COHOMOLGY OF THE NILPOTENT SUBALGEBRAS OF CURRENT LIE ALGEBRAS

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COHOMOLOGY OF THE NILPOTENT SUBALGEBRAS OF CURRENT LIE ALGEBRAS

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Introduction

In this paper we compute the one and two dimensional cohomology of the maximal nilpotent subalgebras of affine Lie algebras with coefficients in the adjoint representation. We also prove one of the possible analogies of the Bott-Kostant theorem for current Lie algebras. This article is an enlarged version of the note [4].

Let $\mathfrak{g}$ be a complex semisimple finite-dimensional Lie algebra, $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ its Cartan decomposition, $\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t,t^{-1}]$ the corresponding current Lie algebra, i.e. the Lie algebra of functions $S^1 \longrightarrow \mathfrak{g}$, having a finite Laurent expansion, with the bracket given by the formula $[f,g](x) = [f(x),g(x)], f,g \in \hat{\mathfrak{g}}, x \in S^1$. Note that $\hat{\mathfrak{g}}$ admits the natural grading: $\hat{\mathfrak{g}} = \oplus \hat{\mathfrak{g}}_m$ where $\hat{\mathfrak{g}}_m = \mathfrak{g} \otimes t^m$. Let us denote $(\mathfrak{n}_+ \otimes 1) \oplus (\mathfrak{g} \otimes t) \oplus (\mathfrak{g} \otimes t^2) \oplus \ldots$ by $\hat{\mathfrak{n}}_+$ and $\mathfrak{g} \otimes \mathbb{C}[t]$ by $\mathfrak{g}[t]$; $\hat{\mathfrak{n}}_+$ and $\mathfrak{g}[t]$ inherits the grading from $\hat{\mathfrak{g}}$. We shall identify $\mathfrak{g}$ with $\mathfrak{g} \otimes 1 \subset \hat{\mathfrak{g}}$.

Recall that a current algebra is the quotient of an affine Lie algebra by its centre ([10]). Note that the main idea in the investigation of the cohomology of current algebras (as well as the other Kac-Moody algebras) is the analogy with the theory of finite-dimensional semisimple complex Lie algebras. In particular, $\hat{\mathfrak{n}}_+$ is a counterpart of the maximal nilpotent subalgebra of a finite-dimensional semisimple Lie algebra. So, we can use the well-known methods for computing the cohomology with the help of the Laplace operator ([11]), the Bernstein-Gelfand-Gelfand resolvent ([2]) etc. In [9] V. Kac proved that the cohomology space of $\hat{\mathfrak{n}}_+$ with trivial coefficients is in one-one correspondence with the group algebra of the affine Weyl

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group. As a consequence he obtained the Kac-MacDonald's identities.

Another approach to the cohomology of current algebras uses the ideas from the cohomology theory of the Lie algebra of tangent vector fields on a smooth manifold ([8]). Now we are going to use both methods.

In [12] Leger and Luks computed $H^2(\mathfrak{n}_+;\mathfrak{n}_+)$ (for another computation see [17]). They used the following method. The cohomology of $\mathfrak{n}_+$ with coefficients in an irreducible finite-dimensional representation V of $\mathfrak{g}$ is well-known. Namely, the Bott-Kostant Theorem (see, [11],[2]) asserts that $\dim H^i(\mathfrak{n}_+;V)$ is equal to the number of elements of length i in the Weyl group of $\mathfrak{g}$. In particular, we know $H^\cdot(\mathfrak{n}_+;\mathfrak{g})$, where $\mathfrak{g}$ is the adjoint representation. Consider now the exact sequences of $\mathfrak{n}_+$-modules:

$$0 \to \mathfrak{n}_+ \to \mathfrak{g} \to \mathfrak{g}/\mathfrak{n}_+ \to 0\ , \quad 0 \to \mathfrak{h} \to \mathfrak{g}/\mathfrak{n}_+ \to \mathfrak{n}_+^* \to 0 .$$

Here $(\mathfrak{g}/\mathfrak{n}_+)/\mathfrak{h}$ can be identified with $\mathfrak{n}_+^*$ by means of the Killing form. These sequences allow us to reduce the computation of $H^2(\mathfrak{n}_+;\mathfrak{n}_+)$ to that of $H^1(\mathfrak{n}_+;\mathfrak{n}_+^*)$ and this space can be determined directly. In this paper we compute $H^i(\hat{\mathfrak{n}}_+;\hat{\mathfrak{n}}_+)$ for $i=1,2$, generalizing the method in [12]. Another approach to affine algebras is contained in [6].

