**The hypoelliptic Laplacian**

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A differential operator is said to be elliptic if its principle symbol is invertible. An 'hypoelliptic operator' is usually an operator which is not elliptic, but which is almost as good as an elliptic operator from the point of view of the regularity of the solutions of the corresponding partial differential equations.

The purpose of my talk was to introduce a deformation of classical Hodge theory, which interpolates between classical Hodge theory and the geodesic flow. The corresponding Laplacian is an hypoelliptic operator of the cotangent bundle of the considered manifold. The construction has been announced in the notes [1, 2, 3, 4] and developed in the paper [5]. Crucial analytic results have been obtained jointly with Lebeau in [6].

Let $X$ be a compact manifold of dimension $n$, and let $\left(F, \nabla^F, g^F\right)$ be a complex flat vector bundle on $X$. Let $\left(\Omega^{\cdot}\left(X,F\right), d^X\right)$ be the de Rham complex of smooth forms on $X$ with coefficients in $F$, whose cohomology is denoted $H^{\cdot}\left(X,F\right)$.

Let $g^{TX}$ be a Riemannian metric on $TX$, let $g^F$ be a Hermitian metric on $F$. Then $\Omega^{\cdot}\left(X,F\right)$ is equipped with a corresponding $L_2$ Hermitian product. Let $d^{X*}$ be the formal adjoint of $d^X$. Let $\square^X$ be the associated Laplacian,

$$\square^X = d^X d^{X*} + d^{X*} d^X. \tag{1}$$

The Laplacian is an elliptic self-adjoint operator. Let $\mathcal{H} = \ker \square^X$ be the harmonic forms. Then the basic result of Hodge theory asserts that $\mathcal{H} \simeq H^{\cdot}\left(X,F\right)$.

Let $f : X \to \mathbf{R}$ be a smooth Morse function. In [14], Witten has introduced a deformation of Hodge theory. For $T \in \mathbf{R}$, set $d^X_T = e^{-Tf} d^X e^{Tf}$. Let $d^{X*}_T = e^{Tf} d^{X*} e^{-Tf}$ be the formal adjoint of $d^X_T$, and let $\square^X_T$ be the corresponding Laplacian. Set $\mathcal{H}_T = \ker \square^X_T$. Still, $\mathcal{H}_T \simeq H^{\cdot}\left(X,F\right)$. As $T \to +\infty$, all the eigenvalues except a finite family of them tend to $+\infty$, the other eigenvalues are 0 or are exponentially small as $T \to +\infty$. Let $F_T$ be the finite dimensional complex of eigenbundles associated to small eigenvalues. In [14], Witten showed that $F_T$ localizes near the critical points of $f$, which is enough for a proof of the Morse inequalities. Assume that $\nabla f$ is Morse-Smale. Witten argues that $F_T$ converges in the appropriate sense to the combinatorial Thom-Smale complex associated to $\nabla f$. This was proved rigorously by Helffer-Sjöstrand [9]. The Witten deformation was used in [7] to establish the equality of the Ray-Singer and Reidemeister torsions.

We tried to adapt the above formalism to the loop space $LX$ of $X$. On one hand, $LX$ does not have a Hodge theory, in particular because of the lack of a satisfactory $L_2$ scalar product on the de Rham complex. On the other hand, $LX$ carries many natural $S_1$-invariant functionals associated to Lagrangians $L\left(x,\dot{x}\right)$. Prominent among this, there is the energy functional $E\left(x\right) = \frac{1}{2}\int_0^1 \left|\dot{x}\right|^2 dt$. Morse theory has been used successfully on $LX$, in particular by Bott [8] in his proof of Bott periodicity.

Our strategy consists in trying to make sense of the small 'time' asymptotics of the heat kernel $e^{-s\square^{LX}}\left(x,x\right)$ on the diagonal (this heat kernel does not make sense anyway...), by describing it in terms of classical partial differential operators on $T^*X$. To avoid the singularity as $s \to 0$ of the heat kernel, one only considers index theory supertraces, for which such a singularity does not occur.

