C(X)) = \mathrm{Ext}(X)$ [4, 5]. More generally, $K^*(C(X))$ can be identified with Kasparov's KK-group $KK(C(X), \mathbb{C})$; we refer the reader to e.g. [4, 6, 15, 23] for complete details of this theory, but the main ingredients for constructing an element of such a group will be described below.

We will show how classes in $K^*(C(X))$, and hence $\mathrm{Ext}(X)$, can be constructed from subspaces $W \in \mathrm{Gr}(q, B)$ which are images of the Krichever correspondence in §4.2 (these are characterized by the size of the ring B_W [22, Remark 6.3]). As a starting point, we endow the holomorphic line bundle $\mathcal{L} \longrightarrow X$ with a general Hermitian metric (see e.g. [26, III Theorem 1.2]). The latter induces a canonical Hermitian connection ∇ on $\mathcal{L}$ whose $(0,1)$-component ∇'' is taken to be the $\bar{\partial}$–operator on sections (see e.g. [26, III Theorem 2.1]). Before looking at $K^*(C(X))$, we have the more general result:

Theorem 7.1. *To each holomorphic line bundle with connection $\{(\mathcal{L}, \nabla) \longrightarrow X\}$ there corresponds a class $u = u(\mathcal{L}, \nabla) \in KK(C(X), \boldsymbol{\mathcal{A}})$.*

Proof. Let $(\mathcal{L}, \nabla) \longrightarrow X$ be the holomorphic line bundle determined by $\mathcal{L}$ with the endowed Hermitian connection ∇ whose $(0,1)$-component ∇'' is taken to be the $\bar{\partial}$-operator on sections.

Since X has the Kähler property, we have a well–defined Dirac operator $D = \sqrt{2}(\bar{\partial}+\bar{\partial}^*)$ (see e.g. [20, (2.20)]). Next, following [4, 23] we specify a Kasparov bimodule $(\mathfrak{E}, S)$ which up to operator homotopy equivalence will (by definition) yield an element in the group $KK(C(X), \boldsymbol{\mathcal{A}})$.

Firstly, the space of $\boldsymbol{\mathcal{A}}$-valued, L^2-sections $\mathfrak{E} = L^2(X, \mathcal{L}) \otimes \boldsymbol{\mathcal{A}}$ is a $\mathbb{Z}_2$-graded Hilbert $\boldsymbol{\mathcal{A}}$-module acted on by $C(X)$ via a *-homomorphism $\pi : C(X) \longrightarrow \mathcal{L}(\mathfrak{E})$, such that for all $a \in C(X)$, $\pi(a)$ is homogeneous of degree 0. Secondly via tensoring by $\boldsymbol{\mathcal{A}}$, the operator D admits a quasi-inverse D^* such that $DD^* - \mathbf{1},\ D^*D - \mathbf{1} \in \mathcal{K}(\mathfrak{E})$ and leads to an associated operator $S = \begin{bmatrix} 0 & D^* \\ D & 0 \end{bmatrix}$ homogeneous of degree 1, such that [23]:

a) $\pi(a)(S^2 - \mathrm{Id}) \in \mathcal{K}(\mathfrak{E})$;

b) the commutator $[\pi(a), S] \in \mathcal{K}(\mathfrak{E})$.

This leads to the bimodule $(\mathfrak{E}, S)$ which up to operator homotopy equivalence yields the desired element in $KK(C(X), \boldsymbol{\mathcal{A}})$. □

We deduce the result for $K^*(C(X))$, and the special case of $\mathrm{Ext}(X)$, from Theorem 7.1 on setting $\boldsymbol{\mathcal{A}} = \mathbb{C}$:

Corollary 7.1. *To each holomorphic line bundle $\mathcal{L}$, corresponding to a certain subspace of* $\mathrm{Gr}(q, B)$ *via the Krichever correspondence of §4.2, upon endowing it with Hermitian metric and attendant canonical connection $\{(\mathcal{L}, \nabla) \longrightarrow X\}$, there corresponds a class $u = u(\mathcal{L}, \nabla) \in K^*(C(X))$ whose degree-1 component, denoted $u^{[1]}$, defines an element of* $\mathrm{Ext}(X)$.

