ON YOUNG HULLS OF CONVEX CURVE IN R^{2n}

V. Sedykh[1] and B. Shapiro[2]

1
Department of Mathematics
Moscow State University of
Technology "Stankin"
Moscow 101472
Russia

Max-Planck-Institut
für Mathematik
Gottfried-Claren-Str. 26
53225 Bonn

Germany

2
Department of Mathematics
University of Stockholm
S-10691 Stockholm
Sweden

MPI 96-95

ON YOUNG HULLS OF CONVEX CURVE IN $\mathbb{R}^{2n}$

V. Sedykh and B. Shapiro

Department of Mathematics, Moscow State University of Technology "Stankin", Moscow 101472, Russia, sedykh@ium.ips.ras.ru,
Department of Mathematics, University of Stockholm, S-10691, Sweden, shapiro@matematik.su.se

Abstract. For a convex curve in an even-dimensional affine space we introduce a series of convex domains (called Young hulls) describe their structure and give a formulas fo the volume of the biggest of these domains.

Introduction

A simple smooth curve in $\mathbb{R}^m$ is called *convex* if the total multiplicity of its intersection with any affine hyperplane does not exceed m. Note that each convex curve is *nondegenerate*, i.e. has a nondegenerate osculating m-frame at each point.

Nondegenerate curves in projective spaces have naturally arisen in many classical geometrical problems as well as in problems related to linear ordinary differential equations, see e.g. [KN,Shr,Li]. The global topological properties of the spaces of such curves were used in the enumeration of symplectic leaves of the Gelfand-Dickey bracket, [KSh].

The set of all nondegenerate curves (say, with fixed osculating flags at both endpoints) contains the subset of convex curves which minimize the maximal possible number of intersection points with hyperplanes. Convex curves correspond to a special class of linear ODE called disconjugate. Their properties were studied, for example, in the classical papers [Ga,Kr,Po] in connection with Sturmian theory and functional analysis. Recently V. Arnold proposed a generalization of Sturmian theory which in the simplest treated case deals with the estimation of the number of flattening points of a curve in $\mathbb{R}^{2n+1}$ which projects on a curve close to the generalized ellipse (1), see [Ar]. (For the reader interested in the general theory of disconjugate linear ODE and convex curves we recommend the survey books [KN] and [Co].) In the present paper we associate to each closed convex curve in $\mathbb{R}^{2n}$ a number of convex domains and study some of their properties.

Note that a convex curve in an odd-dimensional space can not be closed. A simple example of a closed convex curve in $\mathbb{R}^{2n}$ is the *normalized generalized ellipse*

$$(\sin t, \cos t, 1/2 \sin 2t, 1/2 \cos 2t, \dots, 1/n \sin nt, 1/n \cos nt). \tag{1}$$

Definition. A tangent hyperplane to a curve is called a *support hyperplane* if the curve lies on one side w.r.t. this hyperplane.

1991 *Mathematics Subject Classification.* Primary 52A40, Secondary 52A38.
Key words and phrases. convex curves, Young diagrams and hulls.

Let $\gamma : S^1 \to \mathbb{R}^{2n}$ be a simple smooth closed convex curve in $\mathbb{R}^{2n}$. Then for any point t of γ there exists a support hyperplane passing through this point. For instance, the *osculating* hyperplane to γ at t is a support hyperplane (the multiplicity of the intersection of this hyperplane with γ at t is equal to $2n$).

DEFINITION. The *convex hull* of a curve in an affine space is the intersection of all closed half-spaces contained this curve.

I. Schoenberg [Sch] proved that the convex hull $\mathcal{CH}_\gamma$ of a convex curve γ in $\mathbb{R}^{2n}$ is the intersection of a family of closed half-spaces determined by support hyperplanes which are tangent to γ at n distinct points. Moreover, the Euclidean volume $Vol(\mathcal{CH}_\gamma)$ of its convex hull can be expressed by the integral

$$Vol(\mathcal{CH}_\gamma) = \pm \frac{1}{n!(2n)!} \int_{\mathcal{T}^n} \det\bigl[\gamma(t_1), \ldots, \gamma(t_n), \gamma'(t_1), \ldots, \gamma'(t_n)\bigr] dt_1 \ldots dt_n, \tag{2}$$

where $\mathcal{T}^n = (S^1)^n$ denotes the n-dimensional torus.

Note that the support hyperplanes considered by Schoenberg intersect γ with the maximal possible multiplicity $2n$. But there exist other types of support hyperplanes intersecting γ with such a total multiplicity. These different types are enumerated by the Young diagrams of area n.

DEFINITION. A support hyperplane to a curve γ is said to be a *hyperplane of a given Young type* $\mu = (k_1, ..., k_r)$ or just *a* μ*-hyperplane* if it is tangent to γ at r pairwise different points $\gamma(t_1), ..., \gamma(t_r)$ with multiplicities $2k_1, ..., 2k_r$ resp.

Any Young diagram of area at most n defines in the above mentioned way the family of support hyperplanes to a convex curve γ. Conversely, any support hyperplane to a curve γ is a support hyperplane with some Young diagram of area at most n.

REMARK. Note that for a convex curve $\gamma : S^1 \to \mathbb{R}^{2n}$ and an r-tuple of points $(t_1, ..., t_r)$ the hyperplane tangent to γ at $\gamma(t_1), ..., \gamma(t_r)$ with the multiplicities $2k_1, ..., 2k_r$ resp. where $\sum k_i = n$ is automatically a support hyperplane. This explains many simplifications which take place for convex curves in comparison with the general case.

DEFINITION. The *Young* $(k_1, ..., k_r)$*-hull* of a convex curve in $\gamma : S^1 \to \mathbb{R}^{2n}$ is the intersection of all closed half-spaces containing γ and defined by the support hyperplanes of a given Young type $(k_1, ..., k_r)$.

Denote by $\mathcal{YH}_\gamma(k_1, ..., k_r)$ the Young $(k_1, ..., k_r)$-hull of a curve γ.

REMARK. If the area of the Young diagram $(k_1, ..., k_r)$ is less than n, then $\mathcal{YH}_\gamma(k_1, ..., k_r) = \mathcal{YH}_\gamma(1, ..., 1, k_1, ..., k_r)$ where the area of Young diagram $(1, ..., 1, k_1, ..., k_r)$ is equal to n.

Consider now all Young hulls of a curve γ defined by all Young diagrams of area n. These hulls are enclosed according to the lexicographic order on Young diagrams. Namely, $\mathcal{YH}_\gamma(k_1, ..., k_r) \subset \mathcal{YH}_\gamma(k'_1, ..., k'_{r'})$ iff $(k_1, ..., k_r)$ is smaller than $(k'_1, ..., k'_{r'})$ in the standard lexicographic order.