In Section 1 we prove a Theorem, analogous to the Bott-Kostant Theorem, while in Section 2 we calculate $H^i(\hat{\mathfrak{n}}_+;\hat{\mathfrak{n}}_+)$ for $i=1,2$.

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1. Computation of $H^\cdot(\hat{\mathfrak{n}}_+;\hat{\mathfrak{g}})$

The Bott-Kostant Theorem can be generalized to affine Lie algebras at least in two ways. The most direct generalization is the following one: if V is an irreducible representation of the current algebra with dominant highest weight, then

$\dim H^i(\hat{n}_+;V)$ is equal to the number of elements of length i in the Weyl group. The proof is similar to that of the finite-dimensional case. The adjoint representation however is not a module of highest weight.

In this Section we give another generalization of the Bott-Kostant Theorem, namely we compute the cohomology of $\hat{n}_+$ with coefficients in modules similar to the adjoint module consisting of functions on the circle S^1 with values in the representation space of $\mathfrak{g}$.

Let V be a representation of $\mathfrak{g}$, A a $\mathbb{C}$-algebra and $\varphi:\mathbb{C}[t,t^{-1}] \to A$ a homomorphism. Define a representation of $\hat{\mathfrak{g}}$ in $V \otimes A$ by the formula

$$(x \otimes f)(v \otimes a) = x(v) \otimes \varphi(f)a, \quad x \in \mathfrak{g}\ , \ v \in V, \ f \in \mathbb{C}[t,t^{-1}], \ a \in A.$$

We need two special cases: $A = \mathbb{C}[t,t^{-1}]$, φ is the identity map and $A = \mathbb{C}$, $\varphi(f) = f(1)$. In the first case denote the module $V \otimes A$ by $\hat{V}$ and in the second case by V_1. The elements of $\hat{V}$ are rational functions $\mathbb{C} \longrightarrow V$, regular outside the origin. The mapping, sending a function $\mathbb{C} \longrightarrow V$ to its value at 1, is a homomorphism $\hat{V} \longrightarrow V_1$.

The space $\hat{V}$ is endowed with an obvious module structure over $\mathbb{C}[t,t^{-1}]$ and multiplication by an element of $\mathbb{C}[t,t^{-1}]$ is a $\mathfrak{g}$-endomorphism of the $\hat{\mathfrak{g}}$-module $\hat{V}$. Notice that $\hat{V}$ is a graded $\hat{\mathfrak{g}}$-module, $\hat{V} = \bigoplus_{i \in \mathbb{Z}} \hat{V}_i$ where $\hat{V}_i = V \otimes t^i$.

Now we are going to investigate the cohomology of $\hat{n}_+$ with coefficients in $\hat{V}$. Denote by $C^{\cdot}(\hat{n}_+;\hat{V})$ the cochain complex of $\hat{n}_+$ with coefficients in the $\hat{n}_+$-module $\hat{V}$. The complex $C^{\cdot}(\hat{n}_+;\hat{V})$ is graded by weights: $C^{\cdot}(\hat{n}_+;\hat{V}) = \bigoplus_{m \in \mathbb{Z}} C^{\cdot}_{(m)}(\hat{n}_+;\hat{V})$, where for the cochain $\varphi \in C^q_{(m)}(\hat{n}_+;\hat{V})$ the weight of $\varphi(e_{i_1},\dots,e_{i_q})$ is $m + i_1 + \dots + i_q$ (i_k is the weight of e_{i_k}).

<u>Lemma 1.</u> For all m the complexes $C^{\cdot}_{(m)}(\hat{n}_+;\hat{V})$ are isomorphic to each other and to the complex $C^{\cdot}(\hat{n}_+;V_1)$.

In fact, the composition of the embedding

$C^{\cdot}_{(m)}(\hat{\mathfrak{n}}_+;\hat{V}) \longrightarrow C^{\cdot}(\hat{\mathfrak{n}}_+;\hat{V})$ and of the mapping $C^{\cdot}(\hat{\mathfrak{n}}_+;\hat{V}) \longrightarrow C^{\cdot}(\hat{\mathfrak{n}}_+;V_1)$ induced by the homomorphism $\hat{V} \longrightarrow V_1$ is an isomorphism.

From the above it follows that the space $H^{\cdot}(\hat{\mathfrak{g}};\hat{V})$ is a $\mathbb{C}[t,t^{-1}]$-module. Lemma 1 can be reformulated as follows.

Lemma 2. $H^{\cdot}(\hat{\mathfrak{g}};\hat{V}) \cong \mathbb{C}[t,t^{-1}]\otimes_{\mathbb{C}} H^{\cdot}(\hat{\mathfrak{g}};V_1)$.