Suppose that Y is now taken to be a compact manifold. There is a Y-parametrized version of this last result when X is replaced by the compact manifold $X \times Y$. Let $\pi_X : X \times Y \longrightarrow X$ be the first factor projection.

Corollary 7.2. *Suppose $X \times Y$ has a* spin*-structure and let $(\widetilde{\mathcal{L}}, \widetilde{\nabla}) \longrightarrow X \times Y$ be a Hermitian line bundle with connection such that $\widetilde{\mathcal{L}} = \pi_X^*(\mathcal{L})$ where $\mathcal{L} \longrightarrow X$ is the holomorphic line bundle corresponding to a certain point $W \in$* $\mathrm{Gr}(q, B)$ *and $\widetilde{\nabla}$ is the pullback connection under π_X of the Hermitian connection ∇ on $\mathcal{L}$ as above. Then a Y-parametrized family $\{W_y\}_{y \in Y}$ of such subspaces of* $\mathrm{Gr}(q, B)$ *as above determines an element of $K^*(C(X \times Y))$.*

Proof. The first part follows just as in the proof of Theorem 7.1 for the case $\boldsymbol{\mathcal{A}} = \mathbb{C}$, since the assumption of a spin structure on $X \times Y$ provides a Dirac operator $\widetilde{D} : C^\infty(X \times Y, \widetilde{\mathcal{L}}) \longrightarrow C^\infty(X \times Y, \widetilde{\mathcal{L}})$ from which a homogeneous degree 1 operator $\widetilde{S}$ can be constructed as in the proof of Theorem 7.1. The analogous details apply to yield the corresponding bimodule $(\widetilde{\mathfrak{E}}, \widetilde{S}) = (L^2(X \times Y, \widetilde{\mathcal{L}}), \widetilde{S})$ leading to an element of $KK(C(X \times Y), \mathbb{C})$. The result then follows from the Y-parametrization of (5.10). □

7.3 The C*-algebra of the Jacobian and extensions

We return now to the Jacobian torus $J(X)$ of X and recall that there exists a holomorphic embedding $\mu : X \longrightarrow J(X)$ (see e.g. [13]). The following theorem establishes a relationship between the respective commutative C*-algebras of continuous functions $C(J(X))$ and $C(X)$ and the dual K_*-functor:

Theorem 7.2. *There exists a short exact sequence of C*-algebras*

$$0 \to \mathfrak{I} \xrightarrow{i} C(J(X)) \xrightarrow{p} C(X) \to 0, \tag{7.5}$$

where $\mathfrak{I}$ is a two-sided ideal in $C(J(X))$, i is injective and p is surjective. Furthermore, (7.5) *induces a periodic sequence of K_*–groups:*

$$\begin{tikzcd} K_0(\mathfrak{I}) \arrow[r, "i_*"] & K_0(C(J(X))) \arrow[r, "p_*"] & K_0(C(X)) \arrow[d] \\ K_1(C(X)) \arrow[u] & K_1(C(J(X))) \arrow[l, "p_*"] & K_1(\mathfrak{I}) \arrow[l, "i_*"] \end{tikzcd} \tag{7.6}$$

Proof. Firstly, from the embedding $\mu : X \longrightarrow J(X)$ we identify X as a closed subset of $J(X)$ and by standard principles, the induced map on continuous functions $p : C(J(X)) \longrightarrow C(X)$ is surjective. Alternatively, we note that the transcendence degrees of the rings of meromorphic functions $\mathsf{Mero}(J(X))$ and $\mathsf{Mero}(X)$ are $g_X \geq 1$ and 1, respectively. Thus with respect to a two-sided ideal $\mathfrak{I}_0$ in $\mathsf{Mero}(J(X))$ we have a short exact sequence:

$$0 \to \mathfrak{I}_0 \xrightarrow{i} \mathsf{Mero}(J(X)) \xrightarrow{p} \mathsf{Mero}(X) \to 0, \tag{7.7}$$

where in each case the elements of the ring can be approximated by Laurent polynomials extendable to the continuous functions. Hence (7.5) follows. The last part follows from the periodicity of the K_*-functor [6, 12]. $\square$