Thus, for any Young diagram $(k_1, ..., k_r)$ of area n, we have

$$\mathcal{YH}_\gamma(1^n) \subset \mathcal{YH}_\gamma(k_1, \ldots, k_r) \subset \mathcal{YH}_\gamma(n),$$

where $(1^n) \equiv (1, ..., 1)$ (n times). The smalest Young hull $\mathcal{YH}_\gamma(1^n)$ of a curve γ is its convex hull $\mathcal{CH}_\gamma$. The biggest Young hull $\mathcal{YH}_\gamma(n)$ also denote by $\mathcal{EH}_\gamma$ of a curve γ is called its *elliptic hull.*

The main result of this paper is a characterization of the Young hulls of convex curves in even-dimensional spaces and their duals as the convex hulls of some special 'varieties' as well as a formula for the volume of the elliptic hull $\mathcal{EH}_\gamma$ similar to (2). As an example of the obtained results, let us describe the structure of the elliptic hull $\mathcal{EH}_\gamma$.

Let $L(t)$ be a $(2n-2)$-dimensional affine subspace in $\mathbb{R}^{2n}$ passing through the point $\gamma(t)$ and spanned by the vectors $\gamma'(t), \dots, \gamma^{2(n-1)}(t)$. Then we have a map $\Gamma_\gamma : (\mathcal{T}^n \setminus Diag) \to \mathbb{R}^{2n}$ sending an n-tuple of pairwise different points $(t_1, \dots, t_n) \in \mathcal{T}^n$ to the intersection $L(t_1) \cap \dots \cap L(t_{n-1}) \cap L(t_n)$. Obviusly, Γ_γ is invariant w.r.t. the action of the symmetric group $\mathfrak{S}_n$ on $\mathcal{T}^n$ by permutation of coordinates.

The map Γ extends continuously to the $\mathfrak{S}_n$-invariant map $\Gamma_\gamma : \mathcal{T}^n \to \mathbb{R}^{2n}$ (in particular, $\Gamma_\gamma(t, t, ..., t)$ coincides with $\gamma(t)$).

PROPOSITION. The elliptic hull $\mathcal{EH}_\gamma$ of a curve γ is the convex hull of the image $\Gamma_\gamma(\mathcal{T}^n) = \Gamma_\gamma(\mathcal{T}^n/\mathfrak{S}_n)$ and its Euclidean volume can be expressed by the integral

$$
(3)\quad Vol(\mathcal{EH}_\gamma) = \pm \frac{1}{(2n)!} \int_{\mathcal{T}^n} \det\Big[\Gamma_\gamma(t_1, t_1, ..., t_1), \Gamma_\gamma(t_1, t_2, t_2, ..., t_2), \dots, \Gamma_\gamma(t_1, ..., t_n), \frac{\partial \Gamma_\gamma(t_1, ..., t_n)}{\partial t_1}, \dots, \frac{\partial \Gamma_\gamma(t_1, ..., t_n)}{\partial t_n}\Big] dt_1 ... dt_n.
$$

Recall that using (2), Schoenberg has proved the following interesting isoperimetric inequality for convex curves

$$
(4)\qquad l_\gamma^{2n} \geq (2\pi n)^n n! (2n)! Vol(\mathcal{CH}_\gamma)
$$

where l_γ denotes the Eucledian length of γ. Moreover, he has shown that this estimation is sharp and the equality holds only for the curve (1) up to a parallel shift, a homothety and an orthogonal transformation of $\mathbb{R}^{2n}$.

The main motivation of the present study was an attempt to generalize Schoenberg's inequality for the case of elliptic (and more general Young) hulls. Unfortunately, at the present moment this goal still remains unachieved even for the elliptic hulls. We hope to return to this problem in the future.

CONJECTURE. An estimate of the form

$$
l_\gamma^{2n} \geq \kappa_\mu Vol(\mathcal{YH}_\gamma(\mu))
$$

(where κ_μ is some constant depending on μ only) is valid for all Young hulls of a curve γ. Moreover, the equality holds only for the curve (1) (considered up to a homothety and the group of rigid motions of $\mathbb{R}^{2n}$).

The structure of the paper is as follows. In §1 we prove some basic facts about $\mathcal{YH}_\gamma(\mu)$, in particular, find a special ruled hypersurface bounding $\mathcal{YH}_\gamma(\mu)$ and describe the minimal convex subset spanning $\mathcal{YH}_\gamma(\mu)$ and its dual as its convex hull. In §2 we present an integral formula for $Vol(\mathcal{EH}_\gamma)$. Finally, in §3 we calculate explicitly the volume of the elliptic hull of the generalized normalized ellipse in $\mathbb{R}^4$.

ACKNOWLEDGEMENTS. The first author is grateful to the Swedish Natural Science Research Council for the financial support during his stay at the Department of Mathematics of the University of Stockholm where this project was started. The second author wants to acknowledge the hospitality and nice research atmosphere of the Max-Planck Institute during the preparation of the manuscript.

§1. GENERALITIES ON CONVEX CURVES

Boundness.

We start with the following simple proposition.

1.1. LEMMA. For a convex $\gamma : S^1 \to \mathbb{R}^{2n}$ every Young hull $\mathcal{YH}_\gamma(\mu)$ is a bounded convex domain.

PROOF. It suffices to show that $\mathcal{EH}_\gamma$ is bounded since it contains all other Young hulls. One has to show that for every point p lying outside a sufficiently big ball containing γ there exists an osculating hyperplane to γ passing through p. This is equivalent to finding a hyperplane H_p passing through p and tangent to γ. Taking a pont p as above we can find a hyperplane $\tilde{H}_p$ through p not intersecting γ at all. Now fixing some codimension 2 subspace $\tilde{h}_p \subset \tilde{H}_p$ through p we can take the pencil of hyperplanes containing $\tilde{h}_p$. Since $\tilde{H}_p$ does not intersect γ this family of hyperplanes contains a hyperplane tangent to γ.

□

μ-discriminants.

DEFINITION. Given $\mu = (k_1, ..., k_r)$, $\sum k_i = m$ we call a curve $\gamma : S^1 \to \mathbb{R}^m$ *μ-generic* if for any r-ruple of pairwise different points $(t_1, ..., t_r)$ *the spanning affine space* $L_\mu(t_1, ..., t_r)$ passing through $\gamma(t_1), \gamma(t_2), ..., \gamma(t_r)$ and containing $\gamma'(t_1), ..., \gamma^{(k_1-2)}(t_1), \gamma'(t_2), ..., \gamma^{(k_2-2)}(t_2), ...,$ $\gamma^{(k_r-2)}(t_r)$ has codimension $r+1$.