Let us now deal with the computation of $H^{\cdot}(\hat{\mathfrak{n}}_+;V_1)$. The Lie algebra $\hat{\mathfrak{n}}_+$ is embedded into $\mathfrak{g}[t]$, V_1 is naturally endowed with a $\mathfrak{g}[t]$-module structure, consequently the homomorphism

$$\nu : H^{\cdot}(\mathfrak{g}[t],\mathfrak{g};V_1) \longrightarrow H^{\cdot}(\mathfrak{g}[t];V_1) \longrightarrow H^{\cdot}(\hat{\mathfrak{n}}_+;V_1)$$

is defined.
Let τ be the homomorphism

$$H^{\cdot}(\hat{\mathfrak{n}}_+) \otimes H^{\cdot}(\mathfrak{g}[t],\mathfrak{g};V_1) \longrightarrow H^{\cdot}(\hat{\mathfrak{n}}_+;V_1) ,$$

sending $u\otimes v$ to the cohomology class $u\nu(v)$.

Proposition 1. If V is a finite dimensional representation of $\mathfrak{g}$ then τ is an isomorphism.

The proof will be given below. Proposition 1 and Lemma 2 imply the basic result of this Section.

Theorem 1. $H^i(\hat{\mathfrak{n}}_+;\hat{\mathfrak{g}}) \cong \mathbb{C}[t,t^{-1}]\otimes_{\mathbb{C}} H^{i-1}(\hat{\mathfrak{n}}_+)$ for any nonnegative integer i.

Indeed, set $V=\mathfrak{g}$. The space $H^i(\hat{\mathfrak{n}}_+,\hat{\mathfrak{g}})$ is a $\mathbb{C}[t,t^{-1}]$-module. It follows from Lemma 2 that $H^i(\hat{\mathfrak{n}}_+,\hat{\mathfrak{g}})$ is a free module of rank equal to $\dim H^i(\hat{\mathfrak{n}}_+,V_1)$. The cohomology of $\hat{\mathfrak{n}}_+$ with trivial coefficients is known (see for instance [7]). Using this result, it is not difficult to find the cohomology $H^{\cdot}$ of $\mathfrak{g}\otimes t\oplus\mathfrak{g}\otimes t^2\oplus \dots$. We only need the following fact. The space $H^{\cdot}$ is a $\mathfrak{g}$-module, and $\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g},H^i)=0$ if $i\neq 1$ and $\mathbb{C}$ if $i=1$ (see [13]). As $H^i(\mathfrak{g}[t],\mathfrak{g};V) \cong \mathrm{Hom}_{\mathfrak{g}}(V,H^i)$, this gives us that $H^i(\mathfrak{g}[t],\mathfrak{g};\mathfrak{g})=0$ for $i\neq 1$ and is one-dimensional for $i=1$. After this it is enough to apply Proposition 1 and we get

$H^i(\hat{n}_+;V_1) = H^{i-1}(\hat{n}_+)$.

Let us prove now Proposition 1. Introduce two subalgebras of $\hat{\mathfrak{g}}$: $\overline{\mathfrak{g}} = (t-1)\mathfrak{g} \oplus (t-1)^2\mathfrak{g} \oplus \ldots$ and $\overline{n} = \hat{n}_+ \cap \overline{\mathfrak{g}}$. Let G be a compact connected Lie group corresponding to a compact real form of $\mathfrak{g}$.

<u>Lemma 3.</u> $H^\cdot(\overline{n}) \cong H^\cdot(\hat{n}_+) \otimes H^\cdot(\overline{\mathfrak{g}}) \otimes H^\cdot(\Omega G)$.
Here ΩG is the loop space of G.

<u>Proof.</u> Since $\mathfrak{g}[t] = \hat{n}_+ + \overline{\mathfrak{g}}$ and $\overline{n} = \hat{n}_+ \cap \overline{\mathfrak{g}}$ we have $C^\cdot(\overline{n}) = C^\cdot(\hat{n}_+) \otimes_{C^\cdot(\mathfrak{g}[t])} C^\cdot(\overline{\mathfrak{g}})$. Here the tensor product is taken in the category of differential algebras. In such a situation there exists a spectral sequence (Eilenberg-Moore, see [15]), connecting the cohomology of these four differential algebras. This spectral sequence generalizes the Künneth formula [14]. It converges to $H^\cdot(\overline{n})$ and the second term is isomorphic to $\mathrm{Tor}_A(H^\cdot(\hat{n}_+), H^\cdot(\overline{\mathfrak{g}}))$, where $A = H^\cdot(\mathfrak{g}[t])$. We remark that $H^\cdot(\mathfrak{g}[t]) \cong H^\cdot(\mathfrak{g})$ (see e.g. [3]) and $H^\cdot(\mathfrak{g})$ acts trivially on $H^\cdot(\hat{n}_+)$ and on $H^\cdot(\overline{\mathfrak{g}})$ (for $H^\cdot(\overline{\mathfrak{g}})$ this is trivial and for $H^\cdot(\hat{n}_+)$ this follows from the fact that the composition $H^\cdot(\mathfrak{g}) \longrightarrow H^\cdot(n_+) \longrightarrow H^\cdot(\hat{n}_+)$ is trivial). It follows from this that the second term of the spectral sequence is isomorphic to $H^\cdot(\hat{n}_+) \otimes H^\cdot(\overline{\mathfrak{g}}) \otimes \mathrm{Tor}_A(\mathbb{C},\mathbb{C})$.