The periodicity sequence in (7.6) corresponds to the analogous sequence in the K-theory of spaces:

$$\begin{tikzcd} K^0(J(X)\backslash \mathrm{Im}\ \mu) \arrow[r] & K^0(J(X)) \arrow[r] & K^0(X) \arrow[d] \\ K^1(X) \arrow[u] & K^1(J(X)) \arrow[l] & K^1(J(X)\backslash \mathrm{Im}\ \mu) \arrow[l] \end{tikzcd} \tag{7.8}$$

Now we return to K-homology and in particular, in degree 1 the following establishes a relationship between the 'Ext' classes of X and $J(X)$ on introducing the homology Chern character [2, 6]. Note that this is the case in which the BDF theory guarantees that elements of Ext correspond to *-monomorphisms of the C-algebra to the Calkin algebra, and that the set of monomorphisms of the Burchnall-Chaundy algebra into the algebras $B_W \subset \mathcal{L}(H_{\mathcal{A}})$ is indeed the Jacobian, as proved by Burchnall and Chaundy: from this point of view, the Jacobian can be mapped to elements of Ext.

Theorem 7.3. *The following diagram is commutative*

$$\begin{array}{ccc} K^1(C(X)) & \xrightarrow{\hat{\mu}_*} & K^1(C(J(X))) \\ {\scriptstyle \mathrm{ch}_1}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{ch}_1} \\ H_1(X,\mathbb{Q}) & \xrightarrow{\mu_* \cong} & H_1(J(X),\mathbb{Q}) \end{array} \tag{7.9}$$

Here the map μ_* is an isomorphism and the map $\hat{\mu}_*$ is injective. In particular, when $g_X = 1$, the map $\hat{\mu}_*$ is an isomorphism.

Proof. Applying the functor K^* to (7.5), the resulting long exact sequence yields an injective map $\hat{\mu}_* : K^1(C(X)) \longrightarrow K^1(C(J(X)))$. The commutativity of the diagram arises from applying the homology Chern character homomorphism to each side. Regarding the lower horizontal arrow μ_*, we recall some elementary facts concerning X and its Jacobian $J(X)$ (see e.g. [13]). Setting the genus $g_X = g$, if $\delta_1, \ldots, \delta_{2g}$ are 1–cycles in X forming a (canonical) basis for $H_1(X,\mathbb{Z})$, then $H_1(X,\mathbb{Z}) \cong \mathbb{Z}\{\delta_1, \ldots, \delta_{2g}\}$. Next, we identify $J(X) = \mathbb{C}^g/\Lambda$ where $\Lambda \subset \mathbb{C}^g$ is a discrete lattice of maximal rank $2g$. Accordingly, $H_1(J(X),\mathbb{Z}) \cong \Lambda \cong \mathbb{Z}\{\lambda_1, \ldots, \lambda_{2g}\}$, for a basis $\lambda_1, \ldots, \lambda_{2g}$ for Λ. Thus, from the (one-to-one) assignment $\delta_i \mapsto \lambda_i$, $1 \leq i \leq 2g$, it follows that we have an isomorphism $H_1(X,\mathbb{Z}) \cong H_1(J(X),\mathbb{Z})$. Since both X and $J(X)$ are quotients of their respective covering spaces by torsion-free discrete groups, then on tensoring the respective integral homology groups by $\mathbb{Q}$, it also follows that $H_1(X,\mathbb{Q}) \cong H_1(J(X),\mathbb{Q})$. In the case of genus $g_X = 1$, X is, up to the choice of a base point, an elliptic curve (a complex 1-dimensional torus), the map μ is an isomorphism and consequently induces an isomorphism $\hat{\mu}_* : K^1(C(X)) \longrightarrow K^1(C(J(X)))$, in other words, $\mathrm{Ext}(X) \cong \mathrm{Ext}(J(X))$. □

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References

[1] A. Álvarez Vásquez, J. M. Munoz Porras and F. J. Plaza Martin: The algebraic formalism of soliton equations over arbitrary base fields, in 'Workshop on Abelian varieties and Theta Functions' (Morelia MX, 1996), 3–40, *Aportaciones Mat. Investig., 13, Soc. Mat. Mexicana*, Mexico 1998.

[2] P. Baum and R. G. Douglas: K–homology and index theory. *Operator algebras and applications, Part 1, Proc. Symp. Pure Math.* **38**, Amer. Math. Soc, Providence RI, 1982, 117—173.