One can easily check that μ-generic curves form an open dense subset in the space of all maps $S^1 \to \mathbb{R}^m$.

DEFINITION. Given a Young diagram $\mu = (k_1 \geq k_2 \geq ... \geq k_r)$, $\sum_i k_i = m$ we call by *the μ-discriminant* $D_\gamma(\mu)$ of a μ-generic curve $\gamma : S^1 \to \mathbb{R}^m$ the ruled hypersurface obtained as the closure of the union of all spanning affine subspaces $L_\mu(t_1, ..., t_r)$.

REMARK. If $\mu = (m)$ then we get the standard discriminant D_γ (also called the swallowtail or the front of γ), i.e. the hypersurface ruled out by the family of $(m-2)$-dimensional osculating subspaces. In a neighborhood of a generic point of γ (where all m derivatives $\gamma'(t), \dots, \gamma^{(m)}(t)$ are linearly independent) a germ of D is diffeomorphic to a germ of the usual discriminant consisting of all polynomials of degree m with multiple zeros. If γ is globally nondegenerate, i.e. $\gamma'(t), \dots, \gamma^{(m)}(t)$ are linearly independent for any $t \in S^1$ then for any point $p \in \mathbb{R}^m \setminus D_\gamma$ and any hyperplane $\tilde{H}_p$ through p the local multiplicity of intersection of $\tilde{H}_p$ and γ does not exceed 2.

1.2. PROPOSITION. Given a convex curve $\gamma : S^1 \to \mathbb{R}^{2n}$ and a diagram $\mu = (k_1, ..., k_r)$, $\sum k_i = n$ one gets that $\mathcal{YH}_\gamma(\mu)$ coincides with a convex connected component of the complement $\mathbb{R}^{2n} \setminus D_\gamma(2\mu)$ where $2\mu = (2k_1, ..., 2k_r)$ is the 'doubled' diagram of area $2n$.

PROOF. Proposition follows immediately from lemmas 1.3-1.4.

□

DEFINITION. The intersection of the Young μ-hull $\mathcal{YH}_\gamma(\mu)$ of a curve γ with a given support hyperplane is called the *characteristic set* of this hyperplane.

The characteristic set of a support hyperplane is convex (as the intersection of 2 convex sets).

1.3. LEMMA. The boundary of the Young μ-hull $\mathcal{YH}_\gamma(\mu)$ of γ is the closure of the union of characteristic sets of all support μ-hyperplanes to γ.

PROOF. The characteristic set X of a given hyperplane π can not contain interior points of $Y = \mathcal{YH}_\gamma(\mu)$ since Y lies on one side of π. Hence, X belongs to the boundary ∂Y.

Conversely, let $p \in \partial Y$. Then there exists a sequence of points $p_i \notin Y$ and μ-hyperplanes π_i, $i = 1, ...,$ such that p_i and Y lie on the different sides of π_i and $\lim_{i\to\infty} p_i = p$.

Taking subsequences of hyperplanes we can assume that there exists $\lim_{i\to\infty}\pi_i = \pi$. Then the hyperplane π is a support hyperplane to γ. The point p and the set Y have to lie on different sides of π. Hence, $p \in \pi \cap Y$ and $p \in Y$.

□

1.4. LEMMA. The characteristic set of a support μ-hyperplane tangent to γ at $\gamma(t_1), ..., \gamma(t_r)$ belongs to the spanning affine subspace $L_\mu(t_1, ..., t_r)$.

PROOF. Let $D[u_1, ..., u_r, v]$ be the determinant of order $2n$ of the matrix consisting of $\gamma(t_2) - \gamma(t_1), ..., \gamma(t_r) - \gamma(t_1), \gamma'(t_1), ..., \gamma^{(2k_1-2)}(t_1), ..., \gamma(t_r), ..., \gamma^{(2k_r-2)}(t_r), u_1, ..., u_r, v$. Then for any small $\Delta_1, ..., \Delta_r$ the support μ-hyperplane tangent to γ at $\gamma(t_1+\Delta_1), ..., \gamma(t_r+\Delta_r)$ defined by the equation $F(x, \Delta_1, ..., \Delta_r) = 0$ where $x \in \mathbb{R}^{2n}$ and

$$\begin{aligned}F(x, \Delta_1, ..., \Delta_r) &= D[\gamma^{(2k_1-1)}(t_1), ..., \gamma^{(2k_r-1)}(t_r), x - \gamma(t_1)]\\ &+ D[\gamma^{(2k_1)}(t_1), \gamma^{(2k_2-1)}(t_2), ..., \gamma^{(2k_r-1)}(t_r), x - \gamma(t_1)]\Delta_1 + ...\\ &+ D[\gamma^{(2k_1-1)}(t_1), ..., \gamma^{(2k_{r-1}-1)}(t_{r-1}), \gamma^{(2k_r-1)}(t_r), x - \gamma(t_1)]\Delta_r + ...\end{aligned}$$

If x lies on the support hyperplane π tangent to γ at $\gamma(t_1), ..., \gamma(t_r)$ and does not belong to the subspace $L_\mu(t_1, ..., t_r)$ then $F(x, 0, ..., 0) \neq 0$ and there exists $i \in [1, ..., r]$ such that $\partial F/\partial\Delta_i(x, 0, ..., 0) \neq 0$ (since γ is convex and the area of μ equals n). Hence one can find $\Delta_i^- < 0$ and $\Delta_i^+ > 0$ such that

$$F(x, 0, ..., 0, \Delta_i^-, 0, ..., 0)F(x, 0, ..., 0, \Delta_i^+, 0, ..., 0) < 0.$$

Therefore x can not belong to the characteristic set of π.

□

REMARK. Appearently for a convex $\gamma : S^1 \to \mathbb{R}^{2n}$ the complement $\mathbb{R}^{2n} \setminus D_\gamma(2\mu)$ contains a unique convex connected component.

EXAMPLE 1 (useful although the source is $\mathbb{R}$ instead of S^1). Consider the usual rational normal curve $\xi_m : \mathbb{R} \to \mathbb{R}^m$, $t \to (mt, \binom{m}{2}t^2, \dots, \binom{m}{2}t^{m-1}, t^m)$. Identifying $\mathbb{R}^m$ with the space $\mathfrak{Pol}_m$ of all monic polynomials $x^m + a_1x^{m-1} + \cdots + a_m$ in the obvious way one identifies the curve ξ_m with the 1-parameter family of polynomials $(x+t)^m$. The elliptic hull $\mathcal{EH}_{\xi_m}$ consists of all strictly elliptic polynomials, i.e. polynomials with no real zeros. ($\mathcal{EH}_{\xi_m}$ is nonempty only for even m.) The last observation motivates the term 'elliptic hull of γ'.