We will show now that $\mathrm{Tor}_A(\mathbb{C},\mathbb{C}) \cong H^\cdot(\Omega G)$. Indeed, the cohomology algebra of $\mathfrak{g}$ with trivial coefficients coincides with the cohomology algebra of G and by the Hopf Theorem it is commutative and free [16]. Using the computation of $\mathrm{Tor}_A(\mathbb{C},\mathbb{C})$ for the free commutative algebra A (Proposition 7.3

from [15] and see also [1]) and the connection between the cohomology of G and ΩG, we obtain the isomorphism $\mathrm{Tor}_A(\mathbb{C},\mathbb{C}) \cong H^\cdot(\Omega G)$.

Now it can be shown that the spectral sequence degenerates (e.g. by indicating explicit cycles of $C^\cdot(\bar{\mathfrak{n}})$ which represent the generators of E_2, which we shall do at the end of this Section). Lemma 3 is proved.

The Lie algebra $\bar{\mathfrak{n}}$ is an ideal in $\hat{\mathfrak{n}}_+$ and $\hat{\mathfrak{n}}_+/\bar{\mathfrak{n}} \cong \mathfrak{g}$. In virtue of this, $\mathfrak{g}$ acts on $H^\cdot(\bar{\mathfrak{n}})$. The algebra $\mathfrak{g}$ acts trivially on $H^\cdot(\hat{\mathfrak{n}}_+)$ and on $H^\cdot(\Omega G)$, but on $H^\cdot(\bar{\mathfrak{g}})$ it acts in the standard way ($\bar{\mathfrak{g}} \cong \mathfrak{g}\otimes t \oplus \mathfrak{g}\otimes t^2 \oplus \ldots$ is an ideal of $\mathfrak{g}[t]$, $\mathfrak{g}[t]/\bar{\mathfrak{g}} \cong \mathfrak{g}$, so $\mathfrak{g}$ acts on $\bar{\mathfrak{g}}$ naturally and the action of $\mathfrak{g}$ on $H^\cdot(\bar{\mathfrak{g}})$ is semisimple).

Now to finish the proof of Theorem 1 let us consider the Serre-Hochshild spectral sequence, associated with $\hat{\mathfrak{n}}_+$, its ideal $\bar{\mathfrak{n}}$ and the module V_1, converging to $H^\cdot(\hat{\mathfrak{n}}_+;V_1)$. The algebra $\bar{\mathfrak{n}}$ acts on V_1 trivially. The second term of this spectral sequence is the following:

$$H^\cdot(\mathfrak{g};H^\cdot(\bar{\mathfrak{n}},V_1)) \cong H^\cdot(\mathfrak{g};H^\cdot(\bar{\mathfrak{n}}) \otimes V_1) \cong$$

$$\cong H^\cdot(\mathfrak{g};H^\cdot(\hat{\mathfrak{n}}_+) \otimes H^\cdot(\bar{\mathfrak{g}}) \otimes H^\cdot(\Omega G) \otimes V_1) \cong$$

$$\cong H^\cdot(\hat{\mathfrak{n}}_+) \otimes H^\cdot(\Omega G) \otimes H^\cdot(\mathfrak{g};H^\cdot(\bar{\mathfrak{g}}) \otimes V_1) .$$

As $\mathfrak{g}$ is semisimple, $H^\cdot(\bar{\mathfrak{g}}) \otimes V_1$ is the direct sum of finite-dimensional representations, i.e.

