ON AFFINE ALGEBRAS

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These notes contain a unified approach, via bimodules, to a number of results of Artin-Tate type. Throughout we will keep the following notation:

k is a commutative ring (with 1), and
R and S are k-algebras.

As is customary and convenient, (R,S)-bimodules V are assumed to have identical k-operations on both sides: $\xi v = v\xi$ $(v \in V, \xi \in k)$.

1. BIMODULES AND AFFINE ALGEBRAS

LEMMA 1. Let $0 \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow 0$ be a short exact sequence of (R,S)-bimodules and assume that V_S and ${}_RW$ are finitely generated, say $V = Rv_1+\ldots+Rv_m+U$ for suitable $v_i \in V$.

If S is affine over k, then there exists an affine k-subalgebra $R' \subseteq R$ and a finitely generated (R',S)-subbimodule $U' \subseteq U$ such that

$$V = R'v_1 + \ldots + R'v_m + U' .$$

PROOF. Write $V = w_1S + \ldots + w_nS$ and let $x_1,\ldots,x_t \in S$ be k-algebra generators for S. Then

$$w_i = \sum_{h=1}^{m} r_{ih} v_h + u_i , \quad v_i x_j = \sum_{h=1}^{m} r_{ijh} v_h + u_{ij}$$

for suitable r_{ih}, $r_{ijh} \in R$ and u_i, $u_{ij} \in U$. Let $R' \subseteq R$ be the k-subalgebra generated by the r_{ih}'s and r_{ijh}'s, and let $U' \subseteq U$ be the (R',S)-subbimodule generated by the u_i's and u_{ij}'s. Then $V' = R'v_1 + \ldots + R'v_m + U'$ contains w_i and $v_h x_j$ for all i,j,h.

Hence

$$V'x_j = \sum_{h=1}^{m} R'v_h x_j + U'x_j \subseteq R'V' + U' = V' \quad .$$

Since $V'k = kV' \subseteq V'$, it follows that $\sum_{i=1}^{n} w_i S \subseteq V'S \subseteq V'$, whence $V' = V$.

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COROLLARY 1. Let $R \subseteq S$ be k-algebras such that S is affine over k and finitely generated as a left module over R . Then S is also finitely generated as a left module over some affine subalgebra $R' \subseteq R$, with the same module generators.

PROOF. Take $U = 0$ and $V = {}_R S_S$ in the lemma.

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Recall that, for a left R-module V , the trace of V in R is defined by

$$\mathrm{Tr}_R(V) = \sum\{\mathrm{Im}\, f \mid f \in \mathrm{Hom}_R(V,R)\} \quad .$$

$\mathrm{Tr}_R(V)$ is a two-sided ideal of R , and V is a generator for R-mod , the category of left R-modules, if and only if $\mathrm{Tr}_R(V) = R$.

LEMMA 2. Let V be an (R,S)-bimodule such that ${}_RV$ and V_S are finitely generated, and assume that S is affine over k . Suppose that R contains an affine k-subalgebra $A \subseteq R$ and a finitely generated left ideal I with $I \subseteq \mathrm{Tr}_R(V)$ and $R = \langle A,I \rangle_{k\text{-algebra}}$ $(=A + IA)$. Then R is affine over k . This happens in particular if ${}_RV$ is a generator for R-mod .

PROOF. By assumption on I , there exist finitely many $f_i \in \mathrm{Hom}_R(V,R)$ with $I \subseteq \sum \mathrm{Im}\, f_i$. After enlarging I if necessary we may therefore assume that a finite direct sum of copies of ${}_RV$ maps onto I . By Lemma 1, with $U = 0$, there exists an affine k-subalgebra

$R' \subseteq R$ such that ${}_{R'}V$ is finitely generated. Hence ${}_{R'}I$ is also finitely generated, and A, R' , and the generators of I over R' together generate $\langle A, I\rangle_{k\text{-algebra}} = R$.

$\blacksquare$

<u>COROLLARY 2</u>. Let $R \subseteq S$ be k-algebras with S affine over k . Assume that S and $Tr_R(S)$ are finitely generated as left modules over R . Then R is affine over k if and only if $R/Tr_R(S)$ is affine over k .

<u>PROOF</u>. Apply Lemma 2 with $V = {}_R S_S$ and $I = Tr_R(S)$.

$\blacksquare$

2. SOME APPLICATIONS

<u>(A) CORNERS OF RINGS</u>. Assume that S is affine over k and let $e = e^2 \in S$. If SeS is finitely generated as left ideal of S , then eSe is affine over k . (Montgomery-Small [6]).

<u>PROOF</u>. By [6, Lemma 1], eS is finitely generated as left module over eSe . Now take $V = eS$ and $R = eSe$ in Lemma 2 and note that ${}_R V$ maps onto ${}_R R$ via $es \longmapsto ese \ (s \in S)$.

$\blacksquare$

<u>(B) MORITA EQUIVALENCE</u>. If A and B are Morita equivalent rings, then there exists an (A,B)-bimodule P such that ${}_A P$ and P_B are finitely generated projective generators for A-mod , resp. mod-B . In case A and B are k-algebras, and the left and right k-operations on P agree, we conclude from Lemma 2 that A is affine over k if and only if B is affine over k . (Wadsworth, cf.[6,Acknowledgement]).

