DEFORMATION THEORY OF REPRESENTATIONS OF PROP(ERAD)S

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Abstract. For any prop(erad) $\mathcal{P}$ admitting a minimal model we construct a L_∞-algebra controlling deformations of strongly homotopy $\mathcal{P}$-(al)gebras and illustrate it with some particular examples. We also construct higher operations on the deformation complex of algebraic structures generalizing the braces operations on Hochschild cochain complex involved in the proof of Deligne's conjecture for instance.

Introduction

The theory of props and properads provides us with a universal language to describe many algebraic, topological and differential geometric structures. Our main purpose in this paper is to introduce deformation theory of these structures via the associated prop(erad)s. One of our central results associates canonically to a pair, $(\mathcal{F}(V), \partial)$ and (P, d), consisting of a differential graded (dg, for short) quasi-free prop(erad) $(\mathcal{F}(V), \partial)$ on a $\mathbb{Z}$-graded $\mathbb{S}$-bimodule V and an arbitrary dg prop(erad) (P, d), a structure of L_∞-algebra on the (shifted) graded vector space, $s^{-1}\mathrm{Hom}_{\mathbb{Z}}(V, P)$, of morphisms of $\mathbb{Z}$-graded $\mathbb{S}$-bimodules; the Maurer-Cartan elements of this L_∞-algebra are in one-to-one correspondence with the set of all dg morphisms, $\{(\mathcal{F}(V), \partial) \to (P, d)\}$, of prop(erad)s. This canonical L_∞-algebra is used then to define, for any particular morphism $\gamma : (\mathcal{F}(V), \partial) \to (P, d)$, another twisted L_∞-algebra which controls deformation theory of the morphism γ. In the special case when (P, d) is the endomorphism prop(erad), (End_X, d_X), of some dg vector space X, our theory gives L_∞-algebras which control deformation theory of many classical algebraic and geometric structures on X, for example, associative algebra structure, Lie algebra structure, commutative algebra structure, Lie bialgebra structure, associative bialgebra structure, formal Poisson structure, Nijenhuis structure etc. As the case of associative bialgebras has never been rigorously treated in the literature before, we discuss this example in full details; we prove, in particular, that the first term of the canonical L_∞-structure controlling deformation theory of bialgebras is precisely the Gerstenhaber-Schack differential.

We derive the deformation complex and its L_∞-structure from two general methods. First, we define the deformation complex as a total derived functor à la Quillen. We prove that this chain complex is isomorphic, up to a shift of degree, to the space of morphisms of $\mathbb{S}$-bimodule $\mathrm{Hom}^{\mathbb{S}}_{\bullet}(\mathcal{C}, \mathrm{End}_X)$, where $\mathcal{C}$ is a homotopy coprop(erad), that is the dual notion of prop(erad) with relations up to homotopy. Since End_X is a (strict) prop(erad), we prove that the space $\mathrm{Hom}^{\mathbb{S}}_{\bullet}(\mathcal{C}, \mathrm{End}_X)$ has a rich algebraic structure, namely it is a homotopy non-symmetric prop(erad), that is prop(erad) without the action of the symmetric groups and with relations up to homotopy. It is then easy to see that a homotopy non-symmetric prop(erad) induced canonically a L_∞-structure. Moreover, we argue that the higher operations are also interesting. Let consider the first example of this theory, the Koszul non-symmetric operad $\mathcal{A}s$. In this case, the deformation complex is Hochschild cochain complex of an associative algebra. And we have proved once again that this chain complex is a non-symmetric operad. This is algebraic structure is equivalent to the data of braces operations. We recall that these operations play a fundamental role in the proof of Deligne's conjecture (see McClure-Smith [MS02] and Tamarkin [Tam98] for instance).

The paper is organized as follows. In §1 we remind key facts about properads and props and we define the notion of non-symmetric prop(erad). In §2 we introduce and study the convolution prop(erad) canonically associated with a pair, $(\mathcal{C}, \mathcal{D})$, consisting of an arbitrary coprop(erad) $\mathcal{C}$

and an arbitrary prop(erad) $\mathcal{D}$; our main results here are Propositions 5 and 7 constructing a functor from the category of convolution prop(erad)s to the category of Lie-admissible algebras. In §3 we discuss bar and cobar constructions for (co)prop(erad)s, and prove Theorem 16 on bar-cobar resolutions extending thereby earlier results of [Val03] from weight-graded dg properads to arbitrary dg properads. In §4 we recall to the notion and properties of homotopy properads which were first introduced in [Gra06] and we define the notions of homotopy (co)prop(erad). In §5, we recall the definitions of quadratic and minimal models for prop(erad)s. We prove in Theorem 22 that the minimal model of a properad, when it exists, has always a particular form. In §6 we remind geometric interpretation of L_∞-algebras, and then use this geometric language to prove Theorem 27 which associates to pair, $(\mathcal{F}(V), \partial)$ and (P, d), consisting of quasi-free prop(erad) $(\mathcal{F}(V), \partial)$ and an arbitrary dg prop(erad) (P, d), a structure of L_∞-algebra on the (shifted) graded vector space, $s^{-1}\mathrm{Hom}_{\mathbb{Z}}(V, P)$. In §7, we define the deformation complex following Quillen's methods and identify it with $s^{-1}\mathrm{Hom}_{\mathbb{Z}}(V, P)$ in Theorem 29. We show next how this canonical L_∞-algebra gives rise to twisted L_∞-algebras which control deformation theories of particular morphisms $\gamma : (\mathcal{F}(V), \partial) \to (P, d)$ and then illustrate this general construction with several examples from algebra and geometry. In §8 we construct a functor from the category of homotopy properads to the category of L_∞-algebras. In §9 we prove that the space the space of morphisms of $\mathbb{S}$-bimodules from a homotopy prop(erad) to a prop(erad) is a homotopy prop(erad) and thus a L_∞-algebra.

Contents

In this paper, we will always work over a field $\mathbb{K}$ of characteristic 0. By graph we always mean a directed graph without oriented cycles.

1. (Co)Properads, (Co)Props and their non-symmetric versions

In this section, we recall briefly the definitions of (co)properad and (co)prop according to [Val03]. Generalizing the notion of non-symmetric operads to prop(erad), we introduce the notions of *non-symmetric properad* and *non-symmetric prop*. This definition is motivated by the following property : the space of coinvariants, under the action of the symmetric group, of a prop(erad) is canonically a non-symmetric prop(erad). We refer to [Val06] Appendix A for a longer introduction to properads and to [Val03] for a complete exposition.

1.1. **$\mathbb{S}$-bimodules, graphs, composition products.** A *(dg)$\mathbb{S}$-bimodule* is a collection $\{\mathcal{P}(m,n)\}_{m,\,n\in\mathbb{N}}$ of dg modules over the symmetric groups $\mathbb{S}_n$ on the right and $\mathbb{S}_m$ on the left. These two actions are supposed to commute. In the sequel, we will only consider reduced $\mathbb{S}$-bimodules, that is $\mathbb{S}$-bimodules $\mathcal{P}$ such that $\mathcal{P}(m,n)=0$ when $n=0$ or $m=0$. We use the homological convention, that is the degree of differentials is -1. An $\mathbb{S}$-bimodule M is *augmented* when it naturally splits as $M=\overline{M}\oplus I$. We denote the module of morphisms of $\mathbb{S}$-bimodules by $\mathrm{Hom}(M,N)$ and the module of equivariant morphisms, with respect to the action of the symmetric groups, by $\mathrm{Hom}^{\mathbb{S}}(M,N)$.

In this category, we define a two *composition* products $\boxtimes$, based on the composition of operations indexing the vertices of a 2-leveled directed, and $\boxtimes_c$ based on the composition of operations indexing the vertices of a 2-leveled directed connected graph (see [Val03] Figure 1 for an example). Let $\mathcal{G}$ be such a graph with N internal edges between vertices of the two levels. This set of edges between vertices of the first level and vertices of the second level induces a permutation of $\mathbb{S}_N$. Let $\mathcal{P}$ and $\mathcal{Q}$ be two $\mathbb{S}$-bimodules, their composition product is given by the explicit formula

$$\mathcal{P}\boxtimes\mathcal{Q}(m,\,n) := \bigoplus_{N\in\mathbb{N}^*}\left(\bigoplus_{\bar l,\,\bar k,\,\bar\jmath,\,\bar\imath}\mathbb{K}[\mathbb{S}_m]\otimes_{\mathbb{S}_{\bar l}}\mathcal{P}(\bar l,\,\bar k)\otimes_{\mathbb{S}_{\bar k}}\mathbb{K}[\mathbb{S}_N]\otimes_{\mathbb{S}_{\bar\jmath}}\mathcal{Q}(\bar\jmath,\,\bar\imath)\otimes_{\mathbb{S}_{\bar\imath}}\mathbb{K}[\mathbb{S}_n]\right)_{\mathbb{S}_b^{\mathrm{op}}\times\mathbb{S}_a},$$

where the second direct sum runs over the b-tuples $\bar l$, $\bar k$ and the a-tuples $\bar\jmath$, $\bar\imath$ such that $|\bar l|=m$, $|\bar k|=|\bar\jmath|=N$, $|\bar\imath|=n$ and where the coinvariants correspond to the following action of $\mathbb{S}_b^{\mathrm{op}}\times\mathbb{S}_a$:

$$\theta\otimes p_1\otimes\cdots\otimes p_b\otimes\sigma\otimes q_1\otimes\cdots\otimes q_a\otimes\omega\sim$$
$$\theta\,\tau_{\bar l}^{-1}\otimes p_{\tau^{-1}(1)}\otimes\cdots\otimes p_{\tau^{-1}(b)}\otimes\tau_{\bar k}\,\sigma\,\nu_{\bar\jmath}\otimes q_{\nu(1)}\otimes\cdots\otimes q_{\nu(a)}\otimes\nu_{\bar\imath}^{-1}\,\omega,$$

for $\theta\in\mathbb{S}_m$, $\omega\in\mathbb{S}_n$, $\sigma\in\mathbb{S}_N$ and for $\tau\in\mathbb{S}_b$ with $\tau_{\bar k}$ the corresponding block permutation, $\nu\in\mathbb{S}_a$ and $\nu_{\bar\jmath}$ the corresponding block permutation. This product is associative but has not unit. To fix this issue, we restrict to connected graphs.

The permutations of $\mathbb{S}_N$ associated to connected graphs are called *connected* (for more details see Section 1.3 of [Val03]). We denote the set of connected permutations by $\mathbb{S}^c$. We define the *connected composition product* by the following formula

$$\mathcal{P} \boxtimes_c \mathcal{Q}(m,\, n) := \bigoplus_{N\in\mathbb{N}^*} \left(\bigoplus_{\bar{l},\, \bar{k},\, \bar{\jmath},\, \bar{\imath}} \mathbb{K}[\mathbb{S}_m] \otimes_{\mathbb{S}_{\bar{l}}} \mathcal{P}(\bar{l},\, \bar{k}) \otimes_{\mathbb{S}_{\bar{k}}} \mathbb{K}[\mathbb{S}^c_{\bar{k},\, \bar{\jmath}}] \otimes_{\mathbb{S}_{\bar{\jmath}}} \mathcal{Q}(\bar{\jmath},\, \bar{\imath}) \otimes_{\mathbb{S}_{\bar{\imath}}} \mathbb{K}[\mathbb{S}_n] \right)_{\mathbb{S}_b^{\mathrm{op}}\times\mathbb{S}_a}.$$

The unit I for this monoidal product is given by

$$\begin{cases} I(1,\,1) := \mathbb{K}, & \text{and} \\ I(m,\,n) := 0 & \text{otherwise.} \end{cases}$$

We denote by $(\mathbb{S}\text{-biMod}, \boxtimes_c,\, I)$ this monoidal category.

We define the *concatenation product* of two bimodules $\mathcal{P}$ and $\mathcal{Q}$ by

$$\mathcal{P} \otimes \mathcal{Q}(m,\, n) := \bigoplus_{\substack{m'+m''=m\\ n'+n''=n}} \mathbb{K}[\mathbb{S}_{m'+m''}] \otimes_{\mathbb{S}_{m'}\times\mathbb{S}_{m''}} \mathcal{P}(m',\, n') \otimes_{\mathbb{K}} \mathcal{Q}(m'',\, n'') \otimes_{\mathbb{S}_{n'}\times\mathbb{S}_{n''}} \mathbb{K}[\mathbb{S}_{n'+n''}].$$

This product corresponds to take the (horizontal) tensor product of the elements of $\mathcal{P}$ and $\mathcal{Q}$ (see Figure 3 of [Val03] for an example). It is symmetric, associative and unital. On the contrary to the two previous products, it is linear on the left and on the right. We denote by $\mathcal{S}_\otimes(\mathcal{P})$ the free symmetric monoid generated by an $\mathbb{S}$-bimodule $\mathcal{P}$ for the concatenation product (and $\bar{\mathcal{S}}_\otimes(\mathcal{P})$ its augmentation ideal). There is a natural embedding $\mathcal{P} \boxtimes_c \mathcal{Q} \rightarrowtail \mathcal{P} \boxtimes \mathcal{Q}$. And we obtain the composition product from the connected composition product by concatenation, that is $\bar{\mathcal{S}}_\otimes(\mathcal{P} \boxtimes_c \mathcal{Q}) \cong \mathcal{P} \boxtimes \mathcal{Q}$. (From this relation, we can see that $I \boxtimes_c \mathcal{P} = \bar{\mathcal{S}}(\mathcal{P})$ and not $\mathcal{P}$.)

1.2. **Properad.** A *properad* is a monoid in the monoidal category $(\mathbb{S}\text{-biMod}, \boxtimes_c,\, I)$. We denote the set of morphisms of properads by $\mathrm{Mor}(\mathcal{P}, \mathcal{Q})$. A properad $\mathcal{P}$ is *augmented* if there exists a morphism of properads $\varepsilon : \mathcal{P} \to I$. We denote by $\overline{\mathcal{P}}$ the kernel of the augmentation ε and call it the *augmentation ideal*. When $(\mathcal{P}, \mu, \eta,\, \varepsilon)$ is an augmented properad, $\mathcal{P}$ is canonically isomorphic to $I \oplus \overline{\mathcal{P}}$. We denote by $\big(I \oplus \underbrace{\overline{\mathcal{P}}}_{r}\big) \boxtimes_c \big(I \oplus \underbrace{\overline{\mathcal{P}}}_{s}\big)$ the sub-$\mathbb{S}$-bimodule of $\mathcal{P} \boxtimes_c \mathcal{P}$ generated by compositions of s non-trivial elements of $\mathcal{P}$ on the first level with r non-trivial elements of $\mathcal{P}$ on the second level. The corresponding restriction of the composition product μ on this sub-$\mathbb{S}$-bimodule is denoted $\mu_{(r,\,s)}$. The bilinear part of $\mathcal{P} \boxtimes_c \mathcal{P}$ is the $\mathbb{S}$-bimodule $\big(I \oplus \underbrace{\overline{\mathcal{P}}}_{1}\big) \boxtimes_c \big(I \oplus \underbrace{\overline{\mathcal{P}}}_{1}\big)$. It corresponds to the compositions of only 2 non-trivial operations of $\mathcal{P}$. We denote it by $\mathcal{P} \boxtimes_{(1,1)} \mathcal{P}$. The composition of two elements p_1 and p_2 of $\overline{\mathcal{P}}$ is written $p_1 \boxtimes_{(1,1)} p_2$ to lighten the notations. The restriction $\mu_{(1,\,1)}$ of the composition product μ of a properad $\mathcal{P}$ on $\mathcal{P} \boxtimes_{(1,1)} \mathcal{P}$ is called the *partial product*.

1.3. **Connected coproperad.** Dually, we defined the notion of *coproperad*, which is a comonoid in $(\mathbb{S}\text{-biMod}, \boxtimes_c)$. Recall that the partial coproduct $\Delta_{(1,\,1)}$ of a coproperad $\mathcal{C}$ is the projection of the coproduct Δ on $\mathcal{C} \boxtimes_{(1,1)} \mathcal{C} := \big(I \oplus \underbrace{\mathcal{C}}_{1}\big) \boxtimes_c \big(I \oplus \underbrace{\mathcal{C}}_{1}\big)$. More generally, one can define the (r, s)-part of the coproduct by the projection of the image of Δ on $\big(I \oplus \underbrace{\mathcal{C}}_{r}\big) \boxtimes_c \big(I \oplus \underbrace{\mathcal{C}}_{s}\big)$.

Since the dual of the notion of coproduct is the notion of product, we have to be careful with coproperad. For instance, the target space of a morphism of coproperads is a coproduct of modules and not a product. (The same problem appears at the level of algebras and coalgebras). We generalize here the notion of *connected* coalgebra introduced by D. Quillen in [Qui69] Appendix B, Section 3 (see also J.-L. Loday and M. Ronco [LR06] Section 1).
Let $(\mathcal{C},\, \Delta,\, \varepsilon,\, u)$ be an coaugmented (dg)coproperad. Denote by $\overline{\mathcal{C}} := \mathrm{Ker}(\mathcal{C} \xrightarrow{\varepsilon} I)$ its *augmentation*. We have $\mathcal{C} = \overline{\mathcal{C}} \oplus I$ and $\Delta(I) = I \boxtimes_c I$. For $X \in \overline{\mathcal{C}}$, denote by $\overline{\Delta}$ the non-primitive part of

the coproduct, that is $\Delta(X) = I \boxtimes_c X + X \boxtimes_c I + \overline{\Delta}(X)$. The *coradical filtration* of $\mathcal{C}$ is defined inductively as follows

$$\begin{aligned} F_0 &:= \mathbb{K}\, I \\ F_r &:= \{X \in \mathcal{C} \,|\, \overline{\Delta}(X) \in F_{r-1} \boxtimes_c F_{r-1}\}. \end{aligned}$$

An augmented coproperad is *connected* if the coradical filtration is exhaustive $\mathcal{C} = \bigcup_{r\geq 0} F_r$. This condition implies that $\mathcal{C}$ is *conilpotent* which means that for every $X \in \mathcal{C}$, there is an integer n such that $\overline{\Delta}^n(X) = 0$. This assumption is always required to construct morphisms or coderivations between coproperads (see next paragraph and Lemma 12 for instance).

For the same reason, we will sometimes work with the invariant version of the composition product denoted $\mathcal{P} \boxtimes_c^{\mathbb{S}} \mathcal{Q}$ when working with coproperads. It is defined by the same formula than $\boxtimes_c$ but where we consider the invariant elements under the actions of the symmetric groups instead of the coinvariants (see Lemma 4 for instance).

1.4. **Free properad and cofree connected coproperad.** Recall from [Val04] the construction of the free properad. Let V be an $\mathbb{S}$-bimodule. Denote by $V^+ := V \oplus I$ its augmentation and by $V_n := (V^+)^{\boxtimes_c n}$ the n-fold "tensor" power of V^+. This last module can be thought of as n-levelled graphs with vertices indexed by V and I. We define on V_n an equivalence relation $\sim$ by identifying two graphs when one is obtained from the other by moving an operation from a level to an upper or lower level. (Note that this permutation of the place of the operations induces signs). We consider the quotient $\widetilde{V}_n := V_n/\sim$ by this relation. Finally, the free properad $\mathcal{F}(V)$ is given by a particular colimit of the $\widetilde{V}_n$. The dg $\mathbb{S}$-bimodule $\mathcal{F}(V)$ is generated by graphs without levels with vertices indexed elements of V. We denote such graphs by $\mathcal{G}(v_1, \ldots, v_n)$, with $v_1, \ldots, v_n \in V$. Since $\mathcal{G}(v_1, \ldots, v_n)$ represents an equivalence class of leveled graphs, we can chose, up to signs, an order for the vertices. (Any graph $\mathcal{G}$ with n vertices is the quotient by the relation $\sim$ of a graph with n levels and only one non-trivial vertex on each level). The composition product of $\mathcal{F}(V)$ is given by the grafting. It is naturally graded by the number of vertices. This grading is called the *weight*. The part of weight n is denoted by $\mathcal{F}(V)^{(n)}$

The cofree connected coproperad on a dg $\mathbb{S}$-bimodule V has the same underlying space than the free properad, that is $\mathcal{F}^c(V) = \mathcal{F}(V)$. The coproduct is given by pruning the graphs into two parts. This coproperad verifies the universal property only among connected coproperads (see Proposition 2.7 of [Val03])

1.5. **Props.** We would like to define the notion of *prop* as a monoid in the category of $\mathbb{S}$-bimodules with the composition product $\boxtimes$. Since this last one has no unit and is not a monoidal product, strictly speaking, we have to make this definition explicit.

Definition (Prop)**.** A *prop* $\mathcal{P}$ is an $\mathbb{S}$-bimodule endowed with two maps $\mathcal{P} \boxtimes \mathcal{P} \xrightarrow{\mu} \mathcal{P}$ and $I \xrightarrow{\eta} \mathcal{P}$ such that the first is associative and the second one verifies

$$\begin{array}{ccccccccc} I \boxtimes_c \mathcal{P} & \rightarrowtail & I \boxtimes \mathcal{P} & \xrightarrow{\eta \boxtimes \mathcal{P}} & \mathcal{P} \boxtimes \mathcal{P} & \xleftarrow{\mathcal{P} \boxtimes \eta} & \mathcal{P} \boxtimes I & \leftarrowtail & \mathcal{P} \boxtimes_c I \\ & \searrow^{\sim} & & & \downarrow{\scriptstyle \mu} & & & {}^{\sim}\swarrow & \\ & & & & \mathcal{P}. & & & & \end{array}$$

Remark. This definition is not the original definition of Adams and MacLane but is equivalent to it. The original definition consists of two coherent bilinear products, the vertical and horizontal compositions of operations. The definition of the composition product given here includes these two previous compositions at the same time. The partial product $\mathcal{P} \boxtimes_{(1,1)} \mathcal{P} \xrightarrow{\mu_{(1,1)}} \mathcal{P}$ composes two operations. If they are connected by at least one edge, this composition is the vertical composition, otherwise this composition can be seen as the horizontal composition of operations. This presentation will allow us later to define the bar construction, resolutions and minimal models for props.

It is straightforward to extend the results of the preceding subsections to props. There exists notions of augmented props, free prop, coprop and connected cofree coprop. We refer the reader to [Val03] Section 2 for a complete treatment.

1.6. **Non-symmetric prop(erad).** In the sequel, we will have to work with the space of invariant elements of a properad under the action of the symmetric groups (see Section 2). This subspace is not stable under the composition of the properad but we can define on it non-symmetric compositions via the identification with coinvariants. Since this structure is the direct generalization of the notion of non-symmetric operad, we call it *non-symmetric properad.* All the definitions and propositions of this section can be generalize directly to props. For simplicity, we only make them explicit in the case of properads.

Definition. A *(dg)$\mathbb{N}$-bimodule* is a collection $\{\mathsf{P}(m,n)\}_{m,\,n\in\mathbb{N}^*}$ of dg modules.

Definition (Non-symmetric connected composition product)**.** Let P and Q be two $\mathbb{N}$-bimodules, we define their *non-symmetric connected composition product* by the following formula

$$\mathsf{P}\boxtimes_c \mathsf{Q}(m,\,n) := \bigoplus_{N\in\mathbb{N}^*}\left(\bigoplus_{\bar{l},\,\bar{k},\,\bar{\jmath},\,\bar{\imath}} \mathsf{P}(\bar{l},\,\bar{k})\otimes \mathbb{K}[\mathbb{S}^c_{\bar{k},\,\bar{\jmath}}]\otimes \mathsf{Q}(\bar{\jmath},\,\bar{\imath})\right)_{\mathbb{S}_b^{\mathrm{op}}\times\mathbb{S}_a},$$

where the second direct sum runs over the b-tuples $\bar{l}$, $\bar{k}$ and the a-tuples $\bar{\jmath}$, $\bar{\imath}$ such that $|\bar{l}|=m$, $|\bar{k}|=|\bar{\jmath}|=N$, $|\bar{\imath}|=n$ and where the coinvariants correspond to the following action of $\mathbb{S}_b^{\mathrm{op}}\times\mathbb{S}_a$:

$$p_1\otimes\cdots\otimes p_b\otimes\sigma\otimes q_1\otimes\cdots\otimes q_a \sim p_{\tau^{-1}(1)}\otimes\cdots\otimes p_{\tau^{-1}(b)}\otimes \tau_{\bar{k}}\,\sigma\,\nu_{\bar{\jmath}}\otimes q_{\nu(1)}\otimes\cdots\otimes q_{\nu(a)},$$

for $\sigma\in\mathbb{S}^c_{\bar{k},\,\bar{\jmath}}$ and for $\tau\in\mathbb{S}_b$ with $\tau_{\bar{k}}$ the corresponding block permutation, $\nu\in\mathbb{S}_a$ and $\nu_{\bar{\jmath}}$ the corresponding block permutation. Since the context is obvious, we still denote it by $\boxtimes_c$.