Dual domain and convex skeleton.

For any nondegenerate $\gamma : S^1 \to \mathbb{R}^m$ one can define its dual $\gamma^* : S^1 \to (\mathbb{P}^m)^*$ considering $\mathbb{R}^m$ as the standard affine chart in $\mathbb{P}^m$.

DEFINITION. The curve $\gamma^* : S^1 \to (\mathbb{P}^m)^*$ such that $\gamma^*(t)$ is the hyperplane spanned by $\gamma'(t), \dots, \gamma^{(m-1)}(t)$ is called *the dual curve* to the nondegenerate curve γ.

REMARK. The dual to a convex curve is a convex curve, see e.g. [Ar].

Given a convex closed domain $\Omega \subset \mathbb{R}^m \subset \mathbb{P}^m$ (here 'convexity' means that with any two points p_1, p_2 the set Ω contains the whole segment $(p_1, p_2) \subset \mathbb{R}^m$) one defines the dual convex domain as follows.

DEFINITION. *The dual convex domain* $\Omega^* \subset (\mathbb{P}^m)^*$ is the closure of the set of all hyperplanes not intersecting Ω.

REMARK. One can easily check that Ω^* lies in some affine chart of $(\mathbb{P}^m)^*$ and is convex there in the usual sense. Namely, fixing a point p in the interior of Ω we get the corresponding hyperplane $H_p^* \subset (\mathbb{P}^m)^*$. Obviously, $\Omega^* \subset (\mathbb{P}^m)^* \setminus H_p^*$.

Now we will describe the minimal subset the convex hull of which coincides with the domain dual to some Young hull $\mathcal{YH}_\gamma(\mu)$.

DEFINITION. Given a convex $\gamma : S^1 \to \mathbb{R}^{2n}$ and an n-tuple of pairwise different points $(t_1, t_2, \ldots, t_n) \in \mathcal{T}^n \setminus Diag$ we call by *the associated map* $\Gamma_\gamma : \mathcal{T}^n \setminus Diag \to \mathbb{R}^{2n}$ the map sending $(t_1, t_2, \ldots, t_n)$ to the intersection point $\bigcap_{i=1}^n L(t_i)$ where $L(t)$ denotes the $(2n-2)$-dimensional subspace passing through $\gamma(t)$ and spanned by $\gamma'(t), \ldots, \gamma^{(2n-2)}(t)$.

Convexity of γ provides that this intersection is a point for any choice of $(t_1, t_2, \ldots t_n) \in \mathcal{T}^n \setminus Diag$. (Note that $L(t)$ are subspaces ruling out the swallowtail D_γ.)

DEFINITION. *By the convex skeleton* $\mathbf{CSk}_\gamma$ of a convex $\gamma : S^1 \to \mathbb{R}^{2n}$ we denote the closure of the image of the associated map $\overline{\bigcup_{t_1,\ldots,t_n \in \mathcal{T}^n \setminus Diag} \Gamma_\gamma(t_1, t_2, \ldots, t_n)}$.

1.5. LEMMA. The associated map $\Gamma_\gamma : (\mathcal{T}^n \setminus Diag) \to \mathbb{R}^{2n}$ extends continuously to the $\mathfrak{S}_n$-invariant map $\Gamma_\gamma : \mathcal{T}^n \to \mathbb{R}^{2n}$ where as before $\mathfrak{S}_n$ is the symmetric group acting by permutation of coordinates of $\mathcal{T}^n$.

PROOF. One can easily see that convexity of $\gamma : S^1 \to \mathbb{R}^{2n}$ implies (and is in fact equivalent) to the following property. For any r-tuple of points $\gamma(t_1), \ldots, \gamma(t_r)$ there exists and unique hyperplane passing through these points and tangent to γ with the total multiplicity $2n$ distributed in any way between $\gamma(t_1), \ldots, \gamma(t_r)$. In our case using the dual space one gets that the intersection of subspaces $\bigcap_{i=1}^n L(t_i)$ corresponds to the hyperplane $H_{t_1,\ldots,t_n} \subset (\mathbb{P}^{2n})^*$ tangent to the dual curve γ^* at $\gamma^*(t_1), \ldots, \gamma^*(t_n)$ with the multiplicity 2 at each point. Note that if γ is convex then γ^* is also convex. The closure of the set $\bigcup_{(t_1,\ldots,t_n) \in \mathcal{T}^n \setminus Diag} H(t_1, \ldots, t_n)$, obviously, is the closed set of hyperplanes tangent of γ^* with the even local multiplicity at every point. Going back to $\mathbb{R}^{2n}$ we get the required result.

□

EXAMPLE 2. The convex skeleton $\mathbf{CSk}_{\xi_{2n}}$ in the example 1 is the set of all polynomials of the form $(x-t_1)^2 \ldots (x-t_n)^2$.

1.6. COROLLARY. The map $\Gamma_\gamma : \mathcal{T}^n/\mathfrak{S}_n \to \mathbb{R}^{2n}$ sends $\mathcal{T}^n/\mathfrak{S}_n$ homeomorphically onto $\mathbf{CSk}_\gamma$.

PROOF. It suffices to show that $\Gamma_\gamma : \mathcal{T}^n/\mathfrak{S}_n \to \mathbb{R}^{2n}$ is a 1-1-map. This again follows from the fact that for a convex γ^* and a pair $(t'_1, \ldots, t'_{r'}, k'_1, \ldots, k'_{r'})$ and $(t''_1, \ldots, t''_{r''}, k''_1, \ldots, k''_{r''})$, $\sum k'_i = n$ and $\sum k''_i = n$ the hyperplanes H_1 tangent to γ at $(t'_1, \ldots, t'_{r'})$ with the multiplicities $(2k'_1, \ldots, 2k'_{r'})$ and H_2 tangent to γ at $(t''_1, \ldots, t''_{r'})$ with the multiplicities $(2k''_1, \ldots, 2k''_{r''})$ are different unless $(t'_1, \ldots, t'_{r'}, k'_1, \ldots, k'_{r'}) = (t''_1, \ldots, t''_{r''}, k''_1, \ldots, k''_{r''})$.

