# A few comments on N=2 supersymmetric Landau-Ginzburg theories

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In the recent very interesting paper by Cecotti and Vafa [1] they have considered N=2 supersymmetric Landau-Ginzburg theories and have showed that in many cases the metric for supersymmetric ground states for special deformations of this metric satisfies the certain system of PDE's, such for example as Toda equations.

The purpose of the present paper is to give the additional examples of such theories.

**1.** Let us remind first at all some basic facts from N=2 supersymmetric Landau-Ginzburg theory (for more details see [1]). The basic quantities here are the chiral fields $\phi_i$, the vacuum state $|0>$ and the states

$$|j>=\phi_j|0>. \tag{1}$$

The action of $\phi_j$ on this state is given by the formula

$$\phi_i|j>=\phi_i\phi_j|0>=C_{ij}{}^k\phi_k|0>=C_{ij}^k|k>. \tag{2}$$

So the action of the chiral field $\phi_i$ in the subsector of vacuum states is given by the matrix $(C_i)_j^k = C_{ij}{}^k$. Analogously, we have anti-chiral fields $\phi_{\bar{i}}$ and the states $|\bar{j}>$. So we may define two metric tensors

$$\eta_{ij} =< j|i> \tag{3}$$

and

$$g_{i\bar{j}} =< \bar{j}|i>, \tag{4}$$

which should satisfy the condition

$$\eta^{-1}g(\eta^{-1}g)^* = 1. \tag{5}$$

The theory is determined by the superpotential $w(x_a)$ which is holomorphic function of complex variables $x_a$. The superpotential completely determines the chiral ring

$$\mathcal{R} = \mathbf{C}[x_a]/\partial_a w \tag{6}$$

and we may also determine the metric $\eta_{ij}$ by the formula

$$\eta_{ij} =< i|j>= Res_w[\phi_i\phi_j], \tag{7}$$

where

$$Res_w[\phi] = \sum_{dw=0}\phi(x)H^{-1}(x); \quad H = det(\partial_i\partial_j w). \tag{8}$$

As for the metric $g_{i\bar{j}}$, it depends from parameters $t_1, t_2, \ldots$, entering to the superpotential $w(x_a)$. As was shown in [1], it should satisfy the zero-curvature conditions

$$\bar{\partial}_i(g\partial_j g^{-1}) - [C_j, g(C_i)^+ g^{-1}] = 0, \quad \partial_i = \frac{\partial}{\partial t_i}, \bar{\partial}_j = \frac{\partial}{\partial \bar{t}_j}. \tag{9}$$

$$\partial_i C_j - \partial_j C_i + [g(\partial_i g^{-1}), C_j] - [g(\partial_j g^{-1}), C_i] = 0, \tag{10}$$

and also should satisfy the " reality constraint " (5)

In the paper [1] the many interesting examples of Landau-Ginzburg theories were considered. In the next sections we consider two new examples of such theories.

**2.** The model :

$$w(x) = t(e^x - x). \tag{11}$$

Here :

$$w'(x) = t(e^x - 1), \tag{12}$$

and we may identify an element of $\mathcal{R}$ with the set of the values of the function $\phi(x)$ at critical points of $w(x)$:

$$x_j = 2\pi ji, \quad j \in \mathbf{Z}, \quad \phi(x) \in \mathcal{R} \mapsto \{(\phi)_j\}, \quad (\phi)_j = \phi(2\pi ij). \tag{13}$$

The multiplication operation acts componentwise on $\phi$ and we have also

$$w'' = te^x \tag{14}$$

and

$$Res(\phi) = \frac{1}{t}\sum_j (\phi)_j. \tag{15}$$

We choose as basis in $\mathcal{R}$ the elements $a_k \quad (k \in \mathbf{Z})$, such that

$$(a_k)_j = \delta_{kj}. \tag{16}$$

In this basis we have

$$\eta_{kl} = \frac{1}{t}\delta_{kl}. \tag{17}$$

Also

$$(C)_l^k = (1 - 2\pi ik)\delta_l^k. \tag{18}$$

Let us define

$$g_{j\bar{k}} = <\bar{k}|j> . \tag{19}$$

Then we can see that $w(x)$ is quasi-invariant relative to the translation operation:

$$T : f(x) \to f(x + 2\pi i) \tag{20}$$

$$Tw(x) = w(x) - 2\pi i. \tag{21}$$

So the metric $g_{i\bar{k}}$ should be invariant at this transformation

$$g_{j\bar{k}} = g_{j+1,\overline{k+1}}, \tag{22}$$

or

$$g_{j,\overline{j+k}} = g_{0,\overline{k}} = f_k. \tag{23}$$

Now instead the set $\{f_k\}$ we may consider the function

$$f(\theta) = \sum_k f_k e^{2\pi i\theta k}, \tag{24}$$

or

$$g_{k,\bar{l}} = \int_0^1 f(\theta) e^{-2\pi i(k-l)\theta} d\theta. \tag{25}$$

The reality condition (2.9) now take the form

$$|t|^2 |f(\theta)|^2 = 1. \tag{26}$$

Hence

$$f(\theta) = \frac{1}{|t|} exp(i\varphi(t,\bar{t};\theta)). \tag{27}$$

As for equation (3.9) we have

$$\bar{\partial}(f\partial f^{-1}) - [C, fC^+ f^{-1}] = 0, \partial = \frac{\partial}{\partial t}, \bar{\partial} = \frac{\partial}{\partial \bar{t}}. \tag{28}$$

Now

$$C^+ f^{-1} = [C^+, f^{-1}] + f^{-1} C^+. \tag{29}$$

Hence we have

$$\bar{\partial}(f\partial f^{-1}) - [C, f[C^+, f^{-1}]] - [C, C^+] = 0, \tag{30}$$

but

$$[C^+, f^{-1}] = \frac{d}{d\theta}(f^{-1}). \tag{31}$$

Hence we have

$$\bar{\partial}(f\partial f^{-1}) + \frac{d}{d\theta}(f\frac{d}{d\theta}(f^{-1})) = 0 \tag{32}$$

or

$$\bar{\partial}\partial\varphi + \frac{d^2}{d\theta^2}\varphi = 0. \tag{33}$$

Finally:

$$f = f(t,\bar{t};\theta) = \frac{1}{|t|} exp(i\varphi(t,\bar{t};\theta)), \tag{34}$$

$$\Delta_3 \varphi = 0, \tag{35}$$

$$\varphi = \varphi(t,\bar{t};\theta), \quad \varphi(t,\bar{t};\theta+1) = \phi(t,\bar{t};\theta), \tag{36}$$

$$\Delta_3 = \frac{\partial^2}{\partial t_1^2} + \frac{\partial^2}{\partial t_2^2} + \frac{\partial^2}{\partial \theta^2}, \quad t = t_1 + it_2. \tag{37}$$

Note,that the variables may be separated in this equation and if you know $\phi(0,0;\theta)$ or $\phi(t,\bar{t};\theta)$ we may solve this equation $|t| \to \infty$ explicitely.

**3.** The model :

$$w = w_c = t(\frac{1}{2}e^{2x} - 2ce^x + x), \quad c > 1. \tag{38}$$

Here

$$w'(x) = t(e^{2x} - 2ce^x + 1) = 2te^x(\cosh x - c) = 2te^x(\cosh\ x - \cosh\ \gamma), \quad c = \cosh\gamma. \tag{39}$$

We may identify an element of $\mathcal{R}$ with the set of values of the function $\phi(x)$ at critical points $w(x)$:

$$\{x_j\} = \{a_j, b_j\}, \quad a_j = -\gamma + 2\pi ij, \quad b_j = \gamma + 2\pi ij \tag{40}$$

$$\phi(x) \in \mathcal{R} \mapsto \{(\phi)_j^{a,b}\}, (\phi)_j^{a,b} = \phi(\mp\gamma + 2\pi ij). \tag{41}$$

The multiplication operation acts componentwise on $\phi$ and we have also

$$w'' = 2te^x(e^x - c) = 2te^x(e^x - \cosh\gamma). \tag{42}$$

At $x = a_j$ we have $w'' = -2te^{-\gamma}\sinh\gamma$.

At $x = b_j$ we have $w'' = 2te^{\gamma}\sinh\gamma$.

Here

$$Res(\phi) = \frac{1}{2t\sinh\gamma}\sum_j(-e^{\gamma}\phi(a_j) + e^{-\gamma}\phi(b_j)). \tag{43}$$

We choose the basis in $\mathcal{R}$ related to $a_j$ and $b_k$ and in this basis we have

$$\eta_{k,l}^{a,b} = \mp\frac{1}{2t\sinh\gamma}e^{\pm\gamma}\delta_{k,l}. \tag{44}$$

Also

$$(w_j)^{a,b} = t(A \pm B + 2\pi ij), \tag{45}$$

where

$$A = -(1 + \frac{1}{2}\cosh 2\gamma), \quad B = \frac{1}{2}\sinh 2\gamma - \gamma. \tag{46}$$

Hence

$$C = \begin{pmatrix} C^a & 0 \\ 0 & C^b \end{pmatrix}. \tag{47}$$

$$(C^a)_l^k = (A + B + 2\pi ik)\delta_l^k \quad (C^b)_l^k = (A - B + 2\pi ik)\delta_l^k. \tag{48}$$