The definition of the monoidal product for $\mathbb{S}$-bimodule is based on 2-leveled graphs with leaves, inputs and outputs labelled by integers. This definition is based on non-labelled 2-leveled graphs. We define the *non-symmetric composition product* $\boxtimes$ by the same formula with the set of all permutations of $\mathbb{S}_N$ instead of connected permutations.

Proposition 1. *The category* $(\mathbb{N}$-*biMod*$,\boxtimes_c,\,I)$ *of* $\mathbb{N}$-*bimodules with the product* $\boxtimes_c$ *and the unit* I *is a monoidal category.*

Proof. The proof is similar to the one for $\mathbb{S}$-bimodules (see [Val03] Propostion 1.6). □

Definition (Non-symmetric properad)**.** A *non-symmetric properad* (P,μ,η) is a monoid in the monoidal category $(\mathbb{N}\text{-biMod},\boxtimes_c,\,I)$.

Example. A non-symmetric properad P concentrated in arity $(1,n)$, with $n\geq 1$, is the same as a non-symmetric operad.

This definition is motivated by the following property. Let $\mathcal{P}$ be an $\mathbb{S}$-bimodule, we denote by $\mathcal{P}_\mathbb{S}$ the $\mathbb{N}$-bimodule obtained by taking the space of coinvariants of $\mathcal{P}$, that is $\mathcal{P}_\mathbb{S}(m,n) := \mathcal{P}(m,n)_{\mathbb{S}_m^{\mathrm{op}}\times\mathbb{S}_n}$.

Proposition 2. *For two* $\mathbb{S}$-*bimodules* $\mathcal{P}$ *and* $\mathcal{Q}$*, we have* $(\mathcal{P}\boxtimes_c\mathcal{Q})_\mathbb{S}=\mathcal{P}_\mathbb{S}\boxtimes_c\mathcal{Q}_\mathbb{S}$.
If $\mathcal{P}$ *is a properad, its space of coinvariants* $\mathcal{P}_\mathbb{S}$ *is naturally a non-symmetric properad.*

Proof. The first assertion comes directly from the definition of the two composition products. When $(\mathcal{P},\mu,\eta)$ is a properad, we define the non-symmetric product $\bar{\mu}$ on $\mathcal{P}_\mathbb{S}$ by

$$\bar{\mu}\ :\ \mathcal{P}_\mathbb{S}\boxtimes_c\mathcal{P}_\mathbb{S}\cong(\mathcal{P}\boxtimes_c\mathcal{P})_\mathbb{S}\xrightarrow{\mu_\mathbb{S}}\mathcal{P}_\mathbb{S},$$

where the map $\mu_\mathbb{S}$ is when defined since the map μ is equivariant under the action of the symmetric groups. The associativity of μ induced the associativity $\bar{\mu}$. □

In the sequel, we will have to work with the space of invariants, and not coinvariants, of a properad. Since we work over a field of characteristic zero, both are canonically isomorphic. Let V be a vector space with an action of a finite group G. The subspace of invariants is defined by $V^G := \{v\in V\,|\,v.g=v\ ,\,\forall g\in G\}$ and the quotient space of coinvariants is defined by $V_G := V/ <$

$v - v.g,\ \forall (v,g) \in V \times G >$. The map from V^G to V_G is the composite of the inclusion $V^G \rightarrowtail V$ followed by the projection $V \twoheadrightarrow V_G$. The inverse map $V_G \to V^G$ is given by $[v] \mapsto \frac{1}{|G|}\sum_{g\in G} v.g$, where $[v]$ denotes the class of v in V_G. In the case of $\mathbb{S}$-bimodules, we will denote the latter map by $\mathrm{Sym} : \mathcal{P}_{\mathbb{S}} \to \mathcal{P}^{\mathbb{S}}$.

Proposition 3. *Let $\mathcal{P}$ be a properad. Its subspace of invariants $\mathcal{P}^{\mathbb{S}}$ is naturally endowed with a structure of non-symmetric properad.*

Proof. When $(\mathcal{P}, \mu, \eta)$ is a properad, by Proposition 2 the quotient space of coinvariants $\mathcal{P}_{\mathbb{S}}$ is a non-symmetric properad. Since this space $\mathcal{P}_{\mathbb{S}}$ is isomorphic to the subspace of coinvariants $\mathcal{P}_{\mathbb{S}}$, we define the composition product on $\mathcal{P}^{\mathbb{S}}$ by the following composite

$$\mathcal{P}^{\mathbb{S}} \boxtimes_c \mathcal{P}^{\mathbb{S}} \to \mathcal{P}_{\mathbb{S}} \boxtimes_c \mathcal{P}_{\mathbb{S}} \xrightarrow{\bar{\mu}} \mathcal{P}_{\mathbb{S}} \xrightarrow{\mathrm{Sym}} \mathcal{P}^{\mathbb{S}}.$$

That is $\mu^{\mathcal{P}^{\mathbb{S}}}(p_1, \ldots, p_r;\, p'_1, \ldots, p'_s) = \dfrac{1}{n!\, m!} \displaystyle\sum_{\substack{\sigma\in\mathbb{S}_n\\ \tau\in\mathbb{S}_m}} \tau.\mu^{\mathcal{P}}(p_1, \ldots, p_r;\, p'_1, \ldots, p'_s).\sigma.$ ☐

1.7. **Representations of prop(erad)s, gebras.** Let $\mathcal{P}$ and $\mathcal{Q}$ be two prop(erad)s. A morphism $\mathcal{P} \xrightarrow{\Phi} \mathcal{Q}$ of $\mathbb{S}$-bimodules is a *morphism of prop(erad)s* if it commutes with the products and the units of $\mathcal{P}$ and $\mathcal{Q}$. In this case, we say that $\mathcal{Q}$ is a *representation* of $\mathcal{P}$.

We will be mainly interested in representations of the following form. Let X be a dg vector space. We consider the $\mathbb{S}$-bimodule End_X defined by $\mathrm{End}_X(m,n) := \mathrm{Hom}_{\mathbb{K}}(X^{\otimes n}, X^{\otimes m})$. The natural composition of maps provides this $\mathbb{S}$-bimodule with a structure of prop and properad. It is called the *endomorphism prop(erad)* of the space X.

Props and properads are meant to model the operations acting on types of algebras or bialgebras in a generalized sense. When $\mathcal{P}$ is a prop(erad), we call *$\mathcal{P}$-gebra* a dg vector space X with a morphism of prop(erad)s $\mathcal{P} \to \mathrm{End}_X$, that is a representation of $\mathcal{P}$ of the form End_X. When $\mathcal{P}$ is an operad, a $\mathcal{P}$-gebra is an algebra over $\mathcal{P}$. To code operations with multiple inputs and multiple outputs acting on an algebraic structure, we cannot use operads anymore and we need to use prop(erad)s. The categories of Lie bialgebras and (classical) bialgebras, for examples, are categories of gebras over a properad. In these cases, the associated prop is freely obtained from a properad. Therefore, the prop does not model more relations than the properad and the two categories of gebras over the prop and the properad are equal.

2. Convolution prop(erad)

When A is an associative algebra and C a coassociative coalgebra, the space of morphisms $\mathrm{Hom}_{\mathbb{K}}(C,\, A)$ from C to A is naturally an associative algebra with the convolution product. We generalize this property to prop(erad)s, that is the space of morphisms of $\mathbb{S}$-bimodules $\mathrm{Hom}_{\mathbb{S}}(\mathcal{C},\, \mathcal{P})$ from a coprop(erad) $\mathcal{C}$ and a prop(erad) $\mathcal{P}$ is a prop(erad). From this rich structure, we get general operations, the main one will be the *intrinsic* Lie bracket that we will use in deformation theory later in 7.2.

2.1. **Convolution prop(erad).** For two $\mathbb{S}$-bimodules $M = \{M(m,n)\}_{m,n}$ and $N = \{N(m,n)\}_{m,n}$, we denote by $\mathrm{Hom}(M,N)$ the collection $\{\mathrm{Hom}_{\mathbb{K}}\big(M(m,n), N(m,n)\big)\}_{m,n}$ of morphisms of $\mathbb{K}$-modules. It is an $\mathbb{S}$-bimodule with the action by conjugation, that is

$$(\sigma.f.\tau)(x) := \sigma.(f(\sigma^{-1}.x.\tau^{-1})).\tau,$$

for $\sigma \in \mathbb{S}_m$, $\tau \in \mathbb{S}_n$ and $f \in \mathrm{Hom}\big(M,N\big)(m,n)$. An invariant element under this action is an equivariant map from M to N, that is $\mathrm{Hom}(M,N)^{\mathbb{S}} = \mathrm{Hom}^{\mathbb{S}}(M,N)$.

When $\mathcal{C}$ is a coassociative coalgebra and $\mathcal{P}$ is an associative algebra, $\mathrm{Hom}(\mathcal{C},\mathcal{P})$ is an associative algebra known as the *convolution algebra*. When $\mathcal{C}$ is a cooperad and $\mathcal{P}$ is an operad, $\mathrm{Hom}(\mathcal{C},\mathcal{P})$

is an operad called the *convolution operad* by C. Berger and I. Moerdijk in [BM03] Section 1. We extend this construction to properads and props.

Lemma 4. *Let $\mathcal{C}$ be a coprop(erad) and $\mathcal{P}$ be a prop(erad). The space of morphisms* $\mathrm{Hom}(\mathcal{C},\mathcal{P}) = \mathcal{P}^{\mathcal{C}}$ *is a prop(erad).*

Proof. We use the notations of Section 1.1 (see also Section 1.2 of [Val03]). We define an associative and unital map $\mu^{\mathcal{P}^{\mathcal{C}}} : \mathcal{P}^{\mathcal{C}} \boxtimes_{\mathbb{S}} \mathcal{P}^{\mathcal{C}} \to \mathcal{P}^{\mathcal{C}}$ as follow. Let $\mathcal{G}^2(f_1,\ldots,f_r;\, g_1,\ldots,g_s) \in \mathcal{P}^{\mathcal{C}} \boxtimes \mathcal{P}^{\mathcal{C}}(m,n)$ be a 2-levelled graph whose vertices of the first level are labelled by $f_1,\ldots,f_r$ and whose vertices of the second level are labelled by $g_1,\ldots,g_s$. The image of $\mathcal{G}^2(f_1,\ldots,f_r;\, g_1,\ldots,g_s)$ under $\mu^{\mathcal{P}^{\mathcal{C}}}$ is the composite

$$\mathcal{C} \xrightarrow{\Delta^{\mathcal{C}}} \mathcal{C} \boxtimes^{\mathbb{S}} \mathcal{C} \rightarrowtail \mathcal{C} \boxtimes \mathcal{C} \xrightarrow{\widetilde{\mathcal{G}}^2(f_1,\ldots,f_r;\, g_1,\ldots,g_s)} \mathcal{P} \boxtimes \mathcal{P} \twoheadrightarrow \mathcal{P} \boxtimes_{\mathbb{S}} \mathcal{P} \xrightarrow{\mu^{\mathcal{P}}} \mathcal{P},$$

where $\widetilde{\mathcal{G}}^2(f_1,\ldots,f_r;\, g_1,\ldots,g_s)$ applies f_i on the element of $\mathcal{C}$ indexing the i^{th} vertex of the first level and g_j on the element of $\mathcal{C}$ indexing the j^{th} vertex of the second level of an element of $\mathcal{C} \boxtimes \mathcal{C}$. Since the action of the symmetric groups on $\mathcal{P}^{\mathcal{C}}$ is defined by conjugation and since the image of the coproduct lives in the space of invariants, this map factors through the coinvariants, that is $\mathcal{P}^{\mathcal{C}} \boxtimes_{\mathbb{S}} \mathcal{P}^{\mathcal{C}} \to \mathcal{P}^{\mathcal{C}}$.
The unit is given by the map $\mathcal{C} \xrightarrow{\varepsilon} I \xrightarrow{\eta} \mathcal{P}$. The associativity of $\mu^{\mathcal{P}^{\mathcal{C}}}$ comes directly from the coassociativity of $\Delta^{\mathcal{C}}$ and the associativity of $\mu^{\mathcal{P}}$. ☐

Definition. The properad $\mathrm{Hom}(\mathcal{C},\mathcal{P})$ is called the *convolution prop(erad)* and is denoted by $\mathcal{P}^{\mathcal{C}}$.

Assume now that $(\mathcal{C}, d_{\mathcal{C}})$ is a dg coprop(erad) and $(\mathcal{P}, d_{\mathcal{P}})$ is a dg prop(erad). The *derivative* of a graded linear map f from $\mathcal{C}$ to $\mathcal{P}$ is defined as follows

$$D(f) := d_{\mathcal{P}} \circ f - (-1)^{|f|} f \circ d_{\mathcal{C}}.$$

A 0-cycle for this differential is a morphism of chain complexes, that is it commutes with the differentials. In Section 6.3, we give a geometric interpretation of this derivative. The derivative is a derivation for the product of the prop(erad) $\mathrm{Hom}(\mathcal{C},\mathcal{P})$ that verifies $D^2 = 0$. We sum up these relations in the following proposition.

Proposition 5. *When $(\mathcal{C}, d_{\mathcal{C}})$ is a dg coprop(erad) and $(\mathcal{P}, d_{\mathcal{P}})$ is a dg prop(erad),* $(\mathrm{Hom}(\mathcal{C},\mathcal{P}), D)$ *is a dg prop(erad).*

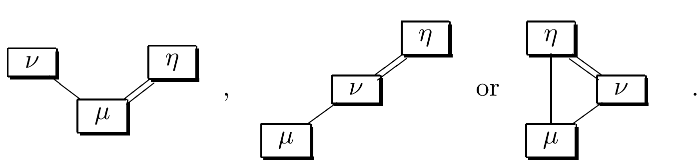
To study the properties of the bar and the cobar construction of (co)properads in Section 3.5, we will need to work in the space of equivariant maps from a coproperad to a properad $\mathrm{Hom}_{\mathbb{S}}(\mathcal{C},\mathcal{P})$. Since this space is equal to the subspace of invariants of the convolution properad $\mathrm{Hom}(\mathcal{C},\mathcal{P})$, it is naturally endowed with a structure of non-symmetric properad by Proposition 3.

Proposition 6. *Let $(\mathcal{C}, d_{\mathcal{C}})$ be a dg coprop(erad) and $(\mathcal{P}, d_{\mathcal{P}})$ be a dg prop(erad), the space of equivariant maps* $(\mathrm{Hom}^{\mathbb{S}}(\mathcal{C},\mathcal{P}), D)$ *is a non-symmetric dg prop(erad).*

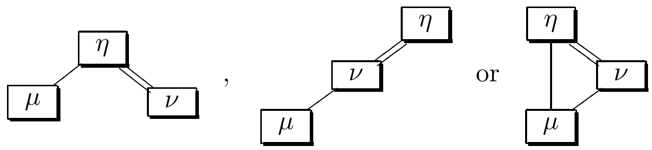
We call it the *non-symmetric convolution prop(erad)*.

2.2. **Lie-admissible product of a properad.** In [KM01], the authors proved that on the total space $\oplus_n \mathcal{P}(n)$ of an operad is endowed with a natural Lie bracket. This Lie bracket is the anti-symmetrization of the preLie product $p \circ q = \sum_i p \circ_i q$ defined by the sum on all possible ways of composing two operations p and q. We generalize this result to properads.

For μ and ν two elements of a (non-symmetric) properad $\mathcal{P}$, denote $\mu \circ \nu$ all the possible compositions of μ by ν along any 2-leveled graph with two vertices in $\mathcal{P}$. For η another element $\mathcal{P}$, the components of $(\mu \circ \nu) \circ \eta$ are of the form



In the same way, the elements of $\mu \circ (\nu \circ \eta)$ are of the form



Denote by $\mu \circ (\nu,\, \eta)$ all the compositions of μ with ν and η above in $\mathcal{P}$ and $(\mu,\, \nu) \circ \eta$ all the compositions of η with μ and ν under. With these notations, we have in $\mathcal{P}$ the following formula

$$(\mu \circ \nu) \circ \eta - \mu \circ (\nu \circ \eta) = \mu \circ (\nu,\, \eta) - (\mu,\, \nu) \circ \eta.$$

When $\mathcal{P} = A$ is concentrated in arity $(1,1)$, it is an associative algebra. In this case, the product $\circ$ is the associative product of A. When $\mathcal{P}$ is an operad, the operation $(\mu,\, \nu) \circ \eta$ vanishes and the product $\mu \circ \nu$ is right symmetric, that is $(\mu \circ \nu) \circ \eta - \mu \circ (\nu \circ \eta) = (\mu \circ \eta) \circ \nu - \mu \circ (\eta \circ \nu)$. Such a product is called *preLie*. In the general case of properads, this product verifies a weaker relation called *Lie-admissible* because its anti-symmetrized bracket verifies the Jacobi identity. Denote by $\mathrm{As}(\mu,\, \nu,\, \eta) := (\mu \circ \nu) \circ \eta - \mu \circ (\nu \circ \eta)$ the associator of μ, ν and η.

Definition (Lie-admissible algebra)**.** A graded vector space A with a binary product $\circ$ is called a (graded) *Lie-admissible* algebra if one has $\sum_{\sigma \in \mathbb{S}_3} \mathrm{sgn}(\sigma) \mathrm{As}(-,\, -,\, -)^\sigma = 0$, where, for instance, $\mathrm{As}(-,\, -,\, -)^{(23)}$ applied to μ, ν and η is equal to $(-1)^{|\nu||\eta|}((\mu \circ \eta) \circ \nu - \mu \circ (\eta \circ \nu))$.

Proposition 7. *Let $\mathcal{P}$ be a dg properad or a non-symmetric dg properad, the space $\bigoplus_{m,n} \mathcal{P}(m,n)$, endowed with the product $\circ$, is a dg Lie-admissible algebra.*

Proof. Let $H = \{id,\, (23)\}$ and $K = \{id,\, (12)\}$ be two subgroups of $\mathbb{S}_3$. We have

$$\begin{aligned}
\sum_{\sigma \in \mathbb{S}_3} \mathrm{sgn}(\sigma) \mathrm{As}(-,\, -,\, -)^\sigma &= \sum_{\sigma \in \mathbb{S}_3} \mathrm{sgn}(\sigma) \big((- \circ (- \circ -))^\sigma - ((- \circ -) \circ -)^\sigma \big) \\
&= \sum_{\tau H \in \mathbb{S}_3 \backslash H} \mathrm{sgn}(\tau) \underbrace{\Big((- \circ (-,\, -))^\tau - (- \circ (-,\, -))^{\tau(23)} \Big)}_{=0} - \\
&\quad \sum_{\omega K \in \mathbb{S}_3 \backslash K} \mathrm{sgn}(\omega) \underbrace{\Big(((-,\, -) \circ -)^\omega - ((-,\, -) \circ -)^{\omega(12)} \Big)}_{=0} \\
&= 0.
\end{aligned}$$

□

For a prop $\mathcal{P}$, we still define the product $p \circ q$ on $\bigoplus_{m,n} \mathcal{P}(m,n)$ by all the possible ways of composing the operations p and q, that is all vertical composites and the horizontal one.

Proposition 8. *Let $\mathcal{P}$ be a dg prop or a non-symmetric dg prop, the space $\bigoplus_{m,n} \mathcal{P}(m,n)$, endowed with the product $\circ$, is a dg associative algebra.*

Proof. We denote by $p \circ_v q$ the sum of all vertical (connected) composites of p and q and by $p \circ_h q$ the horizontal composite. We continue to use the notation $p \circ_v (p, r)$ to represent the composite of an operation p connected to two operations p and r above. We have (in degree 0)

$$\begin{aligned}
&(p \circ q) \circ r = (p \circ_v q + p \circ_h q) \circ r = \\
&p \circ_v q \circ_v r + p \circ_v (q, r) + (p \circ_v q) \circ_h r + (p \circ_v r) \circ_h q + p \circ_h (q \circ_v r) + (p, q) \circ_v r + p \circ_h q \circ_h r,
\end{aligned}$$

and

$$\begin{aligned}
&p \circ (q \circ r) = p \circ (q \circ_v r + q \circ_h r) = \\
&p \circ_v q \circ_v r + (p, q) \circ_v r + p \circ_h (q \circ_v r) + (p \circ_v q) \circ_h r + q \circ_h (p \circ_v r) + p \circ_v (q, r) + p \circ_h q \circ_h r.
\end{aligned}$$

Since the horizontal product is commutative, $(p \circ_v r) \circ_h q$ is equal to $q \circ_h (p \circ_v r)$, which finally implies $(p \circ q) \circ r = p \circ (q \circ r)$. □

The Lie-admissible relation of a product $\circ$ is equivalent to the Jacobi identity of its induced bracket $[\mu, \nu] := \mu \circ \nu - (-1)^{|\mu||\nu|}\nu \circ \mu$. As a consequence, in each case (associative algebra, (non-symmetric) operad, (non-symmetric) properad and (non-symmetric prop)), the product $\circ$ gives rise to a graded Lie bracket, which means that $[[-, -], -] + [[-, -], -]^{(123)} + [[-, -], -]^{(132)} = 0$.

When $\mathcal{P}$ is a properad, we can symmetrize the product $\circ$ to define a Lie-admissible product on the subspace of invariants $\mathcal{P}^{\mathbb{S}}$. For p and q two elements of $\mathcal{P}^{\mathbb{S}}$, it is given by $\mathrm{Sym}(p \circ q)$. By Proposition 3, $\mathcal{P}^{\mathbb{S}}$ is a non-symmetric properad. The symmetrized Lie-admissible product in $\mathcal{P}^{\mathbb{S}}$ is equal to the one defined directly in the non-symmetric properad $\mathcal{P}^{\mathbb{S}}$. (The same result holds for the associative product $\circ$ of a prop).

2.3. **Lie-admissible bracket of a convolution properad.** From the rich structure of non-symmetric properad on $\mathrm{Hom}_{\mathbb{S}}(\mathcal{C}, \mathcal{P})$, we derive several operations. The most interesting one is the Lie bracket induced by the Lie-admissible product. We will use later in our study of deformation theory (see Section 7.2). In the case of the non-symmetric convolution properad, it is equal to the generalization to $\mathrm{Hom}_{\mathbb{S}}(\mathcal{C}, \mathcal{P})$ of the classical convolution product defined on the space of morphisms $\mathrm{Hom}(C, A)$ between a coassociative coalgebra and an associative algebra.

Definition (Convolution product)**.** Let f and g be two elements of $\mathrm{Hom}_{\mathbb{S}}(\mathcal{C}, \mathcal{P})$. Their *convolution product* $f \star g$ is defined by the following composite

$$\mathcal{C} \xrightarrow{\Delta_{(1,1)}} \mathcal{C} \boxtimes_{(1,1)} \mathcal{C} \xrightarrow{f \boxtimes_{(1,1)} g} \mathcal{P} \boxtimes_{(1,1)} \mathcal{P} \xrightarrow{\mu} \mathcal{P}.$$

Since the partial coproduct of a coproperad (or a cooperad) is not coassociative in general, the convolution product is not associative.

Proposition 9. *Let $\mathcal{P}$ be a prop(erad) and $\mathcal{C}$ be a coprop(erad). The convolution product $\star$ on $\mathrm{Hom}_{\mathbb{S}}(\mathcal{C}, \mathcal{P})$ is equal to the product $\circ$ associated to the non-symmetric convolution prop(erad). In the case of (co)properads, it is Lie-admissible and for (co)props, it is associative.*

Proof. The image of the map $\Delta_{(1,1)}$ is a sum over all possible 2-leveled graphs with two vertices indexed by some elements of $\mathcal{C}$. Therefore, the map $\star$ is equal to the sum of all possible compositions of f and g. □

Using the projections $\Delta_{(r,s)}$ of the coproduct, we define more general products with r and s inputs.

Definition (LR-operations)**.** Let $f_1, \dots, f_r$ and $g_1, \dots, g_s$ be elements of $\mathrm{Hom}(\mathcal{C}, \mathcal{P})$. Their *LR-operation* $\{f_1, \dots, f_r\}\{g_1, \dots, g_s\}$ is defined by

$$\mathcal{C} \xrightarrow{\Delta_{(r,s)}} \big(I \oplus \underbrace{\mathcal{C}}_{r}\big) \boxtimes \big(I \oplus \underbrace{\mathcal{C}}_{s}\big) \cong$$
$$\underbrace{\mathcal{C} \otimes \cdots \otimes \mathcal{C}}_{r} \boxtimes \underbrace{\mathcal{C} \otimes \cdots \otimes \mathcal{C}}_{s} \xrightarrow{\{f_1,\dots,f_r\} \boxtimes \{g_1,\dots,g_r\}} \mathcal{P} \boxtimes \mathcal{P} \xrightarrow{\mu} \mathcal{P}.$$

One can see that these operations are symmetric with respect to Koszul sign convention. Notice that the convolution product is equal to $f \star g := \{f\}\{g\}$.