□

The above homeomorphism provides the following natural stratification of $\mathbf{CSk}_\gamma$ into strata enumerated by the Young diagrams of area n. Fixing a partition $\mu = (k_1 \geq k_2 \geq \cdots \geq k_r)$, $\sum k_i = n$ let us denote by $\mathbf{CSk}_\gamma(\mu)$ the closure of the set of all intersections $\bigcap_{i=1}^r L_{k_i}(t_i)$ where $(t_1, \ldots, t_r)$ are pairwise different and $L_{k_i}(t_i)$ is the subspace passing through $\gamma(t)$ and spanned by $\gamma'(t_i), \ldots, \gamma^{2(n-k_i)}(t_i)$. Obviously, $\dim \mathbf{CSk}_\gamma(\mu) = r$ and different strata $\mathbf{CSk}_\gamma(\mu)$ form the stratification of $\mathbf{CSk}_\gamma$ with the adjacency coinciding with the reverse lexicographic partial order on the set of all Young diagrams of area n. Namely, a stratum $\mathbf{CSk}_\gamma(k_1, \ldots, k_{r_1})$ is adjacent to $\mathbf{CSk}_\gamma(k'_1, \ldots, k'_{r_2})$ iff the partition $(k_1, \ldots, k_{r_1})$ is smaller than $(k'_1, \ldots, k'_{r_2})$ in the lexicographic order.

1.7. LEMMA. The convex skeleton $\mathbf{CSk}_\gamma$ (and each stratum $\mathbf{CSk}_\gamma(k_1, .., k_r)$) is a weakly convex set, i.e. it lies on the boundary of its convex hull.

PROOF. We will prove by induction on n that $\mathbf{CSk}_\gamma$ lies on the boundary of $\mathcal{EH}_\gamma$ and therefore is weakly convex. Base of induction, $n = 1$. In this case γ is a convex curve on $\mathbb{R}^2$ and the statement is obvious.

Step of induction. Let us assume that the statement holds for $n \le N$ and consider a convex curve γ in $\mathbb{R}^{2N+2}$. Let us show that all intersections $\bigcap_{i=1}^{N+1} L(t_i)$ belong to $\partial\mathcal{EH}_\gamma$. Fixing t_1 let us take the following curve $\gamma^{t_1} \subset L(t_1)$. Namely, $\gamma^{t_1} : S^1 \to L(t_1)$ obtained as the projection of γ along the family of 2-dimensional subspaces through $\gamma(t)$ containing $\gamma'(t)$ and $\gamma''(t)$ on $L(t_1)$. One can show that γ^{t_1} is a smooth convex curve in $L(t_1)$. Moreover, $\mathcal{EH}_{\gamma^{t_1}}$ coincides with $\mathcal{EH}_\gamma \cap L(t_1)$. This follows from the fact that the intersection of an osculating subspace to γ restricted to $L(t_1)$ coincides with the osculating subspace to γ^{t_1} at the corresponding point. Thus the closure of the union of intersections $\bigcap_{i=1}^{N} L(t_i)$ coincides with $\mathbf{CSk}_{\gamma^{t_1}}$. Since γ^{t_1} is convex in $\mathbb{R}^{2n}$ everything is proved for t_1. Varying t_1 we get the necessary statement.

□

Now we are in position to characterize the dual domains to $\mathcal{YH}_\gamma(\mu)$.

1.8. PROPOSITION. For any convex $\gamma : S^1 \to \mathbb{R}^{2n}$ and any diagram μ of area n the dual domain $\mathcal{YH}^*_\gamma(\mu)$ coincides with the convex hull of the stratum $\mathbf{CSk}_{\gamma^*}(\mu^*)$ where μ^* is the dual Young diagram, i.e. obtained by making rows into columns.

PROOF. By definition the boundary of the dual domain $\mathcal{YH}^*_\gamma(\mu)$ consists of all support hyperplanes to $\mathcal{YH}_\gamma(\mu)$. One checks immediately that the union of all support μ-hyperplanes to γ forms $\mathbf{CSk}_{\gamma^*}(\mu^*)$. Fixing a point p in the interior of $\mathcal{YH}_\gamma(\mu)$ one gets that $\mathcal{YH}^*_\gamma(\mu) \subset (\mathbb{P}^{2n})^* \setminus H^*_p$. For any other support hyperplane to γ its halfspace not containing γ lies in the union of halfspaces cut off by the μ-hyperplanes. This means that the hyperplane itself as the point in the dual space lies in the convex hull of $\mathbf{CSk}_{\gamma^*}(\mu^*)$ in $(\mathbb{P}^{2n})^* \setminus H^*_p$.

□

OBSERVATION. The stratum $\mathbf{CSk}_\gamma(\mu)$ is minimal, i.e. none of its point p lies in the convex hull of $\mathbf{CSk}_\gamma(\mu) \setminus p$.

PROOF. Interpreting a point $p \in \mathbf{CSk}_\gamma(\mu)$ as a support hyperplane of the μ^*-type to γ^* one can easily check that in the case of convex γ a positive linear combination of 2 such hyperplanes is again of μ^*-type if and only if these hyperplanes coincide. (Using interpretation with disconjugate linear ODE, see e.g. [BSh] one can think of support hyperlanes of μ^*-type as of nonnegative 2π-periodic solutions to a certain linear ODE having exactly r zeros of multiplicity $2\mu^*_1, ..., 2\mu^*_r$ at some points $t_1, ..., t_r$ on the period $[0, 2\pi)$. This interpretation makes the observation obvious.)

$\mathcal{YH}_\gamma(\mu)$ as a convex hull.

Finally, we describe $\mathcal{YH}_\gamma(\mu)$ as the convex hull of some special set in the initial space $\mathbb{R}^{2n}$. Let us return to the hypersurface $D_\gamma(2\mu)$ ruled out by codimension $r+1$ subspaces $L_\mu(t_1, ..., t_r)$.

DEFINITION. We call the set of subspaces $L_\mu(t^1_1, ..., t^1_r)$, $L_\mu(t^2_1, ..., t^2_r)$, ..., $L_\mu(t^p_1, ..., t^p_r)$ *essential* if the intersection $\bigcap_{i=1}^{p} L_\mu(t^i_1, ..., t^i_r)$ is a point. This intersection point itself is called *essential* and, finally, the closure of the union of all essential points is called *the essential μ-set* $\mathbf{Ess}_\gamma(\mu)$ of γ.

1.9. PROPOSITION. For any convex $\gamma : S^1 \to \mathbb{R}^{2n}$ and any diagram μ of area n the Young hull $\mathcal{YH}_\gamma(\mu)$ is the convex hull of $\mathbf{Ess}_\gamma(\mu)$.

In particular, one has

1.10. COROLLARY. For any convex $\gamma : S^1 \to \mathbb{R}^{2n}$ the elliptic hull $\mathcal{EH}_\gamma$ is the convex hull of $\mathbf{CSk}_\gamma$.

To prove statement 1.9. we need the following definitions.