The matrix $g$ has now the block form

$$g = \begin{pmatrix} g^{aa} & g^{ab} \\ g^{ba} & g^{bb} \end{pmatrix},$$

$$g^{aa} = \{g_{j,k}^{aa}\}, \dots \quad . \tag{49}$$

The invariance group in this case is generated by the translation

$$T : x \to x + 2\pi i. \tag{50}$$

Hence

$$g_{j+l,k+l}^{aa} = g_{j,k}^{aa}, \dots, \tag{51}$$

and

$$g^{aa}(\theta) = \sum e^{2\pi i(k-j)\theta} g_{j,k}, \dots,$$

$$g(\theta) = \begin{pmatrix} g^{aa}(\theta) & g^{ab}(\theta) \\ g^{ba}(\theta) & g^{bb}(\theta) \end{pmatrix}. \tag{52}$$

In these notations

$$\eta = \frac{1}{2t \sinh \gamma} \begin{pmatrix} -e^{\gamma} & 0 \\ 0 & e^{-\gamma} \end{pmatrix}, \tag{53}$$

$$C = (AI + B\Sigma_3 + \frac{d}{d\theta}), \quad C^+ = (AI + B\Sigma_3 - \frac{d}{d\theta}). \tag{54}$$

The reality condition should be taken in the form

$$(\eta^{-1} g(\theta))(\eta^{-1}(g(-\theta))^* = I. \tag{55}$$

It is easy to show that this condition is equivalent one

$$(\eta_0^{-1} \tilde{g}(\theta))(\eta_0^{-1} \tilde{g}(-\theta))^* = I, \quad \eta_0 = \frac{1}{2t \sinh \gamma}(-\Sigma_3). \tag{56}$$

Here

$$g = D\tilde{g}D, \quad D = \begin{pmatrix} e^{-\gamma/2} & 0 \\ 0 & e^{\gamma/2} \end{pmatrix}. \tag{57}$$

So, up to normalized factor, we may consider that $\tilde{g} \in SU(1,1)$. The equation (3) may be reduced now to the equation for the matrix $\tilde{g}$ :

$$\bar{\partial}(\tilde{g}\partial\tilde{g}^{-1}) - [C, \tilde{g}[C^+, \tilde{g}^{-1}]] = 0, \quad C = \frac{d}{d\theta} + B\Sigma_3, \quad C^+ = -\frac{d}{d\theta} + B\Sigma_3, \tag{58}$$

or

$$\bar{\partial}(\tilde{g}\partial\tilde{g}^{-1}) + [\frac{d}{d\theta} + B\Sigma_3, \tilde{g}[\frac{d}{d\theta} - B\Sigma_3, \tilde{g}^{-1}]] = 0. \tag{59}$$

Let

$$\tilde{g}\partial\tilde{g}^{-1} = A_t, \quad \tilde{g}\frac{d}{d\theta}\tilde{g}^{-1} = A_\theta. \tag{60}$$

We have also

$$\frac{d}{d\theta}\tilde{g}^{-1} = \tilde{g}^{-1}A_\theta, \quad \frac{d}{d\theta}\tilde{g} = -A_\theta\tilde{g}. \tag{61}$$

Finally we have

$$\bar{\partial}A_t + \frac{d}{d\theta}A_\theta - B[(\tilde{g}\Sigma_3\tilde{g}^{-1} - \Sigma_3), A_\theta] - B^2[\Sigma_3, \tilde{g}\Sigma_3\tilde{g}^{-1}] = 0, \tag{62}$$

where

$$A_t = \tilde{g}\partial\tilde{g}^{-1}, \quad A_\theta = \tilde{g}\frac{d}{d\theta}\tilde{g}^{-1}, \quad B = \frac{1}{2}\sinh 2\gamma - \gamma,$$
$$\tilde{g} \in SU(1,1), \quad A_t, A_\theta \in su(1,1), \quad \tilde{g}\Sigma_3\tilde{g}^{-1} \in su(1,1). \tag{63}$$

Note that for $B \to 0$ (this corresponds to the case of one chain with double zeros) we obtain the equation of principal chiral field in 3-dimensions with coordinates $t_1, t_2$ and $\theta$ for the group $SU(1,1)$ (see [1]):

$$\partial_\mu\tilde{g}\partial_\mu\tilde{g}^{-1} = 0, \quad \mu = 1,2,3. \tag{64}$$

Here we considered the case of two chains of zeros. The consideration of arbitrary finite number of chains gives analogous equation for some real simple Lie algebra.

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## References

[ 1] S. Cecotti, C. Vafa, *Nucl. Phys. B367* , *359* (1991)