Corollary 10. *The space* $(\mathrm{Hom}_{\mathbb{S}}(\mathcal{C}, \mathcal{P}), [,])$ *is a dg Lie algebra.*

Proof. It is a direct corollary of Proposition 7 and Proposition 9. □

When $\mathcal{C} = C$ is a coassociative coalgebra and $\mathcal{P} = A$ an associative algebra, the product $\boxtimes$ is equal to $\otimes$ and is bilinear. In this case, the partial coproduct of C is equal to the coproduct of C and is coassociative. (All the $\Delta_{(r,s)}$ are null for $r > 1$ or $s > 1$). In this case, the product $\star$ is the classical convolution product on $\mathrm{Hom}(C, A)$, which is associative.

When $\mathcal{C}$ is a cooperad and $\mathcal{P}$ is an operad. Since we have $\Delta_{(r,s)} = 0$ for $r > 1$, the operations $\{f_1, \dots, f_r\}\{g_1, \dots, g_s\}$ are null unless $r = 1$. The remaining operations $\{f\}\{g_1, \dots, g_s\}$ are graded symmetric *brace* operations coming from the brace relations verified by the operadic product (*cf.* [LM05]). When $\mathcal{C}$ is a non-symmetric operad and $\mathcal{P}$ a non-symmetric operad, we can define non-symmetric braces on $\mathrm{Hom}(\mathcal{C}, \mathcal{P})$ in the same way (see [GV95, Val06]). The convolution product verifies the relation $(f \star g) \star h - f \star (g \star h) = \{f\}\{g,\, h\}$. Therefore, in the operadic framework, the (graded) symmetry of the brace products implies that the associator $(f \star g) \star h - f \star (g \star h)$ is symmetric in g and h. In this case, the convolution product $\star$ on $\mathrm{Hom}(\mathcal{C},\, \mathcal{P})$ is called a graded *preLie* product.

The first definition of this kind of preLie operation appeared in the seminal paper of M. Gerstenhaber [Ger63] in the case of the cohomology of associative algebras. In the case treated by M. Gerstenhaber, the cooperad $\mathcal{C}$ is the Koszul dual cooperad $\mathcal{A}s^{¡}$ of the operad $\mathcal{A}s$ coding associative algebras and the operad $\mathcal{P}$ is the endomorphism operad $\mathrm{End}(A)$. This preLie product is universal, that is there is a preLie product on the cohomology theory for algebras over any Koszul operad. This result was first prove by Markl, Shnider, Stasheff in Proposition 3.111 of [MSS02]. We give a stronger statement here, that is the cochain complex defining the cohomology theory of an algebra over a Koszul operad is a non-symmetric operad (see Section 7.2). The symmetric braces play a fundamental role in the proof of Deligne's conjecture for associative algebras (see McClure-Smith [MS02] and Tamarkin [Tam98] for instance) and in the extension of it to other kind of algebras (see [Val06] Section 5.5).

3. Bar and cobar construction

In this section, we recall the definitions of the bar and cobar constructions for (co)properads and extend it to (co)props. We prove to adjunction between these two constructions using the notion of *twisting morphism*, that is Maurer-Cartan elements in the convolution prop(erad). Finally, we show that the bar-cobar construction provide a canonical (cofibrant) resolution.

3.1. (Co)Derivations.

Let $(\mathcal{P}, \mu^{\mathcal{P}})$ and $(\mathcal{Q},\, \mu^{\mathcal{Q}})$ be two augmented dg prop(erad)s and let $\rho\ :\ \mathcal{P} \to \mathcal{Q}$ be a morphism of augmented dg prop(erad)s of degree 0.

Definition (Derivation). A homogenous morphism $\partial\, :\, \mathcal{P} \to \mathcal{Q}$ is a *homogenous derivation of* ρ if

$$\partial \circ \mu^{\mathcal{P}}_{(1,\,1)}(-,-) = \mu^{\mathcal{Q}}_{(1,\,1)}(\partial(-), \rho(-)) + \mu^{\mathcal{Q}}_{(1,\,1)}(\rho(-), \partial(-)).$$

This formula, applied to elements $p_1 \boxtimes_{(1,1)} p_2$ of $\mathcal{P} \boxtimes_{(1,1)} \mathcal{P}$, where p_1 and p_2 are homogenous elements of $\mathcal{P}$, gives

$$\partial \circ \mu^{\mathcal{P}}(p_1 \boxtimes_{(1,1)} p_2) \quad = \quad \mu^{\mathcal{Q}}\big(\partial(p_1) \boxtimes_{(1,1)} \rho(p_2)\big) + (-1)^{|\partial||p_1|}\mu^{\mathcal{Q}}\big(\rho(p_1) \boxtimes_{(1,1)} \partial(p_2)\big).$$

A *derivation* is a sum of homogenous derivations. The set of homogenous derivations with respect to ρ of degree n is denoted $\mathrm{Der}^n_\rho(\mathcal{P},\, \mathcal{Q})$ and the set of derivations is denoted $\mathrm{Der}^\bullet_\rho(\mathcal{P},\, \mathcal{Q})$ or simply $\mathrm{Der}(\mathcal{P},\, \mathcal{Q})$ when the morphism ρ is obvious (for instance, the identity morphism).

Example. The differential of a dg prop(erad) $\mathcal{P}$ is a derivation of degree -1, that is an element of $\mathrm{Der}^{-1}_{\mathrm{Id}}(\mathcal{P},\, \mathcal{P})$.

Remark. Following the definition of a derivation from the framework of associative algebras, we could define a more general notion. Let M be a bimodule over a prop(erad) $\mathcal{P}$, that is we have two maps $\mathcal{P} \boxtimes_{(1,1)} M \to M$ and $M \boxtimes_{(1,1)} \mathcal{P} \to M$ which verify natural coherence with the associativity of the composition product of the prop(erad). One can define the notion of derivation as a map from P to M with the same kind of formula. The definition given above is then a particular case of this one since a morphism of prop(erad)s $\rho\, :\, \mathcal{P} \to \mathcal{Q}$ defines a natural structure of $\mathcal{P}$-bimodule on $\mathcal{Q}$. In the sequel, we will not need such a generalization.

In the rest of the text, we need the following lemma which gives the form of the derivations on a free prop(erad). For a prop(erad) $(\mathcal{Q}, \mu_{\mathcal{Q}})$, any graph $\mathcal{G}$ of $\mathcal{F}(\mathcal{Q})^{(n)}$ represents a class $\overline{\mathcal{G}}$ of levelled graphs of $\mathcal{Q}^{\boxtimes n}$. Therefore, we can define a morphism of prop(erad)s $\widetilde{\mu}_{\mathcal{Q}}\ :\ \mathcal{F}(\mathcal{Q}) \to \mathcal{Q}$ by the formula $\widetilde{\mu}_{\mathcal{Q}}(\mathcal{G}) := \mu_{\mathcal{Q}}^{\circ(n-1)}(\overline{\mathcal{G}})$. The associativity of $\mu_{\mathcal{Q}}$ and the colimit defining $\mathcal{F}(\mathcal{Q})$ proves that $\widetilde{\mu}_{\mathcal{Q}}$ is well defined. The morphism $\widetilde{\mu}_{\mathcal{Q}}$ is the only morphism of prop(erad)s extending $\mathcal{Q} \xrightarrow{\mathrm{Id}} \mathcal{Q}$.

Lemma 11. *Let $\rho : \mathcal{F}(V) \to \mathcal{Q}$ be a morphism of prop(erad)s of degree 0. Every derivation from the free dg prop(erad) $\mathcal{F}(V)$ to $\mathcal{Q}$ is characterized by its restriction on V, that is there is a canonical one-to-one correspondence $\mathrm{Der}^n_\rho(\mathcal{F}(V),\, \mathcal{Q}) \cong \mathrm{Hom}^{\mathbb{S}}_n(V,\, \mathcal{Q})$.*
For every morphism of dg $\mathbb{S}$-bimodules $\theta : V \to \mathcal{Q}$, denote the unique derivation which extends θ by ∂_θ. The image of an element $\mathcal{G}(v_1, \ldots, v_n)$ of $\mathcal{F}(V)^{(n)}$ under ∂_θ is

$$\partial_\theta(\mathcal{G}(v_1, \ldots, v_n)) = \sum_{i=1}^{n} (-1)^{|\theta|.(|v_1|+\cdots+|v_{i-1}|)} \widetilde{\mu}_{\mathcal{Q}}\big(\mathcal{G}(\rho(v_1), \ldots, \rho(v_{i-1}), \theta(v_i), \rho(v_{i+1}), \ldots, \rho(v_n))\big).$$

Proof. Denote by θ the restriction of the derivation ∂ on V, that is $\theta = \partial_V : V \to \overline{\mathcal{Q}}$. From θ, we can construct the whole derivation ∂ by induction on the weight n of the free prop(erad) $\mathcal{F}(V)$ as follows.
For $n = 1$, we have $\partial^1_\theta = \theta : V \to \mathcal{Q}$. Suppose now that $\partial^n_\theta : \mathcal{F}(V)^{(n)} \to \mathcal{Q}$ is constructed and it is given by the formula of the Lemma. Any simple element of $\mathcal{F}(V)^{(n+1)}$ represented by a graph with $n+1$ vertices indexed by elements of V is the concatenation of a graph with n vertices with an extra vertex from the top or the bottom. In the last case, ∂^{n+1}_θ is given the commutative diagram

$$\begin{array}{ccc} \mathcal{F}(V)^{(n+1)} & \xrightarrow{\quad \partial^{n+1}_\theta \quad} & \mathcal{Q} \\ \uparrow{\scriptstyle \mu_{\mathcal{F}(V)}} & & \uparrow{\scriptstyle \mu_{\mathcal{Q}}} \\ V \boxtimes_{(1,1)} \mathcal{F}(V)^{(n)} & \xrightarrow{\rho \boxtimes \partial^n_\theta + \partial^n_\theta \boxtimes \rho} & \mathcal{Q} \boxtimes_{(1,1)} \mathcal{Q}. \end{array}$$

The other case is dual. It is easy to check that the formula is still true for elements of $\mathcal{F}(V)^{(n+1)}$, that is graphs with $n+1$ vertices. Finally, since ρ is a morphism of prop(erad)s, ∂_θ is well defined and is a derivation. ☐

Example. A differential ∂ on a free prop(erad) $\mathcal{F}(V)$ is a derivation of $\mathrm{Der}^{-1}_{\mathrm{Id}}(\mathcal{F}(V),\, \mathcal{F}(V))$ such that $\partial^2 = 0$.

Definition (quasi-free prop(erad))**.** A dg prop(erad) $(\mathcal{F}(V),\, \partial)$ such that the underlying prop(erad) is free is called a *quasi-free* prop(erad).

Notice that in a quasi-free prop(erad), the differential is not freely generated and is a derivation of the form given above.

Dually, let $(\mathcal{C}, \Delta^{\mathcal{C}})$ and $(\mathcal{D}, \Delta^{\mathcal{D}})$ be two coaugmented dg coprop(erad)s and let $\rho : \mathcal{C} \to \mathcal{D}$ be a morphism of coaugmented dg coprop(erad)s of degree 0.

Definition (Coderivation)**.** A homogenous morphism $d : \mathcal{C} \to \mathcal{D}$ is a *homogenous coderivation* of ρ if the following diagram is commutative

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\quad d \quad} & \mathcal{D} \\ \downarrow{\scriptstyle \Delta^{\mathcal{C}}_{(1,1)}} & & \downarrow{\scriptstyle \Delta^{\mathcal{D}}_{(1,1)}} \\ \mathcal{C} \boxtimes \mathcal{C} & \xrightarrow{d \boxtimes \rho + \rho \boxtimes d} & \mathcal{D} \boxtimes \mathcal{D}. \end{array}$$

A *coderivation* is a sum of homogenous coderivations. The space of coderivations is denoted by $\mathrm{Coder}^\bullet_\rho(\mathcal{C},\, \mathcal{D})$.

Example. The differential of a dg coprop(erad) $\mathcal{C}$ is a coderivation of degree -1.

Remark. For a cooperad $\mathcal{D}$, we can define a more general notion of coderivation form a $\mathcal{D}$-cobimodule to $\mathcal{D}$ by a similar formula. The definition given here is a particular case. Since $\rho : \mathcal{C} \to \mathcal{D}$ is a morphism of coprop(erad)s, it provides $\mathcal{C}$ with a natural structure of $\mathcal{D}$-cobimodule.

As explained in the first section, the dual statement of Lemma 11 holds only for connected coprop(erad)s.

Lemma 12. *Let $\mathcal{C}$ be a connected coprop(erad) and let $\rho : \mathcal{C} \to \mathcal{F}^c(W)$ be a morphism of augmented coprop(erad)s. Every coderivation from $\mathcal{C}$ to the cofree connected coprop(erad) $\mathcal{F}^c(W)$ is characterized by its projection on W, that is there is a canonical one-to-one correspondence $Coder^n_\rho(\mathcal{C}, \mathcal{F}(W)) \cong \mathrm{Hom}^{\mathbb{S}}_n(\overline{\mathcal{C}}, W)$.*

Proof. The proof is similar to the one of Lemma 11 and goes by induction on r, where F_r stands for the coradical filtration of $\mathcal{C}$. The assumption that the coprop(erad) $\mathcal{C}$ is connected ensures that the image of an element X of F_r under d lives in $\bigoplus_{n\le r} \mathcal{F}^c(W)^{(n)}$. Therefore, $d(X)$ is finite and d is well defined. □

We denote by d_ν the unique coderivation which extends a map $\nu : \overline{\mathcal{C}} \to W$.

Example. A differential d on a cofree coprop(erad) $\mathcal{F}^c(W)$ is a coderivation of $\mathrm{Der}^{-1}_{\mathrm{Id}}(\mathcal{F}^c(W), \mathcal{F}^c(W))$ such that $d^2 = 0$. By the preceding lemma, it is characterized by the composite $\mathcal{F}^c(W) \xrightarrow{d} \mathcal{F}^c(W) \twoheadrightarrow W$. Its explicit formula can be found in Lemma 18.

Definition (quasi-cofree coprop(erad))**.** A dg coprop(erad) $(\mathcal{F}^c(W), d)$ such that the underlying coprop(erad) is connected cofree is called a *quasi-cofree* coprop(erad).

3.2. **(De)Suspension.** The homological *suspension* of a dg $\mathbb{S}$-bimodule M is denoted by $sM := \mathbb{K}s \otimes M$ with $|s| = 1$, that is $(sM)_i \cong M_{i-1}$. Dually, the homological *desuspension* of M is denoted by $s^{-1}M := \mathbb{K}s^{-1} \otimes M$ with $|s^{-1}| = -1$, that is $(s^{-1}M)_i \cong M_{i+1}$.

Let $(\mathcal{P}, d)$ be an augmented dg $\mathbb{S}$-bimodule, that is $\mathcal{P} = \overline{\mathcal{P}} \oplus I$. A map of augmented $\mathbb{S}$-bimodules $\mu : \mathcal{F}^c(\overline{\mathcal{P}}) \to \mathcal{P}$ consists of a family of morphisms of dg $\mathbb{S}$-bimodules $\mu_n : \mathcal{F}^c(\mathcal{P})^{(n)} \to \mathcal{P}$ for each integer $n \geq 1$. (For $n = 0$, the map μ is the identity $I \to I$.) There is a one-to-one correspondence between maps $\{\mathcal{F}^c(\overline{\mathcal{P}}) \to \mathcal{P}\}$ and maps $\{\mathcal{F}^c(s\overline{\mathcal{P}}) \to s\mathcal{P}\}$. To each map $\mu : \mathcal{F}^c(\overline{\mathcal{P}}) \to \mathcal{P}$, we associate the map $s\mu : \mathcal{F}^c(s\overline{\mathcal{P}}) \to s\mathcal{P}$ defined as follows for $n \geq 1$,

$$(s\mu)_n : \mathcal{F}^c(s\overline{\mathcal{P}})^{(n)} \xrightarrow{\tau_n} s^n\mathcal{F}^c(\overline{\mathcal{P}})^{(n)} \xrightarrow{s^{-(n-1)}} s\mathcal{F}^c(\overline{\mathcal{P}})^{(n)} \xrightarrow{s\otimes\mu_n} s\mathcal{P}.$$

Since it involves permutations between suspensions s and elements of $\mathcal{P}$, the map τ_n yields signs by Koszul-Quillen rule. Using the fact that an element of $\mathcal{F}^c(\overline{\mathcal{P}})$ is an equivalent class of graphs with levels (see 1.4), one can make these signs explicit. The exact formula between $(s\mu)$ and μ is

$$\mu\big(\mathcal{G}(p_1, \ldots, p_n)\big) = (-1)^{\varepsilon(p_1,\ldots,p_n)} s^{-1}(s\mu)\big(\mathcal{G}(sp_1, \ldots, sp_n)\big),$$

where $\varepsilon(p_1, \ldots, p_n) = (n-1)|p_1| + (n-2)|p_2| + \cdots + |p_{n-1}|$.
The degrees of μ and $s\mu$ are related by the formula $|(s\mu)_n| = |\mu_n| - (n-1)$. Therefore, the degree of μ_n is $n-2$ if and only if the degree of $(s\mu)_n$ is -1.

Dually, for any map of augmented $\mathbb{S}$-bimodules $\delta : \mathcal{C} \to \mathcal{F}(\overline{\mathcal{C}})$, we denote by δ_n the composite $\mathcal{C} \xrightarrow{\delta} \mathcal{F}(\overline{\mathcal{C}}) \twoheadrightarrow \mathcal{F}(\overline{\mathcal{C}})^{(n)}$. There is a one-to-one correspondence between maps $\{\mathcal{C} \to \mathcal{F}(\overline{\mathcal{C}})\}$ and maps $\{s^{-1}\mathcal{C} \to \mathcal{F}(s^{-1}\overline{\mathcal{C}})\}$. To each map $\delta : \mathcal{C} \to \mathcal{F}(\overline{\mathcal{C}})$, we associate the map $s^{-1}\delta : s^{-1}\mathcal{C} \to \mathcal{F}^c(s^{-1}\overline{\mathcal{C}})$ defined as follows, for $n \geq 1$,

$$(s^{-1}\delta)_n : s^{-1}\mathcal{C} \xrightarrow{s^{-(n-1)}\otimes\,\delta_n} s^{-n}\mathcal{F}(\overline{\mathcal{C}})^{(n)} \xrightarrow{\sigma_n} \mathcal{F}(s^{(-1)}\overline{\mathcal{C}})^{(n)}.$$

We have $|(s^{-1}\delta)_n| = |\delta_n| - (n-1)$. The degree of δ_n is $n-2$ if and only if the degree of $(s^{-1}\delta)_n$ is -1.

3.3. **Bar construction.** We recall from [Val03] Section 4, the definition of the *bar construction*, which is a functor

$$B : \{\text{aug. dg prop(erad)s}\} \longrightarrow \{\text{coaug. dg coprop(erad)s}\}.$$

Let $(\mathcal{P}, \mu, \eta, \epsilon)$ be an augmented prop(erad). Denote by $\overline{\mathcal{P}}$ its augmentation ideal $\mathrm{Ker}(\mathcal{P} \xrightarrow{\epsilon} I)$. The prop(erad) $\mathcal{P}$ is naturally isomorphic to $\mathcal{P} = I \oplus \overline{\mathcal{P}}$. The bar construction $B(\mathcal{P})$ of $\mathcal{P}$ is a dg coprop(erad) whose underlying space is the cofree coprop(erad) $\mathcal{F}^c(s\overline{\mathcal{P}})$ on the suspension of $\overline{\mathcal{P}}$.

The partial product of $\mathcal{P}$ induces a map of augmented $\mathbb{S}$-bimodules defined by the composite

$$\mu_2\ :\ \overline{\mathcal{F}}^c(\overline{\mathcal{P}}) \twoheadrightarrow \mathcal{F}^c(\overline{\mathcal{P}})^{(2)} \cong \overline{\mathcal{P}} \boxtimes_{(1,1)} \overline{\mathcal{P}} \xrightarrow{\mu_{(1,1)}} \overline{\mathcal{P}}.$$

We have seen in the previous section that μ_2 induces a map $s\mu_2$. Consider the map $\mathbb{K}s \otimes \mathbb{K}s \xrightarrow{\Pi_s} \mathbb{K}s$ of degree -1 defined by $\Pi_s(s\otimes s) := s$. The map $s\mu_2$ is equal to the composite

$$\begin{aligned} s\mu_2 \quad : \quad & \overline{\mathcal{F}}^c(s\overline{\mathcal{P}}) \twoheadrightarrow \mathcal{F}^c(s\overline{\mathcal{P}})^{(2)} \cong (\mathbb{K}s\otimes \overline{\mathcal{P}}) \boxtimes_{(1,1)} (\mathbb{K}s \otimes \overline{\mathcal{P}}) \\ & \xrightarrow{\mathrm{Id}\otimes\tau\otimes\mathrm{Id}} (\mathbb{K}s\otimes\mathbb{K}s)\otimes(\overline{\mathcal{P}}\boxtimes_{(1,1)}\overline{\mathcal{P}}) \xrightarrow{\Pi_s\otimes\mu_{(1,1)}} \mathbb{K}s\otimes\overline{\mathcal{P}}. \end{aligned}$$

Since $\mathcal{F}^c(s\overline{\mathcal{P}})$ is a cofree connected coprop(erad), by Lemma 12 there exists a unique coderivation $d_2 := d_{s\mu_2}\ :\ \mathcal{F}^c(s\overline{\mathcal{P}}) \to \mathcal{F}^c(s\overline{\mathcal{P}})$ which extends $s\mu_2$. When $(\mathcal{P},\ d_{\mathcal{P}})$ is an augmented dg prop(erad). The differential $d_{\mathcal{P}}$ on $\mathcal{P}$ induces an internal differential d_1 on $\mathcal{F}^c(s\overline{\mathcal{P}})$. The total complex of this bicomplex is the *bar construction* $B(\mathcal{P},\ d_{\mathcal{P}}) := (\mathcal{F}^c(s\overline{\mathcal{P}}),\ d = d_1 + d_2)$ of the augmented dg prop(erad) $(\mathcal{P},\ d_{\mathcal{P}})$.

3.4. **Cobar construction.** Dually, the *cobar construction* ([Val03] Section 4) is a functor

$$\Omega\ :\ \{\text{coaug. dg coprop(erad)s}\} \longrightarrow \{\text{aug. dg prop(erad)s}\}.$$

Let $(\mathcal{C},\ \Delta,\ \varepsilon,\ u)$ be a coaugmented coprop(erad). Denote by $\overline{\mathcal{C}}$ its augmentation $\mathrm{Ker}(\overline{\mathcal{C}} \xrightarrow{\varepsilon} I)$. In this case, $\mathcal{C}$ splits naturally as $\mathcal{C} = I \oplus \overline{\mathcal{C}}$. The cobar construction $\Omega(\overline{\mathcal{C}})$ of $\overline{\mathcal{C}}$ is a dg prop(erad) whose underlying space is the free prop(erad) $\mathcal{F}(s^{-1}\overline{\mathcal{C}})$ on the desuspension of $\overline{\mathcal{C}}$.
The partial coproduct of $\mathcal{C}$ induces a natural map of augmented $\mathbb{S}$-bimodules defined by

$$\Delta_2\ :\ \overline{\mathcal{C}} \xrightarrow{\Delta_{(1,1)}} \overline{\mathcal{C}} \boxtimes_{(1,1)} \overline{\mathcal{C}} \cong \mathcal{F}(\overline{\mathcal{C}})^{(2)} \rightarrowtail \overline{\mathcal{F}}(\overline{\mathcal{C}}).$$

This map gives a map $s^{-1}\Delta_2\ :\ s^{-1}\overline{\mathcal{C}} \to \mathcal{F}(s^{-1}\overline{\mathcal{C}})$. Consider $\mathbb{K}s^{-1}$ equipped with the diagonal map $\mathbb{K}s^{-1} \xrightarrow{\Delta_s} \mathbb{K}s^{-1}\otimes\mathbb{K}s^{-1}$ of degree -1 defined by the formula $\Delta_s(s^{-1}) := s^{-1}\otimes s^{-1}$. The map $s^{-1}\Delta_2$ is equal to

$$\begin{aligned} s^{-1}\Delta_2 \quad : \quad & \mathbb{K}s^{-1}\otimes\overline{\mathcal{C}} \xrightarrow{\Delta_s\otimes\Delta_{(1,1)}} \mathbb{K}s^{-1}\otimes\mathbb{K}s^{-1}\otimes\overline{\mathcal{C}}\boxtimes_{(1,1)}\overline{\mathcal{C}} \xrightarrow{\mathrm{Id}\otimes\tau\otimes\mathrm{Id}} \\ & (\mathbb{K}s^{-1}\otimes\mathcal{C})\boxtimes_{(1,1)}(\mathbb{K}s^{-1}\otimes\mathcal{C}) \cong \mathcal{F}(s^{-1}\overline{\mathcal{C}})^{(2)} \rightarrowtail \mathcal{F}(s^{-1}\overline{\mathcal{C}}). \end{aligned}$$

Since $\mathcal{F}(s^{-1}\overline{\mathcal{C}})$ is a free prop(erad), by Lemma 11 there exists a unique derivation $\partial_2 := \partial_{s^{-1}\Delta_2}$: $\mathcal{F}(s^{-1}\overline{\mathcal{C}}) \to \mathcal{F}(s^{-1}\overline{\mathcal{C}})$ which extends $s^{-1}\Delta_2$. When $(\mathcal{C},\ d_{\mathcal{C}})$ is an augmented dg coprop(erad). The differential $d_{\mathcal{C}}$ on $\mathcal{C}$ induces an internal differential ∂_1 on $\mathcal{F}(s^{-1}\overline{\mathcal{C}})$. The total complex of this bicomplex is the *cobar construction* $\Omega(\mathcal{C},\ d_{\mathcal{C}}) := (\mathcal{F}(s^{-1}\overline{\mathcal{C}}),\ \partial = \partial_1 + \partial_2)$ of the augmented dg coprop(erad) $(\mathcal{C},\ d_{\mathcal{C}})$.