$$H^\cdot(\mathfrak{g};H^\cdot(\bar{\mathfrak{g}}) \otimes V_1) \cong H^\cdot(\mathfrak{g}) \otimes I$$

where I is the invariant space of $H^{\cdot}(\bar{\mathfrak{g}}) \otimes V_1$ (see [7]). Note that $I \cong H^{\cdot}(\mathfrak{g}[t],\mathfrak{g};V_1)$. The differentials in the above sequence act in the following way: they map the generators of the algebra $H^{\cdot}(\Omega G)$ into the generators of $H^{\cdot}(\mathfrak{g})$ and are trivial on $H^{\cdot}(\hat{\mathfrak{n}}_+) \otimes H^{\cdot}(\mathfrak{g}[t],\mathfrak{g};V_1)$. It follows from this that the spectral sequence converges to $H^{\cdot}(\hat{\mathfrak{n}}_+) \otimes H^{\cdot}(\mathfrak{g}[t],\mathfrak{g};V_1)$. Thus our spectral sequence is the product of $H^{\cdot}(\hat{\mathfrak{n}}_+) \otimes H^{\cdot}(\mathfrak{g}[t],\mathfrak{g};V_1)$ with the spectral sequence of the Serre path fibration $EG \to G$; it follows from this that τ is an isomorphism.

Now we explain why the spectral sequence in the proof of Lemma 3 collapses. To define explicitly cycles of $C^{\cdot}(\bar{\mathfrak{n}})$, representing the generators of E_2 we apply the continuous cohomology theory. Let $\mathfrak{n}(0,1)$ be the Lie algebra of infinitely differentiable functions $f:[0,1] \longrightarrow \mathfrak{g}$ such that $f(0) \in \mathfrak{n}$, $f(1) = 0$. Denote by $C_c^{\cdot}(0,1)$ the complex of cochains of $\mathfrak{n}(0,1)$, continuous in the C^{∞}-topology. Let α be a generator of $H^{\cdot}(\mathfrak{g})$ and $\bar{\alpha}$ a cochain representing α. For $p \in [0,1]$ denote by ϕ_p the homomorphism $\bar{\mathfrak{n}} \longrightarrow \mathfrak{g}$, "the value at p": $\phi_p((t-1)g_1,(t-1)^2 g_2,\ldots) = \sum_m (p-1)^m g_m$. Let $\alpha_p = \phi_p^{*}\bar{\alpha}$, $\alpha_p \in C_c^{\cdot}(0,1)$. Choose $\bar{\alpha}$ in such a way that $\alpha_0 = \alpha_1 = 0$. Let $p \neq 0,1$; then we can define the cochain $\frac{\partial\alpha}{\partial x}(p)$ where x is the coordinate on $[0,1]$. It is shown in [3] that $\frac{\partial\alpha}{\partial x}(p)$ is a coboundary $\frac{\partial\alpha}{\partial x}(p) = \delta\omega(p)$ where δ is the differential in $C_c^{\cdot}(0,1)$. Indeed, let $K_p (p \neq 0,1)$ be the cochain complex of $\bar{\mathfrak{n}}$ with support at p. It is proved in the same paper that the cohomology of K_p is isomorphic to $H^{\cdot}(\mathfrak{g})$. Now, K_p

is W_1-module, where W_1 is the Lie algebra of formal vector fields at the point p. But $H^{\cdot}(\mathfrak{g})$ is finite-dimensional and W_1 has no nontrivial finite-dimensional representations. We conclude that if $\omega \in K_p$ and $\delta\nu = 0$ then $\partial/_{\partial x}\nu$ is the differential of some other cocycle $\bar{\nu} \in K_p$.

This means that

$$\alpha_p - \alpha_q = \delta \int_p^q \omega(x)dx \quad .$$

In particular, $\delta \int_0^1 \omega(x)dx = 0$. Suppose that $\alpha' = \int_0^1 \omega(x)dx$. The cochain α' represents a nontrivial cohomology class of $\bar{\mathfrak{n}}$.

The Lie algebras $\hat{\mathfrak{n}}_+$ and $\bar{\mathfrak{g}} = \mathfrak{g} \otimes (t-1) \oplus \mathfrak{g} \otimes (t-1)^2 \oplus \ldots$ are graded. Similarly the cochain complexes are also graded. Note that the cochain complex K_0 of $\mathfrak{n}(0,1)$ with support in 0 is isomorphic to $\oplus\, C_i^{\cdot}(\hat{\mathfrak{n}}_+)$ and the cochain complex K_1 with support in 1 is isomorphic to $\oplus\, C_i^{\cdot}(\bar{\mathfrak{g}})$. It follows from this that the cohomology of K_0 and K_1 is isomorphic to $H^{\cdot}(\hat{\mathfrak{n}}_+)$ and $H^{\cdot}(\bar{\mathfrak{g}})$ respectively.

Recall that $H^{\cdot}(\mathfrak{g})$ is isomorphic to the free graded commutative algebra on generators $\xi_1, \xi_2, \ldots$, $\deg \xi_k = 2k+1$. Using the above construction assign to each ξ_i a representative cocycle ξ_i'.