<u>(C) RESULTS OF ARTIN-TATE TYPE</u>. Let $R \subseteq S$ be k-algebras with S affine over k and ${}_R S$ finitely generated. Then R is affine over k in each of the following cases:

i. R is a finitely generated left module over a commutative subalgebra and k is Noetherian;

ii. S is left Noetherian and $R \subseteq S$ is a finite centralizing extension (i.e., $S = \sum_{i=1}^{n} Rx_i$ with $x_i r = rs_i$ for all $r \in R$);

iii. ${}_R S$ is projective and, for each proper two-sided ideal M of R, $MS \neq S$ (e.g., if ${}_R S$ is free or if ${}_R S$ is projective and maximal ideals of R are localizable);

iv. k is Noetherian and, for some commutative subalgebra $C \subseteq R$, the module $\left(\frac{S}{R}\right)_C$ is Noetherian.

PROOF. (i). One can clearly assume that R itself is commutative. Choose $R' \subseteq R$ as in Corollary 1. Then R' is Noetherian, by the Hilbert basis theorem, and hence ${}_{R'}R$ is finitely generated, as ${}_{R'}S$ is. Thus R is affine.

(ii). Again, Corollary 1 yields $R' \subseteq R$ affine such that $R' \subseteq S$ is a finite centralizing extension. As S is left Noetherian, the Eisenbud-Eakin theorem [3] implies that R' is likewise. Now argue as in (i).

(iii). Set $T = Tr_R(S)$. Then $TS = S$, by the dual basis lemma, and so we must have $T = R$. (Actually, by [2], ${}_R S$ maps onto ${}_R R$.) The result now follows from Corollary 2.

(iv). Let $C \subseteq R$ be commutative with $\left(\frac{S}{R}\right)_C$ Noetherian and set $X = \{r \in R \mid Sr \subseteq R\} = \operatorname{ann}\left(\frac{S}{R}\right)_R$. Then $X = SX \subseteq Tr_R({}_R S)$ and $X = \bigcap_v \text{rt. ann}_R(v+R)$, where v runs over a finite generating set for ${}_R\left(\frac{S}{R}\right)$. Therefore, $\left(\frac{R}{X}\right)_R \hookrightarrow \left(\frac{S}{R}\right)_R^n$ for some n. Since $\left(\frac{S}{R}\right)_C$ is Noetherian, we conclude that $\left(\frac{R}{X}\right)_C$ is Noetherian, and hence $\left(\frac{S}{X}\right)_C$ is also Noetherian. By Lemma 1, with $V = {}_S S_C$ and $W = {}_S\left(\frac{S}{X}\right)_C$ (and with sides interchanged), we can find an affine subalgebra $C' \subseteq C$ and a finitely generated (S,C')-subbimodule $X' \subseteq X$ with $\left(\frac{S}{X'}\right)_{C'}$ finitely generated. Now C' is Noetherian and so $\left(\frac{R}{X'}\right)_{C'}$ is finitely generated too. Moreover, since ${}_R S$ is finitely generated, X' is also finitely generated as (R,C')-bimodule, say

$$X' = \sum_{i=1}^{n} R x_i C' \quad .$$

Now set $I = \sum_{i=1}^{n} R x_i$, so that I is a finitely generated left ideal of R with $I \subseteq Tr_R(S)$, and let $A \subseteq R$ be the subalgebra generated by C' and the generators of $\left(\frac{R}{X'}\right)_{C'}$. Then A is affine and $R = A + X' = A + IA$. Thus Lemma 2 yields the result.

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REMARKS. (i) is a mild generalization of the original Artin-Tate Lemma [1] and has been observed by a number of people.

(ii) is contained in [6]. Using a result of Formanek and Jategaonkar [4] instead of the Eisenbud-Eakin theorem, the same proof yields versions of (ii) which work for certain finite normalizing extensions $R \subseteq S$. For example, if $S = \sum_{i=1}^{n} R x_i$ with $rx_i = x_i r^{\sigma_i}$ for certain automorphismus σ_i of R and if $G = \langle \sigma_1, \ldots, \sigma_n \rangle$ acts locally finitely on R , then the argument goes through, because we can then choose $R' \subseteq R$ to be affine and normalized by x_i's . Also, for any finite normalizing extension $R \subseteq S$, proper right ideals of R extend to proper right ideals of S [5]. Thus (iii) above applies to finite normalizing extensions $R \subseteq S$ with ${}_R S$ projective. The question as to whether the Artin-Tate lemma holds for general finite normalizing extensions $R \subseteq S$, with S left Noetherian, say, was raised in [6] and is still open as far as I know.

For the moment, let $\mathcal{T}$ denote the class of finitely generated left R-modules V such that $Tr_R(V)V = V$. Then the assumptions in (iii) could be replaced by: ${}_R S \in \mathcal{T}$ and, for each proper two-sided ideal M of R , $MS \neq S$. Now $\mathcal{T}$ contains all finitely generated projective modules over R as well as, clearly, all generators of R-mod , and $\mathcal{T}$ is closed under direct sums. But I don't know of an easy characterization of the modules in $\mathcal{T}$.

The prototype of (iv) (with $C = k$) is due to Lance Small (oral communication).

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