3.5. **Twisting morphism.** We generalize the notion of *twisting morphism* (or twisting cochains) of associative algebras (see E. Brown [Bro59]) to prop(erad)s.

As for derivations, a morphism of prop(erad)s is characterized by the image of the indecomposable elements. We recall this fact and the dual statement in the following lemma.

Lemma 13. *Let V be an $\mathbb{S}$-bimodule and let $\mathcal{Q}$ be a prop(erad), there is a canonical one-to-one correspondence* $Mor_{\mathrm{prop(erad)s}}\big(\mathcal{F}(V),\mathcal{Q}\big) \cong \mathrm{Hom}^{\mathbb{S}}(V,\mathcal{Q})$.
Dually, let W be an $\mathbb{S}$-bimodule and let $\mathcal{C}$ be a coprop(erad), there is a canonical one-to-one correspondence $Mor_{\mathrm{coprop(erad)s}}\big(\mathcal{C},\mathcal{F}^c(W)\big) \cong \mathrm{Hom}^{\mathbb{S}}(\mathcal{C},W)$.

Let $(\mathcal{C}, d_{\mathcal{C}})$ be a dg coprop(erad) and $(\mathcal{P}, d_{\mathcal{P}})$ be a dg prop(erad). We would like to apply this result to the bar and the cobar construction of $\mathcal{P}$ and $\mathcal{C}$ respectively, that is we want to describe the space of morphisms of **dg**-prop(erad)s $\mathrm{Mor}_{\mathrm{dg\ prop(erad)s}}\,(\Omega(\mathcal{C}),\ \mathcal{P})$ for instance. By the preceding lemma, this space is isomorphic to the space of morphisms of $\mathbb{S}$-bimodules $\mathrm{Hom}^{\mathbb{S}}_0(s^{-1}\mathcal{C},\mathcal{P})$ of degree 0 whose unique extension commutes with the differentials. Therefore, this space of morphisms is the subspace of $\mathrm{Hom}^{\mathbb{S}}_{-1}(\mathcal{C},\mathcal{P})$ whose elements satisfy a certain relation. To make this equation explicit, we need the structure of dg Lie-admissible (or associative) algebra on the non-symmetric convolution prop(erad) $(\mathrm{Hom}^{\mathbb{S}}(\mathcal{C},\mathcal{P}),\star,D)$ defined in the previous section.

Definition. A morphism $\mathcal{C}\xrightarrow{\alpha}\mathcal{P}$, of degree -1, is called a *twisting morphism* if it is a solution of the *Maurer-Cartan* equation

$$D(\alpha)+\alpha\star\alpha=0.$$

Denote by $\mathrm{Tw}(\mathcal{C},\mathcal{P})$ the set of twisting morphisms in $\mathrm{Hom}^{\mathbb{S}}(\mathcal{C},\mathcal{P})$, that is Maurer-Cartan elements in the convolution prop(erad). Since twisting morphisms have degree -1, it is equivalent for them to be solution of the classical Maurer-Cartan equation in the associated dg Lie algebra, that is $D(\alpha)+\frac{1}{2}[\alpha,\alpha]=0$.

When $\mathcal{P}$ is augmented and $\mathcal{C}$ coaugmented, we will consider either a twisting morphism between $\mathcal{C}$ and $\mathcal{P}$, which sends I to 0, or the associated morphism which sends I to I and $\overline{\mathcal{C}}$ to $\overline{\mathcal{P}}$.

Proposition 14. *For every augmented dg prop(erad) $\mathcal{P}$ and every coaugmented dg coprop(erad) $\mathcal{C}$, there is a canonical one-to-one correspondence $Mor_{dg\ prop(erad)s}(\Omega(\mathcal{C}),\,\mathcal{P})\cong Tw(\mathcal{C},\mathcal{P})$.*
Dually, there is a canonical one-to-one correspondence $Mor_{dg\ coprop(erad)s}(\mathcal{C},\,\mathcal{B}(\mathcal{P}))\cong Tw(\mathcal{C},\mathcal{P})$.

Proof. Since $\Omega(\mathcal{C})=\mathcal{F}(s^{-1}(\overline{\mathcal{C}}))$, by Lemma 13 every morphism φ of $\mathbb{S}$-bimodules in $\mathrm{Hom}^{\mathbb{S}}_0(s^{-1}\mathcal{C},\mathcal{P})$ extends to a unique morphism of prop(erad)s between $\Omega(\mathcal{C})$ and $\mathcal{P}$. The latter one commutes with the differentials if and only if the following diagram commutes

$$\begin{array}{ccccc} s^{-1}\overline{\mathcal{C}} & \xrightarrow{\varphi} & \mathcal{P} & & \\ \downarrow{\scriptstyle \partial} & & & \searrow{\scriptstyle d_{\mathcal{P}}} & \\ \mathcal{F}(s^{-1}\overline{\mathcal{C}})^{(\leqslant 2)} & \xrightarrow{\mathcal{F}(\varphi)} & \mathcal{F}(\mathcal{P}) & \xrightarrow{\widetilde{\mu}^{\mathcal{P}}} & \mathcal{P}. \end{array}$$

For an element $c\in\overline{\mathcal{C}}$, we use Sweedler's notation to denote the image of c under Δ_2, that is $\Delta_2(c)=\sum c'\boxtimes_{(1,1)}c''$. The diagram above corresponds to the relation

$$d_{\mathcal{P}}\circ\varphi(s^{-1}c)=\varphi\circ\partial_1(s^{-1}c)+\mu^{\mathcal{P}}\circ(\varphi\boxtimes_{(1,1)}\varphi)\circ s^{-1}\Delta_2(s^{-1}c).$$

Denote by α the desuspension of φ, that is $\alpha(c)=-\varphi(s^-1c)$. Since $\partial_1(s^{-1}c)=-s^{-1}\partial_{\mathcal{C}}(c)$, the relation becomes

$$-d_{\mathcal{P}}\circ\alpha(c)=\alpha\circ\partial_{\mathcal{C}}(c)+\mu^{\mathcal{P}}\circ(\alpha\boxtimes_{(1,1)}\alpha)\circ\Delta_2(c),$$

which is the Maurer-Cartan equation. □

3.6. **Bar-Cobar adjunction.** A direct corollary of Proposition 14 gives that the bar and cobar constructions form a pair of adjoint functors

$$\Omega\ :\ \{\text{coaug. dg coprop(erad)s}\}\rightleftharpoons\{\text{aug. dg prop(erad)s}\}\ :\ B.$$

This adjunction is given by the set of twisting morphisms.

Proposition 15. *For every augmented dg prop(erad) $\mathcal{P}$ and every coaugmented dg coprop(erad) $\mathcal{C}$, there exists natural isomorphisms*

$$Mor_{aug.\ dg\ prop(erad)s}(\Omega(\mathcal{C}),\,\mathcal{P})\cong\mathrm{Tw}(\mathcal{C},\,\mathcal{P})\cong Mor_{coaug.\ dg\ coprop(erad)s}(\mathcal{C},\,B(\mathcal{P}))\,.$$

We apply the isomorphisms of Proposition 15 to $\mathcal{C}=B(\mathcal{P})$. Associated to the identity on $B(\mathcal{P})$, we get the *counit* of the adjunction $\epsilon\,:\,\Omega(B(\mathcal{P}))\to\mathcal{P}$.

3.7. **Props vs properads.** The main difference for (co)bar construction between props and properads lies on the type of graphs and compositions. The underlying module of the bar construction of a prop $\mathcal{P}$ is spanned by non-necessarily connected graphs whose vertices are labelled with elements of $\mathcal{P}$. The boundary map is the unique coderivation which extends the partial product. It is explicitly given by the sum of the compositions of pair of vertices that are either adjacent (see Section 4.2) or belong to two different connected graphs. Whereas for a properad, the underlying module is spanned by connected labelled graphs and the boundary map just composes adjacent pairs of operations.

3.8. **Bar-cobar resolution.** In [Val03] Theorem 5.8, we proved that the unit of adjunction ϵ is a canonical resolution in the weight graded case. We extend this result to any dg properad here.

Theorem 16. *For every augmented dg properad $\mathcal{P}$, the bar-cobar construction is a resolution of $\mathcal{P}$*

$$\epsilon \ : \ \Omega(B(\mathcal{P})) \xrightarrow{\sim} \mathcal{P}.$$

Proof. The bar-cobar construction of $\mathcal{P}$ is the chain complex defined on the underlying $\mathbb{S}$-bimodule $\mathcal{F}\big(s^{-1}\overline{\mathcal{F}}^c(s\bar{\mathcal{P}})\big)$. The differential d is the sum of three terms $d = \partial_2 + d_2 + d_{\mathcal{P}}$, where $d_{\mathcal{P}}$ is induced by the differential on $\mathcal{P}$, d_2 is induced by the differential of the bar construction $B(\mathcal{P})$ and ∂_2 is the unique derivation which extends the partial coproduct of $\mathcal{F}^c(s\bar{\mathcal{P}})$.
Define the filtration $F_s := \bigoplus_{r\leq s} \mathcal{F}\big(s^{-1}\overline{\mathcal{F}}^c(s\bar{\mathcal{P}})\big)_r$, where r denotes the total number of elements of $\bar{\mathcal{P}}$. Let $E^\bullet_{st}$ be the associated spectral sequence.
This filtration is bounded below and exhaustive. Therefore, we can apply the classical convergence theorem for spectral sequences (see [Wei94]) and prove that $E^\bullet$ converges to the homology of the bar-cobar construction.
We have that $E^0_{st} = \mathcal{F}_{s+t}\big(s^{-1}\mathcal{F}^c(s\bar{\mathcal{P}})\big)_s$, where $s + t$ is the total homological degree. From $d_2(F_s) \subset F_{s-1}$, $d_{\mathcal{P}}(F_s) \subset F_s$ and $\partial_2(F_s) \subset F_s$, we get that $d^0 = \partial_2 + d_{\mathcal{P}}$. The problem is now reduced to the computation of the homology of the cobar construction of the dg cofree connected coproperad $\mathcal{F}^c(s\bar{\mathcal{P}})$ on the dg $\mathbb{S}$-bimodule $s\bar{\mathcal{P}}$. This complex is equal to the bar-cobar construction of the weight graded properad $(\mathcal{P}, \mu')$, where $\mathcal{P}^{(0)} = I$ and $\mathcal{P}^{(1)} = \bar{\mathcal{P}}$, such that the composition μ' is null. We conclude using Theorem 5.8 of [Val03]. □

The bar-cobar resolution gives a canonical cofibrant resolution of any properad.

4. Homotopy (Co)prop(erad)s

An associative algebra is a vector space endowed with a binary product that verifies the strict associative relation. J. Stasheff defined in [Sta63] a lax version of this notion. It is the notion of an associative algebra up to homotopy or (strong) homotopy algebra. Such an algebra is a vector space equipped with a binary product that is associative only up to an infinite sequence of homotopies. In this section, we recall the generalization of this notion, that is the notion of (strong) homotopy properad due to J. Granåker [Gra06]. We extend it to props and we also define in details the dual notion of *(strong) homotopy coprop(erad)*. The notions of *homotopy non-symmetric (co)properad* and *homotopy non-symmetric (co)prop* are obtained in the same way.

4.1. **Definitions.** Following the same ideas as for algebras (associative or Lie, for instance), we define the notion of *homotopy (co)prop(erad)* via (co)derivations and (co)free (co)prop(erad)s.

Definition (Homotopy prop(erad))**.** A structure of *homotopy prop(erad)* on an augmented dg $\mathbb{S}$-bimodule $(\mathcal{P}, \mu^{\mathcal{P}}, d_{\mathcal{P}})$ is a coderivation d of degree -1 on $\mathcal{F}^c(s\overline{\mathcal{P}})$ such that $d^2 = 0$.

A structure of homotopy prop(erad) is equivalent to a structure of quasi-cofree coprop(erad) on $s\overline{\mathcal{P}}$. We call the latter the *(generalized) bar construction of* $\mathcal{P}$ and we denote it by $\mathcal{B}_\infty(\mathcal{P})$. Since $\mathcal{F}^c(s\overline{\mathcal{P}})$ is a cofree connected coprop(erad), by Lemma 12 the coderivation d is characterized by the composite

$$s\mu \ : \ \mathcal{F}^c(s\overline{\mathcal{P}}) \xrightarrow{d} \mathcal{F}^c(s\overline{\mathcal{P}}) \twoheadrightarrow s\mathcal{P},$$

that is $d = d_{s\mu}$. The map $s\mu$ of degree -1 is equivalent to a unique map $\mu \ : \ \mathcal{F}^c(\overline{\mathcal{P}}) \to \mathcal{P}$, such that $\mu_n \ : \ \mathcal{F}^c(\overline{\mathcal{P}})^{(n)} \to \mathcal{P}$ has degree $n-2$. The condition $d^2 = 0$ written with the $\{\mu_n\}_n$ is made explicit in Proposition 19.

Example. A dg prop(erad) is a homotopy prop(erad) such that every map $\mu_n = 0$ for $n \geq 3$. In this case, $(\mathcal{F}^c(s\overline{\mathcal{P}}), d)$ is the bar construction of $\mathcal{P}$.

We define the notion of *homotopy coprop(erad)* by a direct dualization of the previous arguments.

Definition (Homotopy coprop(erad))**.** A structure of *homotopy coprop(erad)* on an augmented dg $\mathbb{S}$-bimodule $(\mathcal{C}, d_{\mathcal{C}})$ is a derivation ∂ of degree -1 on $\mathcal{F}(s^{-1}\overline{\mathcal{C}})$ such that $\partial^2 = 0$.

A structure of homotopy coprop(erad) is equivalent to a structure of quasi-free prop(erad) on $s^{-1}\overline{\mathcal{C}}$. We call the latter the *(generalized) cobar construction of* $\mathcal{C}$ and we denote it by $\Omega_\infty(\mathcal{C})$. By Lemma 11, the derivation ∂ is characterized by its restriction on $s^{-1}\overline{\mathcal{C}}$

$$s^{-1}\delta \ : \ s^{-1}\overline{\mathcal{C}} \rightarrowtail \overline{\mathcal{F}}(s^{-1}\overline{\mathcal{C}}) \xrightarrow{\partial} \overline{\mathcal{F}}(s^{-1}\overline{\mathcal{C}}),$$

that is $\partial = \partial_{s^{-1}\delta}$. The map $s^{-1}\delta$ of degree -1 is equivalent a map $\delta : \mathcal{C} \to \mathcal{F}(\overline{\mathcal{C}})$, such that the component $\delta_n : \mathcal{C} \to \mathcal{F}(\overline{\mathcal{C}})^{(n)}$ has degree $n-2$. The condition $\partial^2 = 0$ is equivalent to relations for the $\{\delta_n\}_n$ that we make explicit in Proposition 20.

Example. A dg coprop(erad) is a homotopy coprop(erad) such that every map $\delta_n = 0$ for $n \geq 3$. In this case, $(\mathcal{F}(s^{-1}\overline{\mathcal{C}}), \partial)$ is the cobar construction of $\mathcal{C}$.

When $\mathcal{P}$ is concentrated in arity $(1,1)$, the definition of a homotopy properad on $\mathcal{P}$ is exactly the same than the definition of an strong homotopy algebra given by J. Stasheff in [Sta63]. Dually, when $\mathcal{C}$ is concentrated in arity (1, 1), we get the notion of strong homotopy coassociative algebra. When $\mathcal{P}$ is concentrated in arity $(1,n)$ for $n \geq 1$, we have the notion of *strong homotopy operad* (see [vdL]). The dual notion gives the definition of a *strong homotopy cooperad*.

Remark. By abstract nonsense, the notion of homotopy prop(erad) should also come from Koszul duality for colored operads (see [van03]). There exists a colored operad those"algebras" are (partial) prop(erad)s. Such a colored operad is quadratic (the associativity relation of the partial product of a prop(erad) is an equation between compositions of two elements. It should be a Koszul colored operad. An "algebra" over the Koszul resolution of this colored operad is exactly a homotopy prop(erad).

4.2. **Admissible subgraph.** Let $\mathcal{G}$ be a connected graph directed by a flow and denote by $\mathcal{V}$ its set of vertices. We define a partial order on $\mathcal{V}$ by the following covering relation : $i \prec j$ if i is below j according to the flow and if there is no vertex between them. In this case, we say that i and j are *adjacent* (see also [Val03] p. 34). Examples of adjacent and non-adjacent vertices can be found in Figure 1.

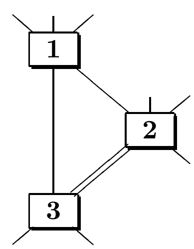
$\mathcal{G} =$ 

Figure 1. The vertices 1, 2 and 2, 3 are adjacent. The vertices 1 and 3 are not adjacent.

Denote this poset by $\Pi_{\mathcal{G}}$ and consider its Hasse diagram $\mathcal{H}(\mathcal{G})$, that is the diagram composed by the elements of the poset with one edge between two of them, when they are related by a covering relation. See Figure 2 for an example.

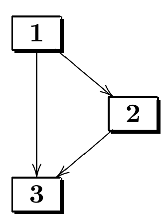
$\mathcal{H}(\mathcal{G}) =$ 

Figure 2. The Hasse diagramm associated to the graph of Figure 1

Actually, $\mathcal{H}(\mathcal{G})$ is obtained from $\mathcal{G}$ be removing the external edges and by replacing several edges between two vertices by only one edge. Since $\mathcal{G}$ is connected and directed by a flow, the Hasse diagram $\mathcal{H}(\mathcal{G})$ has the same properties. A *convex subset* $\mathcal{V}'$ of $\mathcal{V}$ is a set of vertices of $\mathcal{G}$ such that for every pair $i \leq j$ in $\mathcal{V}'$ the interval $[i, j]$ of $\Pi_{\mathcal{G}}$ is included in $\mathcal{V}'$. If $\mathcal{G}$ is a connected graph of genus 0, the set of vertices of any connected subgraph of $\mathcal{G}$ is convex. This property does not hold any more for connected graphs of higher genus.

Lemma 17. *Let $\mathcal{G}$ be a connected directed graph without oriented loops and let $\mathcal{G}'$ be a connected subgraph of $\mathcal{G}$. The set of vertices of $\mathcal{G}'$ is convex if and only if the contraction of $\mathcal{G}'$ inside of $\mathcal{G}$ gives a graph without oriented loops.*

A connected subgraph $\mathcal{G}'$ with this property is called *admissible* in [Gra06]. We denote by $\mathcal{G}/\mathcal{G}'$ the graph obtained by the contraction of $\mathcal{G}$ by $\mathcal{G}'$. See Figure 3 for an example of a admissible subgraph and an example of a non-admissible subgraph of $\mathcal{G}$. By extension, an admissible subgraph of a non-necessarily connected graph is a union of admissible subgraphs (eventually empty) of each connected component.

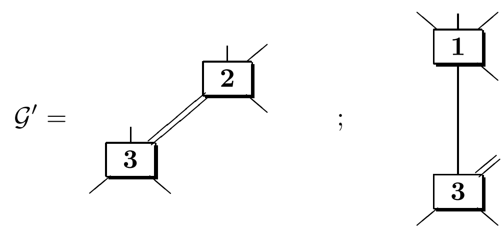


FIGURE 3. Example of a admissible subgraph $\mathcal{G}'$ of $\mathcal{G}$ and an example of a non-admissible subgraph of $\mathcal{G}$.

4.3. **Interpretation in terms of graphs.** Let $\mu : \mathcal{F}^c(\overline{\mathcal{P}}) \to \mathcal{P}$ be a morphism of augmented dg $\mathbb{S}$-bimodules. We denote by $\mu\big(\mathcal{G}(p_1, \ldots, p_n)\big)$ the image of an element $\mathcal{G}(p_1, \ldots, p_n)$ of $\mathcal{F}^c(\overline{\mathcal{P}})^{(n)}$ under μ. Let $\mathcal{G}'$ be a admissible subgraph of $\mathcal{G}$ with k vertices. Denote by $\mathcal{G}/\mu\mathcal{G}'(p_1, \ldots, p_n)$ the element of $\mathcal{F}^c(\overline{\mathcal{P}})^{(n-k+1)}$ obtained by composing $\mathcal{G}'(p_{i_1}, \ldots, p_{i_k})$ in $\mathcal{G}(p_1, \ldots, p_n)$ under μ. When the p_i and μ are not of degree zero, this composition induces natural signs that we make explicit in the following. Let start with a representative element of a class of graph $\mathcal{G}(p_1, \ldots, p_n)$ whose vertices are indexed by elements p_i, that is to say we have chosen an order between the p_i (see Section 1.4). The vertices of $\mathcal{G}'$ are indexed by elements $p_{i_1}, \ldots, p_{i_k}$. We denote by $J = (i_1, \ldots, i_k)$ the associated ordered subset of $[n] = \{1, \ldots, n\}$ and $p_J = p_{i_1}, \ldots, p_{i_k}$. Since $\mathcal{G}'$ is an admissible subgraph, its set of vertices forms a convex subset of the set of vertices of $\mathcal{G}$ (or a disjoint union of convex subsets if $\mathcal{G}$ is not connected). Therefore, it is possible to change the order of the vertices of $\mathcal{G}$ such that the vertices of $\mathcal{G}'$ are next to each others. That is there exists two ordered subsets I_1 and I_2 of $[n]$ such that the underlying subsets I_1, I_2 and J without order form a partition of $[n]$ and such that $\mathcal{G}(p_1, \ldots, p_n) = (-1)^{\varepsilon_1}\mathcal{G}(P_{I_1}, P_J, P_{I_2})$. The sign $(-1)^{\varepsilon_1}$ is given by the Koszul-Quillen sign rule from the permutation of the p_i. Now we can apply μ to get

$$\mathcal{G}/\mu\mathcal{G}'(p_1, \ldots, p_n) = (-1)^{\varepsilon_1+\varepsilon_2}\mathcal{G}/\mathcal{G}'\big(P_{I_1}, \mu(\mathcal{G}'(P_J)), P_{I_2}\big),$$

where $\varepsilon_2 = |P_{I_1}|.|\mu|$. It is an easy exercise to prove that this definition of the signs does not depend on the different choices.

Lemma 18. *Let ν be a map $\mathcal{F}^c(W) \to W$ of degree -1. The unique coderivation $d_\nu \in \mathrm{CoDer}^{-1}_{\mathrm{Id}}(\mathcal{F}^c(W), \mathcal{F}^c(W))$ which extends ν is given by*

$$d_\nu\big(\mathcal{G}(w_1, \ldots, w_n)\big) = \sum_{\mathcal{G}' \subset \mathcal{G}} \mathcal{G}/\nu\mathcal{G}'(w_1, \ldots, w_n),$$

where the sum runs over admissible subgraphs $\mathcal{G}'$ of $\mathcal{G}$.

Proof. This formula defines a coderivation. Since the composite of d_ν with the projection on W is equal to ν, we conclude by the uniqueness property of coderivations of Lemma 12. □

Proposition 19. *A map $\mu : \mathcal{F}^c(\overline{\mathcal{P}}) \to \mathcal{P}$ defines a structure of homotopy prop(erad) on the augmented dg $\mathbb{S}$-bimodule $\mathcal{P}$ if and only if, for every $\mathcal{G}(p_1, \ldots, p_n)$ in $\mathcal{F}^c(\overline{\mathcal{P}})$, we have*

$$\sum_{\mathcal{G}' \subset \mathcal{G}} (-1)^{\varepsilon(\mathcal{G}', p_1, \ldots, p_n)} \mu\big(\mathcal{G}/\mu\mathcal{G}'(p_1, \ldots, p_n)\big) = 0,$$

where the sum runs over admissible subgraphs $\mathcal{G}'$ of $\mathcal{G}$.