<u>Proposition 2.</u> The space $H^{\cdot}(\bar{\mathfrak{n}})$ is generated by the cohomology classes of cochains of form $u \wedge v \wedge P(\xi_1', \xi_2', \ldots)$, where $u \in K_0$, $v \in K_1$ are cocycles, corresponding to the elements

of $H^{\cdot}(\hat{\mathfrak{n}}_+)$ and $H^{\cdot}(\bar{\mathfrak{g}})$ respectively and P is an arbitrary polynomial in generators $\xi_1', \xi_2', \ldots$.

The proof of this Proposition follows from the construction above for continuous cohomology (a similar argument in a more difficult situation was used in [5]). In particular, we have an explicit construction of cochains, representing the generators of E_2 in the proof of Lemma 3., surviving till E_∞.

2. <u>Computation of</u> $H^i(\hat{\mathfrak{n}};\hat{\mathfrak{n}})$ <u>for</u> $i = 1,2$

Let us consider the next exact sequences:

$$0 \to \hat{\mathfrak{n}}_+ \to \hat{\mathfrak{g}} \to \hat{\mathfrak{g}}/\hat{\mathfrak{n}}_+ \to 0; \quad 0 \to \mathfrak{h} \to \hat{\mathfrak{g}}/\hat{\mathfrak{n}}_+ \to \hat{\mathfrak{n}}_+^* \to 0$$

$((\hat{\mathfrak{g}}/\hat{\mathfrak{n}})/\mathfrak{h}$ can be identified with $\hat{\mathfrak{n}}_+^*$ by means of the Killing form). Consider the induced exact cohomology sequences:

$$H^0(\hat{\mathfrak{n}}_+;\hat{\mathfrak{g}}/\hat{\mathfrak{n}}_+) \to H^1(\hat{\mathfrak{n}}_+;\hat{\mathfrak{n}}_+) \to H^1(\hat{\mathfrak{n}}_+;\hat{\mathfrak{g}}) \to H^1(\hat{\mathfrak{n}}_+;\hat{\mathfrak{g}}/\hat{\mathfrak{n}}_+) \to$$
$$\to H^2(\hat{\mathfrak{n}}_+;\hat{\mathfrak{n}}_+) \to H^2(\hat{\mathfrak{n}}_+;\hat{\mathfrak{g}}) \ ;$$
$$H^0(\hat{\mathfrak{n}}_+;\hat{\mathfrak{n}}_+^*) \to H^1(\hat{\mathfrak{n}}_+;\mathfrak{h}) \to H^1(\hat{\mathfrak{n}}_+;\hat{\mathfrak{g}}/\hat{\mathfrak{n}}_+) \to H^1(\hat{\mathfrak{n}}_+;\hat{\mathfrak{n}}_+^*) \to$$
$$\to H^2(\hat{\mathfrak{n}}_+;\mathfrak{h}).$$

The first sequence allows us to compute $H^1(\hat{\mathfrak{n}}_+;\hat{\mathfrak{n}}_+)$ at once. We will state the result, i.e. describe all the derivations of $\hat{\mathfrak{n}}_+$.

Each element $u \in \mathfrak{h}$ defines a cohomology class of $H^1(\hat{\mathfrak{n}}_+;\hat{\mathfrak{n}}_+)$ containing the cocycle: $f \to uf$, $(uf)(t) = [u,f(t)]$, where $f:\mathbb{C} \to \mathfrak{g}$, $f(0) \in \mathfrak{n}_+$. Then to each vector field $tP\,\partial/\partial t$ where P is a polynomial in t, we assign the cocycle

$$\hat{\mathfrak{n}}_+ \to \hat{\mathfrak{n}}_+ : f(t) \mapsto tP\,\partial f(t)/\partial t\,,\ f:\mathbb{C} \to \mathfrak{g}\ ,\ f(0) \in \mathfrak{n}_+ \ .$$

Theorem 2. The mapping, sending the elements of $\mathfrak{h}$ and $t\mathbb{C}[t]\,\partial/\partial t$ to the cohomology classes of the cocycles constructed above gives an isomorphism $\mathfrak{h} \oplus t\mathbb{C}[t]\partial/\partial t \cong H^1(\hat{\mathfrak{n}}_+;\hat{\mathfrak{n}}_+)$.

In other words, an arbitrary derivation of $\hat{\mathfrak{n}}$ is uniquely represented as $u + tP\,\partial/\partial t + q$ where $u \in \mathfrak{h}$, $P \in \mathbb{C}[t]$ and q is an inner derivation.