DEFINITION. A support hyperplane H to a convex domain Ω is called *flexible* if there exists a subspace $h \subset H$ of codimension 2 such that all hyperplanes containing h and lying sufficiently close to H are support hyperplanes. A support hyperplane which is not flexible is called *rigid.*

OBSERVATION. Obviously, the union of all rigid support hyperplanes to Ω is the minimal weakly convex set spanning Ω^* as its convex hull.

PROOF OF 1.10. The boundary of $\mathcal{YH}_\gamma(\mu)$ consists of all support hyperplanes to $\mathcal{YH}^*_\gamma(\mu)$. By proposition 1.8. the set $\mathcal{YH}^*_\gamma(\mu)$ is the convex hull of $\mathbf{CSk}_{\gamma^*}(\mu^*)$. Therefore any hyperplane intersecting $\mathbf{CSk}_{\gamma^*}(\mu^*)$ at some points corresponds to the point in the initial space contained in some intersection $\bigcap_{i=1}^{\tilde{p}} L_\mu(t_1^i, ..., t_r^i)$. Assume now that H_p is a support hyperplane to $\mathcal{YH}^*_\gamma(\mu)$ such that the corresponding intersection $\bigcap_{i=1}^{\tilde{p}} L_\mu(t_1^i, ..., t_r^i)$ is at least 1 -dimensional. Choosing in this intersection any line through p one gets a 1-parameter family of supporting hyperplanes used in the definition of flexibility. Thus H_p is flexible and is not included in the minimal weakly convex set of $\mathcal{YH}^*_\gamma(\mu)$. On the other side using induction similar to the one from 1.7 one can show that each point in $\mathbf{Ess}_\gamma(\mu)$ lies in $\partial\mathcal{YH}_\gamma(\mu)$.

□

EXAMPLE 3. Any elliptic polynomial is a convex linear combination of polynomials of the form $(x-t_1)^2...(x-t_n)^2$.

EXAMPLE 4. The intersection of $L_{(1^n)}(t_1, \dots, t_n)$ with the convex hull $\mathcal{CH}_\gamma$ of a curve γ is the $(n-1)$-dimensional simplex with vertices at points $\gamma(t_1), \dots, \gamma(t_n)$.

EXAMPLE 5. The intersection of $L_{(n)}(t_1) = L(t_1)$ with the elliptic hull $\mathcal{EH}_\gamma$ of a curve γ is the set bounded by the hypersurface in $L(t_1)$ consisting of all intersection points of subspaces $L(t_1) \cap \dots \cap L(t_n)$ for all n-tuple of pairwise different points $(t_1, \dots, t_n)$ on γ with t_1 fixed.

EXAMPLE 6. The intersection of $L_{(1,2)}(t_1, t_2)$ with the Young hull $\mathcal{YH}_\gamma(1,2)$ of a curve γ in $\mathbb{R}^6$ is the set bounded by the surface consisting of all intersection points with $L_{(1,2)}(t_3, t_4)$ where (t_3, t_4) are varying.

§2. INTEGRAL PRESENTATION OF VOLUME OF $\mathcal{EH}_\gamma$

Let as above $\Gamma_\gamma : \mathcal{T}^n \to \mathbb{R}^{2n}$ denote the $\mathfrak{S}_n$-invariant associated map parameterizing the convex skeleton $\mathbf{CSk}_\gamma$. We use the convention that $\Gamma_\gamma(t_1, ..., t_j)$, $j < n$ means that the last variable t_j is repeated $n - j + 1$ times.

2.1. PROPOSITION. The family of simplices spanned by $\Gamma_\gamma(t_1), \Gamma_\gamma(t_1, t_2)$, ..., $\Gamma_\gamma(t_1, ..., t_n)$ where $(t_1, ..., t_n) \in \mathcal{T}^n$ 'parametrizes' the boundary $\partial\mathcal{EH}_\gamma$. Namely, each generic point on $\partial\mathcal{EH}_\gamma$ belongs to exactly $n!$ such simplices.

PROOF. Induction on n. For $n = 1$ each point of a convex γ is passed exactly once when t runs over S^1. Assume that the statement holds for $n \leq N$ and consider a convex curve $\gamma : S^1 \to \mathbb{R}^{2N+2}$. A generic point $p \in \partial\mathcal{EH}_\gamma$ belongs exactly to $N+1$ codimension 2 subspaces $L(t_1), ..., L(t_{N+1})$. Take $L(t_1)$ and the projected curve $\gamma^{t_1} : S^1 \to L(t_1)$ introduced in the proof of 1.7. One has $p \in \partial\mathcal{EH}_{\gamma^{t_1}}$ and the family of $(N-1)$-diimensional simplices for γ^{t_1} is obtained from the family $\Gamma_\gamma(t_1), \Gamma_\gamma(t_1, t_2)$, ..., $\Gamma_\gamma(t_1, ..., t_n)$ where $(t_1, ..., t_n) \in \mathcal{T}^n$ by fixing t_1 and varying $t_2, ..., t_{N+1}$. Therefore by the inductive hypothesis in the restricted family of simplices one covers p exactly $N!$ times. Repeating the argument for $t_2, ..., t_{N+1}$ one gets that the whole family covers p exactly $(N+1)!$ times.

□

2.2. PROPOSITION. One has the following integral presentation for $Vol(\mathcal{EH}_\gamma)$

$$Vol(\mathcal{EH}_\gamma) = \pm\frac{1}{2n!}\int_{\mathcal{T}^n}\det[\Gamma_\gamma(t_1),\Gamma_\gamma(t_1,t_2),...,\Gamma_\gamma(t_1,...,t_n),\frac{\partial\Gamma_\gamma(t_1,...,t_n)}{\partial t_1},$$
$$\frac{\partial\Gamma_\gamma(t_1,...,t_n)}{\partial t_2},...,\frac{\partial\Gamma_\gamma(t_1,...,t_n)}{\partial t_n}]dt_1...dt_n. \tag{5}$$