Proof. By definition, μ induces a structure of homotopy prop(erad) if and only if $d^2_{s\mu} = 0$. This last condition holds if and only if the composite $\mathrm{proj}_{s\mathcal{P}} \circ d^2_{s\mu} = (s\mu) \circ d_{s\mu}$ is zero, where $\mathrm{proj}_{s\mathcal{P}}$ is the projection on $s\mathcal{P}$. From Lemma 18, this is equivalent to

$$\sum_{\mathcal{G}' \subset \mathcal{G}} (s\mu)\big(\mathcal{G}/(s\mu)\mathcal{G}'(sp_1, \ldots, sp_n)\big) = 0,$$

where the sum runs over admissible subgraphs $\mathcal{G}'$ of $\mathcal{G}$. Recall from Section 3.2 that the signs between $(s\mu)$ and μ are

$$\mu\big(\mathcal{G}(p_1, \ldots, p_n)\big) = (-1)^{\varepsilon(p_1, \ldots, p_n)} s^{-1}(s\mu)\big(\mathcal{G}(sp_1, \ldots, sp_n)\big),$$

where $\varepsilon(p_1, \ldots, p_n) = (n-1)|p_1| + (n-2)|p_2| + \cdots + |p_{n-1}|$. Therefore, μ induces a structure of homotopy prop(erad) if and only if

$$\sum_{\mathcal{G}' \subset \mathcal{G}} (-1)^{\varepsilon(\mathcal{G}', p_1, \ldots, p_n)} \mu\big(\mathcal{G}/\mu\mathcal{G}'(p_1, \ldots, p_n)\big) = 0,$$

where $(-1)^{\varepsilon(\mathcal{G}', p_1, \ldots, p_n)}$ is product of the sign coming the composition with $s\mu$ and the sign coming from the formula between μ and $s\mu$. □

Remark. In the case of associative algebras, the graphs involved are branches and we recover exactly the original definition of J. Stasheff [Sta63].

Dually, we have the following characterization of homotopy coprop(erad)s. Let $\mathcal{G}$ be a graph whose i^{th} vertex has n inputs and m outputs. For every graph $\mathcal{G}'$ with n inputs and m outputs, denote by $\mathcal{G} \circ_i \mathcal{G}'$ the graph obtained by inserting $\mathcal{G}'$ in $\mathcal{G}$ at the place of the i^{th} vertex.

Proposition 20. *A map $\delta : \mathcal{C} \to \mathcal{F}(\overline{\mathcal{C}})$ defines a structure of homotopy coprop(erad) on the augmented dg $\mathbb{S}$-bimodule $\mathcal{C}$ if and only if, for every $c \in \overline{\mathcal{C}}$, we have*

$$\sum (-1)^{\rho(\mathcal{G}_i^2, c_1, \ldots, c_l)} \mathcal{G}^1 \circ_i \mathcal{G}_i^2(c_1, \ldots, c_{i-1}, c'_1, \ldots, c'_k, c_{i+1}, \ldots, c_l) = 0,$$

where the sum runs over elements $\mathcal{G}^1(c_1, \ldots, c_l)$ and $\mathcal{G}_i^2(c'_1, \ldots, c'_k)$ such that $\delta(c) = \sum \mathcal{G}^1(c_1, \ldots, c_l)$ and $\delta(c_i) = \sum \mathcal{G}_i^2(c'_1, \ldots, c'_k)$.

Proof. By definition, δ induces a structure of homotopy coprop(erad) if and only if $\partial^2_{s^{-1}\delta} = 0$. Since $\partial_{s^{-1}\delta}$ is a derivation, $\partial^2_{s^{-1}\delta} = 0$ is equivalent to $\partial_{s^{-1}\delta} \circ (s^{-1}\delta)(s^{-1}c) = 0$, for every $c \in \overline{\mathcal{C}}$. Denote $(s^{-1}\delta)(s^{-1}c) = \sum \mathcal{G}^1(s^{-1}c_1, \ldots, s^{-1}c_l)$ and $(s^{-1}\delta)(s^{-1}c_i) = \sum \mathcal{G}_i^2(s^{-1}c'_1, \ldots, s^{-1}c'_k)$. By the explicit formula for $\partial_{s^{-1}\delta}$ given in Lemma 11 applied to $\rho = \mathrm{Id}_{\mathcal{F}(s^{-1}\overline{\mathcal{C}})}$, we have

$$\begin{aligned} \partial_{s^{-1}\delta} \circ (s^{-1}\delta)(s^{-1}c) &= \partial_{s^{-1}\delta}\Big(\sum \mathcal{G}^1(s^{-1}c_1, \ldots, s^{-1}c_l)\Big) \\ &= \sum \mathcal{G}^1 \circ_i \mathcal{G}_i^2(s^{-1}c_1, \ldots, s^{-1}c_{i-1}, s^{-1}c'_1, \ldots, s^{-1}c'_k, s^{-1}c_{i+1}, \ldots, s^{-1}c_l) = 0 \end{aligned}$$

We get back to the map δ with the formula

$$\delta(c) = (-1)^{\varepsilon(c_1, \ldots, c_l)} \sum \mathcal{G}^1(c_1, \ldots, c_l),$$

where $\varepsilon(c_1, \ldots, c_l) = (l-1)|c_1| + (l-2)|c_2| + \cdots + |c_{l-1}|$. We conclude as in proof of Proposition 19. □

4.4. **Homotopy non-symmetric (co)prop(erad).** It is straightforward to generalize the two previous subsections to non-symmetric (co)prop(erad)s. One has just to consider non-labelled graphs instead of graphs with leaves, inputs and outputs labelled by integers. Therefore, there is a bar and a cobar construction between non-symmetric dg prop(erad)s and non-symmetric dg coprop(erad)s. The notion that will be used in the sequel is the notion of *homotopy non-symmetric prop(erad)*. It is defined by a coderivation on the non-symmetric cofree (connected) coprop(erad). Equivalently, we can describe it in terms of non-labelled graphs like in Proposition 19. The chain complex defining the cohomology of a gebra over a prop(erad) has always such a structure. This structure induces a (strong) homotopy Lie algebra which is used to study deformation theory (see Section 7.2).

5. Models

In this section, we recall the definitions of *minimal* and *quadratic model* for properads and we formally extend them to props.

5.1. **Minimal Models.** Recall that a *quasi-free* prop(erad) is a (dg)prop(erad) whose underlying $\mathbb{S}$-bimodule is a free prop(erad) $\mathcal{F}(M)$. It is not necessarily a free prop(erad) since the differential ∂ is not necessarily free.

Definition (Model)**.** Let $\mathcal{P}$ be a prop(erad). A *model* of $\mathcal{P}$ is a quasi-free prop(erad) $(\mathcal{F}(M),\, \partial)$ equipped with a quasi-isomorphism $\mathcal{F}(M) \xrightarrow{\sim} \mathcal{P}$.

Theorem 16 proved that every augmented prop(erad) has a canonical model given by the bar-cobar construction. Some prop(erad)s admit more simple models. The differential ∂ of a quasi-free prop(erad) $\mathcal{F}(M)$ is by definition a derivation. Lemma 11 shows that it is characterized by its restriction $\partial_M\,:\, M \to \mathcal{F}(M)$ on M.

Definition (Decomposable differential)**.** The differential ∂ of a quasi-free prop(erad) is called *decomposable* if the image of its restriction to M, $\partial_M\,:\, M \to \mathcal{F}(M)$, is made of decomposable elements, that is $\mathrm{Im}(\partial_M) \subset \bigoplus_{n\geq 2} \mathcal{F}(M)^{(n)}$.

Definition (Minimal model)**.** A model $(\mathcal{F}(M),\, \partial)$ is called *minimal* if its differential ∂ is decomposable.

5.2. **Form of minimal models.** From Theorem 16, we know that every augmented (dg) properad admits a resolution of the form $\Omega(\mathcal{B}(P))$. A natural way to get a minimal model from this would be to consider the homology of the bar construction, try to endow it with a structure of homotopy coproperad and then take the generalize cobar construction of it. In this section, we prove that when minimal models exist, they are of this form.

Proposition 21. *Let M be an $\mathbb{S}$-bimodule with a map $\delta\,:\, M \to \bar{\mathcal{F}}(M)$ of degree -1. Consider the quasi-free properad $(\mathcal{F}(M),\, \partial_\delta)$, where ∂_δ is the derivation induced by δ. The homology of the bar construction $B(\mathcal{F}(M))$ of $(\mathcal{F}(M),\, \partial)$ is equal to the suspension of M.*

Proof. The bar construction of the dg-properad $\mathcal{P} := \mathcal{F}(M)$ is defined by the underlying $\mathbb{S}$-bimodule $B(\mathcal{P}) := \mathcal{F}^c(s\bar{\mathcal{P}}) = \mathcal{F}^c(s\,\bar{\mathcal{F}}(M))$. The differential d is the sum of two terms $d_0 + \tilde{\partial}$. The component $\tilde{\partial}$ comes from ∂_δ and d_0 is the unique coderivation which extends the partial product of $\mathcal{F}(M)$.

Consider the filtration $F_s := \bigoplus_{r\leq s} \mathcal{F}^c\big(s\bar{\mathcal{F}}(M)\big)_r$, where r is the sum of the degrees of the elements of M. Denote $E^\bullet_{st}$ the associated spectral sequence.

Since the chain complex M is bounded below, this filtration is bounded below $F_{-1} = 0$. It is obviously exhaustive, therefore the classical theorem of convergence of spectral sequences shows that $E^\bullet$ converges to the homology of $B(\mathcal{F}(M))$.

We have $\tilde{\partial}(F_s) \subset F_{s-1}$ and $d_0(F_s) \subset F_s$. Hence, the first term of the spectral sequence is $E^0_{st} = \mathcal{F}^c_{s+t}\big(s\bar{\mathcal{F}}(M)\big)_s$, where $s+t$ is the total homological degree, and $d^0 = d_0$. We have reduced the problem to computing the homology of the bar construction of the free properad on M, which is equal to ΣM by Corollary 5.10 of [Val03] (choose to put each element of M in weight 1). $\square$

The next proposition shows that, when a minimal model of a properad $\mathcal{P}$ exists, it is necessarily given by a quasi-free properad on the homology of the bar construction of $\mathcal{P}$.

Theorem 22. *Let $\mathcal{P}$ be an augmented dg properad and let $(\mathcal{F}(M),\, \partial)$ be a minimal model of $\mathcal{P}$. The $\mathbb{S}$-bimodule sM is isomorphic to the homology of the bar construction of $\mathcal{P}$.*

Proof. In [Val03], we proved in Proposition 4.9 that the bar construction preserves quasi-isomorphisms. Therefore, the bar construction of $\mathcal{F}(M)$ is quasi-isomorphic to the bar construction of $\mathcal{P}$. And we conclude by Proposition 21. □

We denote by $\mathcal{P}^{\text{¡}} := H_{\bullet}(\mathcal{B}(\mathcal{P}))$ the homology of the bar construction of $\mathcal{P}$. When $(\mathcal{F}(s^{-1}\mathcal{P}^{\text{¡}}),\, \partial)$ is a minimal model of $\mathcal{P}$, the derivation ∂ is equivalent to a structure of homotopy coproperad on $\mathcal{P}^{\text{¡}}$ such that $\delta_1 = 0$. That is $(\mathcal{F}(s^{-1}\mathcal{P}^{\text{¡}}),\, \partial)$ is the generalized cobar construction $\Omega_\infty(\mathcal{P}^{\text{¡}})$ of the homotopy coproperad $\mathcal{P}^{\text{¡}}$. As a conclusion, we have that following corollary which gives the form of minimal models.

Corollary 23. *A minimal model of an augmented dg-properad $\mathcal{P}$ is always the generalized cobar construction $\Omega_\infty(\mathcal{P}^{i})$ on the homology of $B(\mathcal{P})$ endowed with a structure of homotopy coproperad.*

In the sequel, we will only consider props freely generated by a properad, in the sense of the horizontal (concatenation) product. The minimal model of such props is given by the generalized cobar construction of the associated homotopy coproperad, viewed as a homotopy coprop. And the result of the preceding lemma still holds.

5.3. **Quadratic models and Koszul duality theory.** In general, it is a difficult problem to find the minimal model of a prop(erad). On has first to compute the homology of the bar construction and then provide it with a structure of homotopy coproperad, that is with higher homotopy cooperations. For weight graded prop(erad)s, there exist simple minimal models which are given by the Koszul duality theory.

Definition (Quadratic differential)**.** The differential ∂ of a quasi-free prop(erad) is called *quadratic* if the image of $\partial_M\,:\, M \to \mathcal{F}(M)$ is in $\mathcal{F}(M)^{(2)}$.

Definition (Quadratic model)**.** A model $(\mathcal{F}(M),\, \partial)$ is called *quadratic* if its differential ∂ is quadratic.

When $\mathcal{P}$ is a weight graded properad, its bar construction splits as a direct sum of finite chain complexes indexed by the weight (cf. [Val03] Section 7.1.1). In this case, we can speak of top dimensional homology groups.

Theorem 24. *Let $\mathcal{P}$ be a weight graded properad concentrated in homological degree* 0*. The following assertions are equivalent.*

(1) *The homology of $B(\mathcal{P})$ is concentrated in top dimension.*
(2) *The $\mathbb{S}$-bimodule $\mathcal{P}^{i}$ is a strict coproperad.*
(3) *The properad $\mathcal{P}$ admits a quadratic model.*

Proof. (1) ⇒ (2) is given by Proposition 7.2 of [Val03].
(2) ⇒ (3) is given by Theorem 5.9 of [Val03]. When $\mathcal{P}^{\text{¡}}$ has a structure of strict coproperad, its cobar construction is a resolution of $\mathcal{P}$ and the differential of it is quadratic.
(3) ⇒ (1) Since $\mathcal{P}$ is isomorphic to $\mathcal{F}(M_0)/(\partial(M_1))$, which ∂ quadratic, this presentation is quadratic. Define an extra weight on M by the formula $\omega(M_n) := n+1$. With this weight, the quasi-isomorphism $\mathcal{F}(M) \xrightarrow{\rho} \mathcal{P}$ is a morphism of weight graded dg-properads. The induced morphism $B(\rho)$ on the bar construction preserves this grading. Therefore we have $H_n(B(\mathcal{P})^{(n)}) = H_n(B(\mathcal{F}(M))^{(n)}) = (sM)_n$ and the homology of the bar construction of $\mathcal{P}$ is concentrated in top dimension. □

In this case, the properad $\mathcal{P}$ is called a *Koszul* properad. The coproperad $\mathcal{P}^{\text{¡}}$ is its Koszul dual and $\mathcal{P}$ has a quadratic model which is the cobar construction on $\mathcal{P}^{\text{¡}}$. In other words, a properad is Koszul when its bar construction is *formal*, that is when $\mathcal{B}(\mathcal{P})$ is quasi-isomorphic to its homology $\mathcal{P}^{\text{¡}}$ as a dg-coproperad. This case is particularly simple but does not apply in general. For instance a Koszul properad is necessarily quadratic (Corollary 7.5 of [Val03]), that is admits a quadratic presentation. This argument implies that the properad of associative bialgebras (cf. Section 7.5.3) can not be Koszul.

6. L_∞-algebras, dg manifolds, dg affine schemes and morphisms of prop(erad)s

6.1. **L_∞-algebras, dg manifolds and dg affine schemes.** Structure of a L_∞-algebra on a $\mathbb{Z}$-graded vector space $\mathfrak{g}$ is, by definition, a degree -1 coderivation, $Q : \odot^{\geq 1} s\mathfrak{g} \to \odot^{\geq 1} s\mathfrak{g}$, of the free cocommutative coalgebra without counit,

$$\odot^{\geq 1} s\mathfrak{g} := \bigoplus_{n\geq 1} \odot^n (s\mathfrak{g}) \ \subset \ \odot^\bullet s\mathfrak{g} := \bigoplus_{n\geq 0} \odot^n (s\mathfrak{g}),$$

which satisfies the condition $Q^2 = 0$. For finite-dimensional $\mathfrak{g}$ the ring $\odot^\bullet(s\mathfrak{g})$ can be geometrically interpreted as the structure ring, $\mathcal{O}_{\mathcal{M}_\mathfrak{g}}$, of smooth functions on the dual space, $\mathcal{M}_\mathfrak{g} := s^{-1}\mathfrak{g}^*$, viewed as a formal graded manifold. The subring $I := \odot^{\geq 1} s\mathfrak{g} \subset \mathcal{O}_{\mathcal{M}_\mathfrak{g}}$ gets then interpreted as the ideal of the distinguished point $0 \in \mathcal{O}_{\mathcal{M}_\mathfrak{g}}$, while the coderivation Q as a degree -1 vector field (denoted by the same letter Q) on $\mathcal{M}_\mathfrak{g}$ which vanishes at the distinguished point (as $QI \subset I$) and satisfies the condition $[Q, Q] = 2Q^2 = 0$. Such vector fields are often called *homological*.

In this geometric picture of L_∞-algebra structures on $\mathfrak{g}$, the subclass of dg Lie algebra structures gets represented by at most quadratic homological vector fields Q, that is, the ones satisfying the equation $Q(I/I^2) \subset I/I^3$. Given a particular dg Lie algebra $(\mathfrak{g}, d, [\ ,\ ])$, the associated homological vector field Q on $\mathcal{M}_\mathfrak{g}$ can be explicitly constructed as follows:

- an arbitrary element $\gamma \in \mathfrak{g}$ defines a linear function on $\mathcal{M}_\mathfrak{g}$ and an arbitrary vector field, Q, is uniquely determined by its values, $Q(\gamma)$, on such linear functions;
- define a degree -1 vector field on $\mathcal{M}_\mathfrak{g}$ by setting

$$Q(\gamma) := d\gamma + \frac{1}{2}[\gamma, \gamma]; \tag{1}$$

  it is clear that Q vanishes at $0 \in \mathcal{M}_\mathfrak{g}$;
- finally check that

$$\begin{aligned} Q^2(\gamma) &= Q\left(d\gamma + \frac{1}{2}[\gamma,\gamma]\right) \\ &= -d(Q(\gamma)) + [Q(\gamma), \gamma] \\ &= -d\left(d\gamma + \frac{1}{2}[\gamma,\gamma]\right) + \left[d\gamma + \frac{1}{2}[\gamma,\gamma], \gamma\right] \\ &= 0. \end{aligned}$$

Notice that the zero locus of Q contains the set of Maurer-Cartan elements in $\mathfrak{g}$.

A serious deficiency of the above literal geometric interpretation of L_∞-algebras is the necessity to work with the dual objects, $(\mathcal{O}_{\mathcal{M}_\mathfrak{g}}, Q)$ which make sense only for finite dimensional $\mathfrak{g}$. So we follow suggestion of Kontsevich [Kon03] and understand from now on a *dg (smooth) manifold* as a pair, $(\odot^{\geq 1} X, Q)$, consisting of a cofree cocommutative algebra on a $\mathbb{Z}$-graded vector space X together with a degree -1 codifferential Q. Note that the dual of $\odot^{\geq 1} X$ is a well defined graded commutative algebra (without assumption on finite-dimensionality of X) and that dual of Q is a well-defined derivation of the latter. We identify from now on Q with its dual and call it a *homological* vector field on the dg manifold[1] X. This abuse of terminology is very helpful — we can use simple and geometrically clear formulae of the type (1) to define (in a mathematically rigorous way!) codifferentials Q on $\odot^{\geq 1} X$. Such codifferentials, $Q : \odot^{\geq 1} X \to \odot^{\geq 1} X$, are completely determined by the associated compositions,

$$Q_{pro} : \odot^{\geq 1} X \xrightarrow{Q} \odot^{\geq 1} X \xrightarrow{proj} X.$$

[1] A warning about shift of grading: according to our definitions, a homological vector field on a graded vector space X is the same as a L_∞-structure on $s^{-1}X$.

The restriction of Q_{pro} to $\odot^n X \subset \odot^{\geq 1} X$ is denoted by Q_n, $n \geq 1$.

If I is a coideal of the coalgebra $\odot^{\geq 1} X$, we denote the associated sub-coalgebra of $\odot^{\geq 1} X$ by $(\mathcal{O}_I := I \setminus \odot^{\geq 1} X, Q)$. When the later is preserved by Q, then the data $(\mathcal{O}_I := I \setminus \odot^{\geq 1} X, Q)$ is naturally a differential graded coalgebra which we often call a *dg affine scheme*. The coideal may not, in general, be homogeneous so the "weight" gradation, $\bigoplus_n \odot^n X$, may not survive in $\mathcal{O}_I$. A generic dg affine scheme by no means corresponds to a L_∞-algebra but, as we shall see below, some interesting examples (with non-trivial and non-homogeneous coideals) do.

6.2. **Maurer-Cartan elements in a filtered L_∞-algebra.** A L_∞-algebra $(\mathfrak{g}, Q = \{Q_n\}_{n\geq 1})$ is called *filtered* if $\mathfrak{g}$ admits a non-negative decreasing Hausdorff filtration,

$$\mathfrak{g}_0 = \mathfrak{g} \supseteq \mathfrak{g}_1 \supseteq \ldots \supseteq \mathfrak{g}_i \supseteq \ldots,$$

such that the linear map $Q_n : \odot^n(s\mathfrak{g}) \to s\mathfrak{g}$ takes values in $s\mathfrak{g}_n$ for all $n \geq n_0$ and some $n_0 \in \mathbb{N}$. In this case Q extends naturally to a coderivation of the cocommutative coalgebra, $\odot^{\geq 1} s\hat{\mathfrak{g}}$, with $\hat{\mathfrak{g}}$ being the completion of $\mathfrak{g}$ with respect to the topology induced by the filtration, and the equation,

$$Q(\sum_{n\geq 1} \frac{1}{n!}\gamma^{\odot n}) = 0,$$

for a degree zero element $\gamma \in s\hat{\mathfrak{g}}$ (i.e. for a degree -1 element in $\hat{\mathfrak{g}}$) makes sense. Its solutions are called *(generalized) Maurer-Cartan elements* (or, shortly, *MC elements*) in $(\mathfrak{g}, Q)$. Geometrically, an MC element is a degree -1 element in $\hat{\mathfrak{g}}$ at which the homological vector field Q vanishes. From now on we do not distinguish between $\mathfrak{g}$ and its completion $\hat{\mathfrak{g}}$.

To every MC element γ in a filtered L_∞-algebra $(\mathfrak{g}, Q)$ there corresponds, by Theorem 2.6.1 in [Mer00], a twisted L_∞-algebra, $(\mathfrak{g}, Q^\gamma)$, with

$$Q^\gamma(\alpha) := Q(\sum_{n\geq 0} \frac{1}{n!}\gamma^{\odot n} \odot \alpha)$$

for an arbitrary $\alpha \in \odot^{\geq 1} s\mathfrak{g}$. (We refer the reader to Lemma 4.4 of [van03] for formulae of the Q_n^γ). The geometric meaning of this twisted L_∞-structure is simple [Mer00]: if a homological vector field Q vanishes at a degree 0 point $\gamma \in s\mathfrak{g}$, then applying to Q a formal diffeomorphism, ϕ_γ, which is a translation sending γ into the origin 0 (and which is nothing but the unit shift, $e^{\mathrm{ad}\gamma}$, along the formal integral lines of the constant vector field $-\gamma$) will give us a new formal vector field, $Q^\gamma := d\phi_\gamma(Q)$, which is *homological* and *vanishes* at the distinguished point; thus Q^γ defines a L_∞ structure on the underlying space $\mathfrak{g}$. In fact, we can apply this "translation diffeomorphism" trick to arbitrary (i.e. not necessarily MC) elements γ of degree 0 in $s\mathfrak{g}$ and get *homological* vector fields, $Q^\gamma := d\phi_\gamma(Q)$, which do not vanish at 0 and hence define generalized L_∞ structures on $\mathfrak{g}$ with "zero term" $Q_0^\gamma \neq 0$.