For the computation of $H^2(\hat{\mathfrak{n}}_+;\hat{\mathfrak{n}}_+)$ by a similar way, we have to know $H^1(\hat{\mathfrak{n}}_+;\hat{\mathfrak{g}}/\hat{\mathfrak{n}}_+)$. This space appears in the second exact sequence and to find it we have to know the isomorphism $H^1(\hat{\mathfrak{n}}_+;\hat{\mathfrak{n}}_+^*) \cong (H_1(\hat{\mathfrak{n}}_+;\hat{\mathfrak{n}}_+))^*$. For this we are going to use the next general construction (see Theorem 4.1 in [12]).

Let L be a Lie algebra, T a derivation of L acting in a semi-simple way and whose eigenvalues are positive. (These restrictions on T can be considerably weakened.) It is clear that such a derivation must be outer. It is easy to see that $\hat{\mathfrak{n}}_+$ has such a derivation. For instance, we can take $u + t\,\partial/\partial t$, where $u \in \mathfrak{h}$, $\langle\gamma,u\rangle \in \mathbb{R}$, $0 < \langle\gamma,u\rangle < \varepsilon$, γ is an arbitrary positive root, ε is a small positive number (we can take $\varepsilon < \frac{1}{2}$).

Let $W(L)$ be the Weyl algebra, associated with L. Recall that $W(L)$ is the standard complex of the differential Lie superalgebra $\overline{L} = L_0 \oplus L_1$, $L_0 \cong L$, L_1 as L_0-module is the adjoint representation and $[x,x] = 0$ if $x \in L_1$. The differential d acts as follows: $d(L_0) = 0$, $d(L_1) \to L_0$ is an isomorphism of L-modules. In other words, $W(L)$ is a differential graded algebra, spanned by L_0^* and L_1^*, $L_0^* \cong L_1^* \cong L^*$ where L_0^* has degree 1 and L_1^* has degree 2, $\phi: L_0^* \to L_1^*$ is the canonical isomorphism. The differential is defined by the formula $\delta\beta = \delta_0\beta + \phi(\beta)$ where $\delta_0\beta$ is the differential of β considered as an element of the standard cochain complex of L. It is not difficult to show that the complex $W(L)$ is acyclic in positive dimensions. The next Lemma states even more.

In $W(L)$ we define a filtration: $W_i = \bigoplus_{j \geq i} \Lambda^*(L_0^*) \otimes S^j(L_1)^*$. Consider the corresponding spectral sequence E.

<u>Lemma 4.</u> The spectral sequence

$$E_2^{p,q} = H^q(L; S^{p/2}L^*) \Longrightarrow H(W(L))$$

is trivial, beginning from E_3.

<u>Remark.</u> For arbitrary algebra L this is of course not true, but in this Lemma we consider such L which satisfies a strong additional assumption: there exists such a semisimple derivation $T: L \longrightarrow L$ for which all the eigenvalues are positive. Such a derivation can only exist in case of nilpotent algebras, and for them also not always. Let us consider in our case such a T.

<u>Proof of Lemma 4.</u> The differential

$$\delta = d_2^{p,q} : H^q(L; S^{p/2}L^*) \to H^{q-1}(L; S^{p/2+1}L^*)$$

is defined on the cochain level by the formula $(\frac{p}{2} = r)$

$$[(\delta\phi)(\ell_1,\dots,\ell_{q-1})](\ell_1',\dots,\ell_{r+1}') = \sum_{j=1}^{r+1} [\phi(\ell_1,\dots,\ell_{q-1},\ell_j')](\ell_1',\dots,\hat{j},\dots,\ell_{r+1}').$$

Define the map $D: H^{q-1}(L; S^{r+1}L^*) \to H^q(L; S^rL^*)$ by the formula

$$[(D\phi)(\ell_1,\dots,\ell_q)](\ell_1',\dots,\ell_r') = \sum_{i=1}^{p} (-1)^{p-i}[\phi(\ell_1,\dots,\hat{i},\dots,\ell_q)](T\ell_i,\ell_1',\dots,\ell_r').$$

It is easy to check that the bracket $[D,\delta]$ coincides with the map, defined in $H^{\cdot}(L;S^*L^*)$ by T. This map has only positive eigenvalues and can be transposed with δ. Define D_0 as $\frac{1}{\lambda}D$ on the λ-eigenspace of T in $H^{\cdot\cdot}(L;S^*L^*)$. Then D_0 is a contracting homotopy in the complex $E_2 = \{H^{\cdot\cdot}(L;S^*L^*),\delta\}$ and this means that $E_3 = 0$.

This lemma implies in particular, that the sequence

$$0 \to H^2(L) \to H^1(L,L^*) \to H^0(L,S^2L^*) \to 0$$

is exact. The arrows here are differentials in the second term of E. We remark that $H^0(L,S^2L^*)$ is exactly the space of invariant bilinear symmetric forms on L.