PROOF. Consider the n-parameter family of $(n-1)$-dimensional simplices $\Delta(t_1,...,t_n)$ spanned by all n-tuples of vertices $\Gamma_\gamma(t_1),\Gamma_\gamma(t_1,t_2),...,\Gamma_\gamma(t_1,...,t_n)$, $(t_1,...,t_n)\in\mathcal{T}^n$. By the above proposition a generic point on $\partial\mathcal{EH}_\gamma$ is covered by this parametrization exactly $n!$ times. Consider now the element $d\omega$ of area of $\partial\mathcal{EH}_\gamma$ svept out by the small deformation of $\Delta(t_1,...,t_n)$ when $(t_1,...,t_n)$ are varying in $\mathcal{T}^n$. Up to infinitesimals of higher order $d\omega$ coincides with the area of the polytope spanned by $\Gamma_\gamma(t_1),\Gamma_\gamma(t_1,t_2),...,\Gamma_\gamma(t_1,...,t_n)$ and another 2^n vertices of the form $\Gamma_\gamma(t_1+\epsilon_1dt_1,t_2+\epsilon_2dt_2,...,t_n+\epsilon_ndt_n)$ where each $\epsilon_1,...,\epsilon_n$ is an arbitrary n-tuple of 0's and 1's. (We do not need to vary the vertices $\Gamma_\gamma(t_1),\Gamma_\gamma(t_1,t_2)$ $,...,\Gamma_\gamma(t_1,...,t_{n-1})$ since their small deformations belong to the subspace L_{t_1} of codimension 2 and contribute to the higher order infinitesimals in $d\omega$.) Now placing the origin 0 inside $\mathcal{EH}_\gamma$ we can present the volume element dV of the cone over $d\omega$ as the volume of the polytope Π spanned by 0 and the above vertices. The last group of 2^n vertices of Π form the n-dimensional parallelepiped up to the higher order infinitesimals. Let us split this parallelepiped into $n!$ n-dimensional simplices and consider $n!$ $2n$-dimensional simplices spanned by each of these simplices and the rest of the vertices $0,\Gamma_\gamma(t_1),...,\Gamma_\gamma(t_1,...,t_n)$. Volumes of these $2n$-dimensional simplices coincide with each other up to the higher order infinitesimals. Take one of these simplices, namely the one spanned by $0,\Gamma_\gamma(t_1),...,\Gamma_\gamma(t_1,...,t_{n-1}),\Gamma_\gamma(t_1,...,t_n)$, $\Gamma_\gamma(t_1+dt_1,t_2,...,t_n),\Gamma_\gamma(t_1,t_2+dt_2,...,t_n),...,\Gamma_\gamma(t_1,t_2,...,t_n+dt_n)$ with the same convention as above. Its volume $Vol(Simp)$ with the appropriate orientation equals

$$Vol(Simp) = \frac{1}{2n!}\det[\Gamma_\gamma(t_1),\Gamma_\gamma(t_1,t_2),...,\Gamma_\gamma(t_1,...,t_n),$$
$$\Gamma_\gamma(t_1+dt_1,...,t_n),\Gamma_\gamma(t_1,...,t_n+dt_n)]. \tag{6}$$

Let us write $\Gamma_\gamma(t_1,...,t_i+dt_i,...,t_n)\approx\Gamma_\gamma(t_1,...,t_i,...,t_n)+dt_i\frac{\partial\Gamma_\gamma(t_1,...,t_i,...,t_n)}{\partial t_i}$ and substitute it in (6). The volume of the whole polytope Π is equal to $n!Vol(Simp)$. Integrating against $\mathcal{T}^n$ and taking into account the fact that our parametrization covers $\partial\mathcal{EH}_\gamma$ exactly $n!$ times we get the above integral formula (5).

□

§3. APPENDIX. VOLUME OF $\mathcal{EH}_\gamma$ FOR NORMALIZED ELLIPSE IN $\mathbb{R}^4$.

PROPOSITION 3.1. If γ is a normalized generalized ellipse in $\mathbb{R}^4$, i.e. $\gamma=(\sin x,\cos x,$ $1/2\sin 2x,\ 1/2\cos x)$ then

$$Vol(\mathcal{EH}_\gamma)=\frac{9\pi^2\sqrt{3}}{64}.$$

PROOF. Let γ be a convex curve in $\mathbb{R}^4$. Then

$$\Gamma(t_1,t_2)=\begin{cases}\gamma(t_1)+F(t_1,t_2)\dfrac{d\gamma}{dt}(t_1)+G(t_1,t_2)\dfrac{d^2\gamma}{dt^2}(t_1), & \text{if } t_2\neq t_1\\ \gamma(t_1), & \text{if } t_2=t_1,\end{cases}$$

where $F = \Delta_1/\Delta$, $G = \Delta_2/\Delta$, and

$$\Delta_1 = \det(\gamma(t_2) - \gamma(t_1), \frac{d^2\gamma}{dt^2}(t_1), \frac{d\gamma}{dt}(t_2), \frac{d^2\gamma}{dt^2}(t_2)),$$

$$\Delta_2 = \det(\frac{d\gamma}{dt}(t_1), \gamma(t_2) - \gamma(t_1), \frac{d\gamma}{dt}(t_2), \frac{d^2\gamma}{dt^2}(t_2)),\ \Delta = \det(\frac{d\gamma}{dt}(t_1), \frac{d^2\gamma}{dt^2}(t_1), \frac{d\gamma}{dt}(t_2), \frac{d^2\gamma}{dt^2}(t_2)).$$

Therefore

$$Vol(\mathcal{EH}_\gamma) = \frac{1}{4!}\int_{T^2} G(F\frac{\partial G}{\partial t_2} - G\frac{\partial F}{\partial t_2})W(t_1)dt_1dt_2$$

where

$$W(t) = \det(\gamma(t), \frac{d\gamma}{dt}(t), \frac{d^2\gamma}{dt^2}(t), \frac{d^3\gamma}{dt^3}(t)).$$

Integration by parts leads to

$$Vol(\mathcal{EH}_\gamma) = -\frac{1}{16}\int_{T^2} G^2\frac{\partial F}{\partial t_2}W(t_1)dt_1dt_2.$$

Let γ be the generalized Lissajoux curve

$$\gamma(t) = (1/k\sin kt, 1/k\cos kt, 1/l\sin lt, 1/l\cos lt)$$

where k, l are natural numbers, $k < l$. Then

$$W(t) \equiv \text{const} = \frac{1}{lk}(l^2 - k^2)^2,$$

$$\begin{aligned}
\Delta_1 &= \frac{l^2-k^2}{lk}\left(k(1-\cos ls)\sin ks - l(1-\cos ks)\sin ls\right)\\
&= \frac{l^2-k^2}{lk}\left(\cos(l-k)t - \cos(l+k)t\right)\left((l+k)\sin(l-k)t - (l-k)\sin(l+k)t\right),\\
\Delta_2 &= \frac{1}{kl}\left(l^2(1+\cos ls)(1-\cos ks) + k^2(1+\cos ks)(1-\cos ls) - 2kl\sin ks\sin ls\right)\\
&= \frac{1}{kl}\left((l+k)\sin(l-k)t - (l-k)\sin(l+k)t\right)^2,
\end{aligned}$$

$$\begin{aligned}
\Delta &= 2kl(1-\cos ks\cos ls) - (l^2+k^2)\sin ks\sin ls\\
&= (l+k)^2\sin^2(l-k)t - (l-k)^2\sin^2(l+k)t,
\end{aligned}$$

where $s = 2t = t_2 - t_1$. The mapping Γ is well-defined (i.e. $\Delta = 0$ iff $s = 0 \bmod 2\pi$) if and only if $l - k = 1$. In the last case

$$Vol(\mathcal{EH}_\gamma) = -\frac{\pi(2k+1)^2}{8k(k+1)}\int_{-\pi/2}^{\pi/2} g^2\frac{df}{dt}dt$$

where

$$
\begin{aligned}
g(t) &= \frac{(2k+1)\sin t - \sin(2k+1)t}{(k(k+1)\left((2k+1)\sin t + \sin(2k+1)t\right)},\\
f(t) &= \frac{(2k+1)\left(\cos t - \cos(2k+1)t\right)}{k(k+1)\left((2k+1)\sin t + \sin(2k+1)t\right)},\\
\frac{df}{dt} &= \frac{4(2k+1)\sin t \sin(2k+1)t}{\left((2k+1)\sin t + \sin(2k+1)t\right)^2}.
\end{aligned}
$$