6.3. **Morphisms of dg props as a dg affine scheme.** Let $(\mathcal{P}, \partial_\mathcal{P})$ and $(\mathcal{Q}, \partial_\mathcal{Q})$ be dg props (or properads[2]) with differentials $\partial_\mathcal{P}$ and $\partial_\mathcal{Q}$ of degree -1. Let $\mathrm{Hom}_\mathbb{Z}(\mathcal{P}, \mathcal{Q})$ denote the graded vector space of all possible morphisms $\mathcal{P} \to \mathcal{Q}$ in category of $\mathbb{Z}$-graded $\mathbb{S}$-bimodules, and let $\mathrm{Mor}(\mathcal{P}, \mathcal{Q})$ denote the set of all possible morphisms $\mathcal{P} \to \mathcal{Q}$ in category of props , (note that we do *not* assume that elements of $\mathrm{Hom}_\mathbb{Z}(\mathcal{P}, \mathcal{Q})$ or $\mathrm{Mor}_(\mathcal{P}, \mathcal{Q})$ respect differentials). It is clear that

$$\mathrm{Mor}(\mathcal{P}, \mathcal{Q}) = \left\{\gamma \in \mathrm{Hom}_\mathbb{Z}(\mathcal{P}, \mathcal{Q}) \,|\, \gamma \circ \mu_\mathcal{P} \left(\mathcal{P} \boxtimes_{(1,1)} \mathcal{P}\right) = \mu_\mathcal{Q} \left(\gamma(\mathcal{P}) \boxtimes_{(1,1)} \gamma(\mathcal{P})\right) \text{ and } |\gamma| = 0\right\}.$$

We need a $\mathbb{Z}$-graded extension of this set,

$$\mathrm{Mor}_\mathbb{Z}(\mathcal{P}, \mathcal{Q}) = \left\{\gamma \in \mathrm{Hom}_\mathbb{Z}(\mathcal{P}, \mathcal{Q}) \,|\, \gamma \circ \mu_\mathcal{P} \left(\mathcal{P} \boxtimes_{(1,1)} \mathcal{P}\right) = \mu_\mathcal{Q} \left(\gamma(\mathcal{P}) \boxtimes_{(1,1)} \gamma(\mathcal{P})\right)\right\},$$

which we define by the same algebraic equations but dropping the assumption on the degree (and even on homogeneity) of γ.

Lemma 25. *The vector space* $\mathrm{Hom}_\mathbb{Z}(\mathcal{P}, \mathcal{Q})$ *is naturally a dg manifold.*

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[2]All what is written in this subsection holds true if reformulated in terms of dg coprop(erad)s so that an implicit assumption in the proof of Theorem 34 below that $(P(m,n)^*)^* = P(m,n)$ can be easily avoided.

Proof. We define a degree -1 coderivation of the free cocommutative coalgebra, $\odot^{\geq 1}\mathrm{Hom}_{\mathbb{Z}}(\mathcal{P},\mathcal{Q})$ by setting (in the dual picture, cf. § 6.1)

$$Q(\gamma) := \partial_{\mathcal{Q}} \circ \gamma - (-1)^{\gamma} \gamma \circ \partial_{\mathcal{P}} \tag{2}$$

for an arbitrary $\gamma \in \mathrm{Hom}_{\mathbb{Z}}(\mathcal{P},\mathcal{Q})$. As

$$\begin{aligned} Q^2(\gamma) &= Q(\partial_{\mathcal{Q}} \circ \gamma - (-1)^{\gamma}\gamma \circ \partial_{\mathcal{P}}) \\ &= -\partial_{\mathcal{Q}} \circ Q(\gamma) - (-1)^{\gamma} Q(\gamma) \circ \partial_{\mathcal{P}}) \\ &= -(-1)^{\gamma}\partial_{\mathcal{Q}} \circ \gamma \circ \partial_{\mathcal{P}} + (-1)^{\gamma}\partial_{\mathcal{Q}} \circ \gamma \circ \partial_{\mathcal{P}} \\ &= 0, \end{aligned}$$

Q is a linear homological field on $\mathrm{Hom}_{\mathbb{Z}}(\mathcal{P},\mathcal{Q})$. (By the way, the zero locus of Q is a linear subspace of $\mathrm{Hom}_{\mathbb{Z}}(\mathcal{P},\mathcal{Q})$ describing morphisms of *complexes*.) □

Proposition 26. *The set $Mor_{\mathbb{Z}}(\mathcal{P},\mathcal{Q})$ is naturally a dg affine scheme.*

Proof. Let I be the coideal in $\odot^{\geq 1}\mathrm{Hom}_{\mathbb{Z}}(\mathcal{P},\mathcal{Q})$ cogenerated by the algebraic relations,

$$\gamma \circ \mu_{\mathcal{P}} \left(\mathcal{P} \boxtimes_{(1,1)} \mathcal{P}\right) - \mu_{\mathcal{Q}} \left(\gamma(\mathcal{P}) \boxtimes_{(1,1)} \gamma(\mathcal{P})\right),$$

on the "variable" $\gamma \in \mathrm{Hom}_{\mathbb{Z}}(\mathcal{P},\mathcal{Q})$. The sub-coalgebra,

$$\mathcal{O}_{\mathrm{Mor}_{\mathbb{Z}}(\mathcal{P},\mathcal{Q})} := I \setminus \odot^{\geq 1}\mathrm{Hom}_{\mathbb{Z}}(\mathcal{P},\mathcal{Q}),$$

of $\odot^{\geq 1}\mathrm{Hom}_{\mathbb{Z}}(\mathcal{P},\mathcal{Q})$ makes the set $\mathrm{Mor}_{\mathbb{Z}}(\mathcal{P},\mathcal{Q})$ into a $\mathbb{Z}$-graded affine scheme. Next we show that the homological vector field Q defined in Lemma 25 is tangent to $\mathrm{Mor}_{\mathbb{Z}}(\mathcal{P},\mathcal{Q})$. Indeed, identifying Q and I with their duals (as in subsection 6.1 and the proof of Lemma 25), we have

$$\begin{aligned} Q\left(\gamma \circ \mu_{\mathcal{P}}\left(\mathcal{P} \boxtimes_{(1,1)} \mathcal{P}\right) - \mu_{\mathcal{Q}}\left(\gamma(\mathcal{P}) \boxtimes_{(1,1)} \gamma(\mathcal{P})\right)\right) &= Q(\gamma) \circ \mu_{\mathcal{P}}\left(\mathcal{P} \boxtimes_{(1,1)} \mathcal{P}\right) - \mu_{\mathcal{Q}}\left(Q(\gamma)(\mathcal{P}) \boxtimes_{(1,1)} \gamma(\mathcal{P})\right) \\ &\quad -(-1)^{|\gamma|}\mu_{\mathcal{Q}}\left(\gamma(\mathcal{P}) \boxtimes_{(1,1)} Q(\gamma)(\mathcal{P})\right). \end{aligned}$$

Consistency of $\partial_{\mathcal{P}}$ and $\partial_{\mathcal{Q}}$ with $\mu_{\mathcal{P}}$ and, respectively, $\mu_{\mathcal{Q}}$ implies,

$$\begin{aligned} Q(\gamma) \circ \mu_{\mathcal{P}}\left(\mathcal{P} \boxtimes_{(1,1)} \mathcal{P}\right) &= \partial_{\mathcal{Q}} \circ \gamma \circ \mu_{\mathcal{P}}\left(\mathcal{P} \boxtimes_{(1,1)} \mathcal{P}\right) - (-1)^{\gamma}\gamma \circ \partial_{\mathcal{P}} \circ \mu_{\mathcal{P}}\left(\mathcal{P} \boxtimes_{(1,1)} \mathcal{P}\right) \\ &= \partial_{\mathcal{Q}} \circ \gamma \circ \mu_{\mathcal{P}}\left(\mathcal{P} \boxtimes_{(1,1)} \mathcal{P}\right) - (-1)^{\gamma}\gamma \circ \mu_{\mathcal{P}}\left(\partial_{\mathcal{P}}(\mathcal{P}) \boxtimes_{(1,1)} \mathcal{P}\right) \\ &\quad -(-1)^{\gamma}\gamma \circ \mu_{\mathcal{P}}\left(\mathcal{P} \boxtimes_{(1,1)} \partial_{\mathcal{P}}(\mathcal{P})\right) \\ &=_{\bmod I} \partial_{\mathcal{Q}} \circ \mu_{\mathcal{Q}}\left(\gamma(\mathcal{P}) \boxtimes_{(1,1)} \gamma(\mathcal{P})\right) - (-1)^{\gamma}\mu_{\mathcal{Q}}\left(\gamma \circ \partial_{\mathcal{P}}(\mathcal{P}) \boxtimes_{(1,1)} \gamma(\mathcal{P})\right) \\ &\quad -\mu_{\mathcal{Q}}\left(\gamma(\mathcal{P}) \boxtimes_{(1,1)} \gamma \circ \partial_{\mathcal{P}}(\mathcal{P})\right) \\ &=_{\bmod I} \mu_{\mathcal{Q}}\left(Q(\gamma)(\mathcal{P}) \boxtimes_{(1,1)} \gamma(\mathcal{P})\right) + (-1)^{|\gamma|}\mu_{\mathcal{Q}}\left(\gamma(\mathcal{P}) \boxtimes_{(1,1)} Q(\gamma)(\mathcal{P})\right). \end{aligned}$$

Thus $Q(I) \subset I$, and hence Q gives rise to a degree -1 codifferential on the coalgebra $\mathcal{O}_{\mathrm{Mor}_{\mathbb{Z}}(\mathcal{P},\mathcal{Q})}$ proving the claim. □

Theorem 27. *Let $(\mathcal{P} = \mathcal{F}(s^{-1}\mathcal{C}), \partial_{\mathcal{P}})$ be a quasi-free prop(erad), that is $\mathcal{C}$ is a homotopy co-prop(erad), on an $\mathbb{S}$-bimodule $s^{-1}\mathcal{C}$ and $(\mathcal{Q}, \partial_{\mathcal{Q}})$ a dg prop (resp., properad). Then*

- (i) *The graded vector space, $s^{-1}\mathrm{Hom}_{\mathbb{Z}}(\mathcal{C},\mathcal{Q})$, is canonically a L_∞-algebra;*
- (ii) *The canonical L_∞-structure in (i) is filtered and its MC elements are morphisms, $(\mathcal{P}, \partial_{\mathcal{P}}) \to (\mathcal{Q}, \partial_{\mathcal{Q}})$, of* dg *props;*
- (iii) *if $\partial_{\mathcal{P}}(s^{-1}\mathcal{C}) \subset \mathcal{F}(s^{-1}\mathcal{C})^{(\leq 2)}$, where $\mathcal{F}(s^{-1}\mathcal{C})^{(\leq 2)}$ is the subspace of $\mathcal{F}(s^{-1}\mathcal{C})$ spanned by decorated graphs with at most two vertices, then $s^{-1}\mathrm{Hom}_{\mathbb{Z}}(\mathcal{C},\mathcal{Q})$ is canonically a dg Lie algebra.*

Proof. (i) If $\mathcal{P}$ is free as a prop, then $\mathcal{O}_{\mathrm{Mor}_{\mathbb{Z}}(\mathcal{P},\mathcal{Q})} = \odot^{\geq 1}s^{-1}\mathrm{Hom}_{\mathbb{Z}}(\mathcal{C},\mathcal{Q})$ and the claim follows from the definition of L_∞-structure in § 6.1. (Corollary 37 provides another proof of this point).

(ii) The canonical L_∞ structure on $s^{-1}\mathrm{Hom}_{\mathbb{Z}}(\mathcal{C},\mathcal{Q})$ is given by the restriction of the homological vector field (2) on $\mathrm{Hom}_{\mathbb{Z}}(\mathcal{P},\mathcal{Q})$ to the subspace $s^{-1}\mathrm{Hom}_{\mathbb{Z}}(\mathcal{C},\mathcal{Q})$. This field is a formal power series in coordinates on $s^{-1}\mathrm{Hom}_{\mathbb{Z}}(\mathcal{C},\mathcal{Q})$ and its part, Q_n, corresponding to monomials of (polynomial) degree n is given precisely by

$$Q_n(\gamma) := \partial_{\mathcal{Q}} \circ \gamma - (-1)^{\gamma}\gamma \circ \partial_{\mathcal{P}}^{(n)}, \tag{3}$$

where $\partial_{\mathcal{P}}^{(n)}$ is the composition,

$$\partial_{\mathcal{P}}^{(n)} : s^{-1}\mathcal{C} \xrightarrow{\partial_{\mathcal{P}}} \mathcal{F}(s^{-1}\mathcal{C}) \xrightarrow{proj} \mathcal{F}(s^{-1}\mathcal{C})^{(n)}.$$

Note that the first summand on the r.h.s. of (3) contributes only to Q_1.

Define an exhaustive increasing filtration on the $\mathbb{S}$-bimodule $\mathcal{C}$ by

$$\mathcal{C}_0 = 0, \quad \mathcal{C}_i := s\bigcap_{n\geq i} \mathrm{Ker}\,\partial_{\mathcal{P}}^{(n)} \text{ for } i \geq 1,$$

and the associated decreasing filtration on $s^{-1}\mathrm{Hom}_{\mathbb{Z}}(\mathcal{C},\mathcal{Q})$ by

$$s^{-1}\mathrm{Hom}_{\mathbb{Z}}(\mathcal{C},\mathcal{Q})_i := \{\gamma \in s^{-1}\mathrm{Hom}_{\mathbb{Z}}(\mathcal{C},\mathcal{Q}) \mid \gamma(v) = 0 \ \forall v \in \mathcal{C}_i\}, \ i \geq 0.$$

Then, for all $n \geq 2$ and any $f_1, \ldots, f_n \in s^{-1}\mathrm{Hom}_{\mathbb{Z}}(\mathcal{C},\mathcal{Q})$, equality (3) implies that the value of the map $Q_n(f_1, \ldots, f_n) \in s^{-1}\mathrm{Hom}_{\mathbb{Z}}(\mathcal{C},\mathcal{Q})$ on arbitrary element of $\mathcal{C}_n \subset \ker \partial_{\mathcal{P}}^{(n)}$ is equal to zero, i.e.

$$Q_n(f_1, \ldots, f_n) \in) \in s^{-1}\mathrm{Hom}_{\mathbb{Z}}(\mathcal{C},\mathcal{Q})_n.$$

Which in turn implies the claim that the canonical L_∞ structure on $s^{-1}\mathrm{Hom}_{\mathbb{Z}}(\mathcal{C},\mathcal{Q})$ is filtered with respect to the constructed filtration. The claim about MC elements follows immediately from the definition (2) of the homological vector field.

(ii) As $\partial_{\mathcal{P}}^{(n)} = 0$ for $n > 2$ we conclude using formula (3) that $Q_n = 0$ for all $n > 2$. □

A special case of the above Theorem when both $\mathcal{P}$ and $\mathcal{Q}$ was operads was proven earlier by van der Laan [vdL] using very different ideas. We shall show below another proof of Theorem 27 which uses Lie_∞ structures associated with convolution prop(erad)s.

6.4. **$\mathcal{P}$-gebra, $\mathcal{P}_{(n)}$-gebra and homotopy $\mathcal{P}$-gebra.** Let $\mathcal{P}$ be a dg prop and $\Omega_\infty(\mathcal{C})$ be its minimal model.

Definition (Homotopy $\mathcal{P}$-gebra)**.** A dg module X endowed with a morphism of dg prop $\Omega_\infty(\mathcal{C}) \to \mathrm{End}_X$ is called a *homotopy $\mathcal{P}$-gebra*.

Any $\mathcal{P}$-gebra is a homotopy $\mathcal{P}$-gebra, $\Omega_\infty(\mathcal{C}) \xrightarrow{\sim} \mathcal{P} \to \mathrm{End}_X$, of particular type. For the Koszul operads $\mathcal{A}s$, $\mathcal{C}om$, $\mathcal{L}ie$, this notion coincides with homotopy associative, commutative, Lie algebras. For the properads $\mathcal{B}i\mathcal{L}ie$ and $\mathcal{B}i\mathcal{A}s$, we get the notions of homotopy Lie bialgebras and homotopy bialgebras.

Theorem 27 shows that a structure of homotopy $\mathcal{P}$-gebra on X is equivalent to a morphism of $\mathbb{S}$-bimodules in $s^{-1}\mathrm{Hom}_0^{\mathbb{S}}(\mathcal{C},\mathrm{End}_X)$ which is a solution to the generalized Maurer-Cartan equation.

Definition ($\mathcal{P}_{(n)}$-gebra)**.** A dg module X endowed with a Maurer-Cartan element γ in $s^{-1}\mathrm{Hom}_0^{\mathbb{S}}(\mathcal{C},\mathrm{End}_X)$ such that $\gamma(c) = 0$ for every $c \in \mathcal{C}_{k>n}$ is called a *$\mathcal{P}_{(n)}$-gebra*.

This notion is the direct generalization of the notion of $A_{(n)}$-algebra of Stasheff in [Sta63]. A $\mathcal{P}_{(n)}$-gebra is a homotopy $\mathcal{P}$-gebra with strict relations from degree n.

7. Deformation theory of morphisms of prop(erad)s

7.1. **Conceptual definition.** Let $(\mathcal{P}, d_{\mathcal{P}}) \xrightarrow{\varphi} (\mathcal{Q}, d_{\mathcal{Q}})$ be a morphism of dg props. We would like to define a chain complex with which we could study the deformation theory of this map. Following Quillen [Qui70], the conceptual method is to take the total right derived functor of the functor Der of derivations from the category of props above $\mathcal{Q}$ (see also [Mar96, vdL]). That is, we consider a cofibrant replacement $(\mathcal{R}, \partial)$ of $\mathcal{P}$ is the category of dg props

$$\begin{array}{ccc} \mathcal{R} & \xrightarrow{\varepsilon} & \mathcal{P} \\ & \overset{\gamma}{\searrow} & \downarrow \varphi \\ & & \mathcal{Q}. \end{array}$$

Lemma 28. *Let $(\mathcal{R},\, \partial)$ be a resolution of $\mathcal{P}$ and let f be an homogenous derivation of degree n in $\mathrm{Der}^n_\rho(\mathcal{R},\, \mathcal{Q})$, the derivative $D(f) = d_\mathcal{Q} \circ f - (-1)^{|f|} f \circ \partial$ is a derivation of degree $n-1$ of $\mathrm{Der}^{n-1}_\rho(\mathcal{R},\, \mathcal{Q})$.*

Proof. The degree of $D(f)$ is $n-1$. It remains to show that it is a derivation. For every pair r_1 and r_2 of homogenous elements of $\mathcal{R}$, $D(f)\left(\mu^\mathcal{R}(r_1 \boxtimes_{(1,1)} r_2)\right)$ is equal to

$$\begin{aligned} D(f)\left(\mu^\mathcal{R}(r_1 \boxtimes_{(1,1)} r_2)\right) &= (d_\mathcal{Q} \circ f - (-1)^n f \circ \partial)\left(\mu^\mathcal{R}(r_1 \boxtimes_{(1,1)} r_2)\right) \\ &= d_\mathcal{Q}\left(\mu^\mathcal{Q}\big(f(r_1) \boxtimes_{(1,1)} \gamma(r_2) + (-1)^{n|r_1|}\gamma(r_1) \boxtimes_{(1,1)} f(r_2)\big)\right) \\ &\quad -(-1)^n f\left(\mu^\mathcal{R}\big(\partial(r_1) \boxtimes_{(1,1)} r_2 + (-1)^{|r_1|} r_1 \boxtimes_{(1,1)} \partial(r_2)\big)\right) \\ &= \mu^\mathcal{Q}\Big((d_\mathcal{Q} \circ f)(r_1) \boxtimes_{(1,1)} \gamma(r_2) + (-1)^{n+|r_1|} f(r_1) \boxtimes_{(1,1)} (d_\mathcal{Q} \circ \gamma)(r_2) \\ &\quad +(-1)^{n|r_1|}(d_\mathcal{Q} \circ \gamma)(r_1) \boxtimes_{(1,1)} f(r_2) + (-1)^{|r_1|(n-1)}\gamma(r_1) \boxtimes_{(1,1)} (d_\mathcal{Q} \circ f)(r_2)\Big) \\ &\quad -(-1)^n \mu^\mathcal{Q}\Big((f \circ \partial)(r_1) \boxtimes_{(1,1)} \gamma(r_2) + (-1)^{n(|r_1|-1)}(\gamma \circ \partial)(r_1)) \boxtimes_{(1,1)} f(r_2) \\ &\quad +(-1)^{|r_1|} f(r_1) \boxtimes_{(1,1)} (\gamma \circ \partial)(r_2)) + (-1)^{(n-1)|r_1|}\gamma(r_1) \boxtimes_{(1,1)} (f \circ \partial)(r_2)\Big). \end{aligned}$$

Since γ is morphism of dg props, it commutes with the differentials, that is $\gamma \circ \partial = d_\mathcal{Q} \circ \gamma$. This gives

$$\begin{aligned} D(f)\left(\mu^\mathcal{R}(r_1 \boxtimes_{(1,1)} r_2)\right) &= \mu^\mathcal{Q}\Big((d_\mathcal{Q} \circ f)(r_1) \boxtimes_{(1,1)} \gamma(r_2) - (-1)^n (f \circ \partial)(r_1) \boxtimes_{(1,1)} \gamma(r_2) \\ &\quad +(-1)^{|r_1|(n-1)}\big(\gamma(r_1) \boxtimes_{(1,1)} (d_\mathcal{Q} \circ f)(r_2) - (-1)^n \gamma(r_1) \boxtimes_{(1,1)} (f \circ \partial)(r_2)\big)\Big) \\ &= \mu^\mathcal{Q}\Big(D(f)(r_1) \boxtimes_{(1,1)} \gamma(r_2) + (-1)^{|r_1|(n-1)}\gamma(r_1) \boxtimes_{(1,1)} D(f)(r_2)\Big). \end{aligned}$$

□

In other words, the space of derivations $\mathrm{Der}(\mathcal{R}, \mathcal{Q})$ is a sub-dg-module of the space of morphisms $\mathrm{Hom}(\mathcal{R}, \mathcal{Q})$. We define the deformation complex of the morphism φ by $C_\bullet(\varphi) := \big(\mathrm{Der}_\bullet(\mathcal{R}, \mathcal{Q}), D\big)$. For the cofibrant replacement, we can always take the bar-cobar resolution by Theorem 16. This will produce a big bicomplex difficult to compute. Instead of that, we will consider the chain complex obtained form the minimal model of $\mathcal{P}$ when it exists. In this sequel, we will focus on the deformation theory for representations of $\mathcal{P}$ of the form End_X, that is $\mathcal{P}$-gebras.

7.2. **Deformation theory of representations of props.** Let $(\mathcal{P}, d_\mathcal{P})$ be a dg prop admitting a minimal model $(\mathcal{P}_\infty := \mathcal{F}(s^{-1}\mathcal{C}) = \Omega_\infty(\mathcal{C}), \partial)$ and (X, d_X) an arbitrary dg $\mathcal{P}$-gebra

$$\begin{array}{ccc} \Omega_\infty(\mathcal{C}) & \xrightarrow{\varepsilon} & \mathcal{P} \\ & \overset{\gamma}{\searrow} & \downarrow \varphi \\ & & \mathrm{End}(X). \end{array}$$

Definition (Deformation complex)**.** We define the *deformation complex of the $\mathcal{P}$-gebra structure of X* by $C_\bullet(\mathcal{P}, X) := \big(\mathrm{Der}_\bullet(\Omega_\infty(\mathcal{C}), \mathrm{End}(X)), D\big)$.

Theorem 29. *The chain complex $\big(\mathrm{Der}_\bullet(\Omega_\infty(\mathcal{C}), \mathcal{Q}), D\big)$ is isomorphic to $s^{-1}\mathrm{Hom}^\mathbb{S}_\bullet(\bar{\mathcal{C}}, \mathcal{Q})$ with $D = Q^\gamma$.*

Proof. Lemma 11 proves the identification between the two spaces. Since γ is a morphism of dg props from a quasi-free prop, it is a solution of the generalized Maurer-Cartan equation $Q(\gamma) = 0$ in the convolution Lie_∞-algebra $s^{-1}\mathrm{Hom}^{\mathbb{S}}_\bullet(\bar{\mathcal{C}}, \mathcal{Q})$ by Theorem 27. Let f be an element of $\mathrm{Hom}^{\mathbb{S}}_n(s^{-1}\bar{\mathcal{C}}, \mathcal{Q})$. Following Lemma 11, we denote by ∂_f the unique derivation of $\mathrm{Der}_n(\Omega_\infty(\mathcal{C}), \mathcal{Q})$ induced by f. We have to show that $D(\partial_f)_{s^{-1}\bar{\mathcal{C}}} = Q^\gamma(f)$. For an element $s^{-1}c \in s^{-1}\mathcal{C}$, we use the Sweedler type notation for $\partial(s^{-1}c) = \sum_{\mathcal{G}} \mathcal{G}(s^{-1}c_1, \ldots, s^{-1}c_n)$. By Lemma 11, we have

$$\partial_f\big(\mathcal{G}(s^{-1}c_1, \ldots, s^{-1}c_n)\big) =$$
$$\sum_{i=1}^{n} (-1)^{n(|c_1|+\cdots+|c_{i-1}|+i-1)} \mu^{\mathcal{Q}}\big(\mathcal{G}(\gamma(s^{-1}c_1), \ldots, \gamma(s^{-1}c_{i-1}), f(s^{-1}c_i), \gamma(s^{-1}c_{i+1}), \ldots, \gamma(s^{-1}c_n)\big).$$

Therefore, $D(\partial_f)_{s^{-1}\bar{\mathcal{C}}}$ is equal to

$$D(\partial_f)(s^{-1}c) = (d_{\mathcal{Q}} \circ \partial_f - (-1)^n \partial_f \circ \partial)(s^{-1}c) = d_{\mathcal{Q}}(f(s^{-1}c)) -$$
$$(-1)^n \sum_{\mathcal{G}} \sum_{i=1}^{n} (-1)^{n(|c_1|+\cdots+|c_{i-1}|+i-1)} \mu^{\mathcal{Q}}\big(\mathcal{G}(\gamma(s^{-1}c_1), \ldots, \gamma(s^{-1}c_{i-1}), f(s^{-1}c_i), \gamma(s^{-1}c_{i+1}), \ldots, \gamma(s^{-1}c_n)\big)$$
$$= Q^\gamma(f).$$

□

Remark.
(1) It is natural to consider the augmentation of this chain complex by $S^{-1}\mathrm{Hom}^{\mathbb{S}}(I, \mathcal{Q})$, that is $s^{-1}\mathrm{Hom}^{\mathbb{S}}_\bullet(\mathcal{C}, \mathcal{Q})$.
(2) In the same way, we define the deformation complex of a $\mathcal{P}_\infty$-gebra $\Omega_\infty(\mathcal{C}) \xrightarrow{\gamma} X$ by $(s^{-1}\mathrm{Hom}^{\mathbb{S}}_\bullet(\bar{\mathcal{C}}, \mathrm{End}_X), Q^\gamma)$.