The functional integral approach is most useful. Indeed let $F = \int_0^1 f(x_t)dt$ be the obvious lift of $f$ to a $S_1$-invariant function on $LX$. Then at least formally, localization of certain eigenforms near the critical points of $f$ as $T \to +\infty$ can be properly understood via the pull-back by $\nabla F$ of the Mathai-Quillen forms [11] of $TLX$ . These are Gaussian shaped closed differential forms of degree $n$, which represent canonically the Thom class of $TLX$.

The idea is now to replace $F$ by $E$. Note that

$$\nabla E = -\ddot{x}. \tag{2}$$

The path integral to be considered takes the form

$$\int_{LX} \exp\left(-\frac{1}{2}\frac{\int_0^1 |\dot{x}|^2 \, dt}{2} - \frac{T^2}{2}\int_0^1 |\ddot{x}|^2 \, dt + \ldots\right). \tag{3}$$

The expression $\ldots$ consists of differential forms on $LX$.

The dynamic interpretation of (3) just says that

$$\dot{x} = p, \qquad \dot{p} = \frac{1}{T}\left(-p + \dot{w}\right), \tag{4}$$

which is equivalent to

$$\ddot{x} = \frac{1}{T}\left(-\dot{x} + \dot{w}\right). \tag{5}$$

In (4), (5), $w$ is a standard Brownian motion along the fibres of $TX$. The second order differential operator on $T^*X$ which describes the dynamic in (4), (5) is given by

$$\frac{1}{2T^2}\left(-\Delta^V + |p|^2 - n\right) + \frac{1}{T}\nabla_{Y^{\mathcal{H}}}. \tag{6}$$

In (6), $\nabla_{Y^{\mathcal{H}}}$ is the Hamiltonian vector field on $T^*X$ associated to the Hamiltonian $\mathcal{H} = \frac{1}{2}|p|^2$, i.e. the generator of the geodesic flow.

Our problem can then be reformulated as follows. Is there a natural deformation of classical Hodge theory, whose Laplacian on $T^*X$ would 'look like' the operator in (6)? The answer to this question is positive. Put $c = 1/T^2$. Let $\pi : T^*X \to X$ be the canonical projection. Let $\omega$ be the symplectic form of $T^*X$. Let $\eta$ be the bilinear form on $T^*X$,

$$\eta\left(U, V\right) = \left\langle \pi_* U, \pi_* V\right\rangle_{g^{TX}} + \omega\left(U, V\right). \tag{7}$$

This bilinear form induces a corresponding bilinear form on $\Omega^{\cdot}\left(T^*X, \pi^*F\right)$. Then we take the adjoint $\overline{d}^{T^*X}_{\phi,\mathcal{H}^c}$ one obtains with respect to this bilinear form, while making a Witten twist with respect to $\mathcal{H}^c = c\mathcal{H}$. The corresponding Laplacian is indeed of the type (6). It is not self-adjoint, and not elliptic. Still it is hypoelliptic by a key result of Hörmander [10]. This Laplacian is indeed self-adjoint with respect to a Hermitian form of signature $(\infty, \infty)$. It interpolates between classical Hodge theory for $c \to +\infty$, and the generator of the geodesic flow for $c \to 0$. It has a number of analytical properties described in joint work with Lebeau [6]. Using a pseudodifferential calculus adapted to the situation, one shows in that the hypoelliptic Laplacian has a smooth heat kernel, that its spectrum is discrete and conjugation-invariant. Also the basic conclusions of classical Hodge theory still hold, except maybe for a discrete family of values of $c$.

Moreover it is shown in [6] that the hypoelliptic Laplacian also has an analytic torsion. By using methods of Quillen [12], this defines in turn a generalized Hermitian metric $\| \, \|^{2,T}_{\lambda}$ on $\lambda = \det H^{\cdot}\left(X, F\right)$. On the other hand let $\| \, \|^0_{\lambda}$ be the classical

Ray-Singer metric [7] on $\lambda$, which is defined via the Ray-Singer analytic torsion of [13]. One key result in [6] is as follows.

**Theorem.** For $T > 0$,

$$\| \, \|_{\lambda}^{T,2} = \| \, \|_{\lambda}^{0,2} . \tag{8}$$

## References


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