Theorem 3. The space of invariant bilinear symmetric forms on $\hat{\mathfrak{n}}_+$ is the direct sum of the following two subspaces intersecting trivally.

a) The first space consists of the forms whose kernel contains $[\hat{\mathfrak{n}}_+,\hat{\mathfrak{n}}_+]$. This space is isomorphic to the space of quadratic forms on $\hat{\mathfrak{n}}_+/[\hat{\mathfrak{n}}_+,\hat{\mathfrak{n}}_+]$ i.e. has dimension $(\ell+1)(\ell+2)/2$ where ℓ is the rank of $\mathfrak{g}$.

b) Let $P(t^{-1})\ \partial/\partial t$ be a vector field, where P is a polynomial without constant term. The second space consists of the forms

$$(x,y) \longmapsto \langle P(t^{-1})\ \partial x/\partial t,y\rangle + \langle P(t^{-1})\ \partial y/\partial t,x\rangle$$

where $x,y \in \hat{\mathfrak{n}}_+$, $\langle,\rangle$ is the Killing form on $\hat{\mathfrak{g}}$.

Proof. Let ω be an invariant bilinear symmetric form on $\hat{\mathfrak{n}}_+$. Let us assign to the quadratic form, associated with ω the mapping $\theta:\hat{\mathfrak{n}}_+ \longrightarrow \hat{\mathfrak{n}}_+^*$;
here $\hat{\mathfrak{n}}_+ = \mathfrak{n}_+ \oplus \mathfrak{g}\otimes t \oplus \mathfrak{g}\otimes t^2 \oplus \dots$, $\hat{\mathfrak{n}}_+^* = \mathfrak{n}_+^* \oplus (\mathfrak{g}\otimes t)^* \oplus (\mathfrak{g}\otimes t^2)^* \oplus \dots$.
Suppose that ω is homogeneous with respect to the grading by the weight of t; the $\hat{\mathfrak{n}}_+$-module $\hat{\mathfrak{n}}_+^*$ is filtrated by the submodules $\hat{\mathfrak{n}}_0^* = \mathfrak{n}_+^*$, $\hat{\mathfrak{n}}_1^* = \mathfrak{n}_+^* \oplus (\mathfrak{g}\otimes t)^*$ etc. Let i be the smallest number such that $\theta(\hat{\mathfrak{n}}_+) \subset \hat{\mathfrak{n}}_i^*$. If $i=0$ then ω lies in the first factor. The assertion that the kernel of this form contains $[\hat{\mathfrak{n}}_+,\hat{\mathfrak{n}}_+]$ follows from Theorem 5.1 in [12]. If $i=1$ then the image of the mapping $\mathfrak{n}_+ \longrightarrow (\mathfrak{g}t)^*$ is either one-dimensional or coincides with $\mathfrak{n}_+ (\mathfrak{n}_+ \subset \mathfrak{g} \cong (\mathfrak{g}\otimes t)^*$, $\mathfrak{g}$ is identified

with $(\mathfrak{g}\otimes t)^*$ by the Killing form). In the first case ω lies in the first factor and in the second case in the second one. These facts follow from the following simple Lemma.

<u>Lemma 5.</u> $\dim \mathrm{Hom}_{\mathfrak{n}_+}(\mathfrak{n}_+,\mathfrak{g}) = 1+\ell\,,\ \ell > 1.$

(This Lemma can be verified for instance by looking over all the simple Lie algebras).

Further, by using Lemma 5, we get that if $i \geq 2$, then the form belongs to the second space. This completes the proof of Theorem 3.

Now, using the exact sequence (3), we compute $H^1(\hat{\mathfrak{n}}_+;\hat{\mathfrak{n}}_+^*)$ and after this we can determine $H^2(\hat{\mathfrak{n}}_+;\hat{\mathfrak{n}}_+)$.

<u>Theorem 4.</u> a) The kernel of the natural mapping

$$\phi : H^1(\hat{\mathfrak{n}}_+) \times H^1(\hat{\mathfrak{n}}_+;\hat{\mathfrak{n}}_+) \longrightarrow H^2(\hat{\mathfrak{n}}_+;\hat{\mathfrak{n}}_+)$$

is $\ell+1$ - dimensional where ℓ is the rank of $\mathfrak{g}$.

b) If $\mathrm{rank}\,\mathfrak{g} > 1$, then $\dim \mathrm{coker}\,\phi = \ell+1+p$ where p is the number of positive roots of $\mathfrak{g}$ representable as the sum of two simple roots.

The case $\mathfrak{g} = sl\,(2,\mathbb{C})$ is not covered by this Theorem. This case is really an exceptional one (see [6]).

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