By Theorem 27 the vector space $s^{-1}\mathrm{Hom}_\bullet(\mathcal{C}, \mathrm{End}_W)$ has a canonical filtered L_∞-structure, Q whose MC elements are morphisms of dg props,

$$\rho : (P_\infty, \partial) \longrightarrow (\mathrm{End}_X, d),$$

that is, representations of (P_∞, ∂) in (X, d). Let γ be any particular P- or P_∞-algebra structure on X, and let Q^γ be the associated twisting of the canonical L_∞-algebra by γ (see §6.2). The following is an immediate corollary to Theorem 27.

Definition-Proposition 30. *(i) The L_∞-algebra Q^γ is said to* control deformations of γ in the class of strongly homotopy P-structures. *The MC elements, Γ, of Q^γ are in one-to-one correspondence with those P_∞-structures, on X,*

$$\rho : (P_\infty, \partial) \longrightarrow (\mathrm{End}_X, d),$$

whose restrictions to the generating space, $s^{-1}\mathcal{C}$, of P_∞ are equal precisely to the sum $\gamma + \Gamma$.

(ii) The cohomology group,

$$H^\bullet_\gamma(X) := H^\bullet(s^{-1}\mathrm{Hom}_{\mathbb{Z}}(V, \mathrm{End}_X), Q^\gamma_1),$$

is independent of the choice of a (minimal) resolution of P and is called homology group of the P_∞-algebra (X, γ).

7.3. **Props versus properads.** All we said above about props and their morphisms and representations holds true for properads — one just replaces in the proofs the product $\boxtimes$ with its connected version $\boxtimes_c$. If (P_∞, δ) is a dg prop originating from a dg properad $(\mathsf{P}_\infty, \delta)$ (see [Val03]), then it is easy to check that L_∞-algebras controlling deformations of P_∞- and P_∞-structures are identical. In such a case we shall work by default with the dg properad $(\mathsf{P}_\infty, \delta)$ rather than with the associated prop (P_∞, δ).

7.4. **Koszul case.** In Theorem 24, we have seen that a properad $\mathcal{P}$ is Koszul if and only if it admits a quadratic model $\Omega(\mathcal{P}^{\text{i}}) \xrightarrow{\sim} \mathcal{P}$, which $\mathcal{P}^{\text{i}}$ the Koszul dual (strict) coproperad. In this case, by Theorem 29, the deformation complex of a $\mathcal{P}_\infty$-gebra $\text{Hom}(\mathcal{P}^{\text{i}}, \text{End}_X)$ is dg-Lie algebra where the boundary map is equal to $D(f) = d(f) + \frac{1}{2}[\gamma, f]$.

This Lie bracket is the *intrinsic Lie bracket* of Stasheff [Sta93], that is it is equal to the Lie bracket of Gerstenhaber [Ger63] on Hochschild cochain complex of associative algebras, the Lie bracket of Nijenhuis-Richardson [NR67] on Chevalley-Eilenberg cochain complex of Lie algebras and the Lie bracket of Stasheff on Harrison cochain complex of commutative algebras. Its proved by Balavoine in [Bal97] that the deformation complex of algebras over any Koszul operad admits a Lie structure. This statement was made more precise by Markl, Shnider and Stasheff in Section 3.9 Part II of [MSS02] where they proves that this Lie bracket comes from a Prelie product. We construct here much more operations on this deformation complex. For every Koszul operad $\mathcal{P}$, the deformation complex of any $\mathcal{P}$-algebra is a non-symmetric operad. As a biproduct, it gives the Prelie bracket. But it also gives the construction of higher braces or LR-operations (see Section 2.3). Notice that the Lie bracket is very useful when we want to interpret the homology groups of this chain complex in terms of (infinitesimal) deformation of the algebraic structure. But the higher braces operations are fundamental, for instance in the proof of Deligne's conjecture (see Section 1.19 Part I of [MSS02] for a good survey on the subject).

This result on the level of operads was proved using the space of coderivations of the cofree $\mathcal{P}^{\text{i}}$-coalgebra, which is shown to be a PreLie algebra. Such a method is impossible to generalize to prop(erad)s simply because there exists no notion of (co)free gebra. As explained here, one has to work with convolution prop(erad) to prove this result. For any Koszul properad $\mathcal{P}$ in the sense of [Val03] and any gebra X over it, the deformation complex of X is non-symmetric properad. From this rich structure, we derive a Lie-admissible and then a Lie bracket which can be used to study the deformations of X. We expect the higher LR-operations to be used in the future for a better understanding of deformation theory. Notice that this Lie bracket was found by hand in one example before this general theory. The properad of Lie bialgebra is Koszul. Therefore, on the deformation (bi)complex of Lie bialgebras, there is a Lie bracket. The construction of this Lie bracket was given by Kosmann-Schwarzbach in [KS91]. (See also Ciccoli-Guerra [CG03] for the interpretation of this bicomplex in terms of deformations.)

7.5. **Examples of deformation theories.**

7.5.1. *Associative algebras.* If P is the properad, Ass, of associative algebras, then its minimal resolution exists and is generated by the $\mathbb{S}$-bimodule $V = \{V(m,n)\}$ with

$$V(m,n) := \mathsf{Ass}^{\text{i}} = \begin{cases} s^{n-2}\mathbb{K}[\mathbb{S}_1] \otimes \mathbb{K}[\mathbb{S}_n] & \text{for } m = 1, n \geq 2 \\ 0 & \text{otherwise.} \end{cases}$$

Hence $s^{-1}\text{Hom}(V, \text{End}_X) = \oplus_{n \geq 2} s^{1-n}\text{Hom}(X^{\otimes n}, X)$ and it is not hard to check that the induced L_∞-algebra, Q on $s\text{Hom}(V, \text{End}_X)$ is precisely the Gerstenhaber Lie algebra, and that Q^γ is the Hochschild dg Lie algebra controlling deformations of a particular associative algebra structure, $\gamma : \mathsf{Ass} \to \text{End}_X$, on a vector space X. The Ass-cohomology of (X, γ) is precisely the Hochschild cohomology.

Analogously one recovers other classical examples — Harrison complex/cohomology and Chevalley-Eilenberg complex/cohomology — from the props of commutative algebras and, respectively, Lie algebras.

7.5.2. *Poisson structures.* A *Lie 1-bialgebra* is, by definition, a graded vector space V together with two linear maps,

$$\begin{array}{rccc} \delta : & V & \longrightarrow & \wedge^2 V \\ & a & \longrightarrow & \sum a_1 \wedge a_2 \end{array} \quad , \quad \begin{array}{rccc} [\,\bullet\,] : & \odot^2 V & \longrightarrow & V \\ & a \otimes b & \longrightarrow & (-1)^{|a|}[a \bullet b] \end{array}$$

of degrees 0 and -1 respectively which satisfy the identities,

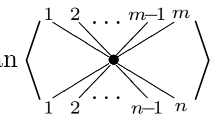
(i) $(\delta\otimes \mathrm{Id})\delta a + \tau(\delta\otimes \mathrm{Id})\delta a + \tau^2(\delta\otimes \mathrm{Id})\delta a = 0$, where τ is the cyclic permutation (123) represented naturally on $V\otimes V\otimes V$ (co-Jacobi identity);

(ii) $[[a\bullet b]\bullet c] = [a\bullet[b\bullet c]] + (-1)^{|b||a|+|b|+|a|}[b\bullet[a\bullet c]]$ (Jacobi identity);

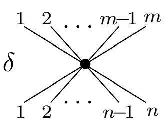
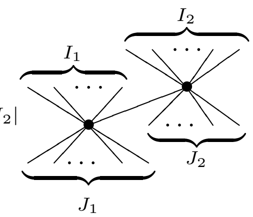
(iii) $\delta[a\bullet b] = \sum a_1\wedge[a_2\bullet b] - (-1)^{|a_1||a_2|}a_2\wedge[a_1\bullet b] + [a\bullet b_1]\wedge b_2 - (-1)^{|b_1||b_2|}[a\bullet b_2]\wedge b_1$ (Leibniz type identity).

This notion of Lie 1-bialgebras is similar to the well-known notion of Lie bialgebras except that in the latter case both operations, Lie and co-Lie brackets, have degree 0.

Let LieB be the properad whose representations are Lie 1-bialgebras. Its minimal resolution, $(\mathsf{LieB}_\infty, \delta)$, exists and is generated by the $\mathbb{S}$-bimodule $V = \{V(m,n)\}_{m,n\geq 1, m+n\geq 3}$ with

$$V(m,n) := s^{m-2} sgn_m \otimes \mathbf{1}_n = \text{span}\left\langle \right.$$  $$\left.\right\rangle,$$

where $sign_m$ stands for the sign representation of $\mathbb{S}_m$ and $\mathbf{1}_n$ for the trivial representation of $\mathbb{S}_n$. The differential is given on generators by [Mer06]

δ  $$= \sum_{\substack{I_1\sqcup I_2=(1,\ldots,m)\\ J_1\sqcup J_2=(1,\ldots,n)\\ |I_1|\geq 0, |I_2|\geq 1\\ |J_1|\geq 1, |J_2|\geq 0}} (-1)^{\sigma(I_1\sqcup I_2)+|I_1||I_2|}$$ 

where $\sigma(I_1\sqcup I_2)$ is the sign of the shuffle $I_1\sqcup I_2=(1,\ldots,m)$.

Hence, for an arbitrary dg vector space X,

$$s^{-1}\mathrm{Hom}(V, \mathrm{End}_X) = \bigoplus_{m,n\geq 1} s^{1-m}\wedge^m X\otimes \odot^n X \simeq \wedge^\bullet T_X,$$

where $\wedge^\bullet T_X$ is the vector space of formal germs of polivector fields at $0\in X$ when we view X as a formal graded manifold. It is not hard to show using the above explicit formular for the differential δ that the canonically induced, in accordance with Theorem 27(i), L_∞-structure on $s^{-1}\mathrm{Hom}(V,\mathrm{End}_X)$ is precisely the classical Schouten Lie algebra structure on polyvector fields. Thus our theory applied to Lie 1-bialgebras reproduces deformation theory of Poisson structures, and LieB-homology is precisely Poisson homology.

In a similar way one can check that our construction of L_∞-algebras applied to the minimal resolution of so called pre-Lie^2-algebras [Mer05] gives rise to another classical geometric object — the Frölicher-Nijenhuis Lie brackets on the sheaf, $T_X\otimes\Omega^\bullet_X$, of tangent vector bundle valued differential forms. Thus the associated deformation theory describes deformations of integrable Nijenhuis structures.

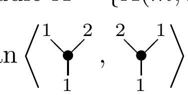
7.5.3. *Associative bialgebras.* As this example has never been rigorously treated in the literature before, we show full details here.

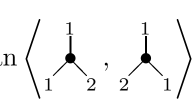
The properad of bialgebras can be defined as a quotient,

$$\mathsf{AssB} := \mathcal{F}(A)/(R)$$

of the free prop, $\mathcal{F}(A)$, generated by the $\mathbb{S}$-bimodule $A=\{A(m,n)\}$,

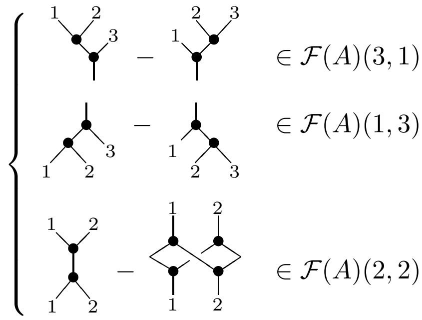
$$A(m,n) := \begin{cases} \mathbb{K}[\mathbb{S}_2]\otimes\mathbb{K}[\mathbb{S}_1] \equiv \text{span}\langle \ \cdot\ ,\ \cdot\ \rangle & \text{if } m=2, n=1,\\ \mathbb{K}[\mathbb{S}_1]\otimes\mathbb{K}[\mathbb{S}_2] \equiv \text{span}\langle \ \cdot\ ,\ \cdot\ \rangle & \text{if } m=1, n=2,\\ 0 & \text{otherwise}\end{cases}$$





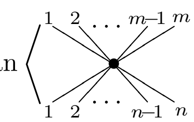
modulo the ideal generated by relations

$R:$

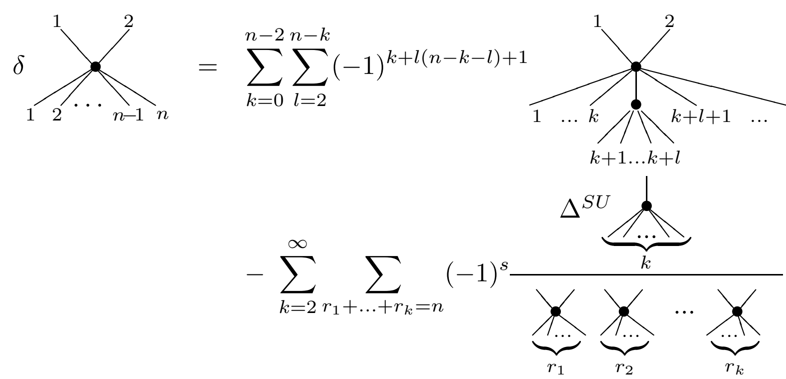


As the ideal contains 4-vertex graphs, the properad AssB is not quadratic. Hence AssB can not be Koszul in the ordinary sense but (as we shall explain elsewhere) it is *homotopy Koszul* so that its minimal resolution, $\mathsf{AssB}_\infty = \mathcal{F}(V),\delta)$ exists, and is freely generated by a relatively small $\mathbb{S}$-bimodule $V = \{V(m,n)\}_{m,n\geq 1, m+n\geq 3}$,

$$V(m,n) := s^{m+n-3}\mathbb{K}[\mathbb{S}_m]\otimes\mathbb{K}[\mathbb{S}_n] = \mathrm{span}\left\langle \quad \right\rangle.$$



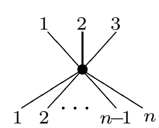
as was first observed in [Mar06]. The differential δ in AssB_∞ is not quadratic and an explicit formula for its value on generic (m,n)-corollas is not known at present. The values of δ on $(1,n)$- and $(m,1)$-corollas are given, of course, by the well-known A_∞-formulae, while



where

$$s = (k-1)(r_1-1) + (k-2)(r_2-1) + \ldots + 1(r_k-1),$$

Δ^{SU} is the Saneblidze-Umble diagonal, and the horizontal line means *fraction* composition from [Mar06]. The meaning of this part of the differential is clear: it describes A_∞-morphisms between an A_∞-structure on X and the associated Saneblidze-Umble diagonal A_∞-structure on $X\otimes X$. The values of δ on corollas of the form



describe a *homotopy* between two natural A_∞-morphisms from X to $X\otimes X\otimes X$, values on corollas with 4 output legs — homotopies between homotopies etc. We conjecture that $(\mathsf{AssB}_\infty,\delta)$ is a one coloured version of a certain $\mathbb{N}$-coloured properad describing A_∞-algebras, morphisms of A_∞-algebras, homotopies between morphisms of A_∞-algebras, homotopies of homotopies etc., and we hope to describe it in a future publication.

It was proven in [Mar06] that there exists a minimal model $(\mathsf{AssB}_\infty,\delta)$ such that the value of δ on a generic corolla is a *finite* sum of decorated connected graphs. Moreover, this δ is a perturbation, $\delta = \delta_0 + \delta_{pert}$, of the differential, δ_0, which describes the minimal resolution, $\frac{1}{2}\mathcal{A}ss\mathcal{B}_\infty$ of the prop of $\frac{1}{2}$-bialgebras. The perturbation part, δ_{pert}, is a linear combination of so called *fractions* and their compositions. We shall assume from now on that δ has all these properties. By checking genus of these fractions one can easily establish the following useful (for our purposes)

Fact 31. *The differential δ_0 is precisely the quadratic part of δ, i.e. it is equal to the composition,*

$$\delta_0 : V \xrightarrow{\delta} \mathcal{F}(V) \xrightarrow{proj} \mathcal{F}(V)^{(2)}.$$

By Theorem 27, the vector space,

$$s^{-1}\mathrm{Hom}(V, \mathrm{End}_X)[-1] = \bigoplus_{\substack{n,m\geq 1\\ m+n\geq 3}} s^{2-m-n}\mathrm{Hom}(X^{\otimes n}, X^{\otimes m}) =: \mathfrak{g}_{GS},$$

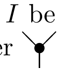
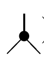
has a canonical L_∞-structure Q (with $Q_1 = 0$), whose MC elements are AssB_∞-structures on X.

If $\gamma : \mathsf{AssB} \to \mathrm{End}_X$ is a bialgebra structure on a vector space X, then, by Definition-Proposition 30, there exists an associated twisted L_∞ structures, $Q^\gamma = \{Q^\gamma_n\}_{n\geq 1}$, on $\mathfrak{g}_{GS}$ which controls deformations of γ in the class of AssB_∞-algebras. As explicit formulae for the differential δ are not yet available, we can not show this L_∞-structure in explicit formulae as well. However, we can show the following,

Theorem 32. *Let $(\mathsf{AssB}_\infty, \delta) \xrightarrow{\pi} \mathsf{AssB}$, be a minimal model of the properad of bialgebras and $\gamma : \mathsf{AssB} \to \mathrm{End}_X$ an arbitrary bialgebra structure on a vector space X. Then the differential,*

$$Q^\gamma_1 = Q \circ e^{\gamma\odot}$$

in the associated to this minimal model twisted L_∞-structure, Q^γ, on $\mathfrak{g}_{GS}$, is isomorphic to the Gerstenhaber-Schack differential. Hence the cohomology of the bialgebra (X, γ) is isomorphic to the Gerstenhaber-Schack cohomology.

Proof. Let $(\mathsf{AssB}_\infty = \mathcal{F}(V), \delta)$ be a minimal model of the properad of bialgebras, and let I be the ideal in $\mathcal{F}(V)$ generated by graphs in $\mathcal{F}(V)^{(\geq 2)}$ with at least two non-binary (i.e. neither 
nor ) vertices, and let

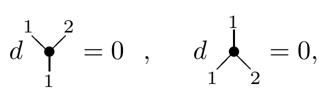
$$\mathsf{B} := \frac{\mathsf{AssB}_\infty}{(I, \delta I)}$$

be the associated quotient dg properad. The induced differential in B we denote by δ_{ind}. It is precisely this quotient part, δ_{ind}, of the total differential δ which completely determines the L_∞-differential differential Q^γ_1. Thus our plan is the following: in the next Lemma we present an explicit, up to an automorphism, form of the differential δ_{ind} (despite the fact that δ is unknown!) and thereafter compare the resulting Q^γ_1 with the Gerstenhaber-Schack definition.

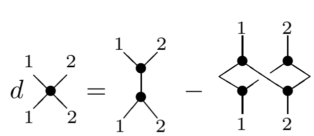
The major step in the proof is the following Lemma (in its formulation we use fraction notations again).

Lemma 33. *(i) The derivation, d, of B given on generators by,*

(4)

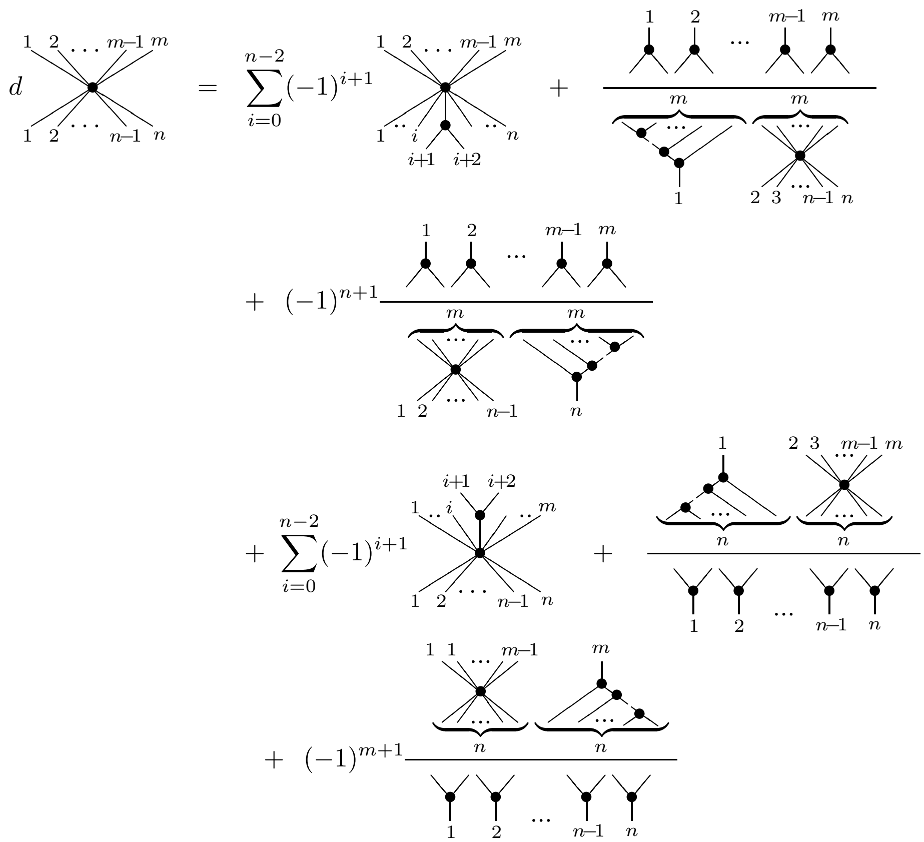


(5)



and, for all other generators with $m+n \geq 4$, by

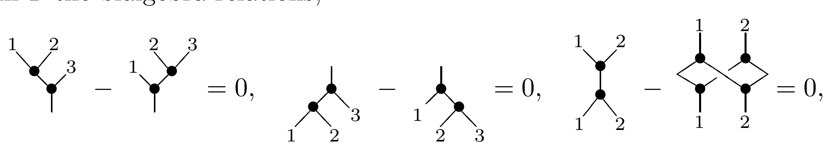
(6)



is a differential.

(ii) The dg properads $(\mathsf{B}, \delta_{ind})$ and (B, d) are isomorphic.

Proof. (i) It is easy to see that among for 2-vertex connected binary graphs[3] attached to any other graph in B the bialgebra relations,



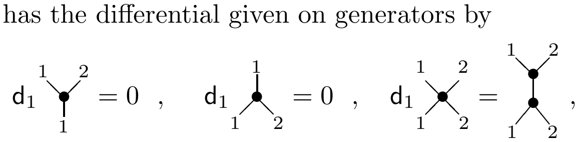
hold. Using this fact it is an easy and straightforward calculation to check that $d^2 = 0$. We omit the details. (In fact we shall show below that d is essentially a graph encoding of the Gerstenhaber-Schack differential d_{GS} so this calculation is essentially identical to the one which establishes $d_{GS}^2 = 0$.)

(ii) We begin our proof of Lemma 33(ii) with the following

Claim 1. *The natural projection $p : (\mathsf{B}, d) \to \mathsf{AssB}$ is a quasi-isomorphism.*

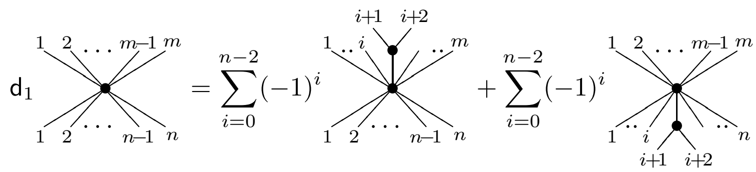
Indeed, the dg properad (B, d) has a natural increasing and bounded above filtration, $\{\mathsf{F}_{-p}\mathsf{B}\}p \geq 0$, with $\mathsf{F}_{-p}\mathsf{B}$ being the span of equivalence classes of graphs which admit a representative in $\mathcal{F}(V)^{(\geq p)}$. Let $(\mathsf{E}_r, \mathsf{d}_r)$ be the associated spectral sequence. The first term $(\mathsf{E}_0, \mathsf{d}_0)$ has trivial differential. The next one, $(\mathsf{E}_1, \mathsf{d}_1)$, has the differential given on generators by

(7)

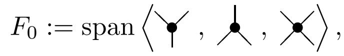


[3]Equivalence classes of graphs in B we call simply graphs for shortness.