[3] D. Beltiţă: *Smooth homogeneous structures in operator theory*, Monographs and Surveys in Pure and Appl. Math. **137**, Chapman and Hall/CRC, Boca Raton Fl. 2006.

[4] B. Blackadar: *K-Theory for Operator Algebras.* Springer Verlag 1986.

[5] L. G. Brown, R. G. Douglas and P. A. Fillmore: Extensions of C*-algebras and K-homology. *Ann. of Math.* **105** (1977), 265–324.

[6] A. Connes: *Noncommutative Geometry.* Academic Press 1994.

[7] G. Corach, H. Porta and L. Recht: Differential geometry of systems of projections in Banach algebras. *Pacific J. Math.* **143** (2), (1990), 209–228.

[8] R. G. Douglas: *Banach Algebra Techniques in Operator Theory* (2nd Ed.). Graduate Texts in Mathematics **179**, Springer Verlag, New York-Berlin, 1998.

[9] M. J. Dupré and P.A. Fillmore: Triviality theorems for Hilbert modules. *Proceedings of the* 5^{th} *Roumanian Conference on Operator Theory*, Birkhäuser Verlag, Boston, Mass, 1981, 71–79.

[10] M. J. Dupré and J. F. Glazebrook: The Stiefel bundle of a Banach algebra. *Integral Equations Operator Theory* **41** No. 3, (2001), 264–287.

[11] M. J. Dupré, J. F. Glazebrook and E. Previato: A Banach algebra version of the Sato Grassmannian and commutative rings of differential operators. *Acta Applicandae Math.* **92** (3) (2006), 241–267.

[12] P. A. Fillmore: *A User's Guide to Operator Algebras.* Wiley–Interscience, New York, 1996.

[13] P. A. Griffiths and J. Harris: *Principles of Algebraic Geometry*, J. Wiley & Sons, New York, 1978.

[14] G. F. Helminck and A. G. Helminck: Hilbert flag varieties and their Kähler structure. *J. Phys. A: Math. Gen.* **35** (2002), 8531–8550.

[15] G. G. Kasparov: The operator K-functor and extensions of C*-algebras. *Math. U.S.S.R. Izv.* **16** (1981) No 3513-572. Translated from *Izv. Akad. Nauk. S.S.S.R. Ser. mat* **44** (1980), 571–636.

[16] I. M. Krichever: Integration of nonlinear equations by the methods of algebraic geometry. *Funkcional. Anal. i Priložen.* **11** No. 1 (1977), 15–31.

[17] E. C. Lance: *Hilbert C*-Modules. London Math. Soc. Lect. Notes* **210**, Cambridge Univ. Press, 1995.

[18] H. Porta and L. Recht: Spaces of projections in a Banach algebra. *Acta Cient. Venez.* **38**, (1987), 408–426.

[19] A. Pressley and G. Segal: *Loop Groups and their Representations.* Oxford Univ. Press, 1986.

[20] J. Roe: *Elliptic operators, topology and asymptotic methods*, Pitman Res. Notes in math. **179**, Longman Scientific/Wiley, New York 1988.

[21] M. Sato: The KP hierarchy and infinite dimensional Grassmann manifolds, *Theta functions – Bowdoin 1987, Part 1* (Brunswick, ME, 1987), 51–66, Proc. Sympos. Pure Math., 49, Part 1, Amer. Math. Soc., Providence, RI, 1989.

[22] G. Segal and G. Wilson: Loop groups and equations of KdV type. *Publ. Math. IHES* **61**, (1985), 5–65.

[23] G. Skandalis: Kasparov's bivariant K-Theory and applications. *Expo. Math.* **9** (1991), 193–250.

[24] J. F. Smith: The p-classes of a Hilbert module. *Proc. Amer. Math. Soc.* **36** (2) (1972), 428–434.

[25] H. Upmeier: *Symmetric Banach Manifolds and Jordan C*-Algebras.* Math. Studies **104**, North Holland, Amsterdam-New York-Oxford, 1985.

[26] R. Wells: *Differential Analysis on Complex manifolds.* Grad. Texts in Math. **65**, Springer-Verlag, New York Heidelberg Berlin, 1980.

[27] M. G. Zaidenberg, S. G. Krein, P. A. Kushment and A. A. Pankov: Banach bundles and linear operators. *Russian Math. Surveys* **30** No. 5, (1975), 115–175.