Note that $\sin(2k+1)t = h(t)\sin t$ where

$$h(t) = \sum_{m=0}^{k} (-1)^m \binom{2k+1}{2m+1} (1-\cos^2 t)^m \cos^{2(k-m)} t.$$

Therefore

$$Vol(\mathcal{EH}_\gamma) = -\frac{\pi(2k+1)^3}{2k^3(k+1)^3} \int_{-\pi/2}^{\pi/2} \frac{(2k+1-h)^2}{(2k+1+h)^4} h dt.$$

Change of variable $z = \tan t$ leads to the formula

$$Vol(\mathcal{EH}_\gamma) = -\frac{\pi(2k+1)^3}{2k^3(k+1)^3} \int_{-\infty}^{+\infty} \frac{\left((2k+1)(1+z^2)^k - P_k(z^2)\right)^2}{\left((2k+1)(1+z^2)^k + P_k(z^2)\right)^4} P_k(z^2)(1+z^2)^{k-1} dz$$

where

$$P_k(x) = \sum_{m=0}^{k} (-1)^m \binom{2k+1}{2m+1} x^m$$

is a polynomial of degree k.

Let us, for example, take $k=1$ (or $l=2$), i.e. consider the normalized ellipse $\gamma = \mathcal{G}_2$ in $\mathbb{R}^4$. Then

$$Vol(\mathcal{EH}_\gamma) = \frac{27\pi}{16} \int_{-\infty}^{+\infty} \frac{(z^2-3)z^4}{(3+z^2)^4} dz = \frac{9\pi\sqrt{3}}{16}(-2I_4 + 5I_3 - 4I_2 + I_1)$$

where

$$I_m = \int_{-\infty}^{+\infty} \frac{dz}{(1+z^2)^m} = \begin{cases} \pi, & \text{if } m=1 \\ \frac{1\cdot3\cdot5\cdot\ldots(2m-3)}{2\cdot4\cdot6\cdot\ldots(2m-2)}\pi, & \text{if } m>1. \end{cases}$$

This gives

$$Vol(\mathcal{EH}_\gamma) = \frac{9\pi^2\sqrt{3}}{64}.$$

□

§4. Final remarks

Below we mention some further questions.

1) Conjecture. For any 2 convex $\gamma_1 : S^1 \to \mathbb{R}^{2n}$ and $\gamma_1 : S^1 \to \mathbb{R}^{2n}$ and a Young diagram μ of area n the pairs $(\mathbb{R}^{2n}, D_{\gamma_1}(\mu))$ and $(\mathbb{R}^{2n}, D_{\gamma_2}(\mu))$ are homeomorphic. In particular, all the standard discriminants D_γ are homeomorphic.

2) Find the projective analog of Schoenberg's inequality for the convex hull of a convex curve, i.e. consider the Fubini-Study metric on $\mathbb{P}^{2n}$ instead of the Eucledian structure.

3) Describe the essential set $\mathbf{Ess}_\gamma(\mu)$ for all μ and obtain a volume formula for all $\mathcal{YH}_\gamma(\mu)$.

4) Find volume formulas for $\mathcal{YH}_\gamma(\mu)$ for the case when the target space is P^n with the standard Fubini-Study metrics;

5) Find volume formulas for the dual domains $\mathcal{YH}^*_\gamma(\mu)$, i.e. for the convex hulls of different strata of $\mathbf{CSk}_\gamma$.

6) Find volume formulas of other components of the complement to $D_\gamma(\mu)$, say, for the standard discriminant D_γ in the Fubini-Stidy metric.

7) The final goal is to prove the conjecture formulated in the introduction that fixing the length of γ one gets the maximal volume of any $\mathcal{YH}_\gamma(\mu)$ only if γ is the generalized normalized ellipse.

References

[Ar] V. I. Arnold, *On the number of flattening points on space curves*, preprint of the Mittag-Leffler Institute (1994/95), no. 1, 1–13.

[Co] W. A. Coppel, *Disconjugacy*, vol. 220, Lecture Notes in Mathematics, 1971.

[Ga] F. R. Gantmaher, *On nonsymmetric Kellog's kernels*, Soviet Doklady **1** (1936), no. 10, 1–3.

[KN] M. G. Krein and A. A. Nudel'man, *Problem of Markov's moments and extremal problems*, Nauka, 1973, pp. 373.

[KSh] B. A. Khesin and B. Z. Shapiro, *Nondegenerate curves on* S^2 *and orbit classification of the Zamolodchikov algebra*, Comm. Math. Phys **145** (1992), 357–362.

[Kr] M. G. Krein, *Oscillation theorems for linear ODE of arbitrary order*, Soviet Doklady **25** (1939), no. 9, 717–720.

[Li] J. Little, *Nondegenerate homotopies of curves on the unit 2-sphere*, J. Diff. Geom **4** (1970), no. 3, 52–69.

[Nu] A. A. Nudelman, *Isoperimetric problems for convex hulls of piecewise linear and smooth curves in multidimensional spaces*, Mat. Sbornik **96** (1975), no. 2, 294–313.

[Po] G. Polya, *On the mean-value theorem corresponding to a given linear homogeneous equation*, Trans. of the AMS **24** (1924), no. 4, 312–324.

[Sch] I. J. Schoenberg, *An isoperimetric inequality for closed curves in even-dimensional Eucledian spaces*, Acta Mathematica **91** (1954), 143–164.

[BSh] B. Z. Shapiro, *Space of linear differential equations and flag manifolds*, Math. USSR - Izv. **36** (1991), no. 1, 183–197.

[Shr] T. L. Sherman, *Conjugate points and simple zeros for ordinary linear differential equations*, Trans. of the AMS **146** (1969), no. 2, 397–411.