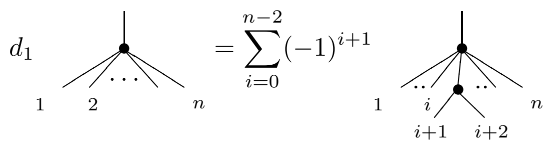
and, for all other generators with $m+n\geq 4$,

(8) 

We want to compute homology, $\mathsf{E}_2 = H^\bullet(\mathsf{E}_1, \mathsf{d}_1)$, of this complex and show that $\mathsf{E}_2 \simeq \mathsf{AssB}$. For this purpose consider a 2-step filtration, $0 \subset F_0 \subset F_1 = \mathsf{E}_1$, of the complex $(\mathsf{E}_1, \mathsf{d}_1)$ with



and let $(\mathcal{E}_r, \partial_r)$ be the associated spectral sequence. The differential ∂_0 is zero on the generators of F_0 and is equal to d_1 on all the other generators. Thus, modulo shifts of gradings, actions of finite groups and tensor products by trivial (i.e. with zero differential) complexes, the complex $(\mathcal{E}_0, \partial_0)$ is isomorphic to the tensor product of two isomorphic operadic complexes (one with "time" flow reversed upside down relative to another) which were studied on page 40 of [MMS] and which have the differential (in notations of that paper) given by



It is shown in [MMS] that the cohomology of this complex concentrated in degree 0 and is isomorphic to the operad of associative algebras. In our context this result immediately implies that $(\mathcal{E}_1, \partial_1)$ is isomorphic to F_0 with the differential ∂_1 given on generators by (7). Its cohomology is obviously concentrated in degree 0 (and is equal, in fact, to the properad, $\frac{1}{2}\mathsf{B}$, of infinitesimal bialgebras). Hence both the spectral sequences $(\mathcal{E}_r, d_r)$ and $(\mathsf{E}, \mathsf{d}_r)$ degenerate at second terms. As they both are convergent (by regularity and boundness of filtrations), we conclude that $H^\bullet(\mathsf{B}, d)$ is also concentrated in degree 0. This in turn implies Claim 1.

Claim 2. *The natural projection* $\pi : (\mathsf{B}, \delta_{ind}) \to \mathsf{AssB}$ *is a quasi-isomorphism.*

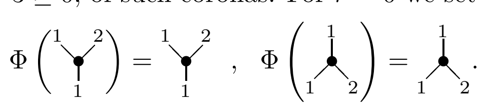
Indeed, the defined above filtration, $\{\mathsf{F}_{-p}\mathsf{B}\}_{p\geq 0}$, by the number of vertices is also compatible with the differential δ_{ind}. Let (E_r, d_r) be the associated spectral sequence. Its first nontrivial term, (E_1, d_1) is, by Fact 31, isomorphic to the complex $(\mathsf{E}_1, \mathsf{d}_1)$ above. Hence we can apply the same reasoning as in the proof of Claim 1.

Claim 3. *The exits a morphism of dg properads* Φ *making the diagram*

$$\begin{array}{ccc} & & (\mathsf{B}, d) \\ & \overset{\Phi}{\nearrow} & \big\downarrow p \\ (\mathsf{AssB}_\infty, \delta) & \xrightarrow[\pi]{} & (\mathsf{AssB}, 0) \end{array}$$

commutative.

As $\mathsf{AssB}_\infty = \mathcal{F}(V)$ is a free properad, a morphism Φ is completely determined by its values on the generating (m,n)-corollas which span the vector space V. We shall construct Φ by induction[4] on the degree, $r := m+n-3 \geq 0$, of such corollas. For $r=0$ we set Φ to be identity, i.e.



Assume we constructed values of Φ on all corollas of degree $r \leq N$. Let e be a generating corolla of non-zero weight $r = N+1$. Note that δe is a linear combination of graphs whose vertices are

[4]this induction is a straightforward analogue of the Whitehead lifting trick in the theory of CW-complexes in algebraic topology.

decorated by corollas of weight $\leq N$ (as differential δ has degee -1). Then, by induction, $\Phi(\delta e)$ is a well-defined element in B. As $\pi(e) = 0$, the element,

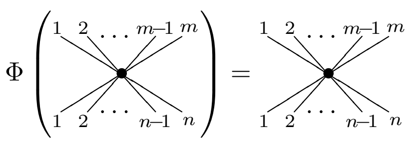
$$\Phi(\delta e)$$

is a closed element in B which projects under p to zero. By Claim 1, the surjection p is a quasi-isomorphism. Hence this element is exact and there exists $\mathfrak{e} \in \mathsf{B}$ such that

$$d\mathfrak{e} = \Phi(\delta e).$$

We set $\Phi(e) := \mathfrak{e}$ completing thereby inductive construction of Φ.

Claim 4. *A morphism Φ can be chosen so that*



+ terms with ≥ 2 number of vertices.

Indeed, the differential d in B has the form,

$$d = \mathsf{d}_1 + d_{rest},$$

where d_1 is the quadratic *differential* in B defined by (8) and the part d_{part} corresponds to graphs lying in $F_{-3}\mathsf{B}$. We shall prove Claim 4 by induction on the degree $r = m+n-3$ of the generating (m,n)-corollas in AssB_∞ (cf. proof of Theorem 43 in [Mar06]). For $r = 0$ the Claim is true. Assume we have already constructed Φ such that the claim is true for values of Φ on corollas with non-zero degree $\leq N$ and consider a generating corolla, e, of degree $N+1$. The value, $\mathfrak{e} := \Phi(e)$, is a solution of the equation,

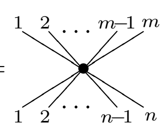
$$\mathsf{d}_1\mathfrak{e} + d_{rest}\mathfrak{e} = \Phi(\delta_0 e) + \Phi(\delta_{pert} e). \tag{9}$$

Let π_1 and π_2 denote projections in B to the subspaces spanned by equivalence classes of graphs with 1 and, respectively, 2 vertices. Then equation (9) implies,

$$\pi_2 \circ \mathsf{d}_1(\mathfrak{e}) = \mathsf{d}_1 \circ \pi_1(\mathfrak{e}) = \pi_2 \circ \Phi(\delta_0 e),$$

as both $d_{rest}\mathfrak{e}$ and $\Phi(\delta_{pert}e)$ are spanned by graphs lying in $F_{-3}\mathsf{B}$. Using now the explicit form for the differential δ_0 (given, e.g., by formula (14) in [Mar06]) and the induction assumption we immediately conclude that

$\pi_1(\mathfrak{e}) =$



completing the proof of Claim 4.

Claim 5. *The morphism Φ induces a dg isomorphism* $(\mathsf{B}, \delta_{ind}) \to (\mathsf{B}, d)$.

Indeed, Φ sends the ideal I to zero. Since Φ respects differentials, it sends the ideal $(I, \delta I)$ to zero as well and hence induces, by Claims 3 and 4, a required isomorphism. This completes proof of Lemma 33. $\square$

Now we continue with the proof of Theorem 32. The differential Q_1^γ in the graded vector space $\mathfrak{g}_{GS} = \oplus_{m,n} s^{2-m-n}\mathrm{Hom}(X^{\otimes n}, X^{\otimes m})$ is completely determined by the quotient differential, δ_{ind}, of the full differential δ in AssB_∞. By Lemma 33, this quotient differential is given, up to automorphims, by formulae (4)-(6). Let us compare these with the Gerstenhaber-Schack differential, d_{GS}, in the bicomplex $\mathfrak{g}_{GS}$ which is defined by [GS90]

$$d_{GS} = \partial_1 + \partial_2,$$

with $\partial_1 : \mathrm{Hom}(X^{\otimes n}, X^{\otimes m}) \to \mathrm{Hom}(X^{\otimes n+1}, X^{\otimes m})$ given on an arbitrary $f \in \mathrm{Hom}(X^{\otimes n}, X^{\otimes m})$ by

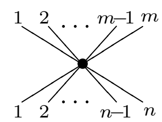
$$\begin{aligned}(\partial_1 f)(a_0, a_1, \ldots, a_n) \quad := \quad & \Delta^m(a_0) \Box f(a_1, a_2, \ldots, a_n) - \sum_{i=0}^{n-1} (-1)^i f(a_1, \ldots, a_i a_{i+1}, \ldots, a_n) \\ & + (-1)^{n+1} f(a_1, a_2, \ldots, a_n) \Box \Delta^m(a_n) \quad \forall \ \ a_i \in X.\end{aligned}$$

Here the multiplication in X is denoted by juxtaposition, the induced multiplication in the algebra $X^{\otimes m}$ by $\Box$, the comultiplication in X by Δ, and

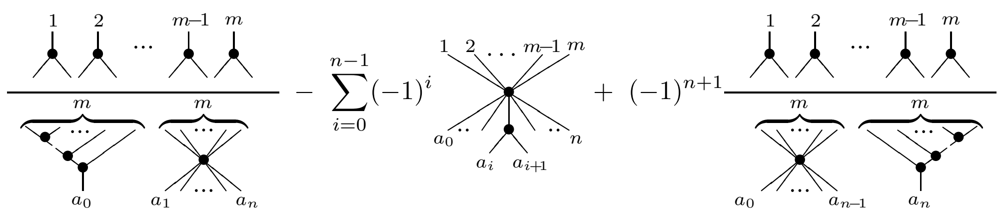
$$\Delta^n : (\Delta \otimes \mathrm{Id}^{\otimes m-2}) \circ (\Delta \otimes \mathrm{Id}^{\otimes m-3}) \circ \ldots \circ \Delta : X \to X^{\otimes m}.$$

The expression for ∂_2 is an obvious "dual" analogue of ∂_1. Now let us represent ∂_1 in graphical terms by associating the graphs

to comultiplication and, respectively, multiplication while the corolla



to f. Then the r.h.s of the formula for ∂_1 reads,



which is precisely the first three summands in (6). The other three terms correspond to ∂_2. The proof of the Theorem 32 is completed. □

8. Homotopy properads and associated homotopy Lie algebras

It was proven in [KM01] that for any operad, $O = \{O(n)\}$, the vector space $\bigoplus_n O(n)$ has a natural structure of Lie algebra which descends to the space of invariants, $\bigoplus_n O(n)_{\mathbb{S}_n}$. In [vdL] this result was generalized to homotopy operads and the associated L_∞-algebras.

In this section we further extend the results of [KM01, vdL] from homotopy operads to homotopy properads. Our approach is, however, completely independent of these two earlier works and, perhaps, provides a conceptual explanation of the phenomenon.

Theorem 34. *Let $\mathcal{P} = \{P(m,n)\}$ be a homotopy properad. Then*

(i) $\bigoplus_{m,n} P(m,n)$ *is canonically a L_∞-algebra;*
(ii) $\bigoplus_{m,n} P(m,n)^{\mathbb{S}_m}$ *is canonically a L_∞-algebra;*
(iii) $\bigoplus_{m,n} P(m,n)^{\mathbb{S}_n}$ *is canonically a L_∞-algebra;*
(iv) $\bigoplus_{m,n} P(m,n)^{\mathbb{S}_m \times \mathbb{S}_n}$ *is canonically a L_∞-algebra;*
(v) *there is a natural commutative diagram of L_∞-morphisms,*

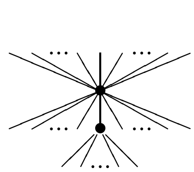
$$
\begin{array}{ccccc}
 & & \bigoplus_{m,n} P(m,n)^{\mathbb{S}_m} & & \\
 & \nearrow & & \searrow & \\
\bigoplus_{m,n} P(m,n) & & & & \bigoplus_{m,n} P(m,n)^{\mathbb{S}_m \times \mathbb{S}_n} \\
 & \searrow & & \nearrow & \\
 & & \bigoplus_{m,n} P(m,n)^{\mathbb{S}_n} & &
\end{array}
$$

Finally, if $\mathcal{P}$ is a dg properad, then all the above data are dg Lie algebras and morphisms of dg Lie algebras.

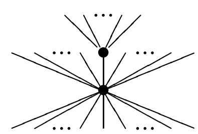
Proof. For a properad $\mathcal{C} = \{\mathcal{C}(m,n)\}$ we denote by $\mathcal{C}^\dagger = \{\mathcal{C}^\dagger(m,n)\}$ the associated "flow reversed" properad with $\mathcal{C}^\dagger(m,n) := \mathcal{C}(n,m)$. Let $\mathcal{A}ss$ be the properad of associative algebras and define the properad, $\mathcal{A}ss^\dagger \bullet \mathcal{A}ss = \{\mathcal{A}ss^\dagger \bullet \mathcal{A}ss(m,n)\}$ by setting

$$\mathcal{A}ss^\dagger \bullet \mathcal{A}ss(m,n) := \mathcal{A}ss^\dagger(m) \otimes \mathcal{A}ss(n) \simeq \mathbb{K}[\mathbb{S}_m] \otimes \mathbb{K}[\mathbb{S}_n]$$

and defining the compositions $\mu_{1,1}$ to be non-zero only on decorated graphs of the form



and



on which it is equal to the operadic compositions in $\mathcal{A}ss$.

Let $\mathcal{C}om$ be the properad of commutative algebras, and define the properads $\mathcal{C}om^\dagger \bullet \mathcal{A}ss$, $\mathcal{A}ss^\dagger \bullet \mathcal{C}om$, and $\mathcal{C}om^\dagger \bullet \mathcal{C}om$ by analogy to $\mathcal{A}ss^\dagger \bullet \mathcal{A}ss$.

Homotopy properad structure on $\mathcal{P}$ is the same as a degree -1 codifferential on the associated bar construction $B(P) = \mathcal{F}^c(sP)$. The dual object, $B(P)^*$, is a free properad, $\mathcal{F}(s^{-1}P^*)$, equipped with the induced differential of degree -1.

Theorem 27(i) applied to $\mathcal{C} = B(P)^*$ and $\mathcal{D} = \mathcal{A}ss^\dagger \bullet \mathcal{A}ss$ asserts that the vector space

$$s^{-1}\mathrm{Hom}_{\mathbb{Z}}\left(s^{-1}P^*, \mathcal{A}ss^\dagger \bullet \mathcal{A}ss\right) = \bigoplus_{m,n} P(m,n)$$

is canonically a L_∞-algebra. Hence the claim (i).

Analogously, claims (ii)-(iv) follow from Theorem 27(i) applied to $\mathcal{C} = B(P)^*$ and $\mathcal{D}$ being $\mathcal{C}om^\dagger \bullet \mathcal{A}ss$, $\mathcal{A}ss^\dagger \bullet \mathcal{C}om$ and, respectively, $\mathcal{C}om^\dagger \bullet \mathcal{C}om$ as

$$s^{-1}\mathrm{Hom}_{\mathbb{Z}}\left(s^{-1}P^*, \mathcal{C}om^\dagger \bullet \mathcal{A}ss\right) = \bigoplus_{m,n} P(m,n)^{\mathbb{S}_m},$$

$$s^{-1}\mathrm{Hom}_{\mathbb{Z}}\left(s^{-1}P^*, \mathcal{A}ss^\dagger \bullet \mathcal{C}om\right) = \bigoplus_{m,n} P(m,n)^{\mathbb{S}_n},$$

and

$$s^{-1}\mathrm{Hom}_{\mathbb{Z}}\left(s^{-1}P^*, \mathcal{C}om^\dagger \bullet \mathcal{C}om\right) = \bigoplus_{m,n} P(m,n)^{\mathbb{S}_m \times \mathbb{S}_n}.$$

The natural morphism of operads,

$$\mathcal{A}ss \longrightarrow \mathcal{C}om,$$

induces a commutative diagram of morphisms of properads,

$$\begin{array}{ccccc} & & \mathcal{C}om^{\dagger} \bullet \mathcal{A}ss & & \\ & \nearrow & & \searrow & \\ \mathcal{A}ss^{\dagger} \bullet \mathcal{A}ss & & & & \mathcal{C}om^{\dagger} \bullet \mathcal{C}om) \\ & \searrow & & \nearrow & \\ & & \mathcal{A}ss^{\dagger} \bullet \mathcal{C}om & & \end{array}$$

which in turn implies the commutative diagram of L_∞ morphisms in claim (v).

Finally, the last claim of the Theorem follows from Theorem 27(iii). □

Proposition 35. *The construction given in the proof of Theorem 34 provides us with four functors*

$$\textsf{Category of dg properads} \longrightarrow \textsf{Category of dg Lie algebras.}$$

and the four endomorphisms of functors.

Analogous statement holds for the category of homotopy properads and the category of L_∞*-algebras.*

Proof is an easy exercise and hence omitted.

9. Homotopy convolution prop(erad)

Theorem 36. *When* $(\mathcal{C}, \delta)$ *is a homotopy coprop(erad) and* $(\mathcal{P}, \mu)$ *is a prop(erad), the convolution prop(erad)* $\mathcal{P}^{\mathcal{C}} = \mathrm{Hom}(\mathcal{C}, \mathcal{P})$ *is a homotopy prop(erad) and* $\mathrm{Hom}^{\mathbb{S}}(\mathcal{C}, \mathcal{P})$ *is a homotopy non-symmetric prop(erad).*

Proof. To an element $\mathcal{G}(f_1, \ldots, f_n)$ of $\mathcal{F}^c(\overline{\mathcal{P}}^{\overline{\mathcal{C}}})^{(n)}$, we consider the map $\widetilde{\mathcal{G}}(f_1, \ldots, f_n) : \mathcal{F}(\overline{\mathcal{C}})^{(n)} \to \mathcal{F}^c(\overline{\mathcal{P}})^{(n)}$ defined by $\mathcal{G}'(c_1, \ldots, c_n) \mapsto (-1)^{\xi} \mathcal{G}(f_1(c_1), \ldots, f_n(c_n))$ if $\mathcal{G}' \cong \mathcal{G}$ and 0 otherwise, where $\xi = \sum_{i=2}^{n} |f_i|(|c_1| + \cdots + |c_{i-1}|)$. We define maps $\mu_n : \mathcal{F}^c(\overline{\mathcal{P}}^{\overline{\mathcal{C}}})^{(n)} \to \mathcal{P}^{\mathcal{C}}$ by the formula

$$\mu_n\big(\mathcal{G}(f_1, \ldots, f_n)\big) := \widetilde{\mu}_{\mathcal{P}} \circ \widetilde{\mathcal{G}}(f_1, \ldots, f_n) \circ \delta_n.$$

The degree of δ_n is $n-2$ and the degree of $\widetilde{\mu}_{\mathcal{P}}$ is zero. Therefore, the degree of μ_n is $n-2$.
The map μ verifies the relation of Proposition 19

$$\begin{aligned} \sum_{\mathcal{G}' \subset \mathcal{G}} \pm \mu\big(\mathcal{G}/\mu\mathcal{G}'(f_1, \ldots, f_n)\big) &= \sum \pm \widetilde{\mu}_{\mathcal{P}} \circ \widetilde{\mathcal{G}/\mathcal{G}'}(f_1, \ldots, \mu_k(\mathcal{G}'(f_{i_1}, \ldots, f_{i_k})), \ldots, f_n) \circ \delta_l \\ &= \sum \pm \widetilde{\mu}_{\mathcal{P}} \circ \widetilde{\mathcal{G}/\mathcal{G}'}(f_1, \ldots, \widetilde{\mu}_{\mathcal{P}} \circ \widetilde{\mathcal{G}}'(f_{i_1}, \ldots, f_{i_k}) \circ \delta_k, \ldots, f_n) \circ \delta_l, \end{aligned}$$

where the sum runs over admissible subgraphs $\mathcal{G}'$ of $\mathcal{G}$. We denote by k the number of vertices of $\mathcal{G}'$ and $l = n-k+1$We use the generic notation i for the new vertex of $\mathcal{G}/\mathcal{G}'$ obtained after contracting $\mathcal{G}'$. For every element $c \in \overline{\mathcal{C}}$, we denote by $\delta(c) = \sum \mathcal{G}^1(c_1, \ldots, c_l)$ and $\delta(c_i) = \sum \mathcal{G}_i^2(c'_1, \ldots, c'_k)$. The associativity of the product of $\mathcal{P}$ gives

$$\begin{aligned} &\sum_{\mathcal{G}' \subset \mathcal{G}} (-1)^{\varepsilon(\mathcal{G}', f_1, \ldots, f_n)} \mu\big(\mathcal{G}/\mu\mathcal{G}'(f_1, \ldots, f_n)\big)(c) = \\ &\widetilde{\mu}_{\mathcal{P}} \circ \widetilde{\mathcal{G}}(f_1, \ldots, f_n) \circ \Big(\sum (-1)^{\rho(\mathcal{G}_i^2, c_1, \ldots, c_l)} \mathcal{G}^1 \circ_i \mathcal{G}_i^2(c_1, \ldots, c'_1, \ldots, c'_k, \ldots, c_l)\Big). \end{aligned}$$

Since $(\mathcal{C}, \delta)$ is a homotopy coprop(erad), the last term vanishes by Proposition 20.
The same statement in the non-symmetric case is proved in the same way. □

Remark. In the particular case when $\mathcal{C}$ is a homotopy coalgebra and $\mathcal{P}$ an associative algebra, $\mathrm{Hom}(\mathcal{C}, \mathcal{P})$ is a homotopy algebra. In the same way, when $\mathcal{C}$ is a homotopy operad and $\mathcal{P}$ an operad, $\mathrm{Hom}(\mathcal{C}, \mathcal{P})$ is an homotopy operad (see Lemma 5.10 of [vdL]).

Denote by $\mathcal{L}ie \xrightarrow{\text{Anti-Sym}} \mathcal{A}s$ the morphism of operads defined by the anti-symmetrization of an associative product to get a Lie bracket. We have the following commutative diagram of dg operads.

$$\begin{array}{ccc} \Omega(\mathcal{A}s^{\mathrm{i}}) & \xrightarrow{\sim} & \mathcal{A}s \\ {\scriptstyle \Omega(\text{Anti-Sym}^{\mathrm{i}})}\uparrow & & \uparrow{\scriptstyle \text{Anti-Sym}} \\ \Omega(\mathcal{L}ie^{\mathrm{i}}) & \xrightarrow{\sim} & \mathcal{L}ie. \end{array}$$

Therefore an $\Omega(\mathcal{A}s^{\mathrm{i}})$-algebra is an $\Omega(\mathcal{L}ie^{\mathrm{i}})$-algebra, that is a homotopy (associative) algebra is a homotopy Lie algebra by anti-symmetrization. This result was extended to homotopy properads in Theorem 34.

Corollary 37. *When $(\mathcal{C}, \delta)$ is a homotopy coprop(erad) and $(\mathcal{P}, \mu)$ is a prop(erad), the convolution prop(erad) $\mathcal{P}^{\mathcal{C}} = \mathrm{Hom}(\mathcal{C}, \mathcal{P})$ and the non-symmetric convolution prop(erad) $\mathrm{Hom}_{\mathbb{S}}(\mathcal{C}, \mathcal{P})$ are homotopy Lie algebras.*

Proof. The proof is a direct corollary of Theorem 34 and Theorem 36. □

In the latter case, the $\mathcal{L}ie_\infty$ 'operations' are explicitly given by the following formula. The image of $f_1, \ldots, f_n \in \mathrm{Hom}_{\mathbb{S}}(\mathcal{C}, \mathcal{P})$ under l_n is given by

$$l_n(f_1, \ldots, f_n) = \sum_{\sigma \in \mathbb{S}_n} (-1)^{\mathrm{sgn}(\sigma, f_1, \ldots, f_n)} \widetilde{\mu}_{\mathcal{P}} \circ (f_{\sigma(1)} \otimes \cdots \otimes f_{\sigma(n)}) \circ \delta_n,$$

where $(-1)^{\mathrm{sgn}(\sigma, f_1, \ldots, f_n)}$ is the Koszul-Quillen sign appearing after permutating the f_i with σ.

In this homotopy Lie algebra, the generalized Maurer-Cartan equation is well defined since the formal infinite sum $Q(\alpha) = \sum_{n \geq 1} \frac{1}{n!} l_n(\alpha, \ldots, \alpha)$ is in fact equal to the composite $\widetilde{\mu}^{\mathcal{P}} \circ \mathcal{F}(\alpha) \circ \